STATS 2107
Statistical Modelling and Inference II
Tutorial 3

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1. Suppose $Y_1, Y_2, \ldots, Y_n$ is a random sample of size n from a gamma-distributed population with parameters $\alpha = 2$ and $\lambda = 1/\beta$, that is, with mean 2β and variance $2\beta^2$.
   (a) Use the method of moment generating functions to show that $X = \frac{2}{\beta}\sum_{i=1}^n Y_i$ is a pivotal quantity and has a χ^2_{4n} distribution. **Recall that if $Y \sim$ Gamma$(\alpha = 2, \lambda = 1/\beta)$, then $M_Y(t) = (1-\beta t)^{-2}$.**
   (b) Use the pivotal quantity X to derive a 95% symmetric confidence interval for β.
   (c) If a sample of size $n = 5$ yields $\bar{y} = 5.39$, use the results from part (b) to give a 95% symmetric confidence interval for β.
2. Consider the independent random variables
$$Y_{ij},\ i = 1, 2;\ j = 1, 2, \ldots, n_i$$
   with $Y_{ij} \sim N(\mu_i, \sigma^2)$. Let
$$S_p^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2}.$$
   (a) Prove that S_p^2 is an unbiased estimator for σ^2.
   (b) Prove that
$$\frac{(n_1+n_2-2)S_p^2}{\sigma^2} \sim \chi^2_{n_1+n_2-2}.$$
   (c) Show that
$$\left(\bar{Y}_1 - \bar{Y}_2 - t_{n_1+n_2-2,\alpha/2} S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}, \bar{Y}_1 - \bar{Y}_2 + t_{n_1+n_2-2,\alpha/2} S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}\right)$$
   is a $100(1-\alpha)\%$ confidence interval for $\mu_1 - \mu_2$ if σ^2 is not known.
3. Suppose we have independent random variables
$$Y_{ij},\ i = 1, 2, 3;\ j = 1, 2, \ldots, n_i$$
   with $Y_{ij} \sim N(\mu_i, \sigma^2)$.We would like to do inference on a linear combination of the mean $\theta = a_1\mu_1 + a_2\mu_2 + a_3\mu_3$. An intuitive estimator for θ is $\hat{\theta} = a_1\bar{Y}_1 + a_2\bar{Y}_2 + a_3\bar{Y}_3$, where $\bar{Y}_i$ is the sample mean of $Y_{i1}, Y_{i2}, \ldots, Y_{in_i}$.
   (a) Find the standard error of the estimator $\hat{\theta}$.
   (b) Find the distribution of the estimator $\hat{\theta}$.
   (c) A pooled estimator for σ^2 is
$$S_p^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2 + (n_3-1)S_3^2}{n_1+n_2+n_3-3},$$
   where S_i^2 is the sample variance of $Y_{i1}, Y_{i2}, \ldots, Y_{in_i}$. State the distribution of
$$W = \frac{(n_1+n_2+n_3-3)S_p^2}{\sigma^2} \quad \text{and} \quad T = \frac{\hat{\theta}-\theta}{S_p\sqrt{\frac{a_1^2}{n_1}+\frac{a_2^2}{n_2}+\frac{a_3^2}{n_3}}}.$$

(d) Using the results from part (c), give a $100(1-\alpha)\%$ confidence interval for θ.

(e) Using the results from part (c), develop a hypothesis test for testing $H_0 : \theta = \theta_0$ vs $H_a : \theta \neq \theta_0$.