Power

Power function

Suppose that H_0 is false in the Example 1.10, i.e. $\underline{\mu \neq \mu_0}$.

Let μ_1 = true value of μ $\quad (\mu_1 \neq \mu_0)$

Power $= 1-\beta = P(\text{reject } H_0 \mid H_0 \text{ false}) = P(|Z^*| \geq Z_{\frac{\alpha}{2}} \mid \mu \neq \mu_0)$

$$Z^* = \frac{\bar{Y}-\mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{\bar{Y}-\mu_1+\mu_1-\mu_0}{\frac{\sigma}{\sqrt{n}}} = \underbrace{\boxed{\frac{\bar{Y}-\mu_1}{\frac{\sigma}{\sqrt{n}}}}}_{\sim N(0,1) \text{ by Lemma 3}} + \underbrace{\boxed{\frac{\mu_1-\mu_0}{\frac{\sigma}{\sqrt{n}}}}}_{\text{constant}}$$

$$\begin{aligned} E[Z^*] &= E\left[\frac{\bar{Y}-\mu_1}{\frac{\sigma}{\sqrt{n}}} + \frac{\mu_1-\mu_0}{\frac{\sigma}{\sqrt{n}}}\right] \\ &= E\left[\frac{\bar{Y}-\mu_1}{\frac{\sigma}{\sqrt{n}}}\right] + \frac{\mu_1-\mu_0}{\frac{\sigma}{\sqrt{n}}} \\ &= 0 + \frac{\mu_1-\mu_0}{\frac{\sigma}{\sqrt{n}}} \\ &= \frac{\mu_1-\mu_0}{\frac{\sigma}{\sqrt{n}}} \end{aligned}$$

$$\therefore Z^* \sim N\left(\frac{\mu_1-\mu_0}{\frac{\sigma}{\sqrt{n}}}, 1\right)$$

$$\text{var}(Z^*) = \text{var}\left(\frac{\bar{Y}-\mu_1}{\frac{\sigma}{\sqrt{n}}}\right) = 1$$

Power function (cont.)

$$\text{Power} = P(|Z^*| \geq z_{\frac{\alpha}{2}} \mid \mu \neq \mu_0)$$

$$= P(Z^* \leq -z_{\frac{\alpha}{2}} \mid \mu \neq \mu_0) + P(Z^* \geq z_{\frac{\alpha}{2}} \mid \mu \neq \mu_0)$$

$$= \Phi\left(-z_{\frac{\alpha}{2}} ; \frac{\mu-\mu_0}{\frac{\sigma}{\sqrt{n}}}, 1\right) + 1 - P(Z^* \leq z_{\frac{\alpha}{2}} \mid \mu \neq \mu_0)$$

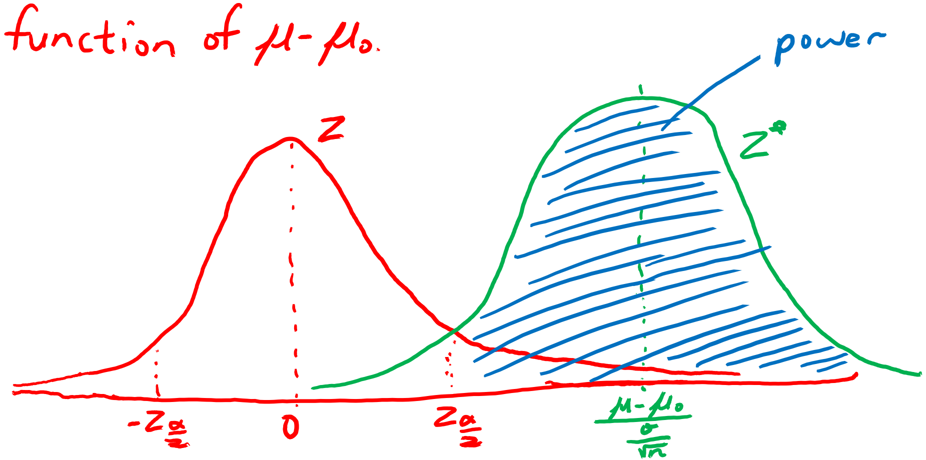
$$= \Phi\left(-z_{\frac{\alpha}{2}} ; \frac{\mu-\mu_0}{\frac{\sigma}{\sqrt{n}}}, 1\right) + 1 - \Phi\left(z_{\frac{\alpha}{2}} ; \frac{\mu-\mu_0}{\frac{\sigma}{\sqrt{n}}}, 1\right)$$

($\Phi(x; \mu, \sigma^2)$ is the traditional notation for the cdf of $N(\mu, \sigma^2)$.

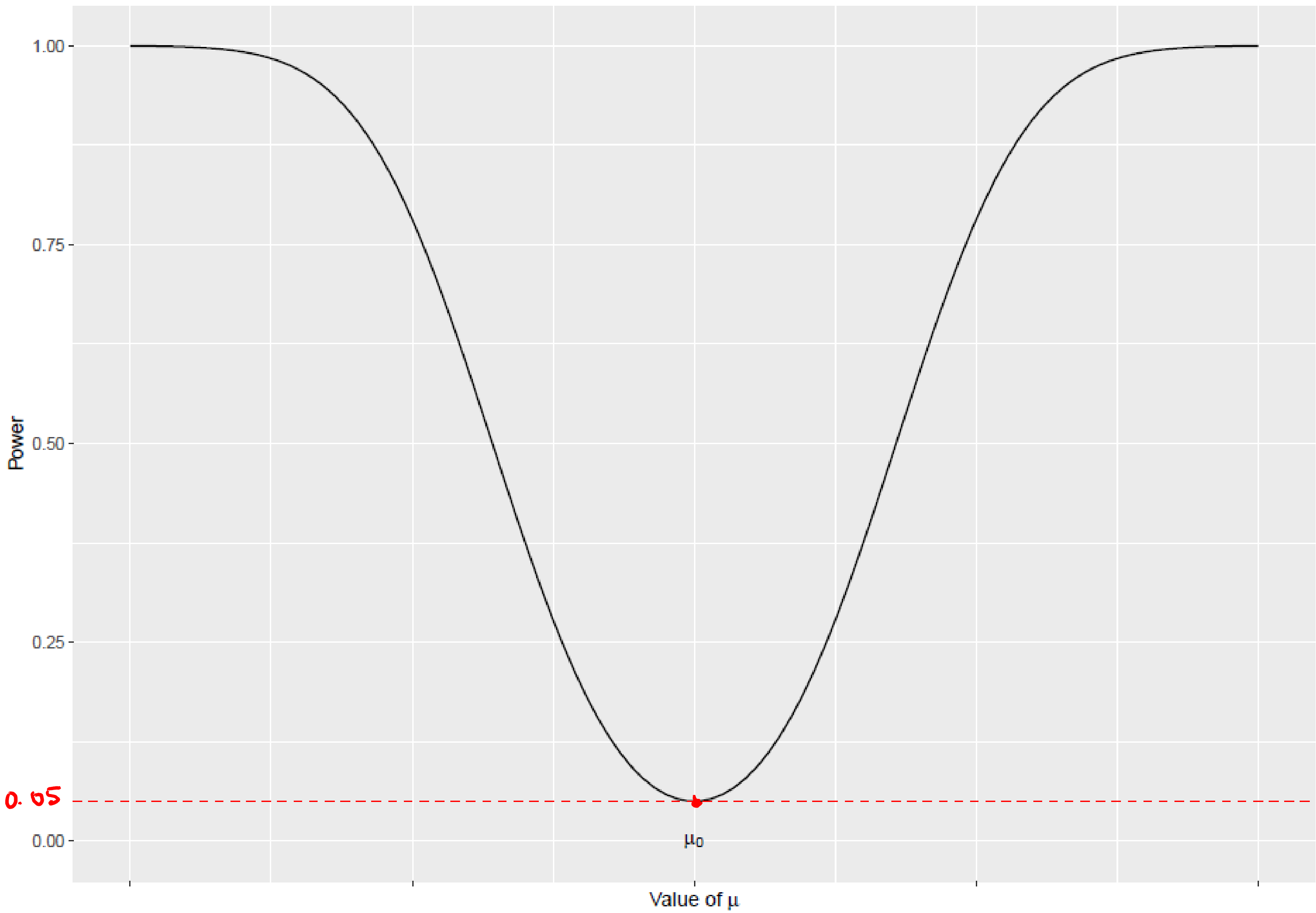
That is, $\Phi(x; \mu, \sigma^2) = P(X \leq x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(t-\mu)^2} dt$,

where $X \sim N(\mu, \sigma^2)$.)

In this case, the power is a function of $\mu - \mu_0$.



Power function (cont.)



One-sided tests

Consider $y_1, y_2, \dots, y_n$ are i.i.d. $N(\mu, \sigma^2)$ observations with σ^2 known.

We can test the one-sided hypothesis

$$H_0: \underline{\mu \leq \mu_0}.$$

$H_a: \mu > \mu_0$

The test statistic is

$$Z = \frac{\bar{Y} - \mu_0}{\sigma/\sqrt{n}}$$

The rule is

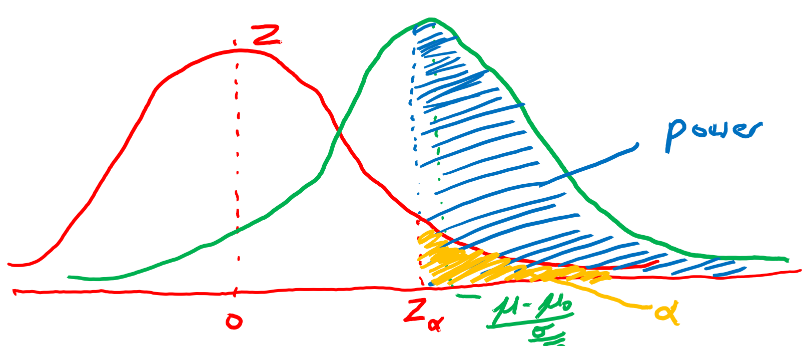
Reject H_0 if $\underline{z \geq z_\alpha}$.

One-sided power function

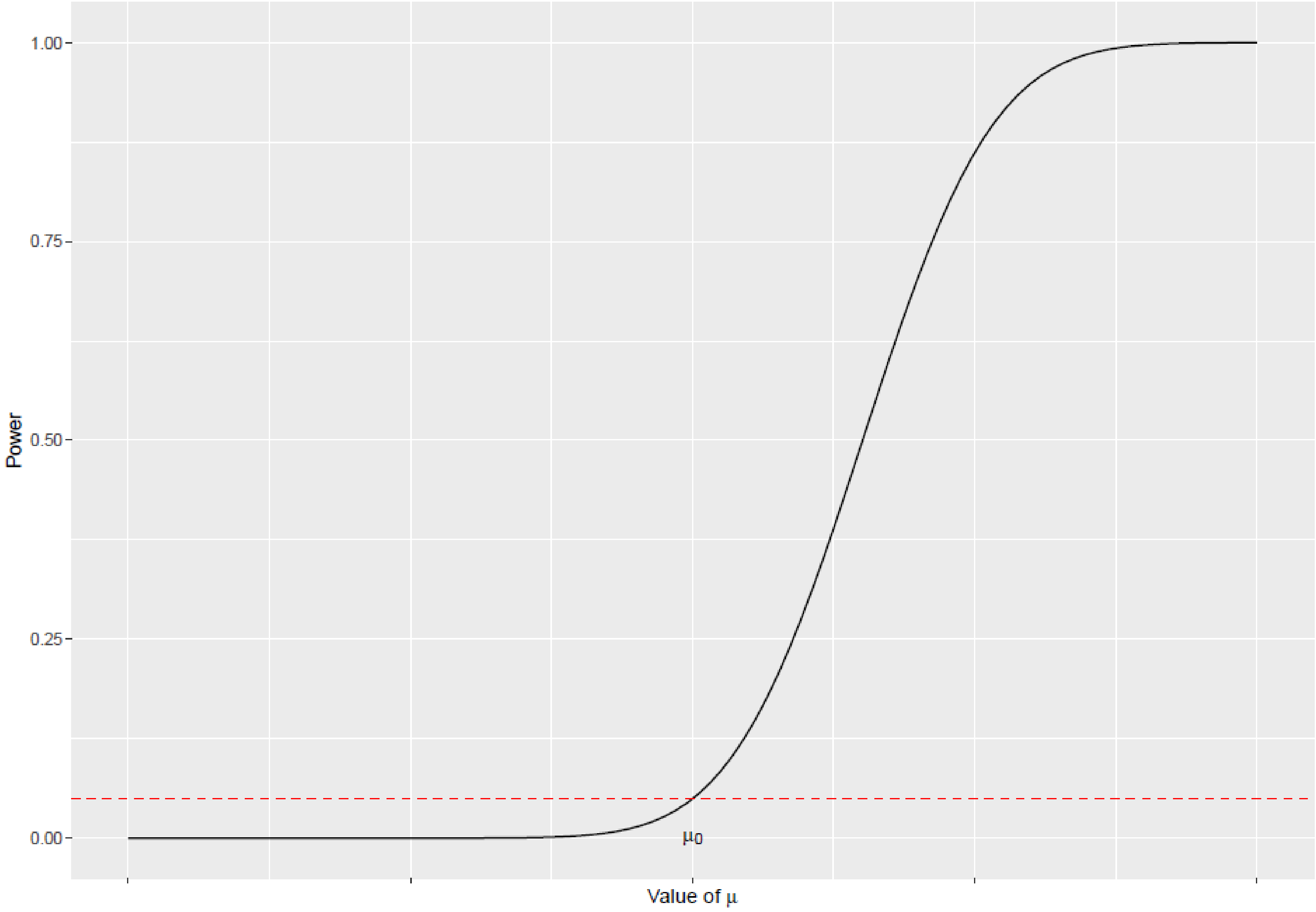
Following the same argument as the two-sided test, we have:

$$\text{If } H_0 \text{ is false (i.e. } \mu > \mu_0\text{), then } Z \sim N\left(\frac{\overset{>0}{\boxed{\mu-\mu_0}}}{\sigma/\sqrt{n}}, 1\right).$$

$$\begin{aligned}
\text{Power} &= P(\text{reject } H_0 \mid H_0 \text{ false}) \\
&= P(Z^* \geq z_\alpha \mid \mu \neq \mu_0) \\
&= 1 - P(Z^* < z_\alpha \mid \mu \neq \mu_0) \\
&= 1 - \Phi\left(z_\alpha ; \frac{\mu-\mu_0}{\frac{\sigma}{\sqrt{n}}}, 1\right)
\end{aligned}$$



One-sided power function



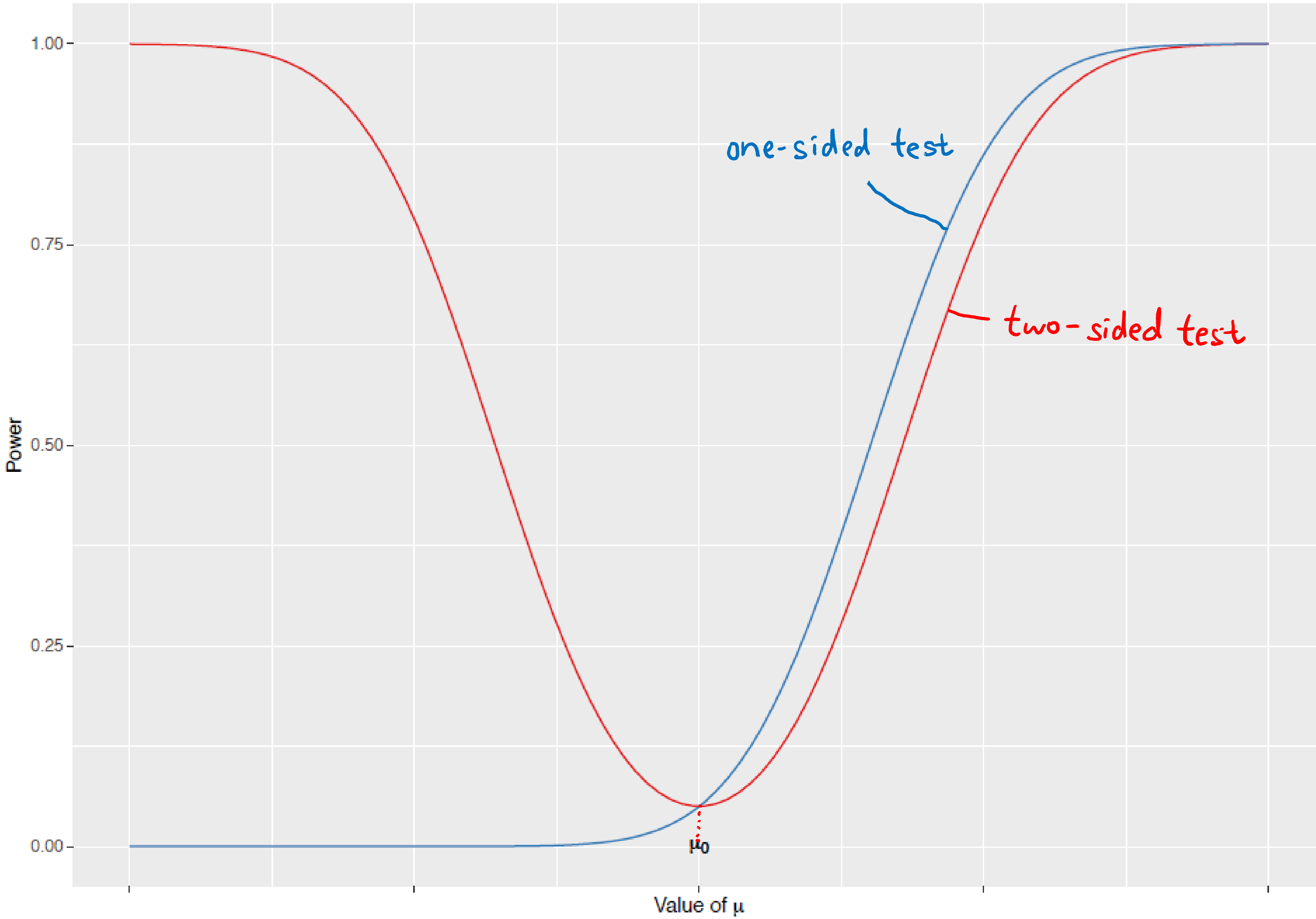
One-sided vs two-sided

For one-sided test, the type I error probability is not a single number, i.e.

$$P(\text{reject } H_0 | H_0 \text{ true}) \le \alpha$$

$$P(Z^* \geq Z_\alpha \mid H_0 \text{ true}) = \begin{cases} \alpha & \text{if } \mu = \mu_0 \\ < \alpha & \text{if } \mu \neq \mu_0 \end{cases}$$

One-sided vs two-sided



One-sided vs two-sided

One-sided test are often subject to an abuse:

Suppose we begin with a problem where a two-sided test is appropriate, but then observed a test statistic such that

$$z_\alpha \leq z_{obs} \leq z_{\alpha/2}$$

One-sided vs two-sided simulation

If we use a one-sided test when in fact a two-sided test should be used, the type I error probability will double.

Let's do a simulation to see this:

1. Generate 100,000 replications of $n = 100$ observations from $N(5, 4)$.
2. Perform one-sided and two-sided tests with $\mu_0 = 5$.
3. Compare the proportion of times where we commit a type I error (i.e. rejecting H_0 in this case).

$Y_i \sim$ iid $N(5,4)$

$H_0: \mu = \mu_0 = 5$

One-sided vs two-sided simulation

Function to perform the Z-test

```
perform_z_test <- function(sample, null, cheat = FALSE, known_var){
# Get sigma
sigma <- sqrt(known_var)
# Get number of obs
n <- length(sample)
# Calculate sample mean
sample_mean <- mean(sample)
# Calculate test statistic
Z <- (sample_mean - null) / (sigma / sqrt(n))
# Decide if to reject or retain
if(abs(Z) > qnorm(0.025, lower.tail = FALSE)){
      return("reject")
}
if(cheat){ #(incorrectly)using the one-sided test
      if(abs(Z) > qnorm(0.05, lower.tail = FALSE)){
             return("reject")
      }
}
return("fail to reject")
}
```

One-sided vs two-sided simulation

Generate the samples

```
library(tidyverse)
sim_data <- 100000 %>%
rerun(rnorm(100, 5, 2))
```

Run the proper test

```
sim_data %>%
      map_chr(perform_z_test, null = 5, known_var = 4) %>%
      table() %>%
      prop.table()
```

Run the wrong test

```
sim_data %>%
      map_chr(perform_z_test, null = 5, known_var = 4, cheat =
      TRUE) %>%
      table() %>%
      prop.table()
```