USAJMO 2014

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by ABCDE, msinghal, djmathman, rrusczyk

Day 1

1 Let a, b, c be real numbers greater than or equal to 1. Prove that

$$\min\left(\frac{10a^2-5a+1}{b^2-5b+10},\frac{10b^2-5b+1}{c^2-5c+10},\frac{10c^2-5c+1}{a^2-5a+10}\right)\leq abc.$$

2 Let $\triangle ABC$ be a non-equilateral, acute triangle with $\angle A = 60^\circ$, and let O and H denote the circumcenter and orthocenter of $\triangle ABC$, respectively.

(a) Prove that line OH intersects both segments AB and AC.

(b) Line OH intersects segments AB and AC at P and Q, respectively. Denote by s and t the respective areas of triangle APQ and quadrilateral $BPQC$. Determine the range of possible values for s/t.

3 Let $\mathbb{Z}$ be the set of integers. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$xf(2f(y)-x)+y^2f(2x-f(y))=\frac{f(x)^2}{x}+f(yf(y))$$

for all $x, y \in \mathbb{Z}$ with $x \neq 0$.

Day 2 April 30th

4 Let $b\geq 2$ be an integer, and let $s_b(n)$ denote the sum of the digits of n when it is written in base b. Show that there are infinitely many positive integers that cannot be represented in the form $n+s_b(n)$, where n is a positive integer.

5 Let k be a positive integer. Two players A and B play a game on an infinite grid of regular hexagons. Initially all the grid cells are empty. Then the players alternately take turns with A moving first. In his move, A may choose two adjacent hexagons in the grid which are empty and place a counter in both of them. In his move, B may choose any counter on the board and remove it. If at any time there are k consecutive grid cells in a line all of which contain a counter, A wins. Find the minimum value of k for which A cannot win in a finite number of moves, or prove that no such minimum value exists.

6 Let ABC be a triangle with incenter I, incircle γ and circumcircle Γ. Let M, N, P be the midpoints of sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ and let E, F be the tangency points of γ with $\overline{CA}$ and $\overline{AB}$, respectively. Let U, V be the intersections of line EF with line MN and line MP, respectively, and let X be the midpoint of arc BAC of Γ.

(a) Prove that I lies on ray CV.

(b) Prove that line XI bisects $\overline{UV}$.
