Sequence and series

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March 24, 2024

Presentation overview

Direct Proof

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Conditional statement

Same in english, we use "if" and "then" in our statement, but in symbol:

Conditional

Statement: If it is raining, then the grass is wet.
Let P (hypothesis) = it is raining, Q (conclusion)= the grass is wet
$P \implies Q$

Example 1

Prove the following statements:

1. If a is odd and b is even, then $a + b$ is odd.
2. If a is odd and b is odd, then ab is odd.

Example 2

Let $p, q \in \mathbb{Z}$ such that p is divisible by 5 and q is divisible by 3. Prove that pq is divisible by 15.

Example 3

Let x and y be positive real numbers. Prove that if $x > y$, then $x^2 > y^2$.

Example 4

Let x and y be any two positive real numbers. Prove that

$$\frac{x+y}{2} \geq \sqrt{xy}$$

Example 5

Different cases work too!
Every person on an island is either a knight or a knave. Knights always tell the truth, and knaves always lie. Alice and Bob are residents on the island. Alice says: ’We are both knaves.’ What are Alice and Bob?

Exercise: 6A

Proof by contrapositive

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The negation of a statement

Negate statement = opposite of the statement
I am a boy $\implies$ I am **not** a boy.

Example 6

1. 2>1
2. 5 is divisible by 3
3. The sum of any two odd numbers is even
4. There are two primes whose product is 12

De Morgan's laws

not (P and Q) is the same as (not P) or (not Q)
not (P or Q) is the same as (not P) and (not Q)

Example 7

Write down each statement and its negation. Which of the statement and its negation is true and which is false?

1. 6 is divisible by 2 and 3
2. 10 is divisible by 2 or 7

Example 8

Every person on an island is either a knight or a knave. Knights always tell the truth, and knaves always lie. Alice and Bob are residents on the island. Alice says: ’I am a knave or Bob is a knight.’ What are Alice and Bob?

Proof by contrapositive

We use negation in contrapositive proof:
Statement: If it is the end of term, then the students are happy.
Contrapositive: If the students are **not** happy, then it is **not** the end of term.

Example 9

Let $n \in \mathbb{Z}$ and consider this statement: If n^2 is even, then n is even.

1. Write down the contrapositive
2. Prove the contrapositive

Example 10

Let $n \in \mathbb{Z}$ and consider this statement: If $n^2 + 4n + 1$ is even, then n is odd.

1. Write the Contrapositive
2. Prove the contrapositive

Example 11

Let x and y be positive real numbers and consider this statement: If $x < y$, then $\sqrt{x} < \sqrt{y}$.

1. Write down the contrapositive
2. Prove the contrapositive

Exercise: 6B

Proof by contradiction

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Example 12

An angle is called reflex if it exceeds 180 . Prove that no quadrilateral has more than one reflex angle.

Example 13

A Pythagorean triple consists of three natural numbers (a,b,c) satisfying:
$a^2 + b^2 = c^2$
Show that if (a,b,c) is a Pythagorean triple, then a, b and c cannot all be odd numbers.

Theorem

$\sqrt{2}$ is irrational

Example 14

Suppose x satisfies $5^x = 2$. Show that x is irrational.

Theorem

There are infinitely many prime numbers.

Exercise: 6C

Proof by contradiction

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The converse of a statement

Statement: If the angle between a and b is 90^o, then $a^2 + b^2 = c^2$
Converse: If $a^2 + b^2 = c^2$, then the angle between a and b is 90^o

Example 15

Let x and y be positive real numbers. Consider the statement: If $x < y$, then $x_2 < y_2$.

1. Write down the converse of this statement.
2. Prove the converse

Example 16

Let m and n be integers. Consider the statement: If m and n are even, then $m+n$ is even.

1. Write down the converse of this statement.
2. Prove the converse

Equivalent statements

P: Heart is beating   Q: You are alive

Since $P \implies Q$ and $Q \implies P$

We can say that $P \iff Q$

Example 17

Let $n \in \mathbb{Z}$. Prove that n is even if and only if $n + 1$is odd.

Exercise: 6D

Disproving statements

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For all ($\forall$) and there exists ($\exists$)

Statement: For all natural numbers n, we have $2n \geq n + 1$.
Statement: There exists an integer m such that $m^2 = 25$.

Example 18

Rewrite each statement using either ’for all’ or ’there exists’:

1. Some real numbers are irrational.
2. Every integer that is divisible by 4 is also divisible by 2.

Example 19

Write down the negation of each of the following statements:

1. Forall natural numbers n, we have $2n \geq n + 1$.
2. There exists an integer m such that $m^2 = 4$ and $m^3 = 8$.

Counterexamples: Example 20

Let $f(n) = n^2 - n + 11$. Disprove this statement: For all n N, the number $f(n)$ is prime.

Example 21

Find a counterexample to disprove this statement: For all $x, y \in \mathbb{R}$, if $x > y$, then $x^2 > y^2$.

Example 22

Disprove this statement: There exists $n \in \mathbb{N}$ such that $n^2 + 13n + 42$ is a prime number.

Example 23

Show that this statement is false: There exists some real number x such that $x^2 = 1$.

Exercise: 6E

Proof by induction

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Induction

Just follow the steps, you should be able to work out most of the induction:

Induction steps

Let $P(n)$ be some proposition about the natural number n

1. Show that P(1) [or the minimum value for other cases]
2. Assume that P(n) is true
3. Prove that P(n+1) is true

Example 24

Prove that
$1 + 3 + 5 + \cdots + (2n - 1) = n^2$
for all $n \in \mathbb{N}$

Example 25

Prove by induction that $7^n - 4$ is divisible by 3 for all $n \in N$.

Example 26

Prove that $3^n > 3 \times 2^n$ for every natural number $n \geq 3$.

Example 27

Given $t_1 = 11$ and $t_{n+1} = 10t_n - 9$, prove that $t_n = 10_n + 1$.

Example 28

Let a_n be the minimum number of moves needed to solve the Tower of Hanoi with n discs.

1. Find a formula for a_{n+1} in terms of a_n.
2. Evaluate an for $n = 1, 2, 3, 4, 5$. Guess a formula for a_n in terms of n.
3. Confirm your formula for a_n using mathematical induction.
4. If $n = 20$,how many days are needed to transfer all the discs to another peg, assuming that one disc can be moved per second?

Exercise: 6F