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by Mrdavid445, rrusczyk

1   A two-pan balance is innacurate since its balance arms are of different lengths and its pans are of different weights. Three objects of different weights A, B, and C are each weighed separately. When placed on the left-hand pan, they are balanced by weights A_1, B_1, and C_1, respectively. When A and B are placed on the right-hand pan, they are balanced by A_2 and B_2, respectively. Determine the true weight of C in terms of A_1, B_1, C_1, A_2, and B_2.

2   Determine the maximum number of three-term arithmetic progressions which can be chosen from a sequence of n real numbers

$$a_1 < a_2 < \cdots < a_n.$$

3   Let $F_r = x^r \sin rA + y^r \sin rB + z^r \sin rC$, where x, y, z, A, B, C are real and $A + B + C$ is an integral multiple of π. Prove that if $F_1 = F_2 = 0$, then $F_r = 0$ for all positive integral r.

4   The inscribed sphere of a given tetrahedron touches all four faces of the tetrahedron at their respective centroids. Prove that the tetrahedron is regular.

5   Prove that for numbers a, b, c in the interval $[0, 1]$,

$$\frac{a}{b + c + 1} + \frac{b}{c + a + 1} + \frac{c}{a + b + 1} + (1 - a)(1 - b)(1 - c) \leq 1.$$

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