CIV_ENV 430 HW#1

Hao Yin

May 18, 2022

2.8 Find the J integral for an infinite strip, of thickness b and width $2h$, with a symmetric semi-infinite crack subjected to imposed zero displacements on its lower face and constant vertical displacement u on its upper face (Fig. 2.1.12; Rice 1968a). Assume linear elasticity and plane strain with known elastic modulus E and Poisson's ratio v.

Solution:

The J-integral as shown in Fig. 1 reads:

$$J = \oint \left(\bar{U} \; \mathrm{d}y - \sigma_{ij} n_j \frac{\partial u_i}{\partial x} \; \mathrm{d}s \right) \tag{1}$$

on Γ_1: $n_1 = -1$, $n_2 = 0$, $\sigma_{ij} n_j = 0$, $\bar{U} = 0 \Longrightarrow J_{\Gamma_1} = 0$

on Γ_2: $n_1 = 0$, $n_2 = -1$, $\mathrm{d}y = 0$, $\sigma_{ij} = 0$, $\partial u_2 / \partial x = 0 \Longrightarrow J_{\Gamma_2} = 0$

on Γ_3: $n_1 = 1$, $n_2 = 0$, $\mathrm{d}s = \mathrm{d}y$, $\sigma_{11} = \sigma_{21} = 0$, $\varepsilon_{11} = \varepsilon_{12} = 0$, $\varepsilon_{22} = u/2h \Longrightarrow J_{\Gamma_3} = \int_{-h}^{h} \bar{U} \mathrm{d}y = 1/2 \int_{-h}^{h} \varepsilon_{22} \sigma_{22} \; \mathrm{d}y$

on Γ_4: x-symmetric with Γ_2, $J_{\Gamma_4} = J_{\Gamma_2} = 0$

on Γ_5: x-symmetric with Γ_1, $J_{\Gamma_5} = J_{\Gamma_1} = 0$

$$J = J_{\Gamma_3} = \frac{1}{2} \int_{-h}^{h} \varepsilon_{22} \sigma_{22} \; \mathrm{d}y \tag{2}$$

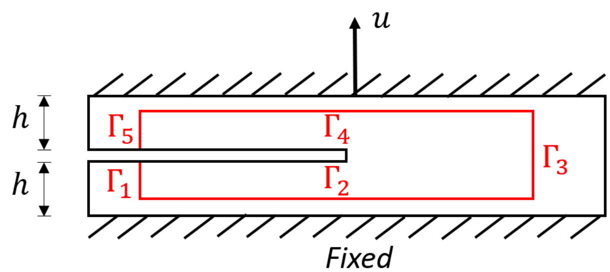

Figure 1: J-integral path of Problem 2.8

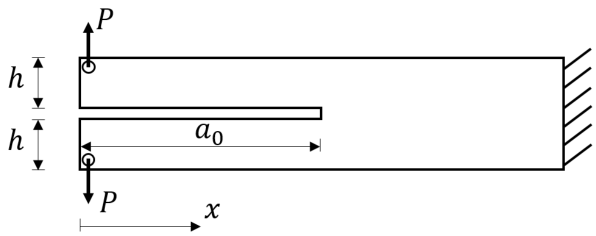

Figure 2: Double cantilever beam in Problem 2.9

For the plane strain problem, one assumes:

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \end{bmatrix} = \frac{1+\nu}{E} \begin{bmatrix} 1-\nu & -\nu \\ -\nu & 1-\nu \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \end{bmatrix} \tag{3}$$

substitute $\varepsilon_{11} = 0$ and $\varepsilon_{22} = \frac{u}{2h}$ back in Eq. (3):

$$\sigma_{22} = \frac{E}{1+\nu} \frac{u}{2h(1-2\nu)}(1-\nu) \tag{4}$$

hence:

$$J = \frac{1}{2}\int_{-h}^{h} \varepsilon_{22}\sigma_{22}\ \mathrm{d}y = J = \frac{1}{2}\int_{-h}^{h} \frac{u}{2h}\frac{E}{1+\nu}\frac{u}{2h(1-2\nu)}(1-\nu)\mathrm{d}y = \frac{u^2E(1-\nu)}{4h(1-2\nu)(1+\nu)} \tag{5}$$

2.9 A double cantilever beam specimen with arm depths $h = 10$ mm, thickness b = 10 mm, and initial crack length $a_0 = 50$ mm, is made of a material with a fracture energy $G_f = 180$ J/m^2 and an elastic modulus E = 250 GPa. The specimen is tested at a controlled displacement rate so that the load goes through the maximum and then decreases, at still increasing displacement, down to 25% of the peak load. When this point is reached, the specimen is completely unloaded. Assuming that LEFM and the beam theory apply, find the $P(u)$ and $\mathcal{G}(a)$ curves. Give the equations of the different arcs and the coordinates of the characteristic points.

Solution:

The displacement at the load point of the double cantilever beam shown in Fig. 2 can be obtained through the principle of virtual work:

$$u = \int_0^a \frac{M\bar{M}}{EI}\mathrm{d}x = \frac{Pa^3}{3EI} \tag{6}$$

where $\bar{M} = x$, $M = Px$. The complementary energy can be calculated as:

$$\Pi^* = 2\frac{Pu}{2} = \frac{P^2a^3}{3EI} \tag{7}$$

the energy release rate reads:

$$\mathcal{G} = \frac{1}{b}\left[\frac{\partial \Pi^*}{\partial a}\right]_P = \frac{P^2a^2}{bEI} \tag{8}$$

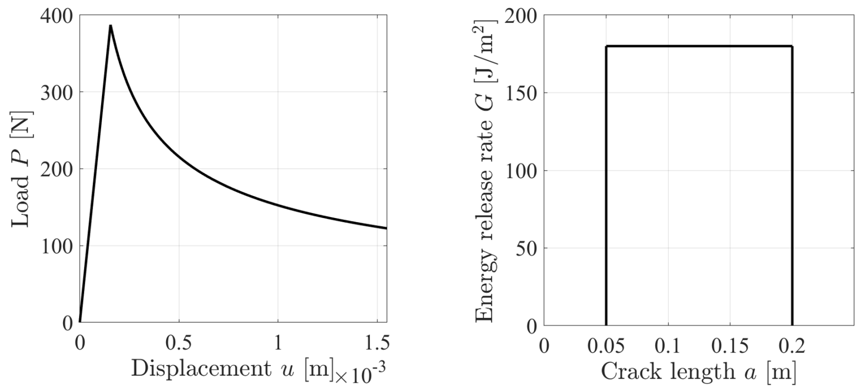

Figure 3: Problem 2.9: $P(u)$ curve (left), $\mathcal{G}(a)$ curve (right)

The external energy reads:

$$U = \frac{1}{2}Pu \tag{9}$$

when crack starts to extend, $\mathcal{G} = G_f$, $a = a_0$, E = 250 GPa, $I = bh^3/12 = 83.33$ mm^4, the corresponding $P_{max} = \sqrt{G_f EIb}/a_0 = 387.298$ N.

when $U = U^*$, combining Eq (7), (8) and Eq (9), one obtains:

$$\begin{aligned} \frac{1}{2}Pu &= \frac{P^2}{3EI}\sqrt{\frac{G_f bEI}{P^2}}\frac{G_f bEI}{P^2} \\ P &= \sqrt{\frac{2}{3}\left(G_f b\right)^{3/2}(EI)^{1/2}u^{-1/2}} = 4.82057u^{-1/2} \end{aligned} \tag{10}$$

The $P(u)$ curve is shown in Fig. 3 left.

The crack length at 25%P reads: $a_{\max} = \sqrt{G_f bEI/\left(0.25P_{\max}\right)^2} = 0.2$ m. The $\mathcal{G}(a)$ curve is shown in Fig. 3 right.

R2 a) Find Π^*, solve $\mathcal{G}$ and K.I.F., assume $h \ll L$.

b) Find Π^*, solve $\mathcal{G}$ and K.I.F., assume $h \ll L$.

Solution:

a) The double simply supported beam in Fig. 4a can be analyzed by half by the midspan-symmetry, it can be then split at the dash line and decomposed to bottom and top layers, as shown in Fig. 4b. The upper beam layer (highlighted in Fig. 4c) is subjected to a uniform moment M over the span, $M_{\text{up}}(x) = M$, the bending moment diagram is shown in Fig. 4d; the lower layer (highlighted in Fig. 4e) is subjected to the moment $-M$ and mid-span load P, $M_{\text{lo}}(x) = Px/2 - M$, the bending moment diagram is shown in Fig. 4f. The elastic beam differential equation reads:

$$\begin{aligned} M(x) &= EI\frac{\mathrm{d}\theta}{\mathrm{d}x} \\ \theta(x) &= \frac{1}{EI}\int_0^x M(x)dx + C \end{aligned} \tag{11}$$

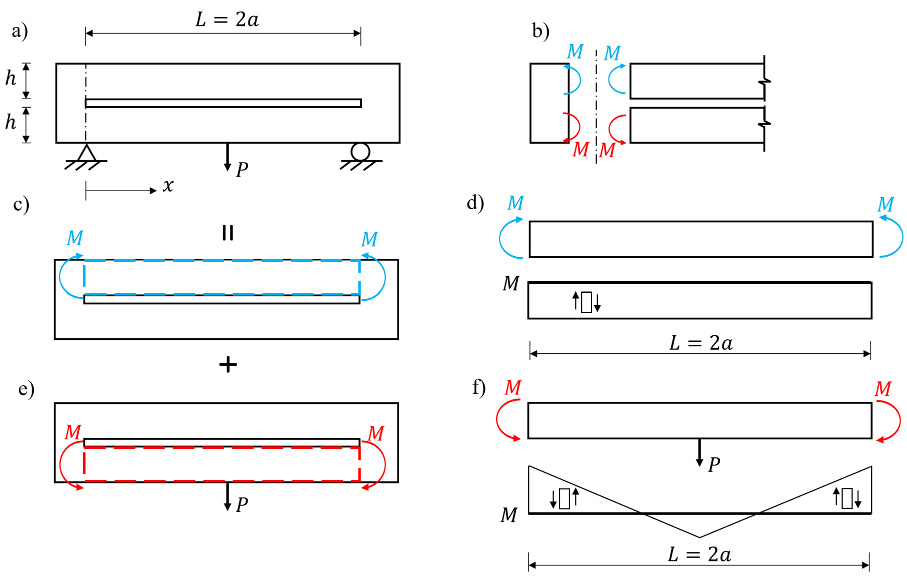

Figure 4: Double simply supported beam in Problem R2a

where E is the elastic modulus, $I = bh^3/12$, b is the width of the beam. For the upper layer, the slope reads:

$$\theta_{\mathrm{up}}(x) = \frac{Mx}{EI} + C_1 \tag{12}$$

by plugging in the boundary conditions $\theta_{\mathrm{up}}(a) = 0$, one get $C_1 = -Ma/EI$. For the lower layer, the slope reads:

$$\theta_{\mathrm{lo}}(x) = \frac{1}{EI}\left(\frac{Px^2}{4} - Mx\right) + C_2 \tag{13}$$

by plugging in the boundary condition $\theta_{\mathrm{lo}}(0) = \theta_{\mathrm{up}}(0)$, one gets $C_2 = -Ma/EI$; By plugging in the boundary condition $\theta_{\mathrm{lo}}(a) = 0$, one gets $M = Pa/8$. The expressions can be summarized as:

$$\begin{aligned} M_{\mathrm{up}}(x) &= \frac{Pa}{8} \\ M_{\mathrm{lo}}(x) &= \frac{P(4x-a)}{8} \\ \theta_{\mathrm{up}}(x) &= \frac{Pa(x-a)}{8EI} \\ \theta_{\mathrm{lo}}(x) &= \frac{P(x-a)(a+2x)}{8EI} \end{aligned} \tag{14}$$

The complementary energy can be calculated as:

$$
\begin{aligned}
\Pi^* &= 2\int_0^a \frac{M_{\mathrm{up}}^2}{2EI}\,\mathrm{d}x + 2\int_0^a \frac{M_{\mathrm{lo}}^2}{2EI}\,\mathrm{d}x \\
&= \frac{5P^2a^3}{96EI} = \frac{5P^2a^3}{8Ebh^3}
\end{aligned} \tag{15}
$$

the energy release rate reads:

$$
\mathcal{G} = \frac{1}{b}\left[\frac{\partial \Pi^*}{\partial a}\right]_P = \frac{15P^2a^2}{8Eb^2h^3} \tag{16}
$$

According to Irwin's relationship, the S.I.F, K_I, has the expression:

$$
K_I = \sqrt{E\mathcal{G}} = \sqrt{\frac{15}{8h}}\frac{Pa}{bh} \tag{17}
$$

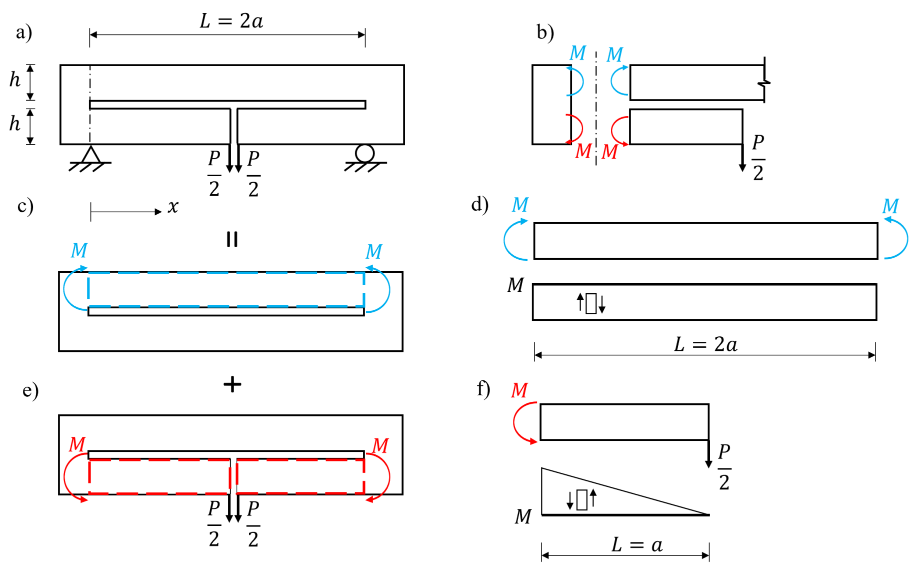

Figure 5: Double simply supported beam in Problem R2b

b) The beam shown in Fig. 5a can be analyzed by half by the midspan-symmetry, by splitting the beam at the dash line, it decomposes to bottom and top layers, as shown in Fig. 5b. The upper beam layer (highlighted in Fig. 5c) is subjected to a uniform moment M over the span, $M_{\mathrm{up}}(x) = M$, the bending moment diagram is shown in Fig. 5d; the lower layer (highlighted in Fig. 5e) is subjected to the moment $-M$ and mid-span load P, $M_{\mathrm{lo}}(x) = -P(a-x)/2$, the bending moment diagram is shown in Fig. 5f. By setting up the boundary condition $M_{\mathrm{up}}(0) = -M_{\mathrm{lo}}(0)$, one gets $M = Pa/2$.

The complementary energy can be directly calculated as:

$$\begin{aligned}\Pi^* &= 2\int_0^a \frac{M_{\mathrm{up}}^2}{2EI}\,\mathrm{d}x + 2\int_0^a \frac{M_{\mathrm{lo}}^2}{2EI}\,\mathrm{d}x \\ &= \frac{P^2a^3}{3EI} = \frac{4P^2a^3}{Ebh^3}\end{aligned} \tag{18}$$

where E is the elastic modulus, $I = bh^3/12$, b is the width of the beam. The energy release rate reads:

$$\mathcal{G} = \frac{1}{b}\left[\frac{\partial \Pi^*}{\partial a}\right]_P = \frac{12P^2a^2}{Eb^2h^3} \tag{19}$$

According to Irwin's relationship, the S.I.F, K_I, has the expression:

$$K_I = \sqrt{E\mathcal{G}} = \sqrt{\frac{12}{h}}\frac{Pa}{bh} \tag{20}$$

1 Appendix

Matlab code for HW#1:

```
clear;clc
%% Problem 2.9
% P-u curve
Gf = 180;
E = 250e9;
a0 = 50e-3;
b = 10e-3;
h = 10e-3;
I = b*h^3/12;

Pmax = sqrt(Gf*E*I*b)/a0;
u0 = (sqrt(2.0/3*(Gf*b)^(1.5)*(E*I)^(0.5))/Pmax)^2;
ue = linspace(0,u0);
Pe = Pmax/u0*ue;
u = linspace(u0,u0*10);
P = sqrt(2.0/3*(Gf*b)^(1.5)*(E*I)^(0.5))./sqrt(u);
figure()
p0 = plot(u,P,'k-','LineWidth',2);
hold on
pe = plot(ue,Pe,'k-','LineWidth',2);
xlabel("Displacement $u$ [m]",'Interpreter','latex')
ylabel('Load $P$ [N]','Interpreter','latex')
grid on;
box on
set(gcf,'color','w');
set(gca,'fontsize',18);
set(gca,'fontname','Times New Roman')
set(gca, 'LooseInset', [0,0,0,0]);
pbaspect([1 1 1])

% G-a curve
Gmax = 180;
```

```
a0 = 0.05;
amax = 0.20;
figure()
p1 = plot([a0,amax],[Gmax,Gmax],'k-','LineWidth',2);
hold on
pa = plot([a0,a0],[0,Gmax],'k-','LineWidth',2,'markers',2);
pb = plot([amax,amax],[0,Gmax],'k-','LineWidth',2,'markers',2);
xlabel("Crack length $a$ [m]",'Interpreter','latex')
ylabel('Energy release rate $G$ [J/m$^2$]','Interpreter','latex')
xlim([0, 0.25])
ylim([0, 200])
grid on;
box on
set(gcf,'color','w');
set(gca,'fontsize',18);
set(gca,'fontname','Times New Roman')
set(gca, 'LooseInset', [0,0,0,0]);
pbaspect([1 1 1])

%% Problem R2a
syms P a b EI x

M_upper = P*a/8;
M_lower = P*x/2 - P*a/8;

% theta_upper = P*a*(x-a)/8/EI;
% theta_lower = P*(a+2*x)*(x-a)/8/EI;

PIstar = 2*int(M_upper^2/2/EI,x,[0,a]) + 2*int(M_lower^2/2/EI,x,[0,a])

% u = int(M_upper*x/EI,0,a) + int(M_lower*x/EI,0,a)
% PIstar = P*u/2;
G = diff(PIstar,a)/b

%% Problem R2b
syms P a b EI x

M_upper = P*a/2;
M_lower = P*(x-a)/2;

PIstar = 2*int(M_upper^2/2/EI,x,[0,a]) + 2*int(M_lower^2/2/EI,x,[0,a])
G = diff(PIstar,a)/b
```