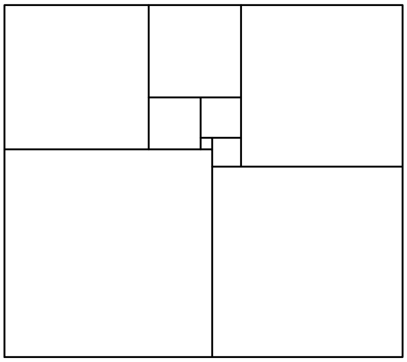
AIME Problems 2000

www.artofproblemsolving.com/community/c4897

by 4everwise, white_horse_king88, joml88, rrusczyk

–	I
–	March 28th
1	Find the least positive integer n such that no matter how 10^n is expressed as the product of any two positive integers, at least one of these two integers contains the digit 0.
2	Let u and v be integers satisfying $0 < v < u$. Let $A = (u, v)$, let B be the reflection of A across the line $y = x$, let C be the reflection of B across the y-axis, let D be the reflection of C across the x-axis, and let E be the reflection of D across the y-axis. The area of pentagon $ABCDE$ is 451. Find $u + v$.
3	In the expansion of $(ax + b)^{2000}$, where a and b are relatively prime positive integers, the coefficients of x^2 and x^3 are equal. Find $a + b$.
4	The diagram shows a rectangle that has been dissected into nine non-overlapping squares. Given that the width and the height of the rectangle are relatively prime positive integers, find the perimeter of the rectangle. 
5	Each of two boxes contains both black and white marbles, and the total number of marbles in the two boxes is 25. One marble is taken out of each box randomly. The probability that both marbles are black is $27/50$, and the probability that both marbles are white is m/n, where m and n are relatively prime positive integers. What is $m + n$?

6 For how many ordered pairs (x, y) of integers is it true that $0 < x < y < 10^6$ and that the arithmetic mean of x and y is exactly 2 more than the geometric mean of x and y?

---

7 Suppose that $x, y,$ and z are three positive numbers that satisfy the equations $xyz = 1, x + \frac{1}{z} = 5,$ and $y + \frac{1}{x} = 29.$ Then $z + \frac{1}{y} = \frac{m}{n},$ where m and n are relatively prime positive integers. Find $m + n.$

---

8 A container in the shape of a right circular cone is 12 inches tall and its base has a 5-inch radius. The liquid that is sealed inside is 9 inches deep when the cone is held with its point down and its base horizontal. When the liquid is held with its point up and its base horizontal, the liquid is $m - n\sqrt[3]{p},$ where $m, n,$ and p are positive integers and p is not divisible by the cube of any prime number. Find $m + n + p.$

---

9 The system of equations

$$\begin{aligned} \log_{10}(2000xy) - (\log_{10} x)(\log_{10} y) &= 4 \\ \log_{10}(2yz) - (\log_{10} y)(\log_{10} z) &= 1 \\ \log_{10}(zx) - (\log_{10} z)(\log_{10} x) &= 0 \end{aligned}$$

has two solutions (x_1, y_1, z_1) and $(x_2, y_2, z_2).$ Find $y_1 + y_2.$

---

10 A sequence of numbers $x_1, x_2, x_3, \ldots, x_{100}$ has the property that, for every integer k between 1 and $100,$ inclusive, the number x_k is k less than the sum of the other 99 numbers. Given that $x_{50} = m/n,$ where m and n are relatively prime positive integers, find $m + n.$

---

11 Let S be the sum of all numbers of the form $a/b,$ where a and b are relatively prime positive divisors of $1000.$ What is the greatest integer that does not exceed $S/10$?

---

12 Given a function f for which

$$f(x) = f(398 - x) = f(2158 - x) = f(3214 - x)$$

holds for all real $x,$ what is the largest number of different values that can appear in the list $f(0), f(1), f(2), \ldots, f(999)$?

---

13 In the middle of a vast prairie, a firetruck is stationed at the intersection of two perpendicular straight highways. The truck travels at 50 miles per hour along the highways and at 14 miles per hour across the prairie. Consider the set of points that can be reached by the firetruck within six minutes. The area of this region is m/n square miles, where m and n are relatively prime positive integers. Find $m + n.$

---

14 In triangle ABC, it is given that angles B and C are congruent. Points P and Q lie on $\overline{AC}$ and $\overline{AB}$, respectively, so that $AP = PQ = QB = BC$. Angle ACB is r times as large as angle APQ, where r is a positive real number. Find the greatest integer that does not exceed $1000r$.

---

15 A stack of 2000 cards is labelled with the integers from 1 to 2000, with different integers on different cards. The cards in the stack are not in numerical order. The top card is removed from the stack and placed on the table, and the next card is moved to the bottom of the stack. The new top card is removed from the stack and placed on the table, to the right of the card already there, and the next card in the stack is moved to the bottom of the stack. The process - placing the top card to the right of the cards already on the table and moving the next card in the stack to the bottom of the stack - is repeated until all cards are on the table. It is found that, reading from left to right, the labels on the cards are now in ascending order: $1, 2, 3, \ldots, 1999, 2000$. In the original stack of cards, how many cards were above the card labelled 1999?

– II

– April 11th

---

1 The number

$$\frac{2}{\log_4 2000^6} + \frac{3}{\log_5 2000^6}$$

can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m+n$.

---

2 A point whose coordinates are both integers is called a lattice point. How many lattice points lie on the hyperbola $x^2 - y^2 = 2000^2$.

---

3 A deck of forty cards consists of four 1's, four 2's,..., and four 10's. A matching pair (two cards with the same number) is removed from the deck. Given that these cards are not returned to the deck, let m/n be the probability that two randomly selected cards also form a pair, where m and n are relatively prime positive integers. Find $m+n$.

---

4 What is the smallest positive integer with six positive odd integer divisors and twelve positive even integer divisors?

---

5 Given eight distinguishable rings, let n be the number of possible five-ring arrangements on the four fingers (not the thumb) of one hand. The order of rings on each finger is significant, but it is not required that each finger have a ring. Find the leftmost three nonzero digits of n.

---

6 One base of a trapezoid is 100 units longer than the other base. The segment that joins the midpoints of the legs divides the trapezoid into two regions whose areas are in the ratio $2:3$. Let x be the length of the segment joining the legs of the trapezoid that is parallel to the bases and that divides the trapezoid into two regions of equal area. Find the greatest integer that does not exceed $x^2/100$.

7 Given that

$$\frac{1}{2!17!}+\frac{1}{3!16!}+\frac{1}{4!15!}+\frac{1}{5!14!}+\frac{1}{6!13!}+\frac{1}{7!12!}+\frac{1}{8!11!}+\frac{1}{9!10!}=\frac{N}{1!18!}$$

find the greatest integer that is less than $\frac{N}{100}$.

8 In trapezoid $ABCD$, leg $\overline{BC}$ is perpendicular to bases $\overline{AB}$ and $\overline{CD}$, and diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular. Given that $AB=\sqrt{11}$ and $AD=\sqrt{1001}$, find BC^2.

9 Given that z is a complex number such that $z+\frac{1}{z}=2\cos 3^\circ$, find the least integer that is greater than $z^{2000}+\frac{1}{z^{2000}}$.

10 A circle is inscribed in quadrilateral $ABCD$, tangent to $\overline{AB}$ at P and to $\overline{CD}$ at Q. Given that $AP=19, PB=26, CQ=37,$ and $QD=23$, find the square of the radius of the circle.

11 The coordinates of the vertices of isosceles trapezoid $ABCD$ are all integers, with $A=(20,100)$ and $D=(21,107)$. The trapezoid has no horizontal or vertical sides, and $\overline{AB}$ and $\overline{CD}$ are the only parallel sides. The sum o f the absolute values of all possible slopes for $\overline{AB}$ is m/n, where m and n are relatively prime positive integers. Find $m+n$.

12 The points A, B and C lie on the surface of a sphere with center O and radius 20. It is given that $AB=13, BC=14, CA=15,$ and that the distance from O to triangle ABC is $\frac{m\sqrt{n}}{k}$, where $m, n,$ and k are positive integers, m and k are relatively prime, and n is not divisible by the square of any prime. Find $m+n+k$.

13 The equation $2000x^6+100x^5+10x^3+x-2=0$ has exactly two real roots, one of which is $\frac{m+\sqrt{n}}{r}$, where m, n and r are integers, m and r are relatively prime, and $r>0$. Find $m+n+r$.

14 Every positive integer k has a unique factorial base expansion $(f_1, f_2, f_3, \ldots, f_m)$, meaning that

$$k=1!\cdot f_1+2!\cdot f_2+3!\cdot f_3+\cdots+m!\cdot f_m,$$

where each f_i is an integer, $0\le f_i\le i$, and $0<f_m$. Given that $(f_1, f_2, f_3, \ldots, f_j)$ is the factorial base expansion of $16!-32!+48!-64!+\cdots+1968!-1984!+2000!$, find the value of $f_1-f_2+f_3-f_4+\cdots+(-1)^{j+1}f_j$.

15 Find the least positive integer n such that

$$\frac{1}{\sin 45^\circ \sin 46^\circ}+\frac{1}{\sin 47^\circ \sin 48^\circ}+\cdots+\frac{1}{\sin 133^\circ \sin 134^\circ}=\frac{1}{\sin n^\circ}.$$