AMC 12/AHSME 2019

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– A

– February 7th, 2019

1 The area of a pizza with radius 4 inches is N percent larger than the area of a pizza with radius 3 inches. What is the integer closest to N?

(A) 25 **(B)** 33 **(C)** 44 **(D)** 66 **(E)** 78

2 Suppose a is 150% of b. What percent of a is $3b$?

(A) 50 **(B)** $66\frac{2}{3}$ **(C)** 150 **(D)** 200 **(E)** 450

3 A box contains 28 red balls, 20 green balls, 19 yellow balls, 13 blue balls, 11 white balls, and 9 black balls. What is the minimum number of balls that must be drawn from the box without replacement to guarantee that at least 15 balls of a single color will be drawn?

(A) 75 **(B)** 76 **(C)** 79 **(D)** 84 **(E)** 91

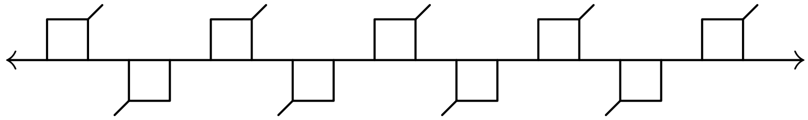
4 What is the greatest number of consecutive integers whose sum is $45?$

(A) 9 **(B)** 25 **(C)** 45 **(D)** 90 **(E)** 120

5 Two lines with slopes $\frac{1}{2}$ and 2 intersect at $(2,2)$. What is the area of the triangle enclosed by these two lines and the line $x+y=10?$

(A) 4 **(B)** $4\sqrt{2}$ **(C)** 6 **(D)** 8 **(E)** $6\sqrt{2}$

6 The figure below shows line ℓ with a regular, infinite, recurring pattern of squares and line segments.



How many of the following four kinds of rigid motion transformations of the plane in which this figure is drawn, other than the identity transformation, will transform this figure into itself?

- some rotation around a point of line ℓ
- some translation in the direction parallel to line ℓ
- the reflection across line ℓ
- some reflection across a line perpendicular to line ℓ

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

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7 Melanie computes the mean μ, the median M, and the modes of the 365 values that are the dates in the months of 2019. Thus her data consist of 12 1s, 12 2s, . . . , 12 28s, 11 29s, 11 30s, and 7 31s. Let d be the median of the modes. Which of the following statements is true?

(A) $\mu < d < M$ **(B)** $M < d < \mu$ **(C)** $d = M = \mu$ **(D)** $d < M < \mu$ **(E)** $d < \mu < M$

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8 For a set of four distinct lines in a plane, there are exactly N distinct points that lie on two or more of the lines. What is the sum of all possible values of N?

(A) 14 **(B)** 16 **(C)** 18 **(D)** 19 **(E)** 21

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9 A sequence of numbers is defined recursively by $a_1 = 1$, $a_2 = \frac{3}{7}$, and

$$a_n = \frac{a_{n-2} \cdot a_{n-1}}{2a_{n-2} - a_{n-1}}$$

for all $n \geq 3$ Then a_{2019} can be written as $\frac{p}{q}$, where p and q are relatively prime positive inegers. What is $p + q$?

(A) 2020 **(B)** 4039 **(C)** 6057 **(D)** 6061 **(E)** 8078

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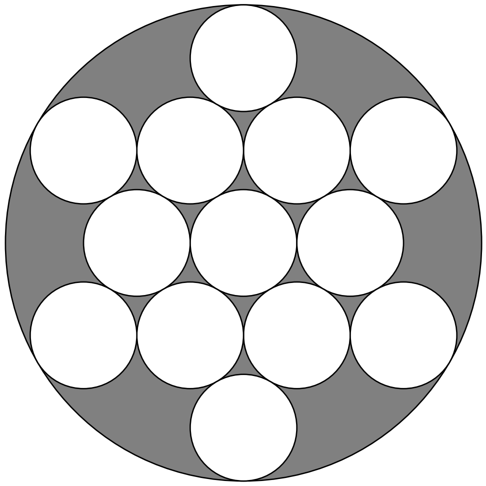
10 The figure below shows 13 circles of radius 1 within a larger circle. All the intersections occur at points of tangency. What is the area of the region, shaded in the figure, inside the larger circle but outside all the circles of radius 1?



(A) $4\pi\sqrt{3}$ **(B)** 7π **(C)** $\pi(3\sqrt{3}+2)$ **(D)** $10\pi(\sqrt{3}-1)$ **(E)** $\pi(\sqrt{3}+6)$

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11 For some positive integer k, the repeating base-k representation of the (base-ten) fraction $\frac{7}{51}$ is $0.\overline{23}_k = 0.232323..._k$. What is k?

(A) 13 **(B)** 14 **(C)** 15 **(D)** 16 **(E)** 17

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12 Positive real numbers $x \neq 1$ and $y \neq 1$ satisfy $\log_2 x = \log_y 16$ and $xy = 64$. What is $(\log_2 \frac{x}{y})^2$?

(A) $\frac{25}{2}$ **(B)** 20 **(C)** $\frac{45}{2}$ **(D)** 25 **(E)** 32

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13 How many ways are there to paint each of the integers $2, 3, \ldots, 9$ either red, green, or blue so that each number has a different color from each of its proper divisors?

(A) 144 **(B)** 216 **(C)** 256 **(D)** 384 **(E)** 432

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14 For a certain complex number c, the polynomial

$$P(x) = (x^2 - 2x + 2)(x^2 - cx + 4)(x^2 - 4x + 8)$$

has exactly 4 distinct roots. What is $|c|$?

(A) 2 **(B)** $\sqrt{6}$ **(C)** $2\sqrt{2}$ **(D)** 3 **(E)** $\sqrt{10}$

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15 Positive real numbers a and b have the property that

$$\sqrt{\log a} + \sqrt{\log b} + \log \sqrt{a} + \log \sqrt{b} = 100$$

and all four terms on the left are positive integers, where log denotes the base 10 logarithm. What is ab?

(A) 10^{52} **(B)** 10^{100} **(C)** 10^{144} **(D)** 10^{164} **(E)** 10^{200}

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16 The numbers $1, 2, \dots, 9$ are randomly placed into the 9 squares of a 3×3 grid. Each square gets one number, and each of the numbers is used once. What is the probability that the sum of the numbers in each row and each column is odd?

(A) $1/21$ **(B)** $1/14$ **(C)** $5/63$ **(D)** $2/21$ **(E)** $1/7$

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17 Let s_k denote the sum of the *k*th powers of the roots of the polynomial $x^3 - 5x^2 + 8x - 13$. In particular, $s_0 = 3$, $s_1 = 5$, and $s_2 = 9$. Let a, b, and c be real numbers such that $s_{k+1} = a\,s_k + b\,s_{k-1} + c\,s_{k-2}$ for $k = 2$, 3, ... What is $a + b + c$?

(A) -6 **(B)** 0 **(C)** 6 **(D)** 10 **(E)** 26

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18 A sphere with center O has radius 6. A triangle with sides of length $15, 15,$ and 24 is situated in space so that each of its sides is tangent to the sphere. What is the distance between O and the plane determined by the triangle?

(A) $2\sqrt{3}$ **(B)** 4 **(C)** $3\sqrt{2}$ **(D)** $2\sqrt{5}$ **(E)** 5

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19 In $\triangle ABC$ with integer side lengths,

$$\cos A = \frac{11}{16}, \qquad \cos B = \frac{7}{8}, \qquad \text{and} \qquad \cos C = -\frac{1}{4}.$$

What is the least possible perimeter for $\triangle ABC$?

(A) 9 **(B)** 12 **(C)** 23 **(D)** 27 **(E)** 44

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20 Real numbers between 0 and 1, inclusive, are chosen in the following manner. A fair coin is flipped. If it lands heads, then it is flipped again and the chosen number is 0 if the second flip is heads and 1 if the second flip is tails. On the other hand, if the first coin flip is tails, then the number is chosen uniformly at random from the closed interval $[0, 1]$. Two random numbers x and y are chosen independently in this manner. What is the probability that $|x - y| > \frac{1}{2}$?

(A) $\frac{1}{3}$ **(B)** $\frac{7}{16}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{9}{16}$ **(E)** $\frac{2}{3}$

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21 Let

$$z = \frac{1+i}{\sqrt{2}}.$$

What is

$$(z^{1^2} + z^{2^2} + z^{3^2} + \dots + z^{12^2}) \cdot \left(\frac{1}{z^{1^2}} + \frac{1}{z^{2^2}} + \frac{1}{z^{3^2}} + \dots + \frac{1}{z^{12^2}}\right)?$$

(A) 18 **(B)** $72 - 36\sqrt{2}$ **(C)** 36 **(D)** 72 **(E)** $72 + 36\sqrt{2}$

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22 Circles ω and γ, both centered at O, have radii 20 and 17, respectively. Equilateral triangle ABC, whose interior lies in the interior of ω but in the exterior of γ, has vertex A on ω, and the line containing side $\overline{BC}$ is tangent to γ. Segments $\overline{AO}$ and $\overline{BC}$ intersect at P, and $\dfrac{BP}{CP} = 3$. Then AB can be written in the form $\dfrac{m}{\sqrt{n}} - \dfrac{p}{\sqrt{q}}$ for positive integers m, n, p, q with $\gcd(m,n) = \gcd(p,q) = 1$. What is $m+n+p+q$?

(A) 42 **(B)** 86 **(C)** 92 **(D)** 114 **(E)** 130

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23 Define binary operations $\diamondsuit$ and $\heartsuit$ by

$$a \diamondsuit b = a^{\log_7(b)} \qquad \text{and} \qquad a \heartsuit b = a^{\frac{1}{\log_7(b)}}$$

for all real numbers a and b for which these expressions are defined. The sequence (a_n) is defined recursively by $a_3 = 3 \heartsuit 2$ and

$$a_n = (n \heartsuit (n-1)) \diamondsuit a_{n-1}$$

for all integers $n \geq 4$. To the nearest integer, what is $\log_7(a_{2019})$?

(A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

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24 For how many integers n between 1 and 50, inclusive, is

$$\frac{(n^2-1)!}{(n!)^n}$$

an integer? (Recall that $0! = 1$.)

(A) 31 **(B)** 32 **(C)** 33 **(D)** 34 **(E)** 35

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25 Let $\triangle A_0B_0C_0$ be a triangle whose angle measures are exactly 59.999°, 60°, and 60.001°. For each positive integer n define A_n to be the foot of the altitude from A_{n-1} to line $B_{n-1}C_{n-1}$. Likewise, define B_n to be the foot of the altitude from B_{n-1} to line $A_{n-1}C_{n-1}$, and C_n to be the foot of the altitude from C_{n-1} to line $A_{n-1}B_{n-1}$. What is the least positive integer n for which $\triangle A_nB_nC_n$ is obtuse?

(A) 10 **(B)** 11 **(C)** 13 **(D)** 14 **(E)** 15

– B

– February 13th, 2019

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1 Alicia had two containers. The first was $\frac{5}{6}$ full of water and the second was empty. She poured all the water from the first container into the second container, at which point the second container was $\frac{3}{4}$ full of water. What is the ratio of the volume of the first container to the volume of the second container?

(A) $\frac{5}{8}$ **(B)** $\frac{4}{5}$ **(C)** $\frac{7}{8}$ **(D)** $\frac{9}{10}$ **(E)** $\frac{11}{12}$

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2 Consider the statement, "If n is not prime, then $n-2$ is prime." Which of the following values of n is a counterexample to this statement?

(A) 11 **(B)** 15 **(C)** 19 **(D)** 21 **(E)** 27

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3 Which one of the following rigid transformations (isometries) maps the line segment $\overline{AB}$ onto the line segment $\overline{A'B'}$ so that the image of $A(-2,1)$ is $A'(2,-1)$ and the image of $B(-1,4)$ is $B'(1,-4)$?

(A) reflection in the y-axis **(B)** counterclockwise rotation around the origin by 90° **(C)** translation by 3 units to the right and 5 units down **(D)** reflection in the x-axis **(E)** clockwise rotation about the origin by 180°

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4 A positive integer n satisfies the equation $(n+1)!+(n+2)!=n!\cdot 440$. What is the sum of the digits of n?

(A) 2 **(B)** 5 **(C)** 10 **(D)** 12 **(E)** 15

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5 Each piece of candy in a store costs a whole number of cents. Casper has exactly enough money to buy either 12 pieces of red candy, 14 pieces of green candy, 15 pieces of blue candy, or n pieces of purple candy. A piece of purple candy costs 20 cents. What is the smallest possible value of n?

(A) 18 **(B)** 21 **(C)** 24 **(D)** 25 **(E)** 28

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6 In a given plane, points A and B are 10 units apart. How many points C are there in the plane such that the perimeter of $\triangle ABC$ is 50 units and the area of $\triangle ABC$ is 100 square units?

(A) 0 **(B)** 2 **(C)** 4 **(D)** 8 **(E)** infinitely many

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7 What is the sum of all real numbers x for which the median of the numbers $4,6,8,17,$ and x is equal to the mean of those five numbers?

(A) -5 **(B)** 0 **(C)** 5 **(D)** $\frac{15}{4}$ **(E)** $\frac{35}{4}$

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8 Let $f(x)=x^2(1-x)^2$. What is the value of the sum

$$f\left(\frac{1}{2019}\right)-f\left(\frac{2}{2019}\right)+f\left(\frac{3}{2019}\right)-f\left(\frac{4}{2019}\right)+\cdots+f\left(\frac{2017}{2019}\right)-f\left(\frac{2018}{2019}\right)?$$

(A) 0 **(B)** $\frac{1}{2019^4}$ **(C)** $\frac{2018^2}{2019^4}$ **(D)** $\frac{2020^2}{2019^4}$ **(E)** 1

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9 For how many integral values of x can a triangle of positive area be formed having side lengths $\log_2 x, \log_4 x, 3$?

(A) 57 **(B)** 59 **(C)** 61 **(D)** 62 **(E)** 63

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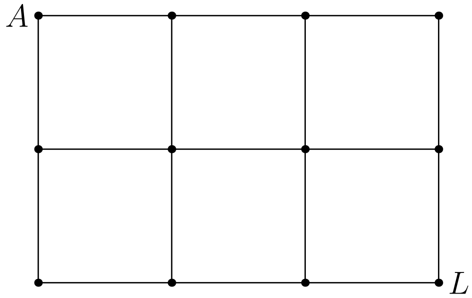
10 The figure below is a map showing 12 cities and 17 roads connecting certain pairs of cities. Paula wishes to travel along exactly 13 of those roads, starting at city A and ending at city L, without traveling along any portion of a road more than once. (Paula *is* allowed to visit a city more than once.) How many different routes can Paula take?



(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

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11 How many unordered pairs of edges of a given cube determine a plane?

(A) 21 **(B)** 28 **(C)** 36 **(D)** 42 **(E)** 66

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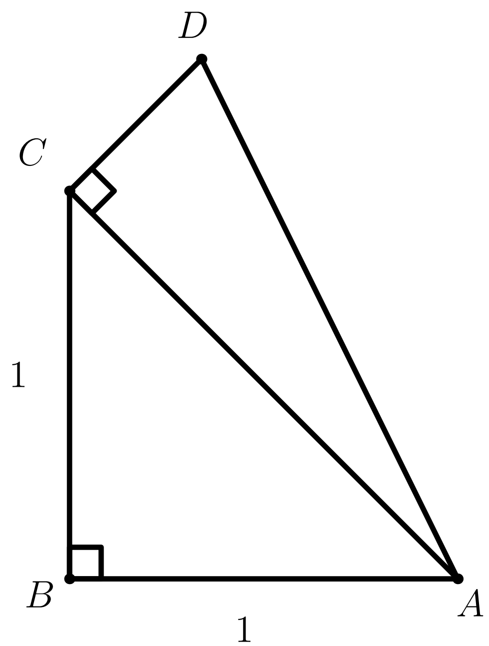
12 Right triangle ACD with right angle at C is constructed outwards on the hypotenuse $\overline{AC}$ of isosceles right triangle ABC with leg length 1, as shown, so that the two triangles have equal perimeters. What is $\sin(2\angle BAD)$?



(A) $\frac{1}{3}$ **(B)** $\frac{\sqrt{2}}{2}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{7}{9}$ **(E)** $\frac{\sqrt{3}}{2}$

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13 A red ball and a green ball are randomly and independently tossed into bins numbered with positive integers so that for each ball, the probability that it is tossed into bin k is 2^{-k} for $k = 1, 2, 3, \ldots$. What is the probability that the red ball is tossed into a higher-numbered bin than the green ball?

(A) $\frac{1}{4}$ **(B)** $\frac{2}{7}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{3}{8}$ **(E)** $\frac{3}{7}$

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14 Let S be the set of all positive integer divisors of $100,000$. How many numbers are the product of two distinct elements of S?

(A) 98 **(B)** 100 **(C)** 117 **(D)** 119 **(E)** 121

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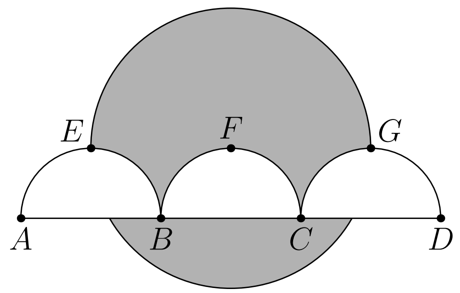
15 As shown in the figure, line segment $\overline{AD}$ is trisected by points B and C so that $AB = BC = CD = 2$. Three semicircles of radius 1, AEB, BFC, and CGD, have their diameters on $\overline{AD}$, and are tangent to line EG at $E, F,$ and G, respectively. A circle of radius 2 has its center on F. The area of the region inside the circle but outside the three semicircles, shaded in the figure, can be expressed in the form

$$\frac{a}{b} \cdot \pi - \sqrt{c} + d,$$

where $a, b, c,$ and d are positive integers and a and b are relatively prime. What is $a + b + c + d$?



(A) 13 **(B)** 14 **(C)** 15 **(D)** 16 **(E)** 17

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16 There are lily pads in a row numbered 0 to 11, in that order. There are predators on lily pads 3 and 6, and a morsel of food on lily pad 10. Fiona the frog starts on pad 0, and from any given lily pad, has a $\frac{1}{2}$ chance to hop to the next pad, and an equal chance to jump 2 pads. What is the probability that Fiona reaches pad 10 without landing on either pad 3 or pad 6?

(A) $\frac{15}{256}$ **(B)** $\frac{1}{16}$ **(C)** $\frac{15}{128}$ **(D)** $\frac{1}{8}$ **(E)** $\frac{1}{4}$

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17 How many nonzero complex numbers z have the property that $0, z,$ and z^3, when represented by points in the complex plane, are the three distinct vertices of an equilateral triangle?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 4 **(E)** infinitely many

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18 Square pyramid $ABCDE$ has base $ABCD$, which measures 3 cm on a side, and altitude $\overline{AE}$ perpendicular to the base, which measures 6 cm. Point P lies on $\overline{BE}$, one third of the way from B to E; point Q lies on $\overline{DE}$, one third of the way from D to E; and point R lies on $\overline{CE}$, two thirds of the way from C to E. What is the area, in square centimeters, of $\triangle PQR$?

(A) $\frac{3\sqrt{2}}{2}$ **(B)** $\frac{3\sqrt{3}}{2}$ **(C)** $2\sqrt{2}$ **(D)** $2\sqrt{3}$ **(E)** $3\sqrt{2}$

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19 Raashan, Sylvia, and Ted play the following game. Each starts with $\$1$. A bell rings every 15 seconds, at which time each of the players who currently have money simultaneously chooses one of the other two players independently and at random and gives $\$1$ to that player. What is the probability that after the bell has rung 2019 times, each player will have $\$1$? (For example, Raashan and Ted may each decide to give $\$1$ to Sylvia, and Sylvia may decide to give her dollar to Ted, at which point Raashan will have $\$0$, Sylvia would have $\$2$, and Ted would have $\$1$, and and that is the end of the first round of play. In the second round Raashan has no money to give, but Sylvia and Ted might choose each other to give their $\$1$ to, and and the holdings will be the same as the end of the second [sic] round.

(A) $\frac{1}{7}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{2}{3}$

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20 Points $A(6,13)$ and $B(12,11)$ lie on circle ω in the plane. Suppose that the tangent lines to ω at A and B intersect at a point on the x-axis. What is the area of ω?

(A) $\frac{83\pi}{8}$ **(B)** $\frac{21\pi}{2}$ **(C)** $\frac{85\pi}{8}$ **(D)** $\frac{43\pi}{4}$ **(E)** $\frac{87\pi}{8}$

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21 How many quadratic polynomials with real coefficients are there such that the set of roots equals the set of coefficients? (For clarification: If the polynomial is $ax^2+bx+c, a \neq 0,$ and the roots are r and $s,$ then the requirement is that $\{a,b,c\}=\{r,s\}$.)

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** infinitely many

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22 Define a sequence recursively by $x_0=5$ and

$$x_{n+1}=\frac{x_n^2+5x_n+4}{x_n+6}$$

for all nonnegative integers $n.$ Let m be the least positive integer such that

$$x_m\leq 4+\frac{1}{2^{20}}.$$

In which of the following intervals does m lie?

(A) $[9,26]$ **(B)** $[27,80]$ **(C)** $[81,242]$ **(D)** $[243,728]$ **(E)** $[729,\infty]$

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23 How many sequences of 0s and 1s of length 19 are there that begin with a 0, end with a 0, contain no two consecutive 0s, and contain no three consecutive 1s?

(A) 55 **(B)** 60 **(C)** 65 **(D)** 70 **(E)** 75

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24 Let $\omega=-\tfrac{1}{2}+\tfrac{1}{2}i\sqrt{3}.$ Let S denote all points in the complex plane of the form $a+b\omega+c\omega^2,$ where $0\leq a\leq 1,0\leq b\leq 1,$ and $0\leq c\leq 1.$ What is the area of S?

(A) $\frac{1}{2}\sqrt{3}$ **(B)** $\frac{3}{4}\sqrt{3}$ **(C)** $\frac{3}{2}\sqrt{3}$ **(D)** $\frac{1}{2}\pi\sqrt{3}$ **(E)** π

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25 Let $ABCD$ be a convex quadrilateral with $BC=2$ and $CD=6.$ Suppose that the centroids of $\triangle ABC,\triangle BCD,$ and $\triangle ACD$ form the vertices of an equilateral triangle. What is the maximum possible value of the area of $ABCD$?

(A) 27 **(B)** $16\sqrt{3}$ **(C)** $12+10\sqrt{3}$ **(D)** $9+12\sqrt{3}$ **(E)** 30
