CIV_ENV 410 HW#3

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Problem 1. Derive the governing equation for circular cylindrical shells.

Solution:

Assumptions

1) Normal segments remain straight.
2) Segments orthogonal to the mid-plane remain orthogonal after deformation.
3) Normal segment has the same length after deformation.
4) $\sigma_z \approx 0$
5) Linear elastic behavior, first order theory.

Kinematics

The coordinate system used for circular cylindrical shell is shown in Figure 1. From symmetry we conclude that the component v of the displacement in the circumferential direction vanishes (here both v and φ represent the circumferential direction, the subscription v and φ are interchangeable in following sections). Thus, we only need to consider the components u and w in the x and z directions, respectively.

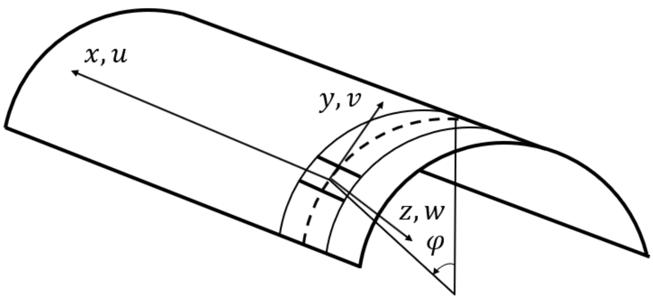


Figure 1: Coordinate system used for circular cylindrical shell

$$\begin{aligned} u(x, y, z) &= u \\ v(x, y, z) &= 0 \\ w(x, y, z) &= w \end{aligned}$$

Strains

$$\varepsilon_x = \frac{du}{dx}$$
$$\varepsilon_\varphi = \frac{ds' - ds}{ds} = \frac{rd\varphi - wd\varphi - r\varphi}{rd\varphi} = -\frac{w}{r}$$

Stresses

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_\varphi)$$
$$\sigma_\varphi = \frac{E}{1-\nu^2}(\varepsilon_\varphi + \nu\varepsilon_x) = E\varepsilon_\varphi = -E\frac{w}{r}$$

Stress Resultants

The diagram of the sign convention of the stress resultants is shown in Figure 2.

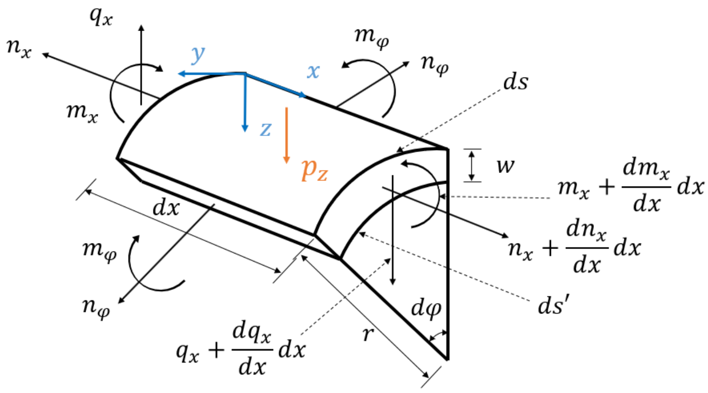


Figure 2: Sign convention of the stress resultants

$$n_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x dz = \frac{Eh}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_\varphi)$$
$$n_\varphi = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_\varphi dz = \frac{Eh}{1-\nu^2}(\varepsilon_\varphi + \nu\varepsilon_x) = -E\frac{w}{r}h$$

Considering the bending moments, we conclude from symmetry that there is no change in curvature in the circumferential direction (membrane behavior only).
The curvature in the x direction is equal to $-\frac{d^2w}{dx^2}$ (flexural behavior only). Using the same equations as for plates, we then obtain

$$m_x = -D\frac{d^2w}{dx^2}$$
$$m_\varphi = \nu m_x$$

where $D = \frac{E}{1-\nu^2} \cdot \frac{h^3}{12}$

Equilibrium

Assuming that the external forces consist only of a pressure normal to the surface, taking the equilibrium of force in x-direction and in z-dirction and moment about y-axis.

(1) Force in x-direction

$$\Sigma F_x = (n_x + \frac{dn_x}{dx}dx)rd\varphi \cdot w - n_x rd\varphi \cdot w = 0$$

Solve the equation

$$\frac{dn_x}{dx} = 0$$

(2) Moment about φ -axis

$$\begin{aligned}\Sigma M_\varphi =& m_x rd\varphi - (m_x + \frac{dm_x}{dx}dx)rd\varphi + p_z rd\varphi \cdot dx \cdot \frac{dx}{2} \\ &+2(n_\varphi \frac{d\varphi}{2})dx \cdot \frac{dx}{2} + (q_x + \frac{dq_x}{dx}dx)rd\varphi dx = 0\end{aligned}$$

Solve the equation

$$q_x = \frac{dm_x}{dx}$$

(3) Force in z-direction

$$\begin{aligned}\Sigma F_z =& - q_x rd\varphi + (q_x + \frac{dq_x}{dx}dx)rd\varphi \\ &+z(n_\varphi \frac{d\varphi}{2})dx + p_z dx \cdot rd\varphi = 0\end{aligned}$$

Solve the equation

$$\frac{dq_x}{dx} + \frac{n_\varphi}{r} + p_z = 0$$

From the first the equation, n_x is constant, and we take them equal to zero in our further discussion. Since $n_x = \frac{Eh}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_\varphi) = 0$, we obtain

$$\varepsilon_x = -\nu\varepsilon_\varphi = \nu\frac{w}{r}$$

and

$$\sigma_x = 0$$

Governing Equations

By solving the equations

$$\begin{cases} q_x = \frac{dm_x}{dx} \\ \frac{dq_x}{dx} + \frac{n_\varphi}{r} + p_z = 0 \\ m_x = -D\frac{d^2w}{dx^2} \end{cases}$$

we can get

$$D\frac{d^4w}{dx^4} + \frac{Eh}{r^2}w = p_z$$

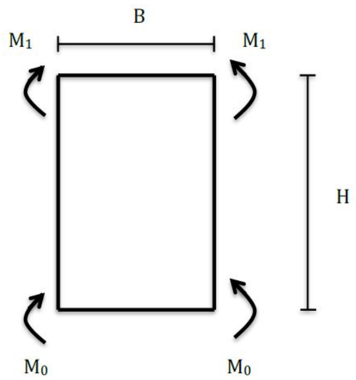
which is the governing equation for circular cylindrical shells.

Problem 2. Compute displacement, rotation, bending moment, and shear force distributions for a cylindrical shell subject to uniform edge moments at both edges by using

(1) the general exact solution (with 4 integration constants and coupled BCs at the two edges), and

(2) the approximated solution with uncoupled BCs at the two edges. Provide the analytical equations of the solution as well as their graphical representation.

Use $M_0 = 2$ kNm/m, $M_1 = 3$ kNm/m, $B = 2.4$ m, $H = 4$ m, thickness $h = 0.1$ m, modulus of elasticity $E_c = 30000$ MPa, Poisson ratio $\nu = 0.2$. Compare and comment on the two solutions.



Solution: The boundary conditions for this problem

$$m_x(x = 0) = M_0 = 2 \times 10^3 \text{ N}\cdot\text{ m/m}$$
$$m_x(x = 4) = -M_1 = -3 \times 10^3 \text{ N}\cdot\text{ m/m}$$
$$q_x(x = 0) = \frac{dm_x}{dx}(x = 0) = 0$$
$$q_x(x = 4) = \frac{dm_x}{dx}(x = 4) = 0$$

(1) The general exact solution (with 4 integration constants and coupled BCs at the two edges)

$$w_0 = e^{-\alpha x}[C_2e^{i\alpha x} + C_3e^{-i\alpha x}] + e^{\alpha x}[C_1e^{i\alpha x} + C_4e^{-i\alpha x}]$$

where

$$\alpha = \sqrt[4]{\frac{Eh}{4Dr^2}} = \frac{\sqrt[4]{3(1-\nu^2)}}{\sqrt{hr}} = \frac{\sqrt[4]{3(1-0.2^2)}}{\sqrt{(0.1)(2.4/2)}} = 3.76$$

Substituting the BCs into the equation above, we obtain

$$\begin{aligned}
C_1 &= -8.3836 \times 10^{-12} + 1.0096i \times 10^{-12} \\
C_2 &= -1.3576 \times 10^{-5} - 1.3576i \times 10^{-5} \\
C_3 &= -1.3576 \times 10^{-5} + 1.3576i \times 10^{-5} \\
C_4 &= -8.3836 \times 10^{-12} - 1.0096i \times 10^{-12}
\end{aligned}$$

Thus, the displacement

$$\Delta = -w_0 = e^{-\alpha x}[C_2 e^{i\alpha x} + C_3 e^{-i\alpha x}] + e^{\alpha x}[C_1 e^{i\alpha x} + C_4 e^{-i\alpha x}]$$

The rotation

$$\gamma = \frac{dw_0}{dx}$$

The bending moment

$$\begin{aligned}
m_x &= -D\frac{d^2 w_0}{dx^2} \\
m_\varphi &= \nu m_x
\end{aligned}$$

The shear force

$$q_x = \frac{dm_x}{dx} = -D\frac{d^3 w_0}{dx^3}$$

The expressions of the rotation, bending moment and the shear force from the exact solution are too long, please see the Appendix I for the full expressions.

(2) The approximated solution with uncoupled BCs at the two edges

Considering M_0 only,

$$w_{0b} = C_b e^{-\alpha x} \sin(\alpha x + \psi_b)$$

Substituting the BCs into the equation above, we obtain

$$\begin{aligned}
C_b &= 3.840 \times 10^{-5} \text{ or } -3.840 \times 10^{-5} \\
\psi_b &= -\frac{\pi}{4} \text{ or } \frac{3\pi}{4}
\end{aligned}$$

Considering M_1 only,

$$w_{0t} = C_t e^{-\alpha(-x)} \sin(\alpha(-x) + \psi_t)$$

Substituting the BCs into the equation above, we obtain

$$\begin{aligned}
C_t &= -1.689 \times 10^{-11} \text{ or } 1.689 \times 10^{-11} \\
\psi_t &= -\frac{\pi}{4} + 15.04 \text{ or } \frac{3\pi}{4} + 15.04
\end{aligned}$$

Here we take the values

$$
\begin{aligned}
C_b &= 3.840 \times 10^{-5} \\
\psi_b &= -\frac{\pi}{4} \\
C_t &= -1.689 \times 10^{-11} \\
\psi_t &= -\frac{\pi}{4} + 15.04
\end{aligned}
$$

By superposing these two solutions,

$$
\begin{aligned}
w_0 &= w_{0b} + w_{0t} \\
&= 3.840 \times 10^{-5} e^{-\alpha x} \sin(\alpha x - \frac{\pi}{4}) \\
&- 1.689 \times 10^{-11} e^{-\alpha(-x)} \sin(\alpha(-x) - \frac{\pi}{4} + 15.04)
\end{aligned}
$$

Thus, the displacement

$$
\begin{aligned}
\Delta &= -w_0 \\
&= -3.840 \times 10^{-5} e^{-\alpha x} \sin(\alpha x - \frac{\pi}{4}) \\
&+ 1.689 \times 10^{-11} e^{-\alpha(-x)} \sin(\alpha(-x) - \frac{\pi}{4} + 15.04)
\end{aligned}
$$

The rotation

$$
\begin{aligned}
\gamma &= \frac{dw_0}{dx} \\
&= (-\sqrt{2}\alpha) 3.840 \times 10^{-5} e^{-\alpha x} \sin(\alpha x - \frac{\pi}{2}) \\
&+ (\sqrt{2}\alpha) 1.689 \times 10^{-11} e^{-\alpha(-x)} \sin(\alpha(-x) - \frac{\pi}{2} + 15.04)
\end{aligned}
$$

The bending moment

$$
\begin{aligned}
m_x &= -D \frac{d^2 w_0}{dx^2} \\
&= (2D\alpha^2) 3.840 \times 10^{-5} e^{-\alpha x} \sin(\alpha x - \frac{3\pi}{4}) \\
&- (2D\alpha^2) 1.689 \times 10^{-11} e^{-\alpha(-x)} \sin(\alpha(-x) - \frac{3\pi}{4} + 15.04)
\end{aligned}
$$

The bending moment

$$
\begin{aligned}
m_\varphi &= \nu m_x \\
&= (2D\nu\alpha^2) 3.840 \times 10^{-5} e^{-\alpha x} \sin(\alpha x - \frac{3\pi}{4}) \\
&- (2D\nu\alpha^2) 1.689 \times 10^{-11} e^{-\alpha(-x)} \sin(\alpha(-x) - \frac{3\pi}{4} + 15.04)
\end{aligned}
$$

The shear force

$$
\begin{aligned}
q_x &= \frac{dm_x}{dx} = -D \frac{d^3 w_0}{dx^3} \\
&= (-2\sqrt{2}D\alpha^3) 3.840 \times 10^{-5} e^{-\alpha x} \sin(\alpha x - \pi) \\
&+ (2\sqrt{2}D\alpha^3) 1.689 \times 10^{-11} e^{-\alpha(-x)} \sin(\alpha(-x) - \pi + 15.04)
\end{aligned}
$$

The plots of the rotation, bending moment and the shear force calculated from the exact solution and the approximate solution are shown in Figure 3-7.

We can observe that the exact solution and the approximate solution are nearly the same, calculating the mean squared error (MSE) over all height span of the exact solution and the approximate solution

$$
\begin{aligned}
\mathrm{MSE}_{\Delta} &= 6.7790 \times 10^{-24} \\
\mathrm{MSE}_{\gamma} &= 1.5149 \times 10^{-22} \\
\mathrm{MSE}_{m_x} &= 8.0236 \times 10^{-9} \\
\mathrm{MSE}_{m_\varphi} &= 1.5149 \times 10^{-22} \\
\mathrm{MSE}_{q_x} &= 4.4530 \times 10^{-7}
\end{aligned}
$$

All MSE are small numbers, which means that the approximation we used for this problem is accurate and practical enough to capture the behavior of circular cylindrical shells.

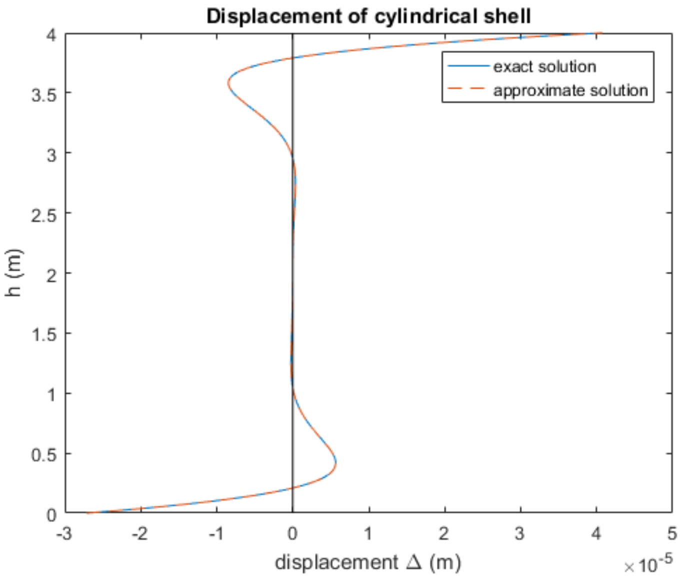


Figure 3: Displacement Δ of the cylindrical shell

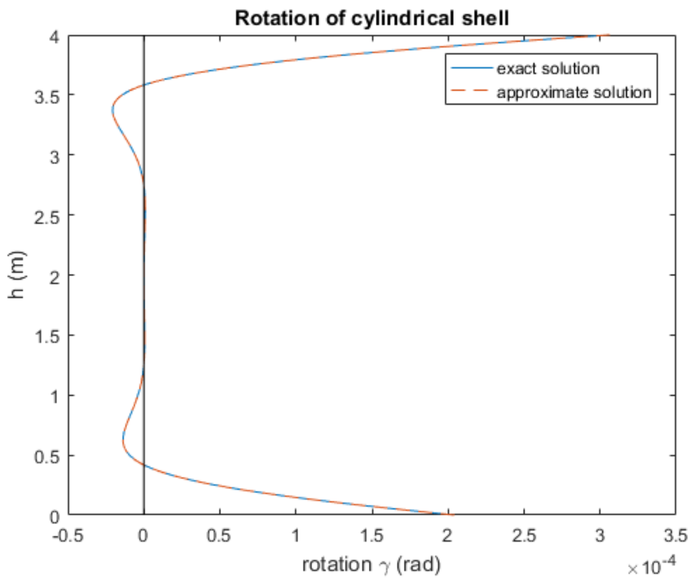


Figure 4: Rotation γ of the cylindrical shell

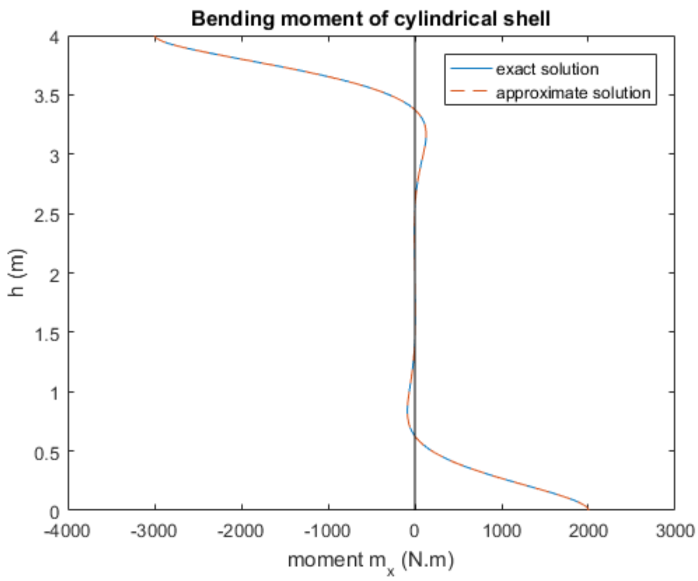


Figure 5: Bending moment m_x of the cylindrical shell

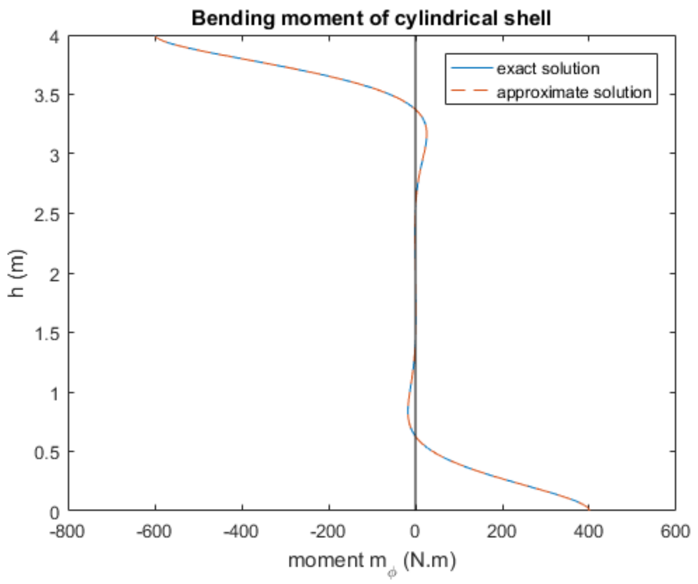


Figure 6: Bending moment m_φ of the cylindrical shell

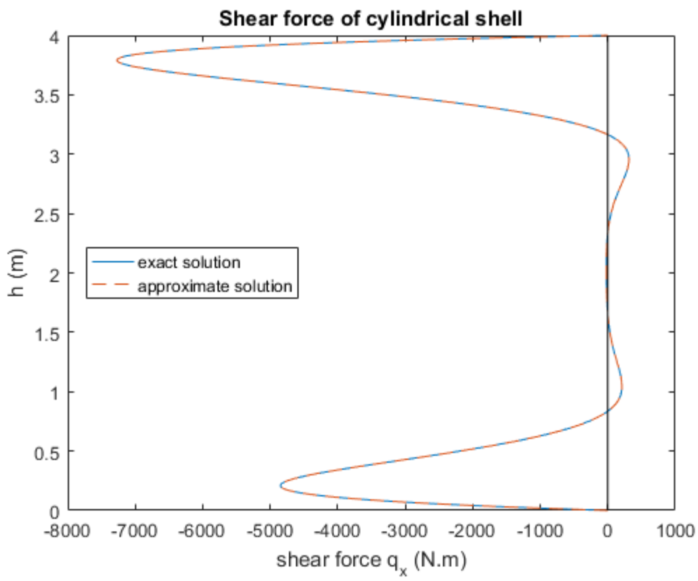


Figure 7: Shear force q_x of the cylindrical shell

Appendix I: Full Expressions

For the exact solution, the displacement:

$$\Delta = e^{-\alpha x}[C_2 e^{i\alpha x} + C_3 e^{-i\alpha x}] + e^{\alpha x}[C_1 e^{i\alpha x} + C_4 e^{-i\alpha x}]$$

The rotation:

$$\begin{aligned}\gamma &= e^{\alpha\, x}\, \left(C_1\, \alpha\, e^{\alpha\, x\, i}\, i - C_4\, \alpha\, e^{-\alpha\, x\, i}\, i\right) + e^{-\alpha\, x}\, \left(C_2\, \alpha\, e^{\alpha\, x\, i}\, i - C_3\, \alpha\, e^{-\alpha\, x\, i}\, i\right)\\ &\quad + \alpha\, e^{\alpha\, x}\, \left(C_1\, e^{\alpha\, x\, i} + C_4\, e^{-\alpha\, x\, i}\right) - \alpha\, e^{-\alpha\, x}\, \left(C_2\, e^{\alpha\, x\, i} + C_3\, e^{-\alpha\, x\, i}\right)\end{aligned}$$

The bending moment:

$$\begin{aligned}m_x &= D\, \left(e^{\alpha\, x}\, \left(C_1\, \alpha^2\, e^{\alpha\, x\, i} + \mathrm{C4}\, \alpha^2\, e^{-\alpha\, x\, i}\right) + e^{-\alpha\, x}\, \left(C_2\, \alpha^2\, e^{\alpha\, x\, i} + C_3\, \alpha^2\, e^{-\alpha\, x\, i}\right)\right.\\ &\quad - 2\, \alpha\, e^{\alpha\, x}\, \left(C_1\, \alpha\, e^{\alpha\, x\, i}\, i - \mathrm{C4}\, \alpha\, e^{-\alpha\, x\, i}\, i\right) + 2\, \alpha\, e^{-\alpha\, x}\, \left(C_2\, \alpha\, e^{\alpha\, x\, i}\, i - C_3\, \alpha\, e^{-\alpha\, x\, i}\, i\right)\\ &\quad - \alpha^2\, e^{\alpha\, x}\, \left(C_1\, e^{\alpha\, x\, i} + \mathrm{C4}\, e^{-\alpha\, x\, i}\right) - \alpha^2\, e^{-\alpha\, x}\, \left(C_2\, e^{\alpha\, x\, i} + C_3\, e^{-\alpha\, x\, i}\right)\end{aligned}$$

The shear force:

$$\begin{aligned}q_x &= D\, \left(e^{\alpha\, x}\, \left(C_1\, \alpha^3\, e^{\alpha\, x\, i}\, i - \mathrm{C4}\, \alpha^3\, e^{-\alpha\, x\, i}\, i\right) + e^{-\alpha\, x}\, \left(C_2\, \alpha^3\, e^{\alpha\, x\, i}\, i - C_3\, \alpha^3\, e^{-\alpha\, x\, i}\, i\right)\right.\\ &\quad - 3\, \alpha^2\, e^{\alpha\, x}\, \left(C_1\, \alpha\, e^{\alpha\, x\, i}\, i - \mathrm{C4}\, \alpha\, e^{-\alpha\, x\, i}\, i\right) - 3\, \alpha^2\, e^{-\alpha\, x}\, \left(C_2\, \alpha\, e^{\alpha\, x\, i}\, i - C_3\, \alpha\, e^{-\alpha\, x\, i}\, i\right)\\ &\quad - \alpha^3\, e^{\alpha\, x}\, \left(C_1\, e^{\alpha\, x\, i} + \mathrm{C4}\, e^{-\alpha\, x\, i}\right) + \alpha^3\, e^{-\alpha\, x}\, \left(C_2\, e^{\alpha\, x\, i} + C_3\, e^{-\alpha\, x\, i}\right)\\ &\quad + 3\, \alpha\, e^{\alpha\, x}\, \left(C_1\, \alpha^2\, e^{\alpha\, x\, i} + \mathrm{C4}\, \alpha^2\, e^{-\alpha\, x\, i}\right) - 3\, \alpha\, e^{-\alpha\, x}\, \left(C_2\, \alpha^2\, e^{\alpha\, x\, i} + C_3\, \alpha^2\, e^{-\alpha\, x\, i}\right)\end{aligned}$$

Appendix II: Matlab Code

```
clear;clc

M0=2000;
M1=3000;
B=2.4;
H=4;
h=0.1;
Ec=30000*10^6;
nu=0.2;
r=B/2;
D=Ec*h^3/(12*(1-nu^2));
alpha=((3*(1-nu^2))^0.25)/(h*r)^0.5;

%% exact solution
syms x C2 C1 C3 C4
w0_syms=exp(-alpha*x)*(C2*exp(1i*alpha*x)+C3*exp(-1i*
    alpha*x))+exp(alpha*x)*(C1*exp(1i*alpha*x)+C4*exp(-1i*
    alpha*x));
dw1_syms=diff(w0_syms,x); % dw0/dx
dw2_syms=diff(dw1_syms); % d2w/dx2
dw3_syms=diff(dw2_syms); % d3w/dx3

mx1_syms=subs(-D*dw2_syms,x,0);
mx2_syms=subs(-D*dw2_syms,x,H);
qx1_syms=subs(-D*dw3_syms,x,0);
qx2_syms=subs(-D*dw3_syms,x,H);

[C1,C2,C3,C4]=solve(mx1_syms==M0,mx2_syms==-M1,qx1_syms
    ==0,qx2_syms==0);
C1_double = double(C1);
C2_double = double(C2);
C3_double = double(C3);
C4_double = double(C4);
w0=exp(-alpha*x)*(C2_double*exp(1i*alpha*x)+C3_double*exp
    (-1i*alpha*x))+exp(alpha*x)*(C1_double*exp(1i*alpha*x)
    +C4_double*exp(-1i*alpha*x));
dw1=diff(w0,x);
dw2=diff(dw1);
dw3=diff(dw2);

w_func_exact = matlabFunction(w0);
gamma_func_exact = matlabFunction(diff(w0,x));
mx_func_exact = matlabFunction(-D*dw2);
qx_func_exact = matlabFunction(-D*dw3);

y = 0:0.01:4;
w_exact = w_func_exact(y);
```

```
gamma_exact = gamma_func_exact(y);
mx_exact = mx_func_exact(y);
qx_exact = qx_func_exact(y);

%% approximate solution
syms x Cb psib
w0b = Cb*exp(-alpha*x)*sin(alpha*x+psib);
dw1b_syms=diff(w0b,x);
dw2b_syms=diff(dw1b_syms);
dw3b_syms=diff(dw2b_syms);
mx1b_syms=subs(-D*dw2b_syms,x,0);
mx2b_syms=subs(-D*dw2b_syms,x,H);
qx1b_syms=subs(-D*dw3b_syms,x,0);
qx2b_syms=subs(-D*dw3b_syms,x,H);
[Cb,psib]=solve(mx1b_syms==M0,qx1b_syms==0);

syms x Ct psit
w0t = Ct*exp(-alpha*(-x))*sin(alpha*(-x)+psit);
dw1t_syms=diff(w0t,x);
dw2t_syms=diff(dw1t_syms);
dw3t_syms=diff(dw2t_syms);
mx1t_syms=subs(-D*dw2t_syms,x,0);
mx2t_syms=subs(-D*dw2t_syms,x,H);
qx1t_syms=subs(-D*dw3t_syms,x,0);
qx2t_syms=subs(-D*dw3t_syms,x,H);
[Ct,psit]=solve(mx2t_syms==-M1,qx2t_syms==0);

% two set of C and psi here
Cb_double = double(Cb(1));
psib_double = double(psib(1));
Ct_double = double(Ct(1));
psit_double = double(psit(1));
%Cb_double = double(Cb(2));
%psib_double = double(psib2));
%Ct_double = double(Ct(2));
%psit_double = double(psit(2));

w_0 = Cb_double*exp(-alpha*x)*sin(alpha*x+psib_double);
w_1 = Ct_double*exp(alpha*x)*sin(-alpha*x+psit_double);
gamma_0 = diff(w_0);
gamma_1 = diff(w_1);
mx_0 = -D*diff(diff(w_0));
mx_1 = -D*diff(diff(w_1));

qx_func = -D*diff(diff(diff(w_0+w_1)));
mx_func = matlabFunction(mx_0+mx_1);
w_func = matlabFunction(w_0+w_1);
qx_func = matlabFunction(qx_func);
gamma_func = matlabFunction(gamma_0+gamma_1);

```

```
y = 0:0.01:4;
w_approx = w_func(y);
gamma_approx = gamma_func(y);
mx_approx = mx_func(y);
qx_approx = qx_func(y);


MSE_disp = mean((w_exact - w_approx).^2)
MSE_gamma = mean((gamma_exact - gamma_approx).^2)
MSE_mx = mean((-mx_exact + mx_approx).^2)
MSE_mphi = mean((gamma_exact - gamma_approx).^2)
MSE_qx = mean((qx_exact - qx_approx).^2)

figure()
plot(w_exact,y,w_approx,y,'--')
line([0 0],[0 4],'Color','k')
xlabel('displacement \Delta (m)')
ylabel('h (m)')
legend('exact solution','approximate solution')
title('Displacement of cylindrical shell')

figure()
plot(gamma_exact,y,gamma_approx,y,'--')
line([0 0],[0 4],'Color','k')
xlabel('gamma_func \gamma (rad)')
ylabel('h (m)')
legend('exact solution','approximate solution')
title('gamma_func of cylindrical shell')

figure()
plot(-mx_exact,y,-mx_approx,y,'--')
line([0 0],[0 4],'Color','k')
xlabel('moment m_x (N.m)')
ylabel('h (m)')
legend('exact solution','approximate solution')
title('Bending moment of cylindrical shell')

figure()
plot(-nu*mx_exact,y,-nu*mx_approx,y,'--')
line([0 0],[0 4],'Color','k')
xlabel('moment m_{\phi} (N.m)')
ylabel('h (m)')
legend('exact solution','approximate solution')
title('Bending moment of cylindrical shell')

figure()
plot(qx_exact,y,qx_approx,y,'--')
line([0 0],[0 4],'Color','k')
xlabel('qx_func force q_x (N.m)')
ylabel('h (m)')
```

```
legend('exact solution','approximate solution','Location'
    ,'West')
title('qx_func force of cylindrical shell')
```