Sample size

Example 1.17

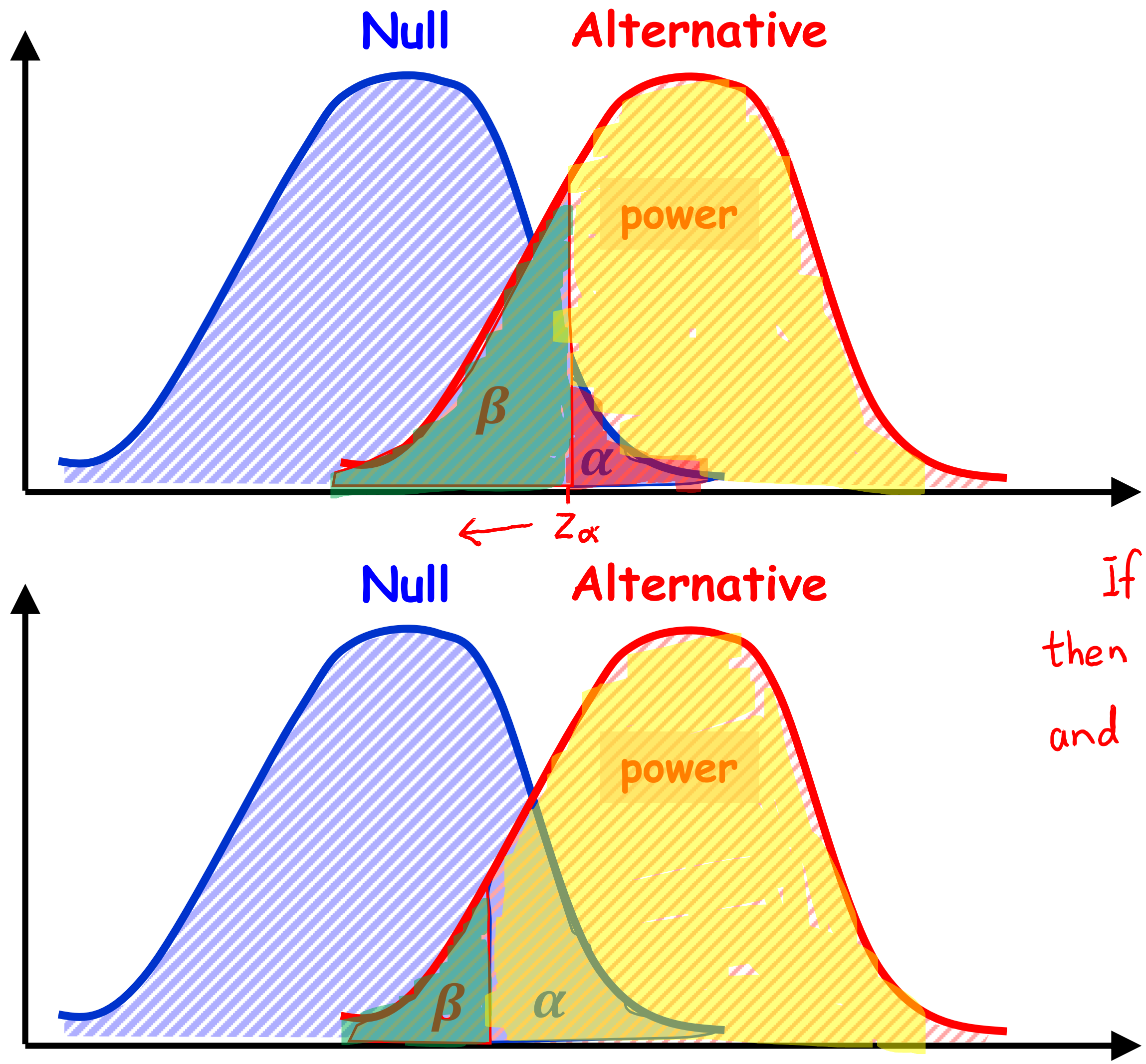
We plan to conduct a study to compare the means of two equally sized treatment groups using a two-sample z-test.

a) Assuming a two-sided test, how many patients are required to detect a difference of 1 unit if the standard deviation is 2, using $\alpha = 0.05$ and $\beta = 0.2$?

b) What is the sample size for a one-sided test with the sample significance level, standard deviation, power and difference?

Factors that affect the power of a test

The effect of α on $1 - \beta$

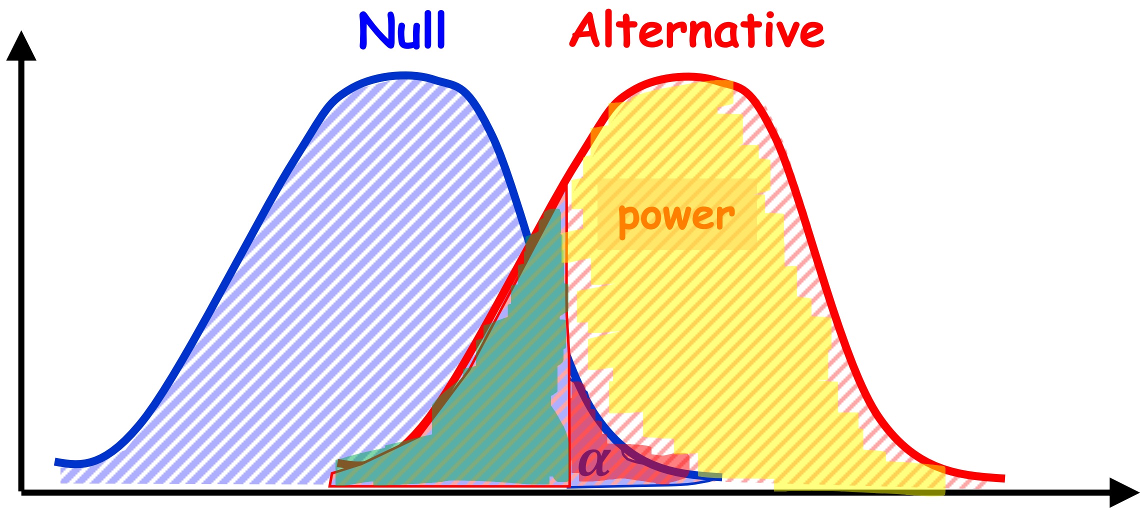


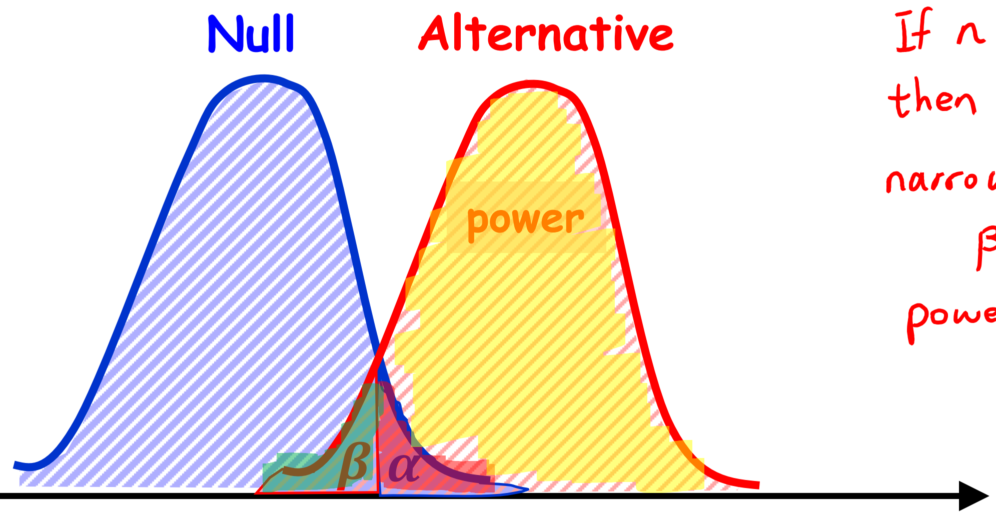
If α increases, then β decreases and power increases.

Factors that affect the power of a test

The effect of standard error on $1 - \beta$

$SE = \frac{\sigma}{\sqrt{n}}$





If n increases,
then SE decreases,
narrower distributions,
β decreases,
power increases

Sample size calculations

We can determine the sample size for a given α and β.

Consider testing $H_0\colon \mu = \mu_0$ against $H_a\colon \mu > \mu_0$.
Given α and β, we want to find the sample size n, for a usual z-test.

We will show that the sample size for this one-sided alternative hypothesis is

$$\boxed{n = \frac{(z_\alpha + z_\beta)^2 \sigma^2}{(\mu_a - \mu_0)^2}},$$

where β is evaluated when $\mu = \mu_a$ and $\mu_a > \mu_0$.

Sample size calculations (proof)

$$\alpha = P(\text{Type I error})$$
$$= P(\text{reject } H_0 \mid H_0 \text{ true})$$
$$= P(\bar{Y} > c \mid \mu = \mu_0)$$
$$= P\left(\frac{\bar{Y}-\mu_0}{\frac{\sigma}{\sqrt{n}}} > \frac{c-\mu_0}{\frac{\sigma}{\sqrt{n}}} \,\middle|\, \mu=\mu_0\right)$$
$$= P\left(Z > \frac{c-\mu_0}{\frac{\sigma}{\sqrt{n}}}\right) \text{ where } Z \sim N(0,1) \text{ under } H_0$$
$$= P(Z > Z_\alpha)$$

Let μ_a = true population mean

$\mu_a > \mu_0$ under H_a

$$\beta = P(\text{Type II error})$$
$$= P(\bar{Y} \leq c \mid \mu = \mu_a)$$
$$= P\left(\frac{\bar{Y}-\mu_a}{\frac{\sigma}{\sqrt{n}}} \leq \frac{c-\mu_a}{\frac{\sigma}{\sqrt{n}}} \,\middle|\, \mu=\mu_a\right)$$
$$= P\left(Z^* \leq \frac{c-\mu_a}{\frac{\sigma}{\sqrt{n}}}\right) \text{ where}$$
$$= P\left(Z^* > -\frac{c-\mu_a}{\frac{\sigma}{\sqrt{n}}}\right) \; Z^* \sim N(0,1)$$
$$= P(Z^* > Z_\beta)$$

Therefore, we have

$$Z_\alpha = \frac{c-\mu_0}{\frac{\sigma}{\sqrt{n}}} \quad \text{and} \quad -Z_\beta = \frac{c-\mu_a}{\frac{\sigma}{\sqrt{n}}}$$

We have two equations in two unknowns (c and n).

We proceed by eliminating c first.

Sample size calculations (proof)

$$Z_\alpha = \frac{c - \mu_0}{\frac{\sigma}{\sqrt{n}}} \quad \text{and} \quad -Z_\beta = \frac{c - \mu_a}{\frac{\sigma}{\sqrt{n}}}$$

$$Z_\alpha \frac{\sigma}{\sqrt{n}} = c - \mu_0 \qquad -Z_\beta \frac{\sigma}{\sqrt{n}} = c - \mu_a$$

$$c = \mu_0 + Z_\alpha \frac{\sigma}{\sqrt{n}} \qquad c = \mu_a - Z_\beta \frac{\sigma}{\sqrt{n}}$$

Equating them:

$$\mu_0 + Z_\alpha \frac{\sigma}{\sqrt{n}} = \mu_a - Z_\beta \frac{\sigma}{\sqrt{n}}$$

$$Z_\alpha \frac{\sigma}{\sqrt{n}} + Z_\beta \frac{\sigma}{\sqrt{n}} = \mu_a - \mu_0$$

$$(Z_\alpha + Z_\beta) \frac{\sigma}{\sqrt{n}} = \mu_a - \mu_0$$

$$\frac{\sigma}{\sqrt{n}} = \frac{\mu_a - \mu_0}{Z_\alpha + Z_\beta}$$

$$\sqrt{n} = \frac{(Z_\alpha + Z_\beta)\sigma}{\mu_a - \mu_0}$$

$$n = \frac{\sigma^2 (Z_\alpha + Z_\beta)^2}{(\mu_a - \mu_0)^2}$$

Some remarks:

1. same expression for the lower one-sided test
2. For the two-sided test, replace Z_α with $Z_{\frac{\alpha}{2}}$.

Example 1.18

How many samples do we need to determine the mean of a population if, from previous knowledge we know $\sigma = 3.1$ and we want to test the hypothesis $H_0: \mu = 5$ against $H_1: \mu = 5.5$? We also want $\alpha = 0.01$ and $\beta = 0.05$.

$\sigma = 3.1, \quad \mu_0 = 5, \quad \mu_a = 5.5, \quad \alpha = 0.01, \quad \beta = 0.05$

$$z_\alpha = z_{0.01} \approx 2.33$$

$$z_\beta = z_{0.05} \approx 1.645$$

$$n = \frac{(z_\alpha + z_\beta)^2 \sigma^2}{(\mu_a - \mu_0)^2} = \frac{(2.33 + 1.645)^2\, 3.1^2}{0.5^2} \approx 607.37$$

So $n = 608$ will provide the desired levels.

Sample size for comparing two means

Consider testing $H_0: \mu_1 = \mu_2$ against $H_a: \mu_1 \neq \mu_2$.
Given α and β, we want to find the sample size n to detect a difference in mean of size δ.

It can be shown that the sample size required (per group) in this case is

$$\boxed{n = \frac{2\sigma^2 \left(Z_{\alpha/2} + z_\beta\right)^2}{\delta^2}.}$$

$\delta = \mu_1 - \mu_2$

Assuming $\sigma_1^2 = \sigma_2^2 = \sigma^2$

$n_1 = n_2 = n$

Example 1.17 (Revisit)

We plan to conduct a study to compare the means of two equally sized treatment groups using a two-sample z-test.

a) Assuming a two-sided test, how many patients are required to detect a difference of 1 unit if the standard deviation is 2, using $\alpha = 0.05$ and $\beta = 0.2$?

b) What is the sample size for a one-sided test with the sample significance level, standard deviation, power and difference?

Example 1.17 Solution

a) $\sigma = 2$, $\alpha = 0.05$, $\beta = 0.2$, $\delta = 1$

$Z_{\frac{\alpha}{2}} = Z_{0.025} = 1.96$

$Z_{\beta} = Z_{0.2} = 0.84$

$$n = \frac{2(2^2)(1.96 + 0.84)^2}{1^2} = 62.72$$

We need 63 patients in each group.

b) $Z_{\beta} = Z_{0.2} = 0.84$

$Z_{\alpha} = Z_{0.05} = 1.64$

$$n = \frac{2(2^2)(1.64 + 0.84)^2}{1^2} = 49.2032$$

We need 50 patients in each group.

Sample size for comparing proportion(s)

Consider testing $H_0: p = p_0$ against $H_a: p \neq p_0$.
Given α and β, the sample size n to detect a population proportion within $\pm\delta$ margin of error is

$$\boxed{n = \frac{z_{\alpha/2}^2 \; p(1-p)}{\delta^2}.}$$

When testing $H_0: p_1 = p_2$ against $H_a: p_1 \neq p_2$, the number of samples n required (per group) to detect a difference in p_1 and p_2, with a given α and β, is

$$\boxed{n = \frac{\left(z_{\alpha/2}+z_\beta\right)^2 [p_1(1-p_1)+p_2(1-p_2)]}{(p_1-p_2)^2}.}$$

For one-sided tests, replace $Z_{\frac{\alpha}{2}}$ with Z_α in the above expressions.

Example 1.19

A researcher plans to conduct a study on the effectiveness of a new drug treatment compared to a standard behavioural treatment for a particular mental illness. It is known that the standard treatment has a cure rate of 70%. Assume a two-sided test is desired, with significance level of 5% and a power of 90%. How many subjects are required to detect an improvement of 10% in the new drug?

$p_1 = 0.7$, $p_2 = 0.8$

$Z_{\frac{\alpha}{2}} = Z_{0.025} = 1.96$ and $Z_{\beta} = Z_{0.1} = 1.28$

$$n = \frac{[0.7(\overset{0.3}{\cancel{0.8}}) + 0.8(0.2)]}{(0.7-0.8)^2}(1.96+1.28)^2 \approx 388.41$$

We need 389 patients per group.