THE STANDARD MODE

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These notes were taken for the *The Standard Model* course taught by Christopher Thomas at the University of Cambridge as part of the Mathematical Tripos Part III in Lent Term 2019. I live-TEXed them using Overleaf, and as such there may be typos; please send questions, comments, complaints, and corrections to itel2@cam.ac.uk.
Many thanks to Arun Debray for the LATEX template for these lecture notes: as of the time of writing, you can find him at https://web.ma.utexas.edu/users/a.debray/.

Contents

Lecture 1.

Friday, January 18, 2019

Note. Here are some preliminary administrative notes on the course. Lecture notes and example sheets are available online at http://www.damtp.cam.ac.uk/user/cet34/teaching/. There are four example sheets and classes, plus a revision class in Easter Term. The instructor's email is c.e.thomas@damtp.cam.ac.uk. This course requires as prerequisites the Quantum Field Theory and Symmetries, Fields and Particles courses from Michaelmas term.

Some useful references are mentioned in the official course notes, including

- Peskin and Schroeder
- Aitchinson and Hey
- Halzen and Martin
- Donoghue, Golowich, and Holstein.

The sign conventions will be mostly in line with the Tong QFT notes, though note the sign of γ^5.

Quantum field theory was originally formulated to reconcile special relativity with quantum mechanics. The prototype for modern quantum field theories is quantum electrodynamics (QED), the quantum theory of light and charge. The *Standard Model* (SM) describes three fundamental forces (EM, weak, and strong) but does not include gravity. The model is an incredibly successful theory, having survived experimental tests up to the 1×10^8 GeV level. However, we know that because it does not include gravity, it must break down somewhere– perhaps at the Planck scale (1×10^{19} GeV).

In the SM, forces are mediated by gauge bosons (spin $= 1$).

- EM (QED): photon, γ (massless)
- Weak force: W boson and Z boson (massive)
- Strong force: gluon (massless)

Of course, our theory wouldn't be very good if we only had forces and no matter. In the SM, matter content is described by spin-1/2 fermions:

- neutrinos: ν_e, ν_μ, ν_τ (weak)
- charged leptons: e, μ, τ (weak and EM)
- quarks: $\begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} t \\ b \end{pmatrix}$.

We notice that there are three "generations" of matter particles where the properties of particles between generations are mostly the same, except the mass goes up in each generation.

Finally, we've got the Higgs boson, H (scalar, spin$=0$). The Higgs is responsible for generating mass of the W and Z bosons as well as all the fermions. This was famously discovered at the Large Hadron Collider in 2012.[1]

Gauge bosons are manifestations of *local* symmetries (as opposed to global symmetries)– we discussed this towards the end of Symmetries last term. The Standard Model gauge group is

$$SU(3)_C \times SU(2)_L \times U(1)_Y.$$

Here, $SU(3)_C$ is the "colour" symmetry of the strong interaction, QCD. The $SU(2)_L$ symmetry is a chiral (handedness) symmetry. And $U(1)_Y$ corresponds to something called hypercharge. It's actually a combination of the $SU(2)_L \times U(1)_Y$ symmetries that gives rise to the $U(1)_{EM}$ gauge symmetry of QED– these two symmetries together govern the electroweak interactions.

Chiral and gauge symmetries As always, we will use natural units in which $\hbar = c = 1$. To discuss *chiral symmetries*, let us consider a spin-1/2 Dirac fermion with a spinor field ψ satisfying the Dirac equation,

$$(i\partial\!\!\!/ - m)\psi = 0. \tag{1.1}$$

We use the Feynman slash notation, such that

$$\partial\!\!\!/ = \partial_\mu \gamma^\mu.$$

The (Dirac) adjoint (bar notation) is defined $\bar{\psi} = \psi^\dagger \gamma^0$, and satisfies

$$\bar{\psi}(-i\partial\!\!\!/^{\leftarrow} - m) = 0, \tag{1.2}$$

where $\partial\!\!\!/^{\leftarrow}$ acts to the left. The Dirac matrices γ^μ are a set of 4×4 matrices which satisfy the Lorentz algebra,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I, \tag{1.3}$$

where we will take $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ (the Minkowski metric with the mostly minus convention) and curly braces denote anticommutators as usual. We also define the γ^5 matrix to be

$$\gamma^5 = +i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{1.4}$$

so that $(\gamma^5)^2 = I, \{\gamma^5, \gamma^\mu\} = 0$. In the *chiral/Weyl basis*, the gamma matrices take the form

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.5}$$

This basis is so named because γ^5 picks out the left- and right-handed components.

Consider the massless limit of the Dirac equation,

$$\partial\!\!\!/\psi = 0 \implies \partial\!\!\!/(\gamma^5\psi) = 0. \tag{1.6}$$

Then we can define the *projection operators*,

$$P_{R,L} = \frac{1}{2}(1 \pm \gamma^5).. \tag{1.7}$$

This allows us to describe the components of a Dirac spinor:

$$\psi_{R,L} \equiv P_{R,L}\psi \implies \gamma^5\psi_{R,L} = \pm\psi_{R,L}. \tag{1.8}$$

These are eigenstates of the chirality operator, and are called "right-handed" or "left-handed" depending on whether they change sign under application of γ^5.

[1]Strictly, a Higgs-like particle which we have since verified many of the other properties of.

These are only properly eigenstates in the massless limit– if the particles are massive, then right-handed and left-handed states can mix (e.g. under Lorentz boosts). In chiral bases, ψ_R (ψ_L) only contains lower (upper) 2-component spinor degrees of freedom.

The effect of the field after projection is that ψ_L (ψ_R) annihilates left-handed (right-handed) chiral particles. Note also that the Dirac adjoint is

$$\bar{\psi}_{R,L} = (P_{R,L}\psi)^\dagger\gamma^0 = \psi^\dagger \frac{1}{2}(1 \pm \gamma^5)\gamma^0 = \bar{\psi}P_{L,R}. \tag{1.9}$$

We now observe that a massless Dirac fermion has a *global* $U(1)_L \times U(1)_R$ chiral symmetry:

$$U(1)_{R,L} : \psi_{R,L} \to e^{i\alpha_{R,L}}\psi_{R,L},$$

as can be seen from the Dirac Lagrangian:

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi = \bar{\psi}_L i\partial\!\!\!/ \psi_L + \bar{\psi}_R i\partial\!\!\!/ \psi_R - m(\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R).$$

However, the mass term explicitly breaks this chiral symmetry (it couples the left- and right-handed eigenstates together). It changes our chiral symmetry to a vector symmetry where $\alpha_L = \alpha_R = \alpha$ so the the field as a whole transforms to

$$U(1)_L \times U(1)_R \to U(1)_V : \psi \to e^{i\alpha}\psi.$$

Lecture 2.

Monday, January 21, 2019

Today we will continue the review of the Dirac field (cf. David Tong's QFT notes).

Review of Dirac field Recall that we can write the Dirac field ψ as a sum over momenta and spin states,

$$\psi(x) = \sum_{p,s}\left[b^s(p)u^s(p)e^{-ip\cdot x} + d^{s\dagger}v^s(p)e^{+ip\cdot x}\right], \tag{2.1}$$

where $s = \pm 1/2$ and $\sum_p \equiv \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}$. Momentum eigenstates are defined as

$$|p\rangle = b^\dagger(p)\,|0\rangle\,,$$

and the relativistic normalization of these momentum eigenstates is $\langle p|\, q\rangle = (2\pi)^3(2E_{\mathbf{p}})\delta^{(3)}(\mathbf{p}-\mathbf{q})$. The identity can be written as $I = \sum_p |p\rangle\langle p|$. Here, $b^\dagger, d^\dagger$ are creation operators for positive and negative frequency modes and u, v are our plane wave solutions to the Dirac equation.

That is, instead of writing a full four-component spinor we can write solutions

$$(p\!\!\!/ - m)u = 0,$$
$$(p\!\!\!/ + m)v = 0,$$

so that in the chiral basis, our plane wave solutions take the form

$$u^s(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s\end{pmatrix}, \quad u^s(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\eta^s\end{pmatrix}. \tag{2.2}$$

Here, $\sigma^\mu = (I_2, \sigma^i)$ and $\bar{\sigma}^\mu = (I_2, -\sigma^i)$. (I write the 2×2 identity matrix as I_2 here to avoid confusion.)

Helicity is defined as the projection of angular momentum onto a linear momentum direction. That is, the helicity operator takes the form

$$h = \mathbf{J}\cdot\hat{\mathbf{p}} = \mathbf{s}\cdot\hat{\mathbf{p}} \tag{2.3}$$

where the angular momentum operator is

$$\mathbf{J} = \mathbf{r}\times\mathbf{p} + \mathbf{s} \tag{2.4}$$

with

$$s_i = \frac{i}{4}\epsilon_{ijk}\gamma^j\gamma^k = \frac{1}{2}\begin{pmatrix}\sigma_i & 0 \\ 0 & \sigma_i\end{pmatrix} \tag{2.5}$$

in the chiral basis.

A massless spinor u then satisfies $\not{p}u = 0$, which means that

$$hu(p) = \frac{\gamma^5}{2}u(p)$$
$$hu_{R,L} = \frac{\gamma^5}{2}u_{R,L} = \pm\frac{1}{2}u_{R,L}.$$

Thus u can be decomposed into a basis of eigenstates u_R, u_L of the chirality operator, where u_R has positive helicity and u_L, negative helicity.

A few notes on chirality:

- Chiral states are only eigenstates of the Dirac equation when $m = 0$ (i.e. they don't mix).
- Helicity is defined for $m = 0$ and $m \neq 0$, but it is not Lorentz invariant when $m \neq 0$. This is because for a massive spinor, we could always imagine Lorentz boosting into a frame where the particle appears to be going the other way (while the direction of its angular momentum is unchanged).
- There is only a 1-1 correspondence between helicity and chirality when $m = 0$.

Review of gauge symmetry (local symmetry) Recall that we had a global symmetry where $\psi \to e^{i\alpha}\psi$, with $\alpha \in \mathbb{C}$. Now suppose we promote α to a function of x, $\alpha = \alpha(x)$ and

$$\psi \to e^{i\alpha(x)}\psi. \tag{2.6}$$

Under this *local* transformation, the old kinetic term is no longer invariant, as it becomes

$$\bar{\psi}i\not{\partial}\psi \to \bar{\psi}i\not{\partial}\psi - (\bar{\psi}\gamma^\mu\psi)(\partial_\mu\alpha(x)). \tag{2.7}$$

The way around this is to introduce a *covariant derivative* D_μ such that

$$D_\mu\psi(x) \to \exp(i\alpha(x))D_\mu\psi(x). \tag{2.8}$$

That is, the derivative transforms like the field itself under a gauge transformation so that our kinetic terms are preserved.

To do this, let us introduce a gauge field $A_\mu(x)$ such that

$$D_\mu\psi = (\partial_\mu + igA_\mu)\psi \quad \text{where } A_\mu(x) \to A_\mu(x) - \frac{1}{g}\partial_\mu\alpha \tag{2.9}$$

so that $\bar{\psi}i\not{D}\psi$ *is* invariant.

We could also introduce a kinetic term for the gauge fields,

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.10}$$

Equivalently $F_{\mu\nu}$ can be defined by a condition on g,

$$igF_{\mu\nu} = [D_\mu, D_0]. \tag{2.11}$$

What other gauge theories can we discuss? The theory of QED has a $U(1)$ gauge symmetry that treats LH and RH fields equivalently ($\alpha_L(x) = \alpha_R(x)$). However, the weak gauge bosons only couple to LH fields, but $U(1)$ is actually not the appropriate symmetry– we will need $SU(2)$. This completes the review of abelian gauge symmetries. We will review non-abelian gauge symmetries a little later.

Types of symmetry Symmetries may manifest themselves in a variety of ways.

(1) We can have a symmetry that is *intact* (unbroken), e.g. the $U(1)_{EM}$ and $SU(3)_C$ gauge symmetries.
(2) The symmetry of $\mathcal{L}$ is broken by an *anomaly* (i.e. it holds classically but when we quantize, something breaks). Not a true symmetry. For example, the global axial $U(1)$ symmetry in the SM.
(3) A symmetry can hold for some terms in the Lagrangian but not others (i.e. the terms which break the symmetry might be small at some relevant energy scale, so we can treat them perturbatively). This is an *explicitly broken* symmetry, though it may be an approximate symmetry if the breaking terms are small. For example, the global *isospin* symmetry relating u and d quarks in QCD.
(4) We might have a *hidden symmetry* which is respected by the Lagrangian but not by the vacuum.
    (a) A *spontaneously broken symmetry* results in a vacuum expectation value (VEV) for one or more scalar fields (cf. Higgs mechanism). In the SM, the $SU(2)_L \times U(1)_\gamma \to U(1)_{EM}$ is a spontaneously broken symmetry.

(b) Even without scalar fields, we can have *dynamical breaking* from quantum effects, e.g. the $SU(2)_L \times SU(2)_R$ global symmetry in QCD (massless quarks).

Discrete symmetries Some discrete symmetries we should be familiar with include

- Parity (P): $(t, \mathbf{x}) \to (t, -\mathbf{x})$
- Time reversal (T): $(t, \mathbf{x}) \to (-t, \mathbf{x})$
- Charge conjugation (C): exchanges particles $\leftrightarrow$ antiparticles.

These first two are spacetime symmetries, while the last is a bit different.

Example 2.12. Let's look at some examples of these symmetries in the Standard Model.

- The $\bar{\psi}\gamma^\mu\psi$ couplings between gauge bosons and fermions, e.g. QED and QCD, are invariant under P and C separately.
- $\bar{\psi}\gamma^\mu(1-\gamma^5)\psi$ couplings to fermions, e.g. the weak interaction, are not.
- The weak interaction violates CP, which implies that T-symmetry is also violated from the CPT theorem (i.e. a system must be invariant under the combination of C, P, and T).

To understand these statements, it will be useful to investigate the consequences of these C, P, T symmetries individually and together.

Lecture 3.

Wednesday, January 23, 2019

Today we'll discuss the consequences of discrete symmetries (CPT).

Symmetry operators We will start by quoting a result proven by Wigner.

Theorem 3.1. *If physics is invariant under some transformation $\Psi \to \Psi'$ (with $\Psi, \Psi' \in$ some Hilbert space), then there is an operator W such that $\Psi' = W\Psi$ and where either W is linear and unitary, or* anti*linear and* anti-*unitary.*

That is, writing the inner product on the hilbert space as $(\cdot, \cdot)$, we have either

- W is unitary and linear,

$$(W\Phi, W\Psi) = (\Phi, \Psi) \text{ and } W(\alpha\Phi + \beta\Psi) = \alpha W\Phi + \beta W\Psi \; (\alpha, \beta \in \mathbb{C}) \tag{3.2}$$

- or W is antiunitary and antilinear,

$$(W\Phi, W\Psi) = (\Phi, \Psi)^* \text{ and } W(\alpha\Phi + \beta\Psi) = \alpha^* W\Phi + \beta^* W\Psi. \tag{3.3}$$

Note that W being antiunitary as an operator is not the same as W being an antiunitary matrix ($W^{-1} = -W^\dagger$).

Now, let us recall the Poincaré transformations, which take

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu. \tag{3.4}$$

In particular we have some improper Lorentz transformations (not of det $= +1$) which are of special importance. There's the parity transformation,

$$\Lambda^\mu{}_\nu = \mathbb{P}^\mu{}_\nu = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \tag{3.5}$$

and also time reversal,

$$\mathbb{T}^\mu{}_\nu = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}. \tag{3.6}$$

Consider an infinitesimal transformation

$$\Lambda^\mu{}_\nu + \delta^\mu{}_\nu + \omega^\mu{}_\nu, a_\mu = \epsilon_\mu. \tag{3.7}$$

Then the corresponding operator can be expanded as

$$W(\Lambda, a) = W(1+\omega, \epsilon) = 1 + \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} - i\epsilon_\mu P^\mu, \tag{3.8}$$

where $J^{\mu\nu}$ is the generator of boosts and rotations and P^μ is a four-momentum operator with $P^0 = H$ the Hamiltonian and $p^i =$ the three-momentum operator.

Thus we can write the parity and time reversal operators as

$$\hat{P} = W(\mathbb{P}, 0)$$
$$\hat{T} = W(\mathbb{T}, 0).$$

From the general composition rule, we can write

$$\hat{P}W(\Lambda, a)\hat{P}^{-1} = W(\mathbb{P}\Lambda\mathbb{P}^{-1}, \mathbb{P}a). \tag{3.9}$$

If we now insert the expansion of W 3.8 on both sides of the equation and compare the coefficients of ϵ_0, we find that

$$\hat{P}iH\hat{P}^{-1} = iH, \tag{3.10}$$

where we recall that $H = P^0$. Similarly,

$$\hat{T}W(\Lambda, a)\hat{T}^{-1} = W(\mathbb{T}\Lambda\mathbb{T}^{-1}, \mathbb{T}a), \tag{3.11}$$

which implies that

$$\hat{T}iH\hat{T}^{-1} = -iH. \tag{3.12}$$

We've been careful not to move the i through the operator $\hat{T}$, since we don't yet know whether the operator is unitary or anti-unitary.

Suppose now Ψ is an energy eigenstate,

$$(\Psi, iH\Psi) = iE.$$

If $\hat{P}$ and $\hat{T}$ are symmetries, then $\hat{P}\Psi$ and $\hat{T}\Psi$ should also be energy eigenstates with energy E.

Suppose $\hat{P}$ is linear. Then we have

$$(\hat{P}\Psi, iH\hat{P}\Psi) = (\hat{P}\Psi, \hat{P}iH\Psi) = (\hat{P}\Psi, \hat{P}iE\Psi) = iE(\hat{P}\Psi, \hat{P}\Psi) = iE, \tag{3.13}$$

by 3.10 and linearity. We could have also run this argument with unitarity instead.

Similarly, suppose $\hat{T}$ is linear. Then

$$(\hat{T}\Psi, iH\hat{T}\Psi) = -(\hat{T}\Psi, \hat{T}iH\Psi) = -iE \tag{3.14}$$

by an equivalent argument using 3.12. But this tells us that $\hat{T}$ has produced an energy eigenstate with energy $-iE$, which is wrong.

Therefore, suppose $\hat{T}$ is anti-linear. Then

$$(\hat{T}\Psi, iH\hat{T}\Psi) = -(\hat{T}\Psi, \hat{T}iH\Psi) = -(\hat{T}\Psi, \hat{T}iE\Psi) = +iE(\hat{T}\Psi, \hat{T}\Psi) = +iE. \tag{3.15}$$

Therefore $\hat{T}$ must be anti-linear and anti-unitary.

To sum up, the parity operator $\hat{P}$ is unitary and linear, while the time reversal operator $\hat{T}$ is antiunitary and antilinear.

Parity Now that we've defined some basic properties of these symmetries, let's consider what parity does to different fields.

For a complex scalar field,

$$\phi(x) = \sum_p \Big[a(p)e^{-ip\cdot x} + c^\dagger(p)e^{+ip\cdot x}\Big], \tag{3.16}$$

where the operator a annihilates a particle and $c^\dagger$ creates an antiparticle.

The operator $\hat{P}$ maps momentum eigenstates $|p\rangle \mapsto \eta^{a*}|p_P\rangle$ where

$$p_P = (p^0, -\mathbf{p}) \tag{3.17}$$
$$X_P = (x^0, -\mathbf{x}) \tag{3.18}$$

and η^{a*} is a complex phase.

Thus
$$\hat{P}a^\dagger(p)\left|0\right\rangle = \eta^{a*}a^\dagger(p_P)\left|0\right\rangle. \tag{3.19}$$
Since $\hat{P}\hat{P}^{-1} = I$ and assuming $\hat{P}\left|0\right\rangle = \left|0\right\rangle$ (the vacuum is invariant under $\hat{P}$), we conclude that
$$\hat{P}a^\dagger(p)\hat{P}^{-1} = \eta^{a*}a^\dagger(p_P). \tag{3.20}$$
To preserve the normalization, we must have
$$\hat{P}a(p)\hat{P}^{-1} = \eta^{a}a(p_P). \tag{3.21}$$
Similarly, we can work out that
$$\hat{P}c^\dagger(p)\hat{P}^{-1} = \eta^{c*}c^\dagger(p_P). \tag{3.22}$$
Now since $\hat{P}$ is linear and unitary, we can write $\hat{P}\phi(x)\hat{P}^{-1}$ as follows:
$$\begin{aligned}
\hat{P}\phi(x)\hat{P}^{-1} &= \sum_p \left[\hat{P}a(p)\hat{P}^{-1}e^{-ip\cdot x} + \hat{P}c^\dagger(p)\hat{P}^{-1}e^{+ip\cdot x}\right] \\
&= \sum_p \left[\eta^a a(p_P)e^{-ip\cdot x} + \eta^{c*}c^\dagger(p_P)e^{+ip\cdot x}\right] \\
&= \sum_{p_P} \left[\eta^a a(p)e^{-ip_P\cdot x} + \eta^{c*}c^\dagger(p)e^{+ip_P\cdot x}\right] \text{ relabeling } p \leftrightarrow p_P \\
&= \sum_{p_P} \left[\eta^a a(p)e^{-ip\cdot x_P} + \eta^{c*}c^\dagger(p)e^{+ip\cdot x_P}\right] \text{ using } p_P\cdot x = p\cdot x_P \\
&= \sum_{p} \left[\eta^a a(p)e^{-ip\cdot x_P} + \eta^{c*}c^\dagger(p)e^{+ip\cdot x_P}\right] \text{ relabeling } \sum_p = \sum_{p_P}.
\end{aligned}$$
Note that this does not "look like" $\phi(x_P)$ unless $\eta^a = \eta^{c*} \equiv \eta_p$ (if you like, we're matching the coefficients of Fourier modes). If you're not convinced by this, notice that we would not in general find the commutator $[\phi(x), \hat{P}\phi^\dagger(y)\hat{P}^{-1}]$ vanishes for spacelike $x - y$.

Lecture 4.

Friday, January 25, 2019

Last time, we argued that the parity transformation takes the form
$$\hat{P}\phi(x)\hat{P}^{-1} = \sum_p \left[\eta_p a(p)e^{-ip\cdot x_p} + \eta_p c^\dagger(p)e^{+ip\cdot x_p}\right], \tag{4.1}$$
with η_p the *intrinsic parity* of ϕ. In this notation, we found that
$$\hat{P}\phi(x)\hat{P}^{-1} = \eta_p\phi(x_p), \tag{4.2}$$
with $x_P = (x^0, -\mathbf{x})$.

Let us make some comments on the parity transformation.

- For a real scalar field, $a = c$ (the particle and antiparticle operators are the same) and so $\eta^a = \eta^{a*} = \eta_p$, which tells us that $\eta_p = \pm 1$. We say that $\eta_p = +1$ is a scalar and -1 a pseudoscalar.
- For a complex scalar field, η_p may not be real, but if there is a conserved charge then we can redefine the operator $\hat{P}$ so that $\eta_p = \pm 1$ (not obvious, but cf. Weinberg §3.3 and 2.2).
- For a vector field,
$$V^\mu(x) = \sum_{p,\lambda}\left[\epsilon^\mu(\lambda, p)a^\lambda(p)e^{-ip\cdot x} + \epsilon^{\mu*}(\lambda, p)c^{\lambda\dagger}(p)e^{+ip\cdot x}\right], \tag{4.3}$$
where $\lambda = -1, 0, +1$ is the helicity (for a massive particle, or else we would not get the zero helicity state). The ϵs are polarization vectors, like for photons. If we use
$$\mathbb{P}^\mu{}_\nu\epsilon^\nu(\lambda, p) = -\epsilon^\mu(\lambda, p_P), \tag{4.4}$$
then by a similar argument as above,
$$\hat{P}V^\mu(x)\hat{P}^{-1} = -\eta_p\mathbb{P}^\mu{}_\nu V^\nu(x_P). \tag{4.5}$$

Vectors have $\eta_p = -$ and axial vectors have $\eta_p = +1$.

The Dirac field For the Dirac field, creation and annihilation operators should behave like those for bosons. The three-momentum reverses direction, but the spin component is unchanged, so

$$\hat{P} b^s(p) \hat{P}^{-1} = \eta^b b^s(p_P) \tag{4.6}$$

and

$$\hat{P} d^{s\dagger}(p) \hat{P}^{-1} = \eta^{d*} d^{s\dagger}(p_P). \tag{4.7}$$

Recalling that

$$\psi(x) = \sum_{p,s} \Big[b^s(p) u^s(p) e^{-ip\cdot x} + d^{s\dagger}(p) v^s(p) e^{+ip\cdot x} \Big], \tag{4.8}$$

we notice that the spinors are just a set of four complex numbers, and four-vector inner products are unchanged by parity, so only the operators $b^s, d^{s\dagger}$ are hit by the parity operator, giving

$$\begin{aligned} \hat{P}\psi(x)\hat{P}^{-1} &= \sum_{p,s} \Big[\eta^b b^s(p_P) u^s(p) e^{-ip\cdot x} + \eta^{d*} d^{s\dagger}(p_P) v^s(p) e^{+ip\cdot x} \Big] \\ &= \sum_{p,s} \Big[\eta^b b^s(p) u^s(p_P) e^{-ip\cdot x_P} + \eta^{d*} d^{s\dagger}(p) v^s(p_P) e^{+ip\cdot x_P} \Big], \end{aligned}$$

We leave it as an exercise to check that $u^s(p_P) = \gamma^0 u^s(p), v^s(p_P) = -\gamma^0 v^s(p)$. Using these relations, it follows that

$$\sum_{p,s} \Big[\eta^b b^s(p) \gamma^0 u^s(p) e^{-ip\cdot x_P} - \eta^{d*} d^{s\dagger}(p) \gamma^0 v^s(p) e^{+ip\cdot x_P} \Big] \implies \eta^b = -\eta^{d*} \equiv \eta_p \tag{4.9}$$

so that

$$\psi^P(x) \equiv \hat{P}\psi(x)\hat{P}^{-1} = \eta_p \gamma^0 \psi(x_P). \tag{4.10}$$

Thus

$$\bar{\psi}^P(x) \equiv \hat{P}\bar{\psi}(x)\hat{P}^{-1} = \eta_p^* \bar{\psi}(x_P)\gamma^0. \tag{4.11}$$

Thus for the Dirac field, parity sends left-handed fields to right-handed fields under

$$\hat{P}\psi_L(x)\hat{P}^{-1} = \eta_p \gamma^0 \psi_R(x_P). \tag{4.12}$$

One should also check that $\psi^P(x)$ satisfies the Dirac equation if $\psi(x)$ does. Thus we can determine the transformation properties of fermion bilinears, e.g.

$$\bar{\psi}(x)\psi(x) \to \hat{P}\bar{\psi}(x)\hat{P}\hat{P}^{-1}\bar{\psi}(x)\hat{P}^{-1} = \bar{\psi}(x_P)\psi(x_P). \tag{4.13}$$

Since we don't pick up a sign flip, we call this a scalar fermion bilinear. Similarly a bit of direct computation yiels the pseudoscalar case:

$$\bar{\psi}(x)\gamma^5\psi(x) \to -\bar{\psi}(x_P)\gamma^5\psi(x_P), \tag{4.14}$$

the vector case:

$$\bar{\psi}(x)\gamma^\mu\psi(x) \to \mathbb{P}^\mu{}_\nu \bar{\psi}(x_P)\gamma^\nu\psi(x_P), \tag{4.15}$$

and the axial vector case:

$$\bar{\psi}(x)\gamma^\mu\gamma^5\psi(x) \to -\mathbb{P}^\mu{}_\nu \bar{\psi}(x_P)\gamma^\nu\gamma^5\psi(x_P), \tag{4.16}$$

Charge conjugation Having thoroughly discussed parity, let us now talk about charge conjugation, $\hat{C}$. The operator $\hat{C}$ is unitary and linear, and it sends particles to antiparticles. Note that Lorentz symmetry constrains the phases, so

$$\hat{C}a(p)\hat{C}^{-1} = \eta_c c(p), \quad \hat{C}c(p)\hat{C}^{-1} = \eta_c^* a(p). \tag{4.17}$$

Thus

$$\hat{C}\left|\text{particle}, p\right\rangle = \hat{C}a^\dagger(p)\left|0\right\rangle = \eta_c^* c^\dagger(p)\left|0\right\rangle = \eta_c^*\left|\text{antiparticle}, p\right\rangle. \tag{4.18}$$

From the decomposition of the field, we find that

$$\begin{aligned} \hat{C}\phi(x)\hat{C}^{-1} &= \eta_c \phi^\dagger(x) \\ \hat{C}\phi^\dagger(x)\hat{C}^{-1} &= \eta_c^* \phi(x). \end{aligned}$$

For a real scalar field, $\phi^\dagger = \phi$ and so $\eta_C = \pm 1$.

This has some important consequences. For instance, the photon field must obey $\hat{C}A_\mu(x)\hat{C}^{-1} = -A_\mu(x)$. Note that the π^0 meson can decay to 2γ, which tells us that $\eta_c^{\pi^0} = (-1)^2 = +1$.

Lecture 5.

Monday, January 28, 2019

Last lecture, we finished parity and began charge conjugation. We said that

$$\hat{C}a(p)\hat{C}^{-1} = \eta_c c(p), \quad \hat{C}c(p)\hat{C}^{-1} = \eta_c^* a(p).$$

For a real scalar field, we have $\phi^\dagger = \phi$ and thus $\eta_c = \pm 1$. On the other hand, for a complex field, η_c is arbitrary. Say

$$\eta_c = e^{2i\beta}, \tag{5.1}$$

with $\beta \in \mathbb{R}$. We can do a global $U(1)$ transformation sending

$$\phi \to \phi' = e^{-i\beta}\phi$$

so that $\eta_c' = 1$. We can do this for a single field, though we can't quite repeat it for arbitrary numbers of fields.

Dirac field For Dirac fields, define a matrix C s.t.

$$(\gamma^\mu C)^T = \gamma^\mu C. \tag{5.2}$$

In the chiral basis where $\gamma^{0T} = \gamma^0, \gamma^{2T} = \gamma^2 . \gamma^{1T} = \gamma^1, \gamma^{3T} = -\gamma^T$, a suitable choice for C is

$$C = -i\gamma^0\gamma^2 = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}. \tag{5.3}$$

We observe that $C = -C^T = -C^\dagger = -C^{-1}$. In addition,

$$(\gamma^\mu)^T = -C^{-1}\gamma^\mu C, \quad \gamma^{5T} = +C^{-1}\gamma^5 C. \tag{5.4}$$

Similarly to the bosonic operators, the fermion operators b^s, d^s are transformed as

$$\begin{aligned} \hat{C}b^s(p)\hat{C}^{-1} &= \eta_c d^s(p) \\ \underbrace{\hat{C}d^{s\dagger}(p)\hat{C}^{-1}}_{\text{in } \psi} &= \underbrace{\eta_c b^{s\dagger}(p)}_{\text{in } \bar{\psi}}. \end{aligned}$$

So charge conjugation indeed has the interpretation of sending particles to antiparticles and vice versa.

Now consider the conjugation of the whole ψ field:

$$\hat{C}\psi(x)\hat{C}^{-1} = \eta_c \sum_{p,s}\Big[d^s(p)u^s(p)e^{-ip\cdot x} + b^{s\dagger}(p)v^s(p)e^{+ip\cdot x}\Big]. \tag{5.5}$$

We can compare this with

$$\bar{\psi}^T(x) = \eta_c \sum_{p,s}\Big[b^{s\dagger}(p)\bar{u}^{sT}(p)e^{+ip\cdot x} + d^s(p)\bar{v}^{sT}(p)e^{-ip\cdot x}\Big]. \tag{5.6}$$

This almost looks like our previous equation, except for the spinor parts. Consider the spinors and take $\eta^s = i\sigma^2\xi^{s*}$ (choose a basis). We can write $V^s(p) = C\bar{u}^{ST}$ and $u^S(p) = C\bar{v}^{sT}(p)$. Therefore

$$\psi^c(x) \equiv \hat{C}\psi(x)\hat{C}^{-1} = \eta_c C\bar{\psi}^T(x) \tag{5.7}$$

By a similar computation,

$$\bar{\psi}^C(x) \equiv \hat{C}\bar{\psi}(x)\hat{C}^{-1} = \eta_c^*\psi^T(x)C = -\eta_c^*\psi^T(x)C^{-1}. \tag{5.8}$$

Note that if $\psi(x)$ satisfies the Dirac equation, then so does $\psi^C(x)$. In particular, for *Majorana fermions*, we have $b^s(p) = d^s(p) \implies$ the particle is its own antiparticle. In this case, $\psi^C = \psi$. This means the particle cannot have charge, and it turns out that the only candidates for Majorana fermions in the SM are neutrinos, but it is not known whether neutrinos are in fact Majorana fermions. A signature of this would be neutrinoless double β decay, which we'll revisit later in the course.

Fermion bilinears Note that we will want to be careful about what is an operator ($\hat{C}$) and what's a matrix in spinor space (C). How do our bilinears transform under charge conjugation?

Example 5.9. Consider the bilinear $j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$. We can conjugate by $\hat{C}$, noting that the γ^μ are just numbers and so unaffected by conjugation. Thus

$$\begin{aligned}\hat{C}j^\mu(x)\hat{C}^{-1} &= \hat{C}\bar{\psi}(x)\hat{C}^{-1}\gamma^\mu\hat{C}\psi(x)\hat{C}^{-1}\\ &= -\eta_C^*\eta_C\psi^T C^{-1}\gamma^\mu C\bar{\psi}^T,\end{aligned}$$

passing to the matrix notation. Adding back the Dirac indices and noting that $|\eta_c|^2 = 1$, we have

$$\begin{aligned}\hat{C}j^\mu(x)\hat{C}^{-1} &= -\psi_\alpha(C^{-1}\gamma^\mu C)_{\alpha\beta}\bar{\psi}_\beta\\ &= +\bar{\psi}_\beta(C^{-1}\gamma^\mu C)_{\alpha\beta}\psi_\alpha\\ &= \bar{\psi}_\beta(C^{-1}\gamma^\mu C)^T_{\beta\alpha}\psi_\alpha\\ &= \bar{\psi}(C^{-1}\gamma^\mu C)^T\psi,\end{aligned}$$

where we used the fact that fermions anticommute in going from the first to second line, took a transpose to make the indices line up, and then dropped the spinor indices. We recognize $(C^{-1}\gamma^\mu C)^T = -\gamma^\mu$, and conclude that

$$\hat{C}j^\mu(x)\hat{C}^{-1} = -\bar{\psi}\gamma^\mu\psi = -j^\mu(x). \tag{5.10}$$

By the transformation properties of A_μ under charge conjugation, we see that $j^\mu A_\mu$ is C-invariant.

By a similar calculation, we can show that

$$\hat{C}j^{\mu 5}\hat{C}^{-1} = +j^{\mu 5}, \tag{5.11}$$

where

$$j^{\mu 5} = \bar{\psi}\gamma^\mu\gamma^5\psi. \tag{5.12}$$

Time reversal T-symmetric theories leave physics unchanged if time runs backwards. That is,

$$x_T^\mu = (-x^0, \mathbf{x}), p_T^\mu = (p^0, -\mathbf{p}). \tag{5.13}$$

Note that $\hat{T}$ is anti-unitary and antilinear! So we will have to be careful about signs.

For the boson field, we have

$$\hat{T}a(p)\hat{T}^{-1} = \eta_T a(p_T), \quad \hat{T}c^\dagger(p)\hat{T}^{-1} = \eta_T c^\dagger(p_T). \tag{5.14}$$

Note that by antiunitarity, $\hat{T}a(p)e^{-ip\cdot x}\hat{T}^{-1} = \hat{T}a(p)\hat{T}^{-1}e^{+ip\cdot x}$. From the decomposition of ϕ, we see that the full field transforms as

$$\hat{T}\phi(x)\hat{T}^{-1} = \sum_p\Big[\hat{T}a(p)\hat{T}^{-1}e^{+ip\cdot x} + \hat{T}c^\dagger(p)\hat{T}^{-1}e^{-ip\cdot x}\Big]. \tag{5.15}$$

One can use the same steps as for the parity transformation and the identity $p_T\cdot x = -p\cdot x_T$ to rewrite the field as

$$\hat{T}\phi(x)\hat{T}^{-1} = \eta_T\sum_p\Big[a(p)e^{-ip\cdot x_T} + c^\dagger(p)e^{+ip\cdot x_T}\Big] = \eta_T\phi(x_T). \tag{5.16}$$

Lecture 6.

Wednesday, January 30, 2019

This lecture, we're going to finish our discussion of time reversal symmetry, culminating in the CPT theorem next lecture.

	$\hat{P}\ldots\hat{P}^{-1}$	$\hat{C}\ldots\hat{C}^{-1}$	$\hat{T}\ldots\hat{T}^{-1}$
$\mathcal{L}_I(x)$	$\mathcal{L}_I(x_P)$	$\mathcal{L}_I(x)$	$\mathcal{L}_I(x_T)$
$V(t)$	$V(t)$	$V(t)$	$V(-t)$
S	S	S	?

TABLE 1. Caption

Dirac field The operator $\hat{T}$ flips the sign of the angular momentum ($\mathbf{r}\times\mathbf{p}$). Therefore the creation and annihilation operators can be taken to transform as

$$\hat{T}b^s(p)\hat{T}^{-1} = \eta_T(-1)^{\frac{1}{2}-s}b^{-s}(p_T) \tag{6.1}$$

$$\hat{T}d^{s\dagger}(p)\hat{T}^{-1} = \eta_T(-1)^{\frac{1}{2}-s}d^{-s\dagger}(p_T). \tag{6.2}$$

Since s takes values $\pm 1/2$, this just says that the spin-up and spin-down states pick up a relative minus sign. It can be shown that the plane wave solutions themselves transform as

$$(-1)^{1/2-s}u^{-s*}(p_T) = -Bu^s(p) \tag{6.3}$$

$$(-1)^{1/2-s}v^{-s*}(p_T) = -Bv^s(p) \tag{6.4}$$

where

$$B \equiv C^{-1}\gamma^5 = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}.$$

Then

$$\hat{T}\psi(x)\hat{T}^{-1} = \eta_T\sum_{p,s}(-1)^{\frac{1}{2}-s}\left[b^{-s}(p_T)u^{s*}(p)e^{ip\cdot x} + d^{-s\dagger}(p_T)v^{s*}(p)e^{-ip\cdot x}\right]. \tag{6.5}$$

We can relabel s to $-s$ in order to get the spinor indices on b and $d^\dagger$ back to s. By playing our usual game of relabeling p and p_T and then changing $p_T\cdot x = -p\cdot x_T$, we get

$$\begin{aligned}\hat{T}\psi(x)\hat{T}^{-1} &= \eta_T\sum_{p,s}(-1)^{\frac{1}{2}-s+1}\left[b^s(p_T)u^{-s*}(p)e^{-ip\cdot x_T} + d^{s\dagger}(p_T)v^{-s*}(p)e^{+ip\cdot x_T}\right] \\ &= \eta_T B\psi(x_T),\end{aligned}$$

where this final result has come from applying 6.3-6.4. Similarly,

$$\hat{T}\bar{\psi}(x)\hat{T}^{-1} = \eta_T^*\bar{\psi}(x_T)B^{-1}. \tag{6.6}$$

What about our bilinears? A quick aside: we can show that $B^{-1}\gamma^{0*}B = \gamma^0, B^{-1}\gamma^{i*}B = -\gamma^i$, which tells us that in general

$$B^{-1}\gamma^{\mu*}B = -\mathbb{T}^\mu{}_\nu\gamma^\nu. \tag{6.7}$$

Using the previous computations, we find that

$$\begin{aligned}\hat{T}\bar{\psi}(x)\psi(x)\hat{T}^{-1} &= \bar{\psi}(x_T)\psi(x_T) \\ \hat{T}\bar{\psi}(x)\gamma^\mu\psi(x)\hat{T}^{-1} &= \bar{\psi}(x_T)B^{-1}\gamma^{\mu*}B\psi(x_T) \\ &= -\mathbb{T}^\mu{}_\nu\bar{\psi}(x_T)\gamma^\nu\psi(\gamma_T)\end{aligned}$$

Note that $\mu = 0$ has the interpretation of a charge density, while $\mu = i$ has the interpretation of a current density.

Scattering S-matrix From Quantum Field Theory, we recall that the S-matrix relates initial and final states by

$$\langle p_1, p_2, \ldots|\, S\,|k_A, k_B, \ldots\rangle = {}_{\text{out}}\langle p_1, p_2, \ldots\,|\, k_A, k_B, \ldots\rangle_{in},$$

where the RHS quantities are asymptotic out and in states. We can expand this in terms of time evolution, i.e.

$$\langle p_1, p_2, \ldots|\, S\,|k_A, k_B, \ldots\rangle = \lim_{T\to\infty}\langle p_1, p_2, \ldots|\,\mathcal{T}e^{-i\int_{-T}^{T}V(t)dt}\,|k_A, k_B, \ldots\rangle$$

where the $\mathcal{T}$ here indicates time-ordering and $V(t) = -\int d^3x\mathcal{L}_I(x)$ is the potential energy term.

For the QED Lagrangian $\mathcal{L}_I(x) = -e\bar{\psi}(x)\gamma^\mu A_\mu(x)\psi(x)$, we get the results in Table 1. What happens to the time-ordered exponential in S under time reversal, though?

$$S = \sum_{n=0}^{\infty} (-i)^n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \ldots \int_{-\infty}^{t_{n-1}} dt_n V(t_1)V(t_2)\ldots V(t_n), \tag{6.8}$$

so then

$$S_T \equiv \hat{T}S\hat{T}^{-1} = \sum_{n=0}^{\infty} (+i)^n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \ldots \int_{-\infty}^{t_{n-1}} dt_n V(-t_1)V(-t_2)\ldots V(-t_n). \tag{6.9}$$

That is, time reversal has switched the sign on t in all the potential terms, and we have picked up factors of i because $\hat{T}$ is anti-unitary. Under the substitution $\tau_i = -t_{n+1-i}$, this string of integrals becomes

$$\int_{+\infty}^{-\infty} (-d\tau_n) \int_{+\infty}^{-t_1} (-d\tau_{n-1}) \ldots \int_{+\infty}^{-t_{n-1}} (-d\tau_1) V(\tau_n)V(\tau_{n-1})\ldots V(\tau_1).$$

Switching the limits of integration, we can get rid of all the minus signs. Finally, we relabel $-t_1 = \tau_n, \ldots, -t_{n-1} = \tau_2$ to find our integrals become

$$\int_{-\infty}^{\infty} d\tau_n \int_{\tau_n}^{\infty} d\tau_{n-1} \ldots \int_{\tau_2}^{\infty} d\tau_1.$$

Recognizing that (geometrically, for example) $\int_{-\infty}^{\infty}\int_{\tau_n}^{\infty} d\tau_{n-1} = \int_{-\infty}^{\infty} d\tau_{n-1} \int_{-\infty}^{\tau_{n-1}} d\tau_n$, we can put everything back in the right order in our new τ variables and find that

$$S_T = \sum_{n=0}^{\infty} (+1)^n \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 \ldots \int_{-\infty}^{\tau_{n-1}} d\tau_n V(\tau_n)V(\tau_{n-1})\ldots V(\tau_1). \tag{6.10}$$

By comparing, we see that

$$\begin{aligned} S^\dagger &= \sum_{n=0}^{\infty} (+1)^n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \ldots \int_{-\infty}^{t_{n-1}} dt_n [V(t_1)V(t_2)\ldots V(t_n)]^\dagger \\ &= \sum_{n=0}^{\infty} (+1)^n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \ldots \int_{-\infty}^{t_{n-1}} dt_n [V(t_n)V(t_{n-1})\ldots V(t_1)], \end{aligned}$$

so we conclude that $S_T = S^\dagger$. Conside

Lecture 7.

Friday, February 1, 2019

S-matrix, continued Recall that we computed $S_T \equiv \hat{T}S\hat{T}^{-1} = S^\dagger$, so equivalently $S = S_T^\dagger$. Consider time-reversed states

$$|\xi_T\rangle = \hat{T}\,|\xi\rangle\,, \quad |\eta_T\rangle = \hat{T}\,|\eta\rangle\,. \tag{7.1}$$

The the inner product is

$$\begin{aligned} \langle \eta_T|\, S\, |\xi_T\rangle &= (\hat{T}\eta, S_T^\dagger \hat{T}\xi) \\ &= (\hat{T}\eta, \hat{T}S^\dagger \xi) \\ &= (\eta, S^\dagger\xi)^* \\ &= (S^\dagger\xi, \eta) \\ &= (\xi, S\eta) \\ &= \langle\xi|\, S\, |\eta\rangle\,. \end{aligned}$$

Therefore if $\hat{T}\mathcal{L}_I(x)\hat{T}^{-1} = \mathcal{L}(x_T)$, S matrix elements are equal for time-reversed processes where the initial and final states are swapped.

CPT theorem

Theorem 7.2. *Any Lorentz-invariant Lagrangian $\mathcal{L}$ with a hermitian Hamiltonian should be invariant under the product of P, C, and T.*

We won't prove this theorem, but details are in Streater and Wightman, "PCT, spin and statistics, and all that" (1989). All observations suggest that CPT is respected in nature. This implies that a particle (positive charge, spin up) propagating forward in time cannot be distinguished from an antiparticle (negative charge, spin down) propagating backwards in time.

Baryogenesis *Baryogenesis* is the generation of a matter-antimatter asymmetry in the universe (i.e. the question of why there is more matter than antimatter in the universe today). According to Sakharov, there are three necessary conditions for baryogenesis:

(a) Baryon number violation, i.e. processes like $X \to Y + B$ where B represents excess baryons (or leptogenesis: lepton number asymmetry $\to$ baryon no. asymmetry through $B + L$ violation)
(b) Non-equilibrium (otherwise $\Gamma(Y + B \to X) = \Gamma(x \to Y + B)$, where Γ is the rate of the process)
(c) C and CP violation, otherwise

$$\frac{dB}{dt} \propto \Gamma(X \to Y + B) - \Gamma(\bar{X} \to \bar{Y} + \bar{B}) = 0$$

under C-symmetry, and

$$\Gamma(x \to nq_L) + \Gamma(x \to nq_R) = \Gamma(\bar{x} \to n\bar{q}_R) + \Gamma(\bar{x} \to n\bar{q}_L)$$

under CP symmetry, with n some number of quarks.

Spontaneous symmetry breaking (SSB) Spontaneous symmetry breaking is a hidden symmetry or symmetries which are present in the Lagrangian $\mathcal{L}$ but not in observables. Let us first discuss the SSB of a discrete symmetry.

Consider a real scalar field $\phi(x)$ with a symmetric potential $V(\phi)$, i.e. $V(\phi) = V(-\phi)$, with a Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi), \tag{7.3}$$

and let us take ϕ^4 theory, with

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4, \quad \lambda > 0. \tag{7.4}$$

In the usual case, we take $m^2 > 0$ (i.e. massive scalar field). Thus $V(\phi)$ has a minimum at $\phi = 0$, and we solve by considering perturbations around the state $\phi = 0$ (for small λ).

However, if $m^2 < 0$, then we can rewrite the potential as

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - V^2)^2 \tag{7.5}$$

where $V \equiv \sqrt{-m^2/\lambda}$ up to an unimportant constant. Now $\phi = 0$ is an *unstable* vacuum and there are two degenerate vacua (minima) at $\phi = \pm V$.

In the $m^2 < 0$ case, we see that ϕ has acquired a non-zero vacuum expectation value (VEV). WLOG, let us study small excitations about the $\phi = +V$ vacuum. Thus we write $\phi(x) = V + f(x)$ in terms of a shifted field f. The Lagrangian in terms of f is then

$$\mathcal{L} = \frac{1}{2}\partial_\mu f \partial^\mu f - \lambda(V^2 f^2 + V f^3 + \frac{1}{4}f^4) + \text{const.} \tag{7.6}$$

We therefore see that f is a scalar field with mass $m_f^2 = 2\lambda V^2 > 0$. However, note that this $\mathcal{L}$ is *not* invariant under $f \to -f$. The $\phi \to -\phi$ symmetry which the original Lagrangian enjoyed is thus broken by the VEV of ϕ.

Lecture 8.

Monday, February 4, 2019

SSB of a continuous (global) symmetry Consider a real N-component scalar field $\phi = (\phi_1, \phi_2, \ldots, \phi_N)^T$, with the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi) \cdot (\partial^\mu \phi) - V(\phi), \tag{8.1}$$

where

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4, \quad \phi^2 = \phi \cdot \phi, \phi^4 = (\phi^2)^2, \lambda > 0. \tag{8.2}$$

This Lagrangian is invariant under global $O(N)$ symmetry, $\phi \to O\phi$ where $O^T O = I$. If $m^2 > 0$ we can expand about the $\phi = 0$ minimum, so here we'll be interested in $m^2 < 0$ again.

In this case,

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - V^2)^2 \tag{8.3}$$

up to an irrelevant constant factor, where $V^2 = -m^2/\lambda > 0$. This is alternately known as the "sombrero," "Mexican hat," or "wine bottle potential."[2]

The minima of $V(\phi)$ now trace out a circle (or a sphere in higher dimensions), $\phi^2 = V^2$. As before, the $\phi = 0$ vacuum is unstable, so we must expand about a stable vacuum in order to study the field. WLOG we'll choose $\phi_0 = (0, 0, \ldots, V)^T$ (so that $V(\phi_0) = 0$) and study small fluctuations about this,

$$\phi(x) = (\pi_1(x), \pi_2(x), \ldots, V + \sigma(x))^T \tag{8.4}$$

where $\pi(x)$ is a real scalar field with $N - 1$ components and $\sigma(x)$ has one component.

In terms of our new fields, the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \pi) \cdot (\partial^\mu \pi) + \frac{1}{2}(\partial^\mu \sigma)(\partial_\mu \sigma) - V(\pi, \sigma). \tag{8.5}$$

Our potential is now modified so that only the field σ has a mass while the other π fields remain massless:

$$V(\pi, \sigma) = \frac{1}{2}m_\sigma^2 \sigma^2 + \lambda V(\sigma^2 + \pi^2)\sigma + \frac{\lambda}{4}(\sigma^2 + \pi^2)^2. \tag{8.6}$$

The σ field has a mass

$$m_\sigma^2 = 2\lambda V^2$$

as before, but the $N - 1$ π fields are massless. We also observe the that potential has a third-order term $O(\sigma^3)$, which tells us that the new Lagrangian does not have the old $\phi \to -\phi$ symmetry when expanded about a stable vacuum. The σ field gives radial excitations in $V(\phi)$, while the π fields are excitations in the azimuthal (angular) direction, i.e. flat directions.

Let's now generalize this analysis to a symmetry group G of a Lagrangian $\mathcal{L}$ which is broken to a subgroup $H \subset G$ by the vacuum. We'll generally be considering normal subgroups. That is, let us transform the field $\phi \to g\phi$ with $g \in G$ in some representation, so that $\mathcal{L}(\phi) = \mathcal{L}(g\phi) \forall g \in G$. Thus g is a symmetry of the entire Lagrangian (and therefore the action).

Assume that G is spontaneously broken, and hence the vacuum is not unique but a manifold[3]

$$\Phi_0 = \{\phi_0 : V(\phi_0) = V_{\text{min}}\}. \tag{8.7}$$

The invariant subgroup (or stability group) $H \subset G$ is

$$H = \{h \in G : h\phi_0 = \phi_0\}. \tag{8.8}$$

That is, choose a vacuum state ϕ_0 and consider all the elements which leave ϕ_0 unchanged. Different vacua are related by $\phi_0' = g\phi_0$, with $\phi_0, \phi_0' \in \Phi_0$. Note that the stability groups for different vacua are isomorphic, i.e. for ϕ_0' the stability group is all elements $h' = ghg^{-1}$, i.e. $H' \simeq gHg^{-1}$.

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[2]Savvy readers may already know this is connected to the Higgs mechanism.

[3]I think we basically get this for free since we are imposing an algebraic constraint on the fields of the form $f(\phi) =$ constant, and the fields without the constraint formed a manifold ($\mathbb{R}^n$, $\mathbb{C}^n$, etc.).

The group elements that map one vacuum to another are in the coset space G/H and fall into the equivalence classes,

$$g_1 \sim g_2 \text{ if } \exists\, h \in H \text{ s.t. } g_1 = g_2 h. \tag{8.9}$$

That is, two elements of the group are equivalent if they are in the same left coset. Thus

$$\phi_0' = g_1\phi_0 = g_2\phi_0 \implies g_2^{-1}g_1 \in H. \tag{8.10}$$

That is, this element is in the stabilizer, and so there is one equivalence class (coset) for each $\phi_0' \in \Phi_0 : \Phi_0 \simeq G/H$. This is itself a group if H is a normal subgroup. That is, if we quotient out by the stabilizer, we get just the elements which take us between vacua.

Let's now consider Lie groups (i.e. groups with manifold structure). In the context of Lie groups, we can look at infinitesimal transformations:

$$g\phi = \phi + \delta\phi, \quad \delta\phi = i\alpha^a t^a \phi \tag{8.11}$$

where $a = 1, \ldots, \dim G$, t^a are the generators of the Lie algebra in the representation acting on ϕ, and α^a are small parameters. The invariance of the potential under G now means that $V(\phi + \delta\phi) = V(\phi)$, or expanding out,

$$V(\phi + \delta\phi) - V(\phi) = i\alpha^a (t^a\phi) r \frac{\partial V}{\partial \phi_r} = 0 \tag{8.12}$$

to first order, where $r = 1, 2, \ldots, N$ indexes over components of Φ in its representation. If ϕ_0 is a minimum of V, then by looking at the second-order terms we get

$$V(\phi + \delta\phi) - V(\phi) = \frac{1}{2}\delta\phi_r \frac{\partial^2 V}{\partial\phi_r \partial\phi_s}\delta\phi_s + \ldots, \tag{8.13}$$

where this second derivative is defined to be the "mass matrix,"

$$M_{rs}^2 \equiv \frac{1}{2}\delta\phi_r \frac{\partial^2 V}{\partial\phi_r \partial\phi_s}\delta\phi_s. \tag{8.14}$$

Lecture 9.

Wednesday, February 6, 2019

Suppose we have a symmetry (Lie) group G of the Lagrangian $\mathcal{L}$. Moreover, suppose it is broken to a subgroup H by the vacuum state, so that the vacuum manifold is a set of states described by $\Phi_0 \simeq G/H$. We can write an infinitesimal transformation in terms of the generators of the Lie algebra, i.e.

$$\delta\phi = i\alpha^i t^i \phi, \quad V(\phi + \delta\phi) - V(\phi) = i\alpha^a (t^a\phi)_r \frac{\partial v}{\partial\phi_r} = 0. \tag{9.1}$$

If we differentiate this equation and evaluate at $\phi = \phi_0$, then

$$\frac{\partial}{\partial\phi_s}\left[(t^a\phi)_r \frac{\partial V}{\partial\phi_r}\right] = \frac{\partial}{\partial r_s}(t^a\phi)_r \frac{\partial V}{\partial\phi_r}|_{\phi_0} + (t^a\phi_0)_r M_{sr}^2 = 0. \tag{9.2}$$

This first term in the product rule is zero by 9.1, while the second term is the mass matrix as defined last time. Thus we have some cases to consider:

- Unbroken symmetry: $g\phi_0 = \phi_0 \forall g \in G \implies \delta\phi = 0 \implies t^a\phi_0 = 0 \forall a$.
- Broken symmetry: there is some $g \in G$ s.t. $\exists a$ with $(t^a\phi_0) \neq 0$. Thus $t^a\phi_0)$ is an eigenstate of M_{rs}^2 with eigenvalue 0. Generators of $H \subset G$ are $\tilde{t}^i$ with $i = 1, \ldots, \dim H$ and $(\tilde{t}^i\phi_0) = 0$.

Recall now from *Symmetries, Fields and particles* that for a compact, semi-simple Lie algebra of G, we can define a group invariant inner product and orthogonality. If we choose a basis for the Lie algebra

$$t^a = \{\tilde{t}^i, \theta^{\tilde{a}}\}$$

where the $\theta^{\tilde{a}}$ are orthogonal to $\tilde{t}^i$ (i.e. $\operatorname{Tr} \tilde{t}^i\theta^{\tilde{a}} = 0$). We therefore learn that $(\theta^{\tilde{a}}\phi_0)$ is a unique zero eigenvector of M_{sr}^2 for $\tilde{a} = 1, 2, \ldots, \dim G - \dim H \implies$ there are $\dim G - \dim H$ massless modes of our theory, which we call *Goldstone bosons* or *Nambu-Goldstone bosons*, and in general there will be $N - (\dim G - \dim H)$ massive modes (at least, these modes are not guaranteed to be massless).

This is the *classical* proof of Goldstone's theorem. For instance, for the $O(N)$ model where we go from $O(N) \to O(N-1)$, the remaining symmetries are $\Phi_0 = S^{N-1}$. Comparing dimensions, we have $\dim O(N) = \frac{1}{2}N(N-1), \dim O(N-1) = \frac{1}{2}(N-1)(N-2)$. We therefore expect $\frac{1}{2}(N-1)(N-(N-2)) = N-1$ massless modes, which is what we found: $N-1$ π fields.

Example 9.3. Suppose a $\mathcal{L}$ written in terms of a complex $N \times N$ matrix field M is invariant under the transformation

$$M \to AMB^{-1} \quad \text{where } A \in U(N), B \in U(N). \tag{9.4}$$

These are still global symmetries, rather than gauge symmetries. Is the symmetry group of $\mathcal{L}$ $U(N) \times U(N)$? There should only be one identity element in the group, e.g. $(I_A, I_B) \in U(N) \times U(N)$ such that $M = I_A M I_B^{-1}$. This must be true when $M = I$, so $I_A I_B^{-1} = I \implies I_A = I_B$. Therefore

$$I_A M = M I_A$$

for arbitrary M.

However, we can then apply Schur's lemma from group theory, which tells us that if $SD(g) = D(g)S \forall g \in G$ where $D(g)$ is an irrep of G, then $S \propto I$. Thus $I_A \propto I$, i.e.

$$I_A = e^{i\theta} I \text{ with } \theta \in \mathbb{R}.$$

The proportionality constant must be a complex phase since the matrix I_A is unitary. These I_As form a $U(1)$ normal subgroup of $U(N)$, so by the uniqueness of the identity, the symmetry group of this system is in fact

$$\frac{U(N) \times U(N)}{U(1)}.$$

Note that running this argument for $SU(N)$ fails because the phase $e^{i\theta}$ is then fixed to be $+1$.

Goldstone's theorem Having presented a classical proof of Goldstone's theorem, let us consider SSB in a fully quantum way. As before, we have a symmetry group G of $\mathcal{L}$ which is spontaneously broken to $H \subset G$. That is, our field ϕ attains a non-zero VEV, $\langle 0 | \phi(x) | 0 \rangle = \phi_0 \neq 0$. This VEV is invariant under $h \in H$, $\langle 0 | h\phi(x) | 0 \rangle = \phi_0$, but not under a general $g' \in G$ where $g' \notin H$.

We can then write the Lie algebras of G and H in terms of their generators $t^a, a \in [1, \dots, \dim G]$ and $\tilde{t}^i, i \in [1, \dots, \dim H]$ respectively. However, recall that if G is a symmetry of $\mathcal{L}$, then by Noether's theorem there exist some conserved currents in our system,

$$j^{a\mu}(x) = i\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} t_a \phi, \tag{9.5}$$

and also charges

$$Q^a = \int d^3x j^{a0}(x) = \int d^3x \pi(x) t_a \phi(x). \tag{9.6}$$

These charges "induce a representation" of the Lie algebra. The variation of the field can then be computed to be

$$\delta\phi(0) = i\alpha^a t^a \phi(0) = i[Q^a, \phi(0)]\alpha^a. \tag{9.7}$$