Two-way ANOVA (no replications) and MLR

- One-way ANOVA: one factor
- Two-way ANOVA: two factors
- E.g. compare the effectiveness of different brands of detergents at removing marks on different types of fabric
- Two cases to consider:
  - The two factors are independent (*additive model*)
  - The two factors are not independent (*interaction model*)

Layout of two-way ANOVA

column factor

row factor	1	2	...	J
1	y_{11}	y_{12}	...	y_{1J}
2	y_{21}	y_{22}	...	y_{2J}
$\vdots$	$\vdots$	$\vdots$		$\vdots$
I	y_{I1}	y_{I2}	...	y_{IJ}

one observation per cell

total is IJ observations.

$$Y_{ij} \sim N(\mu_{ij}, \sigma^2)$$

for $i = 1, 2, \ldots, I$ and $j = 1, 2, \ldots, J$

e.g. Y = yield of crop

row factor: level of fertiliser used

column factor: locations

e.g. Y = fuel efficiency of cars

row factor: type of car

column factor: driving conditions

Write $\underset{\substack{\text{overall}\\\text{mean}}}{\mu_{ij} = \mu} + \underset{\substack{\text{effect of}\\\text{row } i \text{ on } \mu_{ij}}}{\alpha_i} + \underset{\substack{\text{effect of}\\\text{column } j \text{ on } \mu_{ij}}}{\beta_j}$

Assumptions:

1. Observations from the same row have means that differ only by the difference in column effects (β_j)

2. Observations from the same column have means that differ only by the difference in row effect (α_i)

3. Observations from different rows and columns have means that differ by the difference in row and column effects ($\alpha_i + \beta_j$)

Two-way layout (no replications)

Consider the two-way layout with one observation per cell

$$Y_{ij} = \underbrace{\boxed{\mu + \alpha_i + \beta_j}}_{\mu_{ij}} + \epsilon_{ij}$$

with

$$\epsilon_{ij} \sim i.i.d.N(0,\sigma^2) \text{ for } i = 1,2,\ldots,I; j = 1,2,\ldots,J$$

The parameters of this model are μ, α_i, and β_j.

They are not uniquely defined.

$$\cancel{\mu} = \frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J}\mu_{ij} = \frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J}(\mu+\alpha_i+\beta_j) = \cancel{\mu} + \frac{1}{I}\sum_{i=1}^{I}\alpha_i + \frac{1}{J}\sum_{j=1}^{J}\beta_j$$

$$0 = \frac{1}{I}\sum_{i=1}^{I}\alpha_i + \frac{1}{J}\sum_{j=1}^{J}\beta_j$$

$$\frac{1}{I}\sum_{i=1}^{I}\alpha_i = -\frac{1}{J}\sum_{j=1}^{J}\beta_j$$

Constraints

- Zero Sum Constraints:

$$0 = \boxed{\sum_{i=1}^{I} \alpha_i} = \boxed{\sum_{j=1}^{J} \beta_j = 0.}$$

- Reference Category Constraints: used in R

$$\underline{\alpha_1 = \beta_1 = 0.} \qquad \text{(same as } \mu_{11} = \mu\text{)}$$

Hypotheses

Within the present context, there are two hypotheses of interest:

no difference in row means

$$H_1: \alpha_1 = \alpha_2 = \cdots = \alpha_I = 0$$

and

no difference in column means

$$H_2: \beta_1 = \beta_2 = \cdots = \beta_J = 0$$

Zero-sum

$$\alpha_1 + \alpha_2 + \dots + \alpha_I = 0 \quad \Rightarrow \quad \alpha_I = -\alpha_1 - \alpha_2 - \dots - \alpha_{I-1}$$

$$\beta_1 + \beta_2 + \dots + \beta_J = 0 \quad \Rightarrow \quad \beta_J = -\beta_1 - \beta_2 - \dots - \beta_{J-1}$$

Our parameters are $\beta^T = [\mu \ \alpha_1 \ \alpha_2 \ \dots \ \alpha_{I-1} \ \beta_1 \ \beta_2 \ \dots \ \beta_{J-1}]$

$$Y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}$$

I^{th} row: $Y_{Ij} = \mu + \alpha_I + \beta_j + \varepsilon_{Ij} = \mu + (-\alpha_1 - \alpha_2 - \dots - \alpha_{I-1}) + \beta_j + \varepsilon_{Ij}$

J^{th} column: $Y_{iJ} = \mu + \alpha_i + \beta_J + \varepsilon_{iJ} = \mu + \alpha_i + (-\beta_1 - \beta_2 - \dots - \beta_{J-1}) + \varepsilon_{iJ}$

MLR setup (zero sum)

The additive model may also be specified as a MLR as follows:

$$
\underset{(IJ\times 1)}{\boldsymbol{y}} = \begin{bmatrix} y_{11} \\ y_{12} \\ \vdots \\ y_{1J} \\ y_{21} \\ y_{22} \\ \vdots \\ y_{2J} \\ \vdots \\ y_{I1} \\ y_{I2} \\ \vdots \\ y_{IJ} \end{bmatrix}, \quad
\boldsymbol{X} = \begin{array}{c}
\begin{array}{ccccccccc} \mu & \alpha_1 & \alpha_2 & & \alpha_{I-1} & \beta_1 & \beta_2 & & \beta_{J-1} \end{array} \\
\left[\begin{array}{c|cccc|cccc}
1 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
1 & 1 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 \\
\hline
1 & 0 & 1 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 1 & \cdots & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 1 & \cdots & 0 & -1 & -1 & \cdots & -1 \\
\hline
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & & \vdots \\
\hline
1 & -1 & -1 & \cdots & -1 & 1 & 0 & \cdots & 0 \\
1 & -1 & -1 & \cdots & -1 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1
\end{array}\right]
\end{array}
$$

Same pattern as above

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Note that under the zero-sum constraints

$\alpha_I = -\alpha_1 - \alpha_2 - \cdots - \alpha_{I-1}$ and $\beta_J = -\beta_1 - \beta_2 - \cdots - \beta_{J-1}$

MLR setup (reference)

The formulation for the reference category constraints can specified as a MLR as follows:

$$
\boldsymbol{y} = \begin{bmatrix} y_{11} \\ y_{12} \\ \vdots \\ y_{1J} \\ y_{21} \\ y_{22} \\ \vdots \\ y_{2J} \\ \vdots \\ y_{I1} \\ y_{I2} \\ \vdots \\ y_{IJ} \end{bmatrix}, \boldsymbol{X} = \left[\begin{array}{c|ccc|ccc} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 1 \\ \hline 1 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 & 0 & \cdots & 1 \\ \hline \vdots & \vdots & & & \vdots & & \vdots \\ \hline 1 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & & & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 1 & 1 & \cdots & 1 \end{array}\right]
$$

Set $\alpha_1 = \beta_1 = 0$

first row: $Y_{1j} = \mu + \beta_j + \varepsilon_{1j}$

first column: $Y_{i1} = \mu + \alpha_i + \varepsilon_{i1}$

ANOVA table

Source	SS	df	MSE	F
Row Effects	$J\sum_i(\bar{y}_{i\bullet} - \bar{y}_{\bullet\bullet})^2$	I-1	MSA	MSA/MSE
Col Effects	$I\sum_j(\bar{y}_{\bullet j} - \bar{y}_{\bullet\bullet})^2$	J-1	MSB	MSB/MSE
Residual	$\sum_{ij}(y_{ij} - \bar{y}_{i\bullet} - \bar{y}_{\bullet j} + \bar{y}_{\bullet\bullet})^2$	(I-1)(J-1)	MSE	
Total	$\sum_{ij}(y_{ij} - \bar{y}_{\bullet\bullet})^2$	IJ-1	MST	

Although the zero-sum and reference category constraints will give different estimates for α_i and β_j, it can be proved that in both cases $\hat{\mu}_{ij} = \bar{y}_{i.} + \bar{y}_{.j} - \bar{y}_{..}$