Graph theory

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Presentation overview

Heads Up!

Graphs and adjacency matrices

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What is a graph?

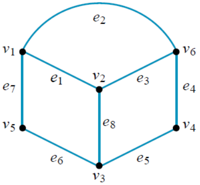
Definition

Let there exists a graph G, it consists of:

1. a finite non-empty set of elements called vertices
2. a finite set of elements called edges
3. an edge-endpoint function that indicates the endpoints of each edge - this function maps each edge to a set of either one or two vertices.

Vertex(Vertices) - Points
Edge - Lines between vertices

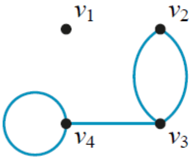
Matrix representation - adjacency matrix



Edge	Endpoints
e_1	$\{v_1, v_2\}$
e_2	$\{v_1, v_6\}$
e_3	$\{v_2, v_6\}$
e_4	$\{v_4, v_6\}$
e_5	$\{v_3, v_4\}$
e_6	$\{v_3, v_5\}$
e_7	$\{v_1, v_5\}$
e_8	$\{v_2, v_3\}$

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

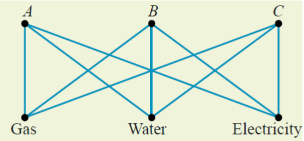
Matrix representation - adjacency matrix



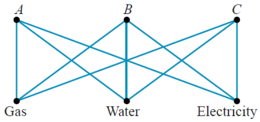
$$\begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

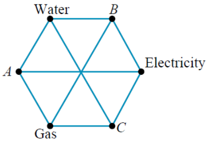
Example

The following graph represents three houses, A, B and C, that are each connected to three utilities, gas (G), water (W) and electricity (E). Construct the adjacency matrix for this graph.



Different pattern(s)

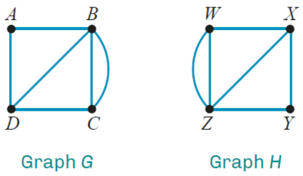




Isomorphism

Definition

There exists Graph H and G if and only if graph H can be obtained from graph G by simply relabelling its vertices.

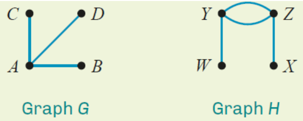


By relabelling, we can see the relations of:

$$A \leftrightarrow Y, \quad B \leftrightarrow Z, \quad C \leftrightarrow W, \quad D \leftrightarrow X$$

Example

Give three reasons why the two graphs shown on the right could not possibly be isomorphic.



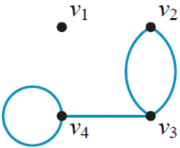
Possible solutions:

1. Graph G does not have multiple edges, while graph H has multiple edges.
2. Graph G has three edges, while graph H has four edges.
3. Graph G has one vertex where three edges meet, while graph H has two such vertices.

Degree of a vertex

Definition

Let v be a vertex of graph G. The degree of v (deg(v))= number of edges that have vertex v as an endpoint.
Each edge with a loop counts twice.



Vertex	Degree
v_1	0
v_2	2
v_3	3
v_4	3

Handshaking Lemma

Note: Lemma is one of a smaller proof to prove on a larger theorem when combined together.

Handshaking Lemma

The total degree of any graph is equal to twice the number of edges of the graph.

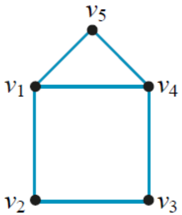
Proof:

Handshaking Lemma

Catogarising Graphs

Simple Graph

A graph with no loops or multiple edges.

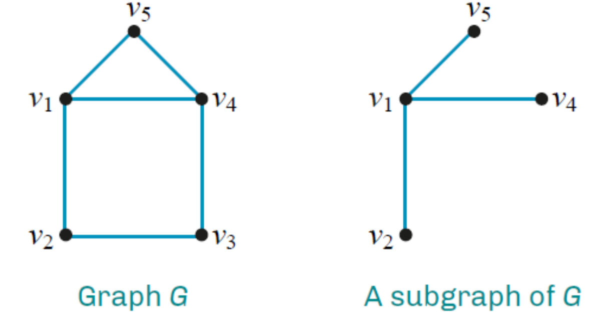


$$\begin{array}{c c} & \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Catogarising Graphs

Subgraph

A graph with no loops or multiple edges.



Exercise 12A

Euler circuits

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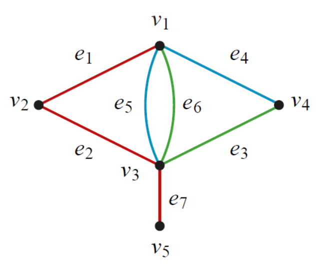
Before we go to Euler circuit

Let's understand what are walks in graph!

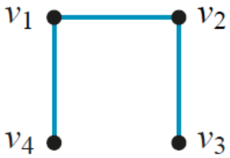
Walks in graphs

A walk in a graph is an alternating sequence of vertices and edges
$v_1, e_1, v_2, e_2, \ldots, v_{n-1}, e_{n-1}, v_n$
where the edge e_i joins the vertices v_i and v_{i+1}.

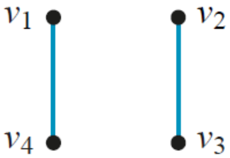
If each pair of adjacent vertices in a walk is joined by only one edge, then the walk can be described by the sequence of vertices $v_1, v_2, \ldots, v_n$.



Connect vs disconnect graphs



A connected graph G



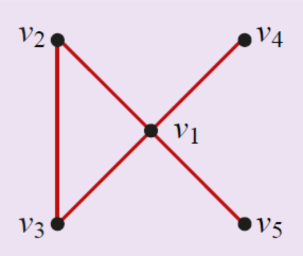
A disconnected graph H

Euler trail

A trail is a walk in a graph that does not use the same edge more than once.

Euler trail

- Uses every edge exactly once
- Some vertices may be visited more than once

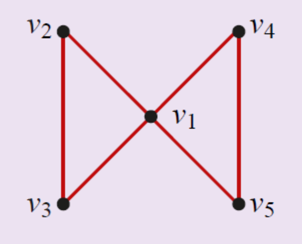


What is the walk?:

Euler circuit

Euler circuit

- uses every edge exactly once
- Starts and ends at the same vertex



What is the walk?:

Euler circuit

Theorem

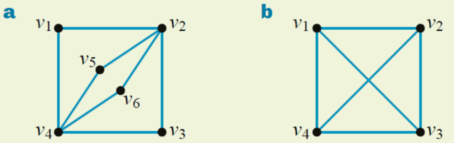
A connected graph has an Euler circuit if and only if the degree of every vertex is even.

Proof:

Euler circuit

Example

For each of the following graphs, name an Euler circuit if one exists:



Euler trail

Theorem

A connected graph has an Euler trail if and only if one of the following holds:

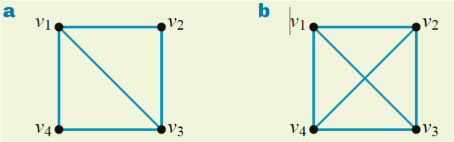
1. every vertex has even degree
2. exactly two vertices have odd degree.

Proof:

Euler trail

Example

For each of the following graphs, name an Euler circuit if one exists:

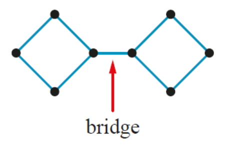


Fleury's algorithm

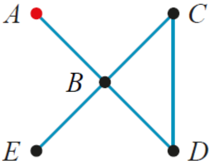
In the real world, there can be many vertices. Hence, we can't just do trial and error. Now Fleury's algorithm comes in handy!

To find an Euler trail in a connected graph such that every vertex has even degree or exactly two vertices have odd degree:

1. If there are two vertices of odd degree, then start from one of them. Otherwise, start from any vertex.
2. Move from the current vertex across an edge to an adjacent vertex. Always choose a non-bridge edge unless there is no alternative.
3. Delete the edge that you have just traversed.
4. Repeat from Step 2 until there are no edges left.



Try this one



Exercise 12B

Hamiltonian cycles

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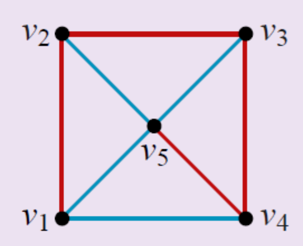
Hamiltonian path

Path - a walk in graph that does not repeat any vertices (also edges)

Hamiltonian path

- a walk in a graph that visits every vertex exactly once



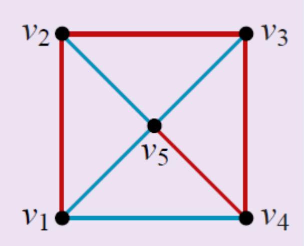
What is the walk?:

Hamiltonian path

Cycle - walk that starts and end at the same vertex, also does not repeat any vertex or edges

Hamiltonian path

- walk that starts and ends at the same vertex and visits every other vertex exactly once

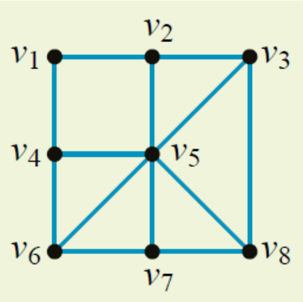


What is the walk?:

Note: Hamilton path or cycle, some edges may not be used

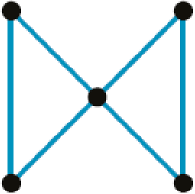
Example

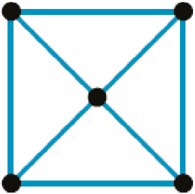
Eight towns are represented by the vertices $v_1, v_2, \ldots, v_8$. The roads that connect these towns are represented as edges. Starting and ending at v_1, how can a salesperson visit every town exactly once?



Difference between Euler and Hamiltonian

1. Euler are defined in terms of Edges
2. If a graph has Euler, then no Hamiltonian
3. If a graph has Hamiltonian, then no Euler





Exercise 12C

Using matrix powers to count walks in graphs

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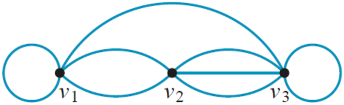
Length of a walk

Definition

Length of a walk in a graph is the number of edges in the walk.
If you use the same edge more than once, then you must count that edge more than once



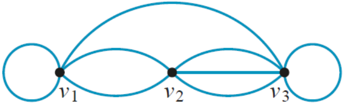
How does length of 1 look like?

$v_1 \longrightarrow v_3$

$v_2 \longrightarrow v_3$

$v_1 \longrightarrow v_2$

Walk of length 2



What is the number of walks of length 2 from $v_1 \longrightarrow v_3$ Case 1: $v_1 \longrightarrow v_1 \longrightarrow v_3$

Case 2: $v_1 \longrightarrow v_2 \longrightarrow v_3$

Case 2: $v_1 \longrightarrow v_3 \longrightarrow v_3$

Matrix power comes in handy!

Walk of length 1:

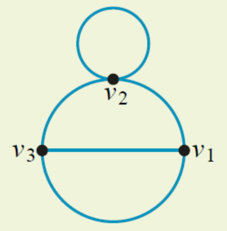
$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 1 \end{bmatrix} \end{array}$$

Walk of length 2:

Example

Note: It is helpful to define $A^0 = I$, where I is the identity matrix. We can then also consider walks of length 0. We say that there is one walk of length 0 from any vertex to itself.

Find the number of walks of length 3 from vertex v1 to vertex v3 in the graph shown.



Exercise 12D

Regular, cycle, comple and bipartite graphs

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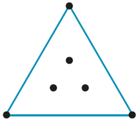
Regular graphs

A graph is said to be regular if all its vertices have the same degree.



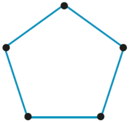
Theorem

If G is a regular graph with n vertices of degree r, then G has $\frac{nr}{2}$ edges.

Proof:

Cycle graphs

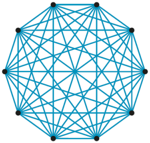
A cycle graph is consisting of a single cycle of vertices and edges. For $n \geq 3$, the cycle graph with n vertices is denoted by C_n.



Every cycle graph C_n is regular, since each vertex has degree 2. The cycle graph C_5 is shown above.

Complete graphs

A complete graph is a simple graph with 1 edge joining each pair of distinct vertices. With n vertices, it is denoted by K_n.



It is $K_1 0$ above! The graph K_n is regular, since each vertex has degree $n - 1$. Adjancency matrix of K_n has 1s in all poitions (except for main diagonal being 0s.)

Theorem

The complete graph K_n has $\frac{n(n-1)}{2}$ edges.

Proof:

Complement of simple graph

Complement of simple graph

If G is a simple graph, complement G is $\bar{G}$:

1. $\bar{G}$ and G have the same set of vertices
2. Two vertices are adjacent in $\bar{G}$ iff the are not adjacent in G.

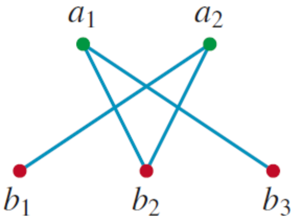
Note: To generate the complement of a simple graph, fill in all the missing edges required to form a complete graph and then remove all the edges of the original graph.

Example

1. How many edges does the complete graph K_5 have?
2. Draw K_5
3. Draw the cycle graph C_5 and draw the complement of C_5.

Bipartite graph

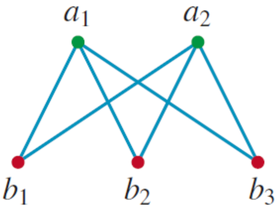
A bipartite graph is a graph whose vertcies can be divided into two disjoint subsets A and B such that every edge of the graph joins a vertex in A to a vertex in B.



Every edge of the graph joins a vertex in $A = a_1, a_2$ to a vertex in $B = b_1, b_2, b_3$.

Complete bipartite graph

A bipartite graph is a simple graph whose vertcies can be divided into two disjoint subsets A and B such that



1. every edge joins a vertex in A to a vertex in B
2. every vertex in A is joined to every vertex in B

The complete bipartite graph where A contains m vertices and B contains n vertices is denoted by $K_{m,n}$.

Example

Draw the complete bipartite graph $K_{2,4}$ and give its adjacency matrix.

Example

1. Show that the cycle graph C_6 is a bipartite graph by dividing the set of vertices into two suitable disjoint subsets.
2. Hence show that C_6 is a subgraph of $K_{3,3}$.

Exercise 12E

Trees

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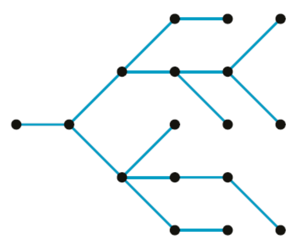
Definition

Tree is a connected graph without any cycle



Properties:

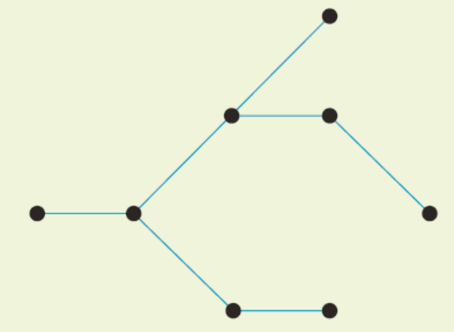
1. A tree with n vertices has $n - 1$ edges.
2. In any tree, there is exactly one path between each pair of distinct vertices.
3. Every tree is a bipartite graph.

Example

Draw the three trees with five vertices.

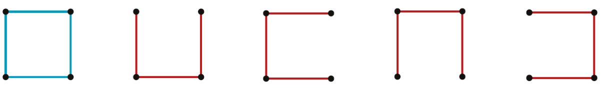
Example

Show that this tree is a bipartite graph by dividing its vertices into two suitable disjoint subsets.



Definition

Let G be a connected graph. Spanning tree of G is a subgraph of G that is a tree with same set of vertices of G.



Theorem

Every connected graph has a spanning tree.

Algorithm for finding a spanning tree

1. If the graph has no cycles, then stop.
2. Choose any edge that belongs to a cycle, and delete the chosen edge.
3. Repeat from Step 1

Example

Find a spanning tree for the complete graph K_5.

Exercise 12F

Euler's formula and the Platonic solids

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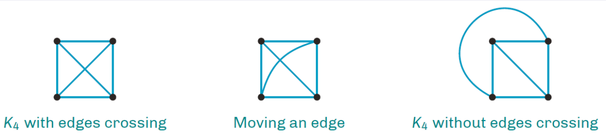
Planar graphs

Definition

A graph G is a planar graph if:

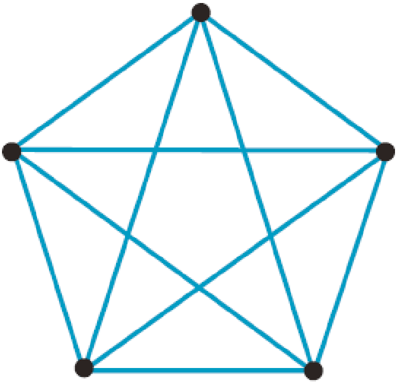
1. Only edges intersect at their endpoints
2. Edges does not interset with each other

This is known as plane drawing of G.



This is not Planar graph!

You can give it a go to see if it can be changed to planar, but there is proof that it is not planar graph!



Subgraphs of planar graphs

Theorem

Any subgraph of a planar graph is also planar

Proof:

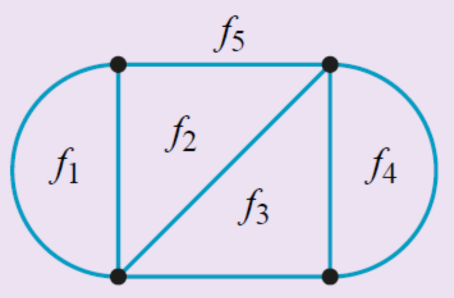
Euler's formula

Faces of a planar graph

If G is a planar graph:

1. Faces - any plane drawing of G divides the plane into regions
2. Bounded faces - faces that are bordered by edges
3. Unbounded face - face that is not bordered by edges

Task: Identify which are bounded and unbounded.



Euler's formula

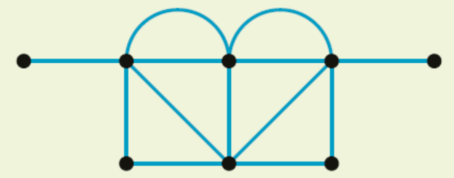
Theorem

If G is a connected planar graph with v vertices, e edges and f faces, then:
$v - e + f = 2$

Proof:

Example

Verify Euler's formula for the planar graph shown.



Edges of planar graphs

Theorem

Let G be a connected simple graph with v vertices and e edges, where $v \geq 3$. If the graph G is planar, then $e \leq 3v - 6$.

Proof:

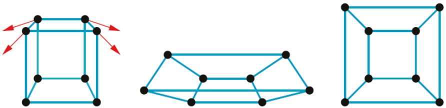
Example

A connected simple graph has 6 vertices and 14 edges. Show that this graph is not planar.

Polyhedral graphs

A polyheadral is a three-dimensional solid formed from a collection of polygons joined along their edges.

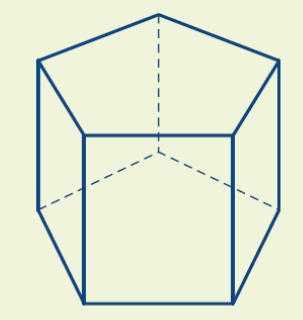
Every convex polyhedron can be drawn as a connected planar graph.



Example

A pentagonal prism is shown.

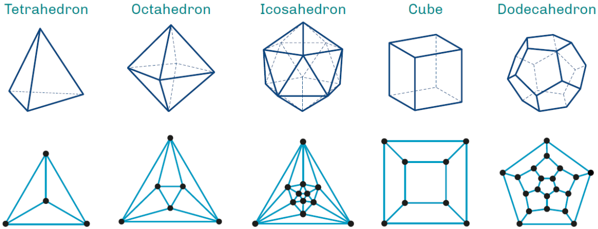
1. Give a plane drawing of the graph that represents the pentagonal prism.
2. Verify Euler's formula for this graph.



Platonic solid

Platonic solid is a convex polyhedron such that:

1. Polygonal faces are all congruent (identical in shape and in size) and regular (all angles and sides are equal)
2. The same number of faces meet at each vertex



Platonic solid

Theorem

There are only five Platonic solids.

Proof:

Exercise 12G


BUT WAIT
WE'RE NOT DONE YET

When every vertex has even degree

Kin Hei Wong

February 15, 2024

This is from Section 12B

Theorem - Euler circuits

A connected graph has an Euler circuit iff the degree of every vertex is even.

We only proved halfway on this theory, we still need to prove if the degree of every vertex is even, then the connected graph has a Euler's circuit.

Theorem

Let G be a graph in which every vertex has even degree. Then G can be split into cycles, no two of which have an edge in common.

Proof:

We can now proof the theorem on Euler's circuit

Theorem

A connected graph has an Euler circuit if every vertex has even degree.

Exercise 12H


I'M FINALLY
DONE
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