Q1 a. $E[X_i X_j]$

- If $i \neq j$

$$E[X_i X_j] = E[X_i]\,E[X_j] \quad \text{(independence)}$$
$$= \mu \cdot \mu$$
$$= \mu^2$$

- If $i = j$:

$$E[X_i X_j] = E[X_i X_i]$$
$$= E[X_i^2] \quad (1)$$
$$Var(X_i) = E[X_i^2] - E[X_i]^2$$
$$\therefore E[X_i^2] = Var(X_i) + E[X_i]^2 \quad (2)$$
$$(1),(2) \Rightarrow E[X_i X_j] = Var(X_i) + E[X_i]^2 \quad (i=j)$$
$$= \sigma^2 + \mu^2$$
$$\therefore E[X_i X_j] = \begin{cases} \mu^2 & \text{if } i \neq j, \\ \mu^2 + \sigma^2 & \text{otherwise} \end{cases}$$

b. $$E[X_i \bar{X}] = E\left[X_i \frac{1}{n}\sum_{j=1}^{n} X_j\right]$$
$$= \frac{1}{n}\sum_{j=1}^{n} E[X_i X_j]$$
$$= \frac{1}{n}\sum_{\substack{j=1 \\ j \neq i}}^{n} E[X_i]\,E[X_j] + \frac{1}{n} E[X_i^2] \quad \text{(from a.)}$$
$$= \frac{n-1}{n}\mu^2 + \frac{1}{n}(\mu^2 + \sigma^2)$$
$$= \mu^2 - \frac{1}{n}\mu^2 + \frac{1}{n}\mu^2 + \frac{1}{n}\sigma^2$$
$$= \mu^2 + \frac{\sigma^2}{n}$$
$$\therefore E[X_i \bar{X}] = \mu^2 + \frac{\sigma^2}{n}$$

c. $$E[S_{XX}] = E\left[\sum_{i=1}^{n}(X_i - \bar{X})^2\right]$$
$$= \sum_{i=1}^{n} E[(X_i - \bar{X})^2]$$
$$= \sum_{i=1}^{n} E[X_i^2 - 2X_i\bar{X} + \bar{X}^2]$$

$$= \sum_{i=1}^{n} \left(E[X_i^2] - 2E[X_i \bar{X}] + E[\bar{X}^2] \right)$$

We have:

$$\bullet \ Var(\bar{X}) = Var\left(\frac{1}{n} \sum_{i=1}^{n} X_i \right)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} Var(X_i) \quad (\text{independence})$$

$$= \frac{1}{n^2} \cdot n \cdot \sigma^2$$

$$= \frac{\sigma^2}{n}$$

$$\bullet \ E[\bar{X}] = E\left[\frac{1}{n} \sum_{i=1}^{n} X_i \right]$$

$$= \frac{1}{n} \sum_{i=1}^{n} E[X_i]$$

$$= \frac{1}{n} \cdot n \cdot \mu$$

$$= \mu$$

$$\bullet \ Var(\bar{X}) = E[\bar{X}^2] - E[\bar{X}]^2$$

$$\therefore E[\bar{X}^2] = Var(\bar{X}) + E[\bar{X}]^2$$

$$= \frac{\sigma^2}{n} + \mu^2$$

~~Hence~~ And from a. and b.:

$$\therefore E[S_{XX}] = \sum_{i=1}^{n} \left(E[X_i^2] - 2E[X_i \bar{X}] + E[\bar{X}^2] \right)$$

$$= \sum_{i=1}^{n} \left[\mu^2 + \sigma^2 - 2\left(\mu^2 + \frac{\sigma^2}{n} \right) + \frac{\sigma^2}{n} + \mu^2 \right]$$

$$= n \left(2\mu^2 + \frac{n\sigma^2 + \sigma^2}{n} - 2\mu^2 - \frac{2\sigma^2}{n} \right)$$

$$= n\sigma^2 - \sigma^2$$

$$= (n-1)\sigma^2$$

$$\therefore E[S_{XX}] = (n-1)\sigma^2$$

d. The sample variance S^2:

$$\bullet \ S^2 = \frac{1}{n-1} \cancel{S_{XX}} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

$$= \frac{1}{n-1} S_{XX}$$

$$\therefore E[S^2] = E\left[\frac{1}{n-1} S_{XX} \right]$$

$$= \frac{1}{n-1} E[S_{XX}]$$

$$= \frac{1}{n-1} \cdot (n-1)\sigma^2 \quad (\text{from c.})$$

$$= \sigma^2$$

We have

$$\bullet \ b_{S^2}(\sigma^2) = E[S^2] - \sigma^2$$

$$= \sigma^2 - \sigma^2$$

$$= 0$$

$\therefore$ The sample variance S^2 is unbiased for σ^2

Q2 a. $E[\hat{p}^2] = E\left[\left(\frac{X}{n}\right)^2\right]$

$$= \frac{1}{n^2} E[X^2] \quad (1)$$

- $Var(X) = E[X^2] - E[X]^2$

$\therefore E[X^2] = Var(X) + E[X]^2$

$$= np(1-p) + (np)^2 \qquad (X \sim Bin(n,p))$$

$$= np - np^2 + n^2p^2 \quad (2)$$

$(1),(2) \Rightarrow E[\hat{p}^2] = \frac{1}{n^2}(np - np^2 + n^2p^2)$

$$= \frac{p-p^2}{n} + p^2$$

$$= \frac{p(1-p)}{n} + p^2$$

- $b_{\hat{p}^2}(p^2) = E[\hat{p}^2] - p^2$

$$= \frac{p(1-p)}{n} + p^2 - p^2$$

$$= \frac{p(1-p)}{n}$$

$\therefore E[\hat{p}^2] = \frac{p(1-p)}{n} + p^2$ and $b_{\hat{p}^2}(p^2) = \frac{p(1-p)}{n}$

$\therefore$ the $\hat{p}^2$ is a biased estimator for p^2

b. We have :

$$\frac{d^k}{dt^k} M_X(t)\Big|_{t=0} = E[X^k]$$

$E[X] = \frac{d}{dt} M_X(t)\Big|_{t=0}$

- $\frac{d}{dt} M_X(t) = \frac{d}{dt}\left[M_X(t)\right]$

$$= \frac{d}{dt}\left[(1-p+pe^t)^n\right]$$

$$= n(1-p+pe^t)^{n-1}(pe^t)$$

$\therefore E[X] = n(1-p+pe^0)^{n-1}(pe^0)$

$$= n(1-p+p)^{n-1}\, p$$

$$= np$$

- $E[X^2] = \frac{d^2}{dt^2} M_X(t)\Big|_{t=0}$

- $\frac{d^2}{dt^2} M_X(t) = n(n-1)(1-p+pe^t)^{n-2}(pe^t)^2 + n(1-p+pe^t)^{n-1}(pe^t)$

$\therefore E[X^2] = n(n-1)(1-p+pe^0)^{n-2}(pe^0)^2 + n(1-p+pe^0)^{n-1}(pe^0)$

$$= n(n-1)(1-p+p)^{n-2}(p)^2 + n(1-p+p)^{n-1}\, p$$

$$= n(n-1)p^2 + np$$

- $E[X^3] = \frac{d^3}{dt^3} M_X(t)\Big|_{t=0}$

- $\frac{d^3}{dt^3} M_X(t) = n(n-1)(n-2)(1-p+pe^t)^{n-3}(pe^t)^3 + 2n(n-1)(1-p+pe^t)^{n-2}(pe^t)^2 + n(n-1)(1-p+pe^t)^{n-2}(pe^t)^2 + n(1-p+pe^t)^{n-1}(pe^t)$

$$\therefore E[X^3] = n(n-1)(n-2)(1-p+pe^0)^{n-3}(pe^0)^3 + 2n(n-1)(1-p+pe^0)^{n-2}(pe^0)^2$$
$$+ n(n-1)(1-p+pe^0)^{n-2}(pe^0)^2 + n(1-p+pe^0)^{n-1}(pe^t)$$
$$= n(n-1)(n-2)\,p^3 + 2n(n-1)\cancel{p}\,p^2 + n(n-1)p^2 + np$$
$$= n(n-1)(n-2)\,p^3 + 3n(n-1)p^2 + np$$

- $\frac{d^3}{dt^3} M_X(t) = n(n-1)(n-2)(1-p+pe^t)^{n-3}(pe^t)^3 + 3n(n-1)(1-p+pe^t)^{n-2}(pe^t)^2$
$$+ n(1-p+pe^t)^{n-1}(pe^t)$$

- $E[X^4] = \frac{d^4}{dt^4} M_X(t)\Big|_{t=0}$

- $\frac{d^4}{dt^4} M_X(t) = n(n-1)(n-2)(n-3)(1-p+pe^t)^{n-4}(pe^t)^4 + 3n(n-1)(n-2)(1-p+pe^t)^{n-3}(pe^t)^3$
$$+ 3n(n-1)(n-2)(1-p+pe^t)^{n-3}(pe^t)^3 + 6n(n-1)(1-p+pe^t)^{n-2}(pe^t)^2$$
$$+ n(n-1)(1-p+pe^t)^{n-2}(pe^t)^2 + n(1-p+pe^t)^{n-1}(pe^t)$$
$$= n(n-1)(n-2)(n-3)(1-p+pe^t)^{n-4}(pe^t)^4 + 6n(n-1)(n-2)(1-p+pe^t)^{n-3}(pe^t)^3$$
$$+ 7n(n-1)(1-p+pe^t)^{n-2}(pe^t)^2 + n(1-p+pe^t)^{n-1}(pe^t)$$

$$\therefore E[X^4] = n(n-1)(n-2)(n-3)(1-p+pe^0)^{n-4}(pe^0)^4 + 6n(n-1)(n-2)(1-p+pe^0)^{n-3}(pe^0)^3$$
$$+ 7n(n-1)(1-p+pe^0)^{n-2}(pe^0)^2 + n(1-p+pe^0)^{n-1}(pe^0)$$
$$= n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$$

c. $MSE_{\hat{p}^2}(p^2) = Var(\hat{p}^2) + \left(b_{\hat{p}^2}(p^2)\right)^2$

We have:

- $Var(\hat{p}^2) = Var\left(\left(\frac{X}{n}\right)^2\right)$
$$= \frac{1}{n^4} Var(X^2)$$
$$= \frac{1}{n^4}\left(E[(X^2)^2] - E[X^2]^2\right)$$
$$= \frac{1}{n^4}\left(E[X^4] - E[X^2]^2\right)$$

- $E[X^4] - E[X^2]^2 = n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$
$$- [n(n-1)p^2 + np]^2$$
$$= (n^4 - 6n^3 + 11n^2 - 6n)p^4 + (6n^3 - 18n^2 + 12n)p^3 + (7n^2 - 7n)p^2 + np$$
$$- n^2(n-1)^2p^4 - 2n^2(n-1)p^3 - n^2p^2$$
$$= (n^4 - 6n^3 + 11n^2 - 6n)p^4 + (6n^3 - 18n^2 + 12n)p^3 + (7n^2 - 7n)p^2$$
$$+ np - (n^4 - 2n^3 + n^2)p^4 - (2n^3 - 2n^2)p^3 - n^2p^2$$

$$= (-4n^3 + 10n^2 - 6n)p^4 + (4n^3 - \overset{16}{\cancel{20}}n^2 + 12n)p^3 + (6n^2 - 7n)p^2 + np$$

$$\therefore Var(\hat{p}^2) = \frac{1}{n^4}\left[(-4n^3 + 10n^2 - 6n)p^4 + (4n^3 - 16n^2 + 12n)p^3 + (6n^2 - 7n)p^2 + np\right]$$

$$= \frac{1}{n^3}\left[(-4n^2 + 10n - 6)p^4 + (4n^2 - 16n + 12)p^3 + (6n\cancel{^2} - 7\cancel{n})p^2 + \cancel{n}p\right]$$

$$\bullet \left(b_{\hat{p}^2}(p^2)\right)^2 = \left(\frac{p(1-p)}{n}\right)^2$$

$$= \frac{p^2(1-p)^2}{n^2}$$

$$= \frac{p^2(1-2p+p^2)}{n^2}$$

$$= \frac{p^2 - 2p^3 + p^4}{n^2} = \frac{1}{n^3}(np^2 - 2np^3 + np^4)$$

$$\therefore MSE_{\hat{p}^2}(p^2) = Var(\hat{p}^2) + \left(b_{\hat{p}^2}(p^2)\right)^2$$

$$= \frac{1}{n^3}\left[(-4n^2 + 10n - 6)p^4 + (4n^2 - 16n + 12)p^3 + (6n - 7)p^2 + p\right]$$

$$+ \frac{1}{n^3}(np^2 - 2np^3 + np^4)$$

$$= \frac{1}{n^3}\left[(-4n^2 + 11n - 6)p^4 + (4n^2 - 18n + 12)p^3 + (7n - 7)p^2 + p\right]$$

$$\therefore MSE_{\hat{p}^2}(p^2) = \frac{1}{n^3}\left[(-4n^2 + 11n - 6)p^4 + (4n^2 - 18n + 12)p^3 + (7n - 7)p^2 + p\right]$$

d. $$E\left[\frac{\hat{p}(1-\hat{p})}{n-1}\right] = \frac{1}{n-1}E[\hat{p}(1-\hat{p})]$$

$$= \frac{1}{n-1}E[\hat{p} - \hat{p}^2]$$

$$= \frac{1}{n-1}\left(E[\hat{p}] - E[\hat{p}^2]\right) \quad (1)$$

We have:

$$\bullet E[\hat{p}] = E\left[\frac{X}{n}\right]$$

$$= \frac{1}{n}E[X]$$

$$= \frac{1}{n}\, n.p$$

$$= p \quad (2)$$

$$\bullet E[\hat{p}^2] = \frac{p(1-p)}{n} + p^2 \quad (3)$$

From (1), (2) and (3)

$$\therefore E\left[\frac{\hat{p}(1-\hat{p})}{n-1}\right] = \frac{1}{n-1}\left(p - \frac{p(1-p)}{n} + p^2\right)$$

$$= \frac{1}{n-1} \cdot \frac{np - p - p^2 + np^2}{n}$$

$$= \frac{p}{n(n-1)} \cdot (n - 1 - p + np)$$

$$= \frac{p}{n(n-1)} (n-1)(1-p)$$

$$= \frac{p(1-p)}{n}$$

$$\therefore E\left[\frac{\hat{p}(1-\hat{p})}{n-1}\right] = \frac{p(1-p)}{n}$$

e. Let T be an unbiased estimator for p^2

$$\therefore b_T(p^2) = E[T] - p^2 = 0$$

$$\therefore E[T] = p^2$$

$$E\left[\frac{\hat{p}(1-\hat{p})}{n-1}\right] = \frac{p(1-p)}{n}$$

$$= \frac{p-p^2}{n}$$

$$\therefore\ p-p^2 = n\,E\left[\frac{\hat{p}(1-\hat{p})}{n-1}\right]$$

$$\left.\begin{aligned} \therefore\ p^2 &= p - E\left[\frac{n\hat{p}(1-\hat{p})}{n-1}\right] \\ E[\hat{p}] &= p\ , \end{aligned}\right\} \Rightarrow p^2 = E[\hat{p}] - E\left[\frac{n\hat{p}(1-\hat{p})}{n-1}\right]$$

$$\therefore\ p^2 = E\left[\hat{p} - \frac{n\hat{p}(1-\hat{p})}{n-1}\right]$$

$$= E\left[\frac{\hat{p}(n-1) - n\hat{p}(1-\hat{p})}{n-1}\right]$$

$$= E\left[\frac{n\hat{p} - \hat{p} - n\hat{p} + n\hat{p}^2}{n-1}\right]$$

$$= E\left[\frac{n\hat{p}^2 - \hat{p}}{n-1}\right]$$

And $E[T] = p^2$

$$\therefore\ E[T] = E\left[\frac{n\hat{p}^2 - \hat{p}}{n-1}\right]$$

$$\therefore\ T = \frac{n\hat{p}^2 - \hat{p}}{n-1}$$

$\therefore\ \frac{n\hat{p}^2 - \hat{p}}{n-1}$ is an unbiased estimator for p^2