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1 (a) Suppose that each square of a 4 x 7 chessboard is colored either black or white. Prove that with *any* such coloring, the board must contain a rectangle (formed by the horizontal and vertical lines of the board) whose four distinct unit corner squares are all of the same color.
(b) Exhibit a black-white coloring of a 4 x6 board in which the four corner squares of every rectangle, as described above, are not all of the same color.

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2 If A and B are fixed points on a given circle and XY is a variable diameter of the same circle, determine the locus of the point of intersection of lines AX and BY. You may assume that AB is not a diameter.

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3 Determine all integral solutions of

$$a^2 + b^2 + c^2 = a^2b^2.$$

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4 If the sum of the lengths of the six edges of a trirectangular tetrahedron $PABC$ (i.e., $\angle APB = \angle BPC = \angle CPA = 90^\circ$) is S, determine its maximum volume.

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5 If $P(x), Q(x), R(x)$, and $S(x)$ are all polynomials such that

$$P(x^5) + xQ(x^5) + x^2R(x^5) = (x^4 + x^3 + x^2 + x + 1)S(x),$$

prove that $x - 1$ is a factor of $P(x)$.
