AMC 10 2019

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–	A
–	February 7th, 2019

1 What is the value of

$$2^{\left(0^{\left(1^9\right)}\right)}+\left(\left(2^0\right)^1\right)^9?$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

2 What is the hundreds digit of $(20!-15!)$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 4 **(E)** 5

3 Ana and Bonita were born on the same date in different years, n years apart. Last year Ana was 5 times as old as Bonita. This year Ana's age is the square of Bonita's age. What is n?

(A) 3 **(B)** 5 **(C)** 9 **(D)** 12 **(E)** 15

4 A box contains 28 red balls, 20 green balls, 19 yellow balls, 13 blue balls, 11 white balls, and 9 black balls. What is the minimum number of balls that must be drawn from the box without replacement to guarantee that at least 15 balls of a single color will be drawn$?$

(A) 75 **(B)** 76 **(C)** 79 **(D)** 84 **(E)** 91

5 What is the greatest number of consecutive integers whose sum is $45?$

(A) 9 **(B)** 25 **(C)** 45 **(D)** 90 **(E)** 120

6 For how many of the following types of quadrilaterals does there exist a point in the plane of the quadrilateral that is equidistant from all four vertices of the quadrilateral?

- a square
-a rectangle that is not a square
- a rhombus that is not a square
- a parallelogram that is not a rectangle or a rhombus
- an isosceles trapezoid that is not a parallelogram

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

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7 Two lines with slopes $\dfrac{1}{2}$ and 2 intersect at $(2,2)$. What is the area of the triangle enclosed by these two lines and the line $x+y=10$?

(A) 4 **(B)** $4\sqrt{2}$ **(C)** 6 **(D)** 8 **(E)** $6\sqrt{2}$

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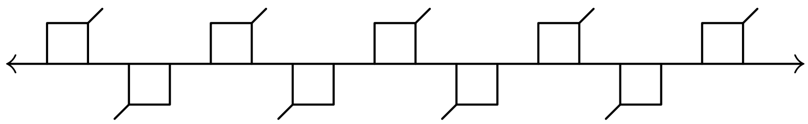
8 The figure below shows line ℓ with a regular, infinite, recurring pattern of squares and line segments.



How many of the following four kinds of rigid motion transformations of the plane in which this figure is drawn, other than the identity transformation, will transform this figure into itself?

- some rotation around a point of line ℓ
- some translation in the direction parallel to line ℓ
- the reflection across line ℓ
- some reflection across a line perpendicular to line ℓ

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

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9 What is the greatest three-digit positive integer n for which the sum of the first n positive integers is $\underline{not}$ a divisor of the product of the first n positive integers?

(A) 995 **(B)** 996 **(C)** 997 **(D)** 998 **(E)** 999

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10 A rectangular floor that is 10 feet wide and 17 feet long is tiled with 170 one-foot square tiles. A bug walks from one corner to the opposite corner in a straight line. Including the first and the last tile, how many tiles does the bug visit?

(A) 17 **(B)** 25 **(C)** 26 **(D)** 27 **(E)** 28

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11 How many positive integer divisors of 201^9 are perfect squares or perfect cubes (or both)?

(A) 32 **(B)** 36 **(C)** 37 **(D)** 39 **(E)** 41

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12 Melanie computes the mean μ, the median M, and the modes of the 365 values that are the dates in the months of 2019. Thus her data consist of 12 1s, 12 2s, . . . , 12 28s, 11 29s, 11 30s, and 7 31s. Let d be the median of the modes. Which of the following statements is true?

(A) $\mu < d < M$ **(B)** $M < d < \mu$ **(C)** $d = M = \mu$ **(D)** $d < M < \mu$ **(E)** $d < \mu < M$

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13 Let ΔABC be an isosceles triangle with $BC = AC$ and $\angle ACB = 40^\circ$. Contruct the circle with diameter $\overline{BC}$, and let D and E be the other intersection points of the circle with the sides $\overline{AC}$ and $\overline{AB}$, respectively. Let F be the intersection of the diagonals of the quadrilateral $BCDE$. What is the degree measure of $\angle BFC$?

(A) 90 **(B)** 100 **(C)** 105 **(D)** 110 **(E)** 120

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14 For a set of four distinct lines in a plane, there are exactly N distinct points that lie on two or more of the lines. What is the sum of all possible values of N?

(A) 14 **(B)** 16 **(C)** 18 **(D)** 19 **(E)** 21

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15 A sequence of numbers is defined recursively by $a_1 = 1$, $a_2 = \frac{3}{7}$, and

$$a_n = \frac{a_{n-2} \cdot a_{n-1}}{2a_{n-2} - a_{n-1}}$$

for all $n \geq 3$ Then a_{2019} can be written as $\frac{p}{q}$, where p and q are relatively prime positive inegers. What is $p + q$?

(A) 2020 **(B)** 4039 **(C)** 6057 **(D)** 6061 **(E)** 8078

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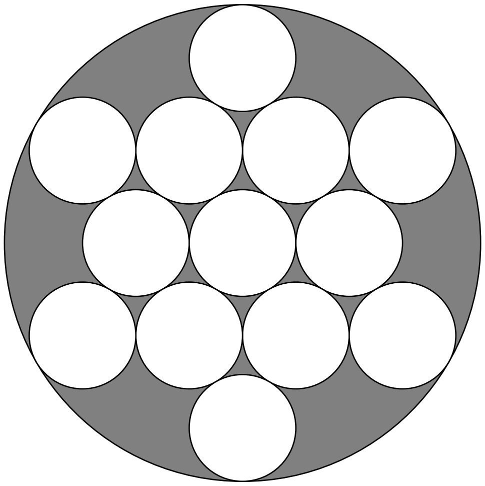
16 The figure below shows 13 circles of radius 1 within a larger circle. All the intersections occur at points of tangency. What is the area of the region, shaded in the figure, inside the larger circle but outside all the circles of radius 1?



(A) $4\pi\sqrt{3}$ **(B)** 7π **(C)** $\pi(3\sqrt{3}+2)$ **(D)** $10\pi(\sqrt{3}-1)$ **(E)** $\pi(\sqrt{3}+6)$

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17 A child builds towers using identically shaped cubes of different color. How many different towers with a height 8 cubes can the child build with 2 red cubes, 3 blue cubes, and 4 green cubes? (One cube will be left out.)

(A) 24 **(B)** 288 **(C)** 312 **(D)** $1,260$ **(E)** $40,320$

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18 For some positive integer k, the repeating base-k representation of the (base-ten) fraction $\frac{7}{51}$ is $0.\overline{23}_k = 0.232323..._k$. What is k?

(A) 13 **(B)** 14 **(C)** 15 **(D)** 16 **(E)** 17

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19 What is the least possible value of

$$(x+1)(x+2)(x+3)(x+4)+2019$$

where x is a real number?

(A) 2017 **(B)** 2018 **(C)** 2019 **(D)** 2020 **(E)** 2021

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20 The numbers $1, 2, \dots, 9$ are randomly placed into the 9 squares of a 3×3 grid. Each square gets one number, and each of the numbers is used once. What is the probability that the sum of the numbers in each row and each column is odd?

(A) $1/21$ **(B)** $1/14$ **(C)** $5/63$ **(D)** $2/21$ **(E)** $1/7$

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21 A sphere with center O has radius 6. A triangle with sides of length $15, 15,$ and 24 is situated in space so that each of its sides is tangent to the sphere. What is the distance between O and the plane determined by the triangle?

(A) $2\sqrt{3}$ **(B)** 4 **(C)** $3\sqrt{2}$ **(D)** $2\sqrt{5}$ **(E)** 5

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22 Real numbers between 0 and 1, inclusive, are chosen in the following manner. A fair coin is flipped. If it lands heads, then it is flipped again and the chosen number is 0 if the second flip is heads and 1 if the second flip is tails. On the other hand, if the first coin flip is tails, then the number is chosen uniformly at random from the closed interval $[0,1]$. Two random numbers x and y are chosen independently in this manner. What is the probability that $|x-y| > \frac{1}{2}$?

(A) $\frac{1}{3}$ **(B)** $\frac{7}{16}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{9}{16}$ **(E)** $\frac{2}{3}$

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23 Travis has to babysit the terrible Thompson triplets. Knowing that they love big numbers, Travis devises a counting game for them. First Tadd will say the number 1, then Todd must say the next two numbers (2 and 3), then Tucker must say the next three numbers ($4, 5, 6$), then Tadd must say the next four numbers ($7, 8, 9, 10$), and the process continues to rotate through the three children in order, each saying one more number than the previous child did, until the number $10,000$ is reached. What is the 2019th number said by Tadd?

(A) 5743 **(B)** 5885 **(C)** 5979 **(D)** 6001 **(E)** 6011

24 Let p, q, and r be the distinct roots of the polynomial $x^3 - 22x^2 + 80x - 67$. It is given that there exist real numbers A, B, and C such that

$$\frac{1}{s^3 - 22s^2 + 80s - 67} = \frac{A}{s-p} + \frac{B}{s-q} + \frac{C}{s-r}$$

for all $s \notin \{p, q, r\}$. What is $\tfrac{1}{A} + \tfrac{1}{B} + \tfrac{1}{C}$?

(A) 243 **(B)** 244 **(C)** 245 **(D)** 246 **(E)** 247

25 For how many integers n between 1 and 50, inclusive, is

$$\frac{(n^2-1)!}{(n!)^n}$$

an integer? (Recall that $0! = 1$.)

(A) 31 **(B)** 32 **(C)** 33 **(D)** 34 **(E)** 35

– B

– February 13th, 2019

1 Alicia had two containers. The first was $\tfrac{5}{6}$ full of water and the second was empty. She poured all the water from the first container into the second container, at which point the second container was $\tfrac{3}{4}$ full of water. What is the ratio of the volume of the first container to the volume of the second container?

(A) $\frac{5}{8}$ **(B)** $\frac{4}{5}$ **(C)** $\frac{7}{8}$ **(D)** $\frac{9}{10}$ **(E)** $\frac{11}{12}$

2 Consider the statement, "If n is not prime, then $n-2$ is prime." Which of the following values of n is a counterexample to this statement?

(A) 11 **(B)** 15 **(C)** 19 **(D)** 21 **(E)** 27

3 In a high school with 500 students, 40% of the seniors play a musical instrument, while 30% of the non-seniors do not play a musical instrument. In all, 46.8% of the students do not play a musical instrument. How many non-seniors play a musical instrument?

(A) 66 **(B)** 154 **(C)** 186 **(D)** 220 **(E)** 266

4 All lines with equation $ax+by=c$ such that a, b, c form an arithmetic progression pass through a common point. What are the coordinates of that point?

(A) $(-1, 2)$ **(B)** $(0, 1)$ **(C)** $(1, -2)$ **(D)** $(1, 0)$ **(E)** $(1, 2)$

5 Triangle ABC lies in the first quadrant. Points A, B, and C are reflected across the line $y = x$ to points A', B', and C', respectively. Assume that none of the vertices of the triangle lie on the line $y = x$. Which of the following statements is *not* always true? (A) Triangle $A'B'C'$ lies in the first quadrant. (B) Triangles ABC and $A'B'C'$ have the same area. (C) The slope of line AA' is -1. (D) The slopes of lines AA' and CC' are the same. (E) Lines AB and $A'B'$ are perpendicular to each other.

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6 A positive integer n satisfies the equation $(n+1)! + (n+2)! = n! \cdot 440$. What is the sum of the digits of n?

(A) 2 **(B)** 5 **(C)** 10 **(D)** 12 **(E)** 15

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7 Each piece of candy in a store costs a whole number of cents. Casper has exactly enough money to buy either 12 pieces of red candy, 14 pieces of green candy, 15 pieces of blue candy, or n pieces of purple candy. A piece of purple candy costs 20 cents. What is the smallest possible value of n?

(A) 18 **(B)** 21 **(C)** 24 **(D)** 25 **(E)** 28

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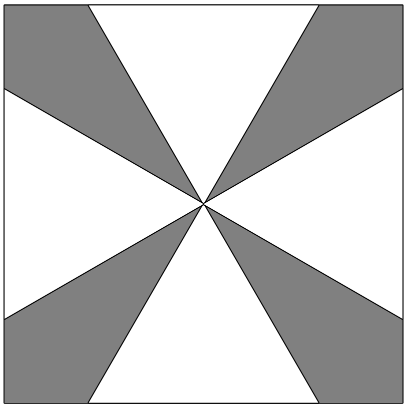
8 The figure below shows a square and four equilateral triangles, with each triangle having a side lying on a side of the square, such that each triangle has side length 2 and the third vertices of the triangles meet at the center of the square. The region inside the square but outside the triangles is shaded. What is the area of the shaded region?



(A) 4 **(B)** $12 - 4\sqrt{3}$ **(C)** $3\sqrt{3}$ **(D)** $4\sqrt{3}$ **(E)** $16 - \sqrt{3}$

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9 The function f is defined by

$$f(x) = \Big\lfloor |x| \Big\rfloor - \Big| \lfloor x \rfloor \Big|$$

for all real numbers x, where $\lfloor r \rfloor$ denotes the greatest integer less than or equal to the real number r. What is the range of f?

(A) $\{-1,0\}$ **(B)** The set of nonpositive integers **(C)** $\{-1,0,1\}$

(D) $\{0\}$ **(E)** The set of nonnegative integers

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10 In a given plane, points A and B are 10 units apart. How many points C are there in the plane such that the perimeter of $\triangle ABC$ is 50 units and the area of $\triangle ABC$ is 100 square units?

(A) 0 **(B)** 2 **(C)** 4 **(D)** 8 **(E)** infinitely many

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11 Two jars each contain the same number of marbles, and every marble is either blue or green. In Jar 1 the ratio of blue to green marbles is 9:1, and the ratio of blue to green marbles in Jar 2 is 8:1. There are 95 green marbles in all. How many more blue marbles are in Jar 1 than in Jar 2?

(A) 5 **(B)** 10 **(C)** 25 **(D)** 45 **(E)** 50

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12 What is the greatest possible sum of the digits in the base-seven representation of a positive integer less than 2019?

(A) 11 **(B)** 14 **(C)** 22 **(D)** 23 **(E)** 27

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13 What is the sum of all real numbers x for which the median of the numbers $4, 6, 8, 17,$ and x is equal to the mean of those five numbers?

(A) -5 **(B)** 0 **(C)** 5 **(D)** $\frac{15}{4}$ **(E)** $\frac{35}{4}$

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14 The base-ten representation for $19!$ is $121,6T5,100,40M,832,H00$, where T, M, and H denote digits that are not given. What is $T+M+H$?

(A) 3 **(B)** 8 **(C)** 12 **(D)** 14 **(E)** 17

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15 Two right triangles, T_1 and T_2, have areas of 1 and 2, respectively. One side length of one triangle is congruent to a different side length in the other, and another side length of the first triangle is congruent to yet another side length in the other. What is the square of the product of the third side lengths of T_1 and T_2?

(A) $\frac{28}{3}$ **(B)** 10 **(C)** $\frac{32}{3}$ **(D)** $\frac{34}{3}$ **(E)** 12

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16 In $\triangle ABC$ with a right angle at C, point D lies in the interior of $\overline{AB}$ and point E lies in the interior of $\overline{BC}$ so that $AC=CD$, $DE=EB$, and the ratio $AC:DE=4:3$. What is the ratio $AD:DB?$

(A) $2:3$ **(B)** $2:\sqrt{5}$ **(C)** $1:1$ **(D)** $3:\sqrt{5}$ **(E)** $3:2$

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17 A red ball and a green ball are randomly and independently tossed into bins numbered with positive integers so that for each ball, the probability that it is tossed into bin k is 2^{-k} for $k = 1, 2, 3, \ldots$. What is the probability that the red ball is tossed into a higher-numbered bin than the green ball?

(A) $\frac{1}{4}$ **(B)** $\frac{2}{7}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{3}{8}$ **(E)** $\frac{3}{7}$

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18 Henry decides one morning to do a workout, and he walks $\frac{3}{4}$ of the way from his home to his gym. The gym is 2 kilometers away from Henry's home. At that point, he changes his mind and walks $\frac{3}{4}$ of the way from where he is back toward home. When he reaches that point, he changes his mind again and walks $\frac{3}{4}$ of the distance from there back toward the gym. If Henry keeps changing his mind when he has walked $\frac{3}{4}$ of the distance toward either the gym or home from the point where he last changed his mind, he will get very close to walking back and forth between a point A kilometers from home and a point B kilometers from home. What is $|A-B|$?

(A) $\frac{2}{3}$ **(B)** 1 **(C)** $1\frac{1}{5}$ **(D)** $1\frac{1}{4}$ **(E)** $1\frac{1}{2}$

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19 Let S be the set of all positive integer divisors of $100,000$. How many numbers are the product of two distinct elements of S?

(A) 98 **(B)** 100 **(C)** 117 **(D)** 119 **(E)** 121

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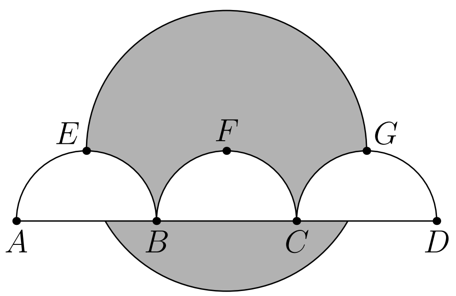
20 As shown in the figure, line segment $\overline{AD}$ is trisected by points B and C so that $AB = BC = CD = 2$. Three semicircles of radius 1, AEB, BFC, and CGD, have their diameters on $\overline{AD}$, and are tangent to line EG at $E, F,$ and G, respectively. A circle of radius 2 has its center on F. The area of the region inside the circle but outside the three semicircles, shaded in the figure, can be expressed in the form

$$\frac{a}{b} \cdot \pi - \sqrt{c} + d,$$

where $a, b, c,$ and d are positive integers and a and b are relatively prime. What is $a+b+c+d$?



(A) 13 **(B)** 14 **(C)** 15 **(D)** 16 **(E)** 17

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21 Debra flips a fair coin repeatedly, keeping track of how many heads and how many tails she has seen in total, until she gets either two heads in a row or two tails in a row, at which point she stops flipping. What is the probability that she gets two heads in a row but she sees a second tail before she sees a second head?

(A) $\frac{1}{36}$ **(B)** $\frac{1}{24}$ **(C)** $\frac{1}{18}$ **(D)** $\frac{1}{12}$ **(E)** $\frac{1}{6}$

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22 Raashan, Sylvia, and Ted play the following game. Each starts with $\$1$. A bell rings every 15 seconds, at which time each of the players who currently have money simultaneously chooses one of the other two players independently and at random and gives $\$1$ to that player. What is the probability that after the bell has rung 2019 times, each player will have $\$1$? (For example, Raashan and Ted may each decide to give $\$1$ to Sylvia, and Sylvia may decide to give her dollar to Ted, at which point Raashan will have $\$0$, Sylvia would have $\$2$, and Ted would have $\$1$, and and that is the end of the first round of play. In the second round Raashan has no money to give, but Sylvia and Ted might choose each other to give their $\$1$ to, and and the holdings will be the same as the end of the second [sic] round.

(A) $\frac{1}{7}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{2}{3}$

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23 Points $A(6,13)$ and $B(12,11)$ lie on circle ω in the plane. Suppose that the tangent lines to ω at A and B intersect at a point on the x-axis. What is the area of ω?

(A) $\frac{83\pi}{8}$ **(B)** $\frac{21\pi}{2}$ **(C)** $\frac{85\pi}{8}$ **(D)** $\frac{43\pi}{4}$ **(E)** $\frac{87\pi}{8}$

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24 Define a sequence recursively by $x_0 = 5$ and

$$x_{n+1} = \frac{x_n^2 + 5x_n + 4}{x_n + 6}$$

for all nonnegative integers n. Let m be the least positive integer such that

$$x_m \leq 4 + \frac{1}{2^{20}}.$$

In which of the following intervals does m lie?

(A) $[9,26]$ **(B)** $[27,80]$ **(C)** $[81,242]$ **(D)** $[243,728]$ **(E)** $[729,\infty]$

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25 How many sequences of 0s and 1s of length 19 are there that begin with a 0, end with a 0, contain no two consecutive 0s, and contain no three consecutive 1s?

(A) 55 **(B)** 60 **(C)** 65 **(D)** 70 **(E)** 75

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