Assignment 1

STATS 2107

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Q1

a. For $E[X_i X_j]$:

- If $i \neq j$ then: $E[X_i X_j] = E[X_i]E[X_j]$

(Because X_i, X_j are independent)

$\therefore E[X_i X_j] = E[X_i]E[X_j] = \mu \cdot \mu = \mu^2$

- If $i = j$ then: $E[X_i X_j] = E[X_i^2]$

$= var(X_i) + E[X_i]^2$

$= \sigma^2 + \mu^2.$

$$\therefore E[X_i X_j] = \begin{cases} \mu^2 & \text{if } i \neq j \\ \mu^2 + \sigma^2 & \text{otherwise.} \end{cases}$$

b. $E[X_i \bar{X}] = E[X_i \frac{1}{n} \sum_{j=1}^{n} X_j]$

$= \frac{1}{n} \sum_{j=1}^{n} [X_i X_j]$

$= \frac{1}{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} E[X_i]E[X_j] + \frac{1}{n} E[X_i^2]$

$= \cancel{\frac{(n-1)}{n} E[X_i X_j]}$

$= \left(\frac{n-1}{n}\right)\mu^2 + \frac{1}{n}(\mu^2 + \sigma^2)$

$= \mu^2 + \frac{\sigma^2}{n}$

c. $E[S_{XX}] = \sum_{i=1}^{n} E[(X_i - \bar{X})^2]$

$$= \sum_{i=1}^{n} E[(X_i^2 - 2X_i\bar{X} + \bar{X}^2)]$$

$$= \sum_{i=1}^{n} (E[X_i^2] - 2E[X_i\bar{X}] + E[\bar{X}^2])$$

we have:

$$var(\bar{X}) = var\left(\frac{X_1 + X_2 + \ldots + X_n}{n}\right)$$

$$= \left(\frac{1}{n}\right)^2 var(X_1 + X_2 + \ldots + X_n)$$

$$= \left(\frac{1}{n}\right)^2 (\sigma^2 + \sigma^2 + \ldots + \sigma^2) \text{ (since } X_1, X_2, \ldots X_n \text{ are i.i.d)}$$

$$= \left(\frac{1}{n}\right)^2 (n\sigma^2) = \frac{\sigma^2}{n}$$

we also have:

$$E[\bar{X}^2] = var(\bar{X}) + E[\bar{X}]^2$$

$$= \frac{\sigma^2}{n} + \mu^2$$

$$\therefore E[S_{XX}] = n\left(\mu^2 + \sigma^2 - 2\left(\mu^2 + \frac{\sigma^2}{n}\right) + \frac{\sigma^2}{n} + \mu^2\right)$$

$$= n\left(\sigma^2 - \frac{2\sigma^2}{n} + \frac{\sigma^2}{n}\right)$$

$$= n\left(\sigma^2 - \frac{\sigma^2}{n}\right)$$

$$= \sigma^2(n-1)$$

$$\therefore E[S_{XX}] = \sigma^2(n-1)$$

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d. We have sample variance is:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$
$$= \frac{1}{n-1} S_{XX}$$

$$\therefore E[S^2] = E\left[\frac{1}{n-1} S_{XX}\right]$$
$$= \frac{1}{n-1} E[S_{XX}]$$
$$= \frac{1}{n-1}(n-1)\sigma^2$$
$$= \sigma^2$$

We have:

$$b_{S^2}(\sigma^2) = E[S^2] - \sigma^2$$
$$= \sigma^2 - \sigma^2$$
$$= 0$$

$\therefore$ The sample variance S^2 is unbiased for σ^2.

Q2

a. $$\hat{p}^2 = \left(\frac{X}{n}\right)^2$$

$$E[\hat{p}^2] = E\left[\left(\frac{X}{n}\right)^2\right]$$
$$= \frac{1}{n^2} E[X^2]$$
$$= \frac{1}{n^2}(\mathrm{var}(X) + E[X]^2)$$
$$= \frac{1}{n^2}(np(1-p) + (np)^2) \quad \text{(Because } X \sim Bin(n,p)\text{)}$$
$$= \frac{np}{n^2}(1-p+np) = \frac{p}{n}(1-p+np)$$

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$$= \frac{p(1-p)}{n} + p^2$$

$$b_{\hat{p}^2}(p^2) = \cancel{\frac{p(1-p)}{n} + p^2} - p^2 \; E[\hat{p}^2] - p^2 = \frac{p(1-p)}{n} + p^2 - p^2$$

$$= \frac{p(1-p)}{n}$$

$$\therefore E[\hat{p}^2] = \frac{p(1-p)}{n} + p^2 \text{ and } b_{\hat{p}^2}(p^2) = \frac{p(1-p)}{n}$$

b. We have:

$$\frac{d^k}{dt^k} M_X(t) \Big|_{t=0} = E[X^k]$$

or we can write:

$$E[X^k] = M_X^{(n)}(0) \text{ where } M_X^{(n)}(t) \text{ is the } n^{th} \text{ derivative of } M_X(t)$$

$\cancel{\therefore E[X] = M_X'}$

$$M'_X(t) = n(1-p+pe^t)^{n-1}(pe^t)$$

$$\therefore M'_X(0) = n(1-p+p)^{n-1}(p) = np$$

$$\therefore E[X] = M'_X(0) = np$$

$$M''_X(t) = n(n-1)(1-p+pe^t)^{n-2}(pe^t)^2 + n(1-p+pe^t)^{n-1}(pe^t)$$

$$\therefore M''_X(0) = n(n-1)(1-p+p)^{n-2}(p)^2 + n(1-p+p)^{n-1}(p)$$

$$= n(n-1)p^2 + np$$

$$\therefore E[X^2] = M''_X(0) = n(n-1)p^2 + np$$

$$M'''_X(t) = n(n-1)(n-2)(1-p+pe^t)^{n-3}(pe^t)^3 + 2n(n-1)(1-p+pe^t)^{n-2}(pe^t)^2 + n(n-1)(1-p+pe^t)^{n-2}(pe^t)^2 + n(1-p+pe^t)^{n-1}(pe^t)$$

$$= n(n-1)(n-2)(1-p+pe^t)^{n-3}(pe^t)^3$$
$$+ 3n(n-1)(1-p+pe^t)^{n-2}(pe^t)^2$$
$$+ n(1-p+pe^t)^{n-1}(pe^t)$$

$$\therefore M'''_X(0) = n(n-1)(n-2)(1-p+p)^{n-3}p^3$$
$$+ \cancel{3n(n-1)6p^2 + np} + 3n(n-1)(1-p+p)^{n-2}p^2 + n(1-p+p)^{n-1}(p)$$
$$= n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$$

$$\therefore E[X^3] = M'''_X(0) = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$$

$$M^{(4)}_X(t) = n(n-1)(n-2)(n-3)(1-p+pe^t)^{n-4}(pe^t)^4$$
$$+ 3n(n-1)(n-2)(1-p+pe^t)^{n-3}(pe^t)^3$$
$$+ 3n(n-1)(n-2)(1-p+pe^t)^{n-3}(pe^t)^3$$
$$+ 6n(n-1)(1-p+pe^t)^{n-2}(pe^t)^2$$
$$+ n(n-1)(1-p+pe^t)^{n-2}(pe^t)^2$$
$$+ n(1-p+pe^t)^{n-1}(pe^t)$$

$$= n(n-1)(n-2)(n-3)(1-p+pe^t)^{n-4}(pe^t)^4$$
$$+ 6n(n-1)(n-2)(1-p+pe^t)^{n-3}(pe^t)^3$$
$$+ 7n(n-1)(1-p+pe^t)^{n-2}(pe^t)^2$$
$$+ n(1-p+pe^t)^{n-1}(pe^t)$$

$$\therefore M^{(4)}_X(0) = n(n-1)(n-2)(n-3)(1-p+p)^{n-4}p^4$$
$$+ 6n(n-1)(n-2)(1-p+p)^{n-3}p^3$$
$$+ 7n(n-1)(1-p+p)^{n-2}p^2$$
$$+ n(1-p+p)^{n-1}p$$
$$= n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3$$
$$+ 7n(n-1)p^2 + np$$

$$\therefore E[X^4] = M^{(4)}_X(0) = n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3$$
$$+ 7n(n-1)p^2 + np$$

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~~$MSE_{\hat{p}^2}(p^2) = E[(\hat{p}^2 - p^2)^2]$~~

c. $MSE_{\hat{p}^2}(p^2) = var(\hat{p}^2) + (b_{\hat{p}^2}(p^2))^2$

$$var(\hat{p}^2) = var\left(\left(\frac{X}{n}\right)^2\right)$$

$$= \frac{1}{n^4} var(X^2)$$

$$= \frac{1}{n^4}\left(E[X^4] - E[X^2]^2\right)$$

$$= \frac{1}{n^4}\left[n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np - (n(n-1)p^2 + np)^2\right]$$

$$= \frac{1}{n^4}\left[(n^4 - 6n^3 + 11n^2 - 6n)p^4 + (6n^3 - 18n^2 + 12n)p^3 + (7n^2 - 7n)p^2 + np - (n^4 - 2n^3 + n^2)p^4 - (2n^3 - 2n^2)p^3 - n^2p^2\right]$$

$$= \frac{1}{n^4}\left[(-4n^3 + 10n^2 - 6n)p^4 + (4n^3 - 16n^2 + 12n)p^3 + (6n^2 - 7n)p^2 + np\right]$$

$$\therefore var(\hat{p}^2) = \frac{1}{n^3}\left[(-4n^2 + 10n - 6)p^4 + (4n^2 - 16n + 12)p^3 + (6n - 7)p^2 + p\right]$$

$$(b_{\hat{p}^2}(p^2))^2 = \left(\frac{p(1-p)}{n}\right)^2 = \frac{p^2(1-p)^2}{n^2} = \frac{p^2(1-2p+p^2)}{n^2}$$

$$= \frac{p^4 - 2p^3 + p^2}{n^2}$$

$$\therefore MSE_{\hat{p}^2}(p^2) = var(\hat{p}^2) + (b_{\hat{p}^2}(p^2))^2$$

$$= \frac{1}{n^3}\left[(-4n^2 + 10n - 6)p^4 + (4n^2 - 16n + 12)p^3 + (6n-7)p^2 + p\right] + \frac{p^4 - 2p^3 + p^2}{n^2}$$

$$\therefore \underline{MSE_{\hat{p}^2}(p^2)} = \frac{1}{n^3}\left[(-4n^2 + 11n - 6)p^4 + (4n^2 - 18n + 12)p^3 + (7n - 7)p^2 + p\right]$$

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d. $E\left[\frac{\hat{p}(1-\hat{p})}{n-1}\right]$

$= \frac{1}{n-1} E[\hat{p}(1-\hat{p})]$

$= \frac{1}{n-1} E[\hat{p} - \hat{p}^2]$

$= \frac{1}{n-1} (E[\hat{p}] - E[\hat{p}^2])$ (1)

$= \cancel{\frac{\quad}{n-1}}$

We have:

$E[\hat{p}] = E\left[\frac{X}{n}\right] = \frac{1}{n} E[X] = \frac{1}{n}(np) = p$ (2)

$\therefore$ ~~$V[\hat{p}]$~~

From (1) and (2) ~~$V[\hat{p}]$~~ and part (a):

$E\left[\frac{\hat{p}(1-\hat{p})}{n-1}\right] = \frac{1}{n-1}\left(p - \frac{p(1-p)}{n} - p^2\right)$

$= \frac{1}{n-1}\left(\frac{np - p + p^2 - p^2 n}{n}\right)$

$= \frac{p}{(n-1)n}(n - 1 + p - np)$

$= \frac{p}{(n-1)n}((n-1) \cancel{+p} - p(n-1))$

$= \frac{p}{(n-1)n}(n-1)(1-p)$

$= \frac{p(1-p)}{n}$

$\therefore E\left[\frac{\hat{p}(1-\hat{p})}{n-1}\right] = \frac{p(1-p)}{n}$

e. We have:

$$E\left[\frac{\hat{p}(1-\hat{p})}{n-1}\right] = \frac{p(1-p)}{n}$$

$$\therefore \cancel{\frac{1}{n-1} E[\hat{p}(1-\hat{p})] = \frac{1}{n}(p(1-p))}$$

$$\therefore \cancel{E[\hat{p} - \hat{p}^2] = \frac{n-1}{n}}$$

$$\therefore E\left[\frac{n}{n-1}(\hat{p} - \hat{p}^2)\right] = p - p^2$$

$$\therefore p^2 = p - E\left[\frac{n}{n-1}(\hat{p} - \hat{p}^2)\right]$$

And we know: $E[\hat{p}] = p$

$$\therefore p^2 = E[\hat{p}] - E\left[\frac{n}{n-1}(\hat{p} - \hat{p}^2)\right]$$

$$= E\left[\frac{\hat{p}(n-1) - n\hat{p} + n\hat{p}^2}{n-1}\right]$$

$$= E\left[\frac{n\hat{p} - \hat{p} - n\hat{p} + n\hat{p}^2}{n-1}\right]$$

$$= E\left[\frac{n}{n-1}\left(\hat{p}^2 - \frac{\hat{p}}{n}\right)\right]$$

$$\therefore E\left[\frac{n}{n-1}\left(\hat{p}^2 - \frac{\hat{p}}{n}\right)\right] - p^2 = p^2 - p^2 = 0$$

$$\therefore b_T(p^2) = 0 \text{ with } T = \frac{n}{n-1}\left(\hat{p}^2 - \frac{\hat{p}}{n}\right)$$

$\therefore \frac{n}{n-1}\left(\hat{p}^2 - \frac{\hat{p}}{n}\right)$ is the unbiased estimator for p^2.

f. We have

$$\operatorname{var}\left(\left(\frac{n}{n-1}\right)\left(\hat{p}^2-\frac{\hat{p}}{n}\right)\right)=\frac{1}{(n-1)^2}\operatorname{var}(n\hat{p}^2-\hat{p})$$

And:

$$\operatorname{var}(n\hat{p}^2-\hat{p}) = E[(n\hat{p}^2-\hat{p})^2]-E[n\hat{p}^2-\hat{p}]^2$$

$$= E[(n\hat{p}^2)^2-2n\hat{p}^3+\hat{p}^2]-(nE[\hat{p}^2]-E[\hat{p}])^2$$

$$= n^2E[\hat{p}^4]-2nE[\hat{p}^3]+E[\hat{p}^2]$$

$$-(nE[\hat{p}^2]-E[\hat{p}])^2.$$

And we know that:

$$E[\hat{p}] = p.$$

$$E[\hat{p}^2]=\frac{p(1-p)}{n}+p^2$$

$$E[\hat{p}^3]=E\left[\left(\frac{X}{n}\right)^3\right]$$

$$=\frac{1}{n^3}E[X^3]$$

And from (b) $\Rightarrow$ $E[\hat{p}^3]=\frac{1}{n^3}[n(n-1)(n-2)p^3+3n(n-1)p^2+np]$

$$=\frac{p}{n^2}[(n^2-3n+2)p^2+(3n-3)p+1]$$

∴ ~~we can conclude~~

We have:

$$nE[\hat{p}^2]-E[\hat{p}] = p(1-p)+np^2-p$$

$$= p-p^2+np^2-p = p^2(n-1)$$

∴

$$E[\hat{p}^4] = E\left[\left(\frac{X}{n}\right)^4\right]$$

$$= \frac{1}{n^4} E[X^4]$$

$$= \frac{1}{n^4}\left[n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np\right]$$

$$= \frac{p}{n^3}\left[(n^3-6n^2+11n-6)p^3 + (6n^2-18n+12)p^2 + (7n-7)p + 1\right]$$

∴ We can conclude:

$$\mathrm{var}(n\hat{p}^2 - \hat{p}) = \frac{p}{n}\left[(n^3-6n^2+11n-6)p^3 + (6n^2-18n+12)p^2 + (7n-7)p + 1\right] - \frac{2p}{n}\left[(n^2-3n+2)p^2 + (3n-3)p + 1\right] + \left[\frac{p}{n}(1-p) + p^2\right] - \left[p^2(n-1)\right]^2$$

$$= \frac{p}{n}\left[(n^3-6n^2+11n-6)p^3 + (4n^2-12n+8)p^2 + (2n-2)p - p^3(n^3-2n^2+n)\right]$$

$$= \frac{p^2}{n}\left[(-4n^2+10n-6)p^2 + (4n^2-12n+8)p + 2(n-1)\right]$$

$$= \frac{2p^2}{n}\left[(-2n^2+5n-3)p^2 + (2n^2-6n+4)p + (n-1)\right]$$

~~Var~~

Since $\frac{n}{n-1}\left(\hat{p}^2-\frac{\hat{p}}{n}\right)$ is the unbiased estimator for p^2.

$$\therefore MSE_{\frac{n}{n-1}\left(\hat{p}^2-\frac{\hat{p}}{n}\right)}(p^2)=var\left[\left(\frac{n}{n-1}\left(\hat{p}^2-\frac{\hat{p}}{n}\right)\right)\right]$$

$$=\frac{1}{(n-1)^2}var\left(n\hat{p}^2-\hat{p}\right)$$

$$=\frac{2p^2}{n(n-1)^2}\left[(-2n^2+5n-3)p^2+\right.$$

$$\left.+\ (2n^2-6n+4)p+(n-1)\right]$$

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