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1 By a *pure repeating decimal* (in base 10), we mean a decimal $0.\overline{a_1 \cdots a_k}$ which repeats in blocks of k digits beginning at the decimal point. An example is $.243243243\cdots = \frac{9}{37}$. By a *mixed repeating decimal* we mean a decimal $0.b_1 \cdots b_m\overline{a_1 \cdots a_k}$ which eventually repeats, but which cannot be reduced to a pure repeating decimal. An example is $.011363636\cdots = \frac{1}{88}$.

Prove that if a mixed repeating decimal is written as a fraction $\frac{p}{q}$ in lowest terms, then the denominator q is divisible by 2 or 5 or both.

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2 The cubic equation $x^3 + ax^2 + bx + c = 0$ has three real roots. Show that $a^2 - 3b \geq 0$, and that $\sqrt{a^2 - 3b}$ is less than or equal to the difference between the largest and smallest roots.

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3 A function $f(S)$ assigns to each nine-element subset of S of the set $\{1, 2, \ldots, 20\}$ a whole number from 1 to 20. Prove that regardless of how the function f is chosen, there will be a ten-element subset $T \subset \{1, 2, \ldots, 20\}$ such that $f(T - \{k\}) \neq k$ for all $k \in T$.

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4 Let I be the incenter of triangle ABC, and let A', B', and C' be the circumcenters of triangles IBC, ICA, and IAB, respectively. Prove that the circumcircles of triangles ABC and $A'B'C'$ are concentric.

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5 A polynomial product of the form

$$(1-z)^{b_1}(1-z^2)^{b_2}(1-z^3)^{b_3}(1-z^4)^{b_4}(1-z^5)^{b_5}\cdots(1-z^{32})^{b_{32}},$$

where the b_k are positive integers, has the surprising property that if we multiply it out and discard all terms involving z to a power larger than 32, what is left is just $1 - 2z$. Determine, with proof, b_{32}.
