STATS 2107
Statistical Modelling and Inference II
Tutorial 5

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1. Suppose $y_1, y_2, \ldots, y_n$ are independent $Po(\lambda_i)$ observations with

$$\lambda_i = \theta x_i$$

where $\theta > 0$ is the unknown parameter and $x_1,\ x_2,\ \ldots,\ x_n$ are given positive constants.

   (a) Find the log-likelihood, $\ell(\theta; \boldsymbol{y})$, and the score function, $S(\theta; \boldsymbol{y})$.

   (b) Find the maximum likelihood estimate, $\hat{\theta}$, and the Fisher information, I_θ.

2. Consider a **single** binomial observation y from $\text{Bin}(n, \theta)$ where the number of trials is n and the probability of success is θ. Assume n is known.

   (a) Give the log-likelihood $\ell(\theta; y)$.

   (b) Find the Score function and the Fisher information about θ.

   (c) Find the MLE $\hat{\theta}$.

   (d) Find expressions for the score test statistic, U, and the log-likelihood ratio test statistic, G^2, for testing the null hypothesis $H_0 : \theta = \theta_0$ versus $H_a : \theta \neq \theta_0$.

   (e) State the asymptotic distributions of U and G^2, respectively, under H_0.

3. Consider the simple linear regression model with no intercept, that is,

$$Y_i = \beta x_i + \epsilon_i, \quad \epsilon_i \overset{i.i.d.}{\sim} N(0, \sigma^2), \quad i = 1, 2, \ldots, n,$$

where $\boldsymbol{\theta} = (\beta, \sigma^2)$ are the unknown parameters.

   (a) Write down the log-likelihood $\ell(\boldsymbol{\theta}; \boldsymbol{y})$

   (b) Find the score vector $S(\boldsymbol{\theta}; \boldsymbol{y})$.

   (c) Find the Fisher information matrix $I_{\boldsymbol{\theta}}$.

   (d) Find the MLEs $\hat{\beta}$ and $\hat{\sigma}^2$.

4. Suppose that $Y_1, Y_2, \ldots, Y_n$ are independent and identically distributed with density function

$$f(y; \theta) = e^{-(y-\theta)}, \quad y \geq \theta$$

and $f(y; \theta) = 0$ otherwise.
Find the MLE of θ. ***Hint: Take note of the region*** $y \geq \theta$ ***where the density is positive.***

5. Suppose $Y_1, Y_2, \ldots, Y_n \overset{i.i.d.}{\sim} \text{Exp}(\lambda)$ so that the density function is

$$f_{Y_i}(y; \lambda) = \lambda e^{-\lambda y_i}\ .$$

Consider the equivalent parameterisation in terms of $\theta = \frac{1}{\lambda}$ where

$$f_{Y_i}(y;\theta) = \frac{1}{\theta} e^{-\frac{1}{\theta} y_i} .$$

By considering the transformation $\Phi(\lambda) = \frac{1}{\lambda} = \theta$, do the following:

(a) Calculate the log-likelihoods $\ell_\lambda(\lambda; \boldsymbol{y})$ and $\ell_\theta(\theta; \boldsymbol{y})$. Verify directly that $\ell_\lambda(\lambda; \boldsymbol{y}) = \ell_\theta(\Phi(\lambda); \boldsymbol{y})$ and $\ell_\theta(\theta; \boldsymbol{y}) = \ell_\lambda(\Phi^{-1}(\theta); \boldsymbol{y})$.

(b) Calculate $\hat{\lambda}$, the maximum likelihood estimate of λ. Hence, calculate the the maximum likelihood estimate of θ.