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Day 1 April 27th

1 Find, with proof, all positive integers n for which $2^n + 12^n + 2011^n$ is a perfect square.

2 Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + (a + b + c)^2 \le 4$. Prove that

$$\frac{ab+1}{(a+b)^2} + \frac{bc+1}{(b+c)^2} + \frac{ca+1}{(c+a)^2} \ge 3.$$

3 For a point $P = (a, a^2)$ in the coordinate plane, let $l(P)$ denote the line passing through P with slope $2a$. Consider the set of triangles with vertices of the form $P_1 = (a_1, a_1^2), P_2 = (a_2, a_2^2), P_3 = (a_3, a_3^2)$, such that the intersection of the lines $l(P_1), l(P_2), l(P_3)$ form an equilateral triangle $\triangle$. Find the locus of the center of $\triangle$ as $P_1P_2P_3$ ranges over all such triangles.

Day 2 April 28th

4 A *word* is defined as any finite string of letters. A word is a *palindrome* if it reads the same backwards and forwards. Let a sequence of words $W_0, W_1, W_2, \ldots$ be defined as follows: $W_0 = a, W_1 = b$, and for $n \ge 2$, W_n is the word formed by writing W_{n-2} followed by W_{n-1}. Prove that for any $n \ge 1$, the word formed by writing $W_1, W_2, W_3, \ldots, W_n$ in succession is a palindrome.

5 Points A, B, C, D, E lie on a circle ω and point P lies outside the circle. The given points are such that (i) lines PB and PD are tangent to ω, (ii) P, A, C are collinear, and (iii) $DE \parallel AC$. Prove that BE bisects AC.

6 Consider the assertion that for each positive integer $n \ge 2$, the remainder upon dividing 2^{2^n} by $2^n - 1$ is a power of 4. Either prove the assertion or find (with proof) a counterexample.

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