

GENERAL RELATIVITY

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These notes were taken for the *General Relativity* course taught by Malcolm Perry at the University of Cambridge as part of the Mathematical Tripos Part III in Michaelmas Term 2018. I live- \TeX ed them using TeXworks, and as such there may be typos; please send questions, comments, complaints, and corrections to itel2@cam.ac.uk.

Many thanks to Arun Debray for the \LaTeX template for these lecture notes: as of the time of writing, you can find him at <https://web.ma.utexas.edu/users/a.debray/>.

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Lecture 1.

Friday, October 5, 2018

Unlike in previous years, this course is intended to be a stand-alone course on general relativity, building up the mathematical formalism needed to construct the full theory and explore some examples of interesting spacetime metrics. It is linked to the Black Holes course taught in Lent term, which I will also be writing notes for.

Some recommended course materials and readings include the following:

- Sean Carroll, Spacetime and Geometry¹
- Misner, Thorne, and Wheeler, Gravitation
- Wald, General Relativity
- Zee, Einstein Gravity in a Nutshell
- Hawking and Ellis, “The Large Scale Structure of Spacetime”

¹I should point out that Sean Carroll’s textbook is based off a set of GR notes which are available for free online. They are really excellent and pedagogically structured, and I have cross-referenced them frequently when revising these notes after lecture. Here is a link to the PDF on the arXiv: <https://arxiv.org/pdf/gr-qc/9712019.pdf>

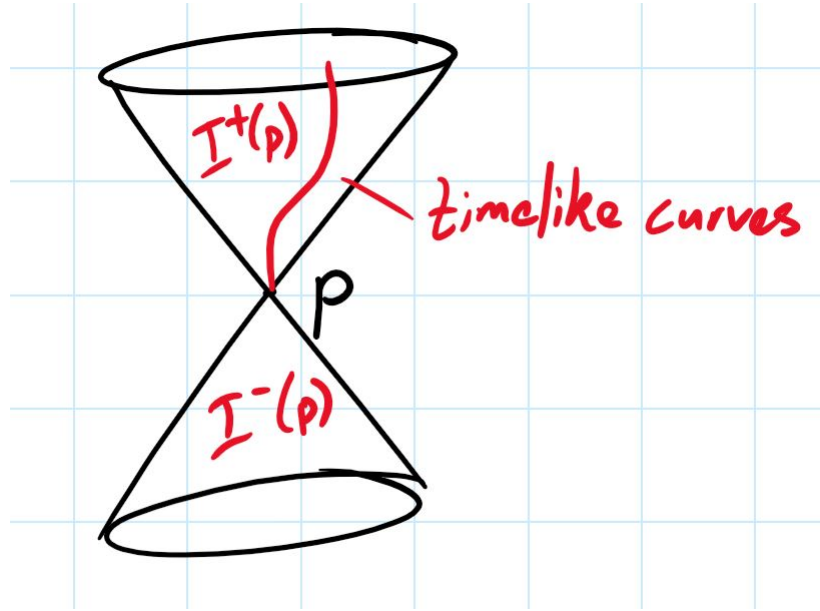


FIGURE 1. An illustration of the light cones from a point p , plus the chronological future I^+ and chronological past I^- . Also depicted in red is a timelike curve (e.g. a possible particle trajectory in spacetime).

In Minkowski² spacetime (flat space) we specify points in spacetime by spatial coordinates in \mathbb{R}^3 , i.e. the Cartesian coordinates (x, y, z) , plus a time coordinate t . The line element (spacetime separation) is given by the metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2.$$

ds is the proper distance between x and $x + dx$, y and $y + dy$, z and $z + dz$, and t and $t + dt$. (As is typical in relativity, we work in units where $c = 1$. Note that the metric sign convention here is flipped from my QFT notes, which uses the “mostly minus” convention—this is arbitrary and so long as one is consistent it makes no difference.) Using the Einstein summation convention, the metric is usually written more compactly as

$$ds^2 = \eta_{\alpha\beta} x^\alpha x^\beta,$$

with $\eta_{\alpha\beta}$ the Minkowski space metric.

Let’s recall from special relativity that we call separations with $ds^2 > 0$ “spacelike,” with $ds^2 < 0$ “timelike,” and $ds^2 = 0$ null (or occasionally lightlike).

Definition 1.1. The *chronological future* of a point p is the set of all points that can be reached from p along future directed timelike lines, and we call this $I^+(p)$. It is the interior of the future-directed light cone. Conversely we have the *chronological past* of p , $I^-(p)$, which is the interior of the past-directed light cone. We also have the *causal future* of p , which is the set of all points that can be reached from p along future-directed timelike or null lines, and we call this $J^+(p)$. Similarly we have the *causal past*, $J^-(p)$. Thus J is the closure of I and is the interior *plus* the light cone itself.

Let $x^a(\tau)$ be a curve in spacetime.³ Then the tangent vector to the curve is $u^a = \frac{dx^a}{d\tau}$. For timelike curves, $u^a u^b \eta_{ab} = -1 \iff \tau$ is the proper time along the curve.⁴ We also know that $\int_p^q d\tau = \Delta\tau$, which just

²I’ve heard some USAmericans pronounce this “min-cow-ski.” In German, it is “min-koff-ski.”

³Evidently we are not using the convention that Greek indices range from 0 to 3 and Latin indices range from 1 to 3. I have copied the lecturer’s convention here, but may change to more traditional notation if it becomes relevant.

⁴The property that $U^a U^b \eta_{ab} = -1$ is easy to prove. See the Special Relativity catch-up sheet found [here](#) for some nice exercises in SR: this is exercise 3. Assuming the result of exercise 2 which states that the four-velocity of a massive particle is $U^\mu = \gamma(1, v^i)$, we then have $U \cdot U = \gamma^2(-1 + v^2) = \frac{v^2-1}{1-v^2} = -1$. Since this is a fully contracted expression (no indices floating around), it is true in all frames.

says that the integral of $d\tau$ along a curve from p to q yields the proper time interval, what a clock actually measures.

We also remark that Minkowski space has some very nice symmetries. Since x, y , and z do not appear explicitly in the metric, our spacetime is invariant under translations. It is also invariant under rotations in \mathbb{R}^3 . It would be nice to extend rotations to include the time coordinate t as well– this is exactly what a Lorentz transformation does.

Lorentz transformations in general involve time– they are defined by the matrices Λ which satisfy

$$\Lambda^T \eta \Lambda = \eta,$$

i.e. they preserve the inner product η in Minkowski space, forming the group $O(3, 1)$. Lorentz transformations consist of rotations in \mathbb{R}^3 and boosts. This is equivalent to the defining property of rotation matrices R that $R^T \delta R = \delta$, meaning that rotation matrices preserve the standard Euclidean inner product in \mathbb{R}^3 and form the group $O(3)$.⁵ Written explicitly, the Lorentz boost in the x -direction to a frame moving with velocity v is

$$\begin{aligned} t \rightarrow t' &= \frac{t - vx}{\sqrt{1 - v^2}} \\ x \rightarrow x' &= \frac{x - vt}{\sqrt{1 - v^2}} \\ y \rightarrow y' &= y \\ z \rightarrow z' &= z \end{aligned}$$

We may also write it in matrix notation,

$$\Lambda^a_b = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where γ is defined in the usual way by $\gamma \equiv \frac{1}{\sqrt{1 - v^2}}$.

Rather than constructing the (in general complicated) Lorentz boost in an arbitrary direction, it is often more convenient to rotate one's frame of reference in \mathbb{R}^3 so the boost is in the new x -direction, perform the Lorentz boost, and then transform back:

$$R^T \Lambda R = \Lambda_R,$$

where Λ_R is a new Lorentz transformation.⁶

Definition 1.2. The Lorentz transformations taken together form the *Lorentz group*. It satisfies the group axioms of identity, unique inverses (since $\det \Lambda \neq 0$), associativity (from associativity of matrix multiplication), and closure (see footnote for proof).⁷

Λ can include reflections in time or space. To avoid such complications, we sometimes refer to the *proper orthochronous Lorentz group*, i.e. to exclude space and time reversals, but often we are more careless and simply call it the Lorentz group.

Definition 1.3. The *Poincaré group* is then the semidirect product of Lorentz transformations and translations. This is the group of symmetries of Minkowski space.

⁵Strictly, $O(3)$ also includes reflections– for matrices which preserve both orientation and the inner product, we must also require that $\det R = +1$, defining the group $SO(3)$. We'll see a similar caveat with the Lorentz group in just a second.

⁶It's easy to check that Λ_R really is a Lorentz transformation– just observe that rotations alone are a subset of Lorentz transformations, since they preserve the inner product on \mathbb{R}^3 and do not affect the time coordinate. In the language of group theory, rotations form an $SO(3)$ subgroup of the full Lorentz group $O(3, 1)$ – see Definition 1.2. Therefore any combination of rotations and Lorentz boosts will form another valid Lorentz transformation by the group closure property.

⁷More precisely, we know that the determinant is nonzero since $-1 = \det \eta = \det(\Lambda^T \eta \Lambda) = \det(\Lambda^T) \det(\eta) \det(\Lambda) = (-1) \det(\Lambda)^2 \implies \det(\Lambda) = \pm 1 \neq 0$. To prove closure, suppose Λ_1, Λ_2 are Lorentz transformations. The product $\Lambda_1 \Lambda_2$ then satisfies $(\Lambda_1 \Lambda_2)^T \eta (\Lambda_1 \Lambda_2) = \Lambda_2^T \Lambda_1^T \eta \Lambda_1 \Lambda_2 = \Lambda_2^T \eta \Lambda_2 = \eta$, so $\Lambda_1 \Lambda_2$ is also a Lorentz transformation.

We have translations defined as

$$x^a \rightarrow x^{a'} = x^a + \Delta x^a$$

and also Lorentz transformations, with the property

$$(\Lambda^T)_a^c \eta_{cd} \Lambda^d_b = \eta_{ab}.$$

Definition 1.4. We also have *contravariant vectors* (indices up) written u^a and their corresponding *covariant* vectors (indices down)

$$u_a \equiv \eta_{ab} u^b,$$

where we have used the metric to lower an index. These are sometimes equivalently called simply vectors and covectors. We can also raise indices using the inverse metric η^{ab} (defined by $\eta^{ab}\eta_{bc} = \delta_c^a$). Thus

$$u^b = \eta^{ba} u_a.$$

We define the Lorentz transformation of a contravariant vector as $u^a \rightarrow u^{a'} = \Lambda^a_{b'} u^b$. For instance, x^a is an example of a contravariant vector.

Definition 1.5. A *scalar* is an object which is invariant under a Lorentz transformation. We saw that a covariant vector transforms with right multiplication by the Lorentz transformation, whereas a contravariant vector transforms by left multiplication.

More generally, a *tensor of type (r, s)* transforms with r copies of the Lorentz transformation on the r up indices and s copies of the Lorentz transformation on the s down indices,

$$T^{\mu_1 \mu_2 \dots \mu_r}_{\nu_1 \nu_2 \dots \nu_s} \rightarrow T^{\alpha_1 \alpha_2 \dots \alpha_r}_{\beta_1 \beta_2 \dots \beta_s} = \Lambda^{\alpha_1}_{\mu_1} \dots \Lambda^{\alpha_r}_{\mu_r} T^{\mu_1 \mu_2 \dots \mu_r}_{\nu_1 \nu_2 \dots \nu_s} \Lambda^{\nu_1}_{\beta_1} \dots \Lambda^{\nu_s}_{\beta_s} \quad (1.6)$$

By this definition, a scalar may be thought of as a type (0,0) tensor, a contravariant vector a type (1,0) tensor, and a covariant vector a type (0,1) tensor.

Lecture 2.

Monday, October 8, 2018

Today, we'll start by remarking that Maxwell's equations can be written compactly in 4-vector format. Recall from a good course on electrodynamics that we define the electromagnetic field strength tensor $F^{\mu\nu}$ as

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}.$$

$F^{\mu\nu}$ is a totally antisymmetric rank two tensor. Defining the four-current $j^\mu \equiv (\rho, \mathbf{j})$ with \mathbf{j} the ordinary current density and ρ the charge density, we see that

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0$$

and

$$\partial_a F^{ab} = -j^b.$$

But there's something strange about this—these equations in their current form hold for cartesian coordinates only. Of course, the laws of physics (i.e. as expressed through observable results in experiments) cannot depend on the coordinate system used.

Example 2.1. The Minkowski metric takes the Cartesian form

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

but if we pass to spherical coordinates, the metric now takes the form

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 = g_{ab} dx^a dx^b.$$

General relativity is thus motivated by a desire to understand how the laws of physics are invariant not just under Lorentz transformations but general coordinate transformations. It is also motivated by the *weak equivalence principle*, which states that inertial mass and gravitational mass are the same thing– the m in $F = ma$ and the m in $F = -\frac{GMm}{r^2}$ are the same mass! This is closely related to the *Einstein equivalence principle*, which states that in a freely falling frame, the laws of physics are those of special relativity. One cannot distinguish between being in freefall under a gravitational field and simply being at rest in no gravitational field.

We consider spacetime to be a 4-dimensional system (3 + 1 dimensions, if you like) and in particular it has a manifold structure. We may make an explicit choice of x^a some coordinates that label points in M , but it would be nice to define vectors in a way that is independent of the coordinates. This will lead us to revisit vectors and covectors.

Consider a curve $\lambda(\tau) : \mathbb{R} \rightarrow M$ a parametrized curve sitting in M . Now take $f = f(x^a)$ a differentiable function of the coordinates, and define an operator that maps f into df/dt : by applying the chain rule, we have

$$df/dt = \frac{\partial x^a}{\partial t} \left(\frac{\partial}{\partial x^a} f \right).$$

Thus a vector is a differential operator that acts on f : explicitly, it is $\frac{\partial x^a}{\partial t} \frac{\partial}{\partial x^a}$, where the $\frac{\partial x^a}{\partial t}$ are the components of the vector.

A general vector may therefore be written in its components in some basis x^a as

$$V = V^a \frac{\partial}{\partial x^a}.$$

Thinking back to our curve $\lambda(\tau)$, we may expand our coordinates locally as $x^a(\tau) = x^a(\tau_0) + V^a(\tau - \tau_0) + O((\tau - \tau_0)^2)$, where V is the tangent vector to some curve through the point τ_0 . (Okay, we're being a bit careless with notation here– the instructor has written $\lambda(t)$, but sometimes t is a coordinate on the manifold.) Therefore we may also interpret (tangent) vectors as describing how our manifold curves locally about a point.

Vectors (somewhat obviously) form a vector space. If W, Y are vectors, α, β real numbers, then $\alpha W + \beta Y$ is another vector with components

$$(\alpha W^a + \beta Y^a) \frac{\partial}{\partial x^a}.$$

As (multi)linear differential operators, vectors obey the Leibniz rule

$$V^a \frac{\partial}{\partial x^a} (fg) = V^a \frac{\partial f}{\partial x^a} g + f V^a \frac{\partial g}{\partial x^a}.$$

So they form a vector space (check the vector space axioms again).

The space of tangent vectors at a point p is called $T_p(M)$. Recall that we defined our tangent vectors with respect to its components in some basis x^a . But if we now change to $\tilde{x}^b = \tilde{x}^b(x^a)$, then by the chain rule our basis vectors $\frac{\partial}{\partial x^a}$ transform as

$$\frac{\partial}{\partial x^a} = \frac{\partial \tilde{x}^b}{\partial x^a} \frac{\partial}{\partial \tilde{x}^b}.$$

But V as an operator is invariant– it does not depend on our choice of coordinates, so only its decomposition into basis vectors can change. This means that if we rewrite V in a different set of coordinates, we find that

$$V = V^a \frac{\partial}{\partial x^a} = \tilde{V}^a \frac{\partial}{\partial \tilde{x}^a} = V^a \frac{\partial \tilde{x}^b}{\partial x^a} \frac{\partial}{\partial \tilde{x}^b},$$

so by comparison the components of V transform as

$$V^a \rightarrow \tilde{V}^{a'} = \frac{\partial \tilde{x}^{a'}}{\partial x^a} V^a.$$

In other words, tangent vectors transform as contravariant vectors, which is a generalization of the formula in special relativity where we had

$$\frac{\partial \tilde{x}^{a'}}{\partial x^a} = \Lambda^{a'}_a$$

with $\Lambda_a^{a'}$ the Lorentz transform.

Definition 2.2. We may also define *one-forms*, which are covariant vectors at some point p . Thus the inner product $\langle \omega, V \rangle$ is a real number, with ω a 1-form and V a vector. The inner product is bilinear: if $V = \alpha Y + \beta W$, then

$$\langle \omega, \alpha Y + \beta W \rangle = \alpha \langle \omega, Y \rangle + \beta \langle \omega, W \rangle$$

and similarly for the first argument, if $\omega = \alpha \eta + \beta \xi$

$$\langle \alpha \eta + \beta \xi, V \rangle = \alpha \langle \eta, V \rangle + \beta \langle \xi, V \rangle.$$

Let us write V in a basis, $V = V^a E_a$ with E_a some set of basis vectors. Then $\omega = \omega_a E^a$ has components in some basis of one forms E^a . We have that $\langle E^a, E_b \rangle = \delta_b^a$, where E^a forms a basis of 1-forms which is dual to the ordinary basis vectors. (Remark: the components V^a of a vector transform like coordinate functions, while the components of a one-form ω_a transform like basis vectors E_a .) We can then compute the inner product of a generic one-form and a vector,

$$\begin{aligned} \langle \omega, V \rangle &= \langle \omega_a E^a, V^b E_b \rangle \\ &= \omega_a V^b \delta_b^a \\ &= \omega_a V^a. \end{aligned}$$

Lecture 3.

Wednesday, October 10, 2018

A quick admin note There is no lecture Monday 15 October. In addition, office hours will be Tuesdays at 4 PM in B1.26. Moving on.

Let us recall that we have a multiplication law on one-forms and vectors,

$$\langle \omega, X \rangle = \omega_a X^a$$

for ω any one-form, X any vector. That is, we can write this product in terms of the components of ω and X .

Definition 3.1. With this in mind, we define the *differential* of a function $f : M \rightarrow \mathbb{R}$ to be the one-form df , such that

$$\langle df, X \rangle = Xf$$

(that is, X as a differential operator acting on f).

Example 3.2. Non-lectured example: consider the function $f = x + y$ in \mathbb{R}^3 and let $X = \frac{\partial}{\partial y}$. (We have chosen a coordinate basis to make the computation clearer.) Then $df = dx + dy$ (a one-form) and now

$$\langle df, X \rangle = Xf = \frac{\partial}{\partial y}(x + y) = 1.$$

Recall we have a basis of 1-forms E^a and a basis of vectors E_b with $\langle E^a, E_b \rangle = \delta_b^a$. In a coordinate basis, the basis vectors take the form

$$E_a = \frac{\partial}{\partial x^a} \text{ and } E^b = dx^b.$$

Thus

$$\langle dx^a, \frac{\partial}{\partial x^b} \rangle = \delta_b^a.$$

Definition 3.3. A one-form is *exact* if it can be written as df for some scalar f . For instance, dt and dr are exact because they are the differentials of t and r , but $r d\theta$ is not exact. However, the one-form $r dr$ is exact, since it can be written $d(r^2/2)$.

In Minkowski space with Cartesian coordinates, the natural basis of one-forms dt, dx, dy, dz forms a coordinate basis since each of these is exact, and the basis of vectors dual to this is $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$.

However, in spherical coordinates the Minkowski metric looks different. It takes the form

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2.$$

The basis of one-forms here,

$$dt, dr, r d\theta, r \sin \theta d\phi$$

is not a coordinate basis because these are not all of the form df . The set of basis vectors dual to the one-forms in spherical coordinates is also kind of bad. They take the form

$$\frac{\partial}{\partial t}, \frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \theta}, \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi},$$

and these are not a coordinate basis because they are not of the form $\frac{\partial}{\partial x^a}$ (equivalently, they are not dual to exact one-forms).

However, we remark that our defining equation for the product of a one-form and vector produces an ordinary scalar, which must be invariant under coordinate transformations:

$$\langle \omega, X \rangle = \omega_a X^a \text{ in any basis.}$$

This determines how the components of a one-form ω_a change under coordinate transformations. In a coordinate basis, we know that the components of a vector transform like coordinate functions:

$$X^a \rightarrow \tilde{X}^{a'} = \frac{\partial \tilde{x}^{a'}}{\partial x^a} X^a.$$

Therefore in a coordinate basis, the components of a one-form must transform in the inverse way,

$$\omega_a \rightarrow \tilde{\omega}_{a'} = \frac{\partial x^a}{\partial \tilde{x}^{a'}} \omega_a.$$

Note where the primed indices lie and which coordinates are the new coordinates \tilde{x} versus the old coordinates x . The factor here $\frac{\partial x^a}{\partial \tilde{x}^{a'}}$ is analogous to how the Lorentz transformation acts (as the Lorentz transformation is a particular coordinate transformation satisfying certain constraints).

Suppose that $\langle df, X \rangle = 0$ for some df . If one is working in n dimensions, this gives one constraint equation on the n components of X . Thus, there are $(n - 1)$ different linearly independent choices of X which solve this equation and therefore span an $n - 1$ -dimensional space. We have put one constraint specified by f on our space of all possible X such that df is the normal to the surface $f = \text{constant}$.

Example 3.4. Again, a non-lectured concrete example. Let us again work in \mathbb{R}^3 and set $f = x$. Then a general X can be written as $X^a \frac{\partial}{\partial x^a}$ and the condition that $\langle df, X \rangle = 0$ can be computed explicitly as

$$\langle df, X \rangle = \left(X^1 \frac{\partial}{\partial x} + X^2 \frac{\partial}{\partial y} + X^3 \frac{\partial}{\partial z} \right) (x) = X^1(1) = 0.$$

Therefore our surface is defined by $X^1 = 0$ but we may choose X^2 and X^3 freely ($n - 1 = 2$ free choices). Indeed, we see that $df = dx$ is normal to the surface $f = x = \text{constant}$.

A tensor of type (r, s) in a basis of 1-forms E^a and vectors E_a takes the form

$$T = T^{a_1 \dots a_r}_{b_1 \dots b_s} E_{a_1} \otimes E_{a_2} \otimes \dots \otimes E_{a_r} \otimes E^{b_1} \otimes \dots \otimes E^{b_s},$$

where \otimes is the tensor product (not just a direct product!).⁸

T is coordinate invariant, so in a coordinate basis the components of T transform as

$$\tilde{T}^{a'_1 \dots a'_r}_{b'_1 \dots b'_s} = \frac{\partial \tilde{x}^{a'_1}}{\partial x^{a_1}} \dots \frac{\partial \tilde{x}^{a'_r}}{\partial x^{a_r}} \frac{\partial x^{b_1}}{\partial \tilde{x}^{b'_1}} \dots \frac{\partial x^{b_s}}{\partial \tilde{x}^{b'_s}} T^{a_1 \dots a_r}_{b_1 \dots b_s}.$$

In a non-coordinate basis, these $\frac{\partial \tilde{x}^{a'}}{\partial x^a}$ are replaced by some general functions $\Phi_a^{a'}$ where $\tilde{x}^{a'} = \Phi_a^{a'} x^a$.

We can perform the symmetrization operation, denoted by putting indices to be symmetrized in parentheses:

$$X_{(a_1 \dots a_r)} \equiv \frac{1}{r!} [\text{sum of all permutations of } a_1 \dots a_r].$$

⁸Tensor products are more complicated than direct products because their addition structure is multilinear, i.e. linear in each argument individually but not all simultaneously. Where it might make sense to add $(2, 1) + (1, 2) = (3, 3)$ in $\mathbb{R} \times \mathbb{R}$, the equivalent tensor product in $\mathbb{R} \otimes \mathbb{R}$ would have $2 \otimes 1 + 1 \otimes 2 = 2 \otimes 1 + 2 \otimes 1 = (2 + 2) \otimes 1 = 4 \otimes 1$. So this is quite a different beast. More info on tensor products and tensors as mathematical constructions can be found at <https://jeremykun.com/2014/01/17/how-to-conquer-tensorphobia/>.

For example, $X_{(ab)} = \frac{1}{2} [X_{ab} + X_{ba}]$. Similarly we have the antisymmetrization operation, denoted by putting indices to be antisymmetrized in square brackets:

$$X_{[a_1 \dots a_r]} = \frac{1}{r!} [\text{sum over all even permutations} - \text{sum of all odd permutations}].$$

For example, $X_{[ab]} = \frac{1}{2} [X_{ab} - X_{ba}]$. Having defined symmetrization and antisymmetrization, we now consider a special class of tensor– the totally antisymmetric $(0, p)$ tensor.

Definition 3.5. A *differential p -form* is a tensor of type $(0, p)$ which is antisymmetric on all indices, i.e. $A_{a_1 \dots a_p} = A_{[a_1 \dots a_p]}$. Some familiar p -forms include the 2-form $F_{\mu\nu}$ from electromagnetism and the Levi-Civita symbol ϵ_{ijk} .

We can describe A in terms of basis vectors E^a using a construction called the wedge product.

Definition 3.6. The *wedge product* is a special kind of antisymmetrizing multiplication of a p -form and a q -form. For a p -form $A = A_{a_1 \dots a_p}$ and a q -form $B = B_{b_1 \dots b_q}$, the wedge product $A \wedge B$ is given by

$$(A \wedge B)_{a_1 \dots a_p b_1 \dots b_q} \equiv A_{[a_1 \dots a_p} B_{b_1 \dots b_q]}.$$

For instance $A \wedge B = (-1)^{pq} B \wedge A$ (this is easy to prove– we simply switch the q indices of B past the p indices of A and pick up the appropriate pq sign flips along the way).

As an invariant object, the p -form A can be written as

$$A = A_{a_1 \dots a_p} E^{a_1} \wedge \dots \wedge E^{a_p},$$

where $A_{a_1 \dots a_p}$ are now the components of the p -form A .

Definition 3.7. We also define the exterior derivative, a generalization of the usual derivative ∂_μ :

$$(dA)_{ba_1 \dots a_p} \equiv \frac{\partial}{\partial x^b} A_{[a_1 \dots a_p]} = \partial_{[b} A_{a_1 \dots a_p]}$$

defines a $p+1$ -form, as it is by definition antisymmetric in its $p+1$ indices. The exterior derivative of a product follows a variation of the Leibniz rule:

$$d(A \wedge B) = dA \wedge B + (-1)^p A \wedge dB.$$

Note that $ddA = 0$, so d is nilpotent (it kills all exact differentials).⁹

The gradient is a simple example of an exterior derivative of a 0-form (AKA a scalar):

$$(d\phi)_\mu = \partial_\mu \phi.$$

From prior experiences with special (or general) relativity, we might have an intuition that the metric has something to do with gravitation. The line element ds (defined by $ds^2 = g_{ab} dx^a dx^b$) is invariant and is therefore a (symmetric) tensor. In a freely falling frame, the metric of Minkowski space is

$$\tilde{\eta}_{a'b'} = \frac{\partial x^a}{\partial \tilde{x}^{a'}} \frac{\partial x^b}{\partial \tilde{x}^{b'}} g_{ab}.$$

Do such $\frac{\partial x^a}{\partial \tilde{x}^{a'}}$ always exist? The answer turns out to be yes– g_{ab} is not degenerate, so one may diagonalize it and then rescale the eigenvalues. Sylvester's theorem states that if g has r positive eigenvalues, s negative eigenvalues, then diagonalizing preserves this.

Therefore given g_{ab} that is non-degenerate, the inverse metric g^{ab} can be define with $g^{ab} g_{bc} = \delta_c^a$ the Kronecker delta. One may use the metric to raise and lower indices: $V_b = g_{bc} V^c$ and $V^a = g^{ab} V_b$.

"There are more unknowns than there are knowns." A brief summary of this course.

⁹Suppose we compute ddA : then we will have two derivatives in our expression $\partial_{[\mu} \partial_{\nu]} A_{a_1 \dots a_p]$. But derivatives commute, so to every $\partial_\alpha \partial_\beta$ term in the antisymmetrization sum there will be a corresponding $-\partial_\beta \partial_\alpha$ term. These terms cancel no matter what A is, so $ddA = 0$ identically.

Lecture 4.

Friday, October 12, 2018

Previously, we defined the exterior derivative, which took a p -form to a $p + 1$ -form. Now we will define the covariant derivative, an operation which in general takes a tensor of type (r, s) to a tensor of type $(r, s + 1)$.

Suppose we start with a scalar field $\phi(x)$. The ordinary derivative is just

$$\partial_a \phi = \frac{\partial \phi}{\partial x^a}.$$

Let us change coordinates to $\tilde{x}^{a'}$ some function of the original coordinates. Then this derivative transforms as

$$\partial_{a'} \phi = \frac{\partial x^a}{\partial \tilde{x}^{a'}} \frac{\partial}{\partial x^a} \phi = \frac{\partial x^a}{\partial \tilde{x}^{a'}} \partial_a \phi.$$

We might ask whether the derivative of a vector transforms in the same way. But instead, we get something a little different.

$$\begin{aligned} \partial_{b'} \tilde{V}^{a'} &= \frac{\partial x^b}{\partial \tilde{x}^{b'}} \frac{\partial}{\partial x^b} \left(\frac{\partial \tilde{x}^{a'}}{\partial x^a} V^a \right) \\ &= \frac{\partial x^b}{\partial \tilde{x}^{b'}} \frac{\partial V^a}{\partial x^b} \frac{\partial \tilde{x}^{a'}}{\partial x^a} + \frac{\partial x^b}{\partial \tilde{x}^{b'}} \frac{\partial^2 \tilde{x}^{a'}}{\partial x^a \partial x^b} V^a. \end{aligned}$$

This first part is tensorial, but the second part is not (it has a term which is a second derivative of the coordinates). In order to get a tensor from the derivative, we need to add a correction term. Let us define

$$\nabla_b V^a \equiv \partial_b V^a + \Gamma_{bc}^a V^c$$

where Γ_{bc}^a is called a *connection*. We can figure out how Γ transforms under coordinate transformations:

$$\tilde{\Gamma}_{b'c'}^{a'} = \frac{\partial \tilde{x}^{a'}}{\partial x^a} \frac{\partial x^a}{\partial \tilde{x}^{b'}} \frac{\partial x^c}{\partial \tilde{x}^{c'}} \Gamma_{bc}^a - \frac{\partial^2 \tilde{x}^{a'}}{\partial x^b \partial x^c} \frac{\partial x^b}{\partial \tilde{x}^{b'}} \frac{\partial x^c}{\partial \tilde{x}^{c'}}.$$

So Γ does *not* transform as a tensor, but that's actually what we want—this correction term allows us to get a proper tensor when we take the covariant derivative of a vector.

We'd like ∇ to be linear and obey the Leibniz rule: for two tensors T, S and two real numbers $\alpha, \beta \in \mathbb{R}$, we should have

$$\nabla(\alpha T + \beta S) = \alpha \nabla T + \beta \nabla S$$

and also

$$\nabla(T \otimes S) = \nabla T \otimes S + T \otimes \nabla S.$$

For a vector V and a one-form W , let $S = V^a W_a$. Then

$$\begin{aligned} \nabla_a S &= \partial_a S \\ &= (\partial_a V^b) W_b + V^b (\partial_a W_b) \\ &= (\nabla_a V^b) W_b - \Gamma_{ac}^b V^c W_b + V^b (\partial_a W_b) \\ &= (\nabla_a V^b) W_b + V^b \nabla_a W_b. \end{aligned}$$

Therefore for the Leibniz rule to hold on the product of a vector and a one-form, it must be that

$$\nabla_b W_a \equiv \partial_b W_a - \Gamma_{ba}^c W_c.$$

Note the sign flip from the vector definition! More generally, we can use Leibniz to deduce what the covariant derivative operator is on a general tensor of type (r, s) .

$$\nabla_c T_{b_1 \dots b_s}^{a_1 \dots a_r} = \partial_c T_{b_1 \dots b_s}^{a_1 \dots a_r} + \Gamma_{cd}^{a_1} T_{b_1 \dots b_s}^{da_2 \dots a_r} + \Gamma_{cd}^{a_2} T_{b_1 \dots b_s}^{a_1 da_3 \dots a_r} + \dots + \Gamma_{cd}^{a_r} T_{b_1 \dots b_s}^{a_1 a_2 \dots d} - \Gamma_{cb_1}^d T_{db_2 \dots b_s}^{a_1 \dots a_r} - \Gamma_{cb_2}^d T_{b_1 d \dots b_s}^{a_1 \dots a_r} - \dots - \Gamma_{cb_s}^d T_{b_1 b_2 \dots d}^{a_1 \dots a_r}.$$

So every upstairs indices we swap out gets a $+\Gamma$ and every downstairs index we swap gets a $-\Gamma$. Let's return to our expression for the transformation of Γ ,

$$\tilde{\Gamma}_{b'c'}^{a'} = \frac{\partial \tilde{x}^{a'}}{\partial x^a} \frac{\partial x^a}{\partial \tilde{x}^{b'}} \frac{\partial x^c}{\partial \tilde{x}^c} \Gamma_{bc}^a - \frac{\partial^2 \tilde{x}^{a'}}{\partial x^b \partial x^c} \frac{\partial x^b}{\partial \tilde{x}^{b'}} \frac{\partial x^c}{\partial \tilde{x}^{c'}}.$$

Note that the second part is symmetric under the interchange of b', c' . Therefore take just the part antisymmetric in b', c' :

$$\Gamma_{b'c'}^{a'} - \Gamma_{c'b'}^{a'} = \frac{\partial \tilde{x}^{a'}}{\partial x^a} \frac{\partial x^b}{\partial \tilde{x}^{b'}} \frac{\partial x^c}{\partial \tilde{x}^{c'}} (\Gamma_{bc}^a - \Gamma_{cb}^a).$$

Definition 4.1. The antisymmetric part of Γ transforms like a tensor, and so we define the torsion tensor

$$T_{bc}^a \equiv \Gamma_{bc}^a - \Gamma_{cb}^a = 2\Gamma_{[bc]}^a.$$

Some definitions define this up to a factor of 2 or with different signs.

Consider an arbitrary scalar S .

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)S = \nabla_a \partial_b S - \nabla_b \partial_a S.$$

If these were just partial derivatives, this would be zero. But working it out explicitly, we see that

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)S = \partial_a \partial_b S - \Gamma_{ab}^c \partial_c S - \partial_b \partial_a S + \Gamma_{ba}^c \partial_c S = T_{ba}^c \partial_c S = T_{ba}^c \nabla_c S.$$

Therefore the torsion measures how much covariant derivatives fail to commute on scalars. In general relativity, the torsion is usually taken to be zero. However, a treatment of fermions naturally requires non-zero torsion, and in local supersymmetry or “superspace formulations of anything,” non-zero torsion is essential.

We haven't yet actually found what the connection is in terms of things we actually care about.

Definition 4.2. Let us define the metric connection as the Γ such that

$$\nabla_c g_{ab} = 0.$$

This will allow us to find a formula for Γ in terms of the metric g .

We'll work it out explicitly.

$$\begin{aligned} \nabla_a g_{bc} &= \partial_a g_{bc} - \Gamma_{ab}^d g_{dc} - \Gamma_{ac}^d g_{bd} = 0, \\ \nabla_b g_{ca} &= \partial_b g_{ca} - \Gamma_{bc}^d g_{da} - \Gamma_{ba}^d g_{cd} = 0, \\ \nabla_c g_{ab} &= \partial_c g_{ab} - \Gamma_{ca}^d g_{bd} - \Gamma_{cb}^d g_{ad} = 0. \end{aligned}$$

If we add the first two of these and subtract the third, we end up

$$\partial_a g_{bc} + \partial_b g_{ca} - \partial_c g_{ab} = 2\Gamma_{ab}^d g_{dc},$$

using the fact that $\Gamma_{bc}^d = \Gamma_{cb}^d$ since we have set the torsion to zero.

Now we simply multiply by g^{ce} to find that

$$\frac{1}{2} g^{ce} (-\partial_c g_{ab} \partial_a g_{bc} + \partial_b g_{ca}) = \Gamma_{ab}^d g_{dc} g^{ce} = \Gamma_{ab}^d \delta_d^e = \Gamma_{ab}^e.$$

This gives us explicitly the metric connection, which we sometimes call the Christoffel connection or Christoffel symbols.¹⁰ Thus

$$\Gamma_{bc}^a = \frac{1}{2} g^{ad} (-\partial_d g_{bc} + \partial_b g_{cd} + \partial_c g_{bd}).$$

It is, as expected, symmetric under exchange $b \leftrightarrow c$ since the metric is symmetric, $g_{ab} = g_{ba}$.

So now on scalars,

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)S = 0,$$

i.e. covariant derivatives commute on scalars. Moreover using the metric connection if we have $V_a = g_{ab} V^b$, then

$$\nabla_c (V_a) = \nabla_c (g_{ab} V^b) = (\nabla_c g_{ab}) V^b + g_{ab} \nabla_c V^b = g_{ab} \nabla_c V^b,$$

¹⁰They are a pain to compute by hand, hence why one professor of mine once referred to them as the “Christ-awful symbols.”

since $\nabla_c g_{ab} = 0$. Therefore with the metric connection, the metric commutes with the covariant derivative. This is also true of the inverse metric– prove this as an exercise.

Exercise 4.3. Prove¹¹ that the covariant derivative of the inverse metric is also zero,

$$\nabla_c g^{ab} = 0.$$

Lecture 5.

Wednesday, October 17, 2018

We defined the symmetric metric connection (i.e. the Christoffel connection) such that

$$\nabla_a g_{bc} = 0 \text{ with } \Gamma_{ab}^c = \Gamma_{ba}^c.$$

We also found that if ϕ is a scalar field, the covariant derivatives commute:

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\phi = \partial_a \partial_b \phi - \Gamma_{ab}^c \partial_c \phi - \partial_b \partial_a \phi + \Gamma_{ba}^c \partial_c \phi = 0$$

where we used the symmetry of the Christoffel symbols to cancel the second and fourth terms. We might then ask if this is true for the covariant derivatives on a 1-form as well.

$$\begin{aligned} (\nabla_a \nabla_b - \nabla_b \nabla_a)V_c &= \nabla_a(\partial_b V_c - \Gamma_{bc}^e V_e) - (a \leftrightarrow b) \\ &= \partial_a \partial_b V_c - \Gamma_{bc}^e \partial_a V_e - (\partial_a \Gamma_{bc}^e) V_e - \Gamma_{ab}^f (\partial_f V_c - \Gamma_{fc}^e V_e) - \Gamma_{ac}^f (\partial_b V_f - \Gamma_{bc}^e V_e) + \Gamma_{bc}^f (\partial_a V_f - \Gamma_{af}^e V_e) \\ &\quad - \partial_b \partial_a V_c + \Gamma_{ac}^e \partial_b V_e + (\partial_b \Gamma_{ac}^e) V_e + \Gamma_{ba}^f (\partial_f V_c - \Gamma_{fc}^e V_e) \end{aligned}$$

It's not quite zero (in general), but what we find after a bit of close inspection is that all the second derivative terms cancel, and all the terms with derivatives of V also cancel.¹² We're left with products of the Christoffel symbols and also derivatives thereof:

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)V_c = (-\partial_a \Gamma_{bc}^e + \partial_b \Gamma_{ac}^e - \Gamma_{bc}^f \Gamma_{af}^e + \Gamma_{ac}^f \Gamma_{bf}^e) V_e.$$

Since the expression on the LHS is a tensor, the RHS must also be a tensor. (We can check this explicitly using the transformation properties of Γ , though I don't recommend it.)

Definition 5.1. We therefore define the curvature tensor $R_{abc}{}^e$ by the following:

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)V_c \equiv R_{abc}{}^e V_e,$$

where $R_{abc}{}^e$ is given explicitly by

$$R_{abc}{}^e = -\partial_a \Gamma_{bc}^e + \partial_b \Gamma_{ac}^e - \Gamma_{bc}^f \Gamma_{fa}^e + \Gamma_{ac}^f \Gamma_{fb}^e.$$

Roughly speaking, the curvature tensor measures how much the covariant derivatives of tensors fail to commute.¹³

On arbitrary tensors $T_{cd}^{ab\dots}$ one can write down a rather long expression for the commutator of the covariant derivatives:

$$(\nabla_e \nabla_f - \nabla_f \nabla_e)T_{cd}^{ab\dots} = R_{efp}{}^a T_{cd}^{pb\dots} + R_{efp}{}^b T_{cd}^{ap\dots} + R_{efc}^p T_{pd}^{ab\dots} + R_{efd}^p T_{cp}^{ab\dots} + \dots$$

similar to our formula for taking individual covariant derivatives. There is no further interesting content in these arbitrary tensors, however– all the interesting physics seems to already be captured in the curvature tensor.

¹¹I think we can do this with Leibniz, actually. $\nabla_c(g_{ab}g^{ab}) = \nabla_c(g_{ab})g^{ab} + g_{ab}\nabla_c g^{ab} = 0$ since the trace of the metric is just a scalar. If g_{ab} is not identically zero, it must be that $\nabla_c g^{ab}$ vanishes.

¹²Actually, we could do this for a general connection without assuming that the connection is symmetric in its lower two indices. If we do so, we pick up a term with the torsion tensor in it.

¹³This is only technically true for torsion-free connections. To quote Sean Carroll, "The Riemann tensor measures that part of the commutator of covariant derivatives that is proportional to the vector field, while the torsion tensor measures the part that is proportional to the covariant derivative of the vector field."

The Riemann tensor (i.e. the curvature tensor with an index lowered) also has some nice symmetries which you may like to check.

$$\begin{aligned} R_{abcd} &= -R_{bacd} \\ R_{abcd} &= -R_{abdc} \\ R_{abcd} &= R_{cdab} \\ R_{abcd} + R_{acdb} + R_{adbc} &= 0 \end{aligned}$$

These can be recovered from the explicit form of the curvature tensor with sufficient patience.

As a consequence of these identities, the Riemann tensor has many components (though they are somewhat constrained by symmetry). In d dimensions, it has $\frac{1}{12}d^2(d^2 - 1)$ components, so in 4 spacetime dimensions there are 20 independent components. In $d = 3$ there are only 6 and in $d = 2$, just 1.

In general there are many, many terms one needs to work out to actually compute the Riemann tensor. There are very nice computer programs like Mathematica which can automate the process, or if you have some time on your hands it is a decent exercise to write the code yourself.

Definition 5.2. We also define the *Ricci tensor*:

$$R_{bd} \equiv R_{abcd}g^{ac}$$

where we contract on the first and third indices of the Riemann tensor. The Ricci tensor is symmetric, $R_{ab} = R_{ba}$.

Definition 5.3. If we contract once more, we get the *Ricci scalar*,

$$R \equiv R^{ab}g_{ab}.$$

In two-dimensional calculations, R is the same as the Gaussian curvature up to a numerical factor. In addition, computing the Ricci scalar for Minkowski space reveals that it is zero—Minkowski space is flat.

We can now discuss geodesics, curves which extremize the proper distance between two endpoints p, q .

Definition 5.4. The proper distance along the line from p to q is given by

$$\int_p^q ds = \int_p^q \sqrt{\left| g_{ab} \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda} \right|} d\lambda$$

since $ds^2 = g_{ab}dx^a dx^b$. This is a functional of the path $x^a(\lambda)$ we take through the space, and when it is extremized¹⁴ we call the resulting path a *geodesic*.

Geodesics generalize the concept of a straight line to curved space. For instance, a great circle is an example of a geodesic for the surface of the Earth.

Extremizing the integral of ds is hard because of the square root, so we usually just extremize $\int_p^q ds^2$ instead. That is, we extremize

$$I = \int_p^q g_{ab} \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda} d\lambda = \int_p^q L d\lambda.$$

By the Euler-Lagrange equations, we can compute

$$\frac{d}{d\lambda} \left(\frac{\partial L}{\partial \dot{x}^a} \right) - \frac{\partial L}{\partial x^a} = 0$$

where $\dot{} = \partial/\partial\lambda$. Therefore

$$\frac{d}{d\lambda} (2g_{ab}\dot{x}^b) - \partial_a (g_{bc}\dot{x}^b \dot{x}^c) = 0$$

Equivalently

$$\frac{d}{d\lambda} (g_{ab}\dot{x}^b) - \frac{1}{2}(\partial_a g_{bc})\dot{x}^b \dot{x}^c = 0.$$

By the chain rule we rewrite the first term as

$$g_{ab}\ddot{x}^b + \left(\frac{\partial}{\partial x^c} g_{ab} \right) \frac{\partial x^c}{\partial \lambda} \frac{\partial x^b}{\partial \lambda} - \frac{1}{2} \partial_a g_{bc} \dot{x}^b \dot{x}^c = 0.$$

¹⁴When this refers to a path length in just space it's minimized, but when we are computing proper time it is maximized.

Regrouping,

$$\begin{aligned} 0 &= g_{ab}\ddot{x}^b + (\partial_c g_{ab} - \frac{1}{2}\partial_a g_{bc})\dot{x}^b\dot{x}^c \\ &= g^{ae}g_{ab}\ddot{x}^b + \frac{1}{2}g^{ae}(\partial_c g_{ab} + \partial_b g_{ac} - \partial_a g_{bc})\dot{x}^b\dot{x}^c. \end{aligned}$$

Recognizing that the first term becomes a delta function δ_b^e and the second term includes a Christoffel symbol, we rewrite this as

$$\frac{d^2 x^e}{d\lambda^2} + \Gamma_{bc}^e \frac{dx^b}{d\lambda} \frac{dx^c}{d\lambda} = 0,$$

which we often call the *geodesic equation*. The name of the game is to construct the tangent vector V^a to the curve, with $V^a = \frac{dx^a}{d\lambda}$. We can also write

$$V^b \nabla_b V^a = 0,$$

which is also sometimes called the geodesic equation.

Lecture 6.

Friday, October 19, 2018

Last time, we wrote down the geodesic equation. That is, we defined curves $x^a(x)$ such that

$$\frac{d^2 x^a}{ds^2} + \Gamma_{bc}^a \frac{dx^b}{ds} \frac{dx^c}{ds} = 0.$$

In terms of the tangent vector to the curve $V^a = \frac{dx^a}{ds}$, we can equivalently write

$$V^a \nabla_a V^b = 0.$$

What happens if we reparametrize the curve? For instance, we could change to some new variables

$$\tilde{s} = \tilde{s}(s).$$

Then $d/ds = \frac{d\tilde{s}}{ds} \frac{d}{d\tilde{s}}$ and $\frac{d^2}{ds^2} = \frac{d^2\tilde{s}}{ds^2} \frac{d}{d\tilde{s}} + \left(\frac{d\tilde{s}}{ds}\right)^2 \frac{d^2}{d\tilde{s}^2}$.

By substituting, we get a new version of the geodesic equation.

$$\left(\frac{d\tilde{s}}{ds}\right)^2 \frac{d^2}{d\tilde{s}^2} x^a + \frac{d^2\tilde{s}}{ds^2} \frac{d}{d\tilde{s}} x^a + \left(\frac{d\tilde{s}}{ds}\right)^2 \Gamma_{bc}^a \frac{dx^b}{d\tilde{s}} \frac{dx^c}{d\tilde{s}} = 0.$$

A little rewriting reveals that in terms of \tilde{s} , we get

$$\frac{d^2 x^a}{d\tilde{s}^2} + \Gamma_{bc}^a \frac{dx^b}{d\tilde{s}} \frac{dx^c}{d\tilde{s}} = - \frac{\frac{d^2\tilde{s}}{ds^2}}{\left(\frac{d\tilde{s}}{ds}\right)^2} \frac{dx^a}{d\tilde{s}}.$$

But \tilde{s} is arbitrary, so in terms of our new tangent vector $\tilde{V}^a = dx^a/d\tilde{s}$, we have

$$\tilde{V}^b \nabla_b \tilde{V}^a = f(s) \tilde{V}^a,$$

a more general form of the geodesic equation.

Definition 6.1. If $f(s) = 0$, we say the geodesic is *affinely parametrized*. If a geodesic is affinely parametrized, then it remains so for \tilde{s} such that $d^2\tilde{s}/ds^2 = 0$, e.g. for $\tilde{s} = as + b$, a, b constants.

If a geodesic is affine, then

$$V^a \nabla_a (V^b V_b) = 2V_b V^a \nabla_a V^b = 0$$

and so $V^b V_b$ is constant along the geodesic— if a curve is initially timelike(/spacelike/null) it will remain timelike(/spacelike/null) along the geodesic.

Recalling how the Christoffel symbol transforms under arbitrary changes of coordinates, we find that under $x^a \mapsto \tilde{x}^{a'} \equiv \tilde{x}^{a'}(x^b)$, we have

$$\Gamma_{b'c'}^{a'} = \frac{\partial \tilde{x}^{a'}}{\partial x^a} \frac{\partial x^b}{\partial \tilde{x}^{b'}} \frac{\partial x^c}{\partial \tilde{x}^{c'}} \Gamma_{bc}^a - \frac{\partial^2 \tilde{x}^{a'}}{\partial x^b \partial x^c} \frac{\partial x^b}{\partial \tilde{x}^{b'}} \frac{\partial x^c}{\partial \tilde{x}^{c'}}.$$

Is it possible to make the new Christoffel symbol vanish in the \tilde{x} coordinates? This is equivalent to the condition that

$$\frac{\partial \tilde{x}^{a'}}{\partial x^a} \Gamma_{bc}^a = \frac{\partial^2 \tilde{x}^{a'}}{\partial x^b \partial x^c}.$$

Suppose we want it to vanish at x_0^a . Let us choose coordinates \tilde{x} defined by

$$\tilde{x}^{a'} = (x^a - x_0^a) + \frac{1}{2} \Gamma_{bc}^a (x^b - x_0^b)(x^c - x_0^c).$$

Taking the first derivative with respect to the original coordinates yields

$$\frac{\partial \tilde{x}^a}{\partial x^e} = \delta_e^a + \Gamma_{bc}^a (x^b - x_0^b) \delta_e^c + \dots$$

where the \dots denotes derivatives of the Christoffel symbols, and similarly

$$\frac{\partial^2 \tilde{x}^a}{\partial x^e \partial x^f} = \Gamma_{ef}^a + \dots$$

We therefore see that

$$\frac{\partial \tilde{x}^{a'}}{\partial x^a} \Gamma_{bc}^a = \Gamma_{bc}^a (\delta_a^{a'} + \Gamma_{b'c'}^a (x^{b'} - x_0^{b'}) \delta_a^{c'}) = \Gamma_{bc}^{a'} = \frac{\partial^2 \tilde{x}^{a'}}{\partial x^b \partial x^c}$$

when $x^b = x_0^b$, so we have found coordinates where the Christoffel symbol vanishes at a point of our choosing.

Definition 6.2. If we choose coordinates so that Γ_{bc}^a vanishes at a point, then those coordinates are called *Gaussian normal coordinates*. In normal coordinates, the metric takes the form

$$g_{ab} = C_{ab} + O(x - x_0)^2,$$

where C_{ab} is some set of constants. These coordinates have forced the terms linear in $x - x_0$ to vanish, and by applying the Lorentz transform and rotations (and possibly a scale transformation) we can diagonalize C_{ab} so that at p ,

$$C_{ab} = \eta_{ab}.$$

What we have learned is that spacetime can always be made to look like Minkowski spacetime at a given point (up to higher-order corrections). We also call such a choice of coordinates *inertial coordinates*.

In normal coordinates, the geodesic equation for an affinely parametrized curve is

$$\frac{d^2 x^a}{ds^2} = 0,$$

the equation of motion for a freely falling particle in Minkowski space. This confirms our intuition that a geodesic is really a generalization of a straight line.

Of course, it's also apparent that this equation is not covariant since it depends on our choice of coordinates x^a . In order to be coordinate-independent, we need to write this as a tensorial equation, which is just

$$V^a \nabla_a V^b = 0.$$

If we further choose $V^a V_a = -1$ (this is true for the trajectory of a massive particle) then the parametrization s is simply the proper time along the curve.

One can now consider families of curve parametrized by both time t along the curve and another parameter s . The tangent vectors to each geodesic are given by

$$T^a(s) \equiv \frac{dx^a(t, s)}{dt}$$

and if we derive with respect to s instead we get a tangent vector relating neighboring geodesics,

$$S^a(t) \equiv \frac{dx^a(t, s)}{ds},$$

sometimes called the deviation vector. How does S^a change as one moves along the geodesics? If we consider

$$V^b \equiv (\nabla_T S)^b = T^a \nabla_a S^b$$

this is like the “relative velocity of geodesics” (where I’ve used ∇_T to represent a directional covariant derivative)– it measures how much the deviation vector S^a points in the direction of the tangent vector T^a .

One can equivalently define the “relative acceleration of geodesics” as

$$A^b \equiv (\nabla_T V)^b = T^a \nabla_a V^b.$$

We’ll revisit this quantity shortly.

Lemma 6.3. Consider the following quantity:

$$S^a \nabla_a T^b - T^a \nabla_a S^b.$$

Expanding out, we find that

$$S^a \nabla_a T^b - T^a \nabla_a S^b = S^a (\partial_a T^b + \Gamma_{ac}^b T^c) - T^a (\partial_a S^b + \Gamma_{ac}^b S^c).$$

The terms with Christoffel symbols cancel by the symmetry of Γ_{ac}^b (since a and c are dummy indices, we can swap them and relabel). But

$$S^a \partial_a T^b = S^a \frac{\partial T^b}{\partial x^a} = \frac{\partial^2 x^b}{\partial s \partial t} = T^a \partial_a S^b,$$

so our expression

$$S^a \nabla_a T^b - T^a \nabla_a S^b = 0.$$

This tells us that we can swap S^a and T^b through the covariant derivative so long as we keep the indices fixed.

Now what is the acceleration between neighboring geodesics,

$$A^a = \frac{d^2 S^a}{dt^2}?$$

Expanding out, we find that it is

$$\begin{aligned} \frac{d^2 S^a}{dt^2} &= T^c \nabla_c (T^b \nabla_b S^a) \\ &= T^c (\nabla_c S^b) (\nabla_b T^a) + T^c S^b \nabla_c \nabla_b T^a \\ &= T^c (\nabla_c S^b) (\nabla_b T^a) + T^c S^b (\nabla_b \nabla_c T^a + R_{cb}{}^a{}_d T^d). \end{aligned}$$

In the next step, we’ll move the T^c inside the ∇_b derivative to get

$$T^c (\nabla_c S^b) (\nabla_b T^a) + S^b \nabla_b (T^c \nabla_c T^a) - S^b (\nabla_b T^c) \nabla_c T^a + T^c S^b R_{cb}{}^a{}_d T^d.$$

But the second term vanishes by the geodesic equation, and the third term can be written as $-S^c \nabla_c T^b \nabla_b T^a = -T^c \nabla_c S^b \nabla_b T^a$ (by the identity we proved earlier) so the third term cancels the first one.

What remains is a function of the Riemann tensor. Using the symmetries of the Riemann tensor, we can rewrite our acceleration equation as

$$\frac{d^2 S^a}{dt^2} = R^a{}_{bcd} T^b T^c S^d,$$

which we call the *equation of geodesic deviation*. If the Riemann tensor is zero, then the relative velocity between geodesics is constant. If the Riemann tensor $\neq 0$, then we get a “stretching force” between neighboring geodesics, sometimes called a tidal force.

Lecture 7.

Monday, October 22, 2018

Today we’ll discuss *parallel transport*. Suppose $x^a(\lambda)$ is some parametrized curve in our space. Then we may take a point on that curve at $\lambda = \lambda_0$ and define a vector defined at $x^a(\lambda = \lambda_0)$, called $V^a(\lambda_0)$. Can we define a way to smoothly slide V^a along this curve?

Take the tangent vector to the curve $x^a(\lambda)$, $l^a = \frac{dx^a}{d\lambda}$. Then define the parallel transport of $V^a(\lambda_0)$ along the curve by the vector V^b satisfying

$$l^a \nabla_a V^b = 0.$$

One may think of this as being a generalization of a geodesic, the solution to the first-order differential equation

$$\frac{dx^a}{d\lambda} \frac{\partial V^b(\lambda)}{\partial x^a} + \frac{dx^a}{d\lambda} \Gamma_{ac}^b V^c(\lambda) = 0$$

or equivalently

$$\frac{\partial V^b(\lambda)}{\partial \lambda} + \frac{dx^a}{d\lambda} \Gamma_{ac}^b V^c(\lambda) = 0. \quad (7.1)$$

If we identify $l^a = V^a$, we see that this is just the geodesic equation, i.e. a geodesic is a curve for which the parallel transport of its tangent vector along the curve is trivial. Equivalently, the directional derivative of the tangent vector to a geodesic along that geodesic is zero, which we could have seen before.

The general solution to Eqn. 7.1 has the form

$$V^b(\lambda) = V^b(\lambda_0) - \int_{\lambda_0}^{\lambda} d\lambda' \Gamma_{ac}^b(\lambda') \frac{dx^a}{d\lambda'} V^c(\lambda'). \quad (7.2)$$

It's easy to check that Eqn. 7.2 solves 7.1 by taking the derivative $\frac{\partial}{\partial \lambda}$ and also checking (trivially) that $V^b(\lambda = \lambda_0) = V^b(\lambda_0)$. Therefore this is a solution (albeit in terms of the function we want, $V^b(\lambda)$) to our first-order differential equation, and this can be solved to whatever order we want by iteration from λ_0 .

Now consider parallel-transporting a tangent vector around a small closed loop starting at $x^a(\lambda_0)$. If the loop is small, we can use Taylor's theorem to look at V and Γ expanded around $x^a(\lambda_0)$ to leading order in powers of $x^a(\lambda) - x^a(\lambda_0)$. I'll change notation from lecture a bit for (what I hope is additional) clarity. Let me define

$$\delta x^a(\lambda) \equiv x^a(\lambda) - x^a(\lambda_0),$$

the leading-order change in x^a with respect to λ about λ_0 .

Explicitly, we have

$$V^a(\lambda) = V^a(\lambda_0) + \frac{\partial V^a}{\partial x^b} \Big|_{\lambda_0} \delta x^b(\lambda) + O(\delta x^2)$$

and similarly

$$\Gamma_{bc}^a(\lambda) = \Gamma_{bc}^a(\lambda_0) + \frac{\partial \Gamma_{bc}^a}{\partial x^d} \Big|_{\lambda_0} \delta x^d(\lambda) + O(\delta x^2).$$

Now let's write the equation of parallel transport at $\lambda = \lambda_0$. From our Taylor expansion, we have

$$\frac{dx^a}{d\lambda} \frac{\partial V^b}{\partial x^a} \Big|_{\lambda_0} + \frac{dx^a}{d\lambda} \Gamma_{ac}^b(\lambda_0) V^c(\lambda_0) = 0$$

which magically becomes¹⁵

$$\frac{\partial V^b}{\partial x^a} \Big|_{\lambda_0} + \Gamma_{ac}^b(\lambda_0) V^c(\lambda_0) = 0.$$

If we now substitute this into our Taylor expansion for V , to lowest order in $\delta x(\lambda)$ we have

$$V^b(\lambda) = V^b(\lambda_0) - \Gamma_{ac}^b(\lambda_0) V^a(\lambda_0) \delta x^c(\lambda). \quad (7.3)$$

We can then plug this into our general solution to find

$$V^b(\lambda) = V^b(\lambda_0) - \int_{\lambda_0}^{\lambda} d\lambda' \frac{dx^a}{d\lambda'} \left[\Gamma_{ac}^b(\lambda_0) + \frac{\partial \Gamma_{ac}^b}{\partial x^d} \Big|_{\lambda_0} \delta x^d(\lambda') + \dots \right] \left[V^c(\lambda_0) - \Gamma_{ef}^c V^e(\lambda_0) \delta x^f(\lambda') \right]$$

Evaluating this to lowest (interesting) order, i.e. terms linear in $x^a(\lambda') - x^a(\lambda_0)$, we get

$$V^b(\lambda) = V^b(\lambda_0) - \int_{\lambda_0}^{\lambda} d\lambda' \frac{dx^a}{d\lambda'} \Big|_{\lambda_0} \left(\Gamma_{ac}^b(\lambda_0) V^c(\lambda_0) + \delta x^d(\lambda') \left[\frac{\partial \Gamma_{ae}^b}{\partial x^d} \Big|_{\lambda_0} V^e(\lambda_0) - \Gamma_{ac}^b(\lambda_0) \Gamma_{de}^c(\lambda_0) V^e(\lambda_0) \right] \right)$$

using the fact that Γ is symmetric in its lower two indices and all indices except for b are dummy indices which can be relabeled.

¹⁵I'm not happy with this step yet. Is it because our path is arbitrary? The next equation seems to say that

If we now integrate around a closed loop, we know that

$$\int_{\lambda_0}^{\lambda} d\lambda' \frac{dx^a}{d\lambda'} = \oint dx^a = x^a|_{\text{initial}}^{\text{final}} = 0,$$

so we can write

$$\int_{\lambda_0}^{\lambda} d\lambda' (x^d(\lambda') - x^d(\lambda_0)) \frac{dx^a}{d\lambda'} = \int_{\lambda_0}^{\lambda} d\lambda' \left(-\frac{dx^d}{d\lambda'} x^a(\lambda') \right) + \text{boundary terms},$$

using integration by parts. But it follows that this first term is zero around a closed loop so

$$\int_{\lambda_0}^{\lambda} d\lambda' (x^d(\lambda') - x^d(\lambda_0)) \frac{dx^a}{d\lambda'} = \int_{\lambda_0}^{\lambda} d\lambda' \left(-\frac{dx^d}{d\lambda'} (x^a(\lambda') - x^a(\lambda_0)) \right).$$

We find that this integral is antisymmetric under interchange $a \leftrightarrow d$. If we antisymmetrize over $a \leftrightarrow d$ inside the square bracket term, we arrive at

$$V^b(\lambda) - V^b(\lambda_0) = -\frac{1}{2} \int d\lambda' [x^d(\lambda') - x^d(\lambda_0)] \frac{dx^a}{d\lambda'} \left[\frac{\partial \Gamma_{ae}^b}{\partial x^d} - \frac{\partial \Gamma_{de}^b}{\partial x^a} - \Gamma_{ac}^b \Gamma_{de}^c + \Gamma_{dc}^b \Gamma_{ae}^c \right]_{\lambda_0} V^e(\lambda_0).$$

But we recognize (okay, we could with enough experience recognize) that the term in square brackets is just the Riemann tensor evaluated at λ_0 , so that our expression simplifies to

$$\Delta V^b = V^b(\lambda) - V^b(\lambda_0) = -\frac{1}{2} R_{cde}^b(\lambda_0) V^c(\lambda_0) \oint x^d \frac{dx^e}{d\lambda'} d\lambda'.$$

We may interpret this equation as telling us that the change in the vector V as we go around a closed loop is in general not zero—instead, it is proportional to the Riemann tensor at the starting point λ_0 and the area of the loop $\oint x^d \frac{dx^e}{d\lambda'} d\lambda'$.

Curvature therefore tells you how vectors change under a general parallel transport. In Minkowski space, we can pick coordinates where the $\Gamma_{bc}^a = 0$, and then vectors do not change under parallel transport.

We may note that the Riemann tensor obeys the *Bianchi identity* (technically the second Bianchi identity). Recall that the Riemann tensor takes the form

$$R_{bcd}^a = \partial_c \Gamma_{db}^a - \partial_d \Gamma_{cb}^a + \Gamma_{bc}^f \Gamma_{fd}^a - \Gamma_{bd}^f \Gamma_{fc}^a.$$

We might want to know how to compute derivatives of the Riemann tensor, e.g. $\nabla_c R_{bcd}^a = ?$

This would be pretty awful to compute by hand, as it would have many terms as we expand out each of the Γ s. Let's be a little more clever. Choose normal coordinates so that $\Gamma_{bc}^a = 0$. Note that this doesn't in general mean that the derivatives $\partial_a \Gamma$ or $\partial^2 \Gamma$ also vanish. Nevertheless, we can see schematically that $\nabla_e R_{bcd}^a$ will have terms like

$$\nabla_e R_{bcd}^a = \partial^2 \Gamma, (\partial \Gamma) \Gamma, \Gamma \partial \Gamma, \Gamma^3.$$

But in normal coordinates, all except the first of these have Γ s in them and must vanish! Therefore we need only consider the terms which are second derivatives of Γ . Namely,

$$\nabla_e R_{bcd}^a = \partial_e \partial_c \Gamma_{db}^a - \partial_e \partial_d \Gamma_{cb}^a.$$

Under relabeling of the indices we get the similar expressions

$$\nabla_c R_{bde}^a = \partial_c \partial_d \Gamma_{eb}^a - \partial_c \partial_e \Gamma_{db}^a$$

and

$$\nabla_d R_{bec}^a = \partial_d \partial_e \Gamma_{cb}^a - \partial_d \partial_c \Gamma_{eb}^a.$$

Adding these up and using the fact that partial derivatives commute, we see that all the terms cancel and we are left with

$$\nabla_e R_{bcd}^a + \nabla_c R_{bde}^a + \nabla_d R_{bec}^a = 0.$$

This vanishes in Gaussian normal coordinates and is a tensor, so it vanishes in all coordinate systems.

We can also contract some indices, for instance a and c . Recall we defined the Ricci tensor, $R_{bd} = R_{bad}^a$. Then contracting, we get

$$\nabla_e R_{bd} + \nabla_a R_{bde}^a - \nabla_d R_{be} = 0,$$

where we've used the antisymmetry of the Riemann tensor in its last two indices, $R_{abcd} = -R_{abdc}$. This... isn't great yet. Let's contract b and e . The last term will give us the Ricci scalar $R = R_{ab}g^{ab}$, and so we'll get the final expression

$$\begin{aligned} 0 &= \nabla_b R_d^b + \nabla_a R_d^a - \nabla_d R \\ &= \nabla_a R_b^a - \frac{1}{2} \nabla_b R \\ &= \nabla_a G^{ab}, \end{aligned}$$

where G^{ab} is the Einstein tensor,

$$G^{ab} \equiv R^{ab} - \frac{1}{2} R g^{ab}.$$

Non-lectured aside: Riemann tensor and parallel transport around a loop I was dissatisfied with the proof that the change in V after parallel transport around a loop is proportional to the Riemann tensor. Therefore, here is an alternate proof which I find more convincing, based on an exercise from Sean Carroll's *Spacetime and Geometry*.¹⁶

Consider the following path. We'd like to compute the parallel transport of a generic vector V from $A \rightarrow B \rightarrow C \rightarrow D \rightarrow A$ and show that the difference between the initial and final vectors is proportional to both the area of the loop $\delta a \delta b$ and the corresponding components of the Riemann tensor. To make our lives easier, we'll just compute the parallel transport from $A \rightarrow B \rightarrow C$ and take advantage of symmetry to recover the other half of the path. That said, how should we go about doing this computation?

First, I'll turn to Carroll's Appendix I on the "parallel propagator" for some useful background and notation. Carroll writes:

We begin by noticing that for some path $\gamma : \lambda \rightarrow x^\sigma(\lambda)$, solving the parallel transport equation for a vector V^μ amounts to finding a matrix $P^\mu{}_\rho(\lambda, \lambda_0)$, which relates the vector at its initial value $V^\mu(\lambda_0)$ to its value somewhere later down the path:

$$V^\mu(\lambda) = P^\mu{}_\rho(\lambda, \lambda_0). \quad (7.4)$$

*Of course the matrix $P^\mu{}_\rho(\lambda, \lambda_0)$, known as the **parallel propagator**, depends on the path γ (although it's hard to find a notation that indicates this without making γ look like an index). If we define*

$$A^\mu{}_\rho(\lambda) = -\Gamma^\mu_{\sigma\rho} \frac{dx^\sigma}{d\lambda},$$

where the quantities on the right-hand side are evaluated at $x^\nu(\lambda)$, then the parallel transport equation becomes

$$\frac{d}{d\lambda} V^\mu = A^\mu{}_\rho V^\rho. \quad (7.5)$$

Since the parallel propagator must work for any vector, substituting 7.4 into 7.5 shows that $P^\mu{}_\rho(\lambda, \lambda_0)$ also obeys this equation:

$$\frac{d}{d\lambda} P^\mu{}_\rho(\lambda, \lambda_0) = A^\mu{}_\sigma(\lambda) P^\sigma{}_\rho(\lambda, \lambda_0).$$

To solve this equation, first integrate both sides:

$$P^\mu{}_\rho(\lambda, \lambda_0) = \delta^\mu_\rho + \int_{\lambda_0}^{\lambda} A^\mu{}_\sigma(\eta) P^\sigma{}_\rho(\eta, \lambda_0) d\eta.$$

The rest of this appendix goes to show that the full solution for the parallel propagator is in fact a path-ordered (cf. time-ordered) exponential— in fact, it is the general relativity version of Dyson's formula from quantum field theory, with the $A^\mu{}_\sigma$ s taking the place of copies of the interaction Hamiltonian. This is a very neat connection but won't actually help us finish the proof, so I direct you to Carroll if you would like the details. For our purposes, the first-order version of P will suffice (i.e. up to a single integral $d\eta$).

¹⁶Chapter 3, Exercise 7, page 148.

What is the parallel propagator $P^\mu{}_\rho(A \rightarrow B)$? By our first-order expression, it is simply

$$\begin{aligned}
 P^\mu{}_\rho(A \rightarrow B) &= \delta^\mu_\rho + \int_{\lambda_0}^\lambda A^\mu{}_\sigma(\eta) P^\sigma{}_\rho(\eta, 0) d\eta \\
 &= \delta^\mu_\rho + \int_0^{\delta a} \left(\Gamma^\mu{}_{\nu\sigma}(A) \frac{dx^\nu}{dx^1} \right) (\delta^\sigma_\rho) dx^1 \\
 &= \delta^\mu_\rho + \int_0^{\delta a} (-\Gamma^\mu{}_{\nu\rho}(A) \delta_1^\nu) dx^1 \\
 &= \delta^\mu_\rho - \Gamma^\mu{}_{1\rho}(A) \delta a,
 \end{aligned}$$

where I have explicitly performed the integration along the path $A \rightarrow B$.

To go from the first line to the second line, I have used the fact that we are only interested in the leading order behavior in δa . Since I know we will already get a factor of δa from the integration, one can see that Taylor expanding the Γ in $A^\mu{}_\sigma$ would end up giving us an $O(\delta a^2)$ term, so it suffices to just take Γ evaluated at A (we'll see in the transport $B \rightarrow C$ an example where this is crucially not the case). The somewhat unusual δ_1^ν comes from the fact I am also using the coordinate x^1 as my integration parameter η .

We can now play the same game to compute the parallel propagator $P^\nu{}_\mu(B \rightarrow C)$: it is

$$\begin{aligned}
 P^\nu{}_\mu(B \rightarrow C) &= \delta^\nu_\mu + \int_0^{\delta b} (-\Gamma^\nu{}_{\sigma\mu}(B) \delta_2^\sigma) dx^2 \\
 &= \delta^\nu_\mu - \int_0^{\delta b} \left[\Gamma^\nu{}_{\sigma\mu}(A) + \frac{\partial}{\partial x^1} \Gamma^\nu{}_{\sigma\mu}(A) \delta a \right] \delta_2^\sigma dx^2 \\
 &= \delta^\nu_\mu - \Gamma^\nu{}_{2\mu}(A) \delta b - \frac{\partial \Gamma^\nu{}_{2\mu}}{\partial x^1} \delta a \delta b.
 \end{aligned}$$

Finally, we put it all together:

$$\begin{aligned}
 V^\nu(C) &= P^\nu{}_\mu(B \rightarrow C) P^\mu{}_\rho(A \rightarrow B) V^\rho \\
 &= P^\nu{}_\mu(B \rightarrow C) \left[\delta^\mu_\rho - \Gamma^\mu{}_{1\rho}(A) \delta a \right] V^\rho \\
 &= P^\nu{}_\mu(B \rightarrow C) \left[V^\mu - \Gamma^\mu{}_{1\rho}(A) V^\rho \delta a \right] \\
 &= \left[\delta^\nu_\mu - \Gamma^\nu{}_{2\mu}(A) \delta b - \frac{\partial \Gamma^\nu{}_{2\mu}}{\partial x^1} \delta a \delta b \right] \left[V^\mu - \Gamma^\mu{}_{1\rho}(A) V^\rho \delta a \right] \\
 &= V^\nu(A) - \Gamma^\nu{}_{2\mu}(A) V^\mu \delta b - \delta^\nu_\mu \Gamma^\mu{}_{1\rho}(A) V^\rho \delta a - V^\mu \frac{\partial \Gamma^\nu{}_{2\mu}}{\partial x^1} \delta a \delta b + \Gamma^\nu{}_{2\mu} \Gamma^\mu{}_{1\rho} V^\rho \delta a \delta b + O(\delta a^2, \delta b^2) \\
 &= V^\nu(A) - \Gamma^\nu{}_{2\mu}(A) V^\mu \delta b - \Gamma^\nu{}_{1\mu}(A) V^\mu \delta a + \delta a \delta b \left(-V^\mu \frac{\partial \Gamma^\nu{}_{2\mu}}{\partial x^1} + \Gamma^\nu{}_{2\mu} \Gamma^\mu{}_{1\rho} V^\rho \right).
 \end{aligned}$$

That seems like a pretty long result! In fact, things are nicer than they seem. Instead of having to repeat this calculation $C \rightarrow D \rightarrow A$ to get us home, we simply note that $A \rightarrow D \rightarrow C$ is just the calculation we did with $x^1 \leftrightarrow x^2$ and $\delta a \leftrightarrow \delta b$. Taking the negative of the resulting expression after this relabeling (if the loop is small) gives us $C \rightarrow D \rightarrow A$. That is,

$$V^\nu(A \rightarrow B \rightarrow C \rightarrow D \rightarrow A) - V^\nu(A) = V^\nu(A \rightarrow B \rightarrow C) - V^\nu(A \rightarrow D \rightarrow C).$$

We see that under the exchange $x^1 \leftrightarrow x^2$ and $\delta a \leftrightarrow \delta b$, the terms linear in $\delta a, \delta b$ will go away, as will the zeroth order term. What is left after a bit of relabeling is

$$V^\nu(A \rightarrow B \rightarrow C \rightarrow D \rightarrow A) = \delta a \delta b V^\mu \left(\partial_2 \Gamma^\nu{}_{1\mu} - \partial_1 \Gamma^\nu{}_{2\mu} + \Gamma^\nu{}_{2\sigma} \Gamma^\sigma{}_{1\mu} - \Gamma^\nu{}_{1\sigma} \Gamma^\sigma{}_{2\mu} \right) = \delta a \delta b V^\mu (-R^\nu{}_{\mu 12}).$$

Just as before, we have found that the difference between parallel transporting in one order versus the other is proportional to the Riemann tensor and to the area of the loop.

Lecture 8.

Wednesday, October 24, 2018

In Newtonian gravity, a matter density ρ instantaneously sources a gravitational potential by Laplace's equation,

$$\nabla^2 \phi = 4\pi G \rho,$$

where $G = 6.67 \times 10^{-8} \text{ cm}^3 \text{ g}^{-1} \text{ s}^{-2}$. Note that this is a second-order differential equation.

In general relativity, we have instead the metric g_{ab} which has 10 components, and T_{ab} , the energy-momentum tensor which also has 10 components, with T_{00} the energy density. In special relativity, the energy-momentum tensor was conserved, $\partial_a T^{ab} = 0$. The equivalent of this in general relativity is the covariant derivative,

$$\nabla_a T^{ab} = 0.$$

In general all our equations of motion are second-order. One needs only to specify two boundary conditions, e.g. position and velocity, and then the equations can be solved. What is the equivalent of the Newtonian equation for gravity? Looking for quantities which transform in the correct way, with T_{ab} as the equivalent of the matter density ρ , the only thing we can cook up is the Einstein tensor:

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}.$$

Note that

$$\nabla_a (R^{ab} - \frac{1}{2} g^{ab} R) = 0$$

by the Bianchi identities, so T^{ab} is indeed conserved up to Λ the cosmological constant. Experimentally, we think that $\Lambda = 111 \times 10^{-56} \text{ cm}^2$. Note that Λ has the effect of globally modifying the Ricci scalar by a constant, which is equivalent to setting the background curvature of our spacetime.

Definition 8.1. Let us introduce the Levi-Civita symbol (also known as the alternating symbol). For n dimensions, we have the tensor

$$\eta_{a_1 \dots a_n}$$

such that

$$\eta_{a_1 \dots a_n} = \begin{cases} +1 & \text{if } a_1 \dots a_n \text{ is an even permutation of } 1, \dots, n \\ -1 & \text{if } a_1 \dots a_n \text{ is an odd permutation of } 1, \dots, n. \end{cases}$$

This is a generalization of our familiar ϵ_{ijk} tensor. Note that for any matrix M_b^a , one can define the determinant in terms of $\eta_{a_1 \dots a_n} \det M = \eta_{b_1 \dots b_n} M_{a_1}^{b_1} M_{a_2}^{b_2} \dots M_{a_n}^{b_n}$.

Let's suppose that $M_a^b = \frac{\partial x^b}{\partial \tilde{x}^a}$ now, e.g. for a change in coordinates. Then

$$\eta_{a_1 \dots a_n} \left[\det \frac{\partial x^c}{\partial \tilde{x}^d} \right] = \eta_{b_1 \dots b_n} \frac{\partial x^{b_1}}{\partial \tilde{x}^{a_1}} \dots \frac{\partial x^{b_n}}{\partial \tilde{x}^{a_n}}.$$

Rearranging a bit, we have

$$\eta_{a_1 \dots a_n} = \left[\det \frac{\partial \tilde{x}^c}{\partial x^d} \right] \eta_{b_1 \dots b_n} \frac{\partial x^{b_1}}{\partial \tilde{x}^{a_1}} \dots \frac{\partial x^{b_n}}{\partial \tilde{x}^{a_n}}.$$

This last bit looks like a good tensorial transformation, but the factor of the determinant spoils it. Therefore η is *not a tensor*; rather, it is a tensor density.

We can fix this, though. Consider the metric tensor g_{ab} . It transforms under an arbitrary coordinate transformation as

$$g_{ab} \mapsto \tilde{g}_{ab} = \frac{\partial x^c}{\partial \tilde{x}^a} \frac{\partial x^d}{\partial \tilde{x}^b} g_{cd}.$$

Taking determinants of both sides we get

$$\det \tilde{g} = \left[\det \frac{\partial x^c}{\partial \tilde{x}^a} \right]^2 (\det g).$$

Taking square roots of both sides (and taking the absolute value, since we don't know the sign of $\det g$) we get

$$\sqrt{|\det \tilde{g}|} = \left| \det \frac{\partial x^c}{\partial \tilde{x}^a} \right| \sqrt{|\det g|}.$$

Thus we see that

$$\sqrt{|\det g|} \eta_{a_1 \dots a_n}$$

is a tensor since the determinant factors then cancel. We may therefore define the alternating tensor

$$\epsilon_{a_1 \dots a_n} = g^{1/2} \eta_{a_1 \dots a_n}$$

where $g^{1/2} \equiv \sqrt{|\det g_{ab}|}$.

Let's also recall that the volume element d^x transforms as

$$d^n x \mapsto d^n \tilde{x} = \left| \det \frac{\partial \tilde{x}^a}{\partial x^b} \right| d^n x.$$

But $d^n x = dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$, which is really a differential n -form— it can be written

$$\frac{1}{n!} \eta_{a_1 \dots a_n} dx^{a_1} \wedge \dots \wedge dx^{a_n}.$$

It's apparent that the volume element then transforms as a tensor density like η . To make it transform as a tensor, we could instead write

$$\frac{1}{n!} \epsilon_{a_1 \dots a_n} dx^{a_1} \wedge \dots \wedge dx^{a_n},$$

where $\epsilon_{a_1 \dots a_n} dx^{a_1} \wedge \dots \wedge dx^{a_n} = g^{1/2} d^n x$ is now invariant. As a result, we call $g^{1/2} d^n x$ the *invariant volume element* and we could integrate a scalar Φ as

$$\int \Phi g^{1/2} d^n x,$$

where the integral is now independent of the coordinate system.

We'll now discuss Stokes's (or Gauss's, or Ostrogradsky's) theorem. Consider the volume integral

$$\int_{\Sigma} \nabla_a V^a g^{1/2} d^n x.$$

We want to write it as an integral over the boundary $\partial \Sigma$. We also choose coordinates such that $\partial \Sigma$ is a surface of x^n constant (we are in n spacetime dimensions), and then take g_{ab} to be of the form

$$g_{ab} = \begin{pmatrix} \gamma_{ij} & 0 \\ 0 & N^2 \end{pmatrix}$$

where $i, j = 1, \dots, n-1$. Now n_a is a unit normal vector to $\partial \Sigma$,

$$n_a \equiv (0, 0, \dots, 0, N)$$

with $n^a = (0, 0, \dots, 1/N)$. Clearly, the inverse metric is

$$g^{ab} = \begin{pmatrix} \gamma^{ij} & 0 \\ 0 & 1/N^2 \end{pmatrix}$$

Now

$$\nabla_a V^a = \partial_a V^a + \nabla_{ac}^a V^c,$$

and since

$$\Gamma_{bc}^a = \frac{1}{2} g^{ad} (-\partial_d g_{bc} + \partial_b g_{cd} + \partial_c g_{bd})$$

we find that

$$\Gamma_{ac}^a = \frac{1}{2} g^{ad} (-\partial_d g_{ac} + \partial_a g_{cd} + \partial_c g_{ad}).$$

These first two terms cancel by the symmetry of g^{ad} , so we're left with

$$\begin{aligned}
 \Gamma_{ac}^a &= \frac{1}{2} g^{ad} \partial_c g_{ad} \\
 &= \frac{1}{2} \text{Tr}[g^{-1} \partial_c g] \\
 &= \frac{1}{2} \text{Tr}[\partial_c \ln g] \\
 &= \frac{1}{2} \partial_c \ln \det g \\
 &= \partial_c \ln \sqrt{|\det g|}
 \end{aligned}$$

where to get from the third to fourth expression, we have used the identity

$$\det g = \exp \text{Tr} \ln g$$

and the fact that ∂_c and the trace commute (since ∂_c is linear). Suppose g is symmetric. Then it can be diagonalized, $D = O^T g O$ with O some orthogonal matrix and D the diagonal matrix of the eigenvalues of g . Equivalently $g = O D O^T$. Then

$$\begin{aligned}
 \exp \text{Tr} \ln O D O^T &= \exp \text{Tr}(\ln O + \ln D + \ln O^T) \\
 &= \exp \text{tr}(\ln(O O^T) + \ln D) \\
 &= \exp \text{Tr}(\ln I + \ln D) \\
 &= \exp \text{Tr}(\ln D) \\
 &= \exp \sum_i \ln \lambda_i \\
 &= \prod_i \lambda_i = \det D = \det O^T g O = \det g O O^T = \det g.
 \end{aligned}$$

Using this identity¹⁷, we see that

$$\Gamma_{ac}^a = \partial_c \ln \sqrt{|\det g|} = \partial_c (\ln g^{1/2}) = \frac{1}{2} \frac{\partial_c \det g}{\det g}.$$

We also need the following result:

$$\partial_a (g^{1/2} V^a) = g^{1/2} \partial_a V^a + V^a \frac{1}{2g^{1/2}} \partial_a (\det g) = g^{1/2} (\partial_a V^a + \Gamma_{ba}^b V^a) = \nabla_a V^a g^{1/2},$$

which will let us turn a covariant derivative into an ordinary partial derivative.

Therefore

$$\int_{\Sigma} \nabla_a V^a g^{1/2} d^n x = \int \partial_a (g^{1/2} V^a) dx^1 dx^2 \dots dx^n.$$

¹⁷Okay, I'm really unsure about this proof. What does it even mean to take the log of a matrix? I guess we could take a power series expansion about the identity? From Symmetries we have the more reasonable version of this identity in terms of exponentials: $\det \exp(g) = \exp(\text{Tr } g)$, which is certainly true. So one can define the log of a matrix A as $B = \log A$, where B is defined such that $\exp B = A$. We've been too cavalier about applying the properties of this matrix logarithm I think, in assuming that all the regular properties of the log hold (although it seems that they do). If this makes you unhappy, repeat the proof with $g' = \exp g$ and then take the log at the end.

Integrating over x^n we get

$$\begin{aligned}
 \int_{\Sigma} \nabla_a V^a g^{1/2} d^n x &= \int_{\Sigma} \partial_a (g^{1/2} V^a) dx^1 dx^2 \dots dx^n \\
 &= \int_{\partial\Sigma} V^{''n''} g^{1/2} dx^1 \dots dx^{n-1} \\
 &= \int_{\partial\Sigma} V^{''n''} N (\det \gamma)^{1/2} dx^1 \dots dx^{n-1} \\
 &= \int_{\partial\Sigma} \left(\frac{1}{N} n_a V^a \right) N (\det \gamma)^{1/2} dx^1 \dots dx^{n-1} \\
 &= \int_{\partial\Sigma} n_a V^a (\det \gamma)^{1/2} dx^1 \dots dx^{n-1},
 \end{aligned}$$

where $V^{''n''}$ is the n th component of V . Here, we have just turned a covariant derivative into a regular partial derivative so we can apply the regular divergence theorem. This equation is covariant, so it is true in general:

$$\int_{\Sigma} \nabla_a V^a g^{1/2} d^n x = \int_{\partial\Sigma} n_a V^a \gamma^{1/2} d^{n-1} x,$$

with $\gamma^{1/2} d^{n-1} x$ the volume element on $\partial\Sigma$.¹⁸

Lecture 9.

Friday, October 26, 2018

Today we'll discuss the action principle for general relativity. Last time, we wrote the Einstein equations,

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}.$$

We can write down an action for our theory: it is

$$I = \underbrace{\frac{1}{16\pi G} \int_M (R - 2\Lambda) g^{1/2} d^4 x}_{\text{Einstein-Hilbert action}} + \underbrace{\int L_{\text{matter}} g^{1/2} d^4 x}_{\text{arbitrary matter contribution}}.$$

We'll need the condition that $\delta I = 0$ as we vary the metric,

$$g_{ab} \mapsto g_{ab} + h_{ab}$$

for h_{ab} a small perturbation. Since h_{ab} is small, we need only keep terms linear in h_{ab} .

The inverse metric is then given by

$$g^{ab} g_{bc} = \delta_c^a,$$

which is the same in any system, so it does not change. In the perturbed metric, we see that the inverse metric must go to

$$g^{ab} - h^{ab}$$

since

$$(g^{ab} - h^{ab})(g_{bc} + h_{bc}) = \delta_c^a - h_c^a + h_c^a + O(h^2) = \delta_c^a.$$

Note that tensor manipulations (e.g. raising and lowering indices) are carried out w.r.t the unperturbed metric g_{ab} .

¹⁸There is a very nice way to write Stokes's theorem in terms of differentials and differential forms. See Carroll Appendix E for the proof.

Now how does $g^{1/2}$ vary? It is

$$\begin{aligned}
 g^{1/2} &= \sqrt{|\det g_{ab}|} \\
 &= \sqrt{|\exp \operatorname{tr} \ln g_{ab}|} \\
 &= \exp \frac{1}{2} \operatorname{tr} \ln g_{ab} \\
 &\rightarrow \exp \frac{1}{2} \operatorname{tr} \ln (g_{ab} + h_{ab}) \\
 &= \exp \frac{1}{2} \operatorname{tr} \ln g_{ac} (\delta_b^c + h_b^c) \\
 &= \exp \frac{1}{2} \operatorname{tr} (\ln g_{ab} + \ln (\delta_b^c + h_b^c)) \\
 &= g^{1/2} \exp \frac{1}{2} \operatorname{tr} (h_b^c) \\
 &= g^{1/2} \exp \frac{1}{2} h \\
 &= g^{1/2} \left(1 + \frac{1}{2} h \right)
 \end{aligned}$$

where we have defined $h \equiv \operatorname{tr} h_b^c$ as the trace of our perturbation h_{ab} and discarded all terms of order h^2 . We conclude that the variation is

$$\delta g^{1/2} = g^{1/2} \frac{1}{2} h.$$

Now how do we vary the Ricci scalar?

$$\delta R = \delta (R_{ab} g^{ab}) = \delta R_{ab} g^{ab} + R_{ab} \delta g^{ab} = \delta R_{ab} g^{ab} - R_{ab} h^{ab}.$$

The variation of the Ricci tensor clearly depends on the variation of the Christoffel symbols Γ_{bc}^a , with

$$R_{ce} = \partial_b \Gamma_{ce}^b - \partial_e \Gamma_{cb}^b + \Gamma_{cf}^b \Gamma_{ce}^f - \Gamma_{ef}^b \Gamma_{cb}^f.$$

The Christoffel symbols in terms of the original metric are

$$\Gamma_{bc}^a = \frac{1}{2} g^{ad} (-\partial_d g_{bc} + \partial_b g_{cd} + \partial_c g_{bd})$$

and under the perturbation $g + h$ we have

$$\begin{aligned}
 \Gamma_{bc}^a(g+h) &= \frac{1}{2} (g^{ad} - h^{ad}) (-\partial_d (g_{bc} + h_{bc}) + \partial_b (g_{cd} + h_{cd}) + \partial_c (g_{bd} + h_{bd})) \\
 &= \Gamma_{bc}^a(g) + \frac{1}{2} \left(g^{ad} (-\partial_d h_{bc} + \partial_b h_{cd} + \partial_c h_{bd}) + h \partial g \right).
 \end{aligned}$$

But since the difference between two connections is a tensor, the whole second term here must be a tensor and in particular the $h \partial g$ term must be made of Christoffel symbols, so it vanishes (equivalently, in normal coordinates the first derivatives of g are all zero at a point).

If we now pretend we were doing this calculation in normal coordinates all along (okay, we change to normal coordinates at each point), the derivatives of g vanish and we can replace the partial derivatives with the covariant derivatives to get

$$\Gamma_{bc}^a(g+h) = \Gamma_{bc}^a(g) + \frac{1}{2} g^{ad} (-\nabla_d h_{bc} + \nabla_b h_{cd} + \nabla_c h_{bd}).$$

Therefore the variation in the Christoffel symbols is in general

$$\delta_{bc}^a = -\frac{1}{2} \nabla^a h_{bc} + \frac{1}{2} \nabla_b h_c^a + \frac{1}{2} \nabla_c h_b^a,$$

which we can easily check is symmetric under $b \leftrightarrow c$.

Then the variation of the Ricci tensor is

$$\begin{aligned}\delta R_{ce} &= \partial_b \delta \Gamma_{ce}^b - \partial_e \delta \Gamma_{bc}^b + (\Gamma \delta \Gamma) \\ &= \nabla_b \delta \Gamma_{ce}^b - \Gamma_e \delta \Gamma_{bc}^b \\ &= \frac{1}{2} (-\nabla_b \nabla^b h_{ce} + \nabla_b \nabla_c h_e^b + \nabla_b \nabla_e h_c^b - \nabla_c \nabla^c h),\end{aligned}$$

where the $\Gamma \delta \Gamma$ terms have vanished since $\Gamma = 0$ in normal coordinates, and we have again promoted ∂_b to ∇_b . We arrive at the variation of the Ricci scalar,

$$\begin{aligned}\delta R &= \delta(R_{ab} g^{ab}) \\ &= \delta R_{ab} g^{ab} + R_{ab} \delta g^{ab} \\ &= \delta R_{ab} g^{ab} - R_{ab} h^{ab} \\ &= \frac{1}{2} g^{ce} (-\nabla_d \nabla^d h_{ce} + \nabla_d \nabla_c h_e^d + \nabla_d \nabla_e h_c^d - \nabla_c \nabla_e h) - R_{ab} h^{ab} \\ &= -\nabla_d \nabla^d h + \nabla_d \nabla_c h^{cd} - R_{ab} h^{ab}.\end{aligned}$$

Putting it all together we have the variation of the action,

$$\delta I_{grav} = \frac{1}{16\pi G} \int_M g^{1/2} g^4 x \left[-\nabla_d (\nabla^d h) + \nabla_d (\nabla_e h^{de}) - R_{de} h^{de} + \frac{1}{2} h R - \Lambda h \right]$$

Using Gauss's theorem, we rewrite the first two terms as

$$\int_{\partial M} d\Sigma (-\nabla^d h + \nabla_e h^{de}) n_d$$

where since these are total derivatives evaluated on the "boundary" of our space, we assume they are irrelevant and vanish as we take the space to be infinitely large.¹⁹ Thus, forcing the variation of the action to vanish, we recover

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 0,$$

the Einstein equations in the absence of matter. Taking the variation of the matter term is much simpler:

$$I_{matter} = \int L_{matter} g^{1/2} d^4x = \int \frac{1}{2} T_{ab} h^{ab} g^{1/2} d^4x,$$

so we get a stress-energy term. This expression defines the energy-momentum tensor of any matter—note that T_{ab} is always symmetric since h^{ab} is symmetric and the indices are fully contracted. In QFT, we defined the energy-momentum tensor a little differently using Noether's theorem for translations in time and space, but that energy-momentum tensor is not guaranteed to be symmetric. If T_{ab} is symmetric, then taking the covariant derivative of the Einstein equations gives

$$8\pi G \nabla_a T_b^a = \nabla^a (R_{ab} - \frac{1}{2} R g_{ab}) + \nabla_b \Lambda = 0$$

by the Bianchi identity and the fact that Λ is a constant.

Example 9.1. One can write a specific action for the matter term. Consider

$$I_{matter} = \int \left[-\frac{1}{2} g^{ab} \partial_a \phi \partial_b \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!} \lambda \phi^4 \right] g^{1/2} d^4x.$$

Note we've chosen the signs here so that in Minkowski space ($g^{ab} = \eta^{ab}$), the kinetic term is positive, $\frac{1}{2} \dot{\phi}^2 > 0$. One may then derive the corresponding energy-momentum tensor,

$$T_{ab} = \partial_c \phi \partial_b \phi - \frac{1}{2} g_{ab} \left[(\partial \phi)^2 + m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right]$$

¹⁹This won't always be valid, but it is here.

Lecture 10.

Monday, October 29, 2018

Last time, we mentioned in passing that there is an action associated to the matter in our space,

$$I_{\text{matter}} = \int g^{1/2} d^4x L_{\text{matter}}.$$

For instance, for a scalar field ϕ in ϕ^4 theory we might have

$$I_{\text{matter}} = \int g^{1/2} d^4x \left[-\frac{1}{2} g^{ab} \partial_a \phi \partial_b \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4} \lambda \phi^4 \right].$$

Now ϕ is invariant under variation of the metric, so

$$\delta I = \int g^{1/2} d^4x \left[\frac{1}{2} h^{ab} \partial_a \phi \partial_b \phi + \frac{1}{2} g_{ab} \left(-\frac{1}{2} g^{cd} \partial_c \phi \partial_d \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4} \lambda \phi^4 \right) \right].$$

The corresponding energy-momentum tensor is therefore

$$T_{ab} = \partial_a \phi \partial_b \phi - \frac{1}{2} g_{ab} (\partial_c \phi \partial_d \phi g^{cd} + m^2 \phi^2 + \frac{1}{2} \lambda \phi^4).$$

The equation for motion for ϕ is

$$-\square \phi + m^2 + \lambda \phi^3 = 0,$$

where the \square operator is $\nabla_a \nabla^a$, a generalization of the Laplacian. (This is sometimes called the d'Alembertian.)

We can calculate $\nabla_a T^{ab}$ to get

$$\square \phi \partial_b \phi + \nabla_a \phi \nabla^a \nabla_b \phi - \nabla_b \phi \nabla^b \nabla_c \phi - m^2 \phi \nabla_b \phi - \lambda \phi^3 \nabla_b \phi,$$

where the second and third terms cancel, so we see that

$$\nabla_a T^{ab} = \nabla^b \phi (-\text{Equation of motion}) = 0.$$

Thus the energy-momentum tensor is conserved by the equations of motion.

Suppose $m^2 < 0$ and look for solutions of the field equations that are constant,

$$\phi^2 = -m^2 / \lambda.$$

Then T_{ab} takes the form

$$T_{ab} = -\frac{1}{2} \left(-m^2 \frac{m^2}{\lambda} + \frac{\lambda}{2} \frac{m^2}{\lambda^2} \right) = -\frac{1}{4} g_{ab} \frac{m^4}{\lambda}.$$

This is what happens in the Higgs mechanism. However, note that the stress-energy tensor is proportional to g_{ab} only and is therefore equivalent to a cosmological constant. That is, from the Einstein equations

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi G T_{ab},$$

we get a contribution to Λ of

$$-8\pi G \left(\frac{1}{4} \frac{m^4}{\lambda} \right)$$

However, Λ has a length scale of 10^{28} cm, whereas m^4/λ has a weak interaction scale of 10^{15} cm, so there is a 43 orders of magnitude conflict between the value of the vacuum energy predicted by the Higgs mechanism and the value experimentally observed. There is no known solution to this problem.

What about the electromagnetic field? For this field, we have an associated action of

$$I_{\text{em}} = -\frac{1}{4} \int d^4x g^{1/2} F_{ab} F^{ab} = -\frac{1}{4} \int d^4x g^{1/2} F_{ab} F_{cd} g^{ac} g^{bd}.$$

Here, the Maxwell tensor is given as usual by $F_{ab} \equiv \partial_a A_b - \partial_b A_a$ with A_a the electromagnetic four-potential. As the metric is varied, we take δA_a to be zero and find the variation

$$\delta I_{\text{em}} = -\frac{1}{4} \int d^4x g^{1/2} [-2h^{ac} g^{bd} F_{ab} F_{cd} + \frac{1}{2} g_{ab} F_{cd} F^{cd} h^{ab}].$$

We find that the corresponding stress-energy tensor is

$$T_{ab} = F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}.$$

It's a simple exercise to check that $\nabla_a T^{ab} = 0$ if $\nabla_a F_{bc} + \nabla_b F_{ca} + \nabla_c F_{ab} = 0$ and $\nabla_a F^{ab} = 0$.

Let's now examine some symmetries of the Einstein equation. In the absence of matter, we have

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 0.$$

If $\Lambda = 0$ as well, then one solution is the Minkowski spacetime,

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2.$$

In Minkowski spacetime, the metric is constant and so all the Christoffel symbols vanish, $\Gamma_{bc}^a = 0$, and thus all the nice tensors and scalars we cooked up also vanish,

$$R^a{}_{bcd} = R_{ab} = R = 0.$$

Note that the Minkowski metric is therefore invariant under translations, spatial rotations, and Lorentz transformations.

What if we consider a spacetime where $\Lambda > 0$? Can we still find a maximally symmetric space for such a Λ ? The answer is yes— it is called de Sitter space, and the solution looks like

$$ds^2 = \Omega^2(t)(-dt^2 + dx^2 + dy^2 + dz^2),$$

where Ω cannot depend on x, y, z to preserve rotational invariance but it may depend on t .

Note that the Einstein equations in vacuum take the nice form

$$\begin{aligned} 0 &= g^{ab}(R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab}) \\ &= R - \frac{1}{2}R(4) + 4\Lambda \\ \implies R &= 4\Lambda, \end{aligned}$$

so the Einstein equations become

$$R_{ab} = \Lambda g_{ab}.$$

Then

$$\square \Omega = -\frac{2}{3}\Lambda \Omega^3$$

where \square is the flat space wave operator

$$\square = -\frac{\partial^2}{\partial t^2} + \nabla^2.$$

Letting $\cdot = \frac{\partial}{\partial t}$, we have

$$\ddot{\Omega} = \frac{2}{3}\Lambda \Omega^3.$$

We can solve by multiplying by $\dot{\Omega}$ and integrating,

$$\dot{\Omega}\ddot{\Omega} = \frac{2}{3}\Lambda \Omega^3 \dot{\Omega} \implies \frac{1}{2}\dot{\Omega}^2 = \frac{1}{2}\Lambda \Omega^4.$$

We conclude that

$$\dot{\Omega} = \sqrt{\frac{\Lambda}{3}}\Omega^2,$$

and integrating again we write

$$\frac{\dot{\Omega}}{\Omega^2} = \pm\sqrt{\Lambda/3} \implies \frac{1}{\Omega} = \mp\sqrt{\Lambda/3}t.$$

It follows that $\Omega^2 = \frac{3}{\Lambda t^2}$, so our de Sitter metric is

$$ds^2 = \frac{3}{\Lambda t^2}[-dt^2 + dx^2 + dy^2 + dz^2].$$

The limit $\Lambda \rightarrow 0$ is pathological, but as $t \rightarrow 0$ the metric coefficients blow up. We call this a *coordinate singularity* (as opposed to an essential singularity, like in a black hole). All this tells us is that this set of coordinates don't work towards $t \rightarrow 0$. The only way to tell in general if a divergence is an essential singularity is to find that one of our curvature tensors or scalars diverges, e.g. $R \rightarrow \infty$. Conversely, we can get rid of coordinate singularities by an appropriate change of coordinates.

Here's an example. Suppose we invent the new coordinate T given by

$$t = \sqrt{\frac{3}{\Lambda}} e^{\sqrt{\Lambda/3}T}.$$

Then the differential is

$$dt = dT e^{\sqrt{\Lambda/3}T}.$$

In the new coordinates, the metric takes the form

$$ds^2 = -dT^2 + e^{-2\sqrt{\Lambda/3}T} (dx^2 + dy^2 + dz^2).$$

Those of you with previous relativity experience might recognize this as a Friedman-Robertson-Walker (FRW) universe—interpreted physically, this is simply a flat-space universe where space contracts as T increases.

But we could have alternatively defined a coordinate \tilde{T} such that

$$t = \sqrt{\frac{3}{\Lambda}} e^{-\sqrt{\Lambda/3}\tilde{T}}.$$

We'd get a similar metric to before, but now space is expanding:

$$ds^2 = -d\tilde{T}^2 + e^{2\sqrt{\Lambda/3}\tilde{T}} (dx^2 + dy^2 + dz^2),$$

which says that space is expanding as \tilde{T} increases. What's gone wrong? All that's happened is that our new "time" coordinates T, \tilde{T} don't describe the entire spacetime. Note that as $T \rightarrow -\infty, t \rightarrow 0$ and as $\tilde{T} \rightarrow +\infty, t \rightarrow 0$.

Our two different metrics, one in T, x, y, z and one in \tilde{T}, x, y, z , both represent solutions of Einstein's equations. But both come from the same original metric, so they seem to describe different parts of spacetime. We'd like to have a single set of coordinates which is good everywhere. Can we do this?

Let us define coordinates $X, Y, Z, V + W$ such that

$$\begin{aligned} T &= \sqrt{\frac{3}{\Lambda}} \ln \frac{V+W}{\sqrt{3/\Lambda}} \\ x &= \sqrt{\frac{3}{\Lambda}} \frac{X}{V+W} \\ Y &= \sqrt{\frac{3}{\Lambda}} \frac{Y}{V+W} \\ Z &= \sqrt{\frac{3}{\Lambda}} \frac{Z}{V+W}. \end{aligned}$$

We also impose the following constraint:

$$-V^2 + W^2 + X^2 + Y^2 + Z^2 = 3/\Lambda,$$

which defines a surface in five-dimensional space. Suppose we look at this constraint in $\mathbb{R}^{1,4}$, flat five-dimensional space with metric

$$ds^2 = -dV^2 + dW^2 + dX^2 + dY^2 + dZ^2.$$

If we eliminate W between these two (by taking dW in the first equation) we find that the metric in our constrained five-dimensional space is precisely the same as what you get by plugging in T, x, y, z as functions of V, W, X, Y, Z into

$$ds^2 = -dT^2 + e^{-2\sqrt{\Lambda/3}T} (dx^2 + dy^2 + dz^2).$$

The proof of this is in Anthony Zee's *Einstein Gravity in a Nutshell* and also in Schrödinger's paper *Properties of expanding universes*.

Lecture 11.

Wednesday, October 31, 2018

Last time, we introduced de Sitter space. de Sitter space is a hyperboloid in $\mathbb{R}^{4,1}$, given by the constraint

$$-V^2 + W^2 + X^2 + Y^2 + Z^2 = 3/\Lambda$$

embedded in a space of metric

$$ds^2 = -dV^2 + dW^2 + dX^2 + dY^2 + dZ^2.$$

To see geometrically what this is, take $V = \text{constant}$ and this puts time as a constant,

$$W^2 + X^2 + Y^2 + Z^2 = \text{const} = k^2,$$

a three-sphere S^3 . We can introduce hyperspherical coordinates, which generalize spherical coordinates for S^2 . They take the form

$$\begin{aligned} W &= k \cos \chi \\ X &= k \sin \chi \cos \theta \\ Y &= k \sin \chi \sin \theta \cos \phi \\ Z &= k \sin \chi \sin \theta \sin \phi, \end{aligned}$$

with $0 \leq \chi \leq \pi, 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi$.

The metric in this space is then

$$d\sigma_{(S^2)}^2 = d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2).$$

That is, spatial sections of de Sitter space are three-spheres.

Thus we can invert the relationships to find

$$\begin{aligned} V &= \sqrt{\frac{3}{\Lambda}} \cosh \tau \sqrt{\frac{\Lambda}{3}} \\ W &= \sqrt{\frac{3}{\Lambda}} \sinh \tau \sqrt{\frac{\Lambda}{3}} \cos \chi \\ X &= \sqrt{\frac{3}{\Lambda}} \sinh \tau \sqrt{\frac{\Lambda}{3}} \sin \chi \cos \theta \\ Y &= \sqrt{\frac{3}{\Lambda}} \sinh \tau \sqrt{\frac{\Lambda}{3}} \sin \chi \sin \theta \cos \phi \\ Z &= \sqrt{\frac{3}{\Lambda}} \sinh \tau \sqrt{\frac{\Lambda}{3}} \sin \chi \sin \theta \sin \phi. \end{aligned}$$

The full metric is then

$$ds^2 = -d\tau^2 + \frac{3}{\Lambda} \cosh^2 \tau \frac{\Lambda}{3} (d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)).$$

de Sitter space thus shrinks from infinite size down to a minimum size $\sqrt{3/\Lambda}$, and then re-expands to infinite size.

So our T, x, y, z coordinates only worked for part of the spacetime, and the same is true of the \tilde{T}, x, y, z coordinates. This is what led us to see only an expanding or contracting universe in the T and \tilde{T} parameters.

Let's also note that in Minkowski spacetime, if you wait long enough, you can see all of space. That is, the light from anywhere in space will eventually reach you (equivalently, null geodesics reach arbitrarily far away in finite time). However, this is not true in de Sitter space. As the universe expands, the light cone is forced to close up as $\tau \rightarrow \infty$. As a result, there are regions of space that can never be seen— we call this a cosmological horizon, a boundary between what you can and cannot see.²⁰

²⁰This is somewhat clearer when we draw the Penrose/conformal diagram for de Sitter space, I think.

Definition 11.1. Recall that the Riemann tensor has 20 independent components in 3 + 1 dimensional spacetime. The Ricci tensor only contains some of the information from the original Riemann tensor, so we define the *Weyl tensor*, C_{abcd} , in terms of the part of R_{abcd} not accounted for by R_{ab} and R :

$$R_{abcd} = C_{abcd} + \frac{1}{2}(g_{ad}R_{bc} - g_{ad}R_{ac} - g_{ac}R_{bd} + g_{bc}R_{ad}) + \frac{1}{6}R(g_{ad}g_{bc} - g_{ac}g_{bd}).$$

Note that since Einstein's equation specifies R_{ab} , R in terms of T_{ab} , Λ ,

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 8\pi GT_{ab},$$

knowing these two quantities determines R_{ab} , R . The degrees of freedom of the gravitational field are described by the Weyl tensor. Note that C has the same symmetries as the Riemann tensor,

$$C_{abcd} = -C_{bacd} = -C_{abdc} = C_{cdab}, \quad C_{a[bcd]} = 0.$$

If we contract a and c (multiply by g^{ac}), we find that

$$C^a{}_{bad} = 0$$

means that C is traceless, and we find that C_{abcd} has 10 independent components.

We'll now observe that the Riemann tensor for de Sitter space is

$$R_{abcd} = \frac{\Lambda}{3}(g_{ac}g_{bd} - g_{ad}g_{bc}).$$

Hence R is completely specified by T_{ab} , Λ and therefore $C_{abcd} = 0$ for de Sitter space.

Conformal transformations Let us define a new metric

$$ds^2 = \Omega^2(-dt^2 + dx^2 + dy^2 + dz^2).$$

This is an example of a *conformal transformation*. A conformal transformation takes g_{ab} to a new metric $\hat{g}_{ab} = \Omega^2 g_{ab}$ where Ω is any function of the coordinates.

Note that Ω^2 is positive, so spacelike/timelike/null curves in g are also spacelike/timelike/null in \hat{g} (the sign of ds^2 for a given curve is unchanged in the new metric). Therefore conformal transformations preserve the causal structure of spacetime. Is it also true that geodesics remain geodesics in the new metric?

We write down the geodesic equation,

$$\frac{d^2 x^a}{ds^2} + \Gamma^a_{bc} \frac{dx^b}{ds} \frac{dx^c}{ds} = 0$$

for an affinely parametrized geodesic. What are the new christoffel symbols? They are

$$\Gamma^a_{bc}(\hat{g}) = \frac{1}{2}\hat{g}^{ad}(-\partial_d \hat{g}_{bc} + \partial_b \hat{g}_{cd} + \partial_c \hat{g}_{bd}).$$

The metric transforms as $g_{ab} \rightarrow \hat{g}_{ab} = \Omega^2 g_{ab}$, and since the inverse metric is given by $\hat{g}^{ab} \hat{g}_{bc} = \delta^a_c$ (the original metric is similar), we find that $\hat{g}^{ab} = \Omega^{-2} g^{ab}$. Therefore plugging in, we get

$$\begin{aligned} \Gamma^a_{bc}(\hat{g}) &= \frac{1}{2}\Omega^{-2}g^{ad}(-\partial_d(\Omega^2 g_{bc}) + \partial_b(\Omega^2 g_{cd}) + \partial_c(\Omega^2 g_{bd})) \\ &= \Gamma^a_{bc}(g) + \frac{1}{\Omega}(-g^{ad}g_{bc}\partial_d \Omega + g^{ad}g_{cd}\partial_b \Omega + g^{ad}g_{bc}\partial_c \Omega) \\ &= \Gamma^a_{bc}(g) + \frac{1}{\Omega} \underbrace{(-g^{ad}g_{bc}\nabla_d \Omega + g^{ad}g_{cd}\nabla_b \Omega + g^{ad}g_{bc}\nabla_c \Omega)}_{\text{tensorial}} \\ &= \Gamma^a_{bc}(g) + \frac{1}{\Omega}(-g_{bc}\nabla^a \Omega + \delta^a_c \nabla_b \Omega + \delta^a_b \nabla_c \Omega). \end{aligned}$$

Note that we've turned partial derivatives into covariant derivatives, since they are equivalent on a scalar like Ω . So we have the original Christoffel symbols plus some junk. Let's put it back into the geodesic equation:

$$\frac{d^2 x^a}{ds^2} + \left[\Gamma^a_{bc}(g) + \frac{1}{\Omega}(-g_{bc}\nabla^a \Omega + \delta^a_c \nabla_b \Omega + \delta^a_b \nabla_c \Omega) \right] \frac{dx^b}{ds} \frac{dx^c}{ds} = 0.$$

If our geodesic $\frac{dx^b}{ds}$ is timelike or spacelike, then $g_{bc} \frac{dx^b}{ds} \frac{dx^c}{ds} = \pm 1$ and this will be hard to solve. But if $\frac{dx^b}{ds}$ is null, then $g_{bc} \frac{dx^b}{ds} \frac{dx^c}{ds} = 0$ gives the geodesic equation

$$\frac{d^2 x^a}{ds^2} + \Gamma_{bc}^a(g) \frac{dx^b}{ds} \frac{dx^c}{ds} = -\frac{2}{\omega} \frac{dx^a}{ds} \left(\nabla_b \Omega \frac{dx^b}{ds} \right).$$

But this is just the geodesic equation for a geodesic that is *not affinely parametrized*. Therefore conformal transformations map null geodesics onto null geodesics.

In general, spacelike and timelike geodesics are not mapped onto geodesics in the new metric, though they are still spacelike and timelike respectively.

Lecture 12.

Friday, November 2, 2018

Let's consider how various quantities of interest transform under covariant transformations, $g_{ab} \rightarrow \hat{g}_{ab} = \Omega^2 g_{ab}$. The Christoffels transform as

$$\Gamma_{bc}^a(\hat{g}) = \Gamma_{bc}^a(g) + \frac{1}{\Omega} (-g_{bc} \nabla^a \Omega + \delta_c^a \nabla_b \Omega + \delta_b^a \nabla_c \Omega),$$

as we saw last time. The Riemann tensor transforms as

$$R_{cd}^{ab}(\hat{g}) = \Omega^{-2} R_{cd}^{ab}(g) + \frac{1}{4} [\delta_c^a \Omega_d^b - \delta_d^a \Omega_c^b - \delta_c^b \Omega_d^a + \delta_d^b \Omega_c^a],$$

with

$$\Omega_b^a \equiv \frac{4}{\Omega} \nabla^a \nabla_b (1/\Omega) - 2 \delta_b^a \nabla_c (1/\Omega) \nabla^c (1/\Omega).$$

The Ricci tensor is also complicated:

$$R_b^a(\hat{g}) = \Omega^{-2} R_b^a(g) + \frac{1}{2\Omega} \nabla_b \nabla^a (1/\Omega) - \frac{1}{2\Omega^4} \square (\Omega^2) \delta_b^a$$

with $\square \equiv \nabla_d \nabla^d$. However, the Ricci scalar and the Weyl tensor transform more nicely:

$$R(\hat{g}) = \Omega^{-2} R(g) - 6\Omega^{-3} \square \Omega \quad \text{and} \quad C_{bcd}^a(\hat{g}) = C_{bcd}^a(g).$$

So in fact the Weyl tensor is invariant under conformal transformations.

Conformal invariance and Maxwell's equations The source-free Maxwell's equations take the form

$$\nabla_a F_{bc} + \nabla_b F_{ca} + \nabla_c F_{ab} = 0, \quad \nabla_a F^{ab} = 0.$$

Writing out the first explicitly, all the Christoffels cancel by symmetries since $\Gamma_{bc}^a = \Gamma_{cb}^a$ and $F_{ab} = -F_{ba}$. Therefore

$$\nabla_a F_{bc} + \nabla_b F_{ca} + \nabla_c F_{ab} = \partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0$$

and our expression is conformally invariant, as it is independent of the connection. What about the equation with the indices down? We have

$$\partial_a F^{ab} + \Gamma_{ac}^a F^{cb} + \Gamma_{ac}^b F^{ac} = 0,$$

where the last term is zero since Γ is symmetric under $a \leftrightarrow c$ and F is antisymmetric. Because $\Gamma_{ac}^a = g^{-1/2} \partial_c (g^{1/2})$, we may rewrite the remaining terms as

$$g^{-1/2} \partial_a (g^{1/2} F^{ab}) = 0.$$

We then have $F^{ab}(g) = g^{ac} g^{bd} F_{cd}(g)$, so each factor of the inverse metric makes us pick up an Ω^{-2} . Therefore

$$F^{ab}(\hat{g}) = \Omega^{-4} F^{ab}(g).$$

Since $g = |\det g_{ab}|$, it follows that

$$\begin{aligned} \hat{g} &= |\det \hat{g}_{ab}| \\ &= |\det g_{ab} \Omega^2| \\ &= \Omega^8 |\det g_{ab}| \\ &= \Omega^8 g. \end{aligned}$$

Thus

$$g^{1/2} \rightarrow \hat{g}^{1/2} \Omega^4 g^{1/2},$$

so indeed the scaling behavior precisely cancels out with the Ω^{-4} attached to F^{ab} . That is,

$$\nabla_a F^{ab} \text{ in } \hat{g} = 0.$$

Note this is only true in 4 dimensions– in N dimensions, we have instead $g^{1/2} \rightarrow \hat{g}^{1/2} = \Omega^N g^{1/2}$.

We have therefore shown that Maxwell's equations are conformally invariant in spacetime dimension 4, but the Einstein equations are not conformally invariant– as we have seen, the Ricci tensor and Ricci scalar transform in some general complicated way. Nevertheless, the conformal invariance of the Weyl tensor tells us that the purely gravitational degrees of freedom are conformally invariant.

Last time, we looked at de Sitter space and remarked that three of the four coordinates took the form of the metric on the three sphere S^3 . We wrote down the full metric

$$ds^2 = -d\tau^2 + \cosh^2 \tau \left(d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2) \right),$$

where we have set the parameter Λ to 3 so that the “radius” $3/\Lambda$ is 1. There is a symmetry which is quite natural to write down for this metric– note that ds^2 does not depend on the coordinate ϕ . Now an *infinitesimal symmetry* is a small change in a coordinate which leaves the line element ds^2 invariant. That is, under the shift $x^a \rightarrow x^a + \zeta^a$,

$$\begin{aligned} ds^2 &= g_{ab} + (x) dx^a dx^b \\ &= g_{ab}(x + \zeta) d(x + \zeta)^a d(x + \zeta)^b \\ &= (g_{ab}(x) + \zeta^e \partial_e g_{ab}(x)) d(x^a + \zeta^a) dx^b + \zeta^b. \end{aligned}$$

However, note that in general, $\zeta^a \equiv \zeta^a(x)$ a function of x . Then we must apply the chain rule when we calculate $d(x^a + \zeta^a(x))$:

$$\begin{aligned} d(x^a + \zeta^a) &= dx^a + \frac{\partial \zeta^a}{\partial x^c} dx^c \\ &= (\delta_c^a + \partial_c \zeta^a) dx^c. \end{aligned}$$

Putting it all together, we find that

$$\begin{aligned} ds^2 &= g_{ab} + (x) dx^a dx^b \\ &= (g_{ab}(x) + \zeta^e \partial_e g_{ab}(x)) (\delta_c^a + \partial_c \zeta^a) dx^c (\delta_d^b + \partial_d \zeta^b) dx^d \\ &= g_{ab} \delta_c^a \delta_d^b dx^c dx^d + (\zeta^e \partial_e g_{ab} \delta_c^a \delta_d^b + g_{ab} \partial_c \zeta^a \delta_d^b + g_{ab} \partial_d \zeta^b \delta_c^a) dx^c dx^d, \end{aligned}$$

where we have collected terms linear in ζ . But now we see that the first term is just $g_{ab} dx^a dx^b$, so if the line element is invariant, then the $dx^c dx^d$ term must vanish. Invariance therefore requires that

$$(\zeta^e \partial_e g_{cd} + g_{ad} \partial_c \zeta^a + g_{cb} \partial_d \zeta^b) dx^c dx^d = 0.$$

Since this is true for arbitrary dx^c , we conclude that the expression in the parentheses is identically zero. That is,

$$\zeta^e \partial_e g_{cd} + g_{ad} \partial_c \zeta^a + g_{cb} \partial_d \zeta^b = 0.$$

This does not look like a tensor a priori, but in fact it is. If we want to replace the derivatives $\partial_c \zeta^a$ and $\partial_d \zeta^b$ with covariant derivatives, we'll have to pay the small price of subtracting off the corresponding Christoffels. Thus

$$\zeta^e \partial_e g_{cd} + g_{ad} \nabla_c \zeta^a - g_{ad} \Gamma_{ce}^a \zeta^e + g_{cb} \nabla_d \zeta^b - g_{cb} \Gamma_{de}^b \zeta^e = 0.$$

If we now write out the Christoffel symbols in terms of derivatives of the metric, we get something that looks kind of messy:

$$\nabla_c \zeta_d + \nabla_d \zeta_c + \zeta^e \partial_e g_{cd} - g_{ad} \frac{1}{2} g^{af} (-\partial_f g_{ce} + \partial_c g_{ef} + \partial_e g_{cf}) \zeta^e - g_{cb} \frac{1}{2} g^{bf} (-\partial_f g_{de} + \partial_d g_{ef} + \partial_e g_{df}) \zeta^e = 0$$

But things are better than they seem. Using the metrics to raise and lower indices appropriately, we get a very nice set of cancellations— all the derivatives of the metric turn out to cancel (including the $\zeta^e \partial_e g_{cd}$ out front), and we are left with the simple expression

$$\nabla_c \zeta_d + \nabla_d \zeta_c = 0$$

We call the solutions ζ *Killing vectors*.²¹ Killing was a German group theorist, and there is in fact a connection to group theory here. Suppose there are two independent Killing vectors k^a, l^a . Consider the commutator of two Killing vectors (i.e. a bracket) defined

$$m^b \equiv k^a \nabla_a l^b - l^a \nabla_a k^b.$$

Expanding out, we see that $m^b = k^a \partial_a l^b + k^a \Gamma_{ac}^b l^c - l^a \partial_a k^b - l^a \Gamma_{ac}^b k^c$, where the Γ terms cancel. Therefore m is independent of the connection. We might ask whether m also satisfies Killing's equation. We find that

$$\begin{aligned} \nabla_a m_b + \nabla_b m_a &= \nabla_a (k^c \nabla_c l_b - l^c \nabla_c k_b) + \nabla_b (k^c \nabla_c l_a - l^c \nabla_c k_a) \\ &= \nabla_a k^c \nabla_c l_b - \nabla_a l^c \nabla_c k_b + \nabla_b k^c \nabla_c l_a - \nabla_b l^c \nabla_c k_a \\ &\quad + k^c \nabla_a \nabla_c l_b - l^c \nabla_a \nabla_c k_b + k^c \nabla_b \nabla_c l_a - l^c \nabla_b \nabla_c k_a. \end{aligned}$$

After a stunning array of cancellations using the fact that $\nabla_a k^c \nabla_c l_b - \nabla^c l_b \nabla_c k_a = 0$ (by Killing's equations) we find that m does indeed satisfy Killing's equations as well (check this as an exercise?). So there is some algebraic structure on Killing vectors.²²

Non-lectured aside: commutator of Killing vectors is a Killing vector The proof in class was a little bit too quick about this, so here is a slower exposition. Let me write a series of brief lemmas, and then put them together to solve the overall problem.

Lemma 12.1. *If k and l are Killing vectors, then $(\nabla_a k^c)(\nabla_c l_b) - (\nabla_b l^c)(\nabla_c k_a) = 0$.*

The proof is simple. First note that whenever we have an index that is contracted over, we can trivially switch which is the up index and which is the down, e.g. $A^\mu B_\mu = g^{\mu\nu} A_\nu B_\mu = A_\nu B^\nu$. In addition, by raising an index in Killing's equations we have

$$\nabla^a k_b + \nabla_b k^a = 0.$$

Then

$$\begin{aligned} \nabla_a k^c \nabla_c l_b &= \nabla_a k_c \nabla^c l_b \\ &= (-\nabla_c k_a)(-\nabla_b l^c) \text{ by Killing's equations} \\ &= (\nabla_b l^c)(\nabla_c k_a). \end{aligned}$$

Thus $(\nabla_a k^c)(\nabla_c l_b) - (\nabla_b l^c)(\nabla_c k_a) = 0$. □

We also recall the following fact, which serves as the definition of the Riemann tensor:

$$[\nabla_a, \nabla_b] V^c = R^c_{dab} V^d.$$

We can lower the index c on the left by metric compatibility and use the fact that $R_{cdab} = R_{abcd}$ to rewrite the commutator in a more useful way:

$$[\nabla_a, \nabla_b] V_c = R_{abcd} V^d. \quad (12.2)$$

This is enough to do the proof. We can explicitly compute

$$\begin{aligned} \nabla_a m_b + \nabla_b m_a &= \nabla_a (k^c \nabla_c l_b - l^c \nabla_c k_b) + \nabla_b (k^c \nabla_c l_a - l^c \nabla_c k_a) \\ &= [(\nabla_a k^c)(\nabla_c l_b) - (\nabla_b l^c)(\nabla_c k_a)] + [(\nabla_b k^c)(\nabla_c l_a) - (\nabla_a l^c)(\nabla_c k_b)] \\ &\quad + k^c \nabla_a \nabla_c l_b - l^c \nabla_a \nabla_c k_b + k^c \nabla_b \nabla_c l_a - l^c \nabla_b \nabla_c k_a. \end{aligned}$$

Here, I have used the product rule and regrouped terms in a suggestive way. Clearly, there are two kinds of terms here: terms of the form $\nabla_a k^c \nabla_c l_b$, with only first derivatives, and terms of the form $k^c \nabla_a \nabla_c l_b$.

²¹"Killing here is not some bizarre ritual but Wilhelm Killing." –Malcolm Perry

²²Killing vectors are vectors by definition, and they certainly seem to form a vector space since the covariant derivative is linear. The fact that we have a bracket on them makes me suspect this is a Lie algebra, $[k, l]^b = k^a \nabla_a l^b - l^a \nabla_a k^b$. It is clearly antisymmetric and bilinear, i.e. $[l, k] = [-k, l]$ and $[k + l, m]^b = (k^a + l^a) \nabla_a m^b - m^a \nabla_a (k^b + l^b) = (k^a \nabla_a m^b - m^a \nabla_a k^b) + (l^a \nabla_a m^b - m^a \nabla_a l^b) = [k, m]^b + [l, m]^b$. I suspect, though have not yet proved, that the Jacobi identity also holds.

However, by applying our lemma 12.1, we see that the pairs of (first derivative) terms in square brackets vanish, so we are left with only the second derivative terms. That is,

$$\nabla_a m_b + \nabla_b m_a = k^c \nabla_a \nabla_c l_b + k^c \nabla_b \nabla_c l_a - l^c \nabla_a \nabla_c k_b - l^c \nabla_b \nabla_c k_a,$$

where I have again anticipated a cancellation by grouping the k^c and l^c terms together.

Now we can rewrite these in terms of their commutators:

$$\begin{aligned} \nabla_a m_b + \nabla_b m_a &= k^c ([\nabla_a, \nabla_c] + \nabla_c \nabla_a) l_b + k^c ([\nabla_b, \nabla_c] + \nabla_c \nabla_b) l_a \\ &\quad - l^c ([\nabla_a, \nabla_c] + \nabla_c \nabla_a) k_b - l^c ([\nabla_b, \nabla_c] + \nabla_c \nabla_b) k_a \\ &= k^c [\nabla_a, \nabla_c] l_b + k^c [\nabla_b, \nabla_c] l_a - l^c [\nabla_a, \nabla_c] k_b - l^c [\nabla_b, \nabla_c] k_a, \end{aligned}$$

where all the not-commutator terms have vanished due to Killing's equations (e.g. $k^c \nabla_c (\nabla_a l_b + \nabla_b l_a) = 0$). We're almost there. Now we apply the definition of the Riemann tensor, Eqn. 12.2, to change commutators into Riemann tensors like so:

$$\begin{aligned} \nabla_a m_b + \nabla_b m_a &= k^c R_{acbd} l^d + k^c R_{bcad} l^d - l^c R_{acbd} k^d - l^c R_{bcad} k^d \\ &= k^c R_{acbd} l^d + k^c R_{bcad} l^d - l^d R_{acbd} k^c - l^d R_{bcad} k^c \\ &= 0, \end{aligned}$$

where we have simply relabeled the dummy indices c and d on the last two terms. Therefore if k and l are Killing vectors, then their commutator is also a Killing vector. \square

Lecture 13.

Monday, November 5, 2018

Last time, we introduced Killing vectors, symmetries of the metric. They obey the equation

$$\nabla_c \zeta_d + \nabla_d \zeta_c = 0.$$

We also introduced a bracket on two Killing vectors k, l defined such that

$$[k, l]^b = m^b = k^a \nabla_a l^b - l^a \nabla_a k^b.$$

If we now plug in m into Killing's equation, we find that $\nabla_a m_b + \nabla_b m_a = 0$, so the commutator of two Killing vectors is itself a Killing vector. We'll simply state without proof that the bracket we've defined also obeys the Jacobi identity,

$$[m, [k, l]] + [k, [l, m]] + [l, [m, k]] = 0.$$

Therefore the set of Killing vectors equipped with this bracket forms a Lie algebra.

From Symmetries, Fields and Particles we recall that the exponential map takes a Lie algebra to its corresponding Lie group. Let's see how that applies here. In Minkowski space, we have the metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

and it turns out that Killing's equations

$$\nabla_a k_b + \nabla_b k_a = 0$$

have 10 linearly independent solutions, which are the maximum allowed in four spacetime dimensions.

How do we find these solutions? Consider the geodesics of a particle, defined by the geodesic equation

$$u^a \nabla_a u^a = 0.$$

Here, u^a is the tangent vector to a particle worldline. Now consider some Killing vector k^a and look at the inner product $k^a u_a$. In particular, let us consider the parallel transport of this quantity

$$u^a \nabla_a (k^b u_b),$$

which describes the evolution of $k^a u_a$ as you move along the worldline of the particle. Expanding this out, we see that

$$u^a \nabla_a (u^b k_b) = u^a u^b \nabla_a k_b + u^a k^b \nabla_a u_b = 0,$$

where the first term is zero by Killing's equation²³ and the second is zero by the geodesic equation. Thus $u^b k_b$ is constant along the particle trajectory, and each Killing vector is associated to a conservation law for free particle motion.

The Killing vectors for Minkowski space include

- $k^a = (1, 0, 0, 0)$, corresponding to time translation symmetry \implies conservation of energy
- $(0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)$, translations in the x, y, z directions \implies conservation of (three components of) linear momentum
- $(0, 0, z, -y), (0, z, 0, -x), (0, y, -x, 0)$, corresponding to spatial rotations \implies conservation of (three components of) angular momentum
- $(x, t, 0, 0), (y, 0, t, 0), (z, 0, 0, t)$ corresponding to Lorentz transformations \implies conservation of the position of the particle at $t = 0$ ²⁴

The Lorentz transformations and rotations form a group $SO(3, 1)$, and the translations also form a group (specifically, an abelian group). The semidirect product of these two results in the Poincaré group, which is the maximal set of symmetries of Minkowski space.

There are two more maximally symmetric four-dimensional spacetimes— they are de Sitter space, which we saw in Lectures 10 and 11, and anti-de Sitter space (AdS), which is similar but has a negative cosmological constant, $\Lambda < 0$. In words, Minkowski space, dS, and AdS are the maximally symmetric $3 + 1$ D spaces with zero, positive, and negative curvature respectively.

Let's take a little detour back to Newtonian gravity. In Newtonian gravity, the gravitational potential ϕ obeys Poisson's equation,

$$\nabla^2 \phi = 2\pi G \rho$$

and the gravitational force is given by

$$\mathbf{g} = -\nabla \phi.$$

For a static spherically symmetric field, outside a gravitating body the potential is simply

$$\phi = -\frac{GM}{r},$$

which reproduces the inverse square law. How does the situation change in general relativity? Suppose we have a metric of the form

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{W(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$

Consider spheres of constant t, r . The Killing vector $m^a = (0, 0, 0, 1)$ corresponds to rotation about the ϕ axis, and there are two more rotational Killing vectors which result in a symmetry group $SO(3)$. The Killing vector $k^a = (1, 0, 0, 0)$ simply results in time translations (with the group structure of $(\mathbb{R}, +)$).

To simplify the calculations, we may take $V = W = C_1 + C_2/r$, where C_1, C_2 are some constants of integration. We choose $C_1 = +1$ so that as $r \rightarrow \infty$, our metric looks like Minkowski space (we say the metric is *asymptotically flat*). If we set $C_2 = -2GM$, this leads to what is known as the *Schwarzschild metric*. Here G is Newton's constant and M has the interpretation of the black hole mass.

There are 4 major experimental tests of general relativity. They are as follows:

- (a) Deflection of starlight
- (b) Determination of orbital shape (circle, ellipse, hyperbola, parabola)
- (c) Gravitational redshift
- (d) Shapiro effect.

We'll see the first three of these in class and work out the last as an exercise on the example sheet, but all four involve finding particle motion in the Schwarzschild geometry.

²³Explicitly, we can write $u^a u^b \nabla_a k_b = \frac{1}{2}(u^a u^b \nabla_a k_b + u^b u^a \nabla_a k_b) = \frac{1}{2}u^a u^b (\nabla_a k_b + \nabla_b k_a) = 0$, switching the dummy indices on the second term and applying Killing's equation.

²⁴In a field theory context, we sometimes say that Lorentz invariance implies the center of energy of the system moves with constant velocity.

We should therefore analyze the geodesics $x^a(s)$ where s is an affine parameter (the proper time τ if the curve is timelike). We define the tangent vector u^a by

$$u^a = \frac{dx^a}{ds} = \dot{x}^a(s).$$

Note that the geodesic equation says that $u^a \nabla_a u^b = 0$, and $u^b u_b = -\epsilon$ for some constant ϵ . This follows since $u^a \nabla_a (u^b u_b) = 2u^b (u^a \nabla_a u_b) = 0$, where the term in parentheses vanishes by the geodesic equation. Now, we could write down the geodesic equation explicitly in terms of Christoffel symbols, but it is often more expedient to write down the action for a free particle,

$$I = \int ds g_{ab} \dot{x}^a \dot{x}^b,$$

and note that geodesics are paths of stationary action, $\delta I = 0$.

Therefore we should write down the action and look at the Euler-Lagrange equations.

$$I = \int ds \left[- \left(1 - \frac{2M}{r} \right) \dot{t}^2 + \left(1 - \frac{2M}{r} \right)^{-1} \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right].$$

Since the metric has no explicit ϕ, t dependence, we can immediately write down two conserved quantities: the energy,

$$E = \left(1 - \frac{2M}{r} \right) \dot{t}$$

and the angular momentum about the ϕ axis,

$$L = r^2 \sin^2 \theta \dot{\phi}.$$

In Newtonian theory, we make things simple by setting $\theta = \pi/2$ so that motion lies in the equatorial plane. If we now write down the θ equation of motion, we have

$$\frac{d}{ds} (r^2 \dot{\theta}) - r^2 \sin \theta \cos \theta \dot{\phi}^2 = 0.$$

If our particle initially lies in the equatorial plane $\theta(0) = \pi/2$ and the initial velocity also lies in the equatorial plane, $\dot{\theta}(0) = 0$, then

$$r^2 \ddot{\theta} + 2r\dot{r}\dot{\theta} = 0 \implies \ddot{\theta}(0) = 0,$$

so just as in the classical case, if our initial motion (position and velocity) lies in the equatorial plane then it is constrained to lie in the equatorial plane at all future times.

Lecture 14.

Wednesday, November 7, 2018

We can write down the action for the Schwarzschild metric: it reads

$$I = \int ds (-V \dot{t}^2 + \dot{r}^2 / V + r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2)).$$

Last time, we found two constants of the motion, the angular momentum and energy given by

$$L = r^2 \sin^2 \theta \dot{\phi},$$

$$E = \dot{t} V$$

where

$$V \equiv 1 - 2M/r.$$

We also showed that for orbits with initial position and velocity in the equatorial plane, $\theta(0) = \pi/2$ and $\dot{\theta}(0) = 0$, such orbits will always stay in the equatorial plane, $\theta(t) = \pi/2$ and $\dot{\theta}(t) = 0$. Therefore by a suitable choice of coordinates, we can always set this initial condition to be true, and it suffices to analyze motion in the equatorial plane (so that the θ dependence drops out of the action and L simplifies to $L = r^2 \dot{\phi}$).

Since the Lagrangian itself is a constant of the motion (because it has no explicit time dependence), we also have

$$-\epsilon = g_{ab}\dot{x}^a\dot{x}^b = -V\dot{t}^2 + \dot{r}^2/V + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2,$$

where we have defined ϵ such that

$$\epsilon = \begin{cases} 0 & \text{for light rays} \\ 1 & \text{for massive particles.} \end{cases}$$

Using the constants of motion and setting $\theta = \pi/2, \dot{\theta} = 0$, we can rewrite this as a radial equation,

$$-\epsilon = -E^2 + \frac{\dot{r}^2}{V} + L^2/r^2. \quad (14.1)$$

This can be solved in terms of elliptic functions, but for those of us who aren't specialists in elliptic functions we have three options:

- Perturbation theory
- Special cases
- Numerical solutions.

We'll consider the first two of these. Equation 14.1 can be written in a suggestive form:

$$\underbrace{\frac{1}{2}\dot{r}^2}_{\text{"kinetic energy"}} + \underbrace{\frac{1}{2}\left(1 - \frac{2M}{r}\right)\left(\frac{L^2}{r^2} + \epsilon\right)}_{\text{effective "potential energy"}} = \underbrace{\frac{1}{2}E^2}_{\text{total energy}}. \quad (14.2)$$

If we now differentiate the rewritten radial equation 14.2 with respect to s and divide by \dot{r} (i.e. we assume a non-circular orbit²⁵), we get a second-order equation in r :

$$\ddot{r} + \frac{\epsilon M}{r^2} - \frac{L^2}{r^3} + \frac{3ML^2}{r^4} = 0. \quad (14.3)$$

The fact that $\epsilon = 0$ for light rays means that light does not experience Newtonian gravity (i.e. an inverse square law).²⁶ There is a centripetal force term, L^2/r^3 , but there is also a correction which does not appear at all in Newtonian mechanics. The $1/r^4$ correction is purely a general relativistic effect.

What can we say about the orbital shapes? In an orbit, we should be able to solve for a relationship between r and ϕ . Let us write this relationship in a somewhat unusual way: let

$$r(\phi) = 1/u(\phi),$$

where $u(\phi)$ is just some function of ϕ (e.g. for a circular orbit, $u(\phi) = 1$) or equivalently

$$u(\phi) = 1/r.$$

By the chain rule, we can compute a derivative of u with respect to ϕ :

$$\begin{aligned} \frac{du}{d\phi} &= \frac{du}{dr} \frac{dr}{ds} \frac{ds}{d\phi} \\ &= -\frac{1}{r^2} \dot{r} \frac{r^2}{L} \\ &= -\frac{\dot{r}}{L}. \end{aligned}$$

If we now rewrite Eqn. 14.2 in terms of u and $du/d\phi$ as

$$L^2 \left(\frac{du}{d\phi} \right)^2 - E^2 + (1 - 2Mu)(\epsilon + L^2 u^2) = 0,$$

²⁵If the orbit is circular, then we have $\dot{r} = 0$ and this just becomes an equation for the orbit radius r in terms of M, E , and L . It also depends on ϵ , i.e. whether we are looking at a massive test particle or a photon. As a fun consequence, it turns out that light can have a stable circular orbit around a black hole, and the radius at which it does so is called the photon sphere. Rotating (Kerr) black holes have more complicated equations of motion and this leads to multiple photon spheres!

²⁶The existence of this M/r^2 term also means that what we have called M (which a priori was just a parameter in our definition of the Schwarzschild metric) really is the mass of the gravitating object. That is, we have reproduced Newton's inverse square law for an object of mass M .

we can divide through by L^2 and differentiate with respect to ϕ ²⁷ to get

$$\frac{d^2 u}{d\phi^2} + u - 3Mu^2 - M\epsilon/L^2 = 0.$$

The first, second, and fourth terms are all Newtonian, but the $-3Mu^2$ term is again a general relativistic correction.

Let's consider a light ray, with $\epsilon = 0$. In Newtonian physics, we would just have

$$\frac{d^2 u}{d\phi^2} + u = 0,$$

whose solution is just a sine,

$$u_0 = \sin \phi / b.$$

²⁸ We've chosen the normalization such that when $\phi = \pi/2$, $u_0 = 1/b \implies b = r_0$, the distance of closest approach. Therefore when

$$\begin{aligned} \phi = 0 &\implies u = 0, r = \infty \\ \phi = \pi &\implies u = 0, r = \infty \\ \phi = \pi/2 &\implies u = 1/b, r = b. \end{aligned}$$

This is equivalent to saying that near a massive body, a light ray travels in a straight line and is not deflected in Newtonian gravity. In contrast, when we introduce the perturbation $-3Mu^2$, we now have

$$\frac{d^2 u}{d\phi^2} + u - 3Mu^2 = 0. \quad (14.4)$$

To solve the perturbed equation, we make the ansatz that $u = u_0 + u_1$ where u_0 solves the original equation and u_1 is a small perturbation. To leading order in u_1 , we get

$$\frac{d^2 u_1}{d\phi^2} + u_1 = \frac{3M \sin^2 \phi}{b^2} \quad (14.5)$$

where we have put back in the explicit solution for u_0 . We've therefore simplified our problem to a linear equation for u_1 , and the quickest way to solve this is the method of variation of parameters. That is, we take the solution to the homogeneous equation and assume that the solution to the inhomogeneous equation ^{14.5} is the homogenous solution multiplied by some function. More concretely, we make the ansatz that

$$u_1 = f(\phi) \sin \phi,$$

where $u_1 = \sin \phi$ solves the homogenous equation $\frac{d^2 u_1}{d\phi^2} + u_1 = 0$.²⁹ Then

$$\frac{du_1}{d\phi} = f \cos \phi + f' \sin \phi$$

and

$$\frac{d^2 u_1}{d\phi^2} = -f \sin \phi + 2f' \cos \phi + f'' \sin \phi.$$

If we substitute this back into Eqn. ^{14.5}, we find that

$$-f \sin \phi + 2f' \cos \phi + f'' \sin \phi + f \sin \phi = \frac{3M \sin^2 \phi}{b^2}.$$

²⁷We must also divide by $du/d\phi$, but this is okay since $du/d\phi = -\dot{r}/L \neq 0$ for non-circular orbits.

²⁸Of course we could solve this with a complex exponential, but since u relates r and ϕ , which are both classical quantities, u had better be a real-valued function.

²⁹Strictly, we should also include the cos solution when we do variation of parameters. I haven't yet worked out the consequences of including this in our analysis.

The order f terms cancel, so we're left with what is effectively a first-order equation in f' . Rewriting in terms of an integration factor (read: a big thing we're taking the derivative of), we get

$$\frac{1}{\sin \phi} \frac{d}{d\phi} (f' \sin^2 \phi) = \frac{3M \sin^2 \phi}{b^2},$$

or equivalently

$$\frac{d}{d\phi} (f' \sin^2 \phi) = \frac{3M \sin^3 \phi}{b^2} = \frac{3M \sin \phi}{b^2} (1 - \cos^2 \phi).$$

An integration and a little algebra reveals that

$$f' = -\frac{2M \cos \phi}{b^2 \sin^2 \phi} - \frac{M \cos \phi}{b^2} + \frac{C}{\sin^2 \phi},$$

but this constant C is just zero by the boundary conditions (f' should not diverge when $\phi \rightarrow 0$ or π). Integrating once more we find that

$$f = \frac{2M}{b^2 \sin \phi} - \frac{M \sin \phi}{b^2},$$

so to leading order

$$u = u_0 + u_1 = \frac{\sin \phi}{b} + \left(\frac{2M}{b^2} - \frac{M \sin^2 \phi}{b^2} \right).$$

Close to $\phi = 0, \pi$ we have $u \rightarrow 0$, so taking $\phi = \epsilon$ small, we get

$$\frac{\sin \epsilon}{b} + \frac{2M}{b^2} - \frac{M \sin^2 \epsilon}{b^2} = 0.$$

Expanding to leading order in ϵ , we find that

$$\epsilon \approx \frac{-2M}{b}.$$

This means that as $r \rightarrow \infty$, $\phi \rightarrow \epsilon \approx -2M/b$. Therefore in a gravitational field, light is deflected from its straight-line path by an angle $4M/b$ (where b is the distance of closest approach).

This has some practical applications. In 1919, Eddington proposed to measure the deflection of light grazing the sun during a solar eclipse to see if this relativistic bending of light could be detected, and indeed this deflection was measured. This was a great success for Einstein's theory. Moreover, the Hubble Space Telescope has made observations of gravitational lensing, i.e. where light is bent around a massive body to produce a secondary image. If the mass is in the way of the light, one sees a ring, called an "Einstein ring."

Finally, one may consider the GR corrections to planetary orbits, taking $\epsilon = 1$ for a massive test particle. One finds that the angular equation now takes the form

$$\left(\frac{du}{d\phi} \right)^2 = \frac{(E^2 - 1)}{L^2} + \frac{2Mu}{L^2} - u^2 + 2Mu^3,$$

where again the u^3 term is the relativistic correction. In the Newtonian theory, this has a solution

$$u_0 = \frac{M}{L^2} (1 + e \cos \phi).$$

Now we again treat the u^3 term as a perturbation, writing

$$u = u_0 + u_1$$

so that

$$2 \frac{du_0}{d\phi} \frac{du_1}{d\phi} = \frac{2Mu_1}{L^2} - 2u_0 u_1 + 2Mu_0^3$$

to first order in the perturbation. Substituting in for u_0 , we get

$$\begin{aligned}\frac{2M^4}{L^6}(1 + e \cos \phi)^3 &= -\frac{2Me \sin \phi}{L^2} \frac{du_1}{d\phi} + \frac{2M}{L^2}(1 + e \cos \phi)u_1 - \frac{2Mu_1}{L^2} \\ &= -\frac{2Me \sin \phi}{L^2} \frac{du_1}{d\phi} + \frac{2M}{L^2}e \cos \phi u_1.\end{aligned}$$

We rewrite as

$$\cos \phi u_1 - \sin \phi \frac{du_1}{d\phi} = \frac{M^3}{eL^4}(1 + e \cos \phi)^3,$$

or in terms of an integration factor,

$$-\sin^2 \phi \frac{d}{d\phi} \left(\frac{u_1}{\sin \phi} \right) = \frac{M^3}{eL^4}(1 + e \cos \phi)^3.$$

One can expand out the $(1 + e \cos \phi)^3$ into its three terms as

$$\frac{d}{d\phi} \left(\frac{u_1}{\sin \phi} \right) = -\frac{M^3}{eL^4 \sin^2 \phi} (1 + 3e \cos \phi + 3e^2 \cos^2 \phi + e^3 \cos^3 \phi).$$

Most of these terms are just periodic in ϕ and only produce small wiggles in u . But note that the $\cos^2 \phi$ term can be rewritten as $3e^2 - 3e^2 \sin^2 \phi$, so in particular the $\sin^2 \phi$ s will cancel and we'll get something that grows without bound in ϕ . Yikes! What's the behavior of this term?

$$\frac{d}{d\phi} \left(\frac{u_1}{\sin \phi} \right) \simeq -\frac{M^3}{3L^4} - 3e^2 \simeq \frac{3eM^3}{L^4}.$$

What we find is that

$$u_1 = \frac{3eM^3}{L^4} \phi \sin \phi$$

and so our solution for u takes the form

$$u = \frac{M}{L^2}(1 + e \cos \phi) + \frac{3eM^3}{L^4} \phi \sin \phi.$$