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1 (a) Prove that

$$[5x]+[5y]\ge [3x+y]+[3y+x],$$

where $x,y\ge 0$ and $[u]$ denotes the greatest integer $\le u$ (e.g., $[\sqrt{2}]=1$).
(b) Using (a) or otherwise, prove that

$$\frac{(5m)!(5n)!}{m!n!(3m+n)!(3n+m)!}$$

is integral for all positive integral m and n.

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2 Let A,B,C, and D denote four points in space and AB the distance between A and B, and so on. Show that

$$AC^2+BD^2+AD^2+BC^2\ge AB^2+CD^2.$$

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3 If $P(x)$ denotes a polynomial of degree n such that $P(k)=\frac{k}{k+1}$ for $k=0,1,2,\dots,n$, determine $P(n+1)$.

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4 Two given circles intersect in two points P and Q. Show how to construct a segment AB passing through P and terminating on the circles such that $AP\cdot PB$ is a maximum.

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5 A deck of n playing cards, which contains three aces, is shuffled at random (it is assumed that all possible card distributions are equally likely). The cards are then turned up one by one from the top until the second ace appears. Prove that the expected (average) number of cards to be turned up is $(n+1)/2$.

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