One-sample t-test

Setup

Suppose $Y_1, Y_2, \ldots, Y_n$ are i.i.d. $N(\mu, \sigma^2)$ random variables with σ^2 unknown.

- The BLUE for μ is $\bar{Y}$
- The estimated standard error for $\bar{Y}$ is $S/\sqrt{n}$

Inference for μ with unknown σ^2

To test

$$H_0: \mu = \mu_0$$
$$H_1: \mu \neq \mu_0$$

the test statistic is

$$t = \frac{\bar{Y} - \mu_0}{S/\sqrt{n}}.$$

Under H_0,
$t \sim t_{n-1}$

We reject H_0 iff

$$|t| \geq t_{n-1,\alpha/2}$$

$P(T > t_\alpha) = \alpha$

This procedure has a significance level of α.

Proof of hypothesis test

Show that $P(\text{this test reject } H_0 \mid H_0 \text{ true}) = \alpha$.

$$P(\text{reject } H_0 \mid H_0 \text{ true})$$

$$= P\left(|t| \geq t_{n-1,\frac{\alpha}{2}} \mid \mu = \mu_0 \right)$$

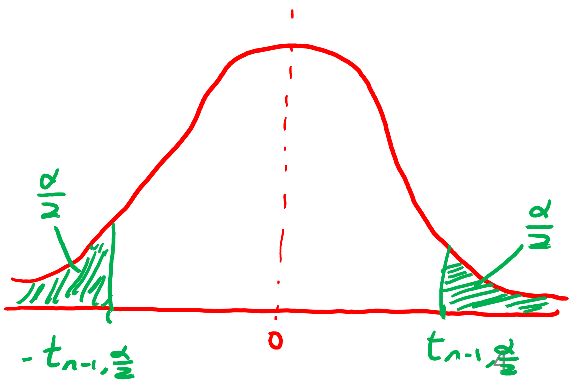
$$= P\left(\left| \frac{\bar{y} - \mu_0}{\frac{s}{\sqrt{n}}} \right| \geq t_{n-1,\frac{\alpha}{2}} \mid \mu = \mu_0 \right)$$

$$= P\left(|T| \geq t_{n-1,\frac{\alpha}{2}} \right) \quad \text{where } T \sim t_{n-1}$$

$$= P\left(T \leq -t_{n-1,\frac{\alpha}{2}} \right) + P\left(T \geq t_{n-1,\frac{\alpha}{2}} \right)$$

$$= \frac{\alpha}{2} + \frac{\alpha}{2}$$

$$= \alpha$$



CI for μ with unknown σ^2

The corresponding $100(1-\alpha)\%$ confidence interval for μ is

$$\left(\bar{Y} - t_{n-1,\frac{\alpha}{2}}\frac{S}{\sqrt{n}}, \bar{Y} + t_{n-1,\alpha/2}\frac{S}{\sqrt{n}}\right).$$

Proof:

$$P\left(\bar{Y} - t_{n-1,\frac{\alpha}{2}}\frac{S}{\sqrt{n}} \le \mu \le \bar{Y} + t_{n-1,\frac{\alpha}{2}}\frac{S}{\sqrt{n}}\right)$$

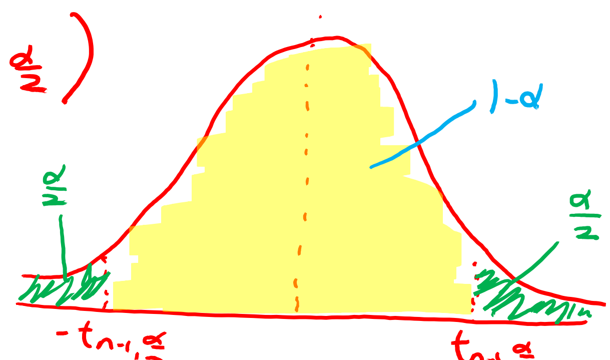
$$= P\left(-t_{n-1,\frac{\alpha}{2}}\frac{S}{\sqrt{n}} \le \mu - \bar{Y} \le t_{n-1,\frac{\alpha}{2}}\frac{S}{\sqrt{n}}\right)$$

$$= P\left(-t_{n-1,\frac{\alpha}{2}}\frac{S}{\sqrt{n}} \le \bar{Y} - \mu \le t_{n-1,\frac{\alpha}{2}}\frac{S}{\sqrt{n}}\right)$$

$$= P\left(-t_{n-1,\frac{\alpha}{2}} \le \boxed{\frac{\bar{Y}-\mu}{\frac{S}{\sqrt{n}}}} \le t_{n-1,\frac{\alpha}{2}}\right)$$

$T \sim t_{n-1}$

$$= 1-\alpha$$



P-value

The corresponding P-value of the hypothesis test is

$$P(|T| \geq |t|)$$

$T \sim t_{n-1}$ observed test statistic

where

$$t = \frac{\bar{y} - \mu_0}{s/\sqrt{n}}$$

is the observed t-statistic and $T \sim t_{n-1}$.

Example 2.6

A new waterjet cutting machine was developed and eight samples were tested. The recorded jet velocities (in m/s) were as follows:

916	891	895	904
913	916	895	885

$\bar{y} = 901.785$

$s = 12.1$, $n = 8$

The developer claims that the new machine produces an average jet velocity of not less than 915m/s. We may assume that jet velocities are approximately normally distributed.

(a) Find a 95% confidence interval for the true average jet velocity μ for machines of this type.

(b) Do the sample data provide sufficient evidence to contradict the developer's claim at 2.5% significance level?

(c) What is the P-value associated with test in part (b)?

Example 2.6 Solution

a) $CI = \bar{y} \pm t_{n-1,\frac{\alpha}{2}} \frac{s}{\sqrt{n}}$ $\qquad t_{7,0.025} = 2.365$

In R: $qt(\cancel{0.95}^{\,0.975}, 7)$

$= 901.815 \pm t_{7,0.025} \frac{12.1}{\sqrt{8}}$

$= 901.815 \pm 2.365 \frac{12.1}{\sqrt{8}}$

$= 901.815 \pm 10.11588$

$\approx (891.76,\ 911.99)$

b) $H_0: \mu = 915$ vs $H_a: \mu < 915$

$t = \dfrac{\bar{y} - \mu_0}{\frac{s}{\sqrt{n}}} = \dfrac{901.875 - 915}{\frac{12.1}{\sqrt{8}}} \approx -3.068$

critical region: $t < -t_{n-1,\alpha} = -t_{7,0.025} = -2.365$

There is sufficient evidence to reject H_0 at the 0.025 level of significance.

c) P-value $= P(T < t)$ where $T \sim t_7$

$= P(T < -3.068)$

≈ 0.0091

In R: $pt(-3.068, \cancel{\text{lower.tail = FALSE}}^{\,7})$

population	σ^2	n	CI
normal	known	small / large	$\bar{y} \pm Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$
normal	not known	small	$\bar{y} \pm t_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}}$
not normal	known	large	$\bar{y} \pm Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$
not normal	not known	large	$\bar{y} \pm t_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}}$
not normal	known	small	no formula exists
not normal	not known	small	no formula exists