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1	In triangle $ABC,$ angle A is twice angle $B,$ angle C is obtuse, and the three side lengths a, b, c are integers. Determine, with proof, the minimum possible perimeter.
2	For any nonempty set S of numbers, let $\sigma(S)$ and $\pi(S)$ denote the sum and product, respectively, of the elements of S. Prove that $$\sum \frac{\sigma(S)}{\pi(S)} = (n^2+2n) - \left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)(n+1),$$ where "Σ" denotes a sum involving all nonempty subsets S of $\{1,2,3,\ldots,n\}$.
3	Show that, for any fixed integer $n \geq 1,$ the sequence $$2,\ 2^2,\ 2^{2^2},\ 2^{2^{2^2}}, \ldots \text{(mod } n)$$ is eventually constant. [The tower of exponents is defined by $a_1 = 2,\ a_{i+1} = 2^{a_i}$. Also a_i (mod n) means the remainder which results from dividing a_i by n.]
4	Let $a = \frac{m^{m+1}+n^{n+1}}{m^m+n^n}$, where m and n are positive integers. Prove that $a^m + a^n \geq m^m + n^n$.
5	Let D be an arbitrary point on side AB of a given triangle $ABC,$ and let E be the interior point where CD intersects the external common tangent to the incircles of triangles ACD and BCD. As D assumes all positions between A and B, prove that the point E traces the arc of a circle.
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