CIV_ENV 424 HW#3

Hao Yin

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2.8.2 Solve p_{cr} for a circular arch whose footings slide radially but cannot rotate (Fig. 2.39b).

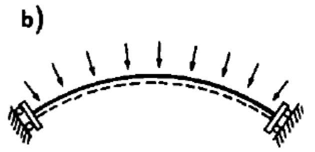

Figure 1: Figure 2.39b in the book

Solution:

From equation 2.8.4 in the book, the governing equation for the buckling of high arches is:

$$\frac{d^2w}{ds^2} + \frac{w}{R^2} = \frac{M}{EI} \tag{1}$$

Note that since we are considering the arc-length s as the variable in this problem, the curvilinear coordinate system (tangential-normal system of reference, $\mathbf{t}-\mathbf{n}$ system) is used here, as shown in Fugre 2.

Solve for the moment equilibrium of the cutting section about the cutting point and the moment equilibrium of the whole arch about its right end in the curvilinear coordinate system as shown in Figure 2 and 3:

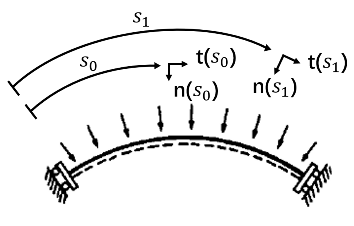

Figure 2: curvilinear coordinate system $\mathbf{t}-\mathbf{n}$

$$\begin{aligned} M_0 + M - P(\delta - w) = 0 \\ M = -M_0 + P(\delta - w) \end{aligned} \tag{2}$$

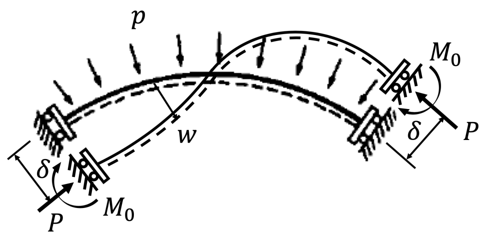

Figure 3: Free body diagram of whole arch

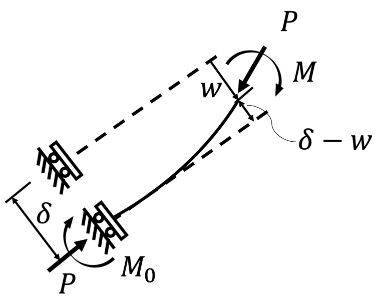

Figure 4: Free body diagram of the cutting section

$$\begin{aligned} M_0 + M_0 - P[\delta - (-\delta)] &= 0 \\ M_0 &= P\delta \end{aligned} \tag{3}$$

By substituting eq. (3) into the eq. (2) gives:

$$\begin{aligned} M_0 + M - P(\delta - w) &= 0 \\ M &= -Pw \end{aligned} \tag{4}$$

By substituting eq. (4) into the eq. (1) gives:

$$\frac{d^2 w}{ds^2} + \left(\frac{1}{R^2} + \frac{P}{EI} \right) w = 0 \tag{5}$$

the general solution of eq. (2) is:

$$w = A \sin ks + B \cos ks \tag{6}$$

consider the following B.Cs:
when $s = 0$,

$$\begin{aligned} & w' = 0 \\ & V = 0 \; (EIw''' + Pw' = 0) \end{aligned} \tag{7}$$

when $s = L$,

$$\begin{aligned} & w' = 0 \\ & V = 0 \; (EIw''' + Pw' = 0) \end{aligned} \tag{8}$$

Solve for $w'(0) = 0$, we have $A = 0$, solve for $w'(L) = 0$, we have

$$\begin{aligned}
-kB \sin kL &= 0 \\
\sin kL &= 0 \\
kL &= \pi, 2\pi, \cdots, n\pi \\
k^2 L^2 &= \pi^2, 4\pi^2, \cdots, n^2\pi^2 \\
\left(\frac{1}{R^2} + \frac{P}{EI}\right) &= \pi^2, 4\pi^2, \cdots, n^2\pi^2 \qquad (9) \\
P_{cr_n} &= \left(\frac{n^2\pi^2}{L^2/R^2} - 1\right)\frac{EI}{R^2} \\
p_{cr_n} &= \left(\frac{n^2\pi^2}{L^2/R^2} - 1\right)\frac{EI}{R^3} \quad n = 1, 2, 3, \cdots
\end{aligned}$$

3.2.2 Solve the problem of the free-standing two-bar massless column in Figure 3.5a for the case that the lower and upper bars have lengths $2l$ and l.

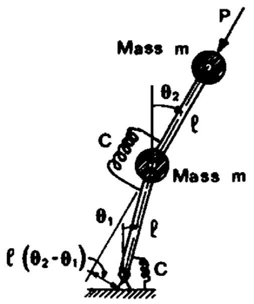

Figure 5: Figure 3.5a in the book

Solution:
The horizontal inertial force acting at the upper and lower point mass are $m2l\ddot{\theta}_1 + ml\ddot{\theta}_2 = ml(2\ddot{\theta}_1 + \ddot{\theta}_2)$ and $m2l\ddot{\theta}_1$.
The moment equilibrium equation of the entire column about the base and of the upper bar about the middle hinge are:

$$\begin{aligned}
ml\left(2\ddot{\theta}_1 + \ddot{\theta}_2\right)3l + 2ml\ddot{\theta}_1 2l + P2l\left(\theta_2 - \theta_1\right) + C\theta_1 &= 0 \\
ml\left(2\ddot{\theta}_1 + \ddot{\theta}_2\right)l + C\left(\theta_2 - \theta_1\right) &= 0
\end{aligned} \qquad (10)$$

the dynamic solution of equation (8) is in the form $\theta_1 = q_1 e^{i\omega t}$ and $\theta_2 = q_2 e^{i\omega t}$, substitute the dynamic solution in equation (8) we have:

$$\begin{bmatrix} (C - 2Pl) - 10ml^2\omega^2 & 2Pl - 3ml^2\omega^2 \\ -C - 2ml^2\omega^2 & C - ml^2\omega^2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (11)$$

set the determinant $\det M = 0$, we have:

$$x = ml^2\omega^2 = \frac{7C - 3Pl}{4} \pm \frac{\sqrt{3}\sqrt{(3C - Pl)(5C - 3Pl)}}{4} \qquad (12)$$

hence

a) For $P < \frac{5C}{3l}$,
   there are four real roots for ω, stable.

b) For $\frac{5C}{3l} < P < \frac{3C}{l}$,
   there are four complex roots for ω, flutter.

c) For $P > \frac{3C}{l}$,
   ω^2 are real but negative, there are four imaginary roots for ω, diverged.

4.4.1 Find the critical load of the structural systems in Figure 4.18a, b, c assuming that the axial force in the bars is less than the Euler load. For Figure 4.18b solve for (a) $C > 0$, EA finite ($< \infty$); and (b) $C = 0$, EA finite. For Figure 4.18c solve (a)$\Pi = \Pi(\theta)$ and $P = P(\theta)$; (b) critical loads; (c) stable regions, (d) limit cases: (1) $C \to 0$, $A_c > 0$; (2) $C > 0$, $A_c \to 0$.

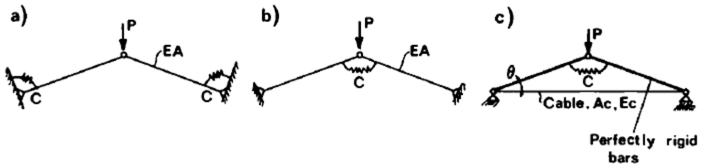

Figure 6: Figure 4.18abc in the book

Solution:

a) As shown in Figure 4, the axial strain of the bars is

$$\varepsilon = \frac{L\cos\alpha/\cos q - L}{L} = \frac{\cos\alpha}{\cos q} - 1 \tag{13}$$

the strain energy can be calculated as

$$\begin{aligned} U &= 2\cdot\frac{1}{2}C(\alpha-q)^2 + 2\int_V \frac{1}{2}E\varepsilon^2 dV \\ &= 2\cdot\frac{1}{2}C(\alpha-q)^2 + 2\int_l\int_A \frac{1}{2}E\varepsilon^2 dAdl \\ &= 2\cdot\frac{1}{2}C(\alpha-q)^2 + \frac{EAL}{\cos q}\varepsilon^2 \\ &= 2\cdot\frac{1}{2}C(\alpha-q)^2 + \frac{EAL}{\cos q}\left(\frac{\cos\alpha}{\cos q} - 1\right)^2 \end{aligned} \tag{14}$$

we find the potential energy is

$$\Pi(q) = U - Pw = 2\cdot\frac{1}{2}C(\alpha-q)^2 + \frac{EAL}{\cos q}\left(\frac{\cos\alpha}{\cos q} - 1\right)^2 - PL(\tan\alpha - \tan q) \tag{15}$$

note the small angle approximation $\sin\theta \approx \tan\theta \approx \theta$, $\cos\theta \approx 1 - \theta^2/2$ and $1/\cos\theta \approx 1 + \theta^2/2$

$$\begin{aligned} \Pi(q) \approx\ & 2\cdot\frac{1}{2}C(\alpha-q)^2 + EAL\left(1 + \frac{q^2}{2}\right)\left[\left(1 - \frac{\alpha^2}{2}\right)\left(1 + \frac{q^2}{2}\right) - 1\right]^2 \\ & - PL(\alpha - q) \end{aligned} \tag{16}$$

the critical load of the structure can be obtained by taking the second derivative of potential energy

$$
\begin{aligned}
&\frac{\partial^2\Pi}{\partial q^2} = 0 \Longrightarrow \\
&P_{cr} = -\frac{C\cos^2 q}{L\tan q} - \frac{6EA\cos^2\alpha}{\cos^2 q\sin q} - \frac{EA}{\sin q} + \frac{EA\cos q}{2\tan q} \\
&\quad -\frac{4EA\cos\alpha}{\tan q} + \frac{6EA\cos\alpha}{\cos q\sin q} + \frac{9EA\cos^2\alpha}{2\sin q} \\
&\quad \approx -\frac{C\cos^2 q}{Lq} - \frac{6EA\cos^2\alpha}{q\cos^2 q} - \frac{EA}{q} + \frac{EA\cos q}{2q} \\
&\quad -\frac{4EA\cos\alpha}{q} + \frac{6EA\cos\alpha}{q\cos q} + \frac{9EA\cos^2\alpha}{2q}
\end{aligned} \tag{17}
$$

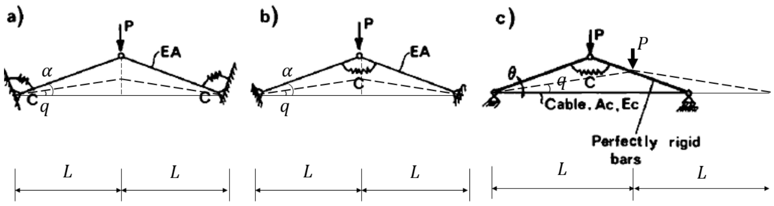

Figure 7: Dimensions of structures

b) Similar to the previous question, the only difference is the potential energy term of the spring, for this question, $C(2\alpha - 2q)^2/2$ instead of $2 \cdot C(\alpha - q)^2/2$. And the potential energy is

$$
\begin{aligned}
\Pi(q) &= U - Pw \\
&= \frac{C(2\alpha - 2q)^2}{2} + \frac{EAL}{\cos q}\left(\frac{\cos\alpha}{\cos q} - 1\right)^2 - PL(\tan\alpha - \tan q) \\
&\approx 2C(\alpha - q)^2 + EAL\left(1 + \frac{q^2}{2}\right)\left[\left(1 - \frac{\alpha^2}{2}\right)\left(1 + \frac{q^2}{2}\right) - 1\right]^2 \\
&\quad -PL(\alpha - q)
\end{aligned} \tag{18}
$$

the critical load of the structure can be obtained by taking the second derivative of potential energy

$$
\begin{aligned}
&\frac{\partial^2\Pi}{\partial q^2} = 0 \Longrightarrow \\
&P_{cr} = -\frac{2C\cos^3 q}{L\sin q} - \frac{6EA\cos^2\alpha}{\cos^2 q\sin q} - \frac{EA}{\sin q} + \frac{EA\cos^2 q}{2\sin q} \\
&\quad -\frac{4EA\cos\alpha\cos q}{\sin q} + \frac{6EA\cos\alpha}{\cos q\sin q} + \frac{9EA\cos^2\alpha}{2\sin q}
\end{aligned} \tag{19}
$$

when $C > 0$

$$\begin{aligned} P_{cr} = & -\frac{2C\cos^3 q}{L\sin q} - \frac{6EA\cos^2\alpha}{\cos^2 q\sin q} - \frac{EA}{\sin q} + \frac{EA\cos^2 q}{2\sin q} \\ & -\frac{4EA\cos\alpha\cos q}{\sin q} + \frac{6EA\cos\alpha}{\cos q\sin q} + \frac{9EA\cos^2\alpha}{2\sin q} \\ \approx & -\frac{2C\cos^2 q}{Lq} - \frac{6EA\cos^2\alpha}{q\cos^2 q} - \frac{EA}{q} + \frac{EA\cos q}{2q} \\ & -\frac{4EA\cos\alpha}{q} + \frac{6EA\cos\alpha}{q\cos q} + \frac{9EA\cos^2\alpha}{2q} \end{aligned} \tag{20}$$

when $C = 0$

$$\begin{aligned} P_{cr} = & -\frac{6EA\cos^2\alpha}{\cos^2 q\sin q} - \frac{EA}{\sin q} + \frac{EA\cos^2 q}{2\sin q} \\ & -\frac{4EA\cos\alpha\cos q}{\sin q} + \frac{6EA\cos\alpha}{\cos q\sin q} + \frac{9EA\cos^2\alpha}{2\sin q} \\ = & -\frac{6\cos^2\alpha}{\cos^2 q} - 1 + \frac{\cos^2 q}{2} \\ & -4\cos\alpha\cos q + \frac{6\cos\alpha}{\cos q} + \frac{9\cos^2\alpha}{2} \end{aligned} \tag{21}$$

c) As shown in Figure 4c, the axial strain of the cable is

$$\varepsilon_c = \frac{2L\cos q - 2L\cos\theta}{2L\cos\theta} = \frac{\cos q}{\cos\theta} - 1 \tag{22}$$

the strain energy can be calculated as

$$\begin{aligned} U &= \frac{C(2\theta - 2q)^2}{2} + 2\int_V \frac{1}{2}E_c\varepsilon^2 dV \\ &= \frac{C(2\theta - 2q)^2}{2} + 2\int_l\int_A \frac{1}{2}E_c\varepsilon_c^2 dAdl \\ &= \frac{C(2\theta - 2q)^2}{2} + E_cA_cL\varepsilon_c^2 \\ &= \frac{C(2\theta - 2q)^2}{2} + E_cA_cL\left(\frac{\cos q}{\cos\theta} - 1\right)^2 \end{aligned} \tag{23}$$

we find the potential energy is

$$\begin{aligned} \Pi(q) &= U - Pw \\ &= \frac{C(2\theta - 2q)^2}{2} + E_cA_cL\left(\frac{\cos q}{\cos\theta} - 1\right)^2 - PL(\tan\theta - \tan q) \\ &\approx 2C(\theta - q)^2 + E_cA_cL\left[\left(1 - \frac{q^2}{2}\right)\left(1 + \frac{\theta^2}{2}\right) - 1\right]^2 \\ &\quad -PL(\theta - q) \end{aligned} \tag{24}$$

when $C \to 0$, $A_c > 0$,

$$\Pi(q) \approx E_cA_cL\left[\left(1 - \frac{q^2}{2}\right)\left(1 + \frac{\theta^2}{2}\right) - 1\right]^2 - PL(\theta - q) \tag{25}$$

when $A_c \to 0$, $C > 0$

$$\Pi(q) \approx 2C(\theta - q)^2 - PL(\theta - q) \tag{26}$$

the equilibrium condition can be obtained by taking the derivative of potential energy:
when $C \to 0$, $A_c > 0$,

$$\begin{aligned} \frac{\partial \Pi}{\partial q} &= 0 \Longrightarrow \\ P &= -2E_cA_cq\left(\frac{\theta^2}{2}+1\right)\left[\left(\frac{q^2}{2}+1\right)\left(\frac{\theta^2}{2}+1\right)+1\right] \end{aligned} \tag{27}$$

when $A_c \to 0$, $C > 0$

$$\begin{aligned} \frac{\partial \Pi}{\partial q} &= 0 \Longrightarrow \\ P &= \frac{4C(\theta - q)}{L} \end{aligned} \tag{28}$$

the critical load of the structure can be obtained by taking the second derivative of potential energy:
when $C \to 0$, $A_c > 0$,

$$\frac{\partial^2 \Pi}{\partial q^2} = 0 \Longrightarrow \frac{E_cA_cL(\theta^2+2)(3q^2\theta^2+6q^2-2\theta^2)}{4} = 0 \tag{29}$$

when $A_c \to 0$, $C > 0$

$$\frac{\partial^2 \Pi}{\partial q^2} = 0 \Longrightarrow 4C = 0, \text{ the structure will not buckle.} \tag{30}$$

The limit points are at which $\partial P/\partial q = 0$:
when $C \to 0$, $A_c > 0$,

$$\begin{aligned} \partial P/\partial q = 0 \Longrightarrow 3q^2\theta^2 + 6q^2 - 2\theta^2 &= 0 \\ q &= \pm\frac{\sqrt{6}\theta}{3\sqrt{\theta^2+2}} \end{aligned} \tag{31}$$

when $A_c \to 0$, $C > 0$

$$\partial P/\partial q = 0 \Longrightarrow -\frac{4C}{L} = 0, \text{ the structure has no limit points.} \tag{32}$$

The stable regions are the parts where $\frac{\partial^2 \Pi}{\partial q^2} > 0$, from the last question, when $C \to 0$, $A_c > 0$,

$$\text{the stable regions are } -\frac{\sqrt{6}\theta}{3\sqrt{\theta^2+2}} < q < \frac{\sqrt{6}\theta}{3\sqrt{\theta^2+2}} \tag{33}$$

when $A_c \to 0$, $C > 0$, the structure is always stable.