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Day 1 April 30th

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1 Are there integers a and b such that a^5b+3 and ab^5+3 are both perfect cubes of integers?

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2 Each cell of an $m\times n$ board is filled with some nonnegative integer. Two numbers in the filling are said to be *adjacent* if their cells share a common side. (Note that two numbers in cells that share only a corner are not adjacent). The filling is called a *garden* if it satisfies the following two conditions:

(i) The difference between any two adjacent numbers is either 0 or 1.
(ii) If a number is less than or equal to all of its adjacent numbers, then it is equal to 0.

Determine the number of distinct gardens in terms of m and n.

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3 In triangle ABC, points P, Q, R lie on sides BC, CA, AB respectively. Let ω_A, ω_B, ω_C denote the circumcircles of triangles AQR, BRP, CPQ, respectively. Given the fact that segment AP intersects ω_A, ω_B, ω_C again at X, Y, Z, respectively, prove that $YX/XZ=BP/PC$.

Day 2 May 1st

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4 Let $f(n)$ be the number of ways to write n as a sum of powers of 2, where we keep track of the order of the summation. For example, $f(4)=6$ because 4 can be written as 4, $2+2$, $2+1+1$, $1+2+1$, $1+1+2$, and $1+1+1+1$. Find the smallest n greater than 2013 for which $f(n)$ is odd.

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5 Quadrilateral $XABY$ is inscribed in the semicircle ω with diameter XY. Segments AY and BX meet at P. Point Z is the foot of the perpendicular from P to line XY. Point C lies on ω such that line XC is perpendicular to line AZ. Let Q be the intersection of segments AY and XC. Prove that

$$\frac{BY}{XP}+\frac{CY}{XQ}=\frac{AY}{AX}.$$

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6 Find all real numbers $x,y,z\geq 1$ satisfying

$$\min(\sqrt{x+xyz},\sqrt{y+xyz},\sqrt{z+xyz})=\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}.$$

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