Example 6.21 Use power series to evaluate $\lim_{x\to 0} \dfrac{1-\cos x}{1+x-e^x}$.

Solution.

$$\begin{aligned}
&\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x} \\
=\ &\lim_{x\to\infty} \frac{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)}{1+x-\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)} \\
=\ &\lim_{x\to\infty} \frac{\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\cdots}{-\frac{x^2}{2!}-\frac{x^3}{3!}-\frac{x^4}{4!}-\cdots} \\
=\ &\lim_{x\to 0} \frac{x^2\left(\frac{1}{2!}-\frac{x^2}{4!}+\frac{x^4}{6!}-\cdots\right)}{x^2\left(-\frac{1}{2!}-\frac{x}{3!}-\frac{x^2}{4!}-\cdots\right)} \\
=\ &\lim_{x\to 0} \frac{\left(\frac{1}{2!}-\frac{x^2}{4!}+\frac{x^4}{6!}-\cdots\right)}{\left(-\frac{1}{2!}-\frac{x}{3!}-\frac{x^2}{4!}-\cdots\right)} \\
=\ &\frac{\frac{1}{2!}}{-\frac{1}{2!}} \\
=\ &-1.
\end{aligned}$$

□

Some comments on Example 6.18.

Using Theorem 7 (The Binomial Series), we have already determined that

$$\frac{1}{\sqrt{4-x}} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!8^n}x^n$$

$$= \sum_{n=0}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!2^{3n+1}}x^n. \tag{1}$$

Strictly speaking, we cannot substitute $n=0$ into $\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!2^{3n+1}}$ to get the coefficient of x^n because the term $2n-1$ would be negative.
Actually the first term (the constant term) on the right hand side is just the value of the function at $x=0$:

$$\frac{1}{\sqrt{4-0}} = \frac{1}{2}.$$

This comes from the definition of the Maclaurin (Taylor series), where the constant term is just $f(0)$ ($f(a)$ in the case of Taylor series centred at a).

Alternatively, we can find this from the binomial series with $k = -\frac{1}{2}$, and with x replaced by $-\frac{x}{4}$ that we used to derive (1). There, we have

$$\frac{1}{\sqrt{4-x}} = \frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(-\frac{x}{4}\right)^n = \frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-1)^n\frac{x^n}{4^n}. \tag{2}$$

The coefficient of x^0 can be read off from the above:

$$\frac{1}{2}\binom{-\frac{1}{2}}{0}(-1)^0\frac{1}{4^0} = \frac{1}{2},$$

since by convention $\binom{k}{0} = 1$ (recall that this is just the first term in the Binomial series $(1+x)^k$.

More comments on Example 6.18

From the solution of Example 6.18, the following Maclaurin series converges on the open interval $(-4, 4)$.

$$\sum_{n=0}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!2^{3n+1}}x^n. \tag{3}$$

The convergence of the series at the endpoints $x = 4, -4$ is still unknown. In general, we deal with this on a case-by-case basis.

REMARK: The following manipulations in bounding a_n are highly nontrivial and will NOT be tested.

Let us now examine the convergence of this series at the endpoints $x = 4, -4$.

Case 1: At $x = -4$, the series becomes

$$\sum_{n=0}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!2^{3n+1}}(-1)^n 4^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n! 2^{n+1}}.$$

Let $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n! 2^{n+1}}$.

Note that

$$\begin{aligned}
0 \leq a_n &= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n! 2^{n+1}} \\
&= \frac{1}{1}\frac{3}{2}\frac{5}{3} \cdots \frac{2n-1}{n} \cdot \frac{1}{2^{n+1}} \\
&= \frac{1}{2(1)}\frac{3}{2(2)}\frac{5}{2(3)} \cdots \frac{2n-1}{2(n)} \cdot \frac{1}{2} \\
&= \left(\prod_{i=1}^{n} \frac{2i-1}{2i} \right) \cdot \frac{1}{2} \\
&= \frac{1}{2} \prod_{i=1}^{n} \left(1 - \frac{1}{2i} \right). \qquad (4)
\end{aligned}$$

Note that

$$\begin{aligned}
\left(\prod_{i=1}^{n} \left(1 + \frac{1}{2i} \right) \right)^2 &= \prod_{i=1}^{n} \left(1 + \frac{1}{2i} \right)^2 \\
&= \prod_{i=1}^{n} \left(1 + \frac{1}{i} + \frac{1}{4i^2} \right) \\
&\geq \prod_{i=1}^{n} \left(1 + \frac{1}{i} \right) \\
&= \prod_{i=1}^{n} \frac{i+1}{i} \\
&= \frac{2}{1}\frac{3}{2}\frac{4}{3} \cdots \frac{n+1}{n} \\
&= n+1.
\end{aligned}$$

Taking square roots on both sides:

$$\prod_{i=1}^{n} \left(1 + \frac{1}{2i} \right) \geq \sqrt{n+1}. \qquad (5)$$

On the other hand,

$$\begin{aligned}
\prod_{i=1}^{n}\left(1-\frac{1}{2i}\right)\left(1+\frac{1}{2i}\right) &= \prod_{i=1}^{n}\left(1-\frac{1}{4i^2}\right) \\
&= \left(1-\frac{1}{4}\right)\prod_{i=2}^{n}\left(1-\frac{1}{4i^2}\right) \\
&\leq \frac{3}{4}. \qquad (6)
\end{aligned}$$

It follows from (5) that

$$\begin{aligned}
\sqrt{n+1}\cdot\prod_{i=1}^{n}\left(1-\frac{1}{2i}\right) &\leq \prod_{i=1}^{n}\left(1-\frac{1}{2i}\right)\prod_{i=1}^{n}\left(1+\frac{1}{2i}\right) \\
&= \prod_{i=1}^{n}\left(1-\frac{1}{2i}\right)\left(1+\frac{1}{2i}\right) \\
&\leq \frac{3}{4} \\
\prod_{i=1}^{n}\left(1-\frac{1}{2i}\right) &\leq \frac{3}{4\sqrt{n+1}} \qquad (7)
\end{aligned}$$

Substituting (7) into (4):

$$0 \leq a_n \leq \frac{1}{2}\cdot\frac{3}{4\sqrt{n+1}}.$$

By Squeeze Theorem, we deduce that

$$\lim_{n\to\infty} a_n = 0.$$

Also, the sequence $\{a_n\}$ is decreasing:

$$a_{n+1} \leq a_n$$

$$\begin{aligned}
&\iff \frac{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}{(n+1)!2^{n+2}} \leq \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!2^{n+1}} \qquad (8) \\
&\iff 2n+1 \leq 2(n+1) \qquad (9)
\end{aligned}$$

which is true for all $n \geq 0$.

Hence, by the Alternating Series Test, the series

$$\sum_{n=0}^{\infty}(-1)^n a_n = \sum_{n=0}^{\infty}(-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n! 2^{n+1}}$$

converges.

Case 2: At $x = 4$, the series becomes

$$\sum_{n=0}^{\infty} a_n,$$

where, as before,

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n! 2^{n+1}}.$$

Now,

$$\begin{aligned}
a_n &= \frac{3}{1}\frac{5}{2}\cdots\frac{2n-1}{n-1}\cdot\frac{1}{n}\cdot\frac{1}{2^{n+1}} \\
&= \frac{3}{2(1)}\frac{5}{2(2)}\cdots\frac{2n-1}{2(n-1)}\cdot\frac{1}{n}\cdot\frac{1}{2^2} \\
&= \left(\prod_{i=1}^{n-1}\frac{2i+1}{2i}\right)\cdot\frac{1}{4n} \\
&= \frac{1}{4n}\left(\prod_{i=1}^{n-1}\left(1+\frac{1}{2i}\right)\right) \\
&\geq \frac{1}{4n}\sqrt{n},
\end{aligned}$$

where the last inequality follows from (5). Thus,

$$a_n \geq \frac{\sqrt{n}}{4n} = \frac{1}{4\sqrt{n}}.$$

The series $\sum \frac{1}{4\sqrt{n}} = \frac{1}{4}\sum \frac{1}{n^{1/2}}$ diverges (this is a p-series with $p \leq 1$). So, by the Comparison Test, the series $\sum a_n$ diverges.