

Q1) a. $E[X_i X_j]$

. If $i \neq j$

$$E[X_i X_j] = E[X_i] E[X_j] \quad (\text{independence})$$

$$= \mu \cdot \mu$$

$$= \mu^2$$

. If $i = j$:

$$E[X_i X_j] = E[X_i X_i]$$

$$= E[X_i^2] \quad (1)$$

$$\text{Var}(X_i) = E[X_i^2] - E[X_i]^2$$

$$\therefore E[X_i^2] = \text{Var}(X_i) + E[X_i]^2 \quad (2)$$

$$\textcircled{1}, \textcircled{2} \Rightarrow E[X_i X_j] = \text{Var}(X_i) + E[X_i]^2 \quad (i=j)$$

$$= \sigma^2 + \mu^2$$

$$\therefore E[X_i X_j] = \begin{cases} \mu^2 & \text{if } i \neq j, \\ \mu^2 + \sigma^2 & \text{otherwise} \end{cases}$$

b. $E[X_i \bar{X}] = E\left[X_i \frac{1}{n} \sum_{j=1}^n X_j\right]$

$$= \frac{1}{n} \sum_{j=1}^n E[X_i X_j]$$

from a. $= \frac{1}{n} \sum_{\substack{j=1 \\ j \neq i}}^n E[X_i] E[X_j] + \frac{1}{n} E[X_i^2] \quad (\text{from a.})$

$$= \frac{n-1}{n} \mu^2 + \frac{1}{n} (\mu^2 + \sigma^2)$$

$$= \mu^2 - \frac{1}{n} \mu^2 + \frac{1}{n} \mu^2 + \frac{1}{n} \sigma^2$$

$$= \mu^2 + \frac{\sigma^2}{n}$$

$$\therefore E[X_i \bar{X}] = \mu^2 + \frac{\sigma^2}{n}$$

c. $E[S_{XX}] = E\left[\sum_{i=1}^n (X_i - \bar{X})^2\right]$

$$= \sum_{i=1}^n E[(X_i - \bar{X})^2]$$

$$= \sum_{i=1}^n E[X_i^2 - 2X_i \bar{X} + \bar{X}^2]$$

$$= \sum_{i=1}^n (E[X_i^2] - 2E[X_i \bar{X}] + E[\bar{X}^2])$$

We have

$$\text{Var}(\bar{X}) = \text{Var}\left(\frac{1}{n} \sum_{i=1}^n X_i\right)$$

$$= \frac{1}{n^2} \sum_{i=1}^n \text{Var}(X_i) \quad (\text{independence})$$

$$= \frac{1}{n^2} \cdot n \cdot \sigma^2$$

$$= \frac{\sigma^2}{n}$$

$$\text{Var}(\bar{X}) = E[\bar{X}^2] - E[\bar{X}]^2$$

$$\therefore E[\bar{X}^2] = \text{Var}(\bar{X}) + E[\bar{X}]^2$$

$$= \frac{\sigma^2}{n} + \mu^2$$

And from a. and b.:

$$\therefore E[S_{XX}] = \sum_{i=1}^n (E[X_i^2] - 2E[X_i \bar{X}] + E[\bar{X}^2])$$

$$= \sum_{i=1}^n \left[\mu^2 + \sigma^2 - 2\left(\mu^2 + \frac{\sigma^2}{n}\right) + \frac{\sigma^2}{n} + \mu^2 \right]$$

$$= n \left(2\mu^2 + \frac{n\sigma^2 + \sigma^2}{n} - 2\mu^2 - \frac{2\sigma^2}{n} \right)$$

$$= n\sigma^2 - \sigma^2$$

$$= (n-1)\sigma^2$$

$$\therefore E[S_{XX}] = (n-1)\sigma^2$$

d. The sample variance S^2 :

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

$$= \frac{1}{n-1} S_{XX}$$

$$\therefore E[S^2] = E\left[\frac{1}{n-1} S_{XX}\right]$$

$$= \frac{1}{n-1} E[S_{XX}]$$

$$= \frac{1}{n-1} \cdot (n-1)\sigma^2 \quad (\text{from c.})$$

$$= \sigma^2$$

We have

$$b_{S^2}(\sigma^2) = E[S^2] - \sigma^2$$

$$= \sigma^2 - \sigma^2$$

$$= 0$$

\therefore The sample variance S^2 is unbiased for σ^2

$$\textcircled{Q2} \text{ a. } E[\hat{p}^2] = E\left[\left(\frac{X}{n}\right)^2\right]$$

$$= \frac{1}{n^2} E[X^2] \quad (1)$$

$$\cdot \text{Var}(X) = E[X^2] - E[X]^2$$

$$\therefore E[X^2] = \text{Var}(X) + E[X]^2$$

$$= np(1-p) + (np)^2 \quad (X \sim \text{Bin}(n, p))$$

$$= np - np^2 + n^2 p^2 \quad (2)$$

$$(1) (2) \Rightarrow E[\hat{p}^2] = \frac{1}{n^2} (np - np^2 + n^2 p^2)$$

$$= \frac{p - p^2}{n} + p^2$$

$$= \frac{p(1-p)}{n} + p^2$$

$$\cdot b_{\hat{p}^2}(p^2) = E[\hat{p}^2] - p^2$$

$$= \frac{p(1-p)}{n} + p^2 - p^2$$

$$= \frac{p(1-p)}{n}$$

$$\therefore E[\hat{p}^2] = \frac{p(1-p)}{n} + p^2 \text{ and } b_{\hat{p}^2}(p^2) = \frac{p(1-p)}{n}$$

~~\therefore The estimator is biased~~ The \hat{p}^2 is a biased estimator for p^2

b. We have:

$$\left. \frac{d^k}{dt^k} M_X(t) \right|_{t=0} = E[X^k]$$

$$\text{We know } E[X] = \left. \frac{d}{dt} M_X(t) \right|_{t=0}$$

$$\cdot \frac{d}{dt} M_X(t) = \frac{d}{dt} [M_X(t)]$$

$$= \frac{d}{dt} [(1-p + pe^t)^n]$$

$$= n(1-p + pe^t)^{n-1} (pe^t)$$

$$\therefore E[X] = n(1-p + pe^0)^{n-1} (pe^0)$$

$$= n(1-p + p)^{n-1} p$$

$$= np$$

$$\cdot E[X^2] = \left. \frac{d^2}{dt^2} M_X(t) \right|_{t=0}$$

$$\cdot \frac{d^2}{dt^2} M_X(t) = n(n-1)(1-p + pe^t)^{n-2} (pe^t)^2$$

$$+ \{n(1-p + pe^t)^{n-1} (pe^t)\}'$$

$$\therefore E[X^2] = n(n-1)(1-p + pe^0)^{n-2} (pe^0)^2$$

$$+ n(1-p + pe^0)^{n-1} (pe^0)'$$

$$= n(n-1)(1-p + p)^{n-2} (p)^2$$

$$+ n(1-p + p)^{n-1} p$$

$$= n(n-1)p^2 + np$$

$$\cdot E[X^3] = \left. \frac{d^3}{dt^3} M_X(t) \right|_{t=0}$$

$$\cdot \frac{d^3}{dt^3} M_X(t) = n(n-1)(n-2)(1-p + pe^t)^{n-3} (pe^t)^3$$

$$+ 2n(n-1)(1-p + pe^t)^{n-2} (pe^t)^2 (pe^t)'$$

$$+ n(n-1)(1-p + pe^t)^{n-2} (pe^t)^2 + n(1-p + pe^t)^{n-1} (pe^t)'$$

$$\begin{aligned}
\therefore E[X^3] &= n(n-1)(n-2)(1-p+pe^0)^{n-3}(pe^0)^3 + 2n(n-1)(1-p+pe^0)^{n-2}(pe^0)^2 \\
&\quad + n(n-1)(1-p+pe^0)^{n-2}(pe^0)^2 + n(1-p+pe^0)^{n-1}(pe^0) \\
&= n(n-1)(n-2)p^3 + 2n(n-1)p^2 + n(n-1)p^2 + np \\
&= n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np
\end{aligned}$$

$$\begin{aligned}
\cdot \frac{d^3}{dt^3} M_X(t) &= n(n-1)(n-2)(1-p+pe^t)^{n-3}(pe^t)^3 + 3n(n-1)(1-p+pe^t)^{n-2}(pe^t)^2 \\
&\quad + n(1-p+pe^t)^{n-1}(pe^t)
\end{aligned}$$

$$\cdot E[X^4] = \frac{d^4}{dt^4} M_X(t) \Big|_{t=0}$$

$$\begin{aligned}
\cdot \frac{d^4}{dt^4} M_X(t) &= n(n-1)(n-2)(n-3)(1-p+pe^t)^{n-4}(pe^t)^4 + 3n(n-1)(n-2)(1-p+pe^t)^{n-3}(pe^t)^3 \\
&\quad + 3n(n-1)(n-2)(1-p+pe^t)^{n-3}(pe^t)^3 + 6n(n-1)(1-p+pe^t)^{n-2}(pe^t)^2 \\
&\quad + n(n-1)(1-p+pe^t)^{n-2}(pe^t)^2 + n(1-p+pe^t)^{n-1}(pe^t) \\
&= n(n-1)(n-2)(n-3)(1-p+pe^t)^{n-4}(pe^t)^4 + 6n(n-1)(n-2)(1-p+pe^t)^{n-3}(pe^t)^3 \\
&\quad + 7n(n-1)(1-p+pe^t)^{n-2}(pe^t)^2 + n(1-p+pe^t)^{n-1}(pe^t)
\end{aligned}$$

$$\begin{aligned}
\therefore E[X^4] &= n(n-1)(n-2)(n-3)(1-p+pe^0)^{n-4}(pe^0)^4 + 6n(n-1)(n-2)(1-p+pe^0)^{n-3}(pe^0)^3 \\
&\quad + 7n(n-1)(1-p+pe^0)^{n-2}(pe^0)^2 + n(1-p+pe^0)^{n-1}(pe^0) \\
&= n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np
\end{aligned}$$

$$c. \text{MSE } \hat{p}^2(p^2) = \text{Var}(\hat{p}^2) + (b_{\hat{p}^2}(p^2))^2$$

We have:

$$\begin{aligned}
\cdot \text{Var}(\hat{p}^2) &= \text{Var}\left(\left(\frac{X}{n}\right)^2\right) \\
&= \frac{1}{n^4} \text{Var}(X^2) \\
&= \frac{1}{n^4} (E[(X^2)^2] - E[X^2]^2) \\
&= \frac{1}{n^4} (E[X^4] - E[X^2]^2)
\end{aligned}$$

$$\begin{aligned}
\cdot E[X^4] - E[X^2]^2 &= n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np \\
&\quad - [n(n-1)p^2 + np]^2 \\
&= (n^4 - 6n^3 + 11n^2 - 6n)p^4 + (6n^3 - 18n^2 + 12n)p^3 + (7n^2 - 7n)p^2 + np \\
&\quad - n^2(n-1)^2p^4 - 2n^2(n-1)p^3 - n^2p^2 \\
&= n^4p^4 - 6n^3p^4 + 11n^2p^4 - 6np^4 + 6n^3p^3 - 18n^2p^3 + 12np^3 + 7n^2p^2 - 7np^2 \\
&\quad + np - (n^4 - 2n^3 + n^2)p^4 - (2n^3 - 2n^2)p^3 - n^2p^2
\end{aligned}$$

$$= (-4n^3 + 10n^2 - 6n)p^4 + (4n^3 - 16n^2 + 12n)p^3 + (6n^2 - 7n)p^2 + np$$

$$\therefore \text{Var}(\hat{p}^2) = \frac{1}{n^4} [(-4n^3 + 10n^2 - 6n)p^4 + (4n^3 - 16n^2 + 12n)p^3 + (6n^2 - 7n)p^2 + np]$$

$$= \frac{1}{n^3} [(-4n^2 + 10n - 6)p^4 + (4n^2 - 16n + 12)p^3 + (6n - 7)p^2 + p]$$

$$\cdot (b_{\hat{p}^2}(p^2))^2 = \left(\frac{p(1-p)}{n} \right)^2$$

$$= \frac{p^2(1-p)^2}{n^2}$$

$$= \frac{p^2(1-2p+p^2)}{n^2}$$

$$= \frac{p^2 - 2p^3 + p^4}{n^2} = \frac{1}{n^3} (np^2 - 2np^3 + np^4)$$

$$\therefore \text{MSE}_{\hat{p}^2}(p^2) = \text{Var}(\hat{p}^2) + (b_{\hat{p}^2}(p^2))^2$$

$$= \frac{1}{n^3} [(-4n^2 + 10n - 6)p^4 + (4n^2 - 16n + 12)p^3 + (6n - 7)p^2 + p]$$

$$+ \frac{1}{n^3} (np^2 - 2np^3 + np^4)$$

$$= \frac{1}{n^3} [(-4n^2 + 11n - 6)p^4 + (4n^2 - 18n + 12)p^3 + (7n - 7)p^2 + p]$$

$$\therefore \text{MSE}_{\hat{p}^2}(p^2) = \frac{1}{n^3} [(-4n^2 + 11n - 6)p^4 + (4n^2 - 18n + 12)p^3 + (7n - 7)p^2 + p]$$

$$d. E\left[\frac{\hat{p}(1-\hat{p})}{n-1}\right] = \frac{1}{n-1} E[\hat{p}(1-\hat{p})]$$

$$= \frac{1}{n-1} E[\hat{p} - \hat{p}^2]$$

$$= \frac{1}{n-1} (E[\hat{p}] - E[\hat{p}^2]) \quad (1)$$

We have:

$$E[\hat{p}] = E\left[\frac{x}{n}\right]$$

$$= \frac{1}{n} E[x]$$

$$= \frac{1}{n} \cdot n \cdot p$$

$$= p \quad (2)$$

$$E[\hat{p}^2] = \frac{p(1-p)}{n} + p^2 \quad (3)$$

From (1), (2) and (3)

$$\therefore E\left[\frac{\hat{p}(1-\hat{p})}{n-1}\right] = \frac{1}{n-1} \left(p - \frac{p(1-p)}{n} + p^2 \right)$$

$$= \frac{1}{n-1} \cdot \frac{np - p - p^2 + np^2}{n}$$

$$= \frac{p}{n(n-1)} \cdot (n-1-p+np)$$

$$= \frac{p}{n(n-1)} \cdot (n-1)(1-p)$$

$$= \frac{p(1-p)}{n}$$

$$\therefore E\left[\frac{\hat{p}(1-\hat{p})}{n-1}\right] = \frac{p(1-p)}{n}$$

a. Let T be an unbiased estimator for p^2

$$\therefore b_T(p^2) = E[T] - p^2 = 0$$

$$\therefore E[T] = p^2$$

$$E\left[\frac{\hat{p}(1-\hat{p})}{n-1}\right] = \frac{p(1-p)}{n}$$

$$= \frac{p-p^2}{n}$$

$$\therefore p-p^2 = n E\left[\frac{\hat{p}(1-\hat{p})}{n-1}\right]$$

$$\therefore p^2 = p - E\left[\frac{n\hat{p}(1-\hat{p})}{n-1}\right] \quad \left. \vphantom{E\left[\frac{n\hat{p}(1-\hat{p})}{n-1}\right]} \right\} \Rightarrow p^2 = E[\hat{p}] - E\left[\frac{n\hat{p}(1-\hat{p})}{n-1}\right]$$

$$E[\hat{p}] = p$$

$$\therefore p^2 = E\left[\hat{p} - \frac{n\hat{p}(1-\hat{p})}{n-1}\right]$$

$$= E\left[\frac{\hat{p}(n-1) - n\hat{p}(1-\hat{p})}{n-1}\right]$$

$$= E\left[\frac{n\hat{p} - \hat{p} - n\hat{p} + n\hat{p}^2}{n-1}\right]$$

$$= E\left[\frac{\cancel{n\hat{p} - \hat{p}} - n\hat{p} + n\hat{p}^2}{n-1}\right]$$

$$\text{And } E[T] = p^2$$

$$\therefore E[T] = E\left[\frac{n\hat{p}^2 - \hat{p}}{n-1}\right]$$

$$\therefore T = \frac{n\hat{p}^2 - \hat{p}}{n-1}$$

$$\therefore \frac{n\hat{p}^2 - \hat{p}}{n-1} \text{ is an unbiased estimator for } p^2$$

f. We have

$$\cdot \text{Var} \left(\frac{n\hat{p}^2 - \hat{p}}{n-1} \right) = \frac{1}{(n-1)^2} \text{Var} (n\hat{p}^2 - \hat{p})$$

$$\begin{aligned} \cdot \text{Var} (n\hat{p}^2 - \hat{p}) &= E[(n\hat{p}^2 - \hat{p})^2] - E[n\hat{p}^2 - \hat{p}]^2 \\ &= E[n^2\hat{p}^4 - 2n\hat{p}^3 + \hat{p}^2] - (nE[\hat{p}^2] - E[\hat{p}])^2 \\ &= n^2 E[\hat{p}^4] - 2n E[\hat{p}^3] + E[\hat{p}^2] - (nE[\hat{p}^2] - E[\hat{p}])^2 \end{aligned}$$

$$\cdot E[\hat{p}] = p \quad (\text{in d.})$$

$$\cdot E[\hat{p}^2] = \frac{p(1-p)}{n} + p^2 \quad (\text{in a.})$$

$$\cdot E[\hat{p}^3] = E \left[\left(\frac{X}{n} \right)^3 \right]$$

$$= \frac{1}{n^3} E[X^3]$$

$$(\text{in b.}) = \frac{1}{n^3} [n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np]$$

$$= \frac{1}{n^2} [(n^2 - 3n + 2)p^3 + (3n - 3)p^2 + p]$$

$$= \frac{p}{n^2} [(n^2 - 3n + 2)p^2 + (3n - 3)p + 1]$$

$$\cdot E[\hat{p}^4] = E \left[\left(\frac{X}{n} \right)^4 \right]$$

$$= \frac{1}{n^4} E[X^4]$$

$$= \frac{1}{n^4} [n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np]$$

$$= \frac{p}{n^3} [(n^3 - 6n^2 + 11n - 6)p^3 + (6n^2 - 18n + 12)p^2 + (7n - 7)p + 1]$$

From all of the above:

$$\therefore \text{Var} (n\hat{p}^2 - \hat{p}) = \frac{p}{n} [(n^3 - 6n^2 + 11n - 6)p^3 + (6n^2 - 18n + 12)p^2 + (7n - 7)p + 1]$$

$$- \frac{2p}{n} [(n^2 - 3n + 2)p^2 + (3n - 3)p + 1] + \frac{p}{n} (1-p) + p^2$$

$$- [p(1-p) + np^2 - p]^2$$

$$= \frac{p}{n} [(n^3 - 6n^2 + 11n - 6)p^3 + (4n^2 - 12n + 8)p^2 + (n-1)p - p] + p^2$$

$$- (p - p^2 + np^2 - p)^2$$

$$= \frac{p^2}{n} [(n^3 - 6n^2 + 11n - 6)p^2 + (4n^2 - 12n + 8)p + n - 2] + p^2$$

$$- \cancel{6n^2 p^4} - \cancel{2np^3} p^4 (n-1)^2$$

$$= \frac{p^2}{n} \left[(n^3 - 6n^2 + 11n - 6)p^2 + (4n^2 - 12n + 8)p + n - 2 + n \right]$$

$$- (n^3 - 2n^2 + n)p^2$$

$$= \frac{p^2}{n} \left[(n^3 - 6n^2 + 11n - 6)p^2 \right]$$

$$= \frac{p^2}{n} \left[(-4n^2 + 10n - 6)p^2 + (4n^2 - 12n + 8)p + 2n - 2 \right]$$

$$= \frac{2p^2}{n} \left[(-2n^2 + 5n - 3)p^2 + (2n^2 - 6n + 4)p + n - 1 \right]$$

$$\bullet \text{MSE } \frac{n\hat{p}^2 - \hat{p}}{n-1} (p^2) = \text{Var} \left(\frac{n\hat{p}^2 - \hat{p}}{n-1} \right) + \left(\frac{n\hat{p}^2 - \hat{p}}{n-1} (p^2) \right)^2$$

And $\frac{n\hat{p}^2 - \hat{p}}{n-1}$ is an unbiased estimator for p^2

$$\therefore \text{MSE } \frac{n\hat{p}^2 - \hat{p}}{n-1} (p^2) = \text{Var} \left(\frac{n\hat{p}^2 - \hat{p}}{n-1} \right)$$

$$= \frac{1}{(n-1)^2} \text{Var} (n\hat{p}^2 - \hat{p})$$

$$= \frac{2p^2}{n(n-1)^2} \left[(-2n^2 + 5n - 3)p^2 + (2n^2 - 6n + 4)p + n - 1 \right]$$