THE UNIVERSITY *of* ADELAIDE

Examination in School of Mathematical Sciences

Semester 2, 2018

104843 STATS 2107 Statistical Modelling & Inference II

Official Reading Time:	10 mins
Writing Time:	120 mins
Total Duration:	130 mins

NUMBER OF QUESTIONS: 5 TOTAL MARKS: 70

Instructions

- Attempt all questions.
- Begin each answer on a new page.
- Examination materials must not be removed from the examination room.

Materials

- 1 Blue book is provided.
- Calculators without remote communications capability are allowed.
- English and foreign-language dictionaries may be used.

DO NOT COMMENCE WRITING UNTIL INSTRUCTED TO DO SO.

1. Let $Y_1, Y_2, \ldots, Y_n$ be independent and identically distributed (*i.i.d.*) random variables with probability density function $f(y;\theta)$ for a real scalar parameter $\theta \in \Theta$, where Θ denotes the parameter space.

   Let $T = T(Y_1, Y_2, \ldots, Y_n)$ be an estimator for θ.

(a) Define the *mean squared error*, $\mathsf{MSE}_T(\theta)$, of T. [1 marks]

Mark Scheme: 1 for definition

Solution:

$$\mathsf{MSE}_T(\theta) = \mathsf{E}[(T-\theta)^2].$$

(b) Define the *bias*, $b_T(\theta)$, of T. [1 marks]

Mark Scheme: 1 for definition

Solution:

$$b_T(\theta) = \mathsf{E}[T] - \theta.$$

(c) Prove that

$$\mathsf{MSE}_T(\theta) = \mathsf{Var}(T) + b_T(\theta)^2.$$

[4 marks]

Mark Scheme: 4 for working

Solution:

$$\begin{aligned}
\mathsf{MSE}_T(\theta) &= \mathsf{E}[(T-\theta)^2] \\
&= \mathsf{E}[(T-\mathsf{E}[T]+\mathsf{E}[T]-\theta)^2] \\
&= \mathsf{E}[(T-\mathsf{E}[T])^2] + \mathsf{E}[(\mathsf{E}[T]-\theta)^2] + 2\mathsf{E}[(T-\mathsf{E}[T])(\mathsf{E}[T]-\theta)] \\
&= \mathsf{Var}(T) + \mathsf{E}[b_T(\theta)^2] + 2(\mathsf{E}[T]-\theta)\mathsf{E}[T-\mathsf{E}[T]] \\
&= \mathsf{Var}(T) + b_T(\theta)^2 + 2(\mathsf{E}[T]-\theta)0 \\
&= \mathsf{Var}(T) + b_T(\theta)^2.
\end{aligned}$$

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(d) Suppose $Y_1, Y_2, \ldots, Y_n$ are independent identically distributed (i.i.d.) $N(\mu, \sigma^2)$ random variables and let

$$\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

be an estimator for μ. Calculate $\mathsf{MSE}_{\bar{Y}}(\mu)$.

[4 marks]

Mark Scheme: 4 for working

Solution:

First calculate bias:

$$\begin{aligned}
b_{\bar{Y}}(\mu) &= \mathsf{E}[\bar{Y}] - \mu \\
&= \mathsf{E}\left[\frac{1}{n}\sum_{i=1}^{n} Y_i\right] - \mu \\
&= \frac{1}{n}\sum_{i=1}^{n} \mathsf{E}[Y_i] - \mu \\
&= \frac{1}{n}\sum_{i=1}^{n} \mu - \mu \\
&= \mu - \mu = 0.
\end{aligned}$$

Next calculate $\mathsf{Var}(\bar{Y})$

$$\begin{aligned}
\mathsf{Var}(\bar{Y}) &= \mathsf{Var}\left[\frac{1}{n}\sum_{i=1}^{n} Y_i\right] \\
&= \frac{1}{n^2}\sum_{i=1}^{n} \mathsf{Var}[Y_i] \quad \text{as independent} \\
&= \frac{1}{n^2}\sum_{i=1}^{n} \sigma^2 \\
&= \frac{\sigma^2}{n}.
\end{aligned}$$

Hence

Please turn over for page 4

$$\begin{aligned}\text{MSE}_{\bar{Y}}(\mu) &= \text{Var}(\bar{Y}) + b_{\bar{Y}}(\mu)^2 \\ &= \frac{\sigma^2}{n}.\end{aligned}$$

[Total: 10]

Core: 6 Adv: 4

2.

(a) Carefully define the t-distribution with k degrees of freedom. [3 marks]

Mark Scheme: 1 for Z, 1 for X, 1 for frac.

Solution:

Suppose $Z \sim N(0,1)$ and $X \sim \chi^2_k$ independently, and let

$$T = \frac{Z}{\sqrt{X/k}},$$

then T is said to have a t-distribution with k degrees of freedom.

(b) Suppose that $Y_1, Y_2, \ldots, Y_n$ are i.i.d. $N(\mu, \sigma^2)$, then prove that

$$\frac{\bar{Y} - \mu}{S/\sqrt{n}} \sim t_{n-1}.$$

You may assume that $\bar{Y}$ and S^2 are independent. [4 marks]

Mark Scheme: 4 for working

Solution:

$$
\begin{aligned}
\frac{\bar{Y}-\mu}{S/\sqrt{n}} &= \frac{\bar{Y}-\mu}{\sigma/\sqrt{n}} \frac{\sigma/\sqrt{n}}{S/\sqrt{n}} \\
&= \frac{\bar{Y}-\mu}{\sigma/\sqrt{n}} \Big/ \sqrt{\frac{S^2}{\sigma^2}} \\
&= \frac{\bar{Y}-\mu}{\sigma/\sqrt{n}} \Big/ \sqrt{\frac{(n-1)S^2}{\sigma^2}(n-1)} \\
&= \frac{Z}{\sqrt{X/(n-1)}} \sim T_{n-1}
\end{aligned}
$$

(c) Let $Z \sim N(0,1)$. Show that the moment generating function of Z^2 is

$$M_{Z^2}(t) = (1-2t)^{-\frac{1}{2}}, \quad t < 1/2.$$

[4 marks]

Mark Scheme: 4 for working

Solution:

$$
\begin{aligned}
M_{Z^2}(t) &= E[e^{tZ^2}] \\
&= \int_{-\infty}^{\infty} e^{tz^2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{z^2(t-1/2)} dz \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2(1-2t)/2} dz \\
&= \frac{1}{(1-2t)^{1/2}} \int_{-\infty}^{\infty} \frac{(1-2t)^{1/2}}{\sqrt{2\pi}} e^{-z^2(1-2t)/2} dz
\end{aligned}
$$

but the integral is one since it is the pdf of $N(0, \frac{1}{1-2t})$. So we have

$$M_{Z^2}(t) = \frac{1}{(1-2t)^{1/2}}.$$

(d) Suppose $Z_1, Z_2, \ldots, Z_k$ are independent and identically distributed $N(0,1)$ random variable and let

$$X = \sum_{i=1}^{k} Z_i^2.$$

Please turn over for page 6

Show that the moment generating function of X is

$$M_X(t) = (1-2t)^{-\frac{k}{2}}, \quad t < 1/2.$$

[4 marks]

Mark Scheme: 4 for working

Solution:

$$\begin{aligned} M_X(t) &= E\left\{\exp(tX)\right\} \\ &= E\left\{\exp\left(t\sum_{i=1}^{k} Z_i^2\right)\right\} \\ &= E\left\{\exp(tZ_1^2)\exp(tZ_2^2)\cdots\exp(tZ_k^2)\right\} \\ &= E\{\exp(tZ_1^2)\}E\{\exp(tZ_2^2)\}\cdots E\{\exp(tZ_k^2)\} \text{ (by independence)} \\ &= (1-2t)^{-1/2}(1-2t)^{-1/2}\cdots(1-2t)^{-1/2} \\ &= \frac{1}{(1-2t)^{k/2}}. \end{aligned}$$

X has the chi-squared distribution with k degrees of freedom.

(e) Hence, suppose that

$$X \sim \chi_k^2,$$

show that

$$\mathsf{E}[X] = k \text{ and } \mathsf{Var}(X) = 2k.$$

[5 marks]

Mark Scheme: 5 for working

Solution:

$$\begin{aligned} E[X] &= \left.\frac{d}{dt}M_x(t)\right|_{t=0} \\ &= \left.\frac{d}{dt}(1-2t)^{-k/2}\right|_{t=0} \\ &= \left.k(1-2t)^{-k/2-1}\right|_{t=0} \\ &= k. \end{aligned}$$

Please turn over for page 7

$$
\begin{aligned}
E[X^2] &= \frac{d^2}{dt^2} M_x(t)\bigg|_{t=0} \\
&= \frac{d}{dt} k(1-2t)^{-k/2-1}\bigg|_{t=0} \\
&= k(k+2)(1-2t)^{-k/2-2}\big|_{t=0} \\
&= k(k+2).
\end{aligned}
$$

Now

$$
\begin{aligned}
\text{Var}(X) &= E[X^2] - E[X]^2 \\
&= k(k+2) - k^2 \\
&= 2k.
\end{aligned}
$$

[Total: 20]

Core: 7 Adv: 13

3. Consider the simple linear regression model

$$Y_i = \beta_0 + \beta_1 x_i + \epsilon_i,$$

where $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are independent with $\epsilon_i \sim N(0, \sigma^2)$ for $i = 1, 2, \ldots, n$.

(a) Consider

$$S_{xy} = \sum_{i=1}^{n} (y_i - \bar{y})(x_i - \bar{x})$$

Show that S_{xy} can be written as

$$S_{xy} = \sum_{i=1}^{n} (x_i - \bar{x}) y_i.$$

[3 marks]

Mark Scheme: 3 for working

Solution:

$$
\begin{aligned}
S_{xy} &= \sum_{i=1}^{n} (y_i - \bar{y})(x_i - \bar{x}) \\
&= \sum_{i=1}^{n} y_i(x_i - \bar{x}) - \sum_{i=1}^{n} \bar{y}(x_i - \bar{x}) \\
&= \sum_{i=1}^{n} y_i(x_i - \bar{x}) - \bar{y} \sum_{i=1}^{n} (x_i - \bar{x}) \\
&= \sum_{i=1}^{n} y_i(x_i - \bar{x}) \qquad\qquad \text{as } \sum_{i=1}^{n} (x_i - \bar{x}) = 0.
\end{aligned}
$$

(b) Find the constants $a_1, a_2, \ldots, a_n$, such that

$$\hat{\beta}_1 = \sum_{i=1}^{n} a_i Y_i.$$

Hint: You may assume that

$$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}}.$$

[2 marks]

Mark Scheme: 2 for working

Solution:

$$
\begin{aligned}
\hat{\beta}_1 &= \frac{S_{xy}}{S_{xx}} \\
&= \frac{\sum_{i=1}^{n} (x_i - \bar{x}) Y_i}{S_{xx}} \\
&= \sum_{i=1}^{n} \frac{(x_i - \bar{x}) Y_i}{S_{xx}} \\
&= \sum_{i=1}^{n} a_i Y_i, \quad \text{where } a_i = \frac{(x_i - \bar{x})}{S_{xx}}
\end{aligned}
$$

(c) Prove that

$$\mathsf{E}[\hat{\beta}_1] = \beta_1 \text{ and } \mathsf{Var}(\hat{\beta}_1) = \frac{\sigma^2}{S_{xx}}.$$

Hint: You may assume that

$$S_{xx} = \sum_{i=1}^{n} (x_i - \bar{x}) x_i.$$

[6 marks]

Mark Scheme: 3 for E; 3 for Var

Solution:

$$\begin{aligned}
\mathsf{E}[\hat{\beta}_1] &= \mathsf{E}\left[\sum_{i=1}^{n} a_i Y_i\right] \\
&= \sum_{i=1}^{n} a_i \mathsf{E}[Y_i] \\
&= \sum_{i=1}^{n} \frac{(x_i - \bar{x})}{S_{xx}} \mathsf{E}[Y_i] \\
&= \sum_{i=1}^{n} \frac{(x_i - \bar{x})}{S_{xx}} (\beta_0 + \beta_1 x_i) \\
&= \sum_{i=1}^{n} \frac{(x_i - \bar{x})}{S_{xx}} \beta_0 + \sum_{i=1}^{n} \frac{(x_i - \bar{x})}{S_{xx}} \beta_1 x_i \\
&= \beta_1 \sum_{i=1}^{n} \frac{(x_i - \bar{x})}{S_{xx}} x_i \qquad\qquad \text{as } \sum_{i=1}^{n} (x_i - \bar{x}) = 0. \\
&= \beta_1 \frac{S_{xx}}{S_{xx}} = \beta_1.
\end{aligned}$$

$$\begin{aligned}
\text{Var}[\hat{\beta}_1] &= \text{Var}\left[\sum_{i=1}^{n} a_i Y_i\right] \\
&= \sum_{i=1}^{n} a_i^2 \text{Var}[Y_i] && \text{as independent} \\
&= \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{S_{xx}^2} \text{Var}[Y_i] \\
&= \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{S_{xx}^2} \sigma^2 \\
&= \frac{S_{xx}}{S_{xx}^2} \sigma^2 \\
&= \frac{\sigma^2}{S_{xx}}.
\end{aligned}$$

[Total: 11]

Core: 0 Adv: 11

4. An analysis of the effect of displacement (`displ`) and drive type (`drv`) on the city fuel efficiency (`cty`) for 38 popular models of car was performed in R. The commands and output are given in Appendix A.
The displacement of a car is volume of the cylinders, while the drive is the type of drive, in this case, we have just three levels - front-wheel drive, rear-wheel drive and four-wheel drive.

(a) Consider the scatterplot of city fuel efficiency against displacement given in Figure 1. Describe the relationship. [3 marks]

Mark Scheme: direction - 1, strength - 1, three lines - 1

Solution:

There is a weak negative non-linear relationship between cty and displ. The three lines do not look parallel.

(b) Consider the separate regression model. Write down the line of best fit for the relationship between displacement and city fuel efficiency for rear-wheel drive cars. [2 marks]

Mark Scheme: 1 for intercept; 1 for slope

Solution:

For rear-wheel drive cars, we have

$$cty = 22.5914 - 3.0124 + (-2.0663 + 1.0039) \times displ$$

(c) Test for a statistically significant interaction term in the separate regression model at the 5% significance level. Remember to include the null and alternative hypotheses, the value of the test statistic, the P-value and your conclusion. [4 marks]

Mark Scheme: 1 for hypotheses; 1 for F; 1 for P; 1 for conclusion

Solution:

$$H_0 : \beta_4 = \beta_5 = 0$$
$$H_a : \text{at least one of } \beta_4, \beta_5 \neq 0$$

where β_4 and β_5 are the coefficients associated with the interactions term.

The value of the test statistic is $F = 9.3963$.

The P-value is 0.0001199.

We reject the null hypothesis at the 5

(d) Using the Akaike's Information Criterion which model fits the data the best? Justify your answer. [2 marks]

Mark Scheme: 1 for separate; 1 for justification.

Solution:

The best model appears to be the separate regression model as this has the smallest AIC with a value of 1051.857.

(e) Assess the assumptions of the linear model used in the separate model. The plots given in Figure 2 may be used where appropriate. [4 marks]

Mark Scheme: 1 for each assumption.

Solution:

Linearity: Residual versus fitted (top left) shows random scatter so reasonable. There are possible outliers (numbers 222, 213, 223).

Homoscedascity: Standardised residual versus fitted (bottom left) shows equal spread as move from left to right so reasonable.

Normality: Residual QQ-plot (top right) is roughly linear so reasonable except for the points at the far right.

Independence: The fuel efficiency of one car should not affect the fuel efficiency of the other cars so this is reasonable.

[Total: 15]

Core: 15 Adv: 0

5. Suppose $y_1, y_2, \ldots, y_n$ are independent Poisson observations with parameter λ, $\lambda > 0$. That is, for $i = 1, 2, \ldots, n$,

$$f(y_i; \lambda) = \frac{e^{-\lambda}\lambda^{y_i}}{y_i!}, y_i > 0.$$

(a) Write down the likelihood. [1 marks]

Mark Scheme: 1 for expanded likelihood

Solution:

$$\prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{y_i}}{y_i!} = \frac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n} y_i}}{\prod_{i=1}^{n} y_i!}$$

(b) Write down the log-likelihood. [1 marks]

Mark Scheme: 1 for formula

Solution:

$$-n\lambda + \sum_{i=1}^{n} y_i \log(\lambda) - \log(\prod_{i=1}^{n} y_i!)$$

(c) Find the maximum likelihood estimate of λ, $\hat{\lambda}$. [3 marks]

Mark Scheme: 1 for diff; 1 for set to zero; 1 for solve.

Solution:

Differentiate the log-likelihood w.r.t. λ

$$\frac{\partial \ell}{\partial \lambda} = -n + \frac{\sum_{i=1}^{n} y_i}{\lambda}$$

Set equal to zero and solve:

$$\begin{aligned} -n + \frac{\sum_{i=1}^{n} y_i}{\lambda} &= 0 \\ \Rightarrow \frac{\sum_{i=1}^{n} y_i}{\lambda} &= n \\ \Rightarrow \hat{\lambda} &= \bar{y}. \end{aligned}$$

(e) Find the Fisher information. [3 marks]

Mark Scheme: 1 for diff; 1 for E; 1 for Fisher

Solution:

$$\begin{aligned}
\frac{\partial^2 \ell}{\partial \lambda^2} &= -\frac{\sum_{i=1}^n y_i}{\lambda^2} \\
I_\lambda &= E\left[-\frac{\partial^2 \ell}{\partial \lambda^2}\right] \\
&= E\left[\frac{\sum_{i=1}^n y_i}{\lambda^2}\right] \\
&= \frac{n\lambda}{\lambda^2} = \frac{n}{\lambda}.
\end{aligned}$$

(f) Let $\phi = \log(\lambda)$. Write down the maximum likelihood estimate, $\hat{\phi}$. [1 marks]

Mark Scheme: 1 for giving answer.

Solution:

We know that

$$\begin{aligned}
\hat{\phi} &= \log(\hat{\lambda}) \\
&= \log(\bar{y})
\end{aligned}$$

[Total: 9]

Core: 3 Adv: 6

6. Haemophilia is a X-chromosome linked, recessive disorder. Suppose a woman has a haemophiliac brother, her father is normal, and her mother is a carrier. Let

$$\theta = \begin{cases} 1 & \text{if the woman is a carrier} \\ 0 & \text{otherwise.} \end{cases}$$

It follows from genetic considerations that the prior distribution is

$$p(\theta) = \begin{cases} \frac{1}{2} & \text{if } \theta = 1, \\ \frac{1}{2} & \text{if } \theta = 0. \end{cases}$$

(a) Suppose the woman has two sons, of which neither have haemophilia. Find the probability the woman is a carrier. [5 marks]

Mark Scheme: 2 for setup; 3 for working

Solution:

Let S be the number of sons with haemophilia. If the woman is not a carrier, then

$$P(S=0|\theta=0)=1,$$

as the sons will only obtain a haemophilia carrying X chromosome X^H from their mother, and if does not have it, she cannot pass it on. If the woman is a carrier she has a probability of 1/2 of passing it on, assuming independence, we have

$$P(S=0|\theta=1)=\frac{1}{4}.$$

Putting this together with Bayes' rule gives

$$\begin{aligned} P(\theta=1|S=0) &= \frac{P(S=0|\theta=1)P(\theta=1)}{P(S=0|\theta=1)P(\theta=1)+P(S=0|\theta=0)P(\theta=0)} \\ &= \frac{1/4\times 1/2}{1/4\times 1/2+1\times 1/2} \\ &= \frac{1}{5}. \end{aligned}$$

(b) Suppose the woman has a third son. Given that the first two sons are not haemophiliacs, what is the probability that the third son is not a haemophiliac? [3 marks]

Mark Scheme: 1 for setup; 2 for working.

Solution:

Let

$$S_0=\begin{cases}1 & \text{if Son 3 is haemophilliac,}\\ 0 & \text{if Son 3 is not haemophilliac.}\end{cases}$$

$$\begin{aligned} P(S_0=1|S=0) &= \sum_\theta P(S_0=1|\theta)P(\theta|S=0) \\ &= P(S_0=1|\theta=1)P(\theta=1|S=0)+P(S_0=1|\theta=0)P(\theta=0|S=0) \\ &= \frac{1}{2}\times\frac{1}{5}+0\times\frac{4}{5}=\frac{1}{10}. \end{aligned}$$

[Total: 8]

Core: 0 Adv: 8

Q	core	adv
1	6	4
2	7	13
3	0	11
4	15	0
5	3	6
6	0	8
total	31	42

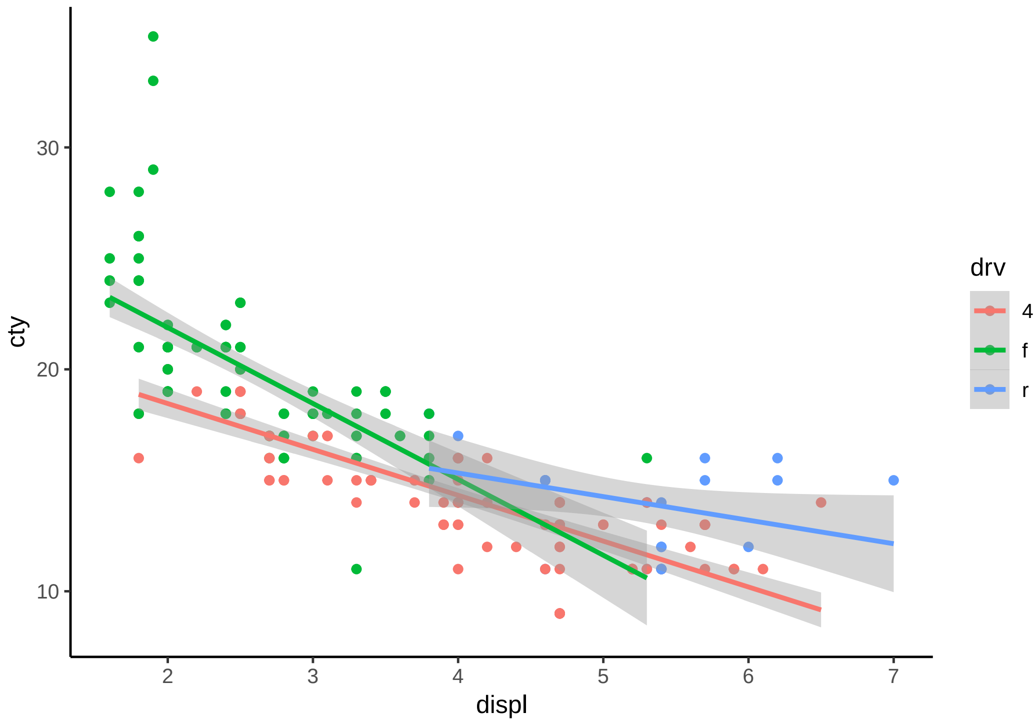

Figure 1: Scatterplot of Fuel efficiency against displacement for the MPG dataset. Colour of points indicates drive (type)

Appendix A

Load the data

```
library(tidyverse)
data(mpg)
theme_set(theme_classic())
```

Visualise data

```
mpg %>%
  ggplot(aes(displ, cty, col = drv)) +
  geom_point() +
  geom_smooth(method = "lm")
```

Fit models

```r
identical  <- lm(cty ~ displ, data = mpg)
parallel  <- lm(cty ~ displ + drv, data = mpg)
separate  <- lm(cty ~ displ * drv, data = mpg)
```

Model Coefficients

```r
summary(separate)
```

```
##
## Call:
## lm(formula = cty ~ displ * drv, data = mpg)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -6.4363 -1.2957 -0.0863  1.1203 12.7768
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  22.5914     0.8136  27.768  < 2e-16 ***
## displ        -2.0663     0.1958 -10.554  < 2e-16 ***
## drvf          6.1284     1.1632   5.269 3.18e-07 ***
## drvr         -3.0124     3.1043  -0.970 0.332872
## displ:drvf   -1.3529     0.3696  -3.661 0.000313 ***
## displ:drvr    1.0039     0.6048   1.660 0.098285 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 2.252 on 228 degrees of freedom
## Multiple R-squared:  0.7261, Adjusted R-squared:  0.7201
## F-statistic: 120.9 on 5 and 228 DF,  p-value: < 2.2e-16
```

```r
anova(separate)
```

```
## Analysis of Variance Table
##
## Response: cty
##            Df  Sum Sq Mean Sq  F value    Pr(>F)
## displ       1 2691.06 2691.06 530.7574 < 2.2e-16 ***
## drv         2  277.99  138.99  27.4136 2.144e-11 ***
## displ:drv   2   95.28   47.64   9.3963 0.0001199 ***
```

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```
## Residuals 228 1156.01    5.07
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```r
anova(separate, parallel)
```

```
## Analysis of Variance Table
##
## Model 1: cty ~ displ * drv
## Model 2: cty ~ displ + drv
##   Res.Df    RSS Df Sum of Sq      F    Pr(>F)
## 1    228 1156.0
## 2    230 1251.3 -2   -95.283 9.3963 0.0001199 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Model Selection

```r
AIC(identical, parallel, separate)
```

```
##           df      AIC
## identical  3 1109.336
## parallel   5 1066.391
## separate   7 1051.857
```

Assumption checking

```r
tmp  <- par(mfrow = c(2,2))
plot(separate)
```

```r
par(tmp)
```

Please turn over for page 20

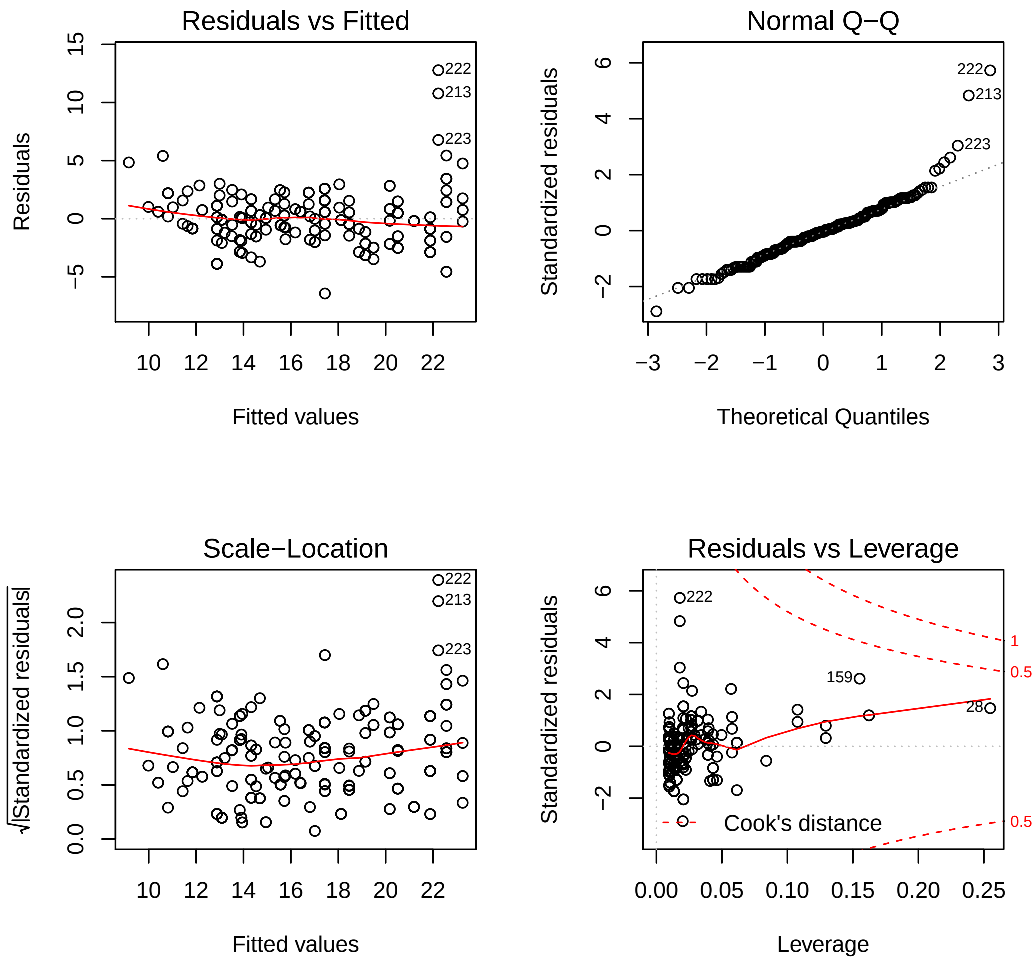

Figure 2: Assumption plot of the separate model for the MPG dataset.

Appendix B

Distribution	Probability mass function / probability density function	Expectation	Variance
Binomial	$p(x) = \binom{n}{x} p^x (1-p)^{n-x}$ for $x = 0, 1, 2, \dots, n$	np	$np(1-p)$
Geometric	$p(x) = p(1-p)^{x-1}$ for $x = 1, 2, \dots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson	$p(x) = \frac{e^{-\lambda}\lambda^x}{x!}$ for $x = 0, 1, 2, \dots$	λ	λ
Uniform	$f(x) = \frac{1}{b-a}$ for $a < x < b$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential	$f(x) = \lambda e^{-\lambda x}$ for $x > 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma	$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$for $x > 0$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$
Normal	$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(1/2\sigma^2)(x-\mu)^2}$ for $-\infty < x < \infty$	μ	σ^2
Beta	$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma\beta} \theta^{\alpha-1}(1-\theta)^{\beta-1}$ for $0 < \theta < 1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$