Errors and Power

Type I and type II errors

	Fail to reject H_0	Reject H_0
H_0 true	Correct conclusion	Type I error
H_0 false	Type II error	Correct conclusion

Remarks:

1. We want both type I and type II error probabilities to be small. But they are in conflict with each other: reducing type I error probability will causes type II error to increases, and vice versa.

2. The usual approach to resolve this conflict is to hold type I error fixed at a small value, then choose a test with type II error probability as small as possible.

Type I and type II errors

	Retain H_0	Reject H_0
H_0 true	Correct conclusion	Type I error (α)
H_0 false	Type II error (β)	Correct conclusion $(1-\beta)$

Significance level:

$$\alpha = P(\text{reject } H_0 | H_0 \text{ true}).$$

level α test

(a test with significance level α)

Power:

$$1 - \beta = P(\text{reject } H_0 \,| H_0 \text{false}) = 1 - P(\text{Type II error})$$

Example 1.11

A toy store chain claims that at least 80% boys under 8 years old prefer Lego over other types of toys. After observing the buying patterns of many boys under 8 years old, we feel that this claim is inflated. In an attempt to disprove this claim, we observed the buying pattern of 20 randomly selected boys under 8 years old. Let x be the number of boys who brought Lego, We wish to test the hypothesis $H_0: p = 0.8$ against $H_a: p < 0.8$. Suppose we decide to reject H_0 if $\{X \leq 12\}$.

(a) Find α.

(b) Find β for $p = 0.6$.

(c) Find β for $p = 0.4$.

(d) Find the critical region of the form $\{X \leq c\}$ such that (i) $\alpha = 0.01$ and (ii) $\alpha = 0.05$.

(e) For the alternative hypothesis $H_a: p = 0.6$, find β for the values of α in part (d).

Example 1.11 Solution

Let X = number of boys who prefer Lego than other toys

$X \sim Bin(n=20, p)$

(a) $\alpha = P(\text{Type I error})$

$= P(\text{reject } H_0 \mid H_0 \text{ true})$

$= P(X \leq 12 \mid p = 0.8)$

$= \sum_{x=0}^{12} \binom{20}{x} 0.8^{x} \, 0.2^{20-x}$

≈ 0.0321

pbinom(12, 20, 0.8)

(b) $\beta = P(\text{Type II error})$

$= P(\text{fail to reject } H_0 \mid H_0 \text{ false})$

$= P(X > 12 \mid p = 0.6)$

$= 1 - P(X \leq 12 \mid p = 0.6)$

$= 1 - \sum_{x=0}^{12} \binom{20}{x} 0.6^{x} \, 0.4^{20-x}$

≈ 0.416

1 - pbinom(12, 20, 0.6)

Example 1.11 Solution

(c) $\beta = P(\text{Type II error})$

$= P(\text{failing to reject } H_0 \mid H_0 \text{ false})$

$= P(X > 12 \mid p = 0.4)$

$= 1 - P(X \leq 12 \mid p = 0.4)$

≈ 0.021

p	0.4	0.6
β	0.021	0.416

$p_0 = 0.8$

As p moves further away from $p_0 = 0.8$, β decreases, and power $(1-\beta)$ increases

(d) $\alpha = P(\text{Type I error})$

$= P(\text{reject } H_0 \mid H_0 \text{ true})$

$= P(X \leq c \mid p = 0.8)$

(i) $\alpha = 0.01 = P(X \leq c \mid p = 0.8)$

qbinom(0.01, 20, 0.8) = 12

$P(X \leq 12 \mid p = 0.8) = 0.0321$

$\boxed{P(X \leq 11 \mid p = 0.8) = 0.00998}$

Choose $c = 11$.

$\therefore$ critical region is $\{X \leq 11\}$.

(ii) $\alpha = 0.05 = P(X \leq c \mid p = 0.8)$

$P(X \leq 13 \mid p = 0.8) = \underline{0.0867}$ exceeds $\alpha = 0.05$

Choose $c = 12$

$\therefore$ critical region is $\{X \leq 12\}$.

Example 1.11 Solution

(e) Given $p = 0.6$. Find β for $\alpha = 0.01$ and $\alpha = 0.05$.

(i) $\alpha = 0.01$, critical region $\{X \leq 11\}$

$$\beta = P(\text{fail to reject } H_0 \mid H_0 \text{ false})$$
$$= P(X > 11 \mid p = 0.6)$$
$$= 1 - P(X \leq 11 \mid p = 0.6)$$
$$\approx 0.596$$

(ii) $\alpha = 0.05$, critical region $\{X \leq 12\}$

$$\beta \approx 0.416$$

α	0.01	0.05
β	0.596	0.416
$1-\beta$	0.414	0.584

As α increases, β decreases, power increases

Example 1.12

Show that in Example 1.9, where we have normal observations with σ^2 known, the test has significance level α.

Check that $P(|Z| \geq Z_{\frac{\alpha}{2}} \mid \mu = \mu_0) = \alpha$.

Test statistic is $Z = \dfrac{\bar{Y} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$.

By Lemma, if $\mu = \mu_0$, then $Z \sim N(0,1)$.

Therefore, under $H_0: \mu = \mu_0$, we know that $Z \sim N(0,1)$.

$$
\begin{aligned}
& P(|Z| \geq Z_{\frac{\alpha}{2}} \mid \mu = \mu_0) \\
&= P(|Z| \geq Z_{\frac{\alpha}{2}}) \quad \text{where } Z \sim N(0,1) \\
&= P(Z \leq -Z_{\frac{\alpha}{2}}) + P(Z \geq Z_{\frac{\alpha}{2}}) \\
&= \frac{\alpha}{2} + \frac{\alpha}{2} \\
&= \alpha
\end{aligned}
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