ADVANCED QUANTUM FIELD THEORY

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These notes were taken for the *Advanced Quantum Field Theory* course taught by Matthew Wingate at the University of Cambridge as part of the Mathematical Tripos Part III in Michaelmas Term 2018. I live-TEXed them using Overleaf, and as such there may be typos; please send questions, comments, complaints, and corrections to itel2@cam.ac.uk. Many thanks to Arun Debray for the LATEX template for these lecture notes: as of the time of writing, you can find him at https://web.ma.utexas.edu/users/a.debray/.

Contents

Lecture 1.

Saturday, January 19, 2019

Note. There will not be official typed course notes, but there will be scanned handwritten notes (which I will link here as they become available). Previous lecturers' notes are currently online (Skinner, Osborn).

Today we introduce path integrals in a QFT context. There are some benefits to working with path integrals– some computations are simplified or more straightforward, and Lorentz invariance is manifest (unlike in the canonical formalism).

Path integrals in quantum mechanics Rather than trying to tackle the full machinery of QFT, we'll start with $0+1$ dimensional non-relativistic quantum mechanics (cf. Osborn § 1.2. We'll set $\hbar = 1$ for now, though we may restore it later in order to make arguments when $\hbar \ll 1$ in a classical limit. In these units,

$$[E][t] = [\hbar] = [p][x]$$

using uncertainty relations.

Let us consider a Hamiltonian in 1 spatial dimension,

$$\hat{H} = H(\hat{x}, \hat{p}) \quad \text{with } [\hat{x}, \hat{p}] = i.$$

We'll further assume for simplicity that the Hamiltonian has a kinetic term and a potential based only on position,

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}).$$

Now the Schrödinger equation takes the form

$$i\frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle \tag{1.1}$$

which has formal solution

$$|\psi(t)\rangle = e^{-i\hat{H}t} |\psi(0)\rangle . \tag{1.2}$$

Let us consider some position eigenstates $|x, t\rangle$ such that

$$\hat{x}(t) |x, t\rangle = x |x, t\rangle , \quad x \in \mathbb{R},$$

where these states obey some normalization

$$\langle x', t | \, x, t \rangle = \delta(x' - x).$$

In the Schrödinger picture, states depend on time, while operators are constant. In terms of fixed (time-independent) eigenstates $\{|x\rangle\}$ of the position operator $\hat{x}$, we may write the wavefunction as

$$\psi(x,t) = \langle x | \, \psi(t) \rangle \,, \tag{1.3}$$

so that applying the Hamiltonian to the wavefunction $\psi(x,t)$ yields

$$\hat{H}\psi(x,t) = (-\frac{1}{2m}\frac{\partial^2}{\partial x^2} + V(x))\psi(x,t). \tag{1.4}$$

This is the traditional presentation of quantum mechanics and the wavefunction. In the path integral formalism, we'll consider a more particle-like treatment, where we express time evolution as a sum over all trajectories (meeting some boundary conditions) appropriately weighted (by an action).

Recall that our formal solution 1.2 tells us what $|\psi(t)\rangle$ is– we can therefore rewrite the wavefunction as

$$\psi(x,t) \, \langle x | \, e^{-i\hat{H}t} \, |\psi(0)\rangle \,. \tag{1.5}$$

By inserting a complete set of (position eigen)states, $1 = \int dx_0 \, |x_0\rangle \langle x_0|$, we get

$$\begin{aligned}\psi(x,t) &= \int dx_0 \, \langle x | \, e^{-i\hat{H}t} \, |x_0\rangle \langle x_0 | \, \psi(0)\rangle \\ &= \int dx_0 \, K(x, x_0; t) \psi(x_0, 0),\end{aligned}$$

where we have defined $K(x, x_0; t) \equiv \langle x | \, e^{-i\hat{H}t} \, |x_0\rangle$. Let us further consider time evolution in discrete steps, with $0 \equiv t_0 < t_1 < t_2 < \ldots < t_n < t_{n+1} \equiv T$ so that

$$e^{-i\hat{H}T} = e^{-i\hat{H}(t_{n+1} - t_n)} \ldots e^{-i\hat{H}(t_1 - t_0)}.$$

As before, we insert complete sets of states, finding that our generic time evolution from any x_0 to an x of our choosing:

$$K(x, x_0; T) = \int \left[\prod_{r=1}^{n} dx_r \, \langle x_{r+1} | \, e^{-i\hat{H}(t_{r+1} - t_r)} \, |x_r\rangle \right] \langle x_1 | \, e^{-i\hat{H}t_1} \, |x_0\rangle \,. \tag{1.6}$$

That is, we integrate over all intermediate positions x_r for each t_r. Naturally, dx_{n+1} must be x.

Let's look at the free theory first to understand what we've done, $V(x) = 0$. Now this weird K_0 object we've defined takes the form

$$K_0(x, x'; t) = \langle x | \, e^{-i\frac{p^2}{2m}t} \, |x'\rangle \,. \tag{1.7}$$

We'll instead insert a complete set of momentum eigenstates $|p\rangle$ with the normalization

$$\int \frac{dp}{2\pi} \, |p\rangle \langle p| = 1,$$

recalling that $\langle x | \, p\rangle = e^{ipx}$ are simply plane waves. Then

$$K_0(x, x'; t) = \int \frac{dp}{2\pi} e^{-ip^2 t/2m} e^{ip(x - x')}.$$

We can compute this– completing the square with a change of variables to $p' = p - \frac{m(x-x')}{t}$, K_0 becomes a gaussian integral,

$$\begin{aligned}K_0(x, x'; t) &= e^{im(x-x')^2/2t} \int_{-\infty}^{\infty} \exp\left[-\frac{i(p')^2 t}{2m} \right] \\ &= e^{im(x-x')^2/2t} \sqrt{\frac{m}{2\pi i t}}.\end{aligned}$$

Note that as $t \to 0$,[1]

$$\lim_{t \to 0} K_0(x, x'; t) = \delta(x - x'),$$

[1] This was more obvious from the original expression for K_0 where $K_0(x, x'; t = 0) = \int \frac{dp}{2\pi} e^{ip(x-x')}$.

which agrees with the fact that $\langle x'|\, x\rangle = \delta(x - x')$.

For $V(\hat{x}) \neq 0$, we still need small time steps but since operators generically do not compute, exponentials don't add in the usual way:

$$e^{\hat{A}}e^{\hat{B}} = \exp(\hat{A} + \hat{B} + \frac{1}{2}[\hat{A}, \hat{B}] + \ldots) \neq e^{\hat{A}+\hat{B}} \quad \text{when } [\hat{A}, \hat{B}] \neq 0.$$

This is the Baker-Campbell-Hausdorff (BCH) formula. However, for small ϵ we can write

$$e^{\epsilon\hat{A}}e^{\epsilon\hat{B}} = \exp(\epsilon\hat{A} + \epsilon\hat{B} + O(\epsilon^2)),$$

or equivalently

$$e^{\epsilon(\hat{A}+\hat{B})} = e^{\epsilon\hat{A}}e^{\epsilon\hat{B}}(1 + O(\epsilon^2)),$$

so we conclude that

$$e^{\hat{A}+\hat{B}} = \lim_{n\to\infty} \left(e^{\hat{A}/n}e^{\hat{B}/n}\right)^n.$$

Suppose now that we divide our time into n time steps so that $t_r - 1 - t_r = \delta t$, with $T = n\delta t$. Then one of the intermediate time evolution steps looks like

$$\begin{aligned}\langle x_{r+1}|\, e^{-i\hat{H}\delta t}\, |x_r\rangle &= e^{-iV(x_r)\delta t}\, \langle x_{r+1}|\, e^{-i\hat{p}^2\delta t/2m}\, |x_r\rangle \\ &= \sqrt{\frac{m}{2\pi i\delta t}} \exp\left[\frac{i}{2}m\left(\frac{x_{r+1} - x_r}{\delta t}\right)^2 \delta t - iV(x_r)\delta t\right].\end{aligned}$$

Taking $T = n\delta t$, we find that the entire K becomes

$$K(x, x_0; T) = \int \left(\prod_{r=1}^{n} dx_r\right) \left(\frac{m}{2\pi i\delta t}\right)^{\frac{n+1}{2}} \exp\left(i\sum_{r=0}^{n}\left[\frac{m}{2}\left(\frac{x_{r+1} - x_r}{\delta t}\right)^2 - V(x_r)\right]\delta t\right). \tag{1.8}$$

Now we take the limit as $n \to \infty, \delta t \to 0$ with T fixed. Then the argument of the exponential becomes

$$\int_0^T \frac{m}{2}\dot{x}^2 - V(x)]dt = \int_0^t L dt, \tag{1.9}$$

where $L(x, \dot{x})$ is the classical Lagrangian and this integral is nothing more than the action. We conclude that

$$K(x, x_0; T) = \langle x|\, e^{-i\hat{H}t}\, |x_0\rangle = \int \mathcal{D}x\, e^{iS[x]}, \tag{1.10}$$

where $S[x] = \int_0^T L(x, \dot{x})dt$ is the classical action and the $\mathcal{D}$ conceals all our sins (the continuum limit) in a cute integration measure. Note that the action has units of energy $\times$ time, so if we restore $\hbar$, we see that this integral becomes

$$K(x, x_0; T) = \int \mathcal{D}x\, e^{iS/\hbar}, \tag{1.11}$$

and in the $\hbar \to 0$ limit (the classical limit), the integral is dominated by paths x which minimize the classical action, and we recognize this as Hamilton's principle from classical mechanics.

Lecture 2.

Monday, January 22, 2019

Last time, we introduced the path integral in quantum mechanics, and we said it took the form

$$\langle x|\, e^{-iHt/\hbar}\, |x_0\rangle = \int \mathcal{D}x e^{iS[x]/\hbar}. \tag{2.1}$$

Let us consider now a "rotation" to imaginary time, $t \to -i\tau$ (Wick rotation). Then our path integral becomes

$$\langle x|\, e^{-H\tau/\hbar}\, |x_0\rangle = \int \mathcal{D}x e^{-S[x]/\hbar}. \tag{2.2}$$

Working with a real exponent has some benefits– the convergence of the integral is more obvious, and in the $\hbar \to 0$ limit we expect the integral to be dominated by the classical path x which minimizes the action $S[x]$.

We can make the observation that 1D quantum mechanics is like a $0+1$D quantum field theory– the field is

$$x(t) : \mathbb{R} \to \mathbb{R}.$$

In fact, 3D quantum mechanics is also like a $0+1$D QFT, where the field is now

$$\mathbf{x}(t) : \mathbb{R} \to \mathbb{R}^3.$$

Given a single spacetime label t, a QM theory gives us a real scalar in $\mathbb{R}$ or a vector in $\mathbb{R}^3$– cf. Srednicki Ch. 1. There are different approaches to quantization, but in the second quantization formalism we demote position $\mathbf{x}$ from an operator to a label on a spacetime point $(\mathbf{x}, t)$. Therefore QFT in $3+1$ dimensions has e.g. a scalar field ϕ which is a map

$$\phi : \mathbb{R}^{1,3} \to \mathbb{R}.$$

Path integral methods Let's begin with the simplest possible case, QFT in zero dimensions.[2] All of spacetime is a single point p,[3] and our (real scalar) field ϕ is a map $\phi : \{p\} \to \mathbb{R}$.

Using our imaginary time (Euclidean signature) convention for the path integral, we write

$$Z = \int_{\mathbb{R}} d\phi \, e^{-S[\phi]/\hbar}. \tag{2.3}$$

We'll take our action $S[\phi]$ to be polynomial in ϕ, with highest power even.

As in statistical field theory, we are interested in correlation functions and expectation values. Given a function $f(\phi)$, we might like to compute the expectation value

$$\langle f \rangle = \frac{1}{Z} \int d\phi \, f(\phi) e^{-S[\phi]/\hbar}. \tag{2.4}$$

For this to have a chance of convergence, f should not grow too rapidly as $|\phi| \to \infty$. Usually the functions we are interested in are polynomial in ϕ.

Free field theory Suppose we have N real scalar fields $\phi_a, a = 1, \ldots, N$. We can compactly write this as a single field

$$\phi : \{p\} \to \mathbb{R}^N, \tag{2.5}$$

and we'd like to compute the integral

$$Z_0 = \int d^N \phi e^{-S[\phi]/\hbar}. \tag{2.6}$$

Now, a free theory simply means that the action is quadratic in our fields. A priori it could have included kinetic terms, but since we are in zero dimensions, there are no derivatives to take and therefore no kinetic terms in this model. Then we can write our action as

$$S(\phi) = \frac{1}{2} \mathcal{M}_{ab} \phi_a \phi_b = \frac{1}{2} \phi^T \mathcal{M} \phi, \tag{2.7}$$

where $\mathcal{M}$ is an $N \times N$ symmetric matrix with $\det M > 0$. So our action could include terms like $\frac{1}{2}\phi_1^2$ and $\frac{5}{2}\phi_1\phi_4$. Since $\mathcal{M}$ is symmetric, we can diagonalize it as

$$\mathcal{M} = P \Lambda P^T$$

for some orthogonal matrix P. But equivalently we could just redefine our fields to some new fields $\phi' = P^T \phi$ so that

$$S(\phi) = \frac{1}{2} \phi'^T \Lambda \phi' = \frac{1}{2} \sum_{i=1}^N \lambda_i (\phi'_i)^2,$$

where λ_i are the eigenvalues of $\mathcal{M}$. Since P is orthogonal, $\det P = 1 \implies d^N \phi = (\det P) d^N \phi' = d^N \phi'$, so our path integral separates into N Gaussian integrals of the form

$$\int_{-\infty}^{\infty} dx \, e^{-\frac{\lambda}{2\hbar} x^2} = \sqrt{\frac{2\pi\hbar}{\lambda}}. \tag{2.8}$$

[2] Cf. Skinner Ch. 2, Srednicki §8,9.

[3] If you're reading my SUSY notes, you should be getting déjà vu right about now.

Thus

$$Z_0 = \int d^N\phi\, e^{-\frac{1}{2\hbar}\phi^T \mathcal{M}\phi} = \prod_{i=1}^{N} d\phi_i\, e^{-\frac{1}{2\hbar}\lambda_i(\phi_i)^2} = \frac{(2\pi\hbar)^{N/2}}{\sqrt{\det \mathcal{M}}}. \tag{2.9}$$

We can now introduce a source term J, modifying our action to

$$S(\phi) = \frac{1}{2}\phi^T \mathcal{M}\phi + J\cdot\phi. \tag{2.10}$$

If we complete the square and make a change of variables $\tilde{\phi} = \phi + \mathcal{M}^{-1}J$, we find that the new path integral with a source is

$$\begin{aligned} Z_0[J] &= \int d^N\phi \, \exp\left[-\frac{1}{2\hbar}\phi^T\mathcal{M}\phi - \frac{1}{\hbar}J\cdot\phi\right] \\ &= \exp\left(\frac{1}{2\hbar}J^T\mathcal{M}^{-1}J\right)\int d^N\tilde{\phi}\, e^{-\frac{1}{2\hbar}\tilde{\phi}^T\mathcal{M}\tilde{\phi}} \\ &= Z_0 \exp\left(\frac{1}{2\hbar}J^T\mathcal{M}^{-1}J\right). \end{aligned}$$

We see that $\frac{\partial}{\partial J}$ derivatives will bring down ϕs, which will allow us to compute correlation functions just like we did in statistical physics with the partition function.

Example 2.11. What is the value of the correlation function $\langle\phi_a\phi_b\rangle$ in this theory? We can compute it directly:

$$\begin{aligned} \langle\phi_a\phi_b\rangle &= \frac{1}{Z_0}\int d^N\phi\,\phi_a\phi_b \exp\left[-\frac{1}{2\hbar}\phi^T\mathcal{M}\phi - \frac{1}{\hbar}J\cdot\phi\right]\Bigg|_{J=0} \\ &= \frac{1}{Z_0}\int d^N\phi\left(-\hbar\frac{\partial}{\partial J_a}\right)\left(-\hbar\frac{\partial}{\partial J_b}\right)\exp\left[-\frac{1}{2\hbar}\phi^T\mathcal{M}\phi - \frac{1}{\hbar}J\cdot\phi\right]\Bigg|_{J=0} \\ &= (-\hbar)^2\frac{\partial}{\partial J_a}\frac{\partial}{\partial J_b}\exp\left[\frac{1}{2\hbar}J^T\mathcal{M}^{-1}J\right]\Bigg|_{J=0} \\ &= \hbar(\mathcal{M}^{-1})_{ab}. \end{aligned}$$

Note that the first J derivative brings down an $\mathcal{M}^{-1}J$ (so our expression is of the form $\mathcal{M}^{-1}J\exp(J^T\mathcal{M}^{-1}J)$), and when we take the second J derivative, we will get two terms, one of the form $\mathcal{M}^{-1}\exp(\dots)$ and another of the form $(\mathcal{M}^{-1}J)^2\exp(\dots)$. The second term is zero when we set $J=0$, and the exponential becomes 1 in both cases, so we are just left with $\mathcal{M}^{-1}$.

What we have calculated is a two-point function, otherwise known as a propagator (though it's a bit silly to call this a propagator when the spacetime is just a single point). We can associate a Feynman diagram to this process:

There is another method we can use to compute propagators (cf. Osborn §1.3):

$$\begin{aligned} \mathcal{M}_{ca}\langle\phi_a\phi_b\rangle &= \frac{1}{Z_0}\int d^N\phi\,\mathcal{M}_{ca}\phi_a\phi_b\exp\left[-\frac{1}{2\hbar}\phi^T\mathcal{M}\phi\right] \\ &= -\frac{\hbar}{Z_0}\int d^N\phi\,\phi_b\frac{\partial}{\partial\phi_c}\exp\left[-\frac{1}{2\hbar}\phi^T\mathcal{M}\phi\right] \\ &= \frac{\hbar}{Z_0}\int d^N\phi\,\frac{\partial\phi_b}{\partial\phi_c}\exp\left[-\frac{1}{2\hbar}\phi^T\mathcal{M}\phi\right] \\ &= \hbar\delta_{bc} \implies \langle\phi_a\phi_b\rangle = \hbar(\mathcal{M}^{-1})_{ab}. \end{aligned}$$

In going from the second to the third line, we have integrated by parts to move the $\frac{\partial}{\partial\phi_c}$ to ϕ_b, and then recognized the remaining integral as Z_0.

More generally, let $l(\phi) = l \cdot \phi = \sum_{a=1}^{N} l_a\phi_a (\neq 0)$ be a linear function of ϕ, with $l_a \in \mathbb{R}$. Then the expected value $\left\langle l_a(\phi) \dots l_p(\phi) \right\rangle$ is given by

$$\left\langle l_a(\phi) \dots l_p(\phi) \right\rangle = (-\hbar)^p \prod_{i=1}^{p} \left(l_i \frac{\partial}{\partial J_i} \right) \left. \frac{Z_0[J]}{Z_0} \right|_{J=0}.$$

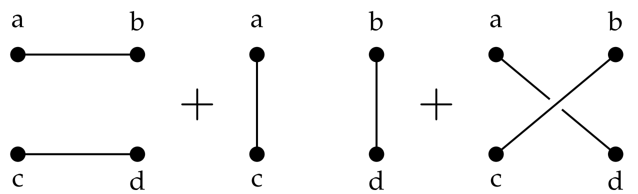
Notice that if we play this game for an odd number of J_i derivatives, all our terms will be of the form $J^p \exp(\dots)$ where p is odd. When we set $J = 0$, all these terms therefore vanish, which tells us that $\left\langle \phi_{a_1} \dots \phi_{a_p} \right\rangle = 0$ for n odd. If we compute it for $p = 2k, k \in \mathbb{N}$, the terms that survive setting $J = 0$ will have k factors of $\mathcal{M}^{-1}$.

Example 2.12. What is the value of the four-point function $\langle \phi_a\phi_b\phi_c\phi_d \rangle$ in free field theory? It is simply

$$\langle \phi_a\phi_b\phi_c\phi_d \rangle = \hbar^2 \left[(\mathcal{M}^{-1})_{ab}(\mathcal{M}^{-1})_{cd} + (\mathcal{M}^{-1})_{ac}(\mathcal{M}^{-1})_{bd} + (\mathcal{M}^{-1})_{ad}(\mathcal{M}^{-1})_{bc} \right].$$

Though we haven't said it, this is effectively a toy version of Wick's theorem– we are taking contractions of the fields using $(\mathcal{M}^{-1})$s as propagators.

We can depict these contractions as connecting some $2k$ dots pairwise with lines using a simplified Feynman diagram notation:



In general, the number of distinct ways we can pair $2k$ elements is

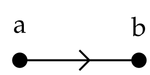
$$\frac{(2k)!}{2^k k!}.$$

The logic here is that we could take all $(2k)!$ permutations of the $2k$ elements, and then take neighboring pairs, e.g. if our elements are $\{a, b, c, d, e, f\}$, one set of pairs is

$$abdcfe \to ab|dc|fe.$$

The order of the 2 elements in each of the k pairs doesn't matter ($ab|dc = ba|dc$), so we've overcounted by a factor of 2^k, and the order of all the k pairs also doesn't matter ($ab|dc = dc|ab$), so we divide by another factor of $k!$ to get the final result.

Example 2.13. One last example– if our free fields are instead complex, $\phi : \{p\} \to \mathbb{C}$, then $\mathcal{M}$ is hermitian. Therefore $(\mathcal{M}^{-1})$ will in general not be symmetric, and so the order of the indices matters. That is, $\left\langle \phi_a\phi_b^* \right\rangle = \hbar(\mathcal{M}^{-1})_{ab}$. Then the associated Feynman diagram has an arrow to indicate direction:



Lecture 3.

Thursday, January 24, 2019

Today, we will continue our exploration of zero-dimensional path integrals in quantum field theory.

Interacting theory Let us consider a single real scalar field $\phi : \{\text{point}\} \to \mathbb{R}$. We choose the action

$$S(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4. \tag{3.1}$$

We'll take $\lambda > 0$ for stability and $m^2 > 0$ such that $\min(S)$ lies at $\phi = 0$ so that we can easily expand around the minimum of S.

The path integral is then

$$Z = \int d\phi \exp\left[-\frac{1}{\hbar}\left(\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right)\right]. \tag{3.2}$$

This will be equivalent to expanding about $\hbar = 0$ (semi-classical limit). We can obviously open up the exponential and rewrite as a series in ϕ and $\hbar$,

$$\begin{aligned} Z &= \int d\phi e^{-\frac{m^2\phi^2}{2\hbar}} \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{\lambda}{\hbar\, 4!}\right)^n \phi^{4n} \\ &= \frac{\sqrt{2\hbar}}{m} \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{\hbar\lambda}{4! m^4}\right)^n \cdot 2^{2n} \int_0^\infty dx e^{-x} x^{2n+\frac{1}{2}-1}, \end{aligned}$$

where we have performed a change of variables $x = \frac{m^2\phi^2}{2\hbar}$. This integral is in fact just a gamma function,

$$\Gamma(2n + \frac{1}{2}) = \frac{(4n)!\sqrt{\pi}}{4^{2n}(2n)!}.$$

Thus our path integral computation using the gamma function is

$$Z = \frac{\sqrt{2\pi\hbar}}{m} \sum_{n=0}^{N} \left(-\frac{\lambda\hbar}{m^4}\right)^n \underbrace{\frac{1}{(4!)^n n!}}_{(1)} \underbrace{\frac{(4n)!}{2^{2n}(2n)!}}_{(2)}. \tag{3.3}$$

Note that from Stirling's approximation, $n! \approx e^{n \log n}$, Thus these two combinatorial-looking terms scale roughly as $e^{n \log n} \approx n!$. The factorial growth of the coefficients means that this path integral actually has zero radius of convergence. This is an asymptotic series– it looks like it is getting better and better, and then everything goes to hell.

In practice the "true" function can differ from the truncated series by some transcendental function which might be small. Cf. Skinner Ch. 2 for more discussion of asymptotic series.

Note that term (1) in the path integral series expansion 3.3 comes from expanding the $\frac{\lambda]4!\hbar}{\phi}^4$ term in the exponent, while term (2) is the number of ways of joining $4n$ elements in distinct pairs (compare our discussion at the end of the previous lecture). We can associate some Feynman diagrams to this– a propagator and a four-point vertex.

Note also that Z has no ϕ dependence, meaning that the Feynman diagrams have no external legs. Let D_n be the set of *labelled* vacuum diagrams with n vertices, so that D_1 is the following set of diagrams, with $|D_1| = 3$.

Then let G_n be the group which permutes each of the 4 fields at each vertex ($(S_4)^n$) and also permutes the n vertices (S_n). The size of this group is

$$|G_n| = |S_4|^n |S_n| = (4!)^n n!.$$

We therefore recognize that

$$\begin{aligned} \frac{Z}{Z_0} &= \sum_{n=0}^{N} \left(-\frac{\lambda\hbar}{m^4}\right)^n \frac{|D_n|}{|G_n|} \\ &= 1 - \frac{\hbar\lambda}{8m^4} + \frac{35}{384}\frac{\hbar^2\lambda^2}{m^8} + \dots \end{aligned}$$

with $Z_0 = \frac{\sqrt{2\pi\hbar}}{m}$. Physically, we can consider $\frac{|D_n|}{|G_n|}$ to be the sum over topologically distinct graphs divided by a symmetry factor. Equivalently, we write

$$\frac{|D_n|}{|G_n|} = \sum_{\Gamma} \frac{1}{S_\Gamma} \tag{3.4}$$

where Γ is a distinct graph free from labels and S_Γ is the number of permutations of lines and vertices leaving Γ invariant. Some examples appear in Fig.

In dimensions > 0, loops correspond to integrals over internal momenta, so these diagrams may have different contributions aside from the symmetry factors.

If we introduce an external source, then our path integral has a generating function

$$Z(J) = \int d\phi \exp -\frac{1}{\hbar}\left(\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + J\phi\right) \tag{3.5}$$

and our correlation functions are modified as before, with $\langle \phi^2 \rangle = \frac{(-\hbar)^2}{Z(0)} \frac{\partial^2}{\partial J^2} Z(J)\Big|_{J=0}$. Source terms correspond to lines terminating on vertices J, so that the expansion of $Z(J)$ involves not only $Z(0)$ vacuum diagrams but also diagrams that terminate with even numbers of source vertices.