AMC 12/AHSME 2005

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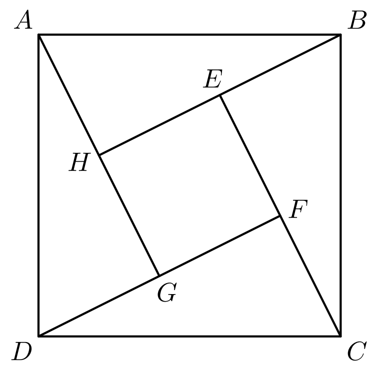
1	Two is 10% of x and 20% of y. What is $x - y$? **(A)** 1 **(B)** 2 **(C)** 5 **(D)** 10 **(E)** 20
2	The equations $2x + 7 = 3$ and $bx - 10 = -2$ have the same solution for x. What is the value of b? **(A)** -8 **(B)** -4 **(C)** -2 **(D)** 4 **(E)** 8
3	A rectangle with a diagonal of length x is twice as long as it is wide. What is the area of the rectangle? **(A)** $\frac{1}{4}x^2$ **(B)** $\frac{2}{5}x^2$ **(C)** $\frac{1}{2}x^2$ **(D)** x^2 **(E)** $\frac{3}{2}x^2$
4	A store normally sells windows at $\$100$ each. This week the store is offering one free window for each purchase of four. Dave needs seven windows and Doug needs eight windows. How many dollars will they save if they purchase the windows together rather than separately? **(A)** 100 **(B)** 200 **(C)** 300 **(D)** 400 **(E)** 500
5	The average (mean) of 20 numbers is 30, and the average of 30 other numbers is 20. What is the average of all 50 numbers? **(A)** 23 **(B)** 24 **(C)** 25 **(D)** 26 **(E)** 27
6	Josh and Mike live 13 miles apart. Yesterday, Josh started to ride his bicycle toward Mike's house. A little later Mike started to ride his bicycle toward Josh's house. When they met, Josh had ridden for twice the length of time as Mike and at four-fifths of Mike's rate. How many miles had Mike ridden when they met? **(A)** 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8
7	Square $EFGH$ is inside the square $ABCD$ so that each side of $EFGH$ can be extended to pass through a vertex of $ABCD$. Square $ABCD$ has side length $\sqrt{50}$ and $BE = 1$. What is the area of the inner square $EFGH$?



(A) 25 **(B)** 32 **(C)** 36 **(D)** 40 **(E)** 42

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8 Let A, M, and C be digits with

$$(100A + 10M + C)(A + M + C) = 2005.$$

What is A?
(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

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9 There are two values of a for which the equation $4x^2 + ax + 8x + 9 = 0$ has only one solution for x. What is the sum of these values of a?
(A) -16 **(B)** -8 **(C)** 0 **(D)** 8 **(E)** 20

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10 A wooden cube n units on a side is painted red on all six faces and then cut into n^3 unit cubes. Exactly one-fourth of the total number of faces of the unit cubes are red. What is n?
(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

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11 How many three-digit numbers satisfy the property that the middle digit is the average of the first and the last digits?
(A) 41 **(B)** 42 **(C)** 43 **(D)** 44 **(E)** 45

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12 A line passes through $A(1, 1)$ and $B(100, 1000)$. How many other points with integer coordinates are on the line and strictly between A and B?
(A) 0 **(B)** 2 **(C)** 3 **(D)** 8 **(E)** 9

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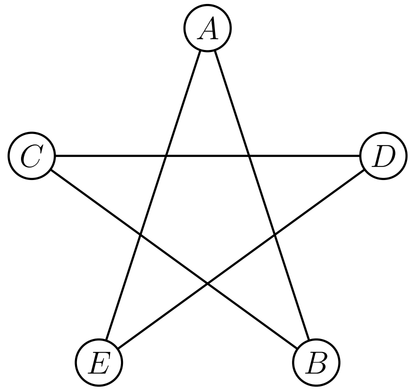
13 In the five-sided star shown, the letters A, B, C, D, and E are replaced by the numbers $3, 5, 6, 7$, and 9, although not necessarily in this order. The sums of the numbers at the ends of the line segments $\overline{AB}$,$\overline{BC}$,$\overline{CD}$,$\overline{DE}$, and $\overline{EA}$ form an arithmetic sequence, although not necessarily in this order. What is the middle term of the arithmetic sequence?



(A) 9 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 13

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14 On a standard die one of the dots is removed at random with each dot equally likely to be chosen. The die is then rolled. What is the probability that the top face has an odd number of dots?

(A) $\frac{5}{11}$ **(B)** $\frac{10}{21}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{11}{21}$ **(E)** $\frac{6}{11}$

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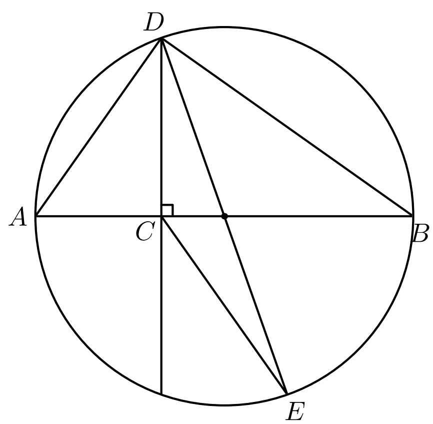
15 Let $\overline{AB}$ be a diameter of a circle and C be a point on $\overline{AB}$ with $2 \cdot AC = BC$. Let D and E be points on the circle such that $\overline{DC} \perp \overline{AB}$ and $\overline{DE}$ is a second diameter. What is the ratio of the area of $\triangle DCE$ to the area of $\triangle ABD$?



(A) $\frac{1}{6}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{2}{3}$

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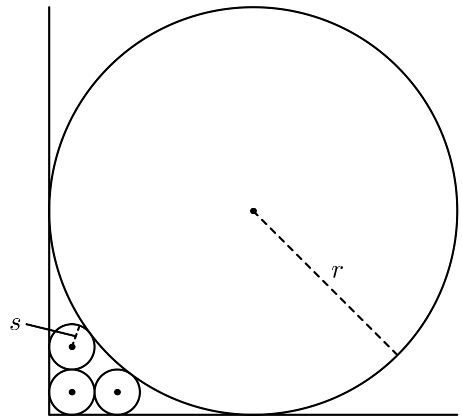
16 Three circles of radius s are drawn in the first quadrant of the xy-plane. The first circle is tangent to both axes, the second is tangent to the first circle and the x-axis, and the third is tangent to the first circle and the y-axis. A circle of radius $r > s$ is tangent to both axes and to the second and third circles. What is r/s?



(A) 5 **(B)** 6 **(C)** 8 **(D)** 9 **(E)** 10

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17 A unit cube is cut twice to form three triangular prisms, two of which are congruent, as shown in Figure 1. The cube is then cut in the same manner along the dashed lines shown in Figure 2. This creates nine pieces. What is the volume of the piece that contains vertex W?

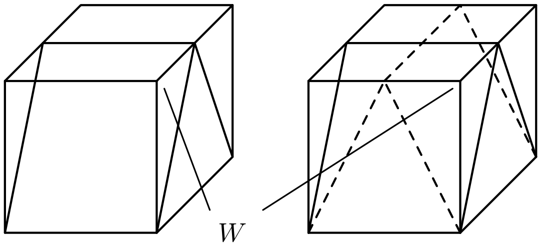

Figure 1 Figure 2

(A) $\frac{1}{12}$ **(B)** $\frac{1}{9}$ **(C)** $\frac{1}{8}$ **(D)** $\frac{1}{6}$ **(E)** $\frac{1}{4}$

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18 Call a number "prime-looking" if it is composite but not divisible by 2, 3, or 5. The three smallest prime-looking numbers are 49, 77, and 91. There are 168 prime numbers less than 1000. How many prime-looking numbers are there less than 1000?
(A) 100 **(B)** 102 **(C)** 104 **(D)** 106 **(E)** 108

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19 A faulty car odometer proceeds from digit 3 to digit 5, always skipping the digit 4, regardless of position. If the odometer now reads 002005, how many miles has the car actually traveled?
(A) 1404 **(B)** 1462 **(C)** 1604 **(D)** 1605 **(E)** 1804

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20 For each x in $[0,1]$, define

$$f(x)=\begin{cases}2x, & \text{if } 0\le x\le \frac{1}{2};\\ 2-2x, & \text{if } \frac{1}{2}<x\le 1.\end{cases}$$

Let $f^{[2]}(x)=f(f(x))$, and $f^{[n+1]}(x)=f^{[n]}(f(x))$ for each integer $n\ge 2$. For how many values of x in $[0,1]$ is $f^{[2005]}(x)=\frac{1}{2}$?
(A) 0 **(B)** 2005 **(C)** 4010 **(D)** 2005^2 **(E)** 2^{2005}

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21 How many ordered triples of integers (a,b,c), with $a\ge 2$, $b\ge 1$, and $c\ge 0$, satisfy both $\log_a b=c^{2005}$ and $a+b+c=2005$?
(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

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22 A rectangular box P is inscribed in a sphere of radius r. The surface area of P is 384, and the sum of the lengths of its 12 edges is 112. What is r?
(A) 8 **(B)** 10 **(C)** 12 **(D)** 14 **(E)** 16

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23 Two distinct numbers a and b are chosen randomly from the set $\{2,2^2,2^3,\dots,2^{25}\}$. What is the probability that $\log_a b$ is an integer?
(A) $\frac{2}{25}$ **(B)** $\frac{31}{300}$ **(C)** $\frac{13}{100}$ **(D)** $\frac{7}{50}$ **(E)** $\frac{1}{2}$

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24 Let $P(x)=(x-1)(x-2)(x-3)$. For how many polynomials $Q(x)$ does there exist a polynomial $R(x)$ of degree 3 such that $P(Q(x))=P(x)\cdot R(x)$?
(A) 19 **(B)** 22 **(C)** 24 **(D)** 27 **(E)** 32

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25 Let S be the set of all points with coordinates (x,y,z), where x, y, and z are each chosen from the set $\{0,1,2\}$. How many equilateral triangles have all their vertices in S?
(A) 72 **(B)** 76 **(C)** 80 **(D)** 84 **(E)** 88

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1 A scout troop buys 1000 candy bars at a price of five for $\$2$. They sell all the candy bars at a price of two for $\$1$. What was their prot, in dollars?

(A) 100 **(B)** 200 **(C)** 300 **(D)** 400 **(E)** 500

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2 A positive number x has the property that $x\%$ of x is 4. What is x?
(A) 2 **(B)** 4 **(C)** 10 **(D)** 20 **(E)** 40

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3 Brianna is using part of the money she earned on her weekend job to buy several equally-priced CDs. She used one fifth of her money to buy one third of the CDs. What fraction of her money will she have left after she buys all the CDs?

(A) $\frac{1}{5}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{2}{5}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{4}{5}$

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4 At the beginning of the school year, Lisas goal was to earn an A on at least 80% of her 50 quizzes for the year. She earned an A on 22 of the first 30 quizzes. If she is to achieve her goal, on at most how many of the remaining quizzes can she earn a grade lower than an A?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

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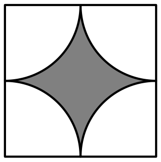
5 An 8-foot by 10-foot floor is tiled with square tiles of size 1 foot by 1 foot. Each tile has a pattern consisting of four white quarter circles of radius $1/2$ foot centered at each corner of the tile. The remaining portion of the tile is shaded. How many square feet of the floor are shaded?



(A) $80-20\pi$ **(B)** $60-10\pi$ **(C)** $80-10\pi$ **(D)** $60+10\pi$ **(E)** $80+10\pi$

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6 In $\triangle ABC$, we have $AC=BC=7$ and $AB=2$. Suppose that D is a point on line AB such that B lies between A and D and $CD=8$. What is BD?
(A) 3 **(B)** $2\sqrt{3}$ **(C)** 4 **(D)** 5 **(E)** $4\sqrt{2}$

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7 What is the area enclosed by the graph of $|3x|+|4y|=12$?
(A) 6 **(B)** 12 **(C)** 16 **(D)** 24 **(E)** 25

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8 For how many values of a is it true that the line $y=x+a$ passes through the vertex of the parabola $y=x^2+a^2$?
(A) 0 **(B)** 1 **(C)** 2 **(D)** 10 **(E)** infinitely many

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9 On a certain math exam, 10% of the students got 70 points, 25% got 80 points, 20% got 85 points, 15% got 90 points, and the rest got 95 points. What is the difference between the mean and the median score on this exam?
(A) 0 **(B)** 1 **(C)** 2 **(D)** 4 **(E)** 5

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10 The first term of a sequence is 2005. Each succeeding term is the sum of the cubes of the digits of the previous terms. What is the 2005th term of the sequence?
(A) 29 **(B)** 55 **(C)** 85 **(D)** 133 **(E)** 250

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11 An envelope contains eight bills: 2 ones, 2 fives, 2 tens, and 2 twenties. Two bills are drawn at random without replacement. What is the probability that their sum is $\$20$ or more?
(A) $\frac{1}{4}$ **(B)** $\frac{2}{7}$ **(C)** $\frac{3}{7}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{2}{3}$

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12 The quadratic equation $x^2 + mx + n = 0$ has roots that are twice those of $x^2 + px + m = 0$, and none of $m, n,$ and p is zero. What is the value of n/p?
(A) 1 **(B)** 2 **(C)** 4 **(D)** 8 **(E)** 16

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13 Suppose that $4^{x_1} = 5, 5^{x_2} = 6, 6^{x_3} = 7, ..., 127^{x_{124}} = 128$. What is $x_1 x_2 \cdots x_{124}$?
(A) 2 **(B)** $\frac{5}{2}$ **(C)** 3 **(D)** $\frac{7}{2}$ **(E)** 4

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14 A circle having center $(0, k)$, with $k > 6$, is tangent to the lines $y = x, y = -x$ and $y = 6$. What is the radius of this circle?
(A) $6\sqrt{2} - 6$ **(B)** 6 **(C)** $6\sqrt{2}$ **(D)** 12 **(E)** $6 + 6\sqrt{2}$

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15 The sum of four two-digit numbers is 221. None of the eight digits is 0 and no two of them are same. Which of the following is **not** included among the eight digits?
(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

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16 Eight spheres of radius 1, one per octant, are each tangent to the coordinate planes. What is the radius of the smallest sphere, centered at the origin, that contains these eight spheres?

(A) $\sqrt{2}$ **(B)** $\sqrt{3}$ **(C)** $1 + \sqrt{2}$ **(D)** $1 + \sqrt{3}$ **(E)** 3

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17 How many distinct four-tuples (a, b, c, d) of rational numbers are there with
$a \log_{10} 2 + b \log_{10} 3 + c \log_{10} 5 + d \log_{10} 7 = 2005$?
(A) 0 **(B)** 1 **(C)** 17 **(D)** 2004 **(E)** infinitely many

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18 Let $A(2, 2)$ and $B(7, 7)$ be points in the plane. Define R as the region in the first quadrant consisting of those points C such that $\triangle ABC$ is an acute triangle. What is the closest integer to the area of the region R?
(A) 25 **(B)** 39 **(C)** 51 **(D)** 60 **(E)** 80

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19 Let x and y be two-digit integers such that y is obtained by reversing the digits of x. The integers x and y satisfy $x^2 - y^2 = m^2$ for some positive integer m. What is $x + y + m$?
(A) 88 **(B)** 112 **(C)** 116 **(D)** 144 **(E)** 154

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20 Let a, b, c, d, e, f, g and h be distinct elements in the set

$$\{-7, -5, -3, -2, 2, 4, 6, 13\}.$$

What is the minimum possible value of

$$(a + b + c + d)^2 + (e + f + g + h)^2$$

(A) 30 **(B)** 32 **(C)** 34 **(D)** 40 **(E)** 50

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21 A positive integer n has 60 divisors and $7n$ has 80 divisors. What is the greatest integer k such that 7^k divides n?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

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22 A sequence of complex numbers $z_0, z_1, z_2, ...$ is defined by the rule

$$z_{n+1} = \frac{iz_n}{\overline{z_n}}$$

where $\overline{z_n}$ is the complex conjugate of z_n and $i^2 = -1$. Suppose that $|z_0| = 1$ and $z_{2005} = 1$. How many possible values are there for z_0?

(A) 1 **(B)** 2 **(C)** 4 **(D)** 2005 **(E)** 2^{2005}

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23 Let S be the set of ordered triples (x, y, z) of real numbers for which

$$\log_{10}(x + y) = z \text{ and } \log_{10}(x^2 + y^2) = z + 1.$$

There are real numbers a and b such that for all ordered triples (x, y, z) in S we have $x^3 + y^3 = a \cdot 10^{3z} + b \cdot 10^{2z}$. What is the value of $a + b$?

(A) $\frac{15}{2}$ **(B)** $\frac{29}{2}$ **(C)** 15 **(D)** $\frac{39}{2}$ **(E)** 24

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24 All three vertices of an equilateral triangle are on the parabola $y = x^2$, and one of its sides has a slope of 2. The x-coordinates of the three vertices have a sum of m/n, where m and n are relatively prime positive integers. What is the value of $m + n$?

(A) 14 **(B)** 15 **(C)** 16 **(D)** 17 **(E)** 18

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25 Six ants simultaneously stand on the six vertices of a regular octahedron, with each ant at a different vertex. Simultaneously and independently, each ant moves from its vertex to one of the four adjacent vertices, each with equal probability. What is the probability that no two ants arrive at the same vertex?

(A) $\frac{5}{256}$ **(B)** $\frac{21}{1024}$ **(C)** $\frac{11}{512}$ **(D)** $\frac{23}{1024}$ **(E)** $\frac{3}{128}$
