Pivotal quantities

Pivotal quantities

The preceding constructions of hypothesis tests and confidence intervals for μ and σ^2 are derived from the key distributional results

Inference for

$$\frac{\bar{Y} - \mu}{\sigma/\sqrt{n}} \sim N(0,1) \qquad \mu$$

$$\frac{\bar{Y} - \mu}{S/\sqrt{n}} \sim t_{n-1} \qquad \mu$$

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \qquad \sigma^2$$

These are all examples of pivotal quantities.

Definition 2.8

A random variable

$$H = H(Y_1, Y_2, \ldots, Y_n, ; \theta)$$

with a known distribution that does not depend on θ is called a pivotal quantity.

① a function of the sample and the unknown parameter θ

(e.g. statistic, estimator for θ)

② θ is the only unknown parameter

③ its distribution does not depend on θ

Pivotal method

① Find constants a and b such that

$$P(a < H(Y;\theta) < b) = 1-\alpha$$

② Rearrange to isolate θ, so that we get

$$P(l(a,b,Y) < \theta < u(a,b,Y)) = 1-\alpha$$

③ The $100(1-\alpha)\%$ confidence interval for θ is

$$(l(a,b,Y),\ u(a,b,Y)).$$

Pivotal quantities

Pivotal quantities are useful for constructing hypothesis tests and confidence intervals for θ . The general construction is similar to the construction of the confidence interval for σ^2.

Although the construction of tests and confidence intervals by this method is often straightforward, it must be kept in mind that the resulting tests and confidence intervals are not guaranteed to have optimal properties. For example, a test derived from pivotal quantity will have specified level of significance, but does not necessarily have high power.

Example 2.11

Suppose $Y_1, Y_2, \dots, Y_n$ is a sample of size n from an exponential distribution with mean θ. $Y_i \sim \text{iid } Exp(\frac{1}{\theta})$

a) Use the method of moment generating functions to show that $\frac{2}{\theta}\sum_{i=1}^{n} Y_i$ is a pivotal quantity and find its distribution.

b) Use the above pivotal quantity to construct a 95% symmetric confidence interval for θ.

c) If a sample size of $n = 7$ yields $\bar{y} = 4.77$, use the results from part (b) to give a 95% confidence interval for θ.

Example 2.11 Solution

a) $X = \frac{2}{\theta}\sum_{i=1}^{n} Y_i$. $\quad Y_i \sim \text{iid } Exp(\frac{1}{\theta})$. $\quad M_{Y_i}(t) = (1-\theta t)^{-1}$ for $t < \frac{1}{\theta}$.

$$M_X(t) = E\left[e^{tX}\right]$$

$$= E\left[e^{t\left(\frac{2}{\theta}\sum_{i=1}^{n} Y_i\right)}\right]$$

$$= E\left[e^{t\left(\frac{2}{\theta}Y_1\right)}\right] E\left[e^{t\left(\frac{2}{\theta}Y_2\right)}\right] \cdots E\left[e^{t\left(\frac{2}{\theta}Y_n\right)}\right] \text{ by independence}$$

$$= M_{Y_1}\left(\frac{2t}{\theta}\right) M_{Y_2}\left(\frac{2t}{\theta}\right) \cdots M_{Y_n}\left(\frac{2t}{\theta}\right)$$

$$= \left[1 - \cancel{\theta}\left(\frac{2t}{\cancel{\theta}}\right)\right]^{-n}$$

$$= (1-2t)^{-n} \quad \text{which is the MGF of } \chi^2_{2n}.$$

$$\therefore \quad X \sim \chi^2_{2n}.$$

Example 2.11 Solution

b) 95 % CI for θ

$$0.95 = P(L \leq \theta \leq U)$$

$$= P\left(\frac{1}{U} \leq \frac{1}{\theta} \leq \frac{1}{L}\right)$$

$$= P\left(\frac{2\sum_{i=1}^{n} Y_i}{U} \leq \frac{2\sum_{i=1}^{n} Y_i}{\theta} \leq \frac{2\sum_{i=1}^{n} Y_i}{L}\right)$$

$$= P\left(\boxed{\frac{2\sum_{i=1}^{n} Y_i}{U}} \leq X \leq \boxed{\frac{2\sum_{i=1}^{n} Y_i}{L}}\right) \quad \text{where } X \sim \chi^2_{2n}$$

Assume we want a symmetric CI:

$0.025 = P(\theta \leq L)$ and $0.025 = P(\theta \geq U)$

We know $P(X \leq \boxed{\chi^2_{2n,\,0.975}}) = 0.025$ and $P(X \geq \boxed{\chi^2_{2n,\,0.025}}) = 0.025$

This implies $\dfrac{2\sum_{i=1}^{n} Y_i}{U} = \chi^2_{2n,\,0.975}$ and $\dfrac{2\sum_{i=1}^{n} Y_i}{L} = \chi^2_{2n,\,0.025}$

$\Rightarrow U = \dfrac{2\sum_{i=1}^{n} Y_i}{\chi^2_{2n,\,0.975}}$ $\qquad L = \dfrac{2\sum_{i=1}^{n} Y_i}{\chi^2_{2n,\,0.025}}$

$\therefore$ The 95% CI for θ is $\left(\dfrac{2\sum_{i=1}^{n} Y_i}{\chi^2_{2n,\,0.025}},\ \dfrac{2\sum_{i=1}^{n} Y_i}{\chi^2_{2n,\,0.975}}\right)$.

Example 2.11 Solution

c) $n = 7$, $\bar{y} = 4.77$

$$\sum_{i=1}^{n} Y_i = n\bar{y} = 7 \times 4.77 = 33.39$$

The critical values are

$$\chi^2_{14,\,0.025} \approx 26.119 \qquad \text{qchisq}(0.975, 14)$$

$$\chi^2_{14,\,0.975} \approx 5.629 \qquad \text{qchisq}(0.025, 14)$$

Therefore, the 95% confidence interval for θ is

$$\left(\frac{2\sum_{i=1}^{n} Y_i}{\chi^2_{2n,\,0.025}} , \frac{2\sum_{i=1}^{n} Y_i}{\chi^2_{2n,\,0.975}} \right)$$

$$= \left(\frac{2 \times 33.39}{26.119} , \frac{2 \times 33.39}{5.629} \right)$$

$$\approx (2.557 , 11.864)$$