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Day 1 April 19th

1 Prove that there are infinitely many distinct pairs (a, b) of relatively prime integers $a > 1$ and $b > 1$ such that $a^b + b^a$ is divisible by $a + b$.

2 Consider the equation

$$(3x^3 + xy^2)(x^2y + 3y^3) = (x - y)^7$$

(a) Prove that there are infinitely many pairs (x, y) of positive integers satisfying the equation.
(b) Describe all pairs (x, y) of positive integers satisfying the equation.

3 Let ABC be an equilateral triangle, and point P on its circumcircle. Let PA and BC intersect at D, PB and AC intersect at E, and PC and AB intersect at F. Prove that the area of $\triangle DEF$ is twice the area of $\triangle ABC$.

Proposed by Titu Andreescu, Luis Gonzales, Cosmin Pohoata

Day 2 April 20th

4 Are there any triples (a, b, c) of positive integers such that $(a-2)(b-2)(c-2)+12$ is a prime number that properly divides the positive number $a^2 + b^2 + c^2 + abc - 2017$?

5 Let O and H be the circumcenter and the orthocenter of an acute triangle ABC. Points M and D lie on side BC such that $BM = CM$ and $\angle BAD = \angle CAD$. Ray MO intersects the circumcircle of triangle BHC in point N. Prove that $\angle ADO = \angle HAN$.

6 Let P_1, P_2, $\ldots$, P_{2n} be $2n$ distinct points on the unit circle $x^2 + y^2 = 1$, other than $(1, 0)$. Each point is colored either red or blue, with exactly n red points and n blue points. Let R_1, R_2, $\ldots$, R_n be any ordering of the red points. Let B_1 be the nearest blue point to R_1 traveling counterclockwise around the circle starting from R_1. Then let B_2 be the nearest of the remaining blue points to R_2 travelling counterclockwise around the circle from R_2, and so on, until we have labeled all of the blue points $B_1, \ldots, B_n$. Show that the number of counterclockwise arcs of the form $R_i \to B_i$ that contain the point $(1, 0)$ is independent of the way we chose the ordering $R_1, \ldots, R_n$ of the red points.

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