Assignment 2 - STATS 2107 a1814824

Q1

a. μ_1 is the true mean dexterity score of piano students

. μ_2 is the true mean dexterity score of singing students

. We have the following hypotheses

$$H_0: \mu_1 = \mu_2 \quad \text{vs} \quad H_a \ \mu_1 \neq \mu_2$$

. Test statistic :

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

(Under the null hypothesis $H_0: \mu_1 = \mu_2$ or $H_0: \mu_1 - \mu_2 = 0$)

$$\therefore z = \frac{37.25 - 35.91}{\sqrt{\frac{(4.34)^2}{137} + \frac{(5.19)^2}{137}}} \simeq 2.32$$

. P-value :

$$\text{P-value} = P(|Z| \geqslant |z|) = P(|Z| \geqslant 2.32)$$

$$\approx 0.0203$$ (2* pnorm(-2.32) in R studio)

$\therefore$ Since P-value $< \alpha$ $(0.0203 < 0.05)$, there is sufficient evidence to reject H_0

$\therefore$ The piano students have a different mean dexterity score than singing students

b. We have

$$z = \frac{\bar{x}_1 - \bar{x}_2 - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} - \frac{\mu_1 - \mu_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \quad (1)$$

. Under the null hypothesis $H_0: \mu_1 = \mu_2$

$$\frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_1}}} \sim N(0,1) \quad (2)$$

. And $$\frac{\mu_1 - \mu_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{3}{\sqrt{\frac{(4.34)^2}{137} + \frac{(5.19)^2}{137}}} \simeq 5.1902$$ (which is a constant) (3)

~~From~~

~~From~~ From ①, ② and ③:

$$.E[z] = E\left[\frac{\bar{x}_1-\bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}} - \frac{\mu_1-\mu_2}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}\right]$$

$$= E\left[\frac{\bar{x}_1-\bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}\right] - E\left[\frac{\mu_1-\mu_2}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}\right]$$

$$= 0 - 5.1902$$

$$= -5.1902$$

$$.Var(z) = Var\left(\frac{\bar{x}_1-\bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}} - \frac{\mu_1-\mu_2}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}\right)$$

$$= Var\left(\frac{\bar{x}_1-\bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}\right) + Var\left(\frac{\mu_1-\mu_2}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}\right)$$

$$= 1 + 0$$

$$= 1$$

$\therefore z \sim N(-5.1902, 1)$

. we have

$$\text{Power} = P(|z| \geq 1.96 \mid \mu_1-\mu_2=3)$$

$$= P(z \leq -1.96 \mid \mu_1-\mu_2=3) + P(z \geq 1.96 \mid \mu_1-\mu_2=3)$$

$$= P(z \leq -1.96 \mid \mu_1-\mu_2=3) + 1 - P(z \leq 1.96 \mid \mu_1-\mu_2=3)$$

$$= \Phi(-1.96; -5.1902, 1) + 1 - \Phi(1.96; -5.1902, 1)$$

(in R) = pnorm(-1.96, -5.1902, 1) + 1 - pnorm(1.96, -5.1902, 1)

≈ 0.9994

$\therefore$ The power of the test is 0.9994

ⓒ. Assume that $\sigma_1^2 = \sigma_2^2 = \sigma^2$ and we can estimate σ^2 with

$$s_p^2 = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}$$

$$= \frac{(137-1)(4.34)^2 + (137-1)(5.19)^2}{137+137-2}$$

$$\approx 22.8859$$

. The sample size needed for the test to achieve a significance level of 0.05 and power of 0.95, when $\mu_1-\mu_2 = 3$:

$$n = \frac{2\sigma^2(z_{\alpha/2}+z_\beta)^2}{\delta^2}$$

$$= \frac{2s_p^2(z_{\alpha/2}+z_\beta)^2}{(\mu_1-\mu_2)^2}$$

$$= \frac{2\times 22.8859(1.96+1.64)^2}{3^2}$$

$$\approx 66.0878$$

$z_{\alpha/2} = z_{0.05/2}$

(in R) = qnorm(0.025, lower.tail = FALSE)

≈ 1.96

$z_\beta = z_{1-0.95}$

(in R) = qnorm(0.05, lower.tail = FALSE)

≈ 1.64

$\therefore$ The sample size needed for the test is 67

Q2

a. The students from the piano group are independent and identically distribubuted (iid) $N(\mu_1, \sigma_1^2)$ random variables.

$\therefore$ Hence, according to Theorem 5, $\frac{(n_1-1)S_1^2}{\sigma_1^2} \sim \chi^2_{n_1-1}$

b. A symmetric 95% confidence interval for σ_1^2, on $\alpha = 0.05$

- Let c_1 and c_2 such that

$P(c_1 < X_1 < c_2) = 0.95$ where $X_1 = \frac{(n_1-1)S_1^2}{\sigma_1^2} \sim \chi^2_{n_1-1}$

$\therefore$ The symmetric 95% confidence interval for σ_1^2:

$$\left(\frac{(n_1-1)S_1^2}{c_2}, \frac{(n_1-1)S_1^2}{c_1}\right)$$

- We have:

$P(X < c_1) = \frac{\alpha}{2} = 0.025$

$\therefore c_1 = \text{qchisq}(\frac{\alpha}{2}, n_1-1) = \text{qchisq}(0.025, 136) \approx 105.6093$ (in R)

$P(X > c_2) = \frac{\alpha}{2}$

$\therefore 1 - P(X > c_2) = 1 - \frac{\alpha}{2}$

$\therefore P(X < c_2) = 1 - \frac{\alpha}{2} = 1 - 0.025 = 0.975$

$\therefore c_2 = \text{qchisq}(1-\frac{\alpha}{2}, n-1) = \text{qchisq}(0.975, 136) \approx 170.1753$ (in R)

$$\therefore \left(\frac{(n_1-1)S_1^2}{c_2}, \frac{(n_1-1)S_1^2}{c_1}\right) = \left(\frac{(137-1)(4.34)^2}{170.1753}, \frac{(137-1)(4.34)^2}{105.6093}\right)$$

$$\approx (15.0530, 24.2558)$$

$\therefore$ The symmetric 95% confidence interval for σ_1^2: $(15.0530, 24.2558)$

c. We have the following hypotheses:

$H_0: \sigma_1^2 - \sigma_2^2 = 0$ vs $H_a: \sigma_1^2 - \sigma_2^2 \neq 0$

- Test statistic

$$F = \frac{\frac{S_1^2}{\sigma_1^2}}{\frac{S_2^2}{\sigma_2^2}} \sim F_{n_1-1, n_2-1}$$

Under the null hypothesis:

$$H_0: \sigma_1^2 - \sigma_2^2 = 0$$

$$\therefore H_0: \sigma_1^2 = \sigma_2^2$$

$\therefore$ Hence the test statistic under the null hypothesis:

$$F = \frac{S_1^2}{S_2^2} \sim F_{n_1-1, n_2-1}$$

$$= \frac{(4.34)^2}{(5.19)^2} \sim F_{137-1, 137-1}$$

$$\approx 0.6993 \sim F_{136, 136}$$

The critical region for this hypothesis test is (with $\alpha = 0.05$)

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- $F > F_{n_1-1, n_2-1, \frac{\alpha}{2}}$

$$F_{n_1-1, n_2-1, \frac{\alpha}{2}} = F_{136, 136, 0.025}$$
$$= qf(1-0.025, 136, 136) \quad \text{(in R)}$$
$$\approx 1.4015$$

- $F < F_{n_1-1, n_2-1, 1-\frac{\alpha}{2}}$

$$F_{n_1-1, n_2-1, 1-\frac{\alpha}{2}} = F_{136, 136, 0.975}$$
$$= qf(1-0.975, 136, 136) \quad \text{(in R)}$$
$$\approx 0.7135$$

$\therefore$ The critical region for this hypothesis test is:

$$CR\{F < 0.7135, F > 1.4015\}$$

- Since $F = 0.6993 < 0.7135$, is in the critical region, there is sufficient evidence to reject H_0

$\therefore$ The two groups have different variance.