

USAMO 2014

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Day 1 April 29th

- 1** Let a, b, c, d be real numbers such that $b - d \geq 5$ and all zeros x_1, x_2, x_3 , and x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$ can take.

- 2** Let \mathbb{Z} be the set of integers. Find all functions $f : \mathbb{Z} \rightarrow \mathbb{Z}$ such that

$$xf(2f(y) - x) + y^2f(2x - f(y)) = \frac{f(x)^2}{x} + f(yf(y))$$

for all $x, y \in \mathbb{Z}$ with $x \neq 0$.

- 3** Prove that there exists an infinite set of points

$$\dots, P_{-3}, P_{-2}, P_{-1}, P_0, P_1, P_2, P_3, \dots$$

in the plane with the following property: For any three distinct integers a, b , and c , points P_a, P_b , and P_c are collinear if and only if $a + b + c = 2014$.

Day 2 April 30th

- 4** Let k be a positive integer. Two players A and B play a game on an infinite grid of regular hexagons. Initially all the grid cells are empty. Then the players alternately take turns with A moving first. In his move, A may choose two adjacent hexagons in the grid which are empty and place a counter in both of them. In his move, B may choose any counter on the board and remove it. If at any time there are k consecutive grid cells in a line all of which contain a counter, A wins. Find the minimum value of k for which A cannot win in a finite number of moves, or prove that no such minimum value exists.

- 5** Let ABC be a triangle with orthocenter H and let P be the second intersection of the circumcircle of triangle AHC with the internal bisector of the angle $\angle BAC$. Let X be the circumcenter of triangle APB and Y the orthocenter of triangle APC . Prove that the length of segment XY is equal to the circumradius of triangle ABC .

- 6** Prove that there is a constant $c > 0$ with the following property: If a, b, n are positive integers such that $\gcd(a + i, b + j) > 1$ for all $i, j \in \{0, 1, \dots, n\}$, then

$$\min\{a, b\} > c^n \cdot n^{\frac{n}{2}}.$$



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