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1 The symbols $(a,b,\ldots,g)$ and $[a,b,\ldots,g]$ denote the greatest common divisor and least common multiple, respectively, of the positive integers $a,b,\ldots,g$. For example, $(3,6,18)=3$ and $[6,15]=30$. Prove that

$$\frac{[a,b,c]^2}{[a,b][b,c][c,a]}=\frac{(a,b,c)^2}{(a,b)(b,c)(c,a)}.$$

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2 A given tetrahedron $ABCD$ is isoceles, that is, $AB=CD$, $AC=BD$, $AD=BC$. Show that the faces of the tetrahedron are acute-angled triangles.

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3 A random selector can only select one of the nine integers $1,2,\ldots,9$, and it makes these selections with equal probability. Determine the probability that after n selections ($n>1$), the product of the n numbers selected will be divisible by 10.

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4 Let R denote a non-negative rational number. Determine a fixed set of integers a,b,c,d,e,f, such that for *every* choice of R,

$$\left|\frac{aR^2+bR+c}{dR^2+eR+f}-\sqrt[3]{2}\right|<\left|R-\sqrt[3]{2}\right|.$$

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5 A given convex pentagon $ABCDE$ has the property that the area of each of five triangles ABC, BCD, CDE, DEA, and EAB is unity *(equal to 1)*. Show that all pentagons with the above property have the same area, and calculate that area. Show, furthermore, that there are infinitely many non-congruent pentagons having the above area property.

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