Logic

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Presentation overview

The algebra of sets

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All subset of a set

It always apply 2^n when a set has n element.
E.g. $\xi = a, b, c, d$

Axioms

Laws of the algebra of sets

For $A, B, C \subseteq \xi$, the following are true:

Primary
- $A \cup A = A$
- $A \cap A = A$
- $A \cup \varnothing = A$
- $A \cap \xi = A$
- $A \cup \xi = \xi$
- $A \cap \varnothing = \varnothing$

Associativity
- $(A \cup B) \cup C = A \cup (B \cup C)$
- $(A \cap B) \cap C = A \cap (B \cap C)$

Commutativity
- $A \cup B = B \cup A$
- $A \cap B = B \cap A$

Distributivity
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Absorption
- $A \cup (A \cap B) = A$
- $A \cap (A \cup B) = A$

Complements
- $A \cup A' = \xi$
- $A \cap A' = \varnothing$
- $\varnothing' = \xi$
- $\xi' = \varnothing$
- $(A \cup B)' = A' \cap B'$
- $(A \cap B)' = A' \cup B'$
- $(A')' = A$

Algebra of sets	$A \cup \varnothing = A$	$A \cap \xi = A$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Algebra of numbers	$a + 0 = a$	$a \times 1 = a$	$a \times (b + c) = a \times b + a \times c$

Example 1

Dual statements

For a given statement about sets, the dual statement is obtained by interchanging:

$$\cup \text{ with } \cap, \quad \varnothing \text{ with } \xi, \quad \subseteq \text{ with } \supseteq$$

Write the dual of $(A \cap B') \cap B = \emptyset$.

Example 2

Equality for sets

When proving that two sets are equal, we can use the following equivalence:
$X \subseteq Y$ and $Y \subseteq X \iff X = Y$

1. Illustrate $(A \cup B)' = A' \cap B'$ with Venn diagram.
2. Prove $(A \cup B)' = A' \cap B'$.

Example 3

Simplify each of the following expressions:

1. $X \cap (Y \cap X')$
2. $X' \cup (Y \cap X)$
3. $[X' \cup (Y \cap X)']$
4. $[(X \cap Y') \cup (X \cap Y)]'$

Exercise 7A

Switching circuits

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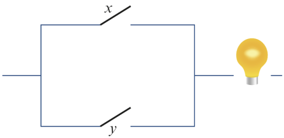
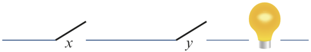
Circuits table



Switches x and y in parallel

x	y	State of system
open	open	open
open	closed	closed
closed	open	closed
closed	closed	closed



Switches x and y in series

x	y	State of system
open	open	open
open	closed	open
closed	open	open
closed	closed	closed

New notation

Open circuit: 0
Closed circuit: 1
$x \cup y = x \vee y$
$x \cap y = x \wedge y$
x' for complement

Example 4

Evaluate $(1 \vee 0) \wedge 1'$.

Example 5

Consider the expression
$(x \wedge y) \vee (z \wedge x')$

1. Draw the switching circuit that is represented by this expression
2. Give a table that describes the operation of this circuit for all possible combinations of switches x, y and z being open (0) and closed (1).

Exercise 7B

Boolean algebra

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More axioms

1. A1(Commutative): $x \vee y = y \vee x \quad | \quad x \wedge y = y \wedge x$
2. A2(Associative): $(x \vee y) \vee z = x \vee (y \vee z) \quad | \quad (x \wedge y) \wedge z = x \wedge (y \wedge z)$
3. A3: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \quad | \quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
4. A4: $x \vee 0 = x \quad | \quad x \wedge 1 = x$
5. A5(Complementation): $x \vee x' = 1 \quad | \quad x \vee x' = 0$

Example 6

Prove that, for each Boolean algebra B and all $x, y, a, b \in B$:

1. $x \vee 1 = 1$
2. $x \wedge 0 = 0$
3. If $a \vee b = 1$ and $a \wedge b = 0$, then $a' = b'$.
4. $(x \vee y)' = x' \wedge y'$

Properties of Boolean algebras

Properties of Boolean algebras

Primary	$x \vee x = x$	$x \wedge x = x$
	$x \vee 0 = x$ (A4)	$x \wedge 1 = x$ (A4)
	$x \vee 1 = 1$	$x \wedge 0 = 0$
Associativity (A2)	$(x \vee y) \vee z = x \vee (y \vee z)$	$(x \wedge y) \wedge z = x \wedge (y \wedge z)$
Commutativity (A1)	$x \vee y = y \vee x$	$x \wedge y = y \wedge x$
Distributivity (A3)	$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$	$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
Absorption	$x \vee (x \wedge y) = x$	$x \wedge (x \vee y) = x$
Complements	$x \vee x' = 1$ (A5)	$x \wedge x' = 0$ (A5)
	$0' = 1$	$1' = 0$
	$(x \vee y)' = x' \wedge y'$	$(x \wedge y)' = x' \vee y'$
	$(x')' = x$	

Example 7

Boolean expressions and functions

$f : 0, 1 \to 0, 1, \qquad f(x) = x \vee 1$
In this case, we have $f(0) = 0 \vee 1 = 1$ and $f(1) = 1 \vee 1 = 1$. In general, a Boolean function has one or more inputs from $0, 1$ and outputs in $0, 1$.

Give the table of values for the Boolean function $f(x, y) = (x \wedge y) \vee y'$.

Example 8

Find a Boolean expression for the Boolean function given by the following table.

	x	y	z	$f(x, y, z)$
1	0	0	0	0
2	0	0	1	0
3	0	1	0	1
4	0	1	1	0
5	1	0	0	1
6	1	0	1	0
7	1	1	0	1
8	1	1	1	1

Example 9

Equivalent Boolean expressions

Two Boolean expressions are **equivalent** if they represent the same Boolean function.

Consider the two Boolean expressions
$((x \vee y) \wedge (x' \vee y)) \vee x$ and $x \vee y$
Show that these two expressions are equivalent by:

1. showing that they represent the same Boolean function
2. using the axioms and properties of Boolean algebras

Exercise 7C

Logical connectives and truth tables

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What is a statement?

A statement is a sentence that is either true or false. Examples of statments are:

1. The boy plays tennis
2. 5 + 7 = 12
3. 5 + 7 = 10

Logical: 'or', 'and', 'not'

Let G be the statement 'n is odd', H be the statement '$n > 10$'

Or (Disjoint) **And (Conjoint)** **Not (Negation)**

Example 10

Write the truth table for $\neg(A \vee B)$.

Example 11

Write the truth table for $(A \wedge B) \wedge (\neg A)$.

Example 12

Show that $\neg(A \wedge B)$ is logically equivalent to $\neg A \vee \neg B$.

Example 13

Show that $(\neg A) \vee (A \vee B)$ is a tautology.

Example 14

Show that $(A \vee B) \wedge (\neg A \wedge \neg B)$ is a contradiction.

Implication

Implication is probably one of the most confusing logical connective.
It understands as If A, then B, notation as: $A \implies B$.
E.g.
A = 'I am elected'
B = 'I will make public tranport free'

Example 15

Give the truth table for $B \implies (A \vee \neg B)$.

Example 16

Give the truth table for $(\neg A \vee \neg B) \iff \neg(A \wedge B)$..

Negation of implication

Try writing down a negation of the conditional statement $A \implies B$.
What do you realise?

Example 17

For each of the following conditional statements:

1. write the converse
2. write the contrapositive
3. write the negation
    1. If you know the password, then you can get in.
    2. Let $n, a, b \in \mathbb{N}$. If n does not divide ab, then n does not divide a and n does not divide b.

The Boolean algebra of statements

We use equivalence ($\equiv$) instead of equality ($=$). E.g.
$A \vee \neg A \equiv T \quad$ and $\quad A \wedge \neg A \equiv F$

Exercise 7D

Valid argument

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Example 18

By constructing a truth table, decide whether or not the following argument is valid.
Premise 1: If Australia is democracy, then Australians have the right to vote.
Premise 2: Australia is a democracy.
Conclusion: Australians have the right to vote.

Example 19

By constructing a truth table, decide whether or not the following argument is valid.
Premise 1: If you invest in Company W, then you get rich.
Premise 2: You did not invest in Company W.
Conclusion: You did not get rich.

Valid arguments with false conclusions

Premise 1: If 2 is odd, then 3 is even.
Premise 2: The number 2 is odd.
Conclusion: The number 3 is even

Invalid argument with true conclusions

Premise 1: If 2 is odd, then 3 is even.
Premise 2: The number 2 is even.
Conclusion: THe number 3 is odd.

Checking for tautology - Example 20

Investigate the validity of each of the following arguments by checking whether or not an appropriate compound statement is a tautology:

1. In March, there are strong winds every day. The wind is not strong today. Therefore it is not March.
2. On Mondays I go swimming. Today is not Monday. Therefore I do not swim today.

Exercise 7E

Logic circuits

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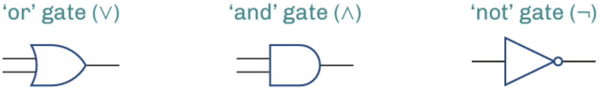
If you are about to study/studying electricity, you are in luck. This is basic for you.



Example 21

Give the gate representation of the Boolean expression $(A \wedge B) \vee C$.

Example 22

1. Give the gate representation of the Boolean expression $A \vee (\neg A \wedge B)$.
2. Describe the operation of this circuit through a truth table.

Example 23

Consider the truth table shown below.

1. Using the technique from Example 8, construct a Boolean expression to match this truth table.
2. Draw a circuit for this expressions.
3. Use the properties of Boolean algebras to simplify the expression, and hence draw a simpler circuit that is equivalent to the circuit from b.

	A	B	Output
1	0	0	1
2	0	1	1
3	1	0	1
4	1	1	0

Exercise 7F

Karnaugh maps

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Minimal representation

Karnaugh maps allow to have more simpler circuit by more compact and cheaper to produce from Boolean expressions.

Minimal representation

Let f be a non-constant Boolean function. Then a **minimal representation** of f is a Boolean expression E which represents f and satisfies the following:

1. The expression E has the form $E_1 \vee E_2 \vee \ldots E_n$, where each E_i is an expression such as $x \wedge y$ or $x' \wedge y' \wedge z$ or $y \wedge z'$.
2. If F is any other expression of this form which also represents f, then the number of terms F_i is greater than or equal to the number of terms E_i.
3. If F and E have the same number of terms, then the number of variables in F is greater than or equal to the number of variables in E.

Two variables

Try writing the truth table for $f(x, y) = (x' \wedge y') \vee (x' \wedge y) \vee (x \wedge y)$.

Karnaugh map calculation

$(x' \wedge y') \vee (x' \wedge y) \vee (x \wedge y)$=

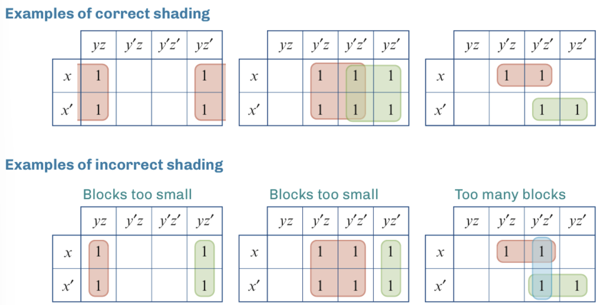
Example 24

Simplify $(x \wedge y) \vee (x \wedge y') \vee (x' \wedge y)$.

Rules for using Karnaugh maps

Blocks in Karnaugh maps

1. You may use an $m \times n$ block in a Karnaugh map if both m and n are powers of 2. (So you may use $1 \times 1, 1 \times 2, 1 \times 4, 2 \times 1, 2 \times 2, 2 \times 4$)
2. You always try to form the biggest blocks that you can, and to use the least number of blocks that you can.



Example 25

Simplify $(x \wedge y \wedge z) \vee (x \wedge y \wedge z') \vee (x \wedge y' \wedge z') \vee (x' \wedge y \wedge z')$.

Example 26

Write a minimal Boolean expression for the following truth table.

x	y	z	$f(x,y,z)$
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

Exercise 7G