A note on the statistical meaning of *f*.....

First define the *exact* distribution describing state of a system of N particles at time t (occupation of states in phase space) as:

$$F(\mathbf{r},\mathbf{v},t) = \sum_{i=1}^{N} \delta(\mathbf{r}-\mathbf{r}_i(t)) \cdot \delta(\mathbf{v}-\mathbf{v}_i(t)),$$

If we know exactely what is where to every single time -> which is not possible with our current measurement methods,
Also we observe everything with a time delay from the travel time of the light carrying the information - So what is further away is more in the past then what is close, so this is not even with a perfect measurement device this is not possible

Then *f* the ensembled-average distribution function, namely the exact distribution function weighted by the probability density of states in phase space:

Therefore we look at a statistical average -> weighted probability
f is the ensemble average -> excat distribution funktion weighted of the probability density of state in space

$$f_1(\mathbf{r},\mathbf{v},t) = \langle F(\mathbf{r},\mathbf{v},t)\rangle = \int F \cdot p \cdot \mathrm{d}\mathbf{r}_1\, \mathrm{d}\mathbf{r}_2 \cdots \mathrm{d}\mathbf{r}_N\, \mathrm{d}\mathbf{v}_1\, \mathrm{d}\mathbf{v}_2 \cdots \mathrm{d}\mathbf{v}_N$$

This is important to remember as it means, in practice, that *f* can be correctly evaluated only after considering many realizations of a system of *N* particles and then taking their weighted average in phase space. We will return to this when dealing with numerical models

This is what we will actually use -> all functions will use this ensemble average since this is just not possible

f can be measured is is most often known

The dynamical equations obeyed by f follow from the notion of conservation of probability density in phase space (formally from Liouville's equation in statistical mechanics).

Conservation of probability in phase space can be understood less formally with the analogy with mass conservation in fluid flow. For an arbitrary volume V in phase space we can define the probability P of finding a particle in V as:

The stars in the galaxy will move -> we can measure the position and velocity at different times. But we know that the stars in the galaxy will stay in the system. Therefore a velocity of the stars in the galaxy is bounded (some range of reasonable values). We also know a system with arbitrary values of positions and system, we can always define a system large enough -> for these positions to always still be in this volume.

$$P = \int_V d^6\mathbf{w}\, f(\mathbf{w}).$$

$\mathbf{w}=(\mathbf{r},\mathbf{v})$

If we now integrate our f over this arbitrary volume, we get the Probability of finding a star in the arbitrary volume -> it has to be conserved so P does not change, which implies the equation below.

While f can evolve P must be conserved, which can then be expressed with a continuity equation *in phase space*:

Conservation law:
Stars can not leave this arbitrary volume -> basically conservation of celestial bodies.

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{w}} \cdot (f \dot{\mathbf{w}}) = 0.$$

This equation follows from dP/dt = 0

This is not important for all -> only for the people who know Hamiltons Equations

Since the collisionless systems that we will model (gravitational) will also be hamiltonian, we can use Hamilton's equations to reformulate in *p, q* coordinates:

r,v --> q,p

w= (q,p)

$$
\begin{aligned}
\frac{\partial}{\partial \mathbf{q}} \cdot (f\dot{\mathbf{q}}) + \frac{\partial}{\partial \mathbf{p}} \cdot (f\dot{\mathbf{p}}) &= \frac{\partial}{\partial \mathbf{q}} \cdot \left(f\frac{\partial H}{\partial \mathbf{p}}\right) - \frac{\partial}{\partial \mathbf{p}} \cdot \left(f\frac{\partial H}{\partial \mathbf{q}}\right) \\
&= \frac{\partial f}{\partial \mathbf{q}} \cdot \frac{\partial H}{\partial \mathbf{p}} - \frac{\partial f}{\partial \mathbf{p}} \cdot \frac{\partial H}{\partial \mathbf{q}} \\
&= \dot{\mathbf{q}} \cdot \frac{\partial f}{\partial \mathbf{q}} + \dot{\mathbf{p}} \cdot \frac{\partial f}{\partial \mathbf{p}},
\end{aligned}
$$

This is the new equation, if we want to use p,q from Hamiltons Equations -> equivalent to what we had before

In the last line one uses the fact that $\partial^2 H/\partial \mathbf{q}\partial \mathbf{p} = \partial^2 H/\partial \mathbf{p}\partial \mathbf{q}$

Rewriting the second term in the left-hand side of the continuity equation using the above result we obtain the Vlasov equation (collisionless Boltzmann):

$$\frac{\partial f}{\partial t} + \dot{\mathbf{q}} \cdot \frac{\partial f}{\partial \mathbf{q}} + \dot{\mathbf{p}} \cdot \frac{\partial f}{\partial \mathbf{p}} = 0, \longrightarrow \frac{df}{dt} = 0$$

(lagrangian derivative)

All three of those derivatives implicate the right one -> distribution function of particles does not change across a lagrangian trajectory

which is a partial differential equation for f as a function of the six phase space coordinates plus time. Question: since we said we discard particle-particle interactions what is the acceleration due to here?

The answer is apparent if we realize that individual particles will still move under the action of the mean field produced by the self-gravity of the galaxy, or the charge distribution in a plasma. In the case of gravitational system we can thus state that what we want to solve is the coupled Vlasov-Poisson system of partial differential equations (PDEs).

Let's first define the mass density of a system of N particles with equal mass m (just for simplicity) as:

only integral about the velocity part of the phase space, then this is equal to the density, if we don't multiply for with the mass, then this is the function density. If all have the same mass then this is the mass density

$$\rho(\mathbf{r},t) = m \int f(\mathbf{r},\mathbf{v},t)\,\mathrm{d}\mathbf{v}.$$

Then the two equations that we want to solve together are:

Then we can write this as a poisson equation using this distribution function

Now we have a system of equations that describe our system. One is the poisson equation and one is the collisionless equation from before

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{r}} - \frac{\partial \Phi}{\partial \mathbf{r}}\frac{\partial f}{\partial \mathbf{v}} = 0,$$

$$\nabla^2 \Phi = 4\pi G m \int f(\mathbf{r},\mathbf{v},t)\,\mathrm{d}\mathbf{v}.$$

The stars are not interacting with each other. Their momement is more directed by the generall gravitational equation

with the acceleration in the Vlasov equation being given by:

$$\mathbf{a} = -\frac{\partial \Phi}{\partial \mathbf{r}}$$

Simply use the total gravitational field of the system -> here is why we are allowed to do this -> collisionless as we don't care about the interaction between the stars

Physical regime of collisionless systems

Physically treating the system as collisionless bears meaning only as long as the potential energy of interaction (eg gravitational binding energy) between individual particles undergoing encounters is much smaller than their kinetic energy, which is equivalent to say that they move under the action of the mean field and not the mutual interaction.

This yields a characteristic distance scale *a* below which the collisionless approximation brakes down. *a* is defined by equating potential and kinetic/thermal energy and must be compared with the typical interparticle separation. So in plasma and galaxies, for example ratios between such two energies are:

$$a_p = \frac{e^2}{4\pi\epsilon_0 k_B T} \qquad a_g = \frac{Gm^2}{\frac{1}{2}mv^2}$$

$$\epsilon_p = a_p n^{1/3} \qquad \epsilon_g = a_g \sigma^{1/2}$$

If these ratios are << 1 then the collisionless approximation works well. (for galaxies we use surface number density σ assuming a flat disk-like configuration)

sigma is the surface density of the 2D system if we ignore vertical expansion

If we equate gravitational and centrifugal acceleration at the edge of the disk we can eliminate v and get

$$a_g \sim (\sigma R)^{-1}$$

Critical distance where E_kin = E_pot an for a 2d stellar disk with surface density sigma

----> $\epsilon_g \sim (\sigma R^2)^{-1/2}$ $\sim 10^{-5} << 1$

A more rigorous criterion deals with timescales, highlighting the notion that a system may or may not be treated as collisionless depending on the timescale over which its evolution is considered.
This is particularly useful in astrophysics where we often deal with timescales of processes and compare them to the age of the Universe (~13.7 billion years). This brings in the notion of *relaxation time* (see Springel's lecture notes pag. 8-10 for derivation):

$$t_{\rm relax} = \frac{N}{8 \ln N} t_{\rm cross}$$

where $t_{cross} \sim R/V_p$ is the typical timescale required by a particle to travel through the system on a straight trajectory and $\Lambda = b_{max}/b_{min}$ is the ratio between largest and smallest impact parameters in particle encounters
The relaxation time expresses the timescale over which the deflection of particle trajectories due to the cumulative gravitational effect of encounters with other particles becomes significant ---> *orbits of particles do not reflect anymore the action of the mean potential* Φ *for* $t > t_{relax}$.
An astrophysical system will be collisionless if its relaxation time is (much) longer than the age of the Universe. As it can be seen, this is a function of *N*, a notion that will extend to superparticle models of physical systems.

()For a galaxy N ~ 10^{11}, t_{cross} ~ ~ 10^8 yr ~ Age of the Universe/100

()For a star cluster (eg Globulal Cluster) clearly the approximation does not work: N ~ 10^5-10^6, t_{cross} ~ 1 Myr ~ Age of the Universe/10^4 !

()For a halo of (cold) dark matter particles surrounding a typical galaxy, assuming the elusive particles have a mass of ~ 100 GeV, since astronomical measurements tell us the halo should have a mass of ~ 10^{12} Mo it follows that N ~ 10^{77}, and t_{cross} ~ Age of the Universe/10 (a dark halo is ~ 10 times larger than its embedded galaxy albeit characteristic velocities are comparable)
----> *a dark matter halo is the ideal case of a collisionless system.*

Weakly Correlated and Strongly Correlated (Fluid) systems

Before we move on with discretization and numerical methods for collisionless systems it is instructive to discuss the more general cases, namely when interactions between particles cannot be neglected anymore. The numerical modeling of such systems will be discussed later in the course.

Let us start from weakly correlated systems, namely those for which a small collisional term must be introduced but the wavelength of the collisional processes is still large compared to the characteristic size of the system (or the relaxation time is still relatively long).

$$f\,(\mathbf{r},\mathbf{v},\mathbf{r}',\mathbf{v}', t) = f_1'\,(\mathbf{r},\mathbf{v},t)\; f_2'\,(\mathbf{r}',\mathbf{v}',t) + g\,(\mathbf{r},\mathbf{v},\mathbf{r}',\mathbf{v}',t)$$

This would be the case for star clusters in astrophysics or weakly collisionless plasma where electromagnetic forces between individual ions cannot be discarded.

The Vlasov equation becomes thus the generalized Boltzmann equation, which in lagrangian form is simply:

$$\frac{df}{dt} = -\int \frac{\mathbf{F}}{m} \cdot \frac{\partial}{\partial \mathbf{v}} g \, d\mathbf{x}' \, d\mathbf{v}' = \left(\frac{\partial f}{\partial t}\right)_c$$

This approximation is used a lot in astroparticle physics and Big Bang cosmology, where g is complex and contains various types of interactions between particles in the standard model of physics. It can also be used to described photons and their interaction with matter, which leads to the radiative transfer equation (we will get back to this at the end of the course)

If the interactions are happening on very short timescales, or the wavelength associated with them is much smaller than the size of the system (which can also happen in photon gases, for example when absorption/scattering is very efficient), then one can move to a moment-based approximation to reduce the dimensionality of the problem and make the calculation computationally feasible.

The Fluid Approximation

This is a moment-based approach to the Boltzmann equation in which the interaction wavelength is much smaller than the system's size and leads to establish rapidly (and restore rapidly) a thermal equilibrium as described by the Maxwellian distribution function. In its simplest form one neglects viscosity, thermal conduction and shear stresses between particles, which originate mainly from electromagnetic forces *(ideal fluid)*, and considers only *isotropic pressure (perfect fluid)*

One starts from taking velocity moments of the Boltzmann equation.

Zeroth order moment gives continuity equation

First order moment gives momentum equation

Second order moment gives energy equation

Zeroth moment

$$\int d\mathbf{v}\, m\left\{\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}}+\frac{\mathbf{F}}{m}\cdot\frac{\partial f}{\partial \mathbf{v}}\right\}=\int d\mathbf{v}\, m\left(\frac{\partial f}{\partial t}\right)_c$$

with

$$\int d\mathbf{v}\, mf = mn = \rho \qquad \int d\mathbf{v}\, m\mathbf{v}f = \rho\mathbf{u}$$

"mean" fluid velocity

yields continuity equation for f ⟶ 0 as v ⟶ 0 and conservation of particle number, which sets right-end side to zero

$$\frac{\partial \rho}{\partial t}+\nabla\cdot\rho\mathbf{u}=0$$

First moment from integral of continuity equation multiplied by m**v**

$$\frac{\partial}{\partial t}\rho\mathbf{u}+\nabla\cdot[\rho\mathbf{u}\mathbf{u}+\mathbf{P}]-n\mathbf{F}=\int d\mathbf{v}\, m\mathbf{v}\left(\frac{\partial f}{\partial t}\right)_c$$

with pressure tensor

$$\mathbf{P}=\int d\mathbf{v}\, mf\,(\mathbf{v}-\mathbf{u})(\mathbf{v}-\mathbf{u})$$

that would need second moment

but can be reduced by assuming scalar (isotropic) pressure and an associated equation of state (eg ideal gas)

$$\mathbf{P} \approx p\mathbf{I} \qquad p = \rho R T = n k_B T.$$

The two above ansatz correspond to a *perfect ideal fluid*

$$\rho \frac{d\mathbf{u}}{dt} = \rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + n\mathbf{F} + \int d\mathbf{v}\, m\mathbf{v}\left(\frac{\partial f}{\partial t}\right)_c$$

Assuming a maxwellian for the distribution function (=thermodynamical equilibrium) the above equation reduces to the well known Euler equation for perfect inviscid fluids since the rightmost term cancels out (the Maxwellian does not depend on time!)

Most astrophysical systems that cannot be described by the collisionless or (weakly) collisional approximation can be well treated as ideal perfect fluids.
The approximation can be extended also to charged fluids with magnetic fields for which the magnetohydrodynamical equations can be derived by coupling fluid and Maxwell's equations. *Examples are*:

Stars, accretion discs around stars and black holes, planets, stellar explosions, interstellar gas in galaxies, star forming clouds....and many others!

Now that you should be convinced that a "Boltzmann-type" equation is the general mathematical model applicable to nearly all astrophysical systems we can finally start discussing numerical discretization of the equations and their integration.

For collisionless systems, rather than solving the Vlasov-Poisson system in the "mean field" regime we will forcefully recover the particle nature of the system by discretizing using superparticles. The coarse discretization of the f will make sense as long as the relaxation timescale in the discretized representation of the system is still long enough to enforce the collisionless behaviour,
---> *N of superparticles should be large enough*!

N-Body models of collisionless systems

In a gravitational system one solves a simpler, alternative system of ordinary differential equations for the mean potential Φ rather than solving directly the Vlasov-Poission equation. The resulting trajectories in phase space should coarsely sample *f* as long as the relaxation time of the N-Body model is long compare to the duration of the simulation. However a single N-Body experiment with a computer will only represent a noisy representation of *f* because no ensemble averaging is performed (one should run many experiments and then average out the resulting forces to get trajectories in the mean potential)

$$\ddot{\mathbf{x}}_i = -\nabla_i \Phi(\mathbf{r}_i),$$

$$\Phi(\mathbf{r}) = -G \sum_{j=1}^{N} \frac{m_j}{[(\mathbf{r}-\mathbf{r}_j)^2 + \varepsilon^2]^{1/2}}$$

The gravitational softening ε is introduced for both computational efficiency and to enforce numerical robustness (=smoothness) of the model for the collisionless fluid. To avoid correlation between particles

$$\langle v^2 \rangle \gg \frac{Gm}{\varepsilon}$$