Vectors in the plane

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Presentation overview

Introduction to vectors

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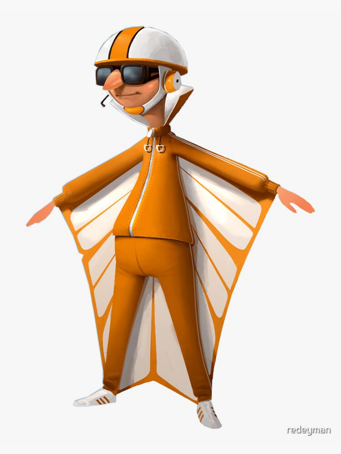
What is vector?



What is vector?

Definition

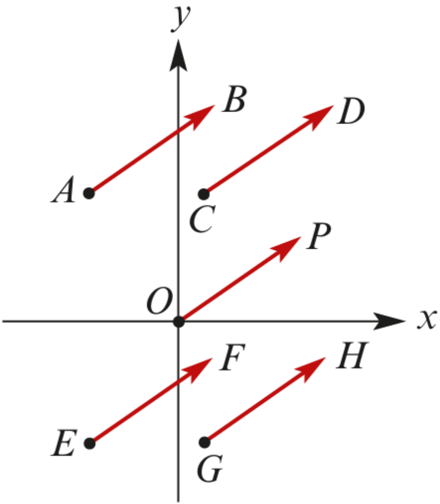
Vector: things that are containing both magnitude AND direction
E.g. Force, velocity, acceleration

Type of representation

Vector notation	Column vector	Component vector
$\vec{AB}$	$\begin{bmatrix} a \\ b \end{bmatrix}$	$\alpha i + \beta j$

Note: We use O to represent origin in Catersian plane, e.g. $\vec{OA}$ = Point of origin (0,0) to point A.
Here, we say A is the position vector (direction relative to the origin)

Drawing vector



Note: $\vec{AB} = \vec{CD} = \vec{OP} = \vec{EF} = \vec{GH}$

Examples

Example 1

Draw a directed line segment corresponding to $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$

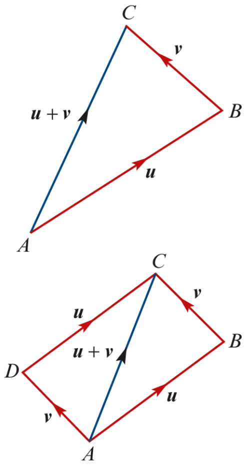
Example 2

The vector u is defined by the directed line segment from (2,6) to (3,1).

If $u = \begin{bmatrix} a \\ b \end{bmatrix}$, find a and b.

Vector addition - Geometric version

For column vector, it's like adding matrices. However, how does it look like geometrically?



Q: What if I want to go from C to A (or $\vec{CA}$)?

Axioms of vectors

Let vectors $u, v, w \in$ vector space$V$ and $\alpha, \beta \in \mathbb{R}$

1. $u + v = v + u$ (commutative)
2. $(u + v) + w = u + (v + w)$ (associativity)
3. There exists a vector $-v$, s.t. $v + (-v) = \vec{0}$ (additive inverse)
4. $\alpha(\beta v) = (\alpha\beta)v$ (associativity: scalar multiplication)
5. $1v = v$ (multiplication by unit scalar)
6. $0v = \vec{0}$

Example 3

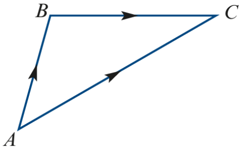
For the vectors $u = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$, find:

1. $2u + 3v$
2. $2u - 3v$

Polygon of vector

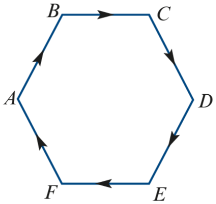
- For two vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$, we have

  $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$



- For a polygon $ABCDEF$, we have

  $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF} + \overrightarrow{FA} = \mathbf{0}$



Parallel vs non-parallel vector

Parallel vector

Two non-zero vectors u and v are parallel if there is some $k \in \mathbb{R} \setminus \{0\}$ such that $u = kv$

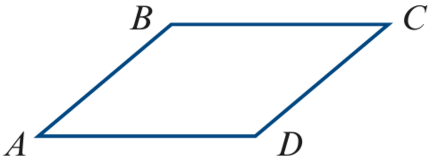
Non-parallel vector

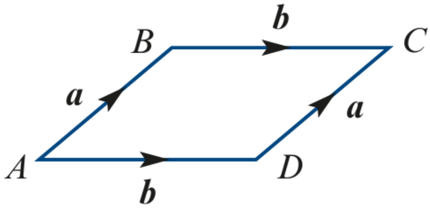
Let there exists $a$and$b$ vectors that are not parallel, then

$$ma + nb = pa + qb$$

Proof:

Parallelograms





Midpoint example

Example 4

Let A, B and C be the vertices of a triangles, and let D be the midpoint of BC
Let $a = \vec{AB}$ and $b = \vec{BC}$.
Find each of the following in terms of a and b:

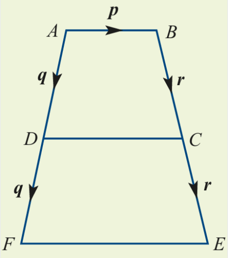
1. $\vec{BD}$
2. $\vec{DC}$
3. $\vec{AC}$
4. $\vec{AD}$
5. $\vec{CA}$

Harder example

Example 5

In this figure, $\vec{DC} = kp$ where $k \in \mathbb{R}\{0\}$.

1. Express p in terms of k, q and r
2. Express $\vec{FE}$ in terms of k and p to show that FE is parallel to DC
3. If $\vec{FE} = 4\vec{AB}$, find the value of k



Exercise 21A

Component of vectors

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Standard unit vector in $\mathbb{R}^2$

Column vector vs Component vector

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = i \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix} = j$$

Component vector

$$u = \begin{bmatrix} x \\ y \end{bmatrix}, \text{ then: } u = xi + yj.$$

Furthermore:
Magnitude of $u = |u| = \sqrt{x^2 + y^2} \longleftarrow$ This is the same as what you learnt in VCE Methods
Note: Some of the other textbooks use $||u||$ to avoid confusion between absolute and magnitude. Hence, if you are using absolute sign in vector it is for magnitude; if it is scalar, it is for absolute.

The axioms still work!

1. $(xi + yj) + (mi + nj) = (x + m)i + (y + n)j$
2. $k(xi + yj) = kxi + kyi$
3. Two vectors are equal iff their components are equal:
$$xi + yi = mi + nj \text{ iff } x = m \text{ and } y = n$$

Example 6

1. Find $\vec{AB} if \vec{OA} = 3i$ and $\vec{OB} = 2i - j$
2. Find $|2i - 3j|$

Example 7

Let A and B be points in the Catersian plane s.t. $\vec{OA} = 2i + j$ and $\vec{OB} = i - 3j$. Find $\vec{AB}$ and $|\vec{AB}|$.

Here comes the most confusing part. . .

Unit vector

Unit vector is a vector of length one unit

Formula: $\hat{u} = \frac{u}{|u|}$

What does it actually mean?

Unit vector is to represent any vector that does not have any unit. It always have the length of 1

Why is it important?

This allows us to seperate the magnitude and the direction, hence, we can use it to analyse on the directional properties.

This is also important we use it in unit cirlce to help us determine angles without calculators!

Let's try it out

Example 8

Let $a = 3i + 4j$

Find $|a|$, the magnitude of a, and hence find the unit vector in the direction of a.

Exercise 21B

Scalar product of vectors

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Scalar product

Definition

Scalar (dot) product of two vectors $a = a_i + a_2 j$ and $b = b_1 i + b2_j$ by

$$a \cdot b = a_1 b_1 + a_2 b_2$$

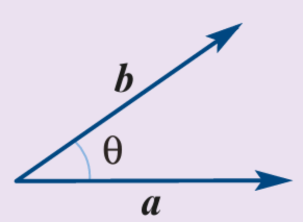
$(2i + 3j) \cdot (i - 4j) =$
If $a = 0$ or $b = 0$, then $a \cdot b = 0$

Goemetric description of the scalar product

For vectors a and b, we have

$$a \cdot b = |a||b|cos\theta$$

where θ is the angle between a and b.



Understand better

Proof:

Examples

Example 9

1. If $|a| = 4$, $|b| = 5$, and the angle between a and b is $30°$, find $a \cdot b$.
2. If $|a| = 4$, $|b| = 5$, and the angle between a and b is $150°$, find $a \cdot b$.

Properties of scalar product

1. $a \cdot b = b \cdot a$
2. $k(a \cdot b) = ka \cdot b = a \cdot (kb)$
3. $a \cdot 0 = 0$
4. $a \cdot (b + c) = a \cdot b + a \cdot c$
5. $a \cdot a = |a|^2$
6. If the vectors a and b are perpendicular, then $a \cdot b = 0$
7. If $a \cdot b = 0$ for non-zero vectors a and b, then the vectors a and b are perpendicular.

Note: In more advanced maths, we say a and b are orthogonal if $a \cdot b = 0$. But don't worry, you won't need this.

Example

Example 10

A, B and C are points defined by the position vectors a, b and c respectively, where

$$a = i + 3j, \qquad b = 2i + j, \qquad c = i - 2j$$

Find the magnitude of $\angle ABC$

Exercise 21C

Vector projections

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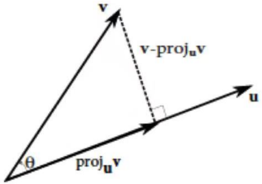
Vector projections

Assume there are two vectors v and u, where $u \neq 0$, the projection of v onto u is given by:

$$proj_u v = \frac{u \cdot v}{|u|^2} u = (\hat{u} \cdot v)\hat{u}$$



$$\begin{aligned} proj_{\mathbf{u}} \mathbf{v} &= \|\mathbf{v}\| \cos\theta \hat{\mathbf{u}} \\ &= \|\hat{\mathbf{u}}\| \|\mathbf{v}\| \cos\theta \hat{\mathbf{u}} \\ &= (\hat{\mathbf{u}} \cdot \mathbf{v})\hat{\mathbf{u}} \end{aligned}$$

Straight to examples

Example 11

Let $a = i + 3j$ and $b = i - j$. Find the vector resolute of:

1. a in the direction of b
2. b in the direction of a

Straight to examples 2

Example 12

Find the scalar resolute of $a = 2i + 2j$ in the direction of $b = -i + 3j$.

Straight to examples 3

Example 13

Resolve $i + 3j$ into rectangular components, one of which is parallel to $2i - 2j$.

Exercise 21D

Geoemetric proofs

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Proofs incoming - but not as scary as before

Definitions to know before proving

1. Collinear points: Three or more points lie on **one** single line.
2. Concurrent lines: Three or more lines lie on **one** single point.

Also some properties

Let there exist vector $\vec{u}$ and $\vec{v}$, and scalar vector $k \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}n\{0\}$.

Parallel vectors

1. $k\vec{u}$ has the same direction as $\vec{u}$, and $-k\vec{u}$ has opposite direction. BUT, magnitude are the same here.
2. For two non-zero vectors $\vec{u}$ and $\vec{v}$ are parallel iff for some $\vec{v}$=$\alpha\vec{u}$
3. If $\vec{u}$ and $\vec{v}$ are parallel with one point in common, then they lie on the same line

Scalar products

1. For two non-zero vectors $\vec{u}$ and $\vec{v}$ are perpendicular iff $\vec{u} \cdot \vec{v} = 0$
2. $\vec{u} \cdot \vec{u} = |u|^2$

Another one

Linear combinations of non-parallel vectors

1. For Two non-parallel and non-zero vectors $\vec{u}$ and $\vec{v}$, if $m\vec{u} + n\vec{v} = p\vec{u} + q\vec{v}$, then

$$m = p \qquad n = q$$

Testing 1 2

Example 14

Three points P, Q and R have position vectors $\vec{p}, \vec{q}$ and $k(2\vec{p} + \vec{q})$ respectively, relative to a fixed origin O. The points O, P and Q are not collinear. Find the value of k if:

1. $\vec{QR}$ is parallel to $\vec{p}$
2. $\vec{PR}$ is parallel to $\vec{q}$
3. P, Q and R are collinear.

Exercise 21E

Application of vectors: displacement and velocity

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Important notes: vector vs scalar

Recall that:
Vector: things that are containing both magnitude AND direction
Scalar: things that are containing only magnitude

Vector vs Scalar

Vector: displacement, velocity, force.
Scalar: distance, time, speed and mass.

Representing displacement with vectors

Example 15

A particle moves from point $A(2, 2)$ to point $B(-1, 3)$. Express the displacement vector of the particle in component form.

What is velocity?

Definition

Velocity: the rate of change of position with respect to time.
What does it mean?
How fast a car can go in respect to time. E.g. If a car can travel 2m in 1s, then we say velocity = $2ms^{-1}$

Some velocity representation listed below:

1. 80 kmh^{-1} in the direction north
2. 10 kmh^{-1} on a bearing of 080°
3. $3i + 3j\ ms^{-1}$

Formula - constant velocity

If an object moves with a constant velocity of $v\ ms^{-1}$ for t seconds, then displacmeent vector, s m:

$$s = tv$$

Example 16

A particle starts at the point A with position vector $\vec{OA} = i + 3j$, where the unit is metres. The particle begins moving with a constant velocity of $2i + 4jms^{-1}$. Find the position vector of the particle after:

1. 5 seconds
2. t seconds

Apply and apply

Example 17

Particle A starts moving from point O with a constant velocity of $v_A = 3i + 4jms^{-1}$. Three seconds after, particle B starts from O and moves in the same directionas A with a constant speed of $7ms^{-1}$. When and where will B catch up to A?

Apply

Example 18

A particle starts from O with a constant velocity of $v_1 = 3i + 4jms^{-1}$. At the same time, a second particle starts moving with constant velocity from the point B, where $\vec{OB} = 25j$.
Given that the two particles meet and their paths are at right angles, find:

1. the position vector of the point where they meet
2. the velocity of the second particle

Exercise 21F

Application of vectors: relative velocity

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Resultant velocity

Resultant is the sume all vectors. $\therefore$ Resultant velocity is all velocity vectors summing up!

Example 19

A river is flowing north at $5kmh^1$. Mila can swim at $2kmh^{-1}$ in still water. She dives in from the west bank of the river and swim towards the oppsite bank.

1. In which direction does she travel?
2. What is her actual speed?

Things are getting saucy here

Relative velocity: velocity that is relative from one things to another.
From example above: We are able to conclude
$v_{s|b} = v_{s|w} + v_{w|b}$
General formula: $v_{a|b} = v_a - v_b$
If we are measuring according to Earth or sometimes origin, this is called **true velocities** or **actual velocities**

Make some relative motions

Example 20

A train is moving with a constant velocity of $80kmh^{-1}$ north. A passenger walks straight across a carriage from the west side to the east side at $3kmh^{-1}$. What is the true velocity of the passenger?

Another one

Example 21

Car A is moving with a velocity of $50kmh^{-1}$ due north, while car B is moving with a velocity of $120kmh^{-1}$ due west. What is the velocity of car A relative to car B?

Jet incoming

Note: Airspeed: speed relative to air

Example 22

A light aircraft has an airspeed of $250kmh^{-1}$. The pilot sets a course due north. If the wind is blowing from the north-west at $80kmh^{-1}$, what is the true speed and direction of the aircraft?

Another one

Example 23

An aeroplane is scheduled to travel from a point P to a point Q, which is $1000km$ due west of P. The aeroplane's airspeed is $500kmh^{-1}$ and the wind is blowing from the south-west at $100kmh^{-1}$.

1. In which direction should the pilot set the course?
2. How long will the flight take?

Exercise 21G

Application of vectors: forces and equilibrium

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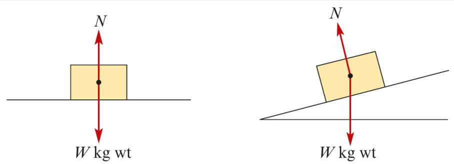
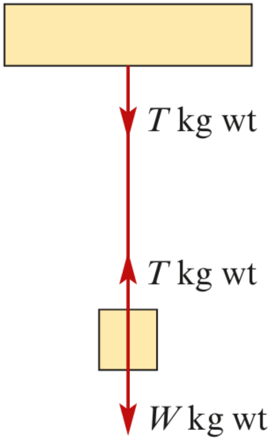
Physicists' favourite

We now introduce forces, where it is always using $F = ma$.
However, when solving forces, we always need to consider where the equilibrium is at.

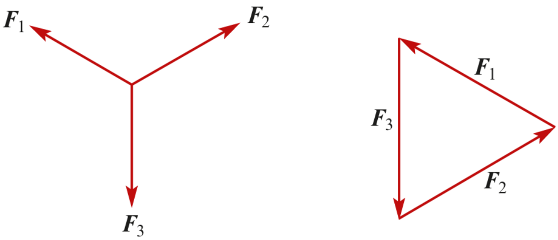
Normal Force



Tension force



Tension force



Example 24

A particle of mass 8kg is suspended by two strings attached to two points in the same horizontal plane. If the two strings make angles of $30°$ and $40°$ to the horizontal, find the tension in each string.

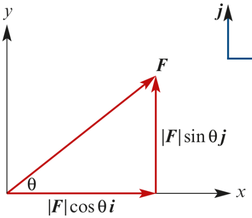
Example 25

A particle of mass 15 kg is suspended vertically from a point P by a string. The particle is pulled horizontal by a force of F kg wt so that the string makes an angle of $30 degree$ with the vertical.
Find the value of F and the tension of string.

Example 26

A body mass 20kg is placed on a smooth plane inclined at 30° to the horizontal. A string is attached to a point further up the plane which prevents the body from moving. Find the tension in the string and the magnitude of the force exerted on the body by the plane.

Resolution of forces



$F = |F|cos\theta i + |F|sin\theta j$

If the particle is in equilibrium, then sum of all i or j components are zero

Example 27

A particle of mass 8kg is suspended by two strings attached to two points in the same horizontal plane. If the two strings make angle of $30°$ and $60°$ to the horizontal, find the tension in each string.

Example 28

A body of mass 10kg is held at rest on a smooth plane inclined at 20° by a string with tension 5kg wt.

Find an angle between string and the inclined plane.

Exercise 21H

Vectors in three dimensions

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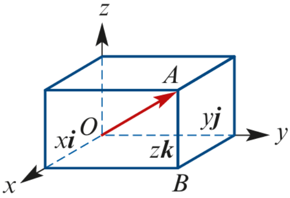
Same format but with k

Type of representation

Vector notation	Column vector	Component vector
$\vec{AB}$	$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$	$\alpha i + \beta j + \gamma k$

More on vector



$\vec{OA}^2 = \vec{OB}^2 + \vec{BA}^2$ (continue to derive)

Example 29

Let $a = i + j - k$ and $b = i + 7k$. Find:

1. $a + b$
2. $b - 3a$
3. $|a|$

Example 30

$OABCDEFG$ is a cuboid s.t.$\vec{OA} = 3j$, $\vec{OC} = k$ and $\vec{OD} = i$.

1. Express each of the following in terms of i, j, and k:
    1. $\vec{OE}$
    2. $\vec{OF}$
    3. $\vec{GF}$
    4. $\vec{GB}$
2. Let M and N be the midpoints of OD and GF respectively. Find MN.

Example 31

If $a = 3i + 2j + 2k$, find $\hat{a}$.

Exercise 21I