Q2 ⓐ We have

. $X^* = XA$

$\therefore \det(X^*) = \det(XA)$

$\therefore \det(X^*) = \det(X)\det(A) \quad (1)$

. Assuming the columns of X^* are not linearly independent, or in another words, the columns of X^* are linearly dependent

$\therefore \det(X^*) = 0 \quad (2)$

. From (1) and (2), we have

$\therefore \det(X)\det(A) = 0$

. Since A is an invertible $p \times p$ matrix, $\det(A)$ is nonzero ($\det(A) \neq 0$)

$\therefore \det(X) = 0$

$\therefore$ The ~~columns~~ columns of X are linearly dependent, which contradict with the ~~information~~ provided information

$\therefore$ The columns of X^* are linearly independent (prove by contradiction)

ⓑ We have

$$
\begin{aligned}
. \; X^*(X^{*T}X^*)^{-1}X^{*T} &= XA((XA)^T(XA))^{-1}(XA)^T && (X^* = XA) \\
&= XA(A^TX^TXA)^{-1}A^TX^T && ((XA)^T = A^TX^T) \\
&= XA\,A^{-1}X^{-1}(X^T)^{-1}(A^T)^{-1}A^TX^T && ((XA)^{-1} = A^{-1}X^{-1}) \\
&= X\,I\,X^{-1}(X^T)^{-1}\,I\,X^T && (AA^{-1} = I) \\
&= X\,X^{-1}(X^T)^{-1}X^T \\
&= X(X^TX)^{-1}X^T
\end{aligned}
$$

$\therefore X^*(X^{*T}X^*)^{-1}X^{*T} = X(X^TX)^{-1}X^T$

ⓒ We have

. $\hat{\eta}^* = X^*\hat{\beta}^*$ and $\hat{\eta} = X\hat{\beta}$

. $M: Y = X\beta + \varepsilon$

$\therefore \hat{\beta} = (X^TX)^{-1}X^TY$

. $M^*: Y = X^*\beta^* + \varepsilon$

$\therefore \hat{\beta}^* = (X^{*T}X^*)^{-1}X^{*T}Y$

- $\hat{\eta}^* = X^*\hat{\beta}^*$

$$= X^* (X^{*T}X^*)^{-1} X^{*T} Y \qquad (\beta^* = (X^{*T}X^*)^{-1} X^{*T} Y)$$

$$= X(X^TX)^{-1}X^TY \qquad (\text{from part (b)})$$

$$= X\hat{\beta} \qquad (\hat{\beta} = (X^TX)^{-1}X^TY)$$

$$= \hat{\eta}$$

$\therefore \hat{\eta}^* = \hat{\eta}$

Q1 (a) We have

- $Y = \dfrac{\alpha x}{\delta + x}$

$$\therefore \frac{1}{Y} = \frac{\delta + x}{\alpha x}$$

$$= \frac{1}{\alpha} + \frac{\delta}{\alpha} \frac{1}{x}$$

- Taking $Y^* = \dfrac{1}{Y}$, $x^* = \dfrac{1}{x}$, $\beta_0 = \dfrac{1}{\alpha}$ and $\beta_1 = \dfrac{\delta}{\alpha}$

$$\therefore Y^* = \beta_0 + \beta_1 x^*$$

- ~~Linearis~~ Linearised model: $Y^* = \beta_0 + \beta_1 x^*$

with $Y^* = \dfrac{1}{Y}$, $x^* = \dfrac{1}{x}$, $\beta_0 = \dfrac{1}{\alpha}$ and $\beta_1 = \dfrac{\delta}{\alpha}$

(b) We have $Q(\beta_0, \beta_1) = \sum_{i=1}^{n} (y_i^* - (\beta_0 + \beta_1 x_i^*))^2$

or $Q(\alpha, \delta) = \sum_{i=1}^{n} \left(\dfrac{1}{y_i} - \left(\dfrac{1}{\alpha} + \dfrac{\delta}{\alpha}\dfrac{1}{x}\right)\right)^2$

- $\beta_0 = \dfrac{1}{\alpha}$

$$\therefore \hat{\beta}_0 = \frac{1}{\hat{\alpha}}$$

$$\therefore \hat{\alpha} = \frac{1}{\hat{\beta}_0}$$

- $\beta_1 = \dfrac{\delta}{\alpha}$

$$\therefore \hat{\beta}_1 = \frac{\hat{\delta}}{\hat{\alpha}}$$

$$\therefore \hat{\delta} = \hat{\beta}_1 \hat{\alpha}$$

$$\therefore \hat{\delta} = \frac{\hat{\beta}_1}{\hat{\beta}_0} \qquad \left(\hat{\alpha} = \frac{1}{\hat{\beta}_0}\right)$$

- Given the least-squares estimate of the parameters of the linearised model $(\hat{\beta}_0, \hat{\beta}_1)$, we can construct: $\hat{\alpha} = \dfrac{1}{\hat{\beta}_0}$ and $\hat{\delta} = \dfrac{\hat{\beta}_1}{\hat{\beta}_0}$

(c) We have:

- The ~~satured~~ saturated growth equation: $Y = \dfrac{\alpha x}{\delta + x}$

$\therefore$ Least-square estimate: $Q_1(\alpha, \delta) = \sum_{i=1}^{n} \left(y_i - \dfrac{\alpha x_i}{\delta + x_i}\right)^2$ (1)

• The linearised model: $Y^* = \beta_0 + \beta_1 x^*$ or $\frac{1}{Y} = \frac{1}{\alpha} + \frac{\delta}{\alpha}\frac{1}{x}$ (from (b))

∴ Least-square estimation: $Q_2(\alpha, \beta) = \sum_{i=1}^{n} \left(\frac{1}{y_i} - \left(\frac{1}{\alpha} + \frac{\delta}{\alpha}\frac{1}{x_i}\right)\right)^2$ (2)

~~• According to Lemma 5.~~

~~(c) For (2) is uniquely minimised~~

• In order to linearise the saturated model, we transform it by taking the inverse of the saturated model. Hence, we have 2 different functions: (1) and (2)

• For each ~~of~~ all y_i and x_i ~~use~~ in (1), we take ~~the~~ the inverse of them in (2). In another words, function (1) is different from function (2)

∴ As a result, ~~the approach in~~ fitting the linearised model ~~directly~~ using the method of least squares is different to fitting the saturated model directly.

(d) (i) We have

. From the R output: $df = 13$.

• $df = n - p$ (n is the number of observations)

$= n - (r + 1)$

• r is the number of predictor variables, and in this case, $r = 1$

∴ $df = n - 1 - 1 = n - 2$

and $df = 13$

∴ $13 = n - 2$

∴ $n = 15$

∴ There are 15 observations in the data.

(ii) From the linearised model, we know that we take the inverse of ~~the Length and Age in the transfor...~~ the Length (Y) and Age (x). Hence, we have the following table.

Length.trans	Age.trans
1/489	1/4
1/476	1/3
1/513	1/7
1/382	1

∴ ~~the respon~~ In the linearised model:

. The response vector $y = \begin{bmatrix} 1/489 \\ 1/476 \\ 1/513 \\ 1/382 \end{bmatrix}$

. The design matrix $X = \begin{bmatrix} 1 & 1/4 \\ 1 & 1/3 \\ 1 & 1/7 \\ 1 & 1 \end{bmatrix}$

(iii) From the R output, we have

. $\hat{\beta}_0 = 6.075 \times 10^{-3}$

. $\hat{\beta}_1 = t\text{-value}(\hat{\beta}_1) \times SE(\hat{\beta}_1) = 8.423 \times 2.897 \times 10^{-4} \approx 2.440 \times 10^{-3}$

. $\hat{\alpha} = \dfrac{1}{\hat{\beta}_0} = \dfrac{1}{6.075 \times 10^{-3}} \approx 164.609$

. $\hat{\delta} = \dfrac{\hat{\beta}_1}{\hat{\beta}_0} = \dfrac{2.440 \times 10^{-3}}{6.075 \times 10^{-3}} \approx 0.402$

∴ The best fitting growth curve: $\hat{Y} = \dfrac{\hat{\alpha} x}{\hat{\delta} + x} = \dfrac{164.609 x}{0.402 + x}$

(iv) We have 90% confidence interval for the intercept term ($\hat{\beta}_0$) ~~of the~~ in the linearised model

. $\alpha = 100\% - 90\% = 10\% = 0.1$

∴ $z_{\alpha/2} = \text{qnorm}\left(\frac{0.1}{2}, \text{lower.tail} = \text{FALSE}\right) \approx 1.6449$

. The 90% confidence interval for β_0:

$(\hat{\beta}_0 - z_{\alpha/2} \times SE(\hat{\beta}_0), \ \hat{\beta}_0 + z_{\alpha/2} \times SE(\hat{\beta}_0))$

$= (6.075 \times 10^{-3} - 1.6449 \times 9.825 \times 10^{-5}, \ 6.075 \times 10^{-3} - 1.6449 \times 9.825 \times 10^{-5})$

$\approx (5.913 \times 10^{-3}, \ 6.237 \times 10^{-3})$

∴ The 90% CI for β_0: $(5.913 \times 10^{-3}, \ 6.237 \times 10^{-3})$

(v) From the R ~~op~~ output, we have

- $fit = 0.006481724$
- $lwr = 0.006023172$

. Let Y_0 is the length of a seal age 6 years

$\therefore upr = fit + (fit - lwr)$

$= 0.006481724 + (0.006481724 - 0.006023172)$

$= 0.006940276$

. The linearised model: $Y^* = \beta_0 + \beta_1 x^*$

$\therefore$ The 90 % prediction interval based on the linearised model.

$L^* < Y_0^* < U^*$

$= lwr < fit < upr$.

$\therefore Y_0^* = fit$; $L^* = lwr$ and $U^* = upr$

. By taking the inverse of the above, we will get the corresponding 90 % prediction interval for the length of a seal aged 6 years (the saturated model)

$L^* < Y_0^* < U^*$

$\therefore \frac{1}{L^*} > \frac{1}{Y_0^*} > \frac{1}{U^*}$

$\therefore U > Y_0 > L$

$\therefore Y_0 = \frac{1}{Y_0^*} = \frac{1}{fit} = \frac{1}{0.006481724} \approx 154.280$

$U = \frac{1}{L^*} = \frac{1}{lwr} = \frac{1}{0.006023172} \approx 166.025$

$L = \frac{1}{U^*} = \frac{1}{upr} = \frac{1}{0.006940276} \approx 144.087$

$\therefore$ The 90% P I for the length of a seal aged 6 years:

$(144.087, 166.025)$