Confidence intervals

Definition 1.7

If $Y_1, Y_2, \ldots, Y_n \sim$ iid $N(\mu, \sigma^2)$ with σ^2 known, then

$$\bar{y} \pm 1.96 \frac{\sigma}{\sqrt{n}}$$

is the 95% confidence interval for μ.

A random interval $\underline{(L, U)}$ is called a $100(1-\alpha)\%$ confidence interval for the parameter θ if it satisfies:

$$\underline{P(\underset{\text{lower confidence limit}}{L} < \theta < \underset{\text{upper confidence limit}}{U}) = \boxed{1-\alpha.}}$$

coverage probability
or
confidence coefficient

Remarks

- Note that the endpoints L and U are random: not θ.

  e.g. $Y_1, Y_2, \ldots, Y_n \sim \text{iid } N(\mu, \sigma^2)$ with σ^2 known,
  then the confidence limits $L = \bar{Y} - 1.96 \frac{\sigma}{\sqrt{n}}$ and $U = \bar{Y} + 1.96 \frac{\sigma}{\sqrt{n}}$ are random variables.

- The quantity $(1 - \alpha)$ is called the coverage probability. It lies between 0 and 1.

- A confidence interval gives the range of *plausible* values for θ. To be useful, it should
  - be narrow
  - have high coverage probability $(1 - \alpha)$

- To resolving these two conflicting requirements, the useful approach is to prescribe a small value for α, and then obtain the narrowest possible confidence interval.

What do we mean by confidence?

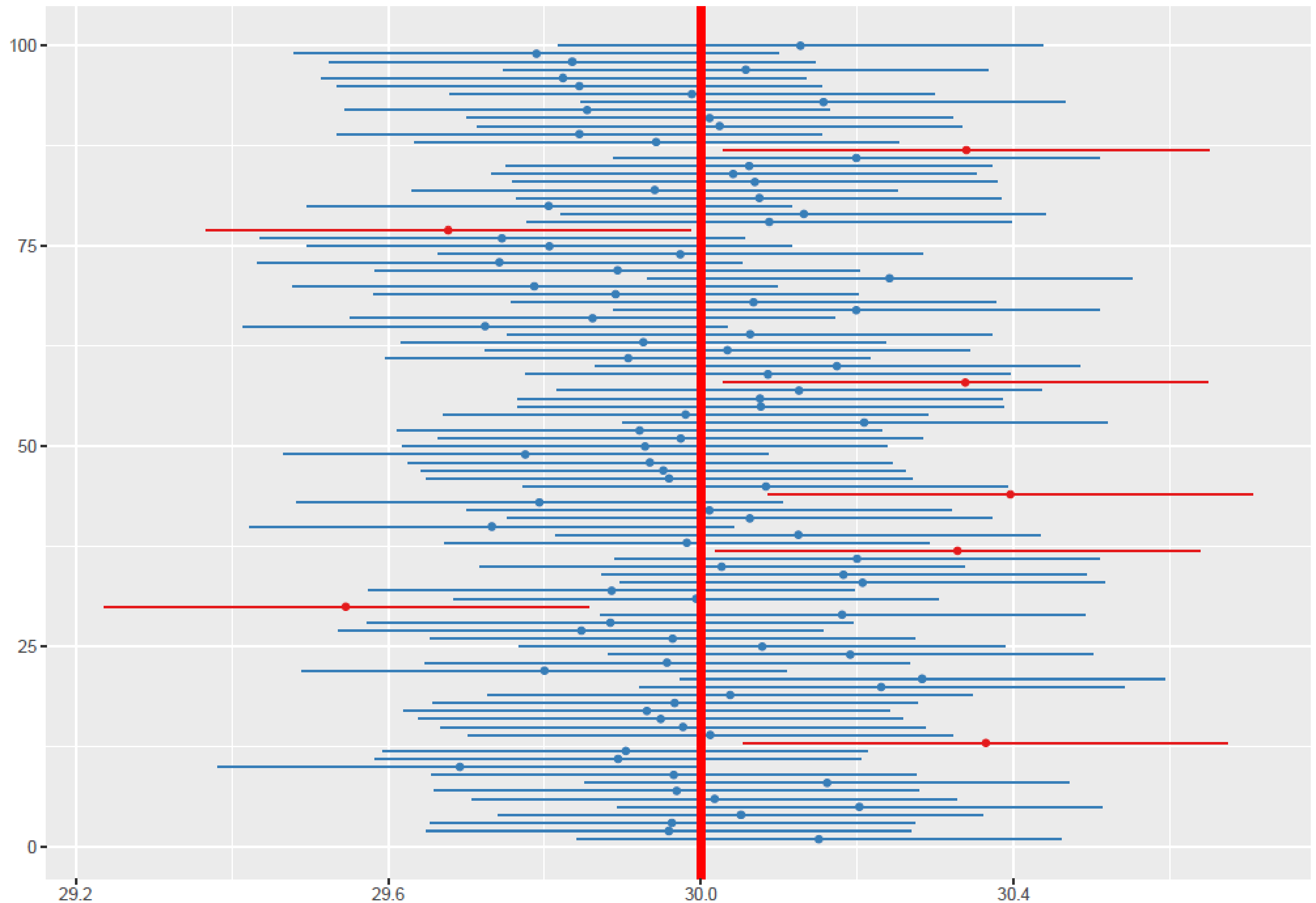
Suppose $\alpha = 0.05$.

✗ There is a 95% chance that the true value of θ falls within our confidence interval.

✗ The true value of θ will fall within our confidence interval 95% of the time.

✓ If samples of the same size are drawn repeatedly from the same population, and a confidence interval is calculated for each sample (using the same method), then it is expected that 95% of these intervals to contain the true value of θ.

What do we mean by confidence?



Lemma 3

Suppose $Y_1, Y_2, \ldots, Y_n$ are i.i.d. $N(\mu, \sigma^2)$, and let

$$Z = \frac{\bar{Y} - \mu}{\sigma/\sqrt{n}},$$

then

$$Z \sim \mathrm{N}(0,1).$$

Proof

Take $a_i = \frac{1}{n}$, then by Lemma 1, $\bar{Y} \sim N(\mu, \frac{\sigma^2}{n})$.

Standardizing $\bar{Y}$ gives $Z = \frac{\bar{Y} - E(\bar{Y})}{sd(\bar{Y})} = \frac{\bar{Y} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$.

(See next slide regarding standardising a normal random variable.)

Standardising a normal random variable

If $Y \sim N(\mu, \sigma^2)$, then $Z = \frac{Y-\mu}{\sigma} \sim N(0,1)$.

Proof:

We will use the method of transformation.

The density of Y is $f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(y-\mu)^2}$

The transformation $Z = h(Y) = \frac{Y-\mu}{\sigma}$ is a one-to-one function of Y.

The inverse function is $Y = h^{-1}(Z) = \sigma Z + \mu$.

Observe that $\left|\frac{dh^{-1}(z)}{dz}\right| = \sigma$.

Then $f_Z(z) = f_Y(h^{-1}(z)) \left|\frac{dh^{-1}(z)}{dz}\right|$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(\sigma z+\mu-\mu)^2} |\sigma|$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$ which is the standard normal distribution.

So $Z \sim N(0,1)$.

Example 1.6

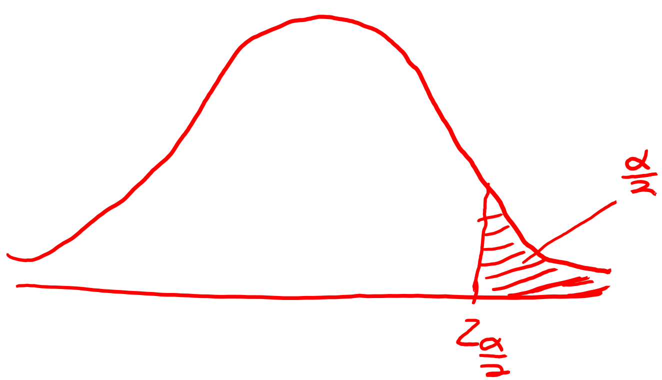
Suppose that $Y_1, Y_2, \ldots, Y_n$ are i.i.d. $N(\mu, \sigma^2)$ with σ^2 known, then

$$\left(\bar{y} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{y} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right),$$

where

$$P\left(Z > Z_{\alpha/2}\right) = \frac{\alpha}{2}, \text{for } Z \sim N(0,1),$$

is a $100(1-\alpha)\%$ confidence interval for μ.



Example 1.6 Proof

Show that $P(\bar{Y} - Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} < \mu < \bar{Y} + Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}) = 1 - \alpha$.

$$P(\bar{Y} - Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} < \mu < \bar{Y} + Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}})$$

$$= P\left(-Z_{\frac{\alpha}{2}} < \frac{\mu - \bar{Y}}{\frac{\sigma}{\sqrt{n}}} < Z_{\frac{\alpha}{2}}\right)$$

$$= P\left(-Z_{\frac{\alpha}{2}} < \frac{\bar{Y} - \mu}{\frac{\sigma}{\sqrt{n}}} < Z_{\frac{\alpha}{2}}\right)$$

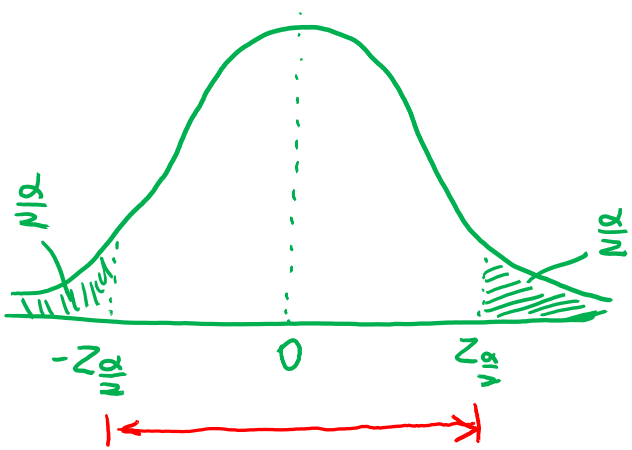
$$= P(-Z_{\frac{\alpha}{2}} < Z < Z_{\frac{\alpha}{2}}) \quad \text{where } Z \sim N(0,1).$$

$$= 1 - \frac{\alpha}{2} - \frac{\alpha}{2}$$

$$= 1 - \alpha$$

$\therefore$ This interval estimator has the correct coverage probability.

Hence $\bar{Y} \pm Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$ is the $100(1-\alpha)\%$ confidence interval for μ, as required.



Definition 1.8

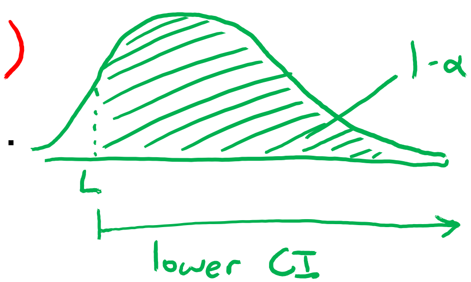
The random interval (L, U) in Definition 1.7, where

$$P(L < \theta < U) = 1 - \alpha$$

is a two-sided confidence interval.

It is possible to form a one-sided confidence interval such that

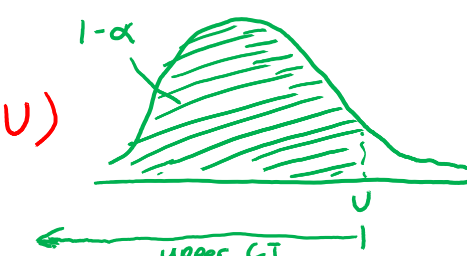
$$P(L < \theta) = 1 - \alpha. \quad (L, \infty)$$



This is the lower one-sided confidence interval.

Similarly, the upper one-sided confidence interval can be formed such that

$$P(\theta < U) = 1 - \alpha. \quad (-\infty, U)$$



Example 1.7

Suppose that $Y_1, Y_2, \ldots, Y_n$ are i.i.d. $N(\mu, \sigma^2)$ with σ^2 known. What is the $100(1-\alpha)\%$ upper and lower one-sided confidence interval for μ?

$$1-\alpha = P(\mu > L)$$
$$= P\left(\frac{\mu - \bar{Y}}{\frac{\sigma}{\sqrt{n}}} > \frac{L - \bar{Y}}{\frac{\sigma}{\sqrt{n}}}\right)$$
$$= P\left(\frac{\bar{Y} - \mu}{\frac{\sigma}{\sqrt{n}}} < \frac{\bar{Y} - L}{\frac{\sigma}{\sqrt{n}}}\right)$$
$$= P\left(Z < \frac{\bar{Y} - L}{\frac{\sigma}{\sqrt{n}}}\right) \text{ where } Z \sim N(0,1)$$
$$= P(Z < Z_\alpha)$$
$$\Rightarrow \quad Z_\alpha = \frac{\bar{Y} - L}{\frac{\sigma}{\sqrt{n}}}$$
$$L = \bar{Y} - Z_\alpha \frac{\sigma}{\sqrt{n}}$$

$$1-\alpha = P(\mu < U)$$
$$= P\left(\frac{\mu - \bar{Y}}{\frac{\sigma}{\sqrt{n}}} < \frac{U - \bar{Y}}{\frac{\sigma}{\sqrt{n}}}\right)$$
$$= P\left(\frac{\bar{Y} - \mu}{\frac{\sigma}{\sqrt{n}}} > \frac{\bar{Y} - U}{\frac{\sigma}{\sqrt{n}}}\right)$$
$$= P\left(Z > \frac{\bar{Y} - U}{\frac{\sigma}{\sqrt{n}}}\right)$$
$$= P(Z > -Z_\alpha)$$
$$\Rightarrow \quad -Z_\alpha = \frac{\bar{Y} - U}{\frac{\sigma}{\sqrt{n}}}$$
$$U = \bar{Y} + Z_\alpha \frac{\sigma}{\sqrt{n}}$$

$\therefore$ lower $(1-\alpha)100\%$ one-sided confidence interval for μ is $\left(\bar{Y} - Z_\alpha \frac{\sigma}{\sqrt{n}}, \infty\right)$

upper $(1-\alpha)100\%$ one-sided confidence interval for μ is $\left(-\infty, \bar{Y} + Z_\alpha \frac{\sigma}{\sqrt{n}}\right)$.

Example 1.8

The shopping times of $n = 64$ randomly selected customer at a supermarket were recorded. The mean shopping time of this sample is 33 minutes. Suppose that the variance of shopping time is 256 minutes, what is the 90% two-sided confidence interval for the true average shopping time per customer?

Let Y = shopping time per customer

μ = true average shopping time per customer

$n = 64$, $\sigma^2 = 256$, $\bar{y} = 33$, $1-\alpha = 0.90$

The two-sided CI for μ is $\left(\bar{y} - Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}, \bar{y} + Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\right)$

$$= \left(33 - Z_{\frac{\alpha}{2}} \sqrt{\frac{256}{33}}, 33 + Z_{\frac{\alpha}{2}} \sqrt{\frac{256}{33}}\right)$$

$$= (33 - Z_{0.05}\, 2, 33 + Z_{0.05}\, 2)$$

$$\approx (29.71, 36.29)$$

```
> qnorm(1-0.05)
[1] 1.644854
```

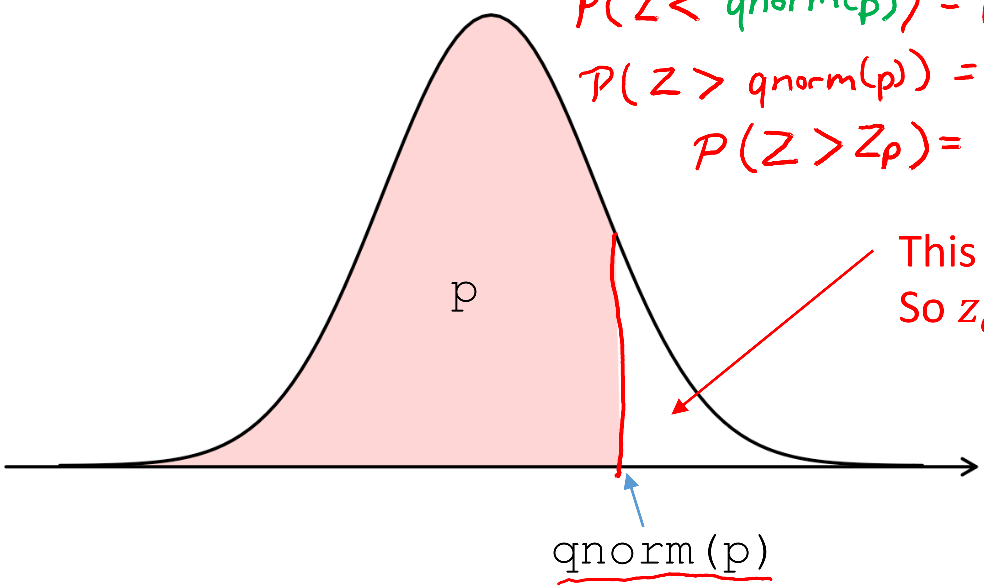
Some notes about using R to compute z_α

The R command `qnorm(p)` computes the *p*th percentile (i.e. the inverse of the cdf) of the normal distribution.

$P(Z < \text{qnorm}(p)) = p$

$P(Z > \text{qnorm}(p)) = 1-p$

$P(Z > Z_p) = 1-p$



This area is α (same as $1 - p$).
So z_α is given by `qnorm(1-α)`.

In Example 1.8, we had $1 - \alpha = 0.90$, implying $\alpha = 0.10$ and $\alpha/2 = 0.05$.
Hence, our $z_{\alpha/2}$ can be computed using `qnorm(0.95)`.

Alternatively, use `qnorm(0.05, lower.tail=FALSE)`