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Day 1 April 19th

1 The isosceles triangle $\triangle ABC$, with $AB = AC$, is inscribed in the circle ω. Let P be a variable point on the arc $\overset{\frown}{BC}$ that does not contain A, and let I_B and I_C denote the incenters of triangles $\triangle ABP$ and $\triangle ACP$, respectively.

Prove that as P varies, the circumcircle of triangle $\triangle PI_BI_C$ passes through a fixed point.

2 Prove that there exists a positive integer $n < 10^6$ such that 5^n has six consecutive zeros in its decimal representation.

Proposed by Evan Chen

3 Let $X_1, X_2, \ldots, X_{100}$ be a sequence of mutually distinct nonempty subsets of a set S. Any two sets X_i and X_{i+1} are disjoint and their union is not the whole set S, that is, $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$, for all $i \in \{1, \ldots, 99\}$. Find the smallest possible number of elements in S.

Day 2 April 20th

4 Find, with proof, the least integer N such that if any 2016 elements are removed from the set $1, 2, ..., N$, one can still find 2016 distinct numbers among the remaining elements with sum N.

5 Let $\triangle ABC$ be an acute triangle, with O as its circumcenter. Point H is the foot of the perpendicular from A to line $\overleftrightarrow{BC}$, and points P and Q are the feet of the perpendiculars from H to the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$, respectively.

Given that
$$AH^2 = 2 \cdot AO^2,$$
prove that the points $O, P,$ and Q are collinear.

6 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,
$$(f(x) + xy) \cdot f(x - 3y) + (f(y) + xy) \cdot f(3x - y) = (f(x + y))^2.$$

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