

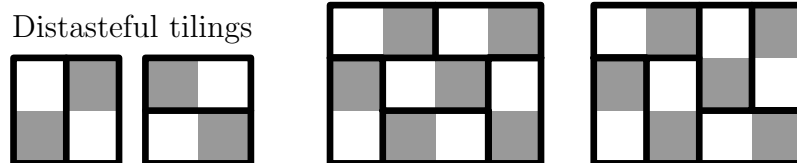
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Day 1 April 28th

- 1 Given circles ω_1 and ω_2 intersecting at points X and Y , let ℓ_1 be a line through the center of ω_1 intersecting ω_2 at points P and Q and let ℓ_2 be a line through the center of ω_2 intersecting ω_1 at points R and S . Prove that if P, Q, R and S lie on a circle then the center of this circle lies on line XY .
- 2 Let n be a positive integer. Determine the size of the largest subset of $\{-n, -n+1, \dots, n-1, n\}$ which does not contain three elements a, b, c (not necessarily distinct) satisfying $a + b + c = 0$.
- 3 We define a *chessboard polygon* to be a polygon whose sides are situated along lines of the form $x = a$ or $y = b$, where a and b are integers. These lines divide the interior into unit squares, which are shaded alternately grey and white so that adjacent squares have different colors. To tile a chessboard polygon by dominoes is to exactly cover the polygon by non-overlapping 1×2 rectangles. Finally, a *tasteful tiling* is one which avoids the two configurations of dominoes shown on the left below. Two tilings of a 3×4 rectangle are shown; the first one is tasteful, while the second is not, due to the vertical dominoes in the upper right corner.



- a) Prove that if a chessboard polygon can be tiled by dominoes, then it can be done so tastefully.
- b) Prove that such a tasteful tiling is unique.

Day 2 April 29th

- 4 For $n \geq 2$ let a_1, a_2, \dots, a_n be positive real numbers such that

$$(a_1 + a_2 + \dots + a_n) \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right) \leq \left(n + \frac{1}{2} \right)^2.$$

Prove that $\max(a_1, a_2, \dots, a_n) \leq 4 \min(a_1, a_2, \dots, a_n)$.

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- 5 Trapezoid $ABCD$, with $\overline{AB} \parallel \overline{CD}$, is inscribed in circle ω and point G lies inside triangle BCD . Rays AG and BG meet ω again at points P and Q , respectively. Let the line through G parallel to \overline{AB} intersect \overline{BD} and \overline{BC} at points R and S , respectively. Prove that quadrilateral $PQRS$ is cyclic if and only if \overline{BG} bisects $\angle CBD$.
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- 6 Let s_1, s_2, s_3, \dots be an infinite, nonconstant sequence of rational numbers, meaning it is not the case that $s_1 = s_2 = s_3 = \dots$. Suppose that t_1, t_2, t_3, \dots is also an infinite, nonconstant sequence of rational numbers with the property that $(s_i - s_j)(t_i - t_j)$ is an integer for all i and j . Prove that there exists a rational number r such that $(s_i - s_j)r$ and $(t_i - t_j)/r$ are integers for all i and j .
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