

USAMO 2012

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Day 1 April 24th

1 Find all integers $n \geq 3$ such that among any n positive real numbers a_1, a_2, \dots, a_n with $\max(a_1, a_2, \dots, a_n) \leq n \cdot \min(a_1, a_2, \dots, a_n)$, there exist three that are the side lengths of an acute triangle.

2 A circle is divided into 432 congruent arcs by 432 points. The points are colored in four colors such that some 108 points are colored Red, some 108 points are colored Green, some 108 points are colored Blue, and the remaining 108 points are colored Yellow. Prove that one can choose three points of each color in such a way that the four triangles formed by the chosen points of the same color are congruent.

3 Determine which integers $n > 1$ have the property that there exists an infinite sequence a_1, a_2, a_3, \dots of nonzero integers such that the equality

$$a_k + 2a_{2k} + \dots + na_{nk} = 0$$

holds for every positive integer k .

Day 2 April 25th

4 Find all functions $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ (where \mathbb{Z}^+ is the set of positive integers) such that $f(n!) = f(n)!$ for all positive integers n and such that $m - n$ divides $f(m) - f(n)$ for all distinct positive integers m, n .

5 Let P be a point in the plane of $\triangle ABC$, and γ a line passing through P . Let A', B', C' be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, AC, AB respectively. Prove that A', B', C' are collinear.

6 For integer $n \geq 2$, let x_1, x_2, \dots, x_n be real numbers satisfying

$$x_1 + x_2 + \dots + x_n = 0, \quad \text{and} \quad x_1^2 + x_2^2 + \dots + x_n^2 = 1.$$

For each subset $A \subseteq \{1, 2, \dots, n\}$, define

$$S_A = \sum_{i \in A} x_i.$$

(If A is the empty set, then $S_A = 0$.)

Prove that for any positive number λ , the number of sets A satisfying $S_A \geq \lambda$ is at most $2^{n-3}/\lambda^2$. For which choices of $x_1, x_2, \dots, x_n, \lambda$ does equality hold?



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