STATS 2107
Statistical Modelling and Inference II

Workshop 10:
Transformations in MLR

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The MLR approach

The data

We will look at the `trees` dataset built into R. This data set provides measurements of the diameter, height and volume of timber in 31 felled black cherry trees

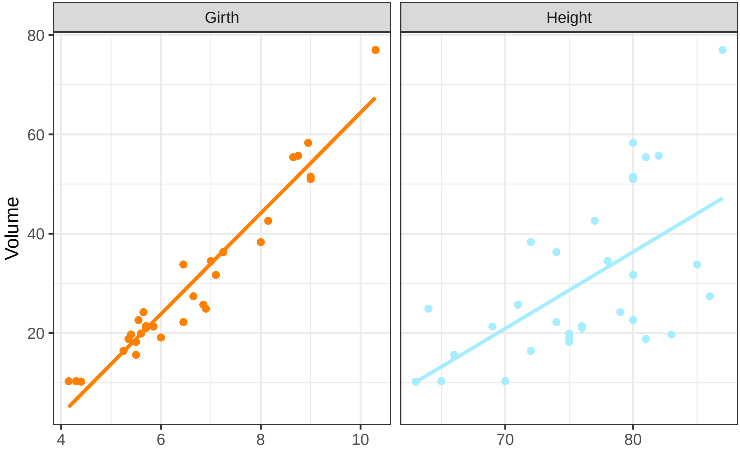
Variable	Description
Girth	Tree diameter in inches
Height	Tree height in feet
Volume	Tree volume of timber in cubic feet

Load the data

```
data("trees")
trees <- trees %>%
  as_tibble() %>%
  mutate(Girth = Girth/2) # Let's look at radius, not diameter
head(trees)
```

```
## # A tibble: 6 x 3
##   Girth Height Volume
##   <dbl>  <dbl>  <dbl>
## 1  4.15     70   10.3
## 2  4.3      65   10.3
## 3  4.4      63   10.2
## 4  5.25     72   16.4
## 5  5.35     81   18.8
## 6  5.4      83   19.7
```

Look at the data



Let's fit a linear model

To predict `Volume`, let's fit the following model

```r
trees_lm1 <- lm(Volume ~ Height + Girth, data = trees)
summary(trees_lm1)
```

```
##
## Call:
## lm(formula = Volume ~ Height + Girth, data = trees)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -6.4065 -2.6493 -0.2876  2.2003  8.4847
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) -57.9877     8.6382  -6.713 2.75e-07 ***
## Height        0.3393     0.1302   2.607   0.0145 *
## Girth         9.4163     0.5285  17.816  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 3.882 on 28 degrees of freedom
## Multiple R-squared:  0.948,  Adjusted R-squared:  0.9442
## F-statistic:   255 on 2 and 28 DF,  p-value: < 2.2e-16
```

Your turn

What to do

1. Does this model meet the assumptions of linear regression?
2. Look at the residuals versus each predictor. Does this tell you anything?

A more informed approach

What is a tree?

Question: What is a good way to estimate the volume of a tree?

The volume of a cylinder

Let's estimate the volume of timber we will get from a tree as:

$$V = 2\pi h r^2$$

where

- V is the volume of the tree
- r is the radius of the tree
- h is the height of the tree

How do we linearise this?

Linearising the volume

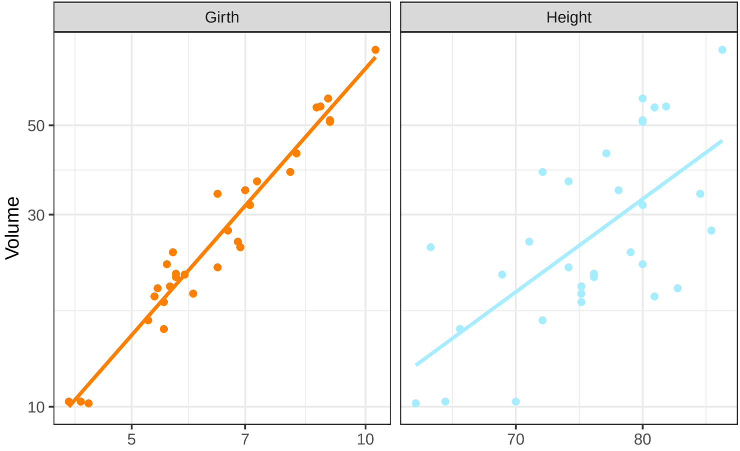
$$\log(V) = \log(2\pi) + \log(h) + 2\log(r)$$

So let's consider the linear regression

$$\log(V) = \beta_0 + \beta_1 \log(h) + \beta_2 \log(r) + \varepsilon$$

where $\varepsilon \sim N(0, \sigma^2)$.

Does the picture look right?



DISCLAIMER: This is actually $\log_{10}$ scale.

Fit the model

```
trees_lm2 <- lm(log(Volume) ~ log(Height) + log(Girth), data = trees)
summary(trees_lm2)
```

```
##
## Call:
## lm(formula = log(Volume) ~ log(Height) + log(Girth), data = trees)
##
## Residuals:
##       Min        1Q    Median        3Q       Max
## -0.168561 -0.048488  0.002431  0.063637  0.129223
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) -5.25735    0.81926  -6.417 6.00e-07 ***
## log(Height)  1.11712    0.20444   5.464 7.81e-06 ***
## log(Girth)   1.98265    0.07501  26.432  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.08139 on 28 degrees of freedom
## Multiple R-squared:  0.9777, Adjusted R-squared:  0.9761
## F-statistic: 613.2 on 2 and 28 DF,  p-value: < 2.2e-16
```

Your turn

What to do

1. Does this model meet the assumptions of linear regression?
2. Look at the residuals versus each predictor. Does this tell you anything?
3. To see if our model is a smart choice, test the hypotheses (at the 5% level):

$$H_0 : \beta_1 = 1 \quad \text{vs} \quad H_a : \beta_1 \neq 1\,,$$

and

$$H_0 : \beta_2 = 2 \quad \text{vs} \quad H_a : \beta_2 \neq 2\,,$$

Let's talk about intervals

Think about prediction

Consider data $Y_1, Y_2, \ldots, Y_n$ and the multiple linear regression model (in matrix form)

$$Y = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}\,.$$

Then $Y_i \sim N(\boldsymbol{x}_i^T\boldsymbol{\beta}, \sigma^2)$ independently for each $i = 1, 2, \ldots, n$.

Think about prediction

Consider a new independent observation Y_0 with predictor $\boldsymbol{x}_0$, then $Y_0 \sim N(\boldsymbol{x}_0^T \boldsymbol{\beta}, \sigma^2)$.

Our best guess of Y_0 is our *predicted value* $\hat{Y}_0 = \boldsymbol{x}_0^T \hat{\boldsymbol{\beta}}$, then $\hat{Y}_0 \sim N(\boldsymbol{x}_0^T \boldsymbol{\beta}, \sigma^2 \boldsymbol{x}_0^T (X^T X)^{-1} \boldsymbol{x}_0)$.

What is a confidence interval?

A $(1-\alpha)\times 100\%$ **confidence interval** for our new data point Y_0 is an interval describing how sure we are about the *mean value of* Y_0.

That is, it is an interval about $\mathrm{E}[Y_0] = \boldsymbol{x}_0^T\boldsymbol{\beta}$.

How to construct the confidence interval

We are trying to get an idea about $\mathrm{E}[Y_0]$, so our best guess at this value is $\hat{Y}_0$.

So how far off is $\hat{Y}_0$ from $\mathrm{E}[Y_0]$? How much to we expect $\hat{Y}_0$ to vary from $\mathrm{E}[Y_0]$?

How to construct the confidence interval

$$\hat{Y}_0 - \mathrm{E}[Y_0] \sim N(0, \sigma^2 \boldsymbol{x}_0^T (X^T X)^{-1} \boldsymbol{x}_0)$$

hence our $(1-\alpha)\times 100\%$ CI is

$$\hat{Y}_0 \pm z_{\alpha/2}\sigma\sqrt{\boldsymbol{x}_0^T (X^T X)^{-1} \boldsymbol{x}_0}$$

if σ^2 is known (replace σ^2 be s_e^2 and $z_{\alpha/2}$ by the appropriate t-critical value if σ^2 unknown.)

Can we back transform this?

Suppose $Y_0 = f(W_0)$ where f is increasing monotonic. When we get a confidence interval for $\mathrm{E}[Y_0]$, can we say anything about $\mathrm{E}[W_0]$?

Can we back transform this?

In general, no. This is because our CI is

$$L = \hat{Y}_0 - z_{\alpha/2}\sigma\sqrt{\mathbf{x}_0^T(X^TX)^{-1}\mathbf{x}_0} < \mathrm{E}[Y_0] < \hat{Y}_0 + z_{\alpha/2}\sigma\sqrt{\mathbf{x}_0^T(X^TX)^{-1}\mathbf{x}_0} = U$$

Applying f^{-1} gives

$$f^{-1}(L) < f^{-1}(\mathrm{E}[Y_0]) < f^{-1}(U)$$

and in general $f^{-1}(\mathrm{E}[Y_0]) \neq \mathrm{E}[f^{-1}(Y_0)] = \mathrm{E}[W_0]$.

What is a prediction interval?

A $(1-\alpha) \times 100\%$ **prediction interval** for our new data point Y_0 is an interval describing how sure we are about the *value of* Y_0, not it's mean!

That is, it is an interval about Y_0 itself.

Recall that Y_0 is random, with $Y_0 \sim N(\boldsymbol{x}_0^T \boldsymbol{\beta}, \sigma^2)$

How to construct the prediction interval

We are trying to get an idea about Y_0, so our best guess at this value is $\hat{Y}_0$.

So how far off is $\hat{Y}_0$ from Y_0? How much to we expect $\hat{Y}_0$ to vary from Y_0?

How to construct the prediction interval

$$\hat{Y}_0 - Y_0 \sim N(0, \sigma^2(1 + \boldsymbol{x}_0^T(X^TX)^{-1}\boldsymbol{x}_0))$$

hence our $(1-\alpha)\times 100\%$ PI is

$$\hat{Y}_0 \pm z_{\alpha/2}\sigma\sqrt{1 + \boldsymbol{x}_0^T(X^TX)^{-1}\boldsymbol{x}_0}$$

if σ^2 is known (replace σ^2 be s_e^2 and $z_{\alpha/2}$ by the appropriate t-critical value if σ^2 unknown.)

Can we back transform this?

Suppose $Y_0 = f(W_0)$ where f is increasing monotonic. When we get a prediction interval for Y_0, can we say anything about W_0?

Can we back transform this?

Yes we can! This is because our PI is

$$L = \hat{Y}_0 - z_{\alpha/2}\sigma\sqrt{1 + \boldsymbol{x}_0^T(X^TX)^{-1}\boldsymbol{x}_0} < Y_0 < \hat{Y}_0 + z_{\alpha/2}\sigma\sqrt{1 + \boldsymbol{x}_0^T(X^TX)^{-1}\boldsymbol{x}_0} = U$$

Applying f^{-1} gives

$$f^{-1}(L) < f^{-1}(Y_0)(= W_0) < f^{-1}(U)$$

Your turn

What to do

1. Using the transformed model from before, obtain a 95% confidence interval for a cherry tree with height 80 feet, and radius of 8 inches.
2. Using the transformed model from before, obtain a 95% prediction interval for a cherry tree with height 80 feet, and radius of 8 inches.
3. What can we say about the volume of the cherry tree from these intervals?