CRACK JEE

2019 - 2020

MATHEMATICS

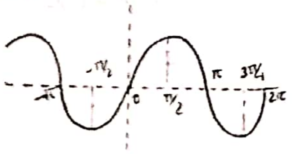
HANDWRITTEN REVISION NOTES

FOR JEE MAIN 2020

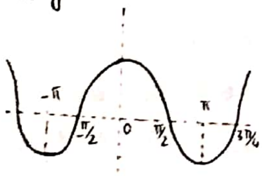
Trigonometry

Graphs:

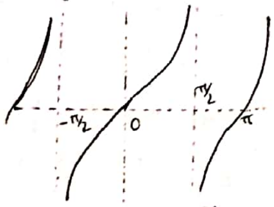
- $\sin x = y$



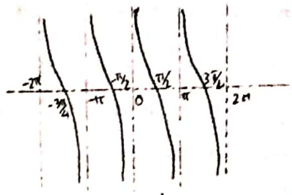
- $y = \cos x$



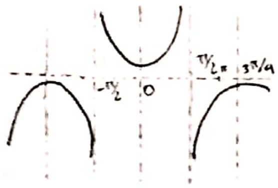
- $y = \tan x$



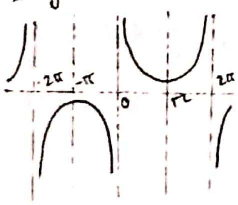
- $y = \cot x$



- $y = \sec x$



- $y = \operatorname{cosec} x$



Sum and Difference of two angles:

- $\sin(A+B) = \sin A\cos B + \cos A\sin B$
- $\sin(A-B) = \sin A\cos B - \cos A\sin B$
- $\cos(A+B) = \cos A\cos B - \sin A\sin B$
- $\cos(A-B) = \cos A\cos B + \sin A\sin B$
- $\tan(A\pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$
- $\cot(A\pm B) = \dfrac{\cot A\cot B \mp 1}{\cot A + \cot B}$

Imp Results:

- $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \sin^2 A$
- $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$
- $\sin(A+B+C) = \sin A\cos B\cos C + \cos A\sin B\cos C + \cos A\cos B\sin C - \sin A\sin B\sin C$
- $\cos(A+B+C) = \cos A\cos B\cos C - \cos A\sin B\sin C - \sin A\cos B\sin C - \sin A\sin B\cos C$
- $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A}$

Range of $f(\theta) = a\sin\theta + b\cos\theta$

$$-\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}$$

Product into sum and difference

- $2\sin A\sin B = \sin(A+B) + \sin(A-B)$
- $2\cos A\sin B = \sin(A+B) - \sin(A-B)$
- $2\cos A\cos B = \cos(A+B) + \cos(A-B)$
- $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

Multiple angles

- $\sin 2A = 2\sin A\cos A = \dfrac{2\tan A}{1+\tan^2 A}$
- $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1 = \dfrac{1-\tan^2 A}{1+\tan^2 A}$
- $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$
- $\sin 3A = 3\sin A - 4\sin^3 A$
- $\cos 3A = 4\cos^3 A - 3\cos A$
- $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$
- $\tan^2 A = \dfrac{1-\cos 2A}{1+\cos 2A}$
- $\cot 3A = \dfrac{\cot^3 A - 3\cot A}{3\cot^2 A - 1}$

Sum or difference into product.

- $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$
- $\sin C - \sin D = 2\sin\left(\frac{C-D}{2}\right)\cos\left(\frac{C+D}{2}\right)$
- $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$
- $\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

Special Trios:

- $\cos A\cos(60-A)\cos(60+A) = \dfrac{\cos 3A}{4}$
- $\sin A\sin(60-A)\sin(60+A) = \dfrac{\sin 3A}{4}$
- $\tan A\tan(60-A)\tan(60+A) = \tan 3A$

AP of Angles:

- $\sin\alpha + \sin(\alpha+\beta) + \dots + \sin[\alpha+(n-1)\beta] = \dfrac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\left[\sin\left(\alpha + \frac{(n-1)\beta}{2}\right)\right]$
- $\cos\alpha + \cos(\alpha+\beta) + \cos[\alpha+2\beta] + \dots + \cos[\alpha+(n-1)\beta] = \dfrac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cos\left[\alpha + \frac{(n-1)\beta}{2}\right]$

Conditional Identities $(A+B+C = \pi)$

- $\tan A + \tan B + \tan B = \tan A\tan B\tan C$
- $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$
- $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$
- $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
- $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

GP of Angles:

$$\cos A\cos 2A\cos 2^2A\cos 2^3A \cdots \cos 2^{n-1}A = \frac{\sin 2^n A}{2^n\sin A}$$

$A+B+C = \frac{\pi}{2}$

- $\sin^2 A + \sin^2 B + \sin^2 C = 1 - 2\sin A\sin B\sin C$
- $\cos^2 A + \cos^2 B + \cos^2 C = 2 + 2\sin A\sin B\sin C$

Inequalities: (ΔABC)

- $\tan A + \tan B + \tan C \ge 3\sqrt{3}$ [All acute]
- $\cos A + \cos B + \cos C \le \frac{3}{2}$
- $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{1}{8}$
- $\sec A + \sec B + \sec C \ge 6$ [Acute]
- $\operatorname{cosec}\frac{A}{2} + \operatorname{cosec}\frac{B}{2} + \operatorname{cosec}\frac{C}{2} \ge 6$

In ΔABC

- $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} \mp \cos^2\frac{C}{2} = 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$
- $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = 2 + 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
- $\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \le \frac{3}{2}$
- $\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} \ge 1$

- $\tan 3A - \tan 2A - \tan A = \tan 3A\tan 2A\tan A$

Progression & Series

- Arithemic Progression- $a, a+d, a+2d, \ldots\ldots, a+(n-1)d$ • nth term (General term): $t_n = a+(n-1)d$, $t_n = l-(n-1)d$
- 3 terms in AP consideration - $a-d, a, a+d$ • 4 terms in A.P.- $a-3d, a-d, a+d, a+3d$
- If $a_1, a_2, a_3, \ldots\ldots, a_{n-2}, a_{n-1}, a_n$ are in A.P. , $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \ldots\ldots = a_r + a_{n-r+1}$
- Sum of n terms in an AP: $S_n = \frac{n}{2}[2a+(n-1)d] = \frac{n}{2}[a+l]$ • If $S_n = an^2+bn+c$, $t_n = S_n - S_{n-1}$
- Arithmetic mean: $AM(x_1, x_2) = \frac{x_1+x_2}{2}$, $AM(x_1, x_2, x_3, \ldots, x_n) = \frac{x_1+x_2+x_3+\ldots+x_n}{n}$
- $A_1 + A_2 + \ldots + A_n = n\left(\frac{a+b}{2}\right)$

- Geometric Progression- $a, ar, ar^2, \ldots, a^{n-1}$. • nth term (General term): $t_n = ar^{n-1}$
- Sum of n terms: $S_n = \frac{a(1-r^n)}{1-r}$ $[1>r]$ $= \frac{a(r^n-1)}{r-1}$ $[r>1]$ • If $x_1, x_2, x_3 \ldots$ are in GP, $\log x_1, \log x_2 \ldots$ are in AP
- 3 terms in GP: $\frac{a}{r}, a, ar$ • 4 terms in GP: $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$ • Sum of Infinite GP. $S_\infty = \frac{a}{1-r}$ $[|r|<1]$
- Geometric means: $GM(x_1, x_2) = (x_1 x_2)^{1/2}$, $GM(x_1, x_2, x_3, \ldots, x_n) = (x_1 x_2 x_3 \ldots x_n)^{1/n}$
- $G_1 G_2 G_3 \ldots G_n = (\sqrt{ab})^n$

- Harmonic progression: $a_1, a_2, a_3 \ldots$ are in H.P. if $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots$ are in AP.
- nth term (General term): $t_n = \frac{1}{t_n \text{ of } AP}$ • Harmonic Mean: $HM(x_1, x_2) = \frac{2x_1x_2}{x_1+x_2}$, $HM(x_1, x_2 \ldots x_n) = \frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n}}$

- Inequalities: $A \geq G \geq H$ • $G^2 = AH$

- Special series: $\sum n = \frac{n(n+1)}{2}$, $\sum n^2 = \frac{n(n+1)(2n+1)}{6}$, $\sum n^3 = \frac{n^2(n+1)^2}{4}$

- Arithmetico Geometric Progression (AGP):- $S = a + (a+d)r + (a+2d)r^2 + \ldots$

$$
\begin{aligned}
S &= a + (a+d)r + (a+2d)r^2 + \ldots \\
-\ rS &= \quad\ \ ar \quad + (a+d)r^2 + \ldots \\
\hline
S(1-r) &= a + d(r+r^2+\ldots)
\end{aligned}
$$

- Difference Series:

$$
\begin{aligned}
S &= 1+2+4+7+11+16+\ldots\ldots + t_n \\
-\ S &= \quad 1+2+4+7+11+\ldots\ldots + t_n \\
\hline
0 &= 1+(1+2+3+\ldots\ldots) - t_n
\end{aligned}
$$

(differences: 1, 2, 3, 4, 5)

- $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)(r+3)} = \sum_{r=1}^{n} \frac{1}{3}\left[\frac{(r+3)-r}{r(r+1)(r+2)(r+3)}\right]$ • $\sum_{r=1}^{n} r(r+1)(r+2)(r+3) = \sum_{r=1}^{n} \frac{1}{5}\left((r+4)-(r-1)\right)\left(r(r+1)(r+2)(r+3)\right)$

Weighted mean:

- $\frac{a_1^m + a_2^m + \ldots + a_n^m}{n} > \left(\frac{a_1+a_2+a_3+\ldots+a_n}{n}\right)^m$ If $m<0$ or $m>1$
- $\frac{a_1^m + a_2^m + \ldots + a_n^m}{n} < \left(\frac{a_1+a_2+\ldots+a_n}{n}\right)^m$ If $0<m<1$

Permutation & Combination

- Let p be a given prime and n, any positive integer, Then the maximum power of p present in $n!$ is
$\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]+\cdots$ Where $[\cdot] \rightarrow$ Greatest Integer function
- ${}^nC_r \times r! = {}^nP_r$
- Number of permutations of n different things taken r at a time $\rightarrow {}^nP_r = \frac{n!}{(n-r)!}$
- Number of permutations of n different things taken all at a time $\rightarrow n!$
- Number of permutations of n things [p are alike, q are alike, r are alike] $\rightarrow \frac{n!}{p!\,q!\,r!}$
- Number of combinations (selections) of n different things taking r at a time $\rightarrow {}^nC_r = \frac{n!}{r!(n-r)!}$
- ${}^nC_r = {}^nC_{n-r}$, ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$, $r \cdot {}^nC_r = n\,{}^{n-1}C_{r-1}$, $\frac{{}^nC_r}{r+1} = \frac{{}^{n+1}C_{r+1}}{n+1}$, $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$
- When n is even, max value of ${}^nC_r \longrightarrow {}^nC_{n/2}$
When n is odd, max value of ${}^nC_r \longrightarrow {}^nC_{\frac{n-1}{2}}$ or ${}^nC_{\frac{n+1}{2}}$
- No of ways of arranging n different things in circular manner $\rightarrow (n-1)!$
- When ACW/CW doesn't matter (e.g - necklace, garland), circular arrangement $\rightarrow \frac{(n-1)!}{2}$
- Total no of combination of n things taken 1 or more at a time $\rightarrow {}^nC_1 + {}^nC_2 + {}^nC_3 + \cdots + {}^nC_n = 2^n - 1$
- Total no of selections of n things, [p similar, q similar, r alike] (including 0) $\rightarrow (p+1)(q+1)(r+1)$
- If $N = p_1^a \times p_2^b \times p_3^c \times \cdots$ where $a, b, c, \ldots$ are ~~non~~ non-negative integers, $p_1, p_2, p_3 \ldots$ are prime no.
Then $\rightarrow$
  - Total No of Divisors $= (a+1)(b+1)(c+1)\cdots\cdots$
  - Sum of all divisors $= \left(\frac{p_1^{a+1}-1}{p_1-1}\right)\times\left(\frac{p_2^{b+1}-1}{p_2-1}\right)\times\left(\frac{p_3^{c+1}-1}{p_3-1}\right)\cdots$
  - All the divisors excluding 1 and N are called proper divisors
  - No of ways of writing N as a product of two natural nos $\rightarrow \begin{cases} \left[\frac{1}{2}(a+1)(b+1)(c+1)\cdots\right] & \text{if N isn't a perfect square} \\ \left[\frac{1}{2}(a+1)(b+1)(c+1)\cdots+1\right] & \text{if N is a perfect square} \end{cases}$
  - N is a perfect square if $a, b, c \ldots$ all are even
  - N is a perfect cube if $a, b, c \ldots$ all are multiples of 3.
  - $N = 2^a \times 3^b \times 5^c \times \cdots$
    - If N is odd, $a = 0$, $b, c, d \ldots \geq 0$
    - If N is even, $a \geq 1$, $b, c, \ldots\ldots \geq 0$
- No of Non negative integral solⁿ of the eqⁿ $x_1 + x_2 + x_3 + \cdots + x_r = n$ is $\rightarrow {}^{n+r-1}C_{r-1}$
- No of positive integral solⁿ of the eqⁿ $x_1 + x_2 + x_3 + \cdots + x_r = n$ is $\rightarrow {}^{n-1}C_{r-1}$
- Sum of all n-digit numbers formed using n digits $= (n-1)!\,(\text{Sum of all } n \text{ digits}) \times (\underbrace{111\ldots\ldots1}_{n\text{ times}})$
- No of diagonals of n sided polygon $\rightarrow {}^nC_2 - n = \frac{n(n-3)}{2}$
- No of squares in two system of ~~perpendicular~~ parallel lines (When 1st set contains m lines and 2nd set contains n lines) is equal to $\longrightarrow \sum_{r=1}^{m-1}(m-r)(n-r)$; $(m<n)$

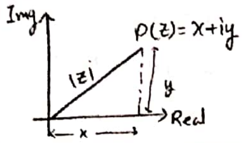
- Dearrangements : No of ways so that no letter goes to the correct address.

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \cdots\cdots + (-1)^n \frac{1}{n!}\right]$$

Complex Number

- $z = x + iy$, $x, y \in R$ and $i = \sqrt{-1}$ ▪ $Re(z) = x$, $Im(z) = y$ ▪ $\sqrt{-a} = i\sqrt{a}$
- The property $\sqrt{a}\sqrt{b} = \sqrt{ab}$ is valid only if at least one of a and b is non negative, if a and b are both negative, then $\sqrt{a}\sqrt{b} = -\sqrt{|a||b|}$
  - $a + ib > c + id$ is meaningful only if $b = d = 0$ ● If $a + ib = c + id$, $a = c$, $b = d$
- In real no system, $a^2 + b^2 = 0$, $a = b = 0$. But $z_1^2 + z_2^2 = 0$ does NOT mean $z_1 = z_2 = 0$.
- $i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$; $i^{4n+1} = i$, $i^{4n+2} = -1$, $i^{4n+3} = -i$, $i^{4n} = 1$
- **Square Root of a Complex No:** $\sqrt{a+ib} = x + iy \Rightarrow a = x^2 - y^2$; $2xy = b$ solve.
  Sign of b decides whether x and y are of same sign or opposite sign.

Modulus of CN: $|z| = r = \sqrt{x^2+y^2}$



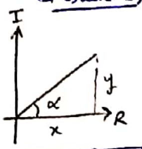
Amplitude of CN

Argument / amplitude of CN, $\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{Im(z)}{Re(z)}\right)$

↑ From the real axis. $arg(z) \in [-\pi, \pi]$

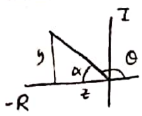
Principle Argument

▪ Quad I / $(x>0, y>0)$



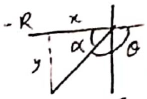
$arg(z) = \alpha$
$= \tan^{-1}\left|\frac{y}{x}\right|$

▪ Qud II / $(x<0, y>0)$



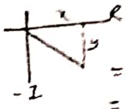
$arg(z) = \theta$
$= \pi - \alpha$
$= \pi - \tan^{-1}\left|\frac{y}{x}\right|$

▪ Qud III / $(x<0, y<0)$



$arg(z) = \theta$
$= \alpha - \pi$
$= \tan^{-1}\left|\frac{y}{x}\right| - \pi$

▪ Qud IV /



$arg(z) = \theta$
$= -\alpha$
$= -\tan^{-1}\left|\frac{y}{x}\right|$

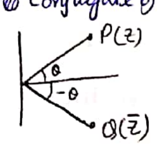
Polar form

$z = x + iy = r(\cos\theta + i\sin\theta)$

Euler's form $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$\therefore e^{i\theta} = \cos\theta + i\sin\theta$

Conjugate of a CN:



$z = x + iy$
$\bar{z} = x - iy$

$z + \bar{z} = 2\,Re(z)$
$z - \bar{z} = 2\,Im(z)$

Properties of Conjugates

- # $\overline{(\bar{z})} = z$ # if $z = \bar{z}$, z is purely real   # $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$
- # $z + \bar{z} = 0$, z is purely imaginary   # $\overline{z^n} = (\bar{z})^n$ # $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$
- # $z \cdot \bar{z} = z^2$ # $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

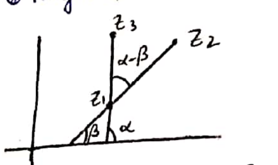
Properties of modulus

- # $|z| = 0 \Rightarrow z = 0 = Im(z) = Re(z)$
- # $|z| = |\bar{z}| = |-z| = |-\bar{z}|$
- # $-|z| \leq Re|z| \leq |z|$
- # $-|z| \leq Im|z| \leq |z|$
- # $z \cdot \bar{z} = |z|^2$ # $|z^n| = |z|^n$
- # $|z_1 z_2 \cdots z_n| = |z_1||z_2| \cdots |z_n|$
- # $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- # $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2Re(z_1\bar{z}_2)$
- # $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2Re(z_1\bar{z}_2)$
- # $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$
- # $|z_1 - z_2| \rightarrow$ dist b/w z_1 & z_2
- # $|z_1 + z_2| \leq |z_1| + |z_2|$
- # $|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$
- # $|z_1 + z_2| \geq ||z_1| - |z_2||$

Properties of Arguments

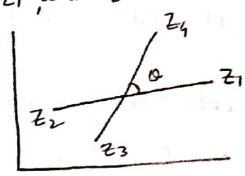
- # $arg(z_1 z_2) = arg(z_1) + arg(z_2)$
- # $arg(z_1 z_2 \cdots z_n) = arg(z_1) + \cdots + arg(z_n)$
- # $arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$
- # $arg(\bar{z}) = -arg(z)$
- # $arg(z^n) = n\,arg(z)$
- # $arg\left(\frac{1}{z}\right) = -arg(z)$
- # If z is purely imaginary, $arg(z) = \pm\frac{\pi}{2}$
- # If z is purely real, $arg(z) = 0/\pi$

Angle b/w 2 lines



$\alpha - \beta = arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$

Angle b/w line joining z_1, z_2 & z_3, z_4



$\theta = arg\left(\frac{z_4 - z_3}{z_1 - z_2}\right)$

- If z_1, z_2 and z_3 are vertices of an equilateral triangle. Then

$$\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$$

- i.e.

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

- **Square Root of $z = a + ib$ are**

$$\begin{cases} \pm\left(\sqrt{\frac{|z|+a}{2}} + i\sqrt{\frac{|z|-a}{2}}\right), & \text{for } b>0 \\ \pm\left(\sqrt{\frac{|z|+a}{2}} - i\sqrt{\frac{|z|-a}{2}}\right), & \text{for } b<0 \end{cases}$$

• If z_1, z_2, z_3 are vertices of an isosceles right angled triangle, w/ right angle at z_3, Then
$(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$

De Moivre's Theorem

$(\cos\theta + i\sin\theta)^n = (re^{i\theta})^n = r^n(e^{in\theta}) = (\cos(n\theta) + i\sin(n\theta))$

• $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$ • $\frac{1}{\cos\theta + i\sin\theta} = (\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta$

• $(\sin\theta \pm i\cos\theta)^n \neq \sin n\theta \pm i\cos\theta$ • $(\cos\theta_1 - i\sin\theta_2)^n \neq \cos n\theta_1 - i\sin n\theta_2$

• $(\sin\theta + i\cos\theta)^n = [\cos(\pi/2 - \theta) + i\sin(\pi/2 - \theta)]^n = [\cos n(\pi/2 - \theta) + i\sin n(\pi/2 - \theta)]$

Cube Roots of Unity

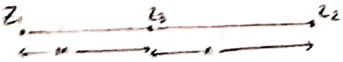
$z = 1^{1/3} = 1, \omega, \omega^2$ where $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, $\omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$
$\omega = e^{i2\pi/3}$, $\omega^2 = e^{-i2\pi/3}$

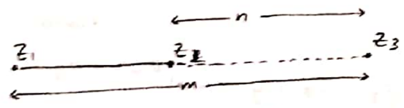
• Sum of roots is 0; $1 + \omega + \omega^2 = 0$ • Product of roots $= 1$; $1\cdot\omega\cdot\omega^2 = 1$

• $\omega = \frac{1}{\omega^2}$, $\omega^2 = \frac{1}{\omega}$ • $\omega = \overline{\omega^2}$, $\omega^2 = \overline{\omega}$ • $\omega^{3n+1} = \omega$, $\omega^{3n+2} = \omega^2$, $\omega^{3n} = 1$

• $1 + \omega^n + \omega^{2n} = \begin{cases} 3, & n \text{ is a multiple of } 3 \\ 0, & n \text{ is not a multiple of } 3 \end{cases}$ • Cube roots of unity represent the vertices of an equilateral triangle on Argand Plane

Section Formula





Internally $z_3 = \frac{mz_2 + nz_1}{m+n}$ Externally, $z_3 = \frac{mz_2 - nz_1}{m-n}$

• Centroid of Δ formed by z_1, z_2 and $z_3 \to \frac{z_1 + z_2 + z_3}{3}$

• If circumcentre of a Δ is origin, then orthocentre $\to z_1 + z_2 + z_3$

nth root of unity

• Sum of all nth roots of unity $= 0$ • Product of all roots $= 1\cdot\alpha\cdot\alpha^2\cdot\alpha^3 \cdots \alpha^{n-1} = \begin{cases} 1, & n \text{ is odd} \\ -1, & n \text{ is even} \end{cases}$

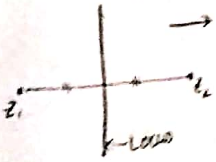
$z = 1^{1/n} \Rightarrow z = (\cos 2k\pi + i\sin 2k\pi)^n = \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^k$

$\therefore \alpha = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$

Locus of a CN (z_1 and z_2 are fixed, z is a variable point)

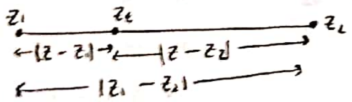
$\to |z - z_1| = |z - z_2|$

$\to$ z lies on perpendicular bisector of z_1z_2



$\to |z - z_1| + |z - z_2| = |z_1 - z_2|$

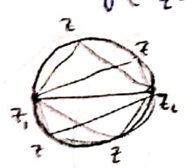
$\to$ z lies on the segment joining z_1 and z_2



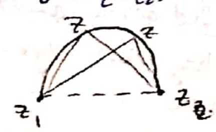
$\to |z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$

$\to$ Circle with z_1 and z_2 as diameter extremities.

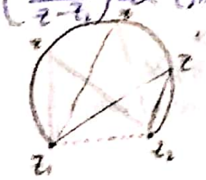
$\to \arg\left(\frac{z - z_1}{z - z_2}\right) = \pm \pi/2$



$\to \arg\left(\frac{z - z_1}{z - z_2}\right) = \pi/2$

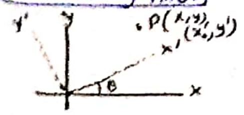


$\to \arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha$ (fixed)



Straight Line

Rotation of Axes



$x = x'\cos\theta - y'\sin\theta$ | $x' = x\cos\theta + y\sin\theta$

$y = x'\sin\theta + y'\cos\theta$ | $y' = -x\sin\theta + y\cos\theta$

	$x\downarrow$	$y\downarrow$
$x'\rightarrow$	$\cos\theta$	$\sin\theta$
$y'\rightarrow$	$-\sin\theta$	$\cos\theta$

Distance formula

$|d| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

- If A, B, C are collinear, $[\Delta ABC] = 0$
- If A, B, C have rational coordinates, ΔABC isn't equilateral.

Area of triangle

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Stair Method

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix} = \frac{1}{2}\left[(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\right]$$
$$= \frac{1}{2}\left[x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2)\right]$$

Area of polygon

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \\ x_1 & y_1 \end{vmatrix}$$

Points must be taken in cyclic order

Section formula

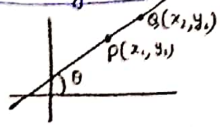
- Internal: $\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$
- External: $\left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}\right)$

Special points:

- Centroid: $\left(\frac{\Sigma x}{3}, \frac{\Sigma y}{3}\right)$
- Incentre: $\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$
- Circumcentre: $\left(\frac{x_1\sin 2A + x_2\sin 2B + x_3\sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \text{similar}\right)$
- O, G, H of an acute angle triangle are collinear

O —— G ———— H, $1 : 2$

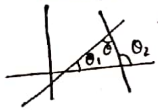
Straight line



Eqⁿ of line → $\boxed{y = mx + c}$

$$\boxed{\frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2}}$$

Angle b/w two lines

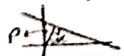


$$\tan\theta = \pm\frac{m_1-m_2}{1+m_1m_2}$$

- If $m_1 = m_2 \rightarrow$ lines are parallel
- If $m_1m_2 = -1 \rightarrow$ lines are perpendicular.

Eqⁿ of line

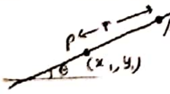
- Parallel to x axis → $y = b$
- Parallel to y axis → $x = a$
- Slope Intercept form → $y = mx + c$
- Point slope form → $y - y_1 = m(x - x_1)$
- Two point form → $\frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2}$
- Intercept form → $\frac{x}{a} + \frac{y}{b} = 1$
- Normal form → $x\cos\alpha + y\sin\alpha = p$



Conditions $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$

- Coincident if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$
- Parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$
- Intersecting if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$
- Perpendicular if $a_1a_2 + b_1b_2 = 0$

Parametric form



$A(x, y) = (x_1 + r\cos\theta, y_1 + r\sin\theta)$

$ax + by + c = 0 \rightarrow$

- ∥ to line → $ax + by + \lambda = 0$
- ⊥ to line → $bx - ay + \mu = 0$

Dist of a point from a line

$$|d| = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$$

Dist b/w two parallel lines

$$d = \frac{|c_1 - c_2|}{\sqrt{a^2+b^2}}$$

Concurrency of three lines

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Angle bisector

of $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$

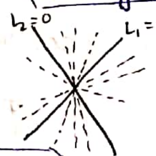
$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2+b_1^2}} = +\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2+b_2^2}} \rightarrow (x)$$

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2+b_1^2}} = -\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2+b_2^2}} \rightarrow (y)$$

- Make c_1, c_2 +ve. Then (x) contains origin

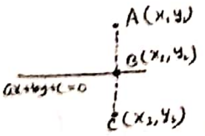
	Acute bisec	Obtuse bisec	
$a_1a_2 + b_1b_2 > 0$	(Y)	(X)	Origin is in obtuse
$a_1a_2 + b_1b_2 < 0$	(X)	(Y)	Origin is in acute.

Family of St. lines



$L_1 + \lambda L_2 = 0$

For Foot of Perpendicular / Image



B/F: $\frac{x_2-x_1}{a} = \frac{y_2-y_1}{b} = -\frac{(ax_1+by_1+c)}{a^2+b^2}$

S/F: $\frac{x_3-x_1}{a} = \frac{y_3-y_1}{b} = -\frac{2(ax_1+by_1+c)}{a^2+b^2}$

Pair of St. lines

$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0$

Bisector of Angle b/w pair of st. line

$ax^2 + 2hxy + by^2 = 0$

$$\frac{x^2-y^2}{a-b} = \frac{xy}{h}$$

Angle b/w pair of st. lines

$m_1 + m_2 = -\frac{2h}{b}$; $m_1m_2 = \frac{a}{b}$ ∴ $\tan\theta = \left|\frac{2\sqrt{h^2-ab}}{a+b}\right|$

- $a + b = 0 \longrightarrow$ lines are perpendicular
- $h^2 = ab \longrightarrow$ lines are ∥ or ~~coinci~~ coincident.

General 2nd degree equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Pair of st. line if

$\Delta(abc + 2fgh - af^2 - bg^2 - ch^2) = 0$

- Point of intersection → $\left(\frac{bg-hf}{h^2-ab}, \frac{af-gh}{h^2-ab}\right)$

→ $\Delta = 0$ (Pair of st. line)

→ $\Delta \neq 0, a = b, h = 0$ (Circle)

→ $\Delta \neq 0, h^2 = ab$ (Parabola)

→ Ellipse ($\Delta \neq 0, h^2 < ab$)

→ $\Delta \neq 0, h^2 > ab$ (Hyperbola)

Circle

Eqn of Circle

$(x-h)^2+(y-k)^2=r^2$

- If centre $(0,0) \rightarrow x^2+y^2=r^2$

- From general equation
  $x^2+y^2+2gx+2fy+c=0$
  - Centre $(-g,-f)$
  - radius $=\sqrt{g^2+f^2-c}$

- From diametre extremeties:
  $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

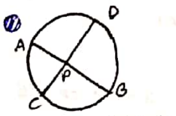
- Intercepts made on axis
  $x^2+y^2+2gx+2fy+c=0$
  x int $\rightarrow 2\sqrt{g^2-c}$
  y int $\rightarrow 2\sqrt{f^2-c}$

- Equation of circumcircle of Δ formed by $ax+by+c=0$ w/ coordinate axis.
  $ab(x^2+y^2)+c(bx+ay)=0$

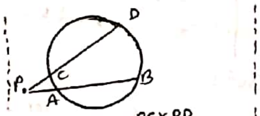
- Two lines $a_1x+b_1y+c_1=0$; $a_2x+b_2y+c_2=0$ cut the coordinate axes at concyclic points. if $m_1m_2=1 \rightarrow a_1a_2=b_1b_2$

- Parametric form: $x^2+y^2=r^2 \rightarrow (r\cos\theta, r\sin\theta)$ $[0\le\theta<2\pi]$
  $(x-h)^2+(y-k)^2=r^2 \longrightarrow (h+r\cos\theta, k+r\sin\theta)$

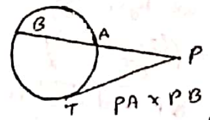
- A line intersects, touches or doesnt intersect the circle if radius is greater than, equal to or less than the length of perpendicular from centre of the circle to the line.



$PA\times PB=PC\times PD$



$PA\times PB=PC\times PD$



$PA\times PB=PT^2$

- If two lines $a_1x+b_1y+c=0$ and $a_2x+b_2y+c_2=0$ meet the axes in four distinct ~~points~~ concyclic points, then $a_1a_2=b_1b_2$ and also the eqn of the circle passing thru those concyclic points is.
  $(a_1x+b_1y+c_1)(a_2x+b_2y+c_1)-(a_1b_2+a_2b_1)xy=0$

Tangents

$$T\ \begin{cases} x^2+y^2+2gx+2fy+c=0 \\ xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0 \end{cases}$$

- for $x^2+y^2=a^2$:
  Point form $\rightarrow xx_1+yy_1-a^2=0$
  Parametric $\rightarrow x\cos\theta+y\sin\theta-a=0$
  Slope form $\rightarrow y=mx\pm a\sqrt{1+m^2}$

- For $x^2+y^2+2gx+2fy+c=0$
  $y+f=m(x+g)\pm\sqrt{g^2+f^2-c}\sqrt{1+m^2}$

- From a point outside
  $(y-y_1)=m(x-x_1)$

- Length of tangent from a point to a circle
  $d=\sqrt{S_1}$

- Pair of tangents (combined eqn)
  $SS_1=T^2$

~~Tangents~~

Normals

$(y-y_1)=\dfrac{y_1+f}{x_1+g}(x-x_1)$

- Chord of Contact $\longrightarrow T=0$
- Eqn of chord bisected at $(x_1,y_1) \longrightarrow S_1=T$

Director circle

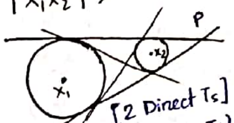
$x^2+y^2=a^2 \xrightarrow{DC} x^2+y^2=2a^2$ $[r_{DC}=\sqrt{2}\cdot r]$

- Angle of Intersection of two circles: $\longrightarrow \cos\theta=\left|\dfrac{r_1^2+r_2^2-d^2}{2r_1r_2}\right|$

- When $\theta=90°$ [orthogonally] $\longrightarrow 2g_1g_2+2f_1f_2=c_1+c_2$

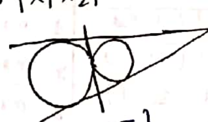
Intersections

- $|x_1x_2|>r_1+r_2$



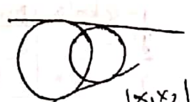
[2 Direct Ts]
[2 Transverse Ts]
[P divided x_1x_2 externally in ratio $r_1:r_2$]

- $|x_1x_2|=r_1+r_2$



[Total 3 Ts]

- $|r_1-r_2|<|x_1x_2|<|r_1+r_2|$

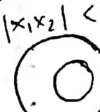


[Total 2 Common Ts]

- $|x_1x_2|=|r_1-r_2|$

[1 CTs]

- $|x_1x_2|<|r_1-r_2|$

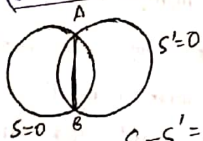


[No CT]

Length of common tangents

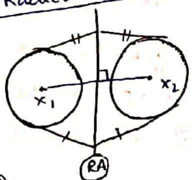
DCT $\rightarrow |AB|=\sqrt{d^2-(r_1-r_2)^2}$
TCT $\rightarrow |CD|=\sqrt{d^2-(r_1+r_2)^2}$
[$d\rightarrow$ dist b/w two centres]

Common Chord



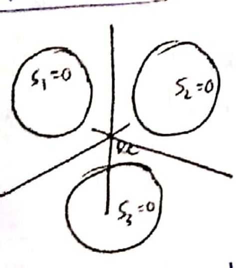
$AB\rightarrow S-S'=0$

Radical Axis



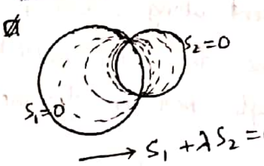
- RA is $\perp$ to line joining the centres
- RA bisects common tangents
- Need not pass thru mp of x_1x_2
- If 2 circles cut a third circle orthogonally, RA of those 2 pass thru 3rd one's centre.

Radical Centre

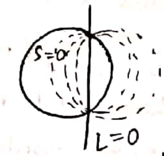


Solve,
$$\left[\begin{array}{l} S_1-S_2=0 \\ S_2-S_3=0 \\ S_3-S_1=0 \end{array}\right]$$

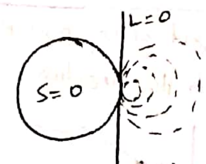
Family of Circles



$\longrightarrow S_1+\lambda S_2=0$



$\longrightarrow S+\lambda L=0$



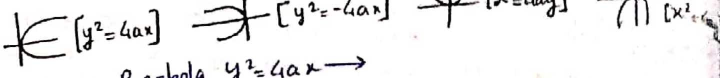
$\longrightarrow S+\lambda L=0$

- Circles touch a line at (x_1,y_1):

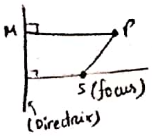
$\longrightarrow (x-x_1)^2+(y-y_1)^2+\lambda L=0$

PARABOLA

Standard forms →



Eccentricity

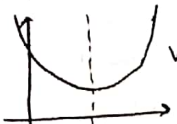


$\frac{PS}{PM} = e$

- $e = 0 \rightarrow$ circle
- $e = 1 \rightarrow$ Parabola
- $e < 1 \rightarrow$ Ellipse
- $e > 1 \rightarrow$ Hyperbola
- $e = \infty \rightarrow$ Pair of st. lines

Position of a point w.r.t. a Parabola $y^2 = 4ax$ →
$y^2 - 4ax < 0 \rightarrow$ Inside / $y^2 - 4ax > 0 \rightarrow$ outside / $y^2 - 4ax = 0 \rightarrow$ on

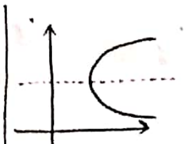
Parabolic curve



$y = Ax^2 + Bx + C$

Vertex $\rightarrow \left(-\frac{B}{2A}, -\frac{D}{2A}\right)$

LR $\rightarrow \frac{1}{|A|}$



$x = Ay^2 + By + C$

Vertex $\rightarrow \left(-\frac{D}{4A}, -\frac{B}{2A}\right)$

Length LR $\rightarrow \frac{1}{|A|}$

Equation of Conic:
Focus (α, β), Directrix $(ax + by + c = 0)$
$$\rightarrow (x-\alpha)^2 + (y-\beta)^2 = e^2\left(\frac{(ax+by+c)^2}{a^2+b^2}\right)$$

Parametric form

$(y-k)^2 = 4a(x-h)$
$\rightarrow x = h + at^2$
$y = k + 2at$

Properties of Focal Chord

$P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ is a focal chord, then $t_1 t_2 = -1$, $Q \equiv \left(\frac{a}{t_1^2}, -\frac{2a}{t_1}\right)$

- Chord joining $P(at^2, 2at)$ has length $a\left(t + \frac{1}{t}\right)^2$
- Focal chord from $P(at^2, 2at)$ has length $4a\operatorname{cosec}^2\theta$
- Focal chord making angle θ with axis has length ... P and Q are extremities of focal chord, $S \rightarrow$ focus
- Semi latus Rectum is HM of SP & SQ where P and Q are extremities of focal chord
- Circle described on the focal length as diameter touches tangent at vertex
- Circle described on the focal chord as diameter touches directrix.

Equation of tangents

- Point form →
  $yy_1 = 2a(x + x_1)$
- Parametric form →
  $ty = x + at^2$ [at $(at^2, 2at)$]
- Slope form →
  $y = mx + \frac{a}{m}$ [at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$]

Pair of tangents: $\rightarrow SS_1 = T^2$

Chord of contact $\rightarrow T = 0$

Equation of chord from mp. $\rightarrow S_1 = T$

Properties of Tangents

- Point of Intersection of tangents at two points $A(at_1^2, 2at_1)$ and $B(at_2^2, 2at_2)$ on the Parabola $y^2 = 4ax$ is $T(at_1t_2, a(t_1 + t_2))$
- Locus of foot of ⊥ from focus upon any tangent is tangent at vertex.
- Length of Tangent b/w POC and POI w/ directrix subtends 90° at the focus.
- Tangents at Extremeties of focal chord are perpendicular ~~at~~ on Directrixs.

Equation of Normal

- Point form →
  $y - y_1 = -\frac{y_1}{2a}(x - x_1)$
- Parametric form →
  $y = -tx + 2at + at^3$
- slope form →
  $y = mx - 2am - am^3$

Parabola	Normal (Parametric)	Normal (Slope)
$y^2 = 4ax$	$y = -tx + 2at + at^3$ $(at^2, 2at)$	$y = mx - 2am - am^3$ $(am^2, -2am)$
$y^2 = -4ax$	$y = tx + 2at + at^3$ $(-at^2, 2at)$	$y = mx + 2am + am^3$ $(-am^2, 2am)$
$x^2 = 4ay$	$x = -ty + 2at + at^3$ $(2at, at^2)$	$y = mx + 2a + \frac{a}{m^2}$ $\left(-\frac{2a}{m}, \frac{a}{m^2}\right)$
$x^2 = -4ay$	$x = ty + 2at + at^3$ $(2at, -at^2)$	$y = mx - 2a - \frac{a}{m^2}$ $\left(\frac{2a}{m}, -\frac{a}{m^2}\right)$

Properties of Normals

- Normals other than axis of Parabola never passes thru focus.
- POI of Normals from $P(at_1^2, 2at_1)$, $Q(at_2^2, 2at_2)$
  $\rightarrow [2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2)]$
- Normal at point $P(t_1)$ meets the curve again at $Q(t_2)$, $t_2 = -t_1 - \frac{2}{t_1}$

Co-normal Points

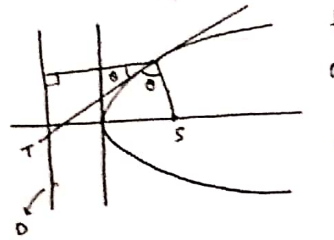
$y = mx - 2am - am^3 \Rightarrow am^3 + (2a - h)m + k = 0$ (cubic eq in m)

$m_1 + m_2 + m_3 = 0$; $m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a-h}{a}$

- Algebraic sum of ordinates of co-normal points = 0
- Centroid of the triangle formed by them lies on axis.
- If three normals drawn on $y^2 = 4ax$ from (h, k) is real $\rightarrow |h > 2a|$

Reflection Property of Parabola

The tangent at any point P to a parabola bisects the angle between the focal chord through P and perpendicular from P to the directrix

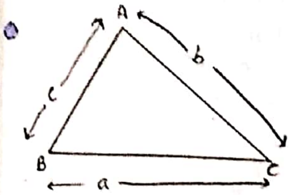


- Thus, if any light ray is sent along a line parallel to the axis of the parabola then the reflected ray passes thru the focus, as the normal bisects the angle between the incident ray and reflected ray.

- Tangents are drawn from the point (x_1, y_1) to the parabola $y^2 = 4ax$, the length of the chord of contact $= \frac{1}{|a|}\sqrt{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}$

- Area of the triangle formed by the tangents drawn from (x_1, y_1) to $y^2 = 4ax$ and their chord of contact is $\frac{(y_1^2 - 4ax_1)^{3/2}}{2a}$

Properties of Triangle / Solution of Triangle



$a+b+c = 2s$ (Perimeter)

$s = \dfrac{a+b+c}{2}$ (semi perimeter)

Sine Rule

$\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} = 2R$

[R → circumradius]

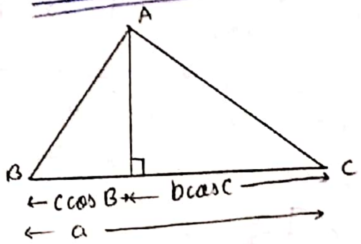
Cosine Rule

\# $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$

\# $\cos B = \dfrac{c^2+a^2-b^2}{2ca}$

\# $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

Projection Formula



\# $a = b\cos C + c\cos B$

\# $b = c\cos A + a\cos C$

\# $c = a\cos B + \cancel{B\cos b}\; b\cos A$

Napier Formula

\# $\tan\left(\dfrac{A-B}{2}\right) = \dfrac{a-b}{a+b}\cot\dfrac{C}{2}$

\# $\tan\left(\dfrac{B-C}{2}\right) = \dfrac{b-c}{b+c}\cot\dfrac{A}{2}$

\# $\tan\left(\dfrac{C-A}{2}\right) = \dfrac{c-a}{c+a}\cot\dfrac{B}{2}$

Half Angle formula

\# $\sin\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$

\# $\sin\dfrac{B}{2} = \sqrt{\dfrac{(s-c)(s-a)}{ac}}$

\# $\sin\dfrac{C}{2} = \sqrt{\dfrac{(s-a)(s-b)}{ab}}$

\# $\cos\dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$

\# $\cos\dfrac{B}{2} = \sqrt{\dfrac{s(s-b)}{ac}}$

\# $\cos\dfrac{C}{2} = \sqrt{\dfrac{s(s-c)}{ab}}$

\# $\tan\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$

\# $\tan\dfrac{B}{2} = \sqrt{\dfrac{(s-c)(s-a)}{s(s-b)}}$

\# $\tan\dfrac{C}{2} = \sqrt{\dfrac{(s-a)(s-b)}{s(s-c)}}$

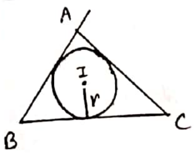
Area of Triangle

\# $\Delta = \frac{1}{2}\cdot b\cdot h$ \# $\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B$ \# $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

\# $\Delta = r\times s$ (r is Inradius) \# $\Delta = 2R^2\sin A\sin B\sin C$.

\# $\Delta = \dfrac{abc}{4R}$ (R is circumcentre)

Inradius



\# $r = \Delta/s$

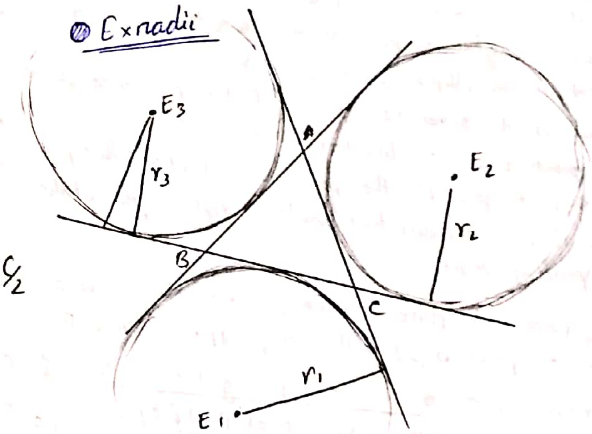
\# $r = (s-a)\tan\dfrac{A}{2}$

$= (s-b)\tan\dfrac{B}{2}$

$= (s-c)\tan\dfrac{C}{2}$

\# $r = 4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

Exradii

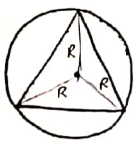


\# $r_1 = \dfrac{\Delta}{s-a} = s\tan\dfrac{A}{2} = 4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

\# $r_2 = \dfrac{\Delta}{s-b} = s\tan\dfrac{B}{2} = 4R\cos\dfrac{A}{2}\sin\dfrac{B}{2}\cos\dfrac{C}{2}$

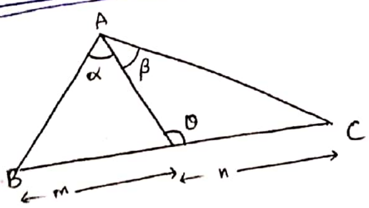
\# $r_3 = \dfrac{\Delta}{s-c} = s\tan\dfrac{C}{2} = 4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\sin\dfrac{C}{2}$

Circumradius



\# $R = \dfrac{abc}{4\Delta}$

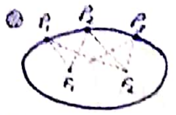
m-n cot Theorem



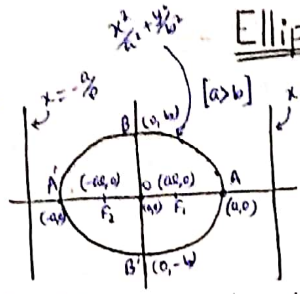
\# $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

\# $(m+n)\cot\theta = n\cot\beta - m\cot C$

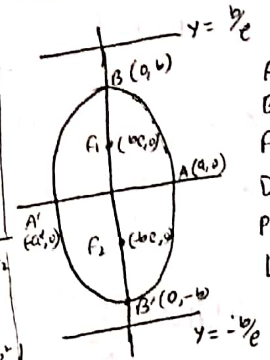
Ellipse



$P_1F_1 + P_1F_2 = P_2F_1 + P_2F_2$
$= P_3F_1 + P_3F_2$

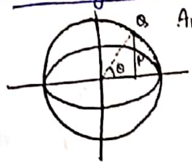


AA' (major axis) $=2a$
BB' (minor axis) $=2b$
Foci $= (\pm ae, 0)$
Directrix $\rightarrow x = \pm \frac{a}{e}$
$PF_1 + PF_2 = 2a$
$b^2 = a^2(1-e^2)$ / $e = \sqrt{1-\frac{b^2}{a^2}}$
Vertices $= (\pm a, 0)$
Latus Rectum $= \frac{2b^2}{a}$
End of LR $= \left(\pm ae, \pm \frac{b^2}{a}\right)$



$[b>a]$
AA' (minor axis) $=2a$
BB' (major axis) $=2b$
Foci $\rightarrow (0, \pm be)$
Directrix $\rightarrow y = \pm \frac{b}{e}$
$PF_1 + PF_2 = 2b$
$LR = \frac{2a^2}{b}$
Ends of $LR = \left(\pm \frac{a^2}{b}, \pm b\right.$

- Two ellipses are similar if they have equal eccentricity.
- Ellipse with axes $\parallel$ to coordinate axes and centre $(h,k) \rightarrow \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$
- Length of LR $=$ (minor axis)^2/major axis $= 2e\left(\frac{a}{e} - ae\right)$
- Position of a point w.r.t. an Ellipse (h,k)
  $\frac{x^2}{a^2} + \frac{k^2}{b^2} - 1 >, =, < 0 \rightarrow$ outside, on, inside.
- Eq^n of an Ellipse referred to two perpendicular lines.
  $L_1: a_1x + b_1y + c_1 = 0$
  $L_2: b_1x - a_1y + c_2 = 0$
  $\Rightarrow \frac{\left(\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}\right)^2}{a^2} + \frac{\left(\frac{b_1x-a_1y+c_2}{\sqrt{a_1^2+b_1^2}}\right)^2}{b^2} = 1$
  - Centre at intersection point of L_1 & L_2
  - Major axis is along L_2 $(a>b)$
- Auxiliary Circle / Eccentric Angle



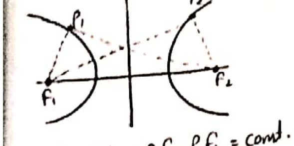
Aux circle: $x^2+y^2=a^2$
$Q(a\cos\theta, a\sin\theta)$
$P(a\cos\theta, b\sin\theta)$
θ is called eccentric angle of point P

- Pair of tangents
  $SS_1 = T^2$
- COC
  $T=0$
- Eq^n of chord with mp (x_1, y_1)
  $T = S_1$
- Properties of ellipse ~~auxiliary circle~~.
  - Area of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab.
  - Ratio of area of any triangle PQR inscribed in ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and that of triangle formed by corresponding points on the aux circle is b/a.
  - Semi LR is HM of segments of focal chord.
  - Circle described on focal length as diameter always touches auxiliary circle.
- Equation of tangent
  - Point form $\rightarrow \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$
  - Parametric form $\rightarrow \frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$
    $(a\cos\theta, b\sin\theta)$
  - Slope form $\rightarrow y = mx \pm \sqrt{a^2m^2+b^2}$
- Director circle

Locus of poi of $\perp$ tangents

- Important Properties related to tangents
  - Locus of feet of perpendiculars from foci upon any tangent is an auxiliary circle.
  - Product of lengths of perpendiculars from foci upon any tangent of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is b^2.
  - Tangents at the extremities of Latus Rectum pass through the corresponding foot of directrix on major axis.
  - Length of tangent b/w the point of contact and the point where it meets the directrix subtends right angle at the corresponding focus.
- Eq^n of Normal
  - Point form $\rightarrow \frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2-b^2$
  - Parametric form $\rightarrow$
    $ax\sec\theta - by\,\mathrm{cosec}\,\theta = a^2-b^2$
- Properties of normals
  - Normal other than major axis never passes through the focus.
  - Normal at the point P bisects angle SPS'
    [Reflection property]
- Co-normal Points: From any point in the plane maximum four normals can be drawn.
  Eccentric angle of all the four point, $\alpha, \beta, \gamma, \delta$ then, $\alpha+\beta+\gamma+\delta = (2n+1)\pi$
- Concyclic Points: $\alpha+\beta+\gamma+\delta = 2n\pi$
- Eq^n of chord joining $P(\alpha)$ & $Q(\beta) \rightarrow \frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$
- POI of tangents at $P(\alpha)$ & $Q(\beta) \rightarrow \left(a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$

Hyperbola



$P_1F_2 - P_1F_1 = P_2F_1 - P_2F_2 = \text{Const.}$

- $LR = \frac{2b^2}{a} = 2e(ae - \frac{a}{e})$

Hyperbolas reffered to two ⊥ lines:

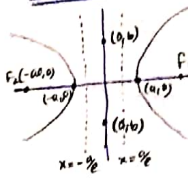
$L_1: lx + my + n = 0$

$L_2: mx - ly + p = 0$

$$\frac{\left(\frac{lx+my+n}{\sqrt{l^2+m^2}}\right)^2}{a^2} - \frac{\left(\frac{mx-ly+p}{\sqrt{m^2+l^2}}\right)^2}{b^2} = 1$$

- Centre is pol of L_1 & L_2
- TA → 2a, CA → 2b
- TA is along $L_2 = 0$

Standard Eqⁿ



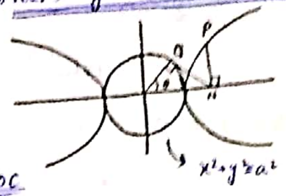
Conjugate Hyperbola

- Hyperbola → $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (e_1)
- conjugate Hyperbola → $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (e_2)
- $\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$
- foci of the hyperbola and conj. are concyclic square.

COC

$T = 0$

Auxiliary Circle and Eccentric Angle



$Q(a\cos\theta, a\sin\theta)$

$P(a\sec\theta, b\tan\theta)$

PoI of tangents from $P(\alpha)$ & $Q(\beta)$

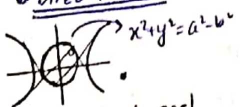
$$\left(a\frac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}},\ b\frac{\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}\right)$$

Eqⁿ of chord joining $P(\alpha)$ and $Q(\beta)$

$$\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right) - \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)$$

Pair of Tangents: $SS_1 = T^2$

Director Circle



- a>b, DC is real
- a=b, DC is point circle
- a<b, no real circle.

Chord with mp given

$T = S_1$

Eqⁿ of Tangent

- Point form: $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$
- Parametric form: $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$
- Slope form: $y = mx \pm \sqrt{a^2m^2 - b^2}$
- at (x_1, y_1) to $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

$$\frac{(x-h)(x_1-h)}{a^2} - \frac{(y-k)(y_1-k)}{b^2} = 1$$

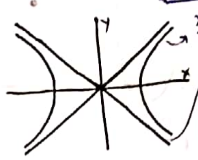
Equation of normal

- point form (x_1, y_1): $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2+b^2$
- parametric form: $ax\cos\theta + by\cot\theta = a^2+b^2$

Properties of Normals

- Normal other than TA never passes through focus.
- Locus of feet of perpendicular drawn from focus upon any target of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is its aux circle i.e. $x^2+y^2=a^2$
- The product of perpendiculars drawn from foci upon any tangent of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is b^2
- The portion of the tangent b/w the poc and the point where it meets the directrix subtends a right angle at corresponding focus.
- The tangent and normal at any point of ~~contact~~ hyperbola bisect the angle b/w focal radii.
- If an ellipse and a hyperbola have same foci, the cut at right angles.
- The foci and the points P & Q in which any tangent meets the tangents at the vertices are concyclic with PQ as diameter.

Asymptotes



$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

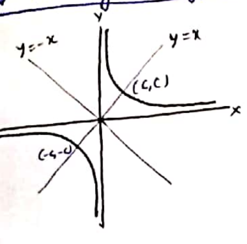
$y = \pm \frac{b}{a}x$

- Asymptotes pass thru the centre of the hyperbola
- The eqⁿ of pair of asymptotes differ from the eqⁿ of hyperbola ~~just~~ by only a constant.
- Asymptotes are diagonals of rectangle formed by lines drawn through extrimities of ~~the~~ each axis parallel to the other.
- For rectangular hyperbola, Asymptotes are at 90° i.e. $y = \pm x$
- At any point of a Asymptote if a st. line is drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted b/w the point and curve is always equal to the square of the semi conjugate axis.
- Perpendiculars from the foci on either asymptote meet it at the same point as the corresponding directrix and common points of intersections lie on aux circle.
- If the asymptotes of a rectangular hyperbola are $x=\alpha$ and $y=\beta$, then its eqⁿ is $(x-\alpha)(y-\beta) = c^2$

Important Points

- If angle b/w asymptotes of hyperbola is 2θ, $e = \sec\theta$
- Angle b/w asymptotes $\theta = \tan^{-1}\left|\frac{2ab}{a^2-b^2}\right|$
- Acute angle b/w asymptotes.
- Hyperbola and its conjugate have same asymptotes.

Rectangular Hyperbola



- Parametric form: $(ct, \frac{c}{t})$
- Equation of tangent at 't': $x + yt^2 - 2ct = 0$
- Eqⁿ of Normal at 't': $xt^3 - yt - ct^4 + c = 0$
- Eqⁿ of tangent at (x_1, y_1): $xy_1 + yx_1 = 2c^2$
- Eqⁿ of normal at (x_1, y_1): $xx_1 - yy_1 = x_1^2 - y_1^2$

- $xy = c^2$, $e = \sqrt{2}$
- Asymptotes, $x=0, y=0$
- TA: $y = x$, CA: $y = -x$
- Vertex $A(c,c)$, $A'(-c,-c)$
- foci $(c\sqrt{2}, c\sqrt{2})$ & $(-c\sqrt{2}, -c\sqrt{2})$
- length of LR $= 2\sqrt{2}c$
- Aux circle → $x^2+y^2 = 2c^2$
- DC → $x^2+y^2=0$
- $x^2-y^2=1$ and $xy=1$ intersect at 90°

Concyclic points on $xy = c^2$

If a circle and a rectangular hyperbola $xy = c^2$ meets at four points $t_1 t_2 t_3 t_4$, then,

- $t_1 t_2 t_3 t_4 = 1$
- Centre of the mean position of the four points bisects the distance b/w the centres of the two curves.

Theory of Equations and Logarithm

Laws of log

- $a^{\log_b x} = x^{\log_b a}$; $a, b > 0 \neq 1$, $x > 0$
- $\log_a x = \dfrac{1}{\log_x a}$
- $\log_a a = 1$, $\log_a 1 = 0$
- $\log_a x = \log_b x \cdot \log_a b = \dfrac{\log_b x}{\log_b a}$
- $\log_a(m^n) = n\log_a m$
- $\log_{a^n}(x) = \frac{1}{n}\log_a x$
- $\log_{a^n} x^m = \left|\frac{m}{n}\right| \log_a x$
- for $x > y > 0$

(i) $\log_a x > \log_a y$, if $a > 1$

(ii) $\log_a x < \log_a y$, if $0 < a < 1$

- $0 < a < 1$ then

(i) $\log_a x > p \Rightarrow 0 < x < a^p$

(ii) $0 < \log_a x < p \Rightarrow a^p < x < 1$

- $a > 1$,

(i) $\log_a x > p \Rightarrow x > a^p$

(ii) $0 < \log_a x < p \Rightarrow 0 < x < a^p$

Relation b/w roots and co-eff

$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1}x + a_n = 0$

- $\sum \alpha_1 = -\frac{a_1}{a_0}$ • $\sum \alpha_1\alpha_2 = \frac{a_2}{a_0}$ .... • $\alpha_1\alpha_2\alpha_3 \cdots \alpha_n = (-1)^n \frac{a_n}{a_0}$

Discriminant & Nature of Roots

$ax^2 + bx + c = 0 \longrightarrow x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

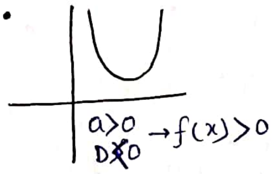
$\boxed{D = b^2 - 4ac}$

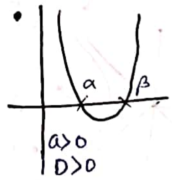
$D > 0 \longrightarrow$ roots are real and distinct.

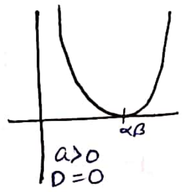
$D = 0 \longrightarrow$ roots are real and equal

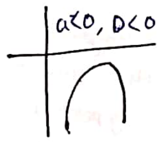
$D < 0 \longrightarrow$ roots are imaginary.

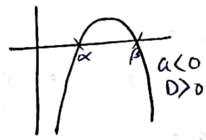
- $f(x) = y = ax^2 + bx + c$

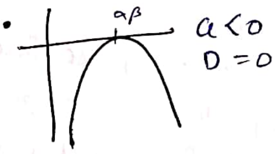


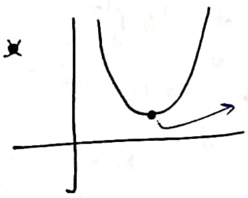












minimum value $\left(-\frac{b}{2a}, \frac{D}{4a}\right)$

Common Roots

- 1 common → $(D_{u2})^2 = P_{a_{3}3} \cdot P_{a_{3}3}$
- 2 common → $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Binomial Theorem

- $(x+a)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1}a + {}^nC_2\, x^{n-2}a^2 + \cdots\cdots + {}^nC_r\, x^{n-r}a^r + \cdots + {}^nC_{n-1}\, x^1 a^{n-1} + {}^nC_n\, x^0 a^n$

→ General Term: $T_{r+1} = {}^nC_r\, x^{n-r}a^r$

- $(x-a)^n = {}^nC_0\, x^n - {}^nC_1\, x^{n-1}a + {}^nC_2\, x^{n-2}a^2 - \cdots\cdots + (-1)^r\, {}^nC_r\, x^{n-r}a^r + \cdots + (-1)^n\, {}^nC_r\, x^{n-r}a^r$

→ General Term: $T_{r+1} = (-1)^r\, {}^nC_r\, x^{n-r}a^r$

- Middle term: (i) $\left(\frac{n}{2}+1\right)$th term, if n is even. $T_{\frac{n}{2}+1} = {}^nC_{n/2}\, x^{n/2}a^{n/2}$

(ii) $\left(\frac{n+1}{2}\right)$th & $\left(\frac{n+3}{2}\right)$th term, if n is odd.

- Greatest term $\frac{T_{r+1}}{T_r} \geq 1$ i.e. $\frac{n-r+1}{r}\left|\frac{a}{x}\right| \geq 1$

- $\sum_{r=0}^{n} (-1)^r\, {}^nC_r = 0$
- ${}^nC_1 - 2\,{}^nC_2 + 3\,{}^nC_3 - \cdots\cdots + n(-1)^{n-1}\, {}^nC_n = 0$
- ${}^nC_1 + 2\cdot{}^nC_2 + 3\,{}^nC_3 + \cdots + n\cdot{}^nC_n = n\cdot 2^{n-1}$
- $C_0C_r + C_1C_{r+1} + \cdots + C_{n-r}C_n = {}^{2n}C_{n-r}$
- ${}^nC_n + {}^{n+1}C_n + \cdots\cdots + {}^{2n-1}C_n = {}^{2n}C_{n+1}$

Properties of Binomial Theorem

- ${}^nC_r = \frac{n}{r}\, {}^{n-1}C_{r-1} = \frac{n}{r}\cdot\frac{n-1}{r-1}\cdot {}^{n-2}C_{r-2}$
- ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
- ${}^nC_r = {}^nC_{n-r}$
- ${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$
- ${}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots = {}^nC_1 + {}^nC_3 + \cdots = 2^{n-1}$

Multinomial Theorem

$$(x_1+x_2+\cdots+x_k)^2 = \sum_{r_1+r_2+r_3+\cdots=n} \frac{n!\, x_1^{r_1} x_2^{r_2}\cdots x_k^{r_k}}{r_1!\, r_2! \cdots r_k!}$$

- no of terms in $(x+y+z)^n$ is ${}^{n+2}C_2$ or $\frac{(n+1)(n+2)}{2}$

Expressions

- $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots + (-x)^r + \cdots \infty$
- $(1-x)^{-1} = 1 + x + x^2 + \cdots\cdots + x^r + \cdots \infty$
- $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots + (r+1)(-x)^r + \cdots \infty$
- $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + (r+1)x^r + \cdots \infty$
- $(1+x)^{-3} = 1 - 3x + 6x^2 - \cdots + \frac{(r+1)(r+2)}{2!}(-x)^r + \cdots \infty$
- $(1-x)^{-3} = 1 + 3x + 6x^2 + \cdots + \frac{(r+1)(r+2)}{2!}(x)^r + \cdots \infty$

Determinants

$$\begin{array}{c} \quad C_1 \quad C_2 \quad C_3 \\ \Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{array}{l} \leftarrow R_1 \\ \leftarrow R_2 \\ \leftarrow R_3 \end{array} \end{array}$$

[3×3]

a_{mn} — m: row, n: column

Minor

- Minor of a_{11}, $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$
- Minor of a_{22}, $M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = a_{11}a_{33} - a_{31}a_{13}$
- Minor of a_{33}, $M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

Co-factor

Co-factor of $a_{ij} = (-1)^{i+j} M_{ij}$

Co-factor of $a_{11} = (-1)^{1+1} M_{11} = M_{11}$

Co-factor of $a_{12} = (-1)^{1+2} M_{12} = -M_{12}$

Signs of cofactors

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Expansion of Determinants

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Properties

- $\begin{vmatrix} 0 & a & b \\ 0 & c & d \\ 0 & e & f \end{vmatrix} = 0$

  Any row/column 0

- $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$

  Whenever we interchange any two row (or column) value of it will be multiplied by '–ve'

- $\begin{vmatrix} a+b & c & d \\ e+f & g & h \\ i+j & k & l \end{vmatrix} = \begin{vmatrix} a & c & d \\ e & g & h \\ i & k & l \end{vmatrix} + \begin{vmatrix} b & c & d \\ f & g & h \\ j & k & l \end{vmatrix}$

- $\begin{vmatrix} ap & bp & cp \\ d & e & f \\ g & h & i \end{vmatrix} = p\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

- $\begin{vmatrix} a & b & c \\ a & b & c \\ d & e & f \end{vmatrix} = 0$ Any two rows or column same.

- Transform

  $\Delta = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \xrightarrow{R_1 = R_1 + pR_2} \Delta = \begin{vmatrix} a+dp & b+ep & c+fp \\ d & e & f \\ g & h & i \end{vmatrix}$

- $\begin{vmatrix} ap & d & g \\ bp & e & h \\ cp & f & i \end{vmatrix} = p\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$

- $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \rightarrow a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 0$

  or

  $a_{11}a_{31} + a_{12}C_{32} + a_{13}C_{33} = 0$

  $a_{12}C_{13} + a_{22}a_{23} + a_{32}a_{33} = 0$

Important Expansion

- $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

- $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

- $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

- $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

System of Evalute

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$\Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

$\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$

$\Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Note

- If $[\Delta \neq 0]$.
  - → Unique Sol^n,
  - → Consistent set of sol^n
- If $[\Delta = 0]$
  - → (a) if any one (or two) of $\Delta_x, \Delta_y, \Delta_z$ is/are Non-zero, Inconsistent System ⇒ No sol^n
  - → (b) $[\Delta_x = \Delta_y = \Delta_z = 0]$ Consistent set of sol^n ⇒ Infinite no of sol^n.

$x = \dfrac{\Delta_x}{\Delta}$

$y = \dfrac{\Delta_y}{\Delta}$

$z = \dfrac{\Delta_z}{\Delta}$

Cramer's Rule ($\Delta \neq 0$)

Trigonometric Equation

- $\sin\theta = 0 \longrightarrow \theta = n\pi,\ n\in Z$
- $\cos\theta = 0 \longrightarrow \theta = (2n+1)\frac{\pi}{2},\ n\in Z$
- $\tan\theta = 0 \longrightarrow \theta = n\pi,\ n\in Z$
- $\sin\theta = 0 \longrightarrow \theta = (4n+1)\frac{\pi}{2},\ n\in Z$
- $\sin\theta = -1 \longrightarrow \theta = (4n-1)\frac{\pi}{2},\ n\in Z$
- $\cos\theta = 1 \longrightarrow \theta = 2n\pi,\ n\in Z$
- $\cos\theta = -1 \longrightarrow \theta = (2n+1)\pi,\ n\in Z$
- $\cot\theta = 0 \longrightarrow \theta = (2n+1)\frac{\pi}{2},\ n\in Z$

- $\sin\theta = \sin\alpha \longrightarrow n\pi + (-1)^n\alpha,\ n\in Z$

$\Rightarrow \sin\theta = k \longrightarrow \theta = n\pi + (-1)^n(\sin^{-1}k),\ n\in Z$
$k\in[-1,1]$

- $\cos\theta = \cos\alpha \longrightarrow \theta = 2n\pi \pm \alpha,\ n\in Z$

$\Rightarrow \cos\theta = k \longrightarrow \theta = 2n\pi \pm (\cos^{-1}k),\ n\in Z,\ k\in[-1,1]$

- $\tan\theta = \tan\alpha \longrightarrow \theta = n\pi + \alpha,\ n\in Z$

$\Rightarrow \tan\theta = k \longrightarrow \theta = n\pi + (\tan^{-1}k),\ k\in R$

- $\sin^2\theta = \sin^2\alpha \,/\, \cos^2\theta = \cos^2\alpha$

$\longrightarrow \theta = n\pi \pm \alpha,\ n\in Z$

- $\tan^2\theta = \tan^2\alpha \longrightarrow \theta = n\pi \pm \alpha,\ n\in Z$

- Solution of the equation of the form $a\cos\theta + b\sin\theta = c$

$\rightarrow$ If $|c| > \sqrt{a^2+b^2}$, then no real solution

$\rightarrow$ If $|c| \le \sqrt{a^2+b^2}$, then divide both sides of the equation by $\sqrt{a^2+b^2}$, then take $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$,

$\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$, equation will reduce to

$\cos(\theta-\alpha) = \cos\beta$, where $\tan\alpha = b/a$

$\cos\beta = \frac{c}{\sqrt{a^2+b^2}}$

- If we take $\sin\alpha = \frac{a}{\sqrt{a^2+b^2}}$,

$\cos\alpha = \frac{b}{\sqrt{a^2+b^2}}$, then the equation will reduce to $\sin(\theta+\alpha) = \sin\beta$,

$\sin\beta = \frac{c}{\sqrt{a^2+b^2}}$

- While solving trigo equation, avoid squaring the equation as far as possible. If squaring is necessary check the solution for extraneous values (similar values following the same pattern.

- Never cancel terms containing unknown terms on the two sides which are in product. It may cause the loss of a genuine solution.

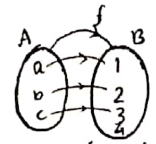
- The answer should not contain such values of angles which make any term undefined or infinite.

- Domain should not change while simplifying the equation. If it changes, necessary corrections must be made.

- Check the denominator is not zero at any stage while solving the equation.

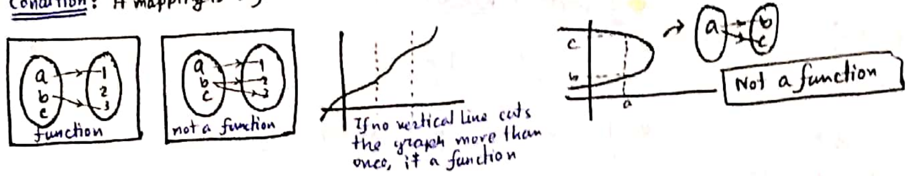
- Extreme values of functions — Keep in mind.

Functions

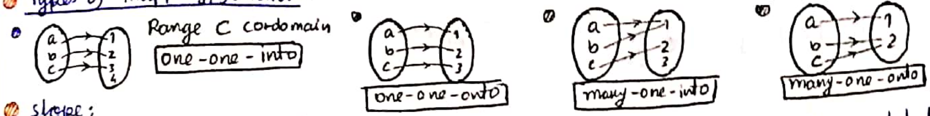


$f: A \to B \equiv \{(a,1),(b,2),(c,3)\}$
Domain of $f: \{a,b,c\} \to$ Input
Range of $f: \{1,2,3\} \to$ output
Co-domain of $f: \{1,2,3,4\} \to$ possible outcome

- Condition: A mapping is a function in each ~~each~~ input have one and only one outputs.

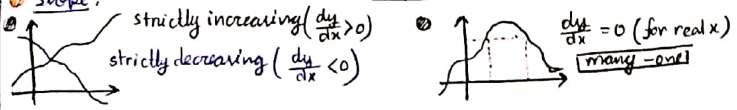


- Types of mapping/function:



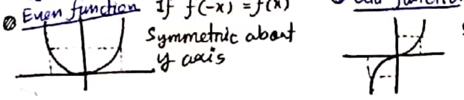
- one-one → Injection
- Onto → Surjection
- one-one-onto → Bijection

- slope:

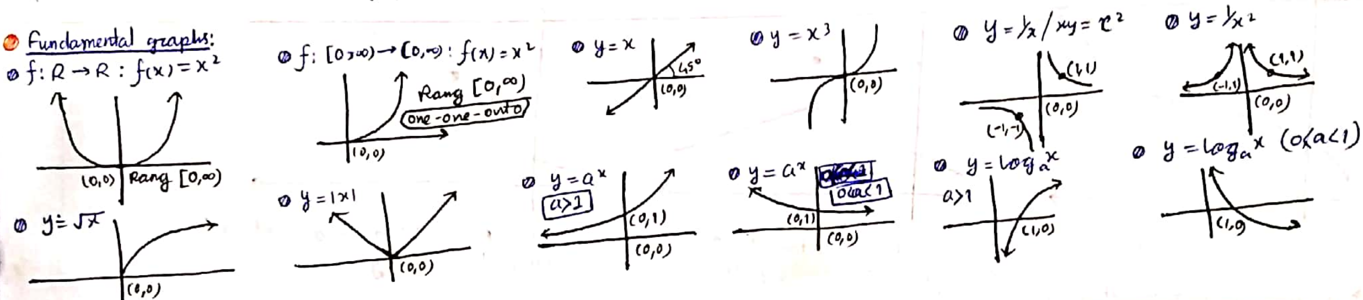


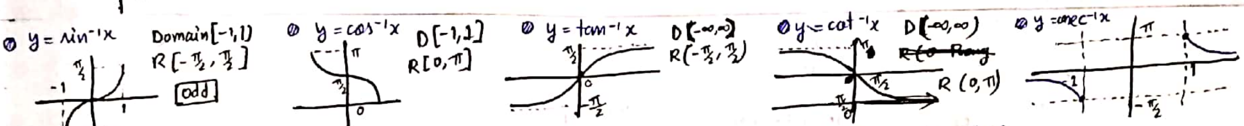
If a horizontal line cuts the graph at more than one point, its a many-one or else one-one

- Type of function:
- Even function If $f(-x) = f(x)$ Symmetric about y axis
- Odd function: If $f(-x) = -f(x)$ Symmetric about origin.



- Fundamental graphs:



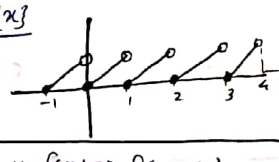


- GIF $[x]$: $y = [x]$

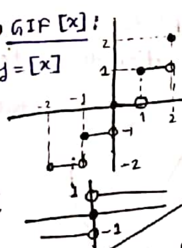
$\boxed{(x-1) < [x] \leq x}$

$[x] + [-x] = \begin{cases} 0, & x \in I \\ -1, & x \notin I \end{cases}$

- Fractional part of $x: \{x\}$
$\{x\} = x - [x]$
$0 \leq \{x\} < 1$

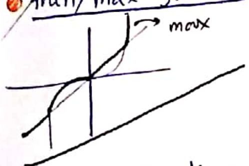


- Signum Function:
$sgn(x) = \frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$



- Graphical Transformation:
- $y = f(x) \to y = -f(x)$: mirror img about x axis.
- $y = f(x) \to y = f(-x)$: mirror img about y axis
- $y = f(x) \to f(x \pm k)$: $-k \to$ right / $+k \to$ left
- $y = f(x) \to f(x) \pm k$: $+k \to$ up / $-k \to$ down.
- $y = f(x) \to |f(x)| = \begin{cases} f(x), & f(x) \geq 0 \\ -f(x), & f(x) < 0 \end{cases}$
- $f(x) \to y = f(|x|) = \begin{cases} f(x), & x \geq 0 \\ f(-x), & x < 0 \end{cases}$

- min/max function:



- Inverse function: $f(x)$ is invertible only if it is one-one-onto / Bijective.
$f^{-1}(f(x)) = x$ | $[f'(f(x))]' f'(x) = 1$ | $f^{-1}(f(x))' = \frac{1}{f'(x)}$ Put $x = 1$, $y = k$ (say)

$\boxed{y^2 - xy^3 - x^3 = 0} \to$ Implicite function

- Periodic Function: If $f(x+T) = f(x)$, $T > 0$ → $f(x)$ is called period function with period T.
smallest value of T is called fundamental period of $f(x)$

→ period of $\sin x, \cos x, \operatorname{cosec} x, \sec x \to 2\pi$ | → $\tan x, \cot x \to \pi$ | $|\sin x|, |\cos x| \ldots \to \pi$

→ $\sin^n x, \cos^n x, \sec^n x, \operatorname{cosec}^n x \to \begin{cases} 2\pi, & n \in odd \\ \pi, & n \in even \end{cases}$ | $\tan^n x, \cot^n x \to \pi$ | If $f(x) \to T \leadsto f(x) \pm k \to T$; $\leadsto f(x \pm k) \to T \leadsto k f(x) \to T$

→ const functions are periodic but period not defined. | $f(x) \to T_1$, $g(x) \to T_2$ → $f(x) \pm g(x) / \frac{f(x)}{g(x)}$ / $f(x) \cdot g(x) \to LCM(T_1, T_2)$

→ $f(x-a) = f(x+a) \to T = 2a$

→ $f(a-x) = f(a+x) \to f(x)$ is symmetric abt. line $x = a$ ~~period 2|a-b|~~

→ $f(a-x) = f(a+x)$ & $f(b-x) = f(b+x)$ → Periodic abt. $2|a-b|$

→ $f(a-x) = f(a+x)$ → $2|a-b|$ → $2|a-c|$
$f(b-x) = f(b+x)$ → $2|b-c|$
$f(c-x) = f(c+x)$

Period of $f(x) = \min\{2|a-b|, 2|b-c|, 2|c-a|\}$

Limits

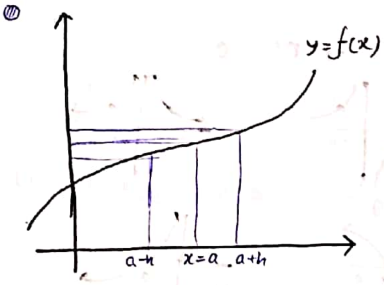
Expansions:

- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$
- $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5$
- $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$
- $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots$
- $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$
- $a^x = 1 + \frac{x\ln a}{1!} + \frac{(x\ln a)^2}{2!} + \frac{(x\ln a)^3}{3!} + \ldots$
- $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$
- $\sin^{-1}x = x + \frac{x^3}{6} + \frac{3}{40}x^5 + \ldots$

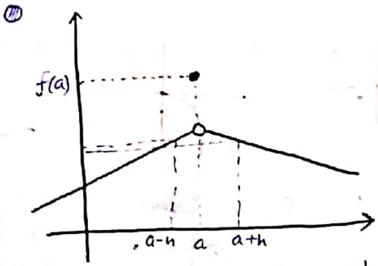
Important Results

- $\lim\limits_{x\to 0}\frac{\sin x}{x} = \lim\limits_{x\to 0}\frac{x}{\sin x} = \lim\limits_{x\to 0}\frac{\tan x}{x} = \lim\limits_{x\to 0}\frac{x}{\tan x} = 1$
- $\lim\limits_{x\to 0}\frac{\sin^{-1}x}{x} = \lim\limits_{x\to 0}\frac{\tan^{-1}x}{x} = 1$
- $\lim\limits_{x\to 0}\frac{e^x-1}{x} = 1$
- $\lim\limits_{x\to 0}\frac{a^x-1}{x} = \ln a$
- $\lim\limits_{x\to 0}\frac{\ln(1+x)}{x} = 1$
- $\lim\limits_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}$

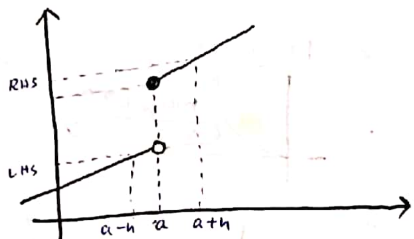
Continuity



If $\underset{x=a}{LHL} = \underset{x=a}{RHL} = f(a)$
$\rightarrow$ $y=f(x)$ is continuous at $x=a$

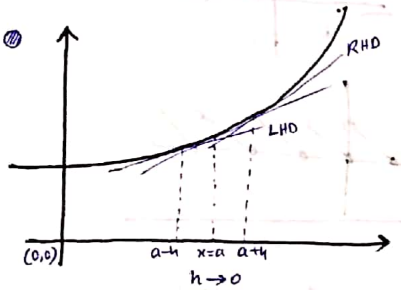


If LHL = RHL $\neq f(a)$, Discontinuous at $x=a$, point discontinuity / Removable discontinuity.



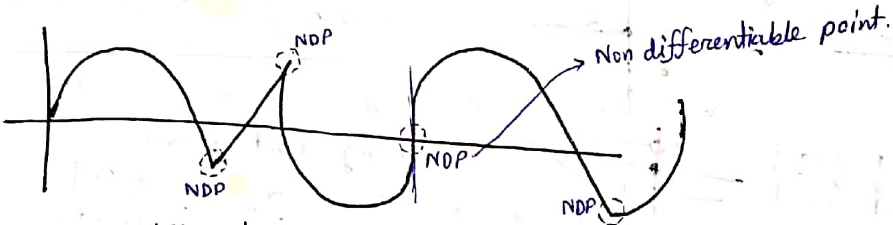
LHL $\neq$ RHL, Discontinuous at $x=a$.
Jump Discontinuity.

Differentiability



If LHD = RHD at $x=a$, $f(x)$ is differentiable at $x=a$

- Sharp turns lead to non-differentiable points.
- Smooth curves are generally differentiable at all points.
- Tangents must have finite slope to make function differentiable.



RHD at $x=a$

$$\boxed{Rf'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}}$$

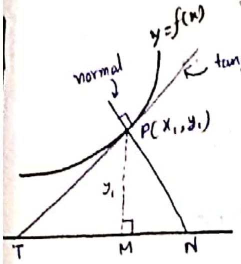
LHD at $x=a$

$$\boxed{Lf'(a) = \lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}}$$

- Discontinuous $\Rightarrow$ non-differentiable
- Differentiable $\Rightarrow$ continuous.

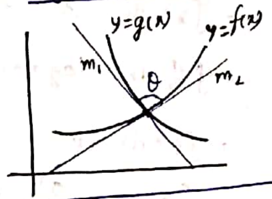
$f(x)\rightarrow$ diff $\rightarrow f'(x)\rightarrow$ cont / $f''(x)\rightarrow$ cont $\rightarrow f'(x)\rightarrow$ diff

Application of Derivatives



- Slope of tangent $= \frac{d}{dx}f(x)\Big|_{(x_1,y_1)} = \tan\theta$
- Slope of normal $= -\frac{dx}{dy}\Big|_{(x_1,y_1)} = -\cot\theta$
- Equation of tangent: $(y-y_1) = \frac{dy}{dx}\Big|_{(x_1,y_1)}(x-x_1)$
- Equation of normal: $(y-y_1) = -\frac{dx}{dy}\Big|_{(x_1,y_1)}(x-x_1)$

Angle b/w 2 curves



$\frac{d}{dx}f(x)\Big|_{(x_1,y_1)} = m_1$

$\frac{d}{dx}g(x)\Big|_{(x_1,y_1)} = m_2$

$$\boxed{\theta = \tan^{-1}\left|\frac{m_1-m_2}{1+m_1m_2}\right|}$$

- $\theta = 90^\circ$ is called orthogonal intersection.

Length of tangent (PT) $= |y|\operatorname{cosec}\theta$, $\boxed{|PT| = |y_1|\sqrt{1+\left(\frac{dx}{dy}\right)^2}}$

Length of normal (PN) $= |y_1|\sec\theta$, $\boxed{|PN| = |y_1|\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$

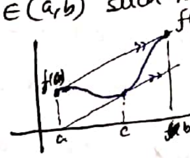
Length of Sub-tangent $= |y_1|\cot\theta$ $\boxed{|TM| = |y_1|\left|\frac{dx}{dy}\right|}$

Length of Sub-normal $= |y_1|\tan\theta$ $\boxed{|MN| = |y_1|\left|\frac{dy}{dx}\right|}$

Lagranges Mean Value Theorem (LMVT)

if $y=f(x)$ is continuous on $[a,b]$ & differentiable on (a,b), There exist at least one such value of $c\in(a,b)$ such that

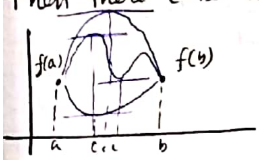
$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$



Rolle's Theorem

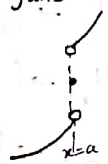
$f(x)$ is cont on $[a,b]$, diff on (a,b) and $f(a)=f(b)$

Then there exist at least one $c\in(a,b)$ so that $f'(c)=0$



Monotonocity:

- $\frac{dy}{dx}>0 \rightarrow y=f(x)$ is an increasing function
- $\frac{dy}{dx}<0 \rightarrow y=f(x)$ is an strictly decreasing function
- $\frac{dy}{dx}\geq 0 \rightarrow y=f(x)$ is a non-decreasing function
- $\frac{dy}{dx}\leq 0 \rightarrow y=f(x)$ is a non-increasing function.



Maxima / Minima:

$\frac{dy}{dx}=0 \Rightarrow x=a,b$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$\frac{d^2y}{dx^2}\Big|_{x=a} < 0$ [$x=a$ is maxima]

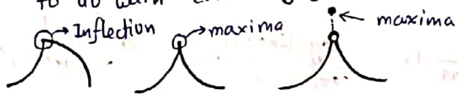
$\frac{d^2y}{dx^2}\Big|_{x=b} > 0$ [$x=b$ is minima]

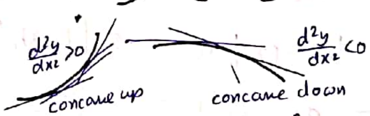
$\frac{d^2y}{dx^2}\Big|_{x=c} = 0$ [$x=c$ is inflection]

For Inflection:

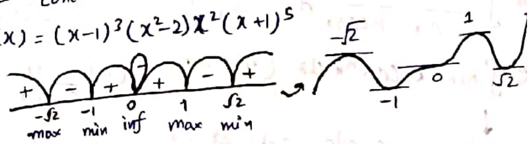
- $\frac{dy}{dx}$ NEED NOT TO BE ZERO
- $\frac{d^2y}{dx^2}=0$
- Around point of inflection, graph changes its concavity.

Monotonocity/Maxima/Minima have NOTHING to do with continuity of the graph.





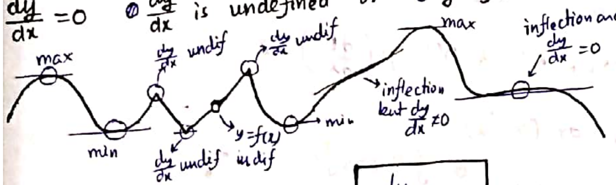
Q» $f'(x) = (x-1)^3(x^2-2)x^2(x+1)^5$



Note: $\frac{d^2y}{dx^2}=0$ at $x=a$ is a point of inflection provided $\frac{d^3y}{dx^3}$ is non zero at $x=a$

Critical points:

$\frac{dy}{dx}=0$ or $\frac{dy}{dx}$ is undefined or $y=f(x)$ is undefined



$\boxed{\frac{dy}{dx}=0}$

- $\boxed{\frac{d^2y}{dx^2}>0}$ (minimum)
- $\boxed{\frac{d^2y}{dx^2}<0}$ (maximum)
- $\boxed{\frac{d^2y}{dx^2}=0}$
  - $\boxed{\frac{d^3y}{dx^3}\neq 0}$ (Inflection)
  - $\boxed{\frac{d^3y}{dx^3}=0}$
    - $\boxed{\frac{d^4y}{dx^4}>0}$ (minimum)
    - $\boxed{\frac{d^4y}{dx^4}<0}$ (maximum)
    - $\boxed{\frac{d^4y}{dx^4}=0}$
      - ⋮

Indefinite Integral

- $\frac{d}{dx} x^n = nx^{n-1} \rightarrow \int x^n dx = \frac{x^{n+1}}{n+1} + c$
- $\frac{d}{dx} \log x = \frac{1}{x} \rightarrow \int \frac{1}{x} dx = \log x + c$
- $\frac{d}{dx} e^x = e^x \rightarrow \int e^x dx = e^x$
- $\frac{d}{dx} a^x = a^x \ln a \rightarrow \int a^x dx = \frac{a^x}{\ln a} + c$
- $\frac{d}{dx} \sin x = \cos x \rightarrow \int \cos x\, dx = \sin x + c$
- $\frac{d}{dx} \cos x = -\sin x \rightarrow$ ~~Integral~~ $\int \sin x\, dx = -\cos x + c$
- $\frac{d}{dx} \tan x = \sec^2 x \rightarrow \int \sec^2 x\, dx = \tan x + c$
- $\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x \rightarrow \int \operatorname{cosec}^2 x\, dx = -\cot x + c$
- $\frac{d}{dx} \sec x = \sec x \tan x \rightarrow \int \sec x \tan x\, dx = \sec x + c$
- $\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cot x \rightarrow \int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + c$
- $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \rightarrow \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$
- $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2} \rightarrow \int \frac{1}{1+x^2} dx = \tan^{-1} x + c$
- $\frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2-1}} \rightarrow \int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c$

- $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$
- $\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$
- $\int \frac{1}{x\sqrt{x^2-a^2}} dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c$
- $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$
- $\int \tan x\, dx = \log|\sec x| + c$
- $\int \cot x\, dx = \log|\sin x| + c$
- $\int \sec x\, dx = \log|\sec x + \tan x| + c$
  $= \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + c$
- $\int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| + c$
  $= \log\left|\tan \frac{x}{2}\right| + c$

ILATE — Inverse, log, Algebraic, Trigo, Expo

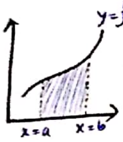
whichever comes first is the Ist function is by parts

- $\frac{d}{dx}$
- By Parts: $\int \mathrm{I}\cdot\mathrm{II}\, dx = \mathrm{I}\int \mathrm{II}\, dx - \int\left(\left(\frac{d}{dx}\mathrm{I}\right)\left(\int \mathrm{II}\, dx\right)\right)dx$
- $\int e^x(f(x) + f'(x)) = e^x f(x) + c$

Forms

- $\int \frac{1}{\text{linear}} dx = \frac{\log|\text{linear}|}{\text{coeff of } x} + c$
- $\int \frac{1}{(\text{linear})^n} dx = \frac{(\text{linear})^{-n+1}}{(-n+1)(\text{coff of } x)} + c$
- $\int \log x\, dx = x\log x - x + c$
- $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$
- $\int (f(x))^n f'(x)\, dx = \frac{(f(x))^{n+1}}{n+1} + c$
- $\int \frac{dx}{\text{Quad}}$
  - $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
  - $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$
  - $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$
- $\int \frac{1}{a\sin x + b\cos x} dx \rightarrow$ put $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$, $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$
- $\int \sin^m x \cos^n x\, dx$ $(m, n \in N)$
  - If $m, n \in$ odd, subs any
  - If one is odd, sub even
  - If both are even, use trigo
  - If both are rational and $\frac{m+n-2}{2}$ is $-$ve int. then sub $\cot x = p$ or $\tan x = p$
- $\int \frac{dx}{a\cos^2 x + b\sin^2 x}$, $\int \frac{dx}{a + b\sin^2 x}$, $\int \frac{dx}{(a\sin x + b\cos x)^2}$, $\int \frac{dx}{a + b\sin^2 x + \cos^2 x} \rightarrow$ divide N^r & D^r by $\cos^2 x$
- $\int \frac{p\cos x + q\sin x + r}{a\cos x + b\sin x + c} dx \rightarrow$ WR, $N^r = \lambda(D^r) + \mu\left(\frac{dD^r}{dx}\right) + \gamma$
- Biquadratic $\rightarrow$ sub $\left(x + \frac{1}{x}\right)$ or $\left(x - \frac{1}{x}\right) = t$
- $\int \text{linear}\sqrt{\text{Quad}}$ $\rightarrow$ $L = m(Q)' + n$
- $\int \frac{px+q}{ax^2+bx+c} dx$, $\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx \rightarrow$ WR $px + q = \frac{d}{dx}(ax^2+bx+c) + \mu$
- $\int \frac{1}{L_1\sqrt{L_2}} dx$, $\int \frac{L_1}{\sqrt{L_2}} dx$, $\int \frac{\sqrt{L_2}}{L_1} dx \rightarrow$ sub $L_2 = t^2$
- $\int \frac{1}{L\sqrt{Q}} dx \rightarrow$ sub $\frac{1}{t} = L$
- $\int \frac{1}{Q_1\sqrt{Q_2}} dx \rightarrow x = \frac{1}{t} \rightarrow$ Integrand will become $\int \frac{t\,dt}{(pt^2+q)\sqrt{rt^2+s}} \rightarrow$ then $u^2 = rt^2 + s$
- $\int \sqrt{\text{Quad}}\, dx \rightarrow$
  - $\int \sqrt{a^2+x^2}\, dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln|x + \sqrt{a^2+x^2}| + c$
  - $\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln|x + \sqrt{x^2-a^2}| + c$
  - $\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$

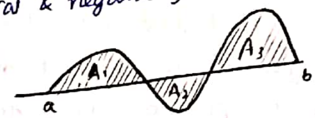
DEFINITE INTEGRATIONS



Area of shaded region $= \int_a^b f(x)\,dx$

- $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$

- Region lying above x axis will give +ve value of integral & negative for the portion lying below x axis.



$\int_a^b f(x)\,dx = A_1 - A_2 + A_3$

- **Properties:**
- $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ [c may or may not belong to (a, b)]
- $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$ [Turning Property]
- $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$
- $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is even func i.e. } f(-x) = f(x) \\ 0, & \text{if } f(x) \text{ is odd func} \end{cases}$
- While substituting, t must be conteneous in the interval

- **Properties related to periodic func:** [if $f(x+T) = f(x)$, period is T]
- $\int_0^{nT} f(x)\,dx = n\int_0^T f(x)\,dx,\ n \in I$
- $\int_a^{a+nT} f(x)\,dx = n\int_0^T f(x)\,dx,\ n \in I$
- $\int_{a+nT}^{b+nT} f(x)\,dx = \int_a^b f(x)\,dx$
- $\int_{mT}^{nT} f(x)\,dx = (n-m)\int_0^T f(x)\,dx,\ n, m \in I$

$$\boxed{\int_0^{\pi/2} \log \sin x\,dx = \int_0^{\pi/2} \log \cos x\,dx = -\frac{\pi}{2}\log 2}$$

- **Newton-Leibnitz 1st Rule:**

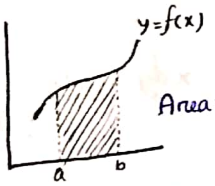
$$\boxed{\frac{d}{dx}\left(\int_{f(x)}^{g(x)} h(t)\,dt\right) = h(g(x)) \times \frac{d}{dx}(g(x)) - h(f(x)) \times \frac{d}{dx}(f(x))}$$

Only for IIT ⚠

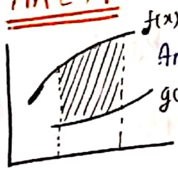
- **Leibnitz 2nd Rule:**

$$\boxed{\text{If } I(\alpha) = \int_a^b f(x,\alpha)\,dx \rightarrow \frac{\partial I}{\partial \alpha} = \int_a^b \frac{\partial f(x,\alpha)}{\partial \alpha}\,dx}$$

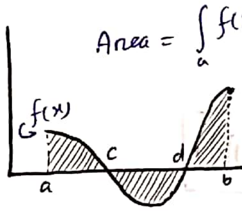
AREA



Area $= \int_a^b f(x)\,dx$

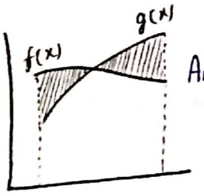


Area $= \int_a^b (f(x) - g(x))\,dx$



Area $= \int_a^c f(x)\,dx + \left|\int_c^d f(x)\,dx\right| + \int_d^b f(x)\,dx$

→ c and d are roots of $f(x) = 0$



Area $= \int_a^c (f(x) - g(x))\,dx + \int_c^b (g(x) - f(x))\,dx$

- **Vertical Strip:**

Area $= \int_{x=a}^{x=b} (\text{upper } y - \text{lower } y)\,dx$

- **Horizontal Strip:**

Area $= \int_{y=c}^{y=d} (\text{Right } x - \text{left } x)\,dy$

DIFFERENTIAL EQUATION

- Eq involving x, y & differentials co-efficient.
- DE represents a family of curves.
- **Order:** Order of highest order derivative present in the eqⁿ is the order of D.E.
- **Degree:** Degree of the highest order derivative present in the eqⁿ is the degree of DE, provided the eqⁿ is polynomial in different co-eff and eqⁿ is free from radicals.

Formation of DE: (Degree of a DE = No of arbitrary constants present in eqⁿ)

- DE of all lines passing thru origin: $y = mx$, $\frac{dy}{dx} = m$, $y = \frac{dy}{dx}x \rightarrow \boxed{x\,dy - y\,dx = 0}$
- DE of all lines: $y = mx + c$, $\frac{dy}{dx} = m$, $\boxed{\frac{d^2y}{dx^2} = 0}$

Solution of DE:

- **Variable-seperable form:** $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

$\Rightarrow \frac{dy}{dx} = \frac{e^x + x^2}{e^y} \Rightarrow \int e^y dy = \int (e^x + x^2)dx$

- **Eqⁿ Reducible to Variable Separable form**

$\frac{dy}{dx} = f(ax+by+c)$, consider, $ax+by+c = t$

- **Homogeneous form:**

$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ where f and g are of same order.

$\frac{dy}{dx} = h(y/x)$ assume $y/x = t$

- **Eqⁿ reducible to Homogenous form:** $\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+D}$

• [If $aB \neq Ab$ or $A+b \neq 0$]

$x = X + h$ $\quad y = Y + k$

$dx = dX$ $\quad dy = dY$

$\therefore \frac{dY}{dX} = \frac{aX + bY + ah + bk + c}{AX + BY + Ah + Bk + D}$

$ah + bk + c = 0$ & $Ah + Bk + D = 0$ → find value of h & k

$\frac{dY}{dX} = \frac{aX + bY}{AX + BY}$ → Homogeneous → In the end, $X = x - h$, $Y = y - k$

• If $aB = Ab \xrightarrow{\text{Assume}} (ax + by = t)$

• If $A + b = 0$ → simply cross multiply & replace $x\,dy + y\,dx$ by $d(xy)$

Linear Differential Eqⁿ:

$\Rightarrow \frac{dy}{dx} + Py = Q$ [P & Q are func of x alone]

$I.F = e^{\int P dx}$

$\rightarrow \boxed{y(I.F) = \int (Q(I.F))dx}$

$\Rightarrow \frac{dx}{dy} + Mx = N$ [M & N are func of y alone]

$I.F. = e^{\int M dy}$

$\rightarrow \boxed{x(I.F.) = \int (N(I.F.))\,dy}$

Barnoulli Eqⁿ

$\frac{dy}{dx} + \frac{y}{x} = y^n$

divide by y^n and then assume co-eff of x as t

here, $t = \frac{1}{y^{n-1}}$

curves.

gree of DE, provided the
ds.

of all lines: $y = mx + c$

$\frac{dy}{dx} = m$, $\boxed{\frac{d^2y}{dx^2} = 0}$

le Seperable form:

consider, $ax + by + c = t$

$\frac{ax + by + c}{Ax + By + D}$

$+C = 0$
$k + D = 0$ } find value of h & k

→ In the end, $X = x - h$
$Y = y - k$

replace $xdy + ydx$ by $d(xy)$

co-eff of x as t.

VECTORS

Angle bisector b/w two vectors:

Internal → $\vec{R} = \lambda(\hat{a} + \hat{b})$

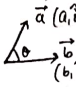
External → $\vec{Q} = \mu(\hat{a} - \hat{b})$

Section formula:

Internal → $\left(\frac{m\vec{b} + n\vec{a}}{m+n}\right)$

External → $\left(\frac{m\vec{b} - n\vec{a}}{m-n}\right)$

Dot (Scalar) Product:



$\vec{a}\,(a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$

$\vec{b}\,(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$

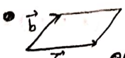
- $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$
- $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$
- $\vec{a} \cdot \vec{a} = |\vec{a}|^2$
- $\vec{a} \cdot \vec{b} = 0$ → perpendicular
- $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

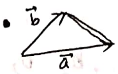
  $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$
- Angle b/w the vectors →

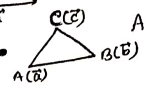
  $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$
- Projection of $\vec{a}$ on $\vec{b}$ →

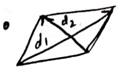
  $\vec{P} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

Cross (Vector) Product:

- $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$
- $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{r}$
- 

  Area $= |\vec{a} \times \vec{b}|$
- 

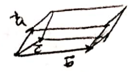
  Area $= \frac{1}{2}|\vec{a} \times \vec{b}|$
- 

  $A = \frac{1}{2}|\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}|$
- 

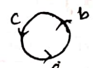
  Area $= \frac{1}{2}|\vec{d_1} \times \vec{d_2}|$

Scalar Triple product / Box Product:

$[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c})$ → volume of parallelop—



- $[k\vec{a}\ \vec{b}\ \vec{c}] = k[\vec{a}\ \vec{b}\ \vec{c}]$
- $[\vec{a}\vec{b}\vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$
- $[\vec{a}\ \vec{b}\vec{c}] = -[\vec{b}\ \vec{a}\ \vec{c}]$



- $[\vec{a}\ \vec{c}\ \vec{d}] + [\vec{b}\ \vec{c}\ \vec{d}] = [\vec{a}+\vec{b}\ \ \vec{c}\ \vec{d}]$
- $[\vec{a}\ \vec{b}\ \vec{b}] = 0$
- $[\vec{a}\ \vec{b}\ \vec{c}] = 0$ → $\vec{a}, \vec{b}, \vec{c}$ are coplanar.
- $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Vector Triple Product:

- $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\cdot\vec{a}$
- $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

Direction Cosines:

$P(x,y,z)$ $\alpha, \beta, \gamma \rightarrow$ direction angles

$\cos\alpha, \cos\beta, \cos\gamma \rightarrow$ direction cosines

$\cos\alpha = l,\ \cos\beta = m,\ \cos\gamma = n$

$\cos\alpha = \frac{x}{r},\ \cos\beta = \frac{y}{r},\ \cos\gamma = \frac{z}{r}$

$\boxed{l^2 + m^2 + n^2 = 1}$

Direction Ratios: Simple ratio of DC.

DR $\rightarrow$ DC >> DR$(a,b,c) \rightarrow$ DC $\left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right)$

DC $\rightarrow$ DR >> DC$\left(\frac{1}{\sqrt3}, -\frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right) \rightarrow$ DR$(1,-1,1)$ or $(2,-2,2)$ or $(\lambda, -\lambda, \lambda)$

(a_1, b_1, c_1) & $(a_2, b_2, c_2) \rightarrow \cos\theta = \frac{a_1a_2 - b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$ $= 0$ if $\theta = 90°$

(l_1, m_1, n_1) & $(l_2, m_2, n_2) \rightarrow \cos\theta = l_1l_2 + m_1m_2 + n_1n_2$

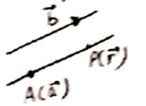
- (l_1, m_1, n_1) & (l_2, m_2, n_2) DC of 2 vectors $\rightarrow$ internal bisector $(l_1+l_2, m_1+m_2, n_1+n_2)$, external bisector $(l_1-l_2, m_1-m_2, n_1-n_2)$
- DR of line joining $A(a_1, b_1, c_1)$ & $B(a_2, b_2, c_2) \rightarrow (a_1 - a_2, b_1 - b_2, c_1 - c_2)$

	2D	3D
$x = 0$	y axis	yz plane
$y = 0$	x axis	xz plane
$x = 0, y = 0$	origin	z axis

- In 3D line is the intersection of 2 planes.

Equation of line in 3D:

(i) Equation of line passing thru a point $\vec a$ and parallel to another vector $\vec b$



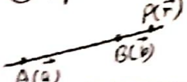
$\boxed{\vec r = \vec a + \lambda \vec b}$ vector form

(ii) $\vec r \equiv (x, y, z)$, $\vec a = (x_1, y_1, z_1)$, $\vec b$ (DR) $\equiv (a, b, c) \rightarrow a\hat i + b\hat j - c\hat k$

$\vec r - \vec a = \lambda \vec b$

$\boxed{\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \lambda}$ Cartesian form

(iii) Equation of line passing thru 2 points.



$\boxed{\vec r = \vec a + \lambda(\vec b - \vec a)}$ vector form

$$\boxed{\begin{aligned} \frac{x - x_1}{x_2 - x_1} &= \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = \lambda \\ &\text{or} \\ \frac{x - x_2}{x_2 - x_1} &= \frac{y - y_2}{y_2 - y_1} = \frac{z - z_2}{z_2 - z_1} = \lambda \end{aligned}}$$ Cartesian form

Angle b/w 2 lines: $\vec r = \vec a + \lambda \vec b$, $\vec r = \vec c + \mu \vec d$

$\cos\theta = \frac{\vec b \cdot \vec d}{|\vec b||\vec d|}$

Shortest distance b/w 2 lines: [Shortest dist = 0 if intersecting]

$\rightarrow$ If parallel $\rightarrow \vec r = \vec a + \lambda \vec b$, $\vec r = \vec c + \mu \vec b$

$\rightarrow$ then shortest dist $= \frac{|(\vec a - \vec c) \times \vec b|}{|\vec b|}$

$\rightarrow$ If skew: $\rightarrow \vec r = \vec a + \lambda \vec b$, $\vec r = \vec c + \mu \vec d$

Shortest dist $= \frac{[(\vec a - \vec c)\, \vec b\, \vec d]}{|\vec b \times \vec d|}$

- If two lines are intersector, then $[(\vec a - \vec c)\, \vec b\, \vec d] = 0$

Cartesian form:

$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ & $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$

$$\text{Shortest dist} = \frac{\left|\begin{matrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{matrix}\right|}{\left|\begin{matrix} \hat i & \hat j & \hat k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{matrix}\right|}$$

Plane passing thru a point $\vec a$ & normal to vector $\hat n$:

$(\vec r - \vec a) \cdot \vec n = 0$

Cartesian form: $r = (x, y, z)$, $a \equiv (x_1, y_1, z_1)$, $\hat n \rightarrow a, b, c$ (DR)

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

- Plane passing thru 3 points $\vec a, \vec b$ & $\vec c$: $[\vec{AP}\ \vec{AB}\ \vec{AC}] = 0$
- Plane passing thru points & parallel to vector $\vec b$ & $\vec c$:

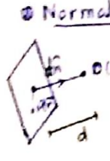
$[(\vec r - \vec a)\, \vec b\, \vec c] = 0$

- Plane passing thru point & a line $\vec r = \vec c + \lambda \vec b$:

$[(\vec r - \vec a)\ (\vec a - \vec c)\ \vec b] = 0$

Plane:

Normal form:



$\vec r \cdot \hat n = d$

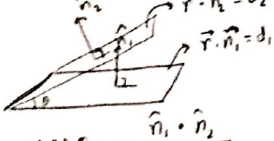
Cartesian form:

$lx + my + nz = 0$

$r = x, y, z$

$\hat n \rightarrow l, m, n$

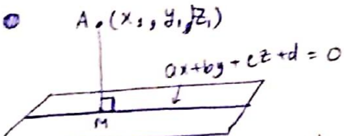
Angle b/w two planes:



$\vec r \cdot \hat n_2 = d_2$

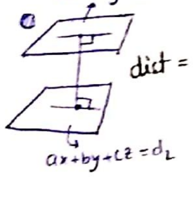
$\vec r \cdot \hat n_1 = d_1$

$\cos\theta = \frac{\hat n_1 \cdot \hat n_2}{|\hat n_1||\hat n_2|}$



$A(x_1, y_1, z_1)$, $ax + by + cz + d = 0$

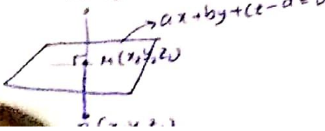
$AM = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$



$ax + by + cz = d_1$, $ax + by + cz = d_2$

$\text{dist} = \left|\frac{d_1 - d_2}{\sqrt{a^2 + b^2 + c^2}}\right|$

Foot of normal and image



$A(x, y, z)$, $ax + by + cz - d = 0$, $M(x_2, y_2, z_2)$

$$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{z_2 - z_1}{c} = -\frac{[ax_1 + by_1 + cz_1 - d]}{a^2 + b^2 + c^2}$$

$$\frac{x_3 - x_1}{a} = \frac{y_3 - y_1}{b} = \frac{z_3 - z_1}{c} = -\frac{2|ax_1 + by_1 + cz_1 - d|}{a^2 + b^2 + c^2}$$

Convert the eq^n of line $(2x - y + z = 1$ & $x + y + 2z = 0)$ in Cartesian form.

$\Rightarrow$ let $z = t$

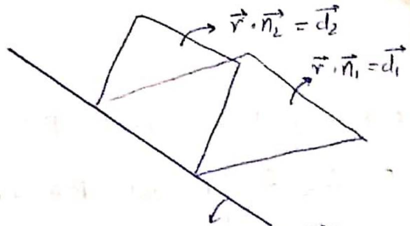
$$\begin{aligned} 2x - y &= 1 - t \\ x + y &= -2t \\ \hline 3x &= 1 - 3t \\ x &= \frac{1-3t}{3} \end{aligned}$$

$y = -\frac{1-3t}{3} \longrightarrow t = \frac{1-3x}{3}$

$t = \frac{-1-3y}{3}$

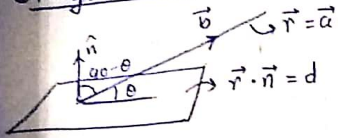
$\therefore$ so $\frac{1-3x}{3} = \frac{-1-3y}{3} = z$

$\Rightarrow \frac{x - \frac{1}{3}}{-1} = \frac{y + \frac{1}{3}}{-1} = \frac{z - 0}{1}$



DR in $(\vec{n_1} \times \vec{n_2})$

Angle b/w plane & Line:



$\vec{r} = \vec{a} + \lambda \vec{d}$

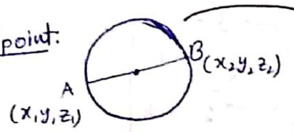
$\vec{r} \cdot \vec{n} = d$

Sphere: $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

centre $(-u, -v, -w)$

radius $= \sqrt{u^2 + v^2 + w^2 - d}$

$\rightarrow$ Diametric point:



$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$

PROBABILITY

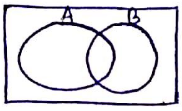
Probability of an event E, $P(E) = \dfrac{\text{Favourable outcomes}}{\text{Total no of outcomes}}$

$\boxed{0 \leq P(E) \leq 1}$

$P(E) = 0 \longrightarrow$ Impossible event

$P(E) = 1 \longrightarrow$ Certain event

$\boxed{P(E) + P(\bar{E}) = 1}$



$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$A \cup B \rightarrow$ A or B | $A \cap B \rightarrow$ A and B

- **Mutually exclusive Events:** Only one of the events can occur at a time. $P(A \cap B) = 0$
- **Mutually independent Events:** Occurence of one event doesn't affect other events. $P(A \cap B) = P(A) \times P(B)$
- **Conditional Probability:** Prob of A given that B has already occured, $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

  Prob of B ——— A ———————, $P(B|A) = \dfrac{P(A \cap B)}{P(A)}$

4 suits → Heart, Spade, Club, Diamond

- **Playing Crads:** 52 cards
  - Black → 13 spade, 13 club
  - Red → 13 hearts, 13 diamonds
  - Each suit → 1 ace, 1 king, 1 queen, 1 Jack, 2-10

Honor card: Ace, king, Queen, Jack $(4 \times 4 = 16)$

Face card: King, Queen, Jack $(3 \times 4 = 12)$

Numbered cards: $(2 \rightarrow 10)$ $9 \times 4 = 36$

- **Baye's Theorem:**

$$P(E/E_2) = \frac{P(E) \times P(E_2/E)}{P(E_1)\,P(E_2/E_1) + P(E)\,P(E_2/E)}$$

$E \longrightarrow$ favoured event

$E_1 \longrightarrow$ opposite of favoured event

$E_2 \longrightarrow$ Already happened event.