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Day 1 April 24th

1 Let n be a positive integer. Define a sequence by setting $a_1 = n$ and, for each $k > 1$, letting a_k be the unique integer in the range $0 \le a_k \le k-1$ for which $a_1 + a_2 + ... + a_k$ is divisible by k. For instance, when $n = 9$ the obtained sequence is $9, 1, 2, 0, 3, 3, 3, ...$. Prove that for any n the sequence $a_1, a_2, ...$ eventually becomes constant.

2 A square grid on the Euclidean plane consists of all points (m, n), where m and n are integers. Is it possible to cover all grid points by an infinite family of discs with non-overlapping interiors if each disc in the family has radius at least 5?

3 Let S be a set containing $n^2 + n - 1$ elements, for some positive integer n. Suppose that the n-element subsets of S are partitioned into two classes. Prove that there are at least n pairwise disjoint sets in the same class.

Day 2 April 25th

4 An *animal* with n *cells* is a connected figure consisting of n equal-sized cells[1].

A *dinosaur* is an animal with at least 2007 cells. It is said to be *primitive* it its cells cannot be partitioned into two or more dinosaurs. Find with proof the maximum number of cells in a primitive dinosaur.

(1) Animals are also called *polyominoes*. They can be defined inductively. Two cells are *adjacent* if they share a complete edge. A single cell is an animal, and given an animal with n cells, one with $n+1$ cells is obtained by adjoining a new cell by making it adjacent to one or more existing cells.

5 Prove that for every nonnegative integer n, the number $7^{7^n} + 1$ is the product of at least $2n + 3$ (not necessarily distinct) primes.

6 Let ABC be an acute triangle with ω, S, and R being its incircle, circumcircle, and circumradius, respectively. Circle ω_A is tangent internally to S at A and tangent externally to ω. Circle S_A is tangent internally to S at A and tangent internally to ω. Let P_A and Q_A denote the centers of ω_A and S_A, respectively. Define points P_B, Q_B, P_C, Q_C analogously. Prove that

$$8P_AQ_A \cdot P_BQ_B \cdot P_CQ_C \le R^3 ,$$

with equality if and only if triangle ABC is equilateral.

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