Combinatorics

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Presentation overview

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2. 9B: Factorial notation and permutations
3. 9C: Permutations with restriction
4. 9D: Permutations of like objects
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6. 9F: Combinations with restriction
7. 9G: Pascal's triangle
8. 9H: The pigeonhole principle

Basic counting methods

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Basic counting methods

Tree diagram:

Example 1

A restaurant has a fixed menu, offering a choice of fish or beef for the main meal, and cake, pudding or ice-cream for dessert. How many different meals can be chosen?

Multipliation principle

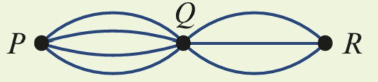
If there are m ways of performing one task and then there are n ways of performing another task, then there are $m \times n$ ways of performing both tasks.

Example 2

Sandra has three different skirts, four different tops and five different pairs of shoes. How many choices does she have for a complete outfit?

Example 3

How many paths are there from point P to R travelling from left to right? [This is also Graph Theory]



Addition principle

Addition principle

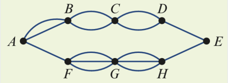
Suppose there are m ways of performing one task and n ways of performing another task. If we cannot perform both tasks, then there are $m + n$ ways to perform one of the tasks.

Example 4

To travel from Melbourne to Sydney tomorrow, Kara has a choice between three different flights and two different trains. How many choices does she have?

Example 5

How many paths are there from point A to E travelling from left to right?



Harder problems involving tree digram

Example 6

A bag contains one blue token, two red tokens and one green token. Three tokens are removed from the bag and placed in a row. How many arrangements are possible?

Exercise 9A

Factorial notation and permutations

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Factorial notation

$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots 2 \cdot 1$
$n! = n \cdot (n-1)!$

Example 7

1. $3!$
2. $\frac{50!}{49!}$
3. $\frac{10!}{2!8!}$

Permutations

Permutation: ordered arrangments of a collections of objects

Example 8

Using a tree diagram, list all the permutations of the letters in the word CAT.

Example 9

How many ways can six different books be arranged on a shelf?

More examples

Example 11

How many four-digit numbers can be formed using the digits 1,2,3 and 4 if:

1. they cannot be repeated
2. they can be repeated?

Number of permutations

Number of permutations

The number of permutations of n objects taken r at a time is denoted by the formula:
${}^{n}P_{r} = \frac{n!}{(n-r)!}$

Proof:

Number of permutations

Example 12

1. Using the letters A,B,C,D and E without repetition, how many different two-letter arrangments are three?
2. Six runners compete in a race. In how many ways can the gold, silver and bronze medals be awarded?

Example 13

How many ways can seven friends sit along a park bench with space for only four people?

Exercise 9B

Permutations with restriction

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Restrictions

When considering permutations with restrictions, we deal with the restrictions first.

Example 14

1. How many arrangements of the word DARWIN begin and end with a vowel?
2. Using the digits 0,1,2,3,4 and 5 without repetition, how many odd four-digit numbers can you form?

Permutations with items grouped together

Example 15

1. How many arrangements of the word EQUALS are there if the vowels are kept together?
2. How many ways can two chemistry, four physics and five biology books be arranged on a shelf if the books of each subject are kept together?

Exercise 9C

Permutations of like objects

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Permutations of like objects

Permutations of like objects

The number of permutations of n objects of which n_1 are alike, n_2 are alike, ... and n_r are alike is given by:
$\frac{n!}{n_1!n_2!\ldots n_r!}$

E.g.: WOOLLOOMOOLOO

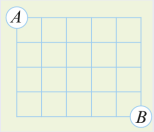
Example

Example 16

1. Find the number of permutations of the letters in the word RIFFRAFF.
2. There are four identical knives, three identical forks and two identical spoons in a drawer. They are taken out of the drawer and lined up in a row. How many ways can this be done?

Example 17

This grid shown consist of unit squares. By travelling only right (R) or down (D) along the grid lines, how many paths are there point A to point B?



Exercise 9D

Combinations

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Combinations

Combination(^nC_r): a selection made regardless of order

Number of combinations

The number of combinations of n objects taken r at a time is given by the formula:
$^nC_r = \frac{n!}{r!(n-r)!}$

Example 18

How many ways can two letters be chosen from the setA,B,C,D?

Example 19

1. A pizza can have three toppings chosen from nine options. How many different pizzas can be made?
2. How many subsets of $1, 2, 3, \ldots, 20$ have exactly two elements?

Examples

Example 21

Consider a group of six students. In how many ways can a group of:

1. two student be selected
2. four students be selected?

Example 22

1. Six points lie on a circle. How many triangles can you make using these points as the vertices?
2. Each of the 20 people at a party shakes hands with every other person. How many handshakes take places?

More and more examples

Example 23

From example 17's grid. By only travelling in right (R) and down (D) along the grid lines, how many paths are there from point A to point B?

Exercise 9E

Combinations with restriction

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Straight to examples

Example 24

1. Grace belongs to a group of eight workers. How many ways can a team of four workers be selected if Grace must be on the team?
2. A hand of card consists of five cards drawn from a deck of 52 playing cards. How many hand contain both the queen and the king of hearts?

Example 25

Four students are to be chosen from a group of eight students for the school tennis team. Two members of the group, Same and Tess, do not get along and cannot both be on the team. How many ways can the team be selected?

Examples... examples... too many...

From seven women and four men in a workplace, how many groups of five can be chosen:

1. without restriction
2. containing three women and two men
3. containing at least one many
4. containing at most one man?

FUSION!

Example 27

1. How many arrangments of the letters in the word DUPLICATE can be made that have two vowels and three consonants?
2. A president, vice-president, secretary and treasurer are to be chosen from a group containing seven women and six men. How many ways can this be done if exactly two women are chosen?

Exercise 9E

Pascal's triangle

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Pascal triangle

Pascal's rule

${}^{n}C_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$, where $1 \leq r < n$

Example 28

Given that ${}^{17}C_2 = 136$ and ${}^{17}C_3 = 680$, evaluate ${}^{18}C_3$.

Example 29

Write down the $n = 6$ row of Pascal's triangle and then write down the value of ${}^{6}C_3$.

Subsets of a set

1. The sum of the entries in row n of Pascal's triangle is 2^n. That is $^nC_0 +^n C_1 +^n C_2 + \cdots +^n C_n - 1 +^n C_n = 2^n$
2. A set of size n has 2^n subsets, including the empty set and the set itself.

Example 30

1. Your friend offers you any of six books that she no longer wants. How many selectiosn are possible assuming that you take at least one book?
2. How many subsets of $1, 2, 3, \ldots, 10$ have at least two elements?

Exercise 9G

The pigeonhole principle

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Pigeonhole principle

WARNING: This principle is very important, such that we will be using this principle again in Graph Theory!

Pigeonhole principle

If $n + 1$ or more objects are placed into n holes, then some hole contains at least two objects.

https://www.youtube.com/watch?v=B2A2pGrDG8I

Examples

Example 31

You have thirteen red, ten blue and eight green socks. How many socks need to be selected at random to ensure that you have a matching pair?

Example 32

1. Show that for any five points chosen inside a 2×2 square, at least two of them will be no more than $\sqrt{2}$ units apart.
2. Seven football tems play 22 games of football. Show that some pair of teams play each other at least twice.

Generalised pigeonhole principle

If at least $mn + 1$ objects are plaecd into n holes, then some hole contains at least $m + 1$ objects.

Example 33

Sixteen natural numbers are written on a whiteboard. Prove that at least four numbers will leave the remainder when divided by 5.

Pigeons in multiple holes

Example 34

Seven people sit at a round table with 10 chairs. Show that there are three consecutive chairs that are occupied.

Exercise 9H

The inclusion - exclusion principle

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Example 35

Consider the three sets of numbers $A = 2, 3, B = 1, 2, 3, 4$and$C = 3, 4, 5$.

1. Find $B \cap C$.
2. Find $A \cup C$.
3. Find $A \cap B \cap C$.
4. Find $A \cup B \cup C$.
5. Find $|A|$.
6. List all the subsets of C.

Addition principle

Addition principle

If A and B are two finite sets of objects such that $A \cap B =$, then
$|A \cup B| = |A| + |B|$

Inclusion-exclusion principle for two sets

If A and B are two finite sets of objects, then
$|A \cup B| = |A| + |B| - |A \cap B|$

Example 36

Each of the 25 students in a Year 11 class studies Physics or Chemistry. Of these students, 15 study Physics and 18 study Chemistry. How many students study both subjects?

Examples

Example 37

A bag contains 100 balls labelled with the numbers from 1 to 100. How many ways can a ball be chosen that is a multiple of 2 or 5?

Example 38

A hand of five cards is dealt from a deck of 52 cards. How many hands contain exactly:

1. two clubs
2. three spades
3. two clubs and three spades
4. two clubs or three spades?

Three sets

Exercise: Try to form the inclusion-exclusion principle for three sets

Examples

Example 39

How many integers from 1 to 140 inclusive are not divisible by 2,5 or 7?

Exercise 9I