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by green_dog_7983, asp211, kootrapali, CantonMathGuy, trumpeter, hwl0304, rrusczyk

– **Day 1** April 17

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1 There are $a+b$ bowls arranged in a row, numbered 1 through $a+b$, where a and b are given positive integers. Initially, each of the first a bowls contains an apple, and each of the last b bowls contains a pear.
A legal move consists of moving an apple from bowl i to bowl $i+1$ and a pear from bowl j to bowl $j-1$, provided that the difference $i-j$ is even. We permit multiple fruits in the same bowl at the same time. The goal is to end up with the first b bowls each containing a pear and the last a bowls each containing an apple. Show that this is possible if and only if the product ab is even.

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2 Let $\mathbb{Z}$ be the set of all integers. Find all pairs of integers (a,b) for which there exist functions $f\colon \mathbb{Z}\to\mathbb{Z}$ and $g\colon \mathbb{Z}\to\mathbb{Z}$ satisfying

$$f(g(x)) = x+a \quad \text{and} \quad g(f(x)) = x+b$$

for all integers x.

Proposed by Ankan Bhattacharya

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3 Let $ABCD$ be a cyclic quadrilateral satisfying $AD^2+BC^2=AB^2$. The diagonals of $ABCD$ intersect at E. Let P be a point on side $\overline{AB}$ satisfying $\angle APD=\angle BPC$. Show that line PE bisects $\overline{CD}$.

Proposed by Ankan Bhattacharya

– **Day 2** April 18

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4 Let ABC be a triangle with $\angle ABC$ obtuse. The [i]A-excircle[/i] is a circle in the exterior of $\triangle ABC$ that is tangent to side BC of the triangle and tangent to the extensions of the other two sides. Let E, F be the feet of the altitudes from B and C to lines AC and AB, respectively. Can line EF be tangent to the A-excircle?

Proposed by Ankan Bhattacharya, Zack Chroman, and Anant Mudgal

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5 Let n be a nonnegative integer. Determine the number of ways that one can choose $(n+1)^2$ sets $S_{i,j}\subseteq\{1,2,\ldots,2n\}$, for integers i,j with $0\leq i,j\leq n$, such that:

- for all $0\leq i,j\leq n$, the set $S_{i,j}$ has $i+j$ elements; and
- $S_{i,j}\subseteq S_{k,l}$ whenever $0\leq i\leq k\leq n$ and $0\leq j\leq l\leq n$.

Proposed by Ricky Liu

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6 Two rational numbers $\frac{m}{n}$ and $\frac{n}{m}$ are written on a blackboard, where m and n are relatively prime positive integers. At any point, Evan may pick two of the numbers x and y written on the board and write either their arithmetic mean $\frac{x+y}{2}$ or their harmonic mean $\frac{2xy}{x+y}$ on the board as well. Find all pairs (m, n) such that Evan can write 1 on the board in finitely many steps.

Proposed by Yannick Yao
