AIME Problems 2020

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–	I
–	March 11, 2020
1	In $\triangle ABC$ with $AB = AC$, point D lies strictly between A and C on side $\overline{AC}$, and point E lies strictly between A and B on side $\overline{AB}$ such that $AE = ED = DB = BC$. The degree measure of $\angle ABC$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
2	There is a unique positive real number x such that the three numbers $\log_8(2x), \log_4 x,$ and $\log_2 x,$ in that order, form a geometric progression with positive common ratio. The number x can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
3	A positive integer N has base-eleven representation $\underline{a}\,\underline{b}\,\underline{c}$ and base-eight representation $\underline{1}\,\underline{b}\,\underline{c}\,\underline{a}$, where a, b, and c represent (not necessarily distinct) digits. Find the least such N expressed in base ten.
4	Let S be the set of positive integers N with the property that the last four digits of N are 2020, and when the last four digits are removed, the result is a divisor of N. For example, $42,020$ is in S because 4 is a divisor of $42,020$. Find the sum of all the digits of all the numbers in S. For example, the number $42,020$ contributes $4 + 2 + 0 + 2 + 0 = 8$ to this total.
5	Six cards numbered 1 through 6 are to be lined up in a row. Find the number of arrangements of these six cards where one of the cards can be removed leaving the remaining five cards in either ascending or descending order.
6	A flat board has a circular hole with radius 1 and a circular hole with radius 2 such that the distance between the centers of the two holes is 7. Two spheres with equal radii sit in the two holes such that the spheres are tangent to each other. The square of the radius of the spheres is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
7	A club consisting of 11 men and 12 women needs to choose a committee from among its members so that the number of women on the committee is one more than the number of men on the committee. The committee could have as few as 1 member or as many as 23 members. Let N be the number of such committees that can be formed. Find the sum of the prime numbers that divide N.

8 A bug walks all day and sleeps all night. On the first day, it starts at point O, faces east, and walks a distance of 5 units due east. Each night the bug rotates 60° counterclockwise. Each day it walks in this new direction half as far as it walked the previous day. The bug gets arbitrarily close to point P. Then $OP^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

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9 Let S be the set of positive integer divisors of 20^9. Three numbers are chosen independently and at random from the set S and labeled $a_1, a_2,$ and a_3 in the order they are chosen. The probability that both a_1 divides a_2 and a_2 divides a_3 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m.$

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10 Let m and n be positive integers satisfying the conditions

- $\gcd(m+n, 210) = 1,$
- m^m is a multiple of n^n, and
- m is not a multiple of n.

Find the least possible value of $m+n$.

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11 For integers a, b, c, and d, let $f(x) = x^2 + ax + b$ and $g(x) = x^2 + cx + d$. Find the number of ordered triples (a, b, c) of integers with absolute values not exceeding 10 for which there is an integer d such that $g(f(2)) = g(f(4)) = 0$.

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12 Let n be the least positive integer for which $149^n - 2^n$ is divisible by $3^3 \cdot 5^5 \cdot 7^7$. Find the number of positive divisors of n.

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13 Point D lies on side BC of $\triangle ABC$ so that $\overline{AD}$ bisects $\angle BAC$. The perpendicular bisector of $\overline{AD}$ intersects the bisectors of $\angle ABC$ and $\angle ACB$ in points E and F, respectively. Given that $AB = 4$, $BC = 5$, $CA = 6$, the area of $\triangle AEF$ can be written as $\frac{m\sqrt{n}}{p}$, where m and p are relatively prime positive integers, and n is a positive integer not divisible by the square of any prime. Find $m+n+p$.

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14 Let $P(x)$ be a quadratic polynomial with complex coefficients whose x^2 coefficient is 1. Suppose the equation $P(P(x)) = 0$ has four distinct solutions, $x = 3, 4, a, b$. Find the sum of all possible values of $(a+b)^2$.

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15 Let ABC be an acute triangle with circumcircle ω and orthocenter H. Suppose the tangent to the circumcircle of $\triangle HBC$ at H intersects ω at points X and Y with $HA = 3$, $HX = 2$, $HY = 6$. The area of $\triangle ABC$ can be written as $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m+n$.

– II (AOIME)

– June 6, 2020

1 Find the number of ordered pairs of positive integers (m, n) such that $m^2n = 20^{20}$.

2 Let P be a point chosen uniformly at random in the interior of the unit square with vertices at $(0,0), (1,0), (1,1)$, and $(0,1)$. The probability that the slope of the line determined by P and the point $\left(\frac{5}{8}, \frac{3}{8}\right)$ is greater than $\frac{1}{2}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

3 The value of x that satisfies $\log_{2^x} 3^{20} = \log_{2^{x+3}} 3^{2020}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

4 Triangles $\triangle ABC$ and $\triangle A'B'C'$ lie in the coordinate plane with vertices $A(0,0)$, $B(0,12)$, $C(16,0)$, $A'(24,18)$, $B'(36,18)$, and $C'(24,2)$. A rotation of m degrees clockwise around the point (x, y), where $0 < m < 180$, will transform $\triangle ABC$ to $\triangle A'B'C'$. Find $m + x + y$.

5 For each positive integer n, let $f(n)$ be the sum of the digits in the base-four representation of n and let $g(n)$ be the sum of the digits in the base-eight representation of $f(n)$. For example, $f(2020) = f(133210_{\text{four}}) = 10 = 12_{\text{eight}}$, and $g(2020) =$ the digit sum of $12_{\text{eight}} = 3$. Let N be the least value of n such that the base-sixteen representation of $g(n)$ cannot be expressed using only the digits 0 through 9. Find the remainder when N is divided by 1000.

6 Define a sequence recursively by $t_1 = 20$, $t_2 = 21$, and

$$t_n = \frac{5t_{n-1}+1}{25t_{n-2}}$$

for all $n \geq 3$. Then t_{2020} can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

7 Two congruent right circular cones each with base radius 3 and height 8 have axes of symmetry that intersect at right angles at a point in the interior of the cones a distance 3 from the base of each cone. A sphere with radius r lies inside both cones. The maximum possible value for r^2 is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

8 Define a sequence of functions recursively by $f_1(x) = |x-1|$ and $f_n(x) = f_{n-1}(|x-n|)$ for integers $n > 1$. Find the least value of n such that the sum of the zeros of f_n exceeds $500{,}000$.

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9 While watching a show, Ayako, Billy, Carlos, Dahlia, Ehuang, and Frank sat in that order in a row of six chairs. During the break, they went to the kitchen for a snack. When they came back, they sat on those six chairs in such a way that if two of them sat next to each other before the break, then they did not sit next to each other after the break. Find the number of possible seating orders they could have chosen after the break.

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10 Find the sum of all positive integers n such that when $1^3+2^3+3^3+\cdots+n^3$ is divided by $n+5$, the remainder is 17.

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11 Let $P(x) = x^2-3x-7$, and let $Q(x)$ and $R(x)$ be two quadratic polynomials also with the coefficient of x^2 equal to 1. David computes each of the three sums $P+Q$, $P+R$, and $Q+R$ and is surprised to find that each pair of these sums has a common root, and these three common roots are distinct. If $Q(0)=2$, then $R(0)=\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

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12 Let m and n be odd integers greater than 1. An $m\times n$ rectangle is made up of unit squares where the squares in the top row are numbered left to right with the integers 1 through n, those in the second row are numbered left to right with the integers $n+1$ through $2n$, and so on. Square 200 is in the top row, and square 2000 is in the bottom row. Find the number of ordered pairs (m,n) of odd integers greater than 1 with the property that, in the $m\times n$ rectangle, the line through the centers of squares 200 and 2000 intersects the interior of square 1099.

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13 Convex pentagon $ABCDE$ has side lengths $AB=5$, $BC=CD=DE=6$, and $EA=7$. Moreover, the pentagon has an inscribed circle (a circle tangent to each side of the pentagon). Find the area of $ABCDE$.

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14 For real number x let $\lfloor x\rfloor$ be the greatest integer less than or equal to x, and define $\{x\}=x-\lfloor x\rfloor$ to be the fractional part of x. For example, $\{3\}=0$ and $\{4.56\}=0.56$. Define $f(x)=x\{x\}$, and let N be the number of real-valued solutions to the equation $f(f(f(x)))=17$ for $0\le x\le 2020$. Find the remainder when N is divided by 1000.

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15 Let $\triangle ABC$ be an acute scalene triangle with circumcircle ω. The tangents to ω at B and C intersect at T. Let X and Y be the projections of T onto lines AB and AC, respectively. Suppose $BT=CT=16$, $BC=22$, and $TX^2+TY^2+XY^2=1143$. Find XY^2.
