STRING THEORY

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These notes were taken for the *String Theory* course taught by R.A. Reid-Edwards at the University of Cambridge as part of the Mathematical Tripos Part III in Lent Term 2019. I live-TEXed them using Overleaf, and as such there may be typos; please send questions, comments, complaints, and corrections to itel2@cam.ac.uk.
Many thanks to Arun Debray for the LATEX template for these lecture notes: as of the time of writing, you can find him at https://web.ma.utexas.edu/users/a.debray/.

Contents

Lecture 1.

Friday, January 18, 2019

Note. This is a 24 lecture course with lectures at 11 AM M/W/F. There will be PDF notes available online somehow (TBD), and also $3+1$ problem sets plus a revision in Easter. The instructor can be reached at rar31@cam.ac.uk. Some recommended course readings[1] include "easier" texts:

- Schomerus[2]
- (Becker)^2 and Schwarz[3]

and "harder" texts:

- Polchinski, Vol 1.[4]
- Lüst and Theisen[5]
- Green, Schwarz, and Witten.[6]

Introduction Here are some of the major topics we'll be covering in this course.

- Classical theory and canonical quantization
- Path integral quantization
- Conformal field theory (CFT) and BRST quantization
- Scattering amplitudes
- Advanced topics (more on this later).

Historically, string theory emerged from ideas in QCD, the theory of the strong force. However, it really took hold as a theory of quantum gravity in the quest to reconcile quantum mechanics with general relativity. A bit of expectation management, first. Some of the motivating ideas which string theory attempts to address are as follows:

- What sets the parameters of the Standard Model?
- What sets the cosmological constant?

[1] Most of these are published by Cambridge University Press. Conspiracy– string theory was invented by CUP to sell textbooks?
[2] Available here for users with access to Cambridge University Press online: https://doi.org/10.1017/9781316672631
[3] Ditto: https://doi.org/10.1017/CBO9780511816086
[4] Here: https://doi.org/10.1017/CBO9780511816079
[5] Possibly available through Springer Link but not a CUP publication. https://link.springer.com/book/10.1007/BFb0113507
[6] Here: rlhttps://doi.org/10.1017/CBO9781139248563

- Failure of perturbative GR (problems in the UV– gravity is non-renormalizable)
- The black hole information paradox (quantum information in gravitational systems)
- How do you quantize a theory in the absence of an existing causal structure? (Most of the causal structure of spacetime is encoded in the metric. But what if it's the metric itself you're trying to quantize?)

There are alternatives to string theory– for instance, one can do QFT in curved spacetime to learn about some limit of quantum gravity. There's also loop quantum gravity and causal set theory, among others, but we won't really discuss those in this course.

What is string theory? We just don't know.

In some sense, string theory is a set of rules which, given a 10-dimensional classical spacetime vacuum, allows us to do quantum perturbation theory around this vacuum. By doing perturbation theory, we seem to arrive at a unique quantum theory (details of this to be discussed more later).

In the popular science conception of string theory, we imagine replacing particles with strings, and the harmonics of these strings correspond to different particles, including the graviton. How do we reconcile this with the idea that gravity is just a function of the curvature of space time? Answer: we assume that we are close to some well-understood solution with metric $\eta_{\mu\nu}$ and take the new metric to be a perturbation,

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x).$$

Now that we have some spacetime structure, we can start to talk about interactions. We might have a propagator for strings, and also interaction vertices with some rules. We might think that an equivalent of Feynman diagrams emerges to tell us how strings can mingle and talk to each other.

In QFT, we were given some Lagrangian and from that Lagrangian, we derived interactions and Feynman rules. But in string theory, the situation is a bit backwards. It's as though we've been given some Feynman rules which do seem to reduce to the particle interactions in some limit, but we don't in some sense know the underlying theory where these rules come from.[7]

Classical theory In quantum mechanics, we have time t as a parameter and position $\hat{\mathbf{x}}$ as an operator. Of course, when we started learning quantum field theory, we were motivated to take our quantum fields $\hat{\phi}(\mathbf{x}, t)$ as operators and demote $\mathbf{x}$ to a simple label, so that $(\mathbf{x}, t)$ are both parameters. Space and time are on equal footing. This is the "second quantization" approach.

However, this isn't the only way we could do it. We could look for a formalism in which $\hat{x}^\mu = (\hat{\mathbf{x}}, \hat{t})$ are operators.

Example 1.1. Consider the *worldline formalism*. Imagine we have a massive particle propagating on a flat spacetime with metric $\eta_{\mu\nu}$. A suitable action for this theory might be

$$S[x] = -m \int_{s_1}^{s_2} ds, \tag{1.2}$$

where we use natural units of $\hbar = c = 1$ and the m is some mass due to dimensional concerns. This has a sort of geodesic interpretation for some integration measure ds. We can parametrize the worldline (e.g. in terms of proper time) such that

$$S[x] = -m \int_{\tau_1}^{\tau_2} d\tau \sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}. \tag{1.3}$$

Here, dots indicate derivatives with respect to proper time. The conjugate momentum is then

$$P_\mu(\tau) = -\frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}}, \tag{1.4}$$

which obeys $P^2 + m^2 = 0$, so this is an "on-shell" formalism. We could then vary $S[x]$ with respect to trajectories $x^\mu(\tau)$ to find the equations of motion. We could imagine doing the same for an extended object and tracing out a "worldsheet" instead.

[7] "There are many reasons to study string theory. I suppose for you lot, you've got nothing better to do between the hours of 11 to 12." –R.A. Reid-Edwards

However, before we do that, let us revisit our action 1.2. In particular, we shall rewrite it as

$$S[x,e] = \frac{1}{2}\int d\tau \left(e^{-1}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - em^2\right). \tag{1.5}$$

This new action has a sensible massless limit, unlike the previous action. For our new action, the $x^\mu(\tau)$ equation of motion is then

$$\frac{d}{d\tau}(e^{-1}\dot{x}^\mu) = 0 \tag{1.6}$$

and the $e(\tau)$ equation of motion gives

$$\dot{x}^2 + e^2m^2 = 0. \tag{1.7}$$

Now $e(\tau)$ appears algebraically, so we can substitute it back into the action to recover our previous formulation 1.2.[8]

Our theory also has some symmetry. If we shift the proper time by a function $\tau \to \tau + \zeta(\tau)$, then x and e change as

$$\delta x^\mu = \zeta\dot{x}^\mu$$
$$\delta e = \frac{d}{d\tau}(\zeta e).$$

We can use the one arbitrary degree of freedom to gauge fix $e(\tau)$ to a convenient value.

There's also a *rigid symmetry* which takes

$$x^\mu(\tau) \to \Lambda^\mu{}_\nu x^\nu(\tau) + a^\mu,$$

which we may recognize as Poincaré invariance in the background spacetime.[9]

Non-lectured aside: reparameterization invariance Here, we'll explicitly show that the action 1.5 is invariant under the transformation

$$\tau \to \tau + \xi(\tau). \tag{1.8}$$

For some reason, this is not spelled out in either David Tong's notes or the standard textbooks I've consulted so far.

We make the assumption as in lecture that x and e change as

$$\delta x^\mu = \zeta\dot{x}^\mu$$
$$\delta e = \frac{d}{d\tau}(\zeta e).$$

If so, then note that

$$\delta(\dot{x}^\mu) = \frac{d}{d\tau}(\delta x^\mu) = \frac{d}{d\tau}(\lambda\dot{x}^\mu) \tag{1.9}$$

and

$$\frac{1}{e+\delta(e)} \sim \frac{1}{e} - \frac{1}{e^2}\delta(e) \implies \delta(e^{-1}) = -\frac{1}{e^2}\delta(e). \tag{1.10}$$

To perform this calculation, we'll also need the equations of motion from lecture, 1.6 and 1.7, reproduced here:

$$\frac{d}{d\tau}(e^{-1}\dot{x}^\mu) = 0$$

and

$$\dot{x}^2 + e^2m^2 = 0.$$

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[8] Explicitly, we see that

$$\begin{aligned} S[x,e] &= \frac{1}{2}\int d\tau(e^{-1}\dot{x}^2 - em^2) \\ &= \frac{1}{2}\int d\tau(e^{-1}(-e^2m^2) - em^2) \\ &= \int d\tau(-em^2) \end{aligned}$$

and by setting $e = 1/m$ we recover 1.2.

[9] We can see that the action respects this symmetry, since it only depends on $\dot{x}^\mu$ and not x^μ (so translational symmetry is preserved) and $\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \to \eta_{\mu\nu}\Lambda^\mu{}_\sigma\dot{x}^\sigma\Lambda^\nu{}_\tau\dot{x}^\tau = \eta_{\sigma\tau}\dot{x}^\sigma\dot{x}^\tau$, so $\dot{x}^2$ is also preserved under Lorentz transformations as it should be.

Let's vary the action!

$$\begin{aligned}
\delta S[x,e] &= \frac{1}{2}\int d\tau \left[\delta(e^{-1})\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + e^{-1}\eta_{\mu\nu}\delta(\dot{x}^\mu)\dot{x}^\nu + e^{-1}\eta_{\mu\nu}\dot{x}^\mu\delta(\dot{x}^\nu) - \delta(e)m^2\right] \\
&= \frac{1}{2}\int d\tau \left[-\frac{1}{e^2}\delta(e)\dot{x}^2 + 2e^{-1}\eta_{\mu\nu}\frac{d}{d\tau}(\lambda\dot{x}^\mu)\dot{x}^\nu - \delta(e)m^2\right] \\
&= \frac{1}{2}\int d\tau \left[-\frac{1}{e^2}\delta(e)(\dot{x}^2 + m^2e^2) + 2(e^{-1}\dot{x}^\nu)\eta_{\mu\nu}\frac{d}{d\tau}(\lambda\dot{x}^\mu)\right] \\
&= \frac{1}{2}\int d\tau \frac{d}{d\tau}(\lambda e^{-1}\dot{x}^2) \\
&= 0.
\end{aligned}$$

In going from the first to the second line, we have explicitly substituted the variations for e^{-1} and for $\dot{x}^\mu$. In going from the second to the third, we simply regrouped terms into $\dot{x}^2 + m^2e^2$, which is zero by the equations of motion, and into $e^{-1}\dot{x}^\nu$, which is constant by the other equation of motion and therefore can be moved inside the total time derivative $\frac{d}{d\tau}$.

We see that after variation, what remains is simply an integral $\int d\tau$ of a total derivative, which is zero when evaluated at the endpoints of the action integral by the boundary conditions. Therefore the action is indeed invariant under reparametrization. ⊠

Lecture 2.

Monday, January 21, 2019

Last time, we introduced a *worldline action* with an einbein e (auxiliary field).

$$S[x,e] = \frac{1}{2}\int d\tau \left(e^{-1}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - em^2\right).$$

In the massless limit, this reduces to

$$S[X,e] = \frac{1}{2}d\tau e^{-1}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu, \tag{2.1}$$

where we have replaced the Minkowski metric with some generic metric. The classical equations of motion for $X^\mu(\tau)$ then give the geodesic equation,

$$\ddot{X}^\mu + \Gamma^\mu_{\nu\lambda}\dot{X}^\nu\dot{X}^\lambda = 0. \tag{2.2}$$

The $e(\tau)$ equations of motion would give some constraints. However, if we attempted to quantize this theory, we would find that the background metric $g_{\mu\nu}$ is not actually deformed in the solutions. Rather than being dynamic as in general relativity, it's sort of a thing that is given to us and sits in the background, unchanging, which is why for a particle this is not a theory of quantum gravity. As we'll see, this is *not* the case for strings.

Strings As a string moves through some flat spacetime $\mathcal{M}$ with metric $\eta_{\mu\nu}$, it sweeps out a worldsheet Σ. Assume that the string is closed, so it has a coordinate σ (along the length of the string, if you like):

$$\sigma \sim \sigma + 2n\pi, n \in \mathbb{Z}.$$

And it moves through time as parametrized by a proper time τ, so the embedding of the worldsheet is given by $X^\mu(\sigma,\tau)$. That is, σ and τ provide good coordinates for the worldsheet in $\mathcal{M}$.

Definition 2.3. We call these X^μ embedding fields. They are maps $X : \Sigma \to \mathcal{M}$ from the worldsheet to the background spacetime manifold.

We also say that the area of the worldsheet Σ is given by

$$\text{area} = \int d\tau d\sigma \sqrt{-\det(\eta_{\mu\nu}\partial_a X^\mu \partial_0 X^\nu)} \tag{2.4}$$

where $\sigma^a = (\tau, \sigma)$ so that $\partial_a = \frac{\partial}{\partial \sigma^a}$. In fact, we shall introduce an extra factor know (for historical reasons) as α' and write

$$S[X] = -\frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{-\det(\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu)}, \tag{2.5}$$

where α' is a free parameter. We often refer to the *string length*,

$$l_s \equiv 2\pi\sqrt{\alpha'} \tag{2.6}$$

or the *tension*

$$T \equiv \frac{1}{2\pi\alpha'}. \tag{2.7}$$

Definition 2.8. The object

$$G_{ab} \equiv \eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \tag{2.9}$$

is an induced metric on Σ, and the action 2.5 is called the *Nambu-Goto action*.

Having just defined this, we won't really do anything with it for the rest of the course. Bummer. However, to make up for it, let's write down a new and improved action, the *Polyakov action*.

Definition 2.10. Consider the action

$$S[X, h] = -\frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{-h}\, h^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu. \tag{2.11}$$

This should remind us of what we did with the einbein last lecture, where we introduced e into our action.

This *Polyakov action* is classically equivalent to the Nambu-Goto action, since this auxiliary h which we have introduced will turn out to be non-dynamical.

The h_{ab} equations of motion are given by a weird variation of the action,

$$-\frac{2\pi}{\sqrt{-h}} \frac{\delta S}{\delta h^{ab}} = 0. \tag{2.12}$$

These equations of motion give the vanishing of the stress tensor, $T_{ab} = 0$, where

$$T_{ab} = -\frac{1}{\alpha'}\left(\partial_a X^\mu \partial_b X_\mu - \frac{1}{2} h_{ab} \partial_c X^\mu \partial_d X_\mu h^{cd}\right). \tag{2.13}$$

Note that in two dimensions, $T_{ab}h^{ab} = 0$, i.e. T_{ab} is traceless. This is our first indication that something is different about two dimensions.

The X^μ equations of motion are

$$\frac{1}{\sqrt{-h}}(\partial_a \sqrt{-h}\, h^{ab} \partial_b X^\mu) = 0, \quad \Box X^\mu = 0. \tag{2.14}$$

Now we could imagine adding a cosmological constant (which would cause the trace of the stress tensor to change) or perhaps some sort of Einstein-Hilbert term to our metric h_{ab}. But we'll see why this might be more complicated than it initially seems.

Symmetries The Polyakov action 2.11 has the following symmetries:

- Rigid (global) symmetry, $X^\mu(\sigma, \tau) \to \Lambda^\mu{}_\nu X^\nu(\sigma, \tau) + a^\mu$ (Poincaré invariance).
- Local symmetries– the physics should be invariant under reparametrizations of the coordinates of the worldsheet, so under transformations $\sigma^a \to \sigma'^a(\sigma, \tau)$. The fields themselves transform as

$$X'^\mu(\sigma', \tau') = X^\mu(\sigma, \tau)$$
$$h_{ab}(\sigma, \tau) = \frac{\partial \sigma'^c}{\partial \sigma^a} \frac{\partial \sigma'^d}{\partial \sigma^b} h'_{cd}(\sigma', \tau').$$

Infinitesimallly, this means that $\sigma^a \to \sigma^a - \xi^a(\sigma, \tau)$, which gives us the variations

$$\begin{aligned}
\delta X^\mu &= \xi^a \partial_a X^\mu \\
\delta h_{ab} &= \xi^c \partial_c h_{ab} + \partial_a \xi^c h_{cb} + \partial_b \xi^c h_{ca} = \nabla_a \xi_b + \nabla_b \xi_a \\
\delta\sqrt{-h} &= \partial_a(\xi^a \sqrt{-h}).
\end{aligned}$$

Note this second variation, δh_{ab}, can be written in terms of some covariant derivatives for an appropriate connection, but we won't usually bother.

- Weyl transformations– we send

$$X'^{\mu}(\sigma,\tau) = X^{\mu}(\sigma,\tau)$$
$$h'_{ab}(\sigma,\tau) = e^{2\Lambda(\sigma,\tau)}h_{ab}(\sigma,\tau).$$

Thus $\delta X^{\mu} = 0$ and $\delta h_{ab} = 2\Lambda h_{ab}$. Under such transformations, we have three arbitrary degrees of freedom in (ξ^a, Λ) (two from the two components of ξ plus one from Λ), and we can use them to fix the three degrees of freedom in h_{ab} (there are three, since h is symmetric and 2×2).

Classical solutions Let us now use reparametrization invariance to fix

$$h_{ab} = e^{2\phi}\eta_{ab}, \quad \eta_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.15}$$

The Polyakov action then becomes

$$S[X] = -\frac{1}{4\pi\alpha'}\int_{\Sigma} d^2\sigma(-\dot{X}^2 + X'^2), \tag{2.16}$$

where

$$\dot{X}^{\mu} \equiv \frac{\partial X^{\mu}}{\partial \tau}, \quad X'^{\mu} \equiv \frac{\partial X^{\mu}}{\partial \sigma} \tag{2.17}$$

and squares are taken by contracting with the metric h_{ab}. In that case, the $X^{\mu}(\sigma,\tau)$ equation of motion becomes the wave equation in 2D, so solutions are of the form

$$X^{\mu}(\sigma,\tau) = X_R^{\mu}(\tau - \sigma) + X_L^{\mu}(\tau + \sigma). \tag{2.18}$$

Moreover, since we have a wave equation it is useful to introduce modes $(\alpha_n^{\mu}, \bar{\alpha}_n^{\mu})$ where

$$X_R^{\mu}(T - \sigma) = \frac{1}{2}x^{\mu} + \frac{\alpha'}{2}p^{\mu}(\tau - \sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^{\mu}e^{-in(\tau-\sigma)}, \tag{2.19}$$

where x^{μ}, p^{μ} are some constants in (τ,σ) and similarly the left-going modes are

$$X_L^{\mu}(T + \sigma) = \frac{1}{2}x^{\mu} + \frac{\alpha'}{2}p^{\mu}(\tau + \sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\bar{\alpha}_n^{\mu}e^{-in(\tau+\sigma)}. \tag{2.20}$$

It's sometimes useful to define a zero-mode,

$$\alpha_0^{\mu} = \bar{\alpha}_0^{\mu} = \sqrt{\frac{\alpha'}{2}}p^{\mu}. \tag{2.21}$$