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1 Given that a, b, c, d, e are real numbers such that

$a+b+c+d+e=8$, $a^2+b^2+c^2+d^2+e^2=16$.

Determine the maximum value of e.

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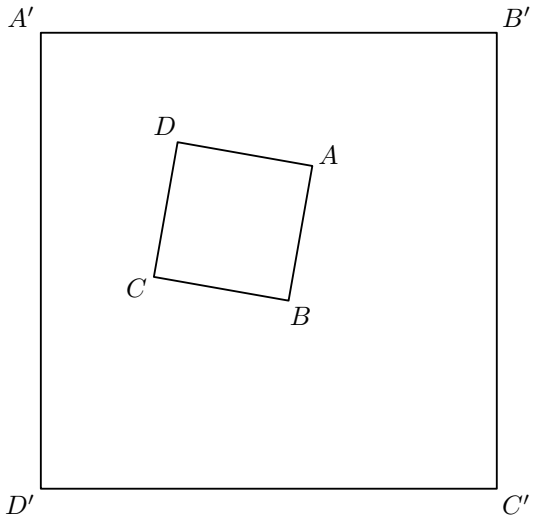
2 $ABCD$ and $A'B'C'D'$ are square maps of the same region, drawn to different scales and superimposed as shown in the figure. Prove that there is only one point O on the small map that lies directly over point O' of the large map such that O and O' each represent the same place of the country. Also, give a Euclidean construction (straight edge and compass) for O.



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3 An integer n will be called *good* if we can write

$$n = a_1 + a_2 + \cdots + a_k,$$

where $a_1, a_2, \ldots, a_k$ are positive integers (not necessarily distinct) satisfying

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = 1.$$

Given the information that the integers 33 through 73 are good, prove that every integer ≥ 33 is good.

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4 (a) Prove that if the six dihedral (i.e. angles between pairs of faces) of a given tetrahedron are congruent, then the tetrahedron is regular.

(b) Is a tetrahedron necessarily regular if five dihedral angles are congruent?

5 Nine mathematicians meet at an international conference and discover that among any three of them, at least two speak a common language. If each of the mathematicians speak at most three languages, prove that there are at least three of the mathematicians who can speak the same language.

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