AIME Problems 2012

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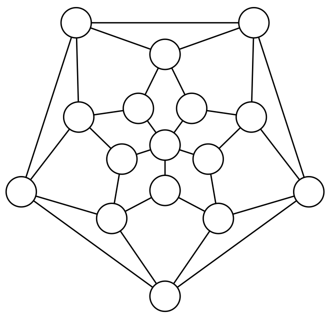
–	I
–	March 15th
1	Find the number of positive integers with three not necessarily distinct digits, abc, with $a \neq 0$, $c \neq 0$ such that both abc and cba are divisible by 4.
2	The terms of an arithmetic sequence add to 715. The first term of the sequence is increased by 1, the second term is increased by 3, the third term is increased by 5, and in general, the kth term is increased by the kth odd positive integer. The terms of the new sequence add to 836. Find the sum of the first, last, and middle terms of the original sequence.
3	Nine people sit down for dinner where there are three choices of meals. Three people order the beef meal, three order the chicken meal, and three order the fish meal. The waiter serves the nine meals in random order. Find the number of ways in which the waiter could serve the meal types to the nine people such that exactly one person receives the type of meal ordered by that person.
4	Butch and Sundance need to get out of Dodge. To travel as quickly as possible, each alternates walking and riding their only horse, Sparky, as follows. Butch begins walking as Sundance rides. When Sundance reaches the first of their hitching posts that are conveniently located at one-mile intervals along their route, he ties Sparky to the post and begins walking. When Butch reaches Sparky, he rides until he passes Sundance, then leaves Sparky at the next hitching post and resumes walking, and they continue in this manner. Sparky, Butch, and Sundance walk at 6, 4, and 2.5 miles per hour, respectively. The first time Butch and Sundance meet at a milepost, they are n miles from Dodge, and have been traveling for t minutes. Find $n+t$.
5	Let B be the set of all binary integers that can be written using exactly 5 zeros and 8 ones where leading zeros are allowed. If all possible subtractions are performed in which one element of B is subtracted from another, find the number of times the answer 1 is obtained.
6	The complex numbers z and w satisfy $z^{13} = w$, $w^{11} = z$, and the imaginary part of z is $\sin\left(\frac{m\pi}{n}\right)$ for relatively prime positive integers m and n with $m < n$. Find n.
7	At each of the sixteen circles in the network below stands a student. A total of 3360 coins are distributed among the sixteen students. All at once, all students give away all their coins by passing an equal number of coins to each of their neighbors in the network. After the trade, all

students have the same number of coins as they started with. Find the number of coins the student standing at the center circle had originally.



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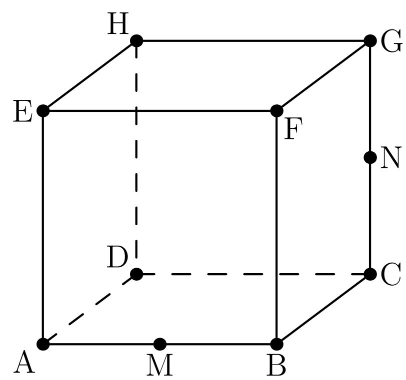
8 Cube $ABCDEFGH$, labeled as shown below, has edge length 1 and is cut by a plane passing through vertex D and the midpoints M and N of $\overline{AB}$ and $\overline{CG}$ respectively. The plane divides the cube into two solids. The volume of the larger of the two solids can be written in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.



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9 Let x, y, and z be positive real numbers that satisfy

$$2\log_{x}(2y) = 2\log_{2x}(4z) = \log_{2x^4}(8yz) \neq 0.$$

The value of xy^5z can be expressed in the form $\frac{1}{2^{p/q}}$, where p and q are relatively prime integers. Find $p+q$.

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10 Let $\mathcal{S}$ be the set of all perfect squares whose rightmost three digits in base 10 are 256. Let $\mathcal{T}$ be the set of all numbers of the form $\frac{x-256}{1000}$, where x is in $\mathcal{S}$. In other words, $\mathcal{T}$ is the set of numbers that result when the last three digits of each number in $\mathcal{S}$ are truncated. Find the remainder when the tenth smallest element of $\mathcal{T}$ is divided by 1000.

11 A frog begins at $P_0 = (0, 0)$ and makes a sequence of jumps according to the following rule: from $P_n = (x_n, y_n)$, the frog jumps to P_{n+1}, which may be any of the points $(x_n + 7, y_n + 2)$, (x_n+2, y_n+7), (x_n-5, y_n-10), or (x_n-10, y_n-5). There are M points (x, y) with $|x|+|y| \le 100$ that can be reached by a sequence of such jumps. Find the remainder when M is divided by 1000.

12 Let $\triangle ABC$ be a right triangle with right angle at C. Let D and E be points on $\overline{AB}$ with D between A and E such that $\overline{CD}$ and $\overline{CE}$ trisect $\angle C$. If $\frac{DE}{BE} = \frac{8}{15}$, then $\tan B$ can be written as $\frac{m\sqrt{p}}{n}$, where m and n are relatively prime positive integers, and p is a positive integer not divisible by the square of any prime. Find $m + n + p$.

13 Three concentric circles have radii 3, 4, and 5. An equilateral triangle with one vertex on each circle has side length s. The largest possible area of the triangle can be written as $a + \frac{b}{c}\sqrt{d}$, where a, b, c and d are positive integers, b and c are relatively prime, and d is not divisible by the square of any prime. Find $a + b + c + d$.

14 Complex numbers a, b and c are the zeros of a polynomial $P(z) = z^3+qz+r$, and $|a|^2+|b|^2+|c|^2 = 250$. The points corresponding to a, b, and c in the complex plane are the vertices of a right triangle with hypotenuse h. Find h^2.

15 There are n mathematicians seated around a circular table with n seats numbered $1, 2, 3, \cdots, n$ in clockwise order. After a break they again sit around the table. The mathematicians note that there is a positive integer a such that

(1) for each k, the mathematician who was seated in seat k before the break is seated in seat ka after the break (where seat $i + n$ is seat i);
(2) for every pair of mathematicians, the number of mathematicians sitting between them after the break, counting in both the clockwise and the counterclockwise directions, is different from either of the number of mathematicians sitting between them before the break.

Find the number of possible values of n with $1 < n < 1000$.

– II

– March 28th

1 Find the number of ordered pairs of positive integer solutions (m, n) to the equation $20m + 12n = 2012$.

2 Two geometric sequences $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3 \ldots$ have the same common ratio, with $a_1 = 27$, $b_1 = 99$, and $a_{15} = b_{11}$. Find a_9.

3 At a certain university, the division of mathematical sciences consists of the departments of mathematics, statistics, and computer science. There are two male and two female professors in each department. A committee of six professors is to contain three men and three women and must also contain two professors from each of the three departments. Find the number of possible comittees that can be formed subject to these requirements.

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4 Ana, Bob, and Cao bike at constant rates of 8.6 meters per second, 6.2 meters per second, and 5 meters per second, respectively. They all begin biking at the same time from the northeast corner of a rectangular field whose longer side runs due west. Ana starts biking along the edge of the field, initially heading west, Bob starts biking along the edge of the field, initially heading south, and Cao bikes in a straight line across the field to a point D on the south edge of the field. Cao arrives at point D at the same time that Ana and Bob arrive at D for the first time. The ratio of the field's length to the field's width to the distance from point D to the southeast corner of the field can be represented as $p : q : r$, where p, q, and r are positive integers with p and q relatively prime. Find $p + q + r$.

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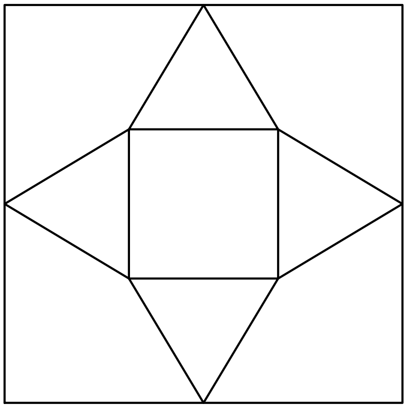
5 In the accompanying figure, the outer square has side length 40. A second square S' of side length 15 is constructed inside S with the same center as S and with sides parallel to those of S. From each midpoint of a side of S, segments are drawn to the two closest vertices of S'. The result is a four-pointed starlike figure inscribed in S. The star figure is cut out and then folded to form a pyramid with base S'. Find the volume of this pyramid.



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6 Let $z = a + bi$ be the complex number with $|z| = 5$ and $b > 0$ such that the distance between $(1 + 2i)z^3$ and z^5 is maximized, and let $z^4 = c + di$.
Find $c + d$.

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7 Let S be the increasing sequence of positive integers whose binary representation has exactly 8 ones. Let N be the 1000^{th} number in S. Find the remainder when N is divided by 1000.

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8 The complex numbers z and w satisfy the system

$$z + \frac{20i}{w} = 5 + i,$$
$$w + \frac{12i}{z} = -4 + 10i.$$

Find the smallest possible value of $|zw|^2$.

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9 Let x and y be real numbers such that $\frac{\sin x}{\sin y} = 3$ and $\frac{\cos x}{\cos y} = \frac{1}{2}$. The value of $\frac{\sin 2x}{\sin 2y} + \frac{\cos 2x}{\cos 2y}$ can be expressed in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

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10 Find the number of positive integers n less than 1000 for which there exists a positive real number x such that $n = x\lfloor x \rfloor$.
Note: $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

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11 Let $f_1(x) = \frac{2}{3} - \frac{3}{3x+1}$, and for $n \geq 2$, define $f_n(x) = f_1(f_{n-1}(x))$. The value of x that satisfies $f_{1001}(x) = x - 3$ can be expressed in the form $\frac{m}{n}$,
where m and n are relatively prime positive integers. Find $m + n$.

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12 For a positive integer p, define the positive integer n to be p-safe if n differs in absolute value by more than 2 from all multiples of p. For example, the set of 10-safe numbers is $\{3, 4, 5, 6, 7, 13, 14, 15, 16, 17, 2$
Find the number of positive integers less than or equal to $10,000$ which are simultaneously 7-safe, 11-safe, and 13-safe.

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13 Equilateral $\triangle ABC$ has side length $\sqrt{111}$. There are four distinct triangles AD_1E_1, AD_1E_2, AD_2E_3, and AD_2E_4, each congruent to $\triangle ABC$, with $BD_1 = BD_2 = \sqrt{11}$. Find $\sum_{k=1}^{4}(CE_k)^2$.

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14 In a group of nine people each person shakes hands with exactly two of the other people from the group. Let N be the number of ways this handshaking can occur. Consider two handshaking arrangements different if and only if at least two people who shake hands under one arrangement do not shake hands under the other arrangement. Find the remainder when N is divided by 1000.

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15 Triangle ABC is inscribed in circle ω with $AB = 5$, $BC = 7$, and $AC = 3$. The bisector of angle A meets side BC at D and circle ω at a second point E. Let γ be the circle with diameter DE. Circles ω and γ meet at E and a second point F. Then $AF^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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