AIME Problems 1997

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1 How many of the integers between 1 and 1000, inclusive, can be expressed as the difference of the squares of two nonnegative integers?

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2 The nine horizontal and nine vertical lines on an 8×8 checkerboard form r rectangles, of which s are squares. The number s/r can be written in the form $m/n,$ where m and n are relatively prime positive integers. Find $m+n.$

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3 Sarah intended to multiply a two-digit number and a three-digit number, but she left out the multiplication sign and simply placed the two-digit number to the left of the three-digit number, thereby forming a five-digit number. This number is exactly nine times the product Sarah should have obtained. What is the sum of the two-digit number and the three-digit number?

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4 Circles of radii 5, 5, 8, and m/n are mutually externally tangent, where m and n are relatively prime positive integers. Find $m+n.$

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5 The number r can be expressed as a four-place decimal $0.abcd,$ where $a, b, c,$ and d represent digits, any of which could be zero. It is desired to approximate r by a fraction whose numerator is 1 or 2 and whose denominator is an integer. The closest such fraction to r is $\frac{2}{7}$. What is the number of possible values for r?

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6 Point B is in the exterior of the regular n-sided polygon $A_1A_2\cdots A_n$, and A_1A_2B is an equilateral triangle. What is the largest value of n for which $A_n, A_1,$ and B are consecutive vertices of a regular polygon?

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7 A car travels due east at $\frac{2}{3}$ mile per minute on a long, straight road. At the same time, a circular storm, whose radius is 51 miles, moves southeast at $\frac{1}{2}\sqrt{2}$ mile per minute. At time $t=0$, the center of the storm is 110 miles due north of the car. At time $t=t_1$ minutes, the car enters the storm circle, and at time $t=t_2$ minutes, the car leaves the storm circle. Find $\frac{1}{2}(t_1+t_2)$.

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8 How many different 4×4 arrays whose entries are all 1's and -1's have the property that the sum of the entries in each row is 0 and the sum of the entires in each column is 0?

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9 Given a nonnegative real number x, let $\langle x\rangle$ denote the fractional part of x; that is, $\langle x\rangle = x - \lfloor x\rfloor$, where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x. Suppose that a is positive, $\langle a^{-1}\rangle = \langle a^2\rangle$, and $2 < a^2 < 3$. Find the value of $a^{12} - 144a^{-1}$.

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10 Every card in a deck has a picture of one shape - circle, square, or triangle, which is painted in one of the three colors - red, blue, or green. Furthermore, each color is applied in one of three shades - light, medium, or dark. The deck has 27 cards, with every shape-color-shade combination represented. A set of three cards from the deck is called complementary if all of the following statements are true:

i. Either each of the three cards has a different shape or all three of the card have the same shape.
ii. Either each of the three cards has a different color or all three of the cards have the same color.
iii. Either each of the three cards has a different shade or all three of the cards have the same shade.

How many different complementary three-card sets are there?

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11 Let $x = \frac{\sum\limits_{n=1}^{44} \cos n^\circ}{\sum\limits_{n=1}^{44} \sin n^\circ}$. What is the greatest integer that does not exceed $100x$?

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12 The function f defined by $f(x) = \frac{ax+b}{cx+d}$. where a, b, c and d are nonzero real numbers, has the properties $f(19) = 19, f(97) = 97$ and $f(f(x)) = x$ for all values except $\frac{-d}{c}$. Find the unique number that is not in the range of f.

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13 Let S be the set of points in the Cartesian plane that satisfy

$$\Big|\big||x|-2\big|-1\Big|+\Big|\big||y|-2\big|-1\Big|=1.$$

If a model of S were built from wire of negligible thickness, then the total length of wire required would be $a\sqrt{b}$, where a and b are positive integers and b is not divisible by the square of any prime number. Find $a+b$.

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14 Let v and w be distinct, randomly chosen roots of the equation $z^{1997} - 1 = 0$. Let m/n be the probability that $\sqrt{2+\sqrt{3}} \le |v+w|$, where m and n are relatively prime positive integers. Find $m+n$.

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15 The sides of rectangle $ABCD$ have lengths 10 and 11. An equilateral triangle is drawn so that no point of the triangle lies outside $ABCD$. The maximum possible area of such a triangle can be written in the form $p\sqrt{q} - r$, where $p, q,$ and r are positive integers, and q is not divisible by the square of any prime number. Find $p+q+r$.

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