SYMMETRIES, FIELDS, AND PARTICLES

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These notes were taken for the *Symmetries, Fields, and Particles* course taught by Nick Dorey at the University of Cambridge as part of the Mathematical Tripos Part III in Michaelmas Term 2018. I live-TEXed them using Overleaf, and as such there may be typos; please send questions, comments, complaints, and corrections to itel2@cam.ac.uk.
Many thanks to Arun Debray for the LATEX template for these lecture notes: as of the time of writing, you can find him at https://web.ma.utexas.edu/users/a.debray/.

Contents

Lecture 1.

Friday, October 5, 2018

Some course materials include:

- Sean Carroll, Spacetime and Geometry
- Misner, Thorne, and Wheeler, Gravitation
- Wald, General Relativity
- Zee, Einstein Gravity in a Nutshell
- Hawking and Ellis, "The Large Scale Structure of Spacetime"

In Minkowski spacetime (flat space) we have spatial coordinates in $\mathbb{R}^3$, the Cartesian coordinates (x, y, z) and a time coordinate t. The line element (spacetime separation) is given by the metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2.$$

ds is the proper distance between x and $x + dx$, y and $y + dy$, z and $z + dz$, and t and $t + dt$. (We work in units where $c = 1$. Note that the metric convention here is flipped from my QFT notes– this is arbitrary.) Using the Einstein summation convention, the metric is usually written as

$$ds^2 = \eta_{\alpha\beta},$$

with $\eta_{\alpha\beta}$ the Minkowski space metric.

Let's recall from special relativity that we call separations with $ds^2 > 0$ "spacelike," with $ds^2 < 0$ "timelike," and $ds^2 = 0$ null (or occasionally lightlike).

Definition 1.1. The *chronological future* of a point p is the set of all points that can be reached from p along future directed timelike lines, and we call this $I^+(p)$. It is the interior of the future-directed light cone. Conversely we have the chronological past of p, $I^-(p)$, which is the interior of the past-directed light cone. We also have the *causal future* of p, which is the set of all points that can be reached from p along future-directed timelike *or* null lines, and we call this $J^+(p)$. Similarly we have the causal past, $J^-(p)$. Thus J is the closure of I and is the interior *plus* the light cone itself.

Let $x^a(\tau)$ be a curve in spacetime. Then the tangent vector to the curve is $u^a = \frac{dx^a}{d\tau}$. For timelike curves, $u^a u^b \eta_{ab} = -1 \iff \tau$ is the proper time along the curve. $\int_p^q d\tau = \Delta\tau$, so the integral of $d\tau$ along a curve from p to q yields the proper time interval, what a clock actually measures.

We also remark that Minkowski space has some very nice symmetries. Since x, y, and z do not appear explicitly in the metric, our spacetime is invariant under translations. It is also invariant under rotations in $\mathbb{R}^3$. It would be nice to extend rotations to include the time coordinate t as well– this is exactly what a Lorentz transformation does.

Lorentz transformations in general involve time– they are defined by the matrices Λ which satisfy

$$\Lambda^T \eta \Lambda = \eta,$$

i.e. they preserve the inner product η in Minkowski space, forming the group $O(3,1)$. Lorentz transformations consist of rotations in $\mathbb{R}^3$ and boosts. This is equivalent to the property $R^T \delta R = \delta$, where R is a rotation and all matrices R satisfying this equation form the group $O(3)$. The Lorentz boost in the x-direction given explicitly is

$$\begin{aligned} t' &= \frac{t - vx}{\sqrt{1-v^2}} \\ x' &= \frac{x - vt}{\sqrt{1-v^2}} \\ y' &= y \\ z' &= z \end{aligned}$$

Rather than constructing the (in general complicated) Lorentz transformation, it is often more convenient to rotate one's frame of reference in $\mathbb{R}^3$ so the boost is in the new x-direction, perform the Lorentz boost, and then transform back:

$$R^T \Lambda R = \Lambda_R,$$

where Λ_R is a new Lorentz transform.

Definition 1.2. The Lorentz transformations taken together form the *Lorentz group*. It satisfies identity, unique inverses (since $\det \Lambda \neq 0$), associativity (from associativity of matrix multiplication), and closure (perhaps good to prove this).[1]

Λ can include reflections in time or space. To avoid such complications, we sometimes refer to the *proper orthochronous Lorentz group*, i.e. to exclude space and time reversals.

Definition 1.3. The Poincaré group is then the semidirect product of Lorentz transformations and translations. This is therefore the group of symmetries of Minkowski space.

We have translations defined as

$$x^a \to x^{a\prime} = x^a + \Delta x^a$$

and also Lorentz transformations, with the property

$$(\Lambda^T)_a{}^c \eta_{cd} \Lambda^d{}_b = \eta_{ab}.$$

Definition 1.4. We also have *contravariant vectors* (indices up) written u^a and their corresponding *covariant* vectors

$$u_a \equiv \eta_{ab} u^b,$$

where we have used the metric to lower an index. We can also raise indices using the inverse metric η^{ab} (defined by $\eta^{ab}\eta_{bc} = \delta^a_c$). Thus

$$u^b = \eta^{ba} u_a.$$

In general the Lorentz transformation of a contravariant vector is given by $u^a \to u^{a\prime} = \Lambda^a{}_b u^b$, where

$$\Lambda^a{}_b = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where γ is given in the usual way by $\gamma = \frac{1}{\sqrt{1-v^2}}$. For instance, x^a is an example of a contravariant vector.

[1] More precisely, we know that the determinant is nonzero since $-1 = \det \eta = \det(\Lambda^T)\det(\eta)\det(\Lambda) = (-1)\det(\Lambda)^2 \implies \det(\Lambda) = \pm 1$.