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1   A certain state issues license plates consisting of six digits (from 0 to 9). The state requires that any two license plates differ in at least two places. (For instance, the numbers 027592 and 020592 cannot both be used.) Determine, with proof, the maximum number of distinct license plates that the state can use.

2   A sequence of functions $\{f_n(x)\}$ is defined recursively as follows:

$$f_1(x) = \sqrt{x^2 + 48}, \quad \text{and}$$
$$f_{n+1}(x) = \sqrt{x^2 + 6f_n(x)} \quad \text{for } n \geq 1.$$

(Recall that $\sqrt{}$ is understood to represent the positive square root.) For each positive integer n, find all real solutions of the equation $f_n(x) = 2x$.

3   Suppose that necklace A has 14 beads and necklace B has 19. Prove that for any odd integer $n \geq 1$, there is a way to number each of the 33 beads with an integer from the sequence

$$\{n, n+1, n+2, \ldots, n+32\}$$

so that each integer is used once, and adjacent beads correspond to relatively prime integers. (Here a "necklace" is viewed as a circle in which each bead is adjacent to two other beads.)

4   Find, with proof, the number of positive integers whose base-n representation consists of distinct digits with the property that, except for the leftmost digit, every digit differs by ± 1 from some digit further to the left. (Your answer should be an explicit function of n in simplest form.)

5   An acute-angled triangle ABC is given in the plane. The circle with diameter AB intersects altitude CC' and its extension at points M and N, and the circle with diameter AC intersects altitude BB' and its extensions at P and Q. Prove that the points M, N, P, Q lie on a common circle.

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