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1	Let p be an odd prime. The sequence $(a_n)_{n\geq 0}$ is defined as follows: $a_0=0, a_1=1, \ldots, a_{p-2}=p-2$ and, for all $n\geq p-1,\ a_n$ is the least positive integer that does not form an arithmetic sequence of length p with any of the preceding terms. Prove that, for all $n,\ a_n$ is the number obtained by writing n in base $p-1$ and reading the result in base p.
2	A calculator is broken so that the only keys that still work are the $\sin$, $\cos$, and $\tan$ buttons, and their inverses (the $\arcsin$, $\arccos$, and $\arctan$ buttons). The display initially shows 0. Given any positive rational number q, show that pressing some finite sequence of buttons will yield the number q on the display. Assume that the calculator does real number calculations with infinite precision. All functions are in terms of radians.
3	Given a nonisosceles, nonright triangle ABC, let O denote the center of its circumscribed circle, and let A_1, B_1, and C_1 be the midpoints of sides BC, CA, and AB, respectively. Point A_2 is located on the ray OA_1 so that OAA_1 is similar to OA_2A. Points B_2 and C_2 on rays OB_1 and OC_1, respectively, are defined similarly. Prove that lines AA_2, BB_2, and CC_2 are concurrent, i.e. these three lines intersect at a point.
4	Suppose $q_0, q_1, q_2, \ldots$ is an infinite sequence of integers satisfying the following two conditions: (i) $m-n$ divides q_m-q_n for $m>n\geq 0,$ (ii) there is a polynomial P such that $\|q_n\| < P(n)$ for all n Prove that there is a polynomial Q such that $q_n=Q(n)$ for all n.
5	Suppose that in a certain society, each pair of persons can be classified as either *amicable* or *hostile*. We shall say that each member of an amicable pair is a *friend* of the other, and each member of a hostile pair is a *foe* of the other. Suppose that the society has n persons and q amicable pairs, and that for every set of three persons, at least one pair is hostile. Prove that there is at least one member of the society whose foes include $q(1-4q/n^2)$ or fewer amicable pairs.