AIME Problems 2010

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–	I
–	March 16th
1	Maya lists all the positive divisors of 2010^2. She then randomly selects two distinct divisors from this list. Let p be the probability that exactly one of the selected divisors is a perfect square. The probability p can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.
2	Find the remainder when $$9 \times 99 \times 999 \times \cdots \times \underbrace{99\cdots9}_{\text{999 9's}}$$ is divided by 1000.
3	Suppose that $y = \frac{3}{4}x$ and $x^y = y^x$. The quantity $x+y$ can be expressed as a rational number $\frac{r}{s}$, where r and s are relatively prime positive integers. Find $r+s$.
4	Jackie and Phil have two fair coins and a third coin that comes up heads with probability $\frac{4}{7}$. Jackie flips the three coins, and then Phil flips the three coins. Let $\frac{m}{n}$ be the probability that Jackie gets the same number of heads as Phil, where m and n are relatively prime positive integers. Find $m+n$.
5	Positive integers a, b, c, and d satisfy $a > b > c > d$, $a+b+c+d = 2010$, and $a^2-b^2+c^2-d^2 = 2010$. Find the number of possible values of a.
6	Let $P(x)$ be a quadratic polynomial with real coefficients satisfying $$x^2 - 2x + 2 \le P(x) \le 2x^2 - 4x + 3$$ for all real numbers x, and suppose $P(11) = 181$. Find $P(16)$.
7	Define an ordered triple (A, B, C) of sets to be minimally intersecting if $\lvert A \cap B\rvert = \lvert B \cap C\rvert = \lvert C \cap A\rvert = 1$ and $A \cap B \cap C = \emptyset$. For example, $(\{1,2\},\{2,3\},\{1,3,4\})$ is a minimally intersecting triple. Let N be the number of minimally intersecting ordered triples of sets for which each set is a subset of $\{1,2,3,4,5,6,7\}$. Find the remainder when N is divided by 1000. **Note**: $\lvert S\rvert$ represents the number of elements in the set S.

8 For a real number a, let $\lfloor a \rfloor$ denominate the greatest integer less than or equal to a. Let $\mathcal{R}$ denote the region in the coordinate plane consisting of points (x, y) such that

$$\lfloor x \rfloor^2 + \lfloor y \rfloor^2 = 25.$$

The region $\mathcal{R}$ is completely contained in a disk of radius r (a disk is the union of a circle and its interior). The minimum value of r can be written as $\frac{\sqrt{m}}{n}$, where m and n are integers and m is not divisible by the square of any prime. Find $m + n$.

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9 Let (a, b, c) be the real solution of the system of equations $x^3 - xyz = 2$, $y^3 - xyz = 6$, $z^3 - xyz = 20$. The greatest possible value of $a^3 + b^3 + c^3$ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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10 Let N be the number of ways to write 2010 in the form

$$2010 = a_3 \cdot 10^3 + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0,$$

where the a_i's are integers, and $0 \le a_i \le 99$. An example of such a representation is $1 \cdot 10^3 + 3 \cdot 10^2 + 67 \cdot 10^1 + 40 \cdot 10^0$. Find N.

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11 Let $\mathcal{R}$ be the region consisting of the set of points in the coordinate plane that satisfy both $|8 - x| + y \le 10$ and $3y - x \ge 15$. When $\mathcal{R}$ is revolved around the line whose equation is $3y - x = 15$, the volume of the resulting solid is $\frac{m\pi}{n\sqrt{p}}$, where m, n, and p are positive integers, m and n are relatively prime, and p is not divisible by the square of any prime. Find $m + n + p$.

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12 Let $M \ge 3$ be an integer and let $S = \{3, 4, 5, \ldots, m\}$. Find the smallest value of m such that for every partition of S into two subsets, at least one of the subsets contains integers a, b, and c (not necessarily distinct) such that $ab = c$.

Note: a partition of S is a pair of sets A, B such that $A \cap B = \emptyset$, $A \cup B = S$.

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13 Rectangle $ABCD$ and a semicircle with diameter AB are coplanar and have nonoverlapping interiors. Let $\mathcal{R}$ denote the region enclosed by the semicircle and the rectangle. Line ℓ meets the semicircle, segment AB, and segment CD at distinct points N, U, and T, respectively. Line ℓ divides region $\mathcal{R}$ into two regions with areas in the ratio $1 : 2$. Suppose that $AU = 84$, $AN = 126$, and $UB = 168$. Then DA can be represented as $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

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14 For each positive integer n, let $f(n) = \sum_{k=1}^{100} \lfloor \log_{10}(kn) \rfloor$. Find the largest value of n for which $f(n) \le 300$.

Note: $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

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15 In $\triangle ABC$ with $AB = 12$, $BC = 13$, and $AC = 15$, let M be a point on $\overline{AC}$ such that the incircles of $\triangle ABM$ and $\triangle BCM$ have equal radii. Let p and q be positive relatively prime integers such that $\frac{AM}{CM} = \frac{p}{q}$. Find $p + q$.

– II

– March 31st

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1 Let N be the greatest integer multiple of 36 all of whose digits are even and no two of whose digits are the same. Find the remainder when N is divided by 1000.

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2 A point P is chosen at random in the interior of a unit square S. Let $d(P)$ denote the distance from P to the closest side of S. The probability that $\frac{1}{5} \le d(P) \le \frac{1}{3}$ is equal to $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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3 Let K be the product of all factors $(b - a)$ (not necessarily distinct) where a and b are integers satisfying $1 \le a < b \le 20$. Find the greatest positive integer n such that 2^n divides K.

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4 Dave arrives at an airport which has twelve gates arranged in a straight line with exactly 100 feet between adjacent gates. His departure gate is assigned at random. After waiting at that gate, Dave is told the departure gate has been changed to a different gate, again at random. Let the probability that Dave walks 400 feet or less to the new gate be a fraction $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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5 Positive numbers x, y, and z satisfy $xyz = 10^{81}$ and $(\log_{10} x)(\log_{10} yz) + (\log_{10} y)(\log_{10} z) = 468$. Find $\sqrt{(\log_{10} x)^2 + (\log_{10} y)^2 + (\log_{10} z)^2}$.

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6 Find the smallest positive integer n with the property that the polynomial $x^4 - nx + 63$ can be written as a product of two nonconstant polynomials with integer coefficients.

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7 Let $P(z) = z^3 + az^2 + bz + c$, where a, b, and c are real. There exists a complex number w such that the three roots of $P(z)$ are $w + 3i$, $w + 9i$, and $2w - 4$, where $i^2 = -1$. Find $|a + b + c|$.

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8 Let N be the number of ordered pairs of nonempty sets $\mathcal{A}$ and $\mathcal{B}$ that have the following properties:

$\mathcal{A} \cup \mathcal{B} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$,
$\mathcal{A} \cap \mathcal{B} = \emptyset$,
The number of elements of $\mathcal{A}$ is not an element of $\mathcal{A}$,
The number of elements of $\mathcal{B}$ is not an element of $\mathcal{B}$.

Find N.

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9 Let $ABCDEF$ be a regular hexagon. Let G, H, I, J, K, and L be the midpoints of sides AB, BC, CD, DE, EF, and AF, respectively. The segments AH, BI, CJ, DK, EL, and FG bound a smaller regular hexagon. Let the ratio of the area of the smaller hexagon to the area of $ABCDEF$ be expressed as a fraction $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m+n$.

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10 Find the number of second-degree polynomials $f(x)$ with integer coefficients and integer zeros for which $f(0)=2010$.

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11 Define a *T-grid* to be a 3×3 matrix which satisfies the following two properties:

(1) Exactly five of the entries are 1's, and the remaining four entries are 0's.
(2) Among the eight rows, columns, and long diagonals (the long diagonals are $\{a_{13},a_{22},a_{31}\}$ and $\{a_{11},a_{22},a_{33}\}$, no more than one of the eight has all three entries equal.

Find the number of distinct T-grids.

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12 Two noncongruent integer-sided isosceles triangles have the same perimeter and the same area. The ratio of the lengths of the bases of the two triangles is $8:7$. Find the minimum possible value of their common perimeter.

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13 The 52 cards in a deck are numbered $1,2,\dots,52$. Alex, Blair, Corey, and Dylan each picks a card from the deck without replacement and with each card being equally likely to be picked, The two persons with lower numbered cards from a team, and the two persons with higher numbered cards form another team. Let $p(a)$ be the probability that Alex and Dylan are on the same team, given that Alex picks one of the cards a and $a+9$, and Dylan picks the other of these two cards. The minimum value of $p(a)$ for which $p(a)\ge\frac{1}{2}$ can be written as $\frac{m}{n}$. where m and n are relatively prime positive integers. Find $m+n$.

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14 In right triangle ABC with right angle at C, $\angle BAC<45$ degrees and $AB=4$. Point P on AB is chosen such that $\angle APC=2\angle ACP$ and $CP=1$. The ratio $\frac{AP}{BP}$ can be represented in the form $p+q\sqrt{r}$, where p,q,r are positive integers and r is not divisible by the square of any prime. Find $p+q+r$.

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15 In triangle ABC, $AC=13, BC=14,$ and $AB=15$. Points M and D lie on AC with $AM=MC$ and $\angle ABD=\angle DBC$. Points N and E lie on AB with $AN=NB$ and $\angle ACE=\angle ECB$. Let P be the point, other than A, of intersection of the circumcircles of $\triangle AMN$ and $\triangle ADE$. Ray AP meets BC at Q. The ratio $\frac{BQ}{CQ}$ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m-n$.

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