Transformation of parameters: confidence intervals

- We will look at how parameter transformation affects confidence intervals
- How can we get the Wald confidence interval for the transformed parameter ϕ based on the information about the original parameter θ?

Confidence interval for $\Phi(\theta)$

Suppose $y_1, y_2, \ldots, y_n$ are independent observations with log-likelihood function $\underline{\ell_\theta(\theta; \boldsymbol{y})}$. Consider the equivalent parameterization $\underline{\phi = \Phi(\theta)}$.

An approximate $100(1-\alpha)\%$ confidence interval for ϕ is given by

$$\left(\Phi(\hat\theta) - z_{\alpha/2}\boxed{\Phi'(\hat\theta)\sqrt{I_{\hat\theta}^{-1}}}, \hat\phi + z_{\alpha/2}\Phi'(\hat\theta)\sqrt{I_{\hat\theta}^{-1}}\right).$$

$$\left(\hat\phi - z_{\alpha/2}\boxed{\sqrt{I_{\hat\phi}^{-1}}},\ \hat\phi + z_{\alpha/2}\sqrt{I_{\hat\phi}^{-1}}\right)$$

(see next slide for explanation about this CI)

During the proof of Theorem 16, we have shown that

$$I_\theta = [\Phi'(\theta)]^2 I_\phi$$

$$[\Phi'(\theta)]^{-2} I_\theta = I_\phi$$

So the asymptotic distribution of $\hat{\phi}$ is

$$\hat{\phi} \longrightarrow N(\phi, I_\phi^{-1})$$

$$Z = \frac{\hat{\phi} - \phi}{\sqrt{I_\phi^{-1}}} \longrightarrow N(0,1)$$

We can use Z as a pivotal quantity to construct a CI for ϕ:

$$CI = \hat{\phi} \pm Z_{\frac{\alpha}{2}} \sqrt{I_\phi^{-1}}$$

$$= \Phi(\hat{\theta}) \pm Z_{\frac{\alpha}{2}} \, \Phi'(\theta) \sqrt{I_\theta^{-1}}$$

$$\approx \Phi(\hat{\theta}) \pm Z_{\frac{\alpha}{2}} \, \Phi'(\hat{\theta}) \sqrt{I_{\hat{\theta}}^{-1}}$$

Example 5.15

Suppose $y_1, y_2, \ldots, y_n$ are $i.i.d.$ Bernoulli observations with probability θ. Consider the log-odds $\phi = \log\left(\frac{\theta}{1-\theta}\right)$.
Find an approximate $100(1-\alpha)\%$ confidence interval for ϕ.

Recall $\hat{\theta} = \bar{y}$ and $S(\theta; y) = \frac{1}{\theta}\left(\sum_{i=1}^{n} y_i\right) - \left(\frac{1}{1-\theta}\right)\left(n - \sum_{i=1}^{n} y_i\right)$

$$\frac{\partial^2 \ell}{\partial \theta^2} = -\frac{1}{\theta^2}\left(\sum_{i=1}^{n} y_i\right) - \left(\frac{1}{1-\theta}\right)^2\left(n - \sum_{i=1}^{n} y_i\right)$$

$$I_\theta = E\left[-\frac{\partial^2 \ell}{\partial \theta^2}\right] = E\left[\frac{1}{\theta^2}\left(\sum_{i=1}^{n} y_i\right) + \left(\frac{1}{1-\theta}\right)^2\left(n - \sum_{i=1}^{n} y_i\right)\right]$$

$$= \frac{1}{\theta^2}\sum_{i=1}^{n} E(Y_i) + \left(\frac{1}{1-\theta}\right)^2\left(n - \sum_{i=1}^{n} E(Y_i)\right)$$

$$= \frac{1}{\theta^2}(n\theta) + \left(\frac{1}{1-\theta}\right)^2(n - n\theta)$$

$$= \frac{n}{\theta} + \frac{n\cancel{(1-\theta)}}{(1-\theta)^{\cancel{2}}}$$

$$= \frac{n}{\theta} + \frac{n}{1-\theta}$$

$$= \frac{n}{\theta(1-\theta)}$$

$$\phi = \Phi(\theta) = \log\left(\frac{\theta}{1-\theta}\right) = \log\theta - \log(1-\theta)$$

$$\Phi'(\theta) = \frac{1}{\theta} + \frac{1}{1-\theta} = \frac{1}{\theta(1-\theta)}$$

$$CI = \Phi(\hat{\theta}) \pm Z_{\frac{\alpha}{2}}\, \Phi'(\hat{\theta}) \sqrt{I_{\hat{\theta}}^{-1}}$$

$$= \Phi(\bar{y}) \pm Z_{\frac{\alpha}{2}}\, \Phi'(\bar{y}) \sqrt{I_{\bar{y}}^{-1}}$$

$$= \log\left(\frac{\bar{y}}{1-\bar{y}}\right) \pm Z_{\frac{\alpha}{2}} \left(\frac{1}{\bar{y}(1-\bar{y})}\right) \sqrt{\frac{\bar{y}(1-\bar{y})}{n}}$$

$$= \log\left(\frac{\bar{y}}{1-\bar{y}}\right) \pm Z_{\frac{\alpha}{2}} \frac{1}{\sqrt{n\bar{y}(1-\bar{y})}}$$