Matrix

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Presentation overview

Matrix notation

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Matrix

A matrix is a rectangular array of numbers. The numbers in the array are called the entries.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Size

The size of the matrix is determined by $m \times n$, where m is the row, n is the columns.

$\therefore$ the size of the matrix above is 3×2.

Use of matrix?

It can be use to store information, as demonstrated below:

Example

Four exporters A, B, C and D sell refrigerators (r), dishwashers (d), microwave ovens (m) and televisions (t). The sales in a particular month can be represented by a 4×4 array of numbers. This array of numbers is called a matrix.

$$\begin{array}{c c} & \begin{array}{cccc} r & d & m & t \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 120 & 95 & 370 & 250 \\ 430 & 380 & 950 & 900 \\ 60 & 50 & 150 & 100 \\ 200 & 100 & 470 & 50 \end{bmatrix} \end{array}$$

Example

A minibus has four rows of seats, with three seats in each row. If 0 indicates that a seat is vacant and 1 indicates that a seat is occupied, write down a matrix to represent:

(a) the 1st and 3rd rows are occupied, but the 2nd and 4th rows are vacant

(b) only the seat at the front-left corner of the minibus is occupied.

Example

There are four clubs in a local football league:

1. Club A has 2 senior teams and 3 junior teams.
2. Club B has 2 senior teams and 4 junior teams.
3. Club C has 1 senior team and 2 junior teams.
4. Club D has 3 senior teams and 3 junior teams.

Represent this information in a matrix.

Entries and equality

If A is a matrix, then a_{ij} will be used to denote the entry that occurs in row i and column j of A.

Thus a 3×4 matrix may be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

Exercise: 11A

Addition, subtraction and multiplication by a real number

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Addition and subtraction

In addition and subtraction, we just add it individually according to the position. HOWEVER, you must make sure that they are in the same size!

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \\ a_{31} + b_{31} & a_{32} + b_{32} \end{bmatrix}$$

Multiplication of a matrix by a real number

If A is any matrix and k is a real number, then the product kA is the matrix obtained by multiplying each entry of A by k.

$$3\begin{bmatrix}2 & -2\\ 0 & 1\end{bmatrix} = \begin{bmatrix}6 & -6\\ 0 & 3\end{bmatrix}$$

Zero matrix

The $m \times n$ matrix with all entries equal to zero is called the zero matrix, and will be denoted by $\mathbf{O}$.

For any $m \times n$ matrix A and the $m \times n$ zero matrix $\mathbf{O}$, we have $A + \mathbf{O} = A$ and $A + (-A) = \mathbf{O}$

Example

Let

$$X = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, Y = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \text{A} = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix} \text{and B} = \begin{bmatrix} 5 & 0 \\ -2 & 4 \end{bmatrix}.$$

Find X + Y, 2X, 4Y + X, X - Y, -3A and 3A + B.

Exercise: 11B

Multiplication of matrices

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Multiplication by matrices is less straightforward!

Multiplication by matrices

If A is an $m \times n$ matrix and B is an $n \times r$ matrix, then the product AB is the $m \times r$ matrix whose entries are determined as follows: To find the entry in row i and column j of AB, single out row i in matrix A and column j in matrix B. Multiply the corresponding entries from the row and column and then add up the resulting products.

TOO LONG TO UNDERSTAND

Just keep in mind of these things:

1. To find first row and first column of the result, multiply the number of first row, first column individually and add together
2. To find first row and second column of the result, multiply the number of first row, second column individually and add together
3. I believe, you see the pattern: but I will demonstrate later
4. If the product AB is only defined iff(if and only if) number of columns of A = number of rows of B

Demonstration

Note: it is always row × column.

Let

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 1\times 5 + 3\times 6 & 1\times 1 + 3\times 3 \\ 4\times 5 + 2\times 6 & 4\times 1 + 2\times 3 \end{bmatrix} = \begin{bmatrix} 23 & 10 \\ 32 & 10 \end{bmatrix}$$

Then

$$BA = \begin{bmatrix} 1\times 5 + 1\times 4 & 5\times 3 + 1\times 2 \\ 6\times 1 + 3\times 4 & 6\times 3 + 3\times 2 \end{bmatrix} = \begin{bmatrix} 9 & 17 \\ 18 & 24 \end{bmatrix}$$

Quickfire question

What do you notice between AB and BA?

Exercise: 11C

Identities, inverses and determinants for 2×2 matrices

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What is identity matrix?

Identiy matrix of n $\times$ n is

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

This assumes that n in finite! Then, if we assume n = 2?

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Try multiplying identity matrix with any matrix, what do you notice? What can we conclude?

Identity matrix property!

Let A be a matrix of n $\times$ n (or we call it square matrix). Then, AI = A = IA. This also means the A is invertible.

Invertible matrix

For an invertible matrix A, we have

$$AA^{-1} = I = A^{-1}A$$

We will start to learn more the next slide.

Inverse matrix

From any non-zero real number x, there is a real number x^{-1}, which has the result of $xx^{-1} = 1$.
It is the same in matrix!

Example

Consider

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} B = \begin{bmatrix} x & y \\ u & v \end{bmatrix}$$

and AB = I. Solve matrix B.

The usual way

General 2×2 matrix

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = \begin{bmatrix} x & y \\ u & v \end{bmatrix}$$

and AB = I implies

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ u & v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Here comes determinant!

Inverse of 2×2 matrix

If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

, then the inverse of A is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{(provided } ad - bc \neq 0\text{)}$$

Determinant

If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then det(A) = $ad - bc$.
Note: 2×2 matrix A has an inverse only if det(A) $\neq 0$.

Application of inverse matrix

A common way of sending an encrypted message is to assign an integer value to each letter of the alphabet and send the message as a string of integers.
E.g. The message: (SEND MONEY) can be sent as 5,8,10,21,7,2,10,8,3.
A more secure way is to use matrix multiplication and matrix inverses!
This is because we can use the inverse matrix to decrypting the message
Slide adapted from UoM: MAST10007 Linear Algebra Lecture Notes 2021

Exercise: 11D

Identities, inverses and determinants for 2×2 matrices

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Simultaneous in matrices

We can represent the simultaneous of:

$3x - 2y = 5$

$5x - 3y = 9$

into

$$\begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

Solving simultaneous in matrices

Note: If determinant of the matrix is 0, then there is no UNIQUE solution.

$$\text{Let A} = \begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix} \therefore, A^{-1} = \begin{bmatrix} -3 & 2 \\ -5 & -3 \end{bmatrix}$$

Example

Let A = $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$, K = $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.Solve the system AX = K, where X = $\begin{bmatrix} x \\ y \end{bmatrix}$

Example

Solve the following simultaneous equations:

$3x - 2y = 6$

$7x + 4y = 7$

Exercise: 11E

Inverse and determinants for $n \times n$ matrices

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Finding inverse of 3×3

Without using a calculator, find the inverse of matrix

$$A = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

Note: There is more efficient method known as Gaussian's elimination.

Finding determinant

We can use the method of Cofactor expansion:

Definition

Let A be a square matrix. The (i,j)-cofactor of A denoted by C_{ij}, is the number given by
$C_{ij} = (-1)^{i+j} det(A(i,j))$
where $A(i,j)$ is the matrix obtained from A by deleting the ith row and jth column.

How do we remember the sign of cofactor?

$$\begin{bmatrix} + & - & + & \dots \\ - & + & - & \dots \\ + & - & + & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Example

Let A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Example

Let $A = \begin{bmatrix} 3 & 2 & 0 \\ 3 & 4 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

Exercise: 11F

Simultaneous linear equations with more than two variables

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Linear equations in three variables

Consider the general system of three linear equations in three variables:

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\text{Let A} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \text{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and B} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Then, AX = B

Example

Use matrix method to solve the following system of three equations in three variables:

$2x + y + z = -1$

$3y + 4z = -7$

$6x + z = 8$

Note: Possible cases for a system of three linear equations in three variables:

1. A unique solution
2. no solution
3. infinitely many solutions

Exercise: 11G

Gaussian elimination

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How to do Gaussian elimination?

Gaussian elimination is a systematic way to reduce a matrix to row-echelon form using row operations.

Gaussian elimination

1. Interchange rows if necessary, to bring a non-zero number to the top of the first column with a non-zero entry.
2. Add suitable multiples of the top row to lower rows so that all entries below the leading entry are zero. (Multiplying a row by a constant is also allowed and is often useful.)
3. Start again in Step 1 applied to the matrix without the first row.

Example of row-echelon

$$\begin{bmatrix} 1 & -2 & 3 & 4 & 5 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ a & b & c \end{array}\right]$$

How is this useful?

This is the most efficient way to solve for:

1. Systematic solver
2. Finding inverse matrix

Example

Make the matrix into row-echelon form

$$\begin{bmatrix} 2 & 5 & 3 \\ 3 & 6 & 9 \\ 4 & 8 & 1 \end{bmatrix}$$

Example

Solve the following system using Gaussian elimination:

$x - 3y + 2z = 11$
$2x - 3y - 2z = 13$
$4x - 2y + 5z = 31$

Example - Inverse matrix

Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

Solution:
Form the augmented matrix $[A|I_3]$ and perform row operations so that A is in reduced row-echelon form.

This comes to the end of Matrix...

Want to learn more? Explore around Linear Algebra or here is a free text website below

https://hefferon.net/