**USAMO 1992**

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**1** Find, as a function of $n$, the sum of the digits of

$$9 \times 99 \times 9999 \times \cdots \times \left(10^{2^n} - 1\right),$$

where each factor has twice as many digits as the previous one.

**2** Prove

$$\frac{1}{\cos 0^\circ \cos 1^\circ} + \frac{1}{\cos 1^\circ \cos 2^\circ} + \cdots + \frac{1}{\cos 88^\circ \cos 89^\circ} = \frac{\cos 1^\circ}{\sin^2 1^\circ}.$$

**3** For a nonempty set $S$ of integers, let $\sigma(S)$ be the sum of the elements of $S$. Suppose that $A = \{a_1, a_2, \ldots, a_{11}\}$ is a set of positive integers with $a_1 < a_2 < \cdots < a_{11}$ and that, for each positive integer $n \leq 1500$, there is a subset $S$ of $A$ for which $\sigma(S) = n$. What is the smallest possible value of $a_{10}$?

**4** Chords $AA'$, $BB'$, $CC'$ of a sphere meet at an interior point $P$ but are not contained in a plane. The sphere through $A$, $B$, $C$, $P$ is tangent to the sphere through $A'$, $B'$, $C'$, $P$. Prove that $AA' = BB' = CC'$.

**5** Let $P(z)$ be a polynomial with complex coefficients which is of degree $1992$ and has distinct zeros. Prove that there exist complex numbers $a_1, a_2, \ldots, a_{1992}$ such that $P(z)$ divides the polynomial

$$\left(\cdots\left((z - a_1)^2 - a_2\right)^2 \cdots - a_{1991}\right)^2 - a_{1992}.$$

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