GAUGE/GRAVITY DUALITY

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LAST UPDATED MAY 20, 2019

These notes were taken for the *Gauge/Gravity Duality* course taught by Aron Wall at the University of Cambridge as part of the Mathematical Tripos Part III in Easter Term 2018. I live-TEXed them using Overleaf, and as such there may be typos; please send questions, comments, complaints, and corrections to `itel2@cam.ac.uk`.
Many thanks to Arun Debray for the LATEX template for these lecture notes: as of the time of writing, you can find him at `https://web.ma.utexas.edu/users/a.debray/`.

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Lecture 1.

Thursday, April 25, 2019

Note. Official course notes are available (as of the time of writing) from `http://www.damtp.cam.ac.uk//user/aw846/AdSCFT.html`.

Size matters... not? To motivate our course, let us start with a story from Galileo. The astronomer Galileo wrote a treatise entitled "Two New Sciences." One of these was the heliocentric model of the solar system, and the other was an early version of the atomic theory. Galileo's work recognized that because of area-volume laws, the laws of physics seem to have a scale dependence. Building a scale model of a cathedral is very different than building a full-sized cathedral because mass scales with volume, whereas the strength of objects (based on local atomic interactions) scales with area.

On the other hand, there are a class of theories which follow the precepts of another great philosopher, Yoda, who stated in The Empire Strikes Back that "Size matters not." These are the so-called *conformal field theories*. We could have some object and then scale it up, and it would behave exactly the same. In fact, we might go so far as to posit that size is an extra *dimension* of our system! We shall call it z.

But maybe we object to this idea on a few grounds.

- *Objection #1.* Real dimensions should have conjugate momenta. In fact, Noether's theorem tells us that under the symmetry $\ln z \to \ln z + C$, we get a conserved quantity p_z corresponding to the dilation symmetry.
- *Objection #2.* We can rotate objects. This would require the group of Poincaré symmetries and dilations to be augmented to some bigger group with d extra generators. Indeed, this happens and we get the special conformal group.
- *Objection #3.* The speed of light is constant. This seems to tell us that we could measure distances by measuring the time that light takes to travel over the same scaled-up object. But by taking a cue from Einstein, we can answer this objection by saying that clocks run slower for bigger objects. That is, there is a redshift factor $ds \sim ds/z$. Our metric would look like

$$ds^2 = \frac{dz^2 + {}^{(d)}\eta_{ij} dx_i dx_j}{z^2}, \tag{1.1}$$

  and in fact this is the unique metric which satisfies the conformal symmetries. This is precisely the metric of Anti-de Sitter space (AdS).
- *Objection #4.* We can't put objects on top of each other without them interfering (e.g. if we scale some things up). Fermions are the obvious case, where we expect to run into trouble with the Pauli exclusion principle. However, there is a loophole. The objects won't interact much *if* there are a large number N of particle species, especially if objects are required to be in singlets (e.g. a gauge theory $SU(N_c)$, with $N \sim N_c^2$).
- *Objection #5.* If N is finite, there will still be a small interaction over large Δz, which implies the existence of a long-range universal force. But this looks like gravity! So things are going pretty well.
- *Objection #6.* When the gauge theory is heated up (e.g. we cram a lot of energy into a small space), we get "deconfinement," leading to a hyperentropic object with huge $O(N)$ entropy. What we've got is none other than a black hole.

This is the motivation for the AdS/CFT duality (originally posited by Maldacena and elaborated by Witten and others).

$$\mathrm{CFT}_d \leftrightarrow \mathrm{AdS}_{d+1} \times F \tag{1.2}$$

where F is a compact fiber. On the left lives an ordinary quantum field theory without gravity, and on the right lives a full theory of quantum gravity (typically a string theory). The large N limit of the QFT corresponds to the classical limit (where the Planck length is much smaller than the curvature scale, $l_p \ll R_{\mathrm{AdS}}$). Moreover, the QFT must be strongly coupled in order to produce local (pointlike) fields on scales below the AdS scale ($l_s \ll R_{\mathrm{Ads}}$, with l_s the string length). Finally, QFT has a set of known axioms (although they are hard to study at strong coupling), whereas we don't know how to treat quantum gravity nonperturbatively. Hence we can either use this to learn about strongly coupled field theories by studying general relativity, or we can try to learn about quantum gravity from the axioms of QFT.

Let's try to elaborate this idea a little more. In a 4d Maxwell theory, we have an action

$$I = \frac{1}{4} \int d^4x \sqrt{-g} F_{ab} F_{cd} g^{ac} g^{bd}, \tag{1.3}$$

which is invariant under the Weyl transformations $g_{ab} \to \Omega^2(x) g_{ab}$. This is because the two factors of the inverse metric each scale as Ω^{-2}, and the determinant of the metric is like a volume. Since g is like a length squared, $\sqrt{-g} \sim \Omega^d$, so these factors will cancel when $d = 4$. If the conformal symmetry holds in a QFT, we call it a CFT.

$$\frac{\delta \ln Z}{\delta \Omega(x)} \sim \langle T \rangle \sim \text{curvature}, \tag{1.4}$$

i.e. variations of the partition function with respect to the factor Ω are proportional to the trace of the stress tensor, which scales with the curvature. For $d = 2, T \sim cR$, while for $d = 4, T \sim a(GB) + cC^2$ where GB is the Gauss-Bonnet term and C is the Weyl tensor.

Conformal symmetry requires that the beta functions of the theory vanish. That is, if our couplings generically depend on scale, $\lambda(z)$, we require that

$$\frac{d\lambda_i}{d \ln z} = \beta_i = 0. \tag{1.5}$$

d translations	$x^a \to x^a + c^a$	$p_a \equiv -i\frac{\partial}{\partial x^a} = -i\partial_a$
$\frac{d(d-1)}{2}$ Lorentz	$x^a \to \Lambda^a{}_b x^b (\Lambda^\dagger \Lambda = 1)$	$M_{ab} = i(x_a \partial_b - x_b \partial_a)$
1 scaling/dilation	$x^a \to \Omega x^a$	$D = -i(x \cdot \partial)$
d special	$x^a \to \frac{x^a - x^2 b^a}{(x-x^2b)^2}, \Omega = \frac{x^2}{(x-x^2b)^2}$	$K_a = i(x^2 \partial_a - 2x_a(x \cdot \partial))$.

TABLE 1. A list of the conformal transformations, how points x^a transform, and their generators written in differential operator form.

These usually represent some special isolated points in the space of theories, except in situations where there is enough supersymmetry to produce e.g. a 1-parameter family of CFTs. Hence many of the theories of interest in AdS/CFT are supersymmetric.

Lecture 2.

Friday, April 26, 2019

Recall from e.g. *General Relativity* or *String Theory* that a Weyl transformation is a rescaling of the metric

$$g_{ab} \to \Omega^2(x) g_{ab}. \tag{2.1}$$

Note that this is slightly different from a conformal symmetry. A *conformal symmetry* is a diffeomorphism ξ^a that preserves g_{ab} up to a Weyl factor Ω.

To describe symmetries we shall need a Killing vector ξ^a, which by definition satisfies $\nabla_a \xi_b + \xi_b \xi_a = 0$. (That is, $\mathcal{L}_\xi g = 0$.) Transformations corresponding to Killing vectors are isometries leaving the metric unchanged. More generally, we might like a *conformal Killing vector*, which satisfies

$$\nabla_a \xi_b + \nabla_b \xi_a - \frac{2}{d} g_{ab} (\nabla \cdot \xi) = 0. \tag{2.2}$$

The factor of $2/d$ comes from the fact that the trace of the metric is $g_{ab}g^{ab} = d$, so that when we take the trace of this, we will get something that is always zero.

In Minkowski, $g_{ab} = \eta_{ab}$. Notice that we can rescale the null coordinates (e.g. the EF coordinates u, v go to $f(u), g(v)$ for some monotonic functions f, g). In $d = 1, 2$, there are infinitely many generators for Minkowski. But for $d > 2$, there are precisely $\frac{1}{2}(d+1)(d+2)$ generators.

For a general d dimensions we have d translations, $\frac{d(d-1)}{2}$ Lorentz transformations, 1 scaling/dilation transformation, and d "special" conformal transformations, as seen in Table 1. Here, we've used an adjoint notation, $\Lambda^\dagger \Lambda^c{}_d g_{ca} g^{db}$. Notice that an inversion is a transformation $x^a \to \frac{x^a}{x^2}, \Omega = 1/x^2$ (since $1/x^a = x^a/x^2$. Hence the special conformal transformations are equivalent to a translation, an inversion, and another translation.

Recall that translation symmetry gives us a conserved stress tensor, $\nabla_a T^{ab} = 0$. We claim the current

$$J^a = T^{ab} \xi_b \tag{2.3}$$

is conserved. This follows since

$$\nabla_a J^a = (\nabla_a Tab) \xi_b + T^{ab} \nabla_a \xi_b \tag{2.4}$$

$$= T^{ab} \nabla_{(a} \xi_{b)}, \tag{2.5}$$

where the first term was zero because of the regular stress-energy conservation. We've symmetrized the second term by the symmetry of the stress tensor, so the second term is just zero for a Killing vector. As the stress tensor is traceless, we can certainly subtract off its trace for free so that

$$T^{ab} \nabla_{(a} \xi_{b)} - \frac{1}{d} T (\nabla \cdot \xi) = 0 \tag{2.6}$$

for a conformal killing vector. We may compute some commutators to get all the relations between the generators:

- Usual Poincaré commutators

- $[D, P_a] = iP_a$
- $[D, K_a] = -iK_a$
- $[K_a, P_b] = 2i(\eta_{ab}D - M_{ab})$
- $[K_a, M_{bc}] = i(\eta_{ab}K_c - \eta - acK_b)$

with all others zero. This is equivalent to the symmetries of $SO(d, 2)$.

We've got to be a little careful about our inversion transformations, since an inversion brings infinity in to the origin and vice versa. So this may not be a real symmetry of Minkowski since it "pushes infinity around." Instead, we should look at the *maximal conformal extension*. For the Euclidean plane $\mathbb{R}^2$, this is a 2-sphere with a "point at infinity." We can see the isomorphism by putting the 2-sphere on the plane and stereographically projecting from the "point at infinity" on the north pole.

On the other hand, the Lorentzian case is more subtle. In Lorentz signature, $|x| = 0$ on the light cone. What happens is we must go to the maximal conformal extension of Minkowski, which is $S_{d-1} \times \mathbb{R}$. It is a cylinder. Our conformal transformations have the effect of flipping or shifting which patch of the cylinder we are looking at.

We might additionally be interested in representations of the conformal group– how do conformal transformations act on fields? We shall focus on unitary, positive-energy irreps, which come from fields (i.e. local operators). In QFT, we construct states by acting on the vacuum with some operator smeared out in spacetime by a suitable test function,

$$|\psi\rangle = \int d^d x\, f(x)\mathcal{O}(x)|0\rangle. \tag{2.7}$$

The operators $\mathcal{O}$ are classified by $SO(d)$ spin and weight of a "primary field" $\mathcal{O}$. By primary field (as we saw in *String Theory*), we mean a field which transforms as $\phi \to \Omega^\Delta \phi$ under conformal transformations, where Δ is called the weight. The derivatives $\partial^n \phi$ are called descendants, and they transform with derivatives of Ω. (This is slightly different from the $2D$ version of primary, where we are additionally interested in Virasoro). A gauge-invariant operator $\mathcal{O}$ must also satisfy "unitarity bounds." The details depend on the dimension d, and some concrete examples are as follows:

$$\Delta_\phi \geq \frac{d-2}{2} \text{ (scalar) except identity,} \tag{2.8}$$

$$\Delta_\psi \geq \frac{d-1}{2} \text{ (spinor)} \tag{2.9}$$

$$\Delta_J \geq d - 1 \text{ (vector)} \tag{2.10}$$

$$\Delta_T \geq d \text{ (symmetric traceless tensor).} \tag{2.11}$$

These bounds are saturated for $\Box\phi = 0, \not{\partial}\psi = 0, \nabla_a J^a = 0, \nabla_a T^{ab} = 0$. Notice once things start interacting, we get anomalous dimensions.

Lecture 3.

Monday, April 29, 2019

Today we shall continue our discussion of conformal symmetry. Note that if you were in *Black Holes*, you might recall that when doing QFT in curved spacetime, the vacuum depends on our choice of reference frame. However, when we do the maximal conformal extension of Minkowski, no such ambiguity is present. The vacuum on the cylinder is the same as the vacuum in the original Minkowski space.

The picture of the cylinder is also the origin of the "operator-state correspondence," in which we may rewrite the time coordinate with Euclidean signature by a Wick rotation $\tau = it$. Hence the cylinder is isomorphic to a punctured plane (e.g. $S_1 \times \mathbb{R} \cong \mathbb{R}^2 \setminus \{0\}$) and we can set up a correspondence between states on the cylinder and operators inserted as boundary conditions on the punctured plane, with time ordering corresponding to radial ordering on the plane.

Recall that last time, we said that scalars have weights obeying

$$\Delta_\phi \geq \frac{d-2}{2} \text{ (scalar)} \tag{3.1}$$

except the identity, which clearly has weight zero. This discussion of weight also requires that we set $\hbar = c = 1$, as is conventional, so that there is only one scale in the game. A free scalar field theory has terms in the action like $\int \partial_\mu \phi \partial^\mu \phi d^d x$. Since ∂_μ has mass dimension $+1$ and $d^d x$ has dimension $-d$, our scalars must have (naive) dimension $\frac{d-2}{2}$.

One may then write down the 2-point (correlation) function, which by dimensional analysis can only look like

$$\langle 0|\phi(y)\phi(x)|0\rangle = \frac{1}{|x-y|^{2\Delta}}. \tag{3.2}$$

That is, it must have dimensions of $d-2$, and it can only depend on the separation between x and y (under translation symmetry). In fact, this needs to be slightly modified to

$$\langle 0|\phi(y)\phi(x)|0\rangle = \frac{1}{|x-y+i\epsilon\hat{t}|^{2\Delta}}, \tag{3.3}$$

where $\hat{t}$ allows us to slightly shift a pole to imaginary time. We haven't fully explained this correction yet, but we'll see why this is the correct correlation function later. The upshot is that the form of the correlation function is scaled by the scaling symmetry.

The three-point function is also fixed by the full conformal symmetry. Note that the fields in the three point function could a priori have different weights $\Delta_1, \Delta_2, \Delta_3$:

$$\langle 0|\phi_3(z)\phi_2(y)\phi_1(x)|0\rangle = \frac{C}{|x-y|^{\Delta_1+\Delta_2-\Delta_3}|x-z|^{\Delta_1+\Delta_3-\Delta_2}|y-z|^{\Delta_2+\Delta_3-\Delta_1}}. \tag{3.4}$$

n.b. the four-point and higher functions cannot be derived directly from the conformal symmetry. Specifying the three-point function is sufficient to completely define the CFT. However, note that in general the problem is overdetermined– if we pick a three-point function at random, it probably won't correspond to a meaningful CFT. The program of trying to determine which three-point functions will give valid CFTs based on self-consistency conditions is known as the *conformal bootstrap*.

Spectral decomposition The spectral decomposition is a Fourier transform which allows us to work in a nicer basis. For a state

$$|\psi\rangle = \int f(x)\phi(x)d^d x, \tag{3.5}$$

we can define the momentum-space wavefunction $\phi(p)$ as

$$\phi(p) = \int e^{ip\cdot x}\phi(x)d^d x. \tag{3.6}$$

Suppose we want to evaluate

$$\langle 0|\phi(-q)\phi(p)|0\rangle \tag{3.7}$$

up to overall normalization. We can do this with dimensional analysis:

$$\langle 0|\phi(-q)\phi(p)|0\rangle \propto \delta^d(p-q)\theta(E)|p|^{2\Delta-d}, \tag{3.8}$$

with $\Delta > \frac{d-2}{2}$. This is because the Fourier transform takes us to a wavefunction which lives in inverse momentum space. Hence we get two contributions of d from the $d^d x$ integrals, one of which is absorbed by the momentum-conserving delta function.

Note that for $\Delta = \frac{d-2}{2}$, we might naively assume that the $|p|^{2\Delta-d}$ dependence turns into $|p|^{-2}$. However, this is wrong. In fact, we get a delta function instead, $\delta(p^2)$. We can think of this as coming from the fact that free fields (which saturate this bound) satisfy $\Box\phi = 0$, which in momentum space corresponds to the constraint $p^2 = 0$. States with $\Delta < \frac{d-2}{2}$ will not satisfy unitarity consistent with some positivity constraint.

Hence for the special case $\Delta = \frac{d-2}{2}$, we only get states on the light cone. Taking a metric

$$ds^2 = -dudv + dy_i^2 \tag{3.9}$$

with $d-2$ coordinates y_i and two null coordinates v, u, we have $\frac{1}{p^2} = \frac{1}{p_{y_i}^2 - 2p_u p_v}$. For states which satisfy and do not saturate the unitarity bound, $\Delta > \frac{d-2}{2}$, our states fill the interior of the future light cone. There are no normalizable states $|\psi\rangle$ with $E^2 = \mathbf{p}^2$ exactly, though states may come arbitrarily close to being null. Thus all the physical states do not obey a wave equation *unless it is in one higher dimension*.

AdS spacetimes Before focusing on AdS, we'll start more generally with maximally symmetric spacetimes in D dimensions (where eventually we shall set $D = d+1$). In Minkowski, we have at most $\frac{D(D+1)}{2}$ Killing vectors in a D-dimensional spacetime. However, we can also set this many constraints on our spacetime by specifying the curvature, $R_{ab} = g_{ab}\Lambda$ for Λ some cosmological constant. (The counting works out since R_{ab} is symmetric.)

There are two interesting cases. For $\Lambda > 0$, we get de Sitter space, which can be thought of as the unit hyperboloid in $D+1$ Minkowski space. However, if we take anti-de Sitter space, we instead have the unit hyperboloid in a signature $(D-1, 2)$ spacetime with two time coordinates.

Lecture 4.

Tuesday, April 30, 2019

Last time, we promised to explain the $i\epsilon\hat{t}$ prescription in the two-point function. Notice that $|x-y|^{2\Delta}$ can vanish when the separation between x and y is null. Hence we need some $i\epsilon$ prescription to push one of the operator insertions slightly into an imaginary direction.

Recall that in momentum space, we have $\langle 0|\phi(-q)\phi(p)|0\rangle \propto \delta^d(p-q)\theta(E)|p|^{2\Delta-d}$. The step function here fixes $E > 0$. Consider now the following integral where WLOG we set $y = 0$:

$$\int_{-\infty}^{\infty} |x|^{-2\Delta} e^{-iEt} dt, \tag{4.1}$$

where $|x|^{2\Delta} = (\mathbf{x}^2 - t^2)^\Delta$. This is just the t part of the Fourier transform for the two-point function (up to normalization). For $E < 0$, the exponential is suppressed as $t \to i\infty$ (the upper half-plane), so adding $+i\epsilon\hat{t}$ pushes the poles at $t = \pm|\mathbf{x}|$ down by a little imaginary part. Note also that these are only honest poles if $\Delta \in \mathbb{Z}$. Otherwise, we have branch cuts which stretch to $-i\infty$ in the $\hat{t}$ direction. Either way, this tells us that we want to focus on the future light cone, and that $[\phi(y), \phi(x)] \neq 0$ when y, x are timelike separated.

Back to AdS As we stated, a maximally symmetric spacetime is one such that

$$R_{ab} = (8\pi G)\frac{2}{D-2} g_{ab}\Lambda, \tag{4.2}$$

where we've restored proportionality constants.

For the de Sitter case, $\Lambda > 0$, we have the unit hyperboloid with one time dimension (Minkowski $D+1$), with metric

$$ds^2 = R_{\text{dS}}^2\left[-d\tau^2 + \cosh^2(\tau) d\Omega_{D-1}^2\right]. \tag{4.3}$$

This just says that spacetime looks like a $D-1$ sphere of radius $R_{\text{dS}}\cosh(\tau)$ with a proper time coordinate τ. The conformal boundary is $S_{D-1} \times S_0$, where by S_0 we just mean two points (one at future timelike infinity and one at past timelike infinity).

On the other hand, for $\Lambda < 0$, we have AdS, which is a unit hyperboloid in a spacetime with two time dimensions, sig$(D-1, 2)$, with symmetry group $SO(D-1, 2)$. We can describe our hyperboloid in de Sitter by $x^2 + y^2 + z^2 + \ldots - t^2 R_{\text{dS}}^2$, so we should make an equivalent construction for AdS. That is,

$$x^2 + y^2 + z^2 + \ldots - t^2 - u^2 = -R_{\text{AdS}}^2 \tag{4.4}$$

where u is our second time coordinate. We may therefore define a metric

$$ds^2 = R_{\text{AdS}}^2\left[d\rho^2 - \cosh^2(\rho) d\tau^2 + \sinh^2(\rho) d\Omega_{D-2}^2\right]. \tag{4.5}$$

This looks like hyperbolic space with a redshift factor. The conformal boundary of AdS is then $S_{D-2} \times S_1$. However, the fact that our time coordinate has a boundary S_1 means that our spacetime has closed timelike curves. Strictly, we can remedy this by instead taking the universal cover of this spacetime and cover the spacetime with many sheets so that each time we go around, we are on a new sheet.[1]

[1] This is a bit like how the universal cover of S_1 is an upward spiral. As we spiral up, we come back to a point which would be projected down to the same point on the circle, but is distinguished by being on a different coil of the spiral.

If we now compactify our ρ coordinate, $\rho = \infty \to \theta = \pi/2$, then our metric takes the form

$$ds^2 = \frac{R^2_{\text{AdS}}}{(\cos\theta)^2}\left[-d\tau^2 + d\theta^2 + \theta^2 d\Omega^2_{d-1}\right]. \tag{4.6}$$

In these coordinates, we see that AdS is conformal to a half-sphere. However, note that our metric blows up as $\theta \to \pm\pi/2$. We can draw this as a "tin can" diagram. In addition, the group center of AdS has the structure of $\mathbb{Z}$, i.e. there are infinitely many discrete antipodal points in AdS. These antipodal points have the property that the represent points where timelike geodesics reconverge. That is, no matter how fast we shoot out an object from the origin, its trajectory eventually reconverges at the next antipodal point. Timelike geodesics are attracted to $\rho = 0$. Moreover, light rays (null geodesics) can actually make it out to $\rho = \infty$, and (with reflecting boundary conditions) can in fact return to $\rho = 0$ after finite time.

The conformal boundary of AdS has the structure of $S_{D-2} \times \mathbb{R} = S_{d-1} \times \mathbb{R}$. Let us also remark that the half-sphere model allows us to make a very simple consistency check– it is not possible to send light faster through the bulk than through the boundary. There are no bulk shortcuts. We can see this since the boundary is an equator of the half-sphere, so trajectories on the boundary are length-minimizing.

It is often convenient to restrict to a patch of the bulk isometric to Mink_d on the CFT side. We call this the Poincaré patch since it has the Poincaré group of symmetries, and it has metric

$$ds^2 = R^2_{\text{AdS}}\left[\frac{dz^2 + (-dt^2 + d\mathbf{x}^2)}{z^2}\right]. \tag{4.7}$$

Hence $t \to \Omega t, \mathbf{x} \to \Omega x, z \to \Omega z$ is a symmetry of this metric and z represents a scale "dimension." The future horizon is the $z \to +\infty$ limit, and in this limit the redshift grows arbitrarily large. This tells us the CFT has arbitrarily low energy.

Lecture 5.

Thursday, May 2, 2019

Scalar fields in AdS-Poincaré Last time, we observed that part of the AdS spacetime is described by the AdS-Poincaré patch, i.e. a region of the bulk whose boundary is conformal (related by a Weyl transformation) to half of a D-dimensional Minkowski space. Thus

$$ds^2 = \frac{dz^2 + {}^{(d)}\eta_{ij}dx^i dx^j}{z^2} \xrightarrow{\text{Weyl}} {}^{(D)}\eta_{ij}dx^i dx^j \tag{5.1}$$

We shall try to solve the scalar wave equation,

$$\overbrace{\Box}^{\nabla_x^2 - \nabla_t^2} \phi = m^2\phi. \tag{5.2}$$

Notice that our current signature, the time derivative comes with a minus sign, and the time eigenvalues (frequencies) are imaginary so that larger mass corresponds to larger oscillations.

The naive scaling dimension of our free field is $\Delta_\phi = \frac{D-2}{2}$, so the normalized field is then

$$\tilde{\phi} = z^{(D-2)/2}\phi. \tag{5.3}$$

However, note that $\Box\phi$ is not a conformal wave equation. That is, since $\phi \to \Omega^\Delta\phi$, $\Box\phi$ will include not only $\Omega^{\Delta+2}\Box\phi$ terms but also $\nabla\Omega, \nabla^2\Omega$. The derivatives will hit our conformal factors, so this operator does not transform as a primary operator.

However, note that R_{ab} is also not a conformal primary. In fact,

$$\left(\Box - \frac{D-2}{4(D-1)}R\right)\phi = 0 \tag{5.4}$$

is conformal. The transformation of R cancels the transformation of $\Box$. In AdS, $R = -D(D-1)$, which we interpret as a mass shift,

$$\tilde{m}^2 = m^2 - \frac{D(D-2)}{4}. \tag{5.5}$$

Maybe we're worried that $m^2\phi$ is dimensionful, and so it seems like we might not be able to make this transform in a nice conformal way. However, we can promote this to a position-dependent mass, $m^2(z) \propto 1/z^2$. Adding back in z dependence, we end up with the following differential equation:

$$\left(\partial_z^2 - \frac{\tilde{m}^2}{z^2} + {}^{(d)}\Box\right)\tilde{\phi} = 0. \tag{5.6}$$

This might be hard to solve exactly (in terms of some hypergeometric function). But something nice happens when we take the $z \to 0$ limit. The ${}^{(d)}\Box$ term becomes subleading– think of taking the Fourier transform of ϕ. Then $\Box$ becomes a constant $-p^2$, while the other two terms scale like $1/z^2$.

We might therefore guess that $\tilde{\phi}$ has some power law dependence on z,

$$\tilde{\phi} \sim z^\nu + O(Z^{\nu+2}). \tag{5.7}$$

Plugging in, we get

$$\nu(\nu-1)z^{\nu-2} - \tilde{m}^2 z^{\nu-2} = 0. \tag{5.8}$$

Collecting coefficients, we get

$$\nu(\nu-1) = \tilde{m}^2. \tag{5.9}$$

In general there are two solutions since this equation is quadratic in ν. This is what we should have expected– it corresponds to a choice of boundary conditions. In the $m = 0$ case, we would have had a "potential" which was zero everywhere, and which we could define in the $z = 0$ limit with either Dirichlet or Neumann boundary conditions. It's therefore not too surprising that we get the same choice in the massive case.

Note that if we take the other limit, $z \to \infty$, the normalizability requires that $p^i p_i < 0$ (timelike normalization). If we tried to put in a tachyonic solution (wrong-sign momentum), the eigenvalue of ${}^d\Box$ has the wrong sign and we get solutions which grow exponentially in time rather than oscillating.

The dictionary Here is our first entry in the "dictionary" of AdS/CFT.

$$\mathcal{O}(x^i) = \lim_{z\to 0} z^{-\nu}\tilde{\phi}(z, x^i) = \lim_{z\to 0} z^{-\Delta_{\mathcal{O}}}\phi(z, x^i). \tag{5.10}$$

That is, an operator on the boundary is equivalent to a field in the bulk in the $z \to 0$ limit. By dimensional analysis, we see that

$$\Delta_{\mathcal{O}} = \Delta_\phi + \nu = \frac{D-2}{2}\nu = \frac{d-2}{2} + \nu + 1/2. \tag{5.11}$$

Hence the unitarity bound is saturated for $\nu = -1/2$, and we have the second equality by the dimensional analysis and the definition of the normalized field.

Let us also observe that our condition $\nu(\nu-1) = \tilde{m}^2$ corresponds to

$$m^2 = \Delta(\Delta - d) \tag{5.12}$$

$$= \left(\nu + \frac{d-1}{2}\right)\left(\nu - \frac{d-1}{2}\right) \tag{5.13}$$

$$= \nu^2 - \nu + \frac{(d-1)(d+1)}{4}. \tag{5.14}$$

This is the conformal mass shift. If we plot this $\tilde{m}^2$ versus Δ, we see that between the zeros at $\nu = 0, \nu = 1$, there is actually a regime with extremum at where $\tilde{m}^2$ goes a bit negative. So our field is actually permitted to be a little bit tachyonic, provided that it is not *too* tachyonic. The minimum which lies at $\nu = 1/2$ is known as the Breitenlohnen-Freedman bound. There is also another bound, the unitary bound at $\nu = -1/2$. For if $\nu = -1/2$, then ϕ^2 scales as z^{-1}, the Klein-Gordon norm of our field goes as $\int \phi^* \partial_t \phi$, which is logarithmically divergent or worse. Hence the unitarity bound at $\nu = -1/2$ is actually strict as a cutoff on *operators*.

Interestingly, we can actually construct different fields on the boundary consistent with the same bulk description by imposing different boundary conditions. We'll see this idea more later. That is, by drawing the figure we see that there are two values of Δ which give the same mass $\tilde{m}$. By convention, we pick the one with larger Δ to be the operator and the one with lower Δ to be the source.

That is, the other solution is a source term in the QFT:

$$\mathcal{Z}_{\text{QFT}} = \int \mathcal{D}x\, e^{-(I[x]+\int J\cdot\mathcal{O}d^dx)} \tag{5.15}$$

Lecture 6.

Friday, May 3, 2019

Last time, we discussed how the scaling dimension of an operator satisfies $\Delta_{\mathcal{O}}(\Delta_{\mathcal{O}} - d) = m_\phi^2$, i.e. how the mass squared represents a shift in the scaling dimension. Only operators with Δ above the unitarity bound of $\frac{d-2}{2}$ are admissible as operators; the other solutions (to a generally quadratic equation) are considered as sources.

By writing as a path integral for the quantum field theory on the boundary, we have the partition function

$$\mathcal{Z}_{\text{QFT}}^{[J]} = \int \mathcal{D}x\, e^{i(I[x]+\int J\cdot\mathcal{O}\, d^dx)} \tag{6.1}$$

$$= \mathcal{Z}_{\text{CFT}} \left\langle \mathcal{T}[e^{-\int J\cdot\mathcal{O}d^dx}] \right\rangle \tag{6.2}$$

$$= \mathcal{Z}_{\text{bulk}}[J]. \tag{6.3}$$

Here, $\mathcal{Z}_{\text{CFT}}$ is the source-free CFT, $J = 0$, and $\mathcal{T}$ indicates time-ordering. Note also that we've written this as $\mathcal{Z}_{\text{QFT}}$, since the sources will generically break conformal invariance. By our duality, the existence of sources corresponds to some boundary conditions for physics in the bulk.

By dimensional analysis, $\Delta_J = d - \Delta_{\mathcal{O}}$, so this explains why smaller values of $\Delta_{\mathcal{O}} < \frac{d-2}{2}$ can still correspond to meaningful source terms. In fact, in $m^2 = 0$, an abvious solution is $\phi =$ constant (under the $\phi \to \phi + c$ symmetry, since the action depends only on derivatives). Hence there is a marginal source $\Delta_J = 0$, corresponding to an operator with $\Delta_{\mathcal{O}} = d$.

Now our operators and sources can be written as

$$\mathcal{O}(x^i) = \lim_{z\to 0}(z^{-\Delta_{\mathcal{O}}}\phi(x^i, z) - J \text{ profile}) \tag{6.4}$$

$$J(x^i) = \lim_{z\to 0}(z^{-\Delta_J}\phi(x^i, z) - \mathcal{O} \text{ profile}), \tag{6.5}$$

where the J profile term is necessary if $\Delta_{\mathcal{O}} > \Delta_J$ and the $\mathcal{O}$ profile term is needed if $\Delta_J > \Delta_{\mathcal{O}}$. Sometimes we may need subleading terms since $\phi \sim z^\nu + \mathcal{O}(z^{\nu+2})$. If it turns out that $\Delta_J - \Delta_{\mathcal{O}} \in 2\mathbb{Z}$, then there may be some coincidences where e.g. $\Box J \sim \mathcal{O}$ are of the same order, leading to log terms.

Let's focus in on sources for a second. There are three cases of sources we might be interested in.

- $\Delta_J > 0$: relevant, i.e. important in the IR of the CFT. These operators matter most in the interior of the bulk, where length scales are large. The QFT makes sense.
- $\Delta_J = 0$: marginal (still a CFT to leading order). Usually, these operators turn out to be marginally relevant or marginally irrelevant at higher order. They are often only exactly marginal in theories with large amounts of SUSY.
- $\Delta_J < 0$: irrelevant. These operators become important in the UV limit of the CFT, i.e. the IR of the AdS. Thus we may have to worry about the UV completion of the theory being potentially ill-defined.

Holographic RG In a nonlinear ϕ theory, note that the z ODE (equations of motion in the bulk) corresponds to an RG flow on the boundary. Hence even though there may be different boundary theories compatible with the one bulk theory, we can still describe those boundary theories looking at how ϕ scales with $\ln z$. These boundary theories are related by an RG flow.

There are some aspects which are scheme-dependent, i.e. they depend on how we do the renormalization. So e.g. the exact beta functions will generically depend on whether we do dim-reg, set our theory on a lattice, add a momentum cutoff, etc. to get $\beta_\alpha = f(\alpha)$. However, there are still universal aspects, e.g. anomalous dimension in CFTs, existence of fixed points, and certain log divergences.

Bulk vector field We can write down an action for a vector field. It takes the form

$$I = \int d^D x \left(\frac{1}{4} F_{ab} F_{cd} g^{ac} g^{bd} \sqrt{-g} + \frac{1}{2} m^2 A_a A_b g^{ab} \sqrt{-g} \right), \tag{6.6}$$

where we have added a Proca mass term. Notice that by the Ω scaling, the first term is conformally invariant only in $D = 4$. However, while the second term is not gauge invariant, adding a mass to the photon gives us an extra degree of freedom (the longitudinal mode). Thus there are $D-2$ degrees of freedom for $m^2 = 0$ and $D-1$ for $m^2 > 0$. There is also a longitudinal ghost for $m^2 < 0$, which is bad news for our theory (so we'd better not set $m^2 < 0$).

Let us specialize to the case of $g_{ab} = \Omega^2 \eta_{ab}$ with $\sqrt{-\eta} = 1$. Hence

$$I = \int d^D x \sqrt{-\eta} \left[\frac{1}{4} \Omega^{D-4} F_{ab} F^{ab} + \frac{1}{2} \Omega^{D-2} m^2 A_a A^a \right], \tag{6.7}$$

where indices have been raised with the Minkowski metric η. The equations of motion are

$$\partial_a (z^{4-D} F^{ab}) = z^{2-D} m^2 A^b. \tag{6.8}$$

Note that in Poincaré-AdS, $\Omega = 1/2$ and hence

$$\partial_i F^{ib} + \partial_z F^{zb} + (4-D) z^{-1} F^{zb} = z^2 m^2 A^b. \tag{6.9}$$

If we try the ansatz $A^i(x^j) = z^\nu J^i(x^j) + \mathcal{O}(z^{\nu+2})$, where i, j are d-indices, we find that $\partial_i F^{ib}$ is subleading,

$$\partial_z F^{zb} \to \partial_z^2 A^i + \ldots = \nu(\nu-1) z^{\nu-2} J^i, \tag{6.10}$$

$$(4-D) z^{-1} F^{zb} \to (3-d) z^{-1} \partial_z A^i \to (3-d) \nu z^{\nu-2} J^1, \tag{6.11}$$

$$z^2 m^2 A^b \to m^2 z^{\nu-2} J^i. \tag{6.12}$$

Matching orders, we arrive at the relation

$$\nu(\nu-1) + (3-d)\nu = m^2 \tag{6.13}$$

and hence

$$A^i = A_j \eta^{ij} = \pm A_i. \tag{6.14}$$

Undoing the rescaling with $x^i \to \Omega x^i, z \to \Omega z$, we find that

$$\Delta_J = 1 + \nu. \tag{6.15}$$

Hence in terms of Δ, our relation on ν gives

$$(\Delta - 1)(\Delta - d + 1) = m^2 \tag{6.16}$$

and so

$$m^2 = 0 \iff \Delta = d - 1 \text{ or } \Delta = 1. \tag{6.17}$$

The $d-1$ case corresponds to a global conserved current in the boundary theory, while $\Delta = 1$ corresponds to a boundary potential. To reiterate, *we started with a $U(1)$ gauge field in the bulk, and it ended up corresponding to a global $U(1)$ symmetry current in the boundary*. In general, gauge fields in the bulk correspond to global symmetries in the boundary.

Lecture 7.

Monday, May 6, 2019

Last time, we studied a massive vector field (Proca field) in the bulk, with action

$$I = \int d^D x \left(\frac{1}{4} F_{ab} F^{ab} + \frac{1}{2} m^2 A_a A_b g^{ab} \right) \sqrt{-g}. \tag{7.1}$$

Let's notice that the equations of motion for this field can be written out as

$$\partial_i F^{ib} + \partial_z F^{zb} + (4-D) z^{-1} F^{zb} = z^{-2} m^2 A^b, \tag{7.2}$$

where indices are raised and lowered with η. Hence under the ansatz that the vector potential A^i scales as $z^\nu J^i$ plus order $z^{\nu+2}$ corrections,

$$A^i = z^\nu J^i + \mathcal{O}(z^{\nu+2}), \tag{7.3}$$

we found that

$$(\Delta - 1)(\Delta - D + 1) = m^2, \tag{7.4}$$

where in the massless case, $m^2 = 0$, we have two solutions. $\Delta = d - 1$ gives us a conserved (global) boundary current and $\Delta = 1$ gives a boundary gauge potential. Of course, we haven't studied the z component yet. Let us do that now.

For A^z, if $m^2 = 0$ then we have a gauge symmetry $\delta A_a = \nabla_a \alpha$ which suggests to us that we can impose a gauge condition. We shall choose a sort of holographic interpretation of the axial gauge, namely

$$A_z = 0. \tag{7.5}$$

This tells us that on lines of constant x_i ($i \neq z$), the field is constant. Our equations of motion then reduce to

$$\partial_i F^{iz} = 0, \tag{7.6}$$

which we may by the antisymmetry of F rewrite as $\partial_z(\partial_i A^i) = 0$. Given that this field should be normalizable in the bulk (i.e. falls off at large z), we can actually conclude that

$$\partial_i J^i = 0 = \nabla_i J^i = 0. \tag{7.7}$$

Note that while we like to think of our conserved currents as vectors, we should actually think of them like tensor densities. This is because the natural thing to integrate over is a $d-1$-dimensional Cauchy slice so that $\sqrt{-g} J^i_{\text{tensor}}$. Moreover, note that we can promote to a covariant derivative in our conservation equation because $m^2 = 0$, so there is no characteristic mass scale in the theory and we therefore do not get $\partial\Omega$ terms when we take the covariant derivative.

What happens if we turn on the mass, $m^2 > 0$? Since we had the gauge symmetry $\delta A_a = \nabla_a \alpha$, we can try varying the action with respect to α. This variation should vanish to first order. We find that

$$\delta_\alpha I = m^2 \int \nabla\alpha \cdot A = -m^2 \int \alpha(\nabla \cdot A). \tag{7.8}$$

Since α is arbitrary, we get a Lorentz gauge-like condition,

$$\nabla \cdot A = 0. \tag{7.9}$$

The difference is that instead of this being a gauge choice, we get it for free by the variation of the action in the Proca action. Thus

$$\nabla_a (A_b g^{ab} \sqrt{g}) = \partial_a (A_b z^{D-2} \eta^{ab} \sqrt{-\eta}) = 0 \tag{7.10}$$

for AdS. We find that

$$A_z = \frac{z^{\Delta_J} (\partial \cdot J)}{\Delta_J - (d-1)}, \tag{7.11}$$

where we've argued that the numerator does not in general vanish once we turn the mass on.

What we learn is that a *boundary global current* $J^i(x)$ corresponds to a *bulk gauge field* $A^a(x,z)$. This is generally true for an abelian theory like $U(1)$– in the case of a non-abelian theory, we would need some gauge index I for a theory like $SU(2)$,

$$J^i_I(x) \leftrightarrow A^a_I(x,z). \tag{7.12}$$

Moreover, every *decent* CFT should have a (traceless) stress-energy tensor. Hence

$$T^{ij}(x) \leftrightarrow g^{ab}(x,z), \tag{7.13}$$

which we recognize as a dynamical graviton, giving us Einstein gravity. Hence the bulk isn't precisely AdS but rather asymptotically AdS, so that our dynamical gravity theory corresponds to an excited state of the CFT. We won't focus on supersymmetry here, but let us just note that

$$\text{SUSY current} \leftrightarrow \text{gravitino}, \tag{7.14}$$

which suggests that in general we would get some sort of supergravity theory in the bulk.

For a conformal irrep $T_{ij}(x)|0\rangle$, a linear spin-2 equation is

$$\nabla^2 h_{ab} - \nabla_{(a}\nabla_{c)} h^c_a + \nabla_a \nabla_c h^c_c + g^{\text{AdS}}_{ab}(\nabla^c\nabla^d h_{cd} - \nabla^2 h^c_c) = m^2(h_{ab} - g^{\text{AdS}}_{ab} h^c_c), \tag{7.15}$$

where this last term is known as the Fierz-Pauli mass. This is how we (self-consistently) turn on the graviton mass. This field equation restricts us to transverse tracefree modes and implies $\nabla^2 h_{ab} = m^2 h_{ab}$.

How many degrees of freedom are left? We have $\frac{(D-2)(D-1)}{2} - 1$ in the $m^2 = 0$ case (the -1 comes from imposing the traceless condition), and $\frac{(D-1)D}{2}$ for $m^2 > 0$. We can also add on a term to an Einstein-Hilbert action (the quadratic piece of GR/Fierz-Pauli) as[2]

$$\int d^D x \sqrt{-g}(R[g] + 2\Lambda) + \frac{1}{2}m^2(h_{ab}h^{ab} - g_{ab}^{\text{AdS}} h_a^a h_b^b). \tag{7.16}$$

We ought to think of this as studying linearized perturbations in massive gravity,

$$g_{ab} = g_{ab}^{\text{AdS}} + h_{ab}. \tag{7.17}$$

To linear order we get the equation on h_{ab} vanishing in pure Einstein gravity (i.e. $m^2 = 0$).

To a perturbation h_{ij} we can associate a boundary term which has the interpretation of a stress-energy tensor,

$$h_{ij}(x, z) = z^{\Delta-2} T_{ij} + \mathcal{O}(z^{\Delta}), \tag{7.18}$$

or equivalently

$$T_{ij} = \lim_{z\to 0} z^{2-\Delta} h_{ij}, \tag{7.19}$$

and we have a scaling dimension relation

$$\Delta(\Delta - d) = m^2. \tag{7.20}$$

As before, the $m^2 = 0$ case gives two solutions, $\Delta = d$ corresponding to a conserved stress-energy tensor and $\Delta = 0$ giving a boundary *metric*, g_{ab}.

Let's reflect on this calculation. Why are we studying linearized equations when GR is generally non-linear? Quantum mechanics is linear, so irreps of the conformal group $\mathcal{H}_{\text{irrep}} \subset \mathcal{H}_{\text{CFT}}$ and thus $T_{ij}|0\rangle =$ 1-graviton state. Hence we get a linear wave equation $D\phi = 0$ (for some differential operator D. In fact, this does not mean that the (linear) CFT is only dual to a linearized version of gravity in the bulk. It is dual to the full non-linear GR, provided that we study n-point functions using something called Witten diagrams (like Feynman diagrams), describing scattering and interactions of gravitons in the bulk. The external edges are given on the boundary of AdS. We have then a correspondence between propagators from the bulk to boundary and from bulk-to-bulk:

$$G_{\text{bulk}\to\text{bdy}} = \lim_{z'\to 0} (z'x')^{-\Delta} G_{\text{bulk}\to\text{bulk}}(z, x, z', x'). \tag{7.21}$$

We're being a bit schematic here, but the idea is that we get a Green's function solving $D\phi = 0$ and more generally we could introduce a boundary source.

Lecture 8.

Tuesday, May 7, 2019

Having treated massive perturbations, let us consider the spin-2 massless case, which is none other than general relativity. For the massive vector field we chose the gauge condition

$$A_z = 0. \tag{8.1}$$

Can we do something similar for the metric? For asymptotically locally AdS spacetimes, we may introduce *Fefferman-Graham coordinates*. In these coordinates, we take

$$g_{zz} = \frac{1}{z^2}, \quad g_{zi} = 0. \tag{8.2}$$

Notice that this gives D conditions on the metric leaving the residual gauge symmetry of diffeomorphisms and Ω (conformal transformations) on the boundary.

We can't do this precisely on $z = 0$, but if we go to a hypersurface of constant $z = \epsilon$ and follow normal geodesics to this surface (i.e. by $\ln z = \ln \epsilon +$ proper distance $\in \sqrt{g_{nn}dn}$ and $z^i =$ const along the normal geodesics). However, such a coordinate system may not work globally, because the geodesics normal to $z = 0$ might converge (cf. conjugate points in *Black Holes*). This is a local construction near the boundary, but it will nevertheless let us find some interesting results.

[2] We sometimes call h a reference metric.

In these coordinates, we can do an FG (Fefferman-Graham) expansion. That is, we solve the nonlinear bulk equations of GR in a.l.[3] AdS. Thus

$$^{(D)}g_{ij} = \frac{1}{z^2}\left[g_{ij}^{(0)} + z^2 g_{ij}^{(2)} + z^4 g_{ij}^{(4)} + \ldots)\right] \tag{8.3}$$

where e.g.

$$g_{ij}^{(0)} = {}^{(d)}g_{ij}, \tag{8.4}$$

$$g_{ij}^{(2)} = \frac{1}{D-2}({}^{(d)}R_{ij} - \frac{1}{2(D-1)}{}^{(d)}R^{(d)}g_{ij}), \tag{8.5}$$

and $g_{ij}^{(4)}$ will include two powers of curvature.

The ... in the expression for the bulk metric depends on whether we are in odd d or even d. For odd d, we have even powers of z up to

$$+ z^d T_{ij} + \mathcal{O}(z^{d+2}), \tag{8.6}$$

whereas in even d, we have

$$+ z^d T_{ij} + \ln(z) z^d h_{ij}^{(d)} \tag{8.7}$$

because the powers coincide. This last term gives us relations

$$-\frac{2}{d} T_{ij}^{\text{trace}} \propto^{(d)} g_{ij}, \tag{8.8}$$

such that in $d = 2$, this is proportional to cR with c the same central charge of the Virasoro algebra, and in $d = 4$ we have $\propto aGB + cW^2$ (the Gauss-Bonnet term and the Weyl tensor).

Suppose the bulk is weakly coupled, i.e. approximately free when $N_{\text{quanta}} \sim 1$, a theory which is classical at the nonlinear regime. For an operator $\phi_i \sim z^{\Delta_i}$ it must be that powers of this operator scale as $\phi^n \sim z^{\sum_i \Delta_i}$, i.e. an operator whose weight is the sum of the weights of its factors.

(a) $\exists$ a collection of operators $\{\mathcal{O}_s\}$ whose spectrum is Fock($\{\mathcal{O}_s\}$).

(b) The expectation value of these operators are approximately Gaussian: $\langle \mathcal{O}_s \mathcal{O}_S \ldots \mathcal{O}_s \mathcal{O}_S \rangle \approx$ Gaussian. In a free field, we only get pairwise couplings in our Witten diagrams which live in the bulk (i.e. Wick contractions when you only have free propagators). This is called a "generalized free field."

(c) However, the CFT itself is not actually free– there is no $\Delta = \frac{d-2}{2}$ field.

Hence we have a theory which is free in the bulk but not in the boundary. It turns out this is actually typical of "single trace" operators in large-N gauge theory. For instance, take a Yang-Mills Lagrangian,

$$\mathcal{L}_{\text{YM}} = \text{tr}\big(F_{\mu\nu} F^{\mu\nu}\big). \tag{8.9}$$

Here, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu]$. This could be $SU(N)$ transforming in the adjoint representation, for example. We take the trace to get a colorless object (something that will be gauge invariant). Now let us take the $N \to \infty$ limit holding $\lambda = g^2 N$ fixed, where λ is called an t'Hooft coupling. This keeps the leading-order loop calculations (which scale as $g^2 N$) constant even as we take the number of colors N to be large.

When we treat something like a gluon, we should think of it as really having two arrows– a color and an anti-color arrow. Hence gluon diagrams will be a set of directed loops.

We can play some Euler characteristic games with our gluon diagramms, associating "half" of an edge to each of the two vertices it connects. With a factor of N for each facte, we have

$$g \sim N^{-1/2} \text{ for 3-vertex} \tag{8.10}$$

and

$$g^2 \sim N^{-1} \text{ for 4-vertex.} \tag{8.11}$$

One may conclude that the amplitude scales as

$$\text{Amp} \sim N^{F-E+V} \sim N^{\chi}. \tag{8.12}$$

[3] asymptotically local

This tells us precisely that the low-genus diagrams dominate, e.g. spheres, torii, $n = 2$ handlebodies, etc. And we can introduce interactions/sources by adding e.g. punctures into our sphere, torus, etc. And this starts to look a lot like we're doing string theory! This is because in string theory, we also added punctures, which increase the Euler characteristic of the surface.

Now, just trying to start with a string theory doesn't exactly work– we must have conformal invariance on the boundary, and our basic Yang-Mills theory we've written down won't be conformally invariant until we add in some matter fields. But once we have a proper CFT we get something that looks like weakly coupled strings. There are some special operators $\mathcal{O}_s$ which are the single-trace operators of the form

$$\mathrm{tr}(FFFF\ldots), \tag{8.13}$$

as opposed to double trace $\mathrm{tr}(\ldots)\mathrm{tr}(\ldots)$ or higher trace operators $\mathrm{tr}()^n$, which create (or annihilate) n strings more generally.

Note that for a derivative term, $\mathrm{tr}(FD_\mu^n F)$, we get a tower of higher spin fields such that in the $\lambda \ll 1$ limit, $\Delta \approx$ the naive ("engineering") dimension plus small corrections. On the other hand, in the strong coupling $\lambda \gg 1$ limit, we have Δs which may get big, meaning that we remove Δ to high energies except a small number of operators which are *protected*. In the next lecture, we will start to present examples of this duality and discuss how they were derived originally.

Lecture 9.

Thursday, May 9, 2019

Today we'll discuss how people found actual examples of AdS/CFT using string theory. Now, there are many versions of the duality known, but we'll start with the classic version. As a quick convention note, from now on D is the *total* bulk dimension including Kaluza-Klein (compactified) dimensions, so it will not be generally true that $D = d + 1$. Now, if you took the *String Theory* course, you might recall that string theories are filled with *p-form fields*, i.e. generalizations of the Maxwell equations with extra indices.

Instead on just a vector field A_a, we might start with a p-form A_{abcd}, where there are p indices, and such that taking the exterior derivative $dA^{(p)} = F^{(p+1)}$ yields a $p+1$-form representing an associated curvature. Note also that the Hodge star (dual) operation takes us from a p-form to a $D-p$-form. Hence $F^{(p+1)} \overset{*}{\longleftrightarrow} F^{(D-p-1)}$, so that

$$dA^{(p)} = F^{(p+1)} \overset{*}{\longleftrightarrow} d\tilde{A}^{(D-p-2)} = F^{(D-p-1)}. \tag{9.1}$$

We don't usually work with both A and $\tilde{A}$ at the same time, but we have some freedom in how to choose which is e.g. our electric and magnetic field.

The form $dA^{(p)}$ is identified with a $p-1$-brane (where the number associated to the brane counts the spatial dimensions only, for historical reasons) and similarly $d\tilde{A}^{(D-p-2)}$ is assoicated to a $(D-p-3)$-brane. Perhaps the best-known case is in $D-4$, where the electric and magnetic fields both couple to 0-branes since A_a is a one-form.

	NS	RR
IIA (10D)	$B^{(2)} \to F1, NS5$	$C^{(1)} \to D0, D6$ and $C^{(3)} \to D2, D4$
IIB (10D)	$B^{(2)} \to F1, NS5$	$C^{(0)} \to D(-1), D7$ and $C^{(2)} \to D1, D5$ and $C(4) \to D3$, $F^{(5)} = *F^{(5)}$

These B and C fields are gauge fields of the corresponding degree, and the objects $F1, D0$, etc. are the branes to which they couple. The classic example takes the D3 brane, which involves the geometry $\mathrm{AdS}_5 \times S_5$. Another key example comes from M-theory in 11 dimensions, where $A^{(3)}$ couples to $M2, M5$ branes corresponding to $\mathrm{AdS}_3 \times S_7$ and $\mathrm{AdS}_7 \times S_3$ respectively.

It's worth noting that when $T = 0$ (in conformal symmetry), we get $R = 0$ and hence the dilaton coupling $e^{-2\phi}R$ in the action becomes trivial. In general, we have to take multiple branes to construct the duality, e.g. by taking D1 and D5 to get $\mathrm{AdS}_3 \times S_3 \times T_4$.

Suppose we have a *stack* of N coincident D3 branes. At weak coupling, $g_S N \ll 1$, we can have some open strings which start on one brane and connect back to another brane in the stack. We ought to assign the endpoints some color indices i, j. In particular we may describe adjoints with a $U(N) = SU(N) \times U(1)$ symmetry. The $U(1)$ symmetry is abelian and just describes the center of mass. But something interesting happens in the low energy limit.

In this limit, the closed strings decouple– the theory we get is $d = 4, \mathcal{N} = 4$ super Yang-Mills with a coupling $g^2 = 4\pi g_s$. This is the maximal number of supersymmetries we can have without taking us past spin 1. In such a theory, we can define helicities:

spin	-1	$-1/2$	0	$1/2$	-1
helicities	1	4	6	4	1
field	A^-	ψ^*	ϕ	ψ	A^+.

Our action takes the form

$$I = \int d^4x \,\mathrm{tr}\left(\frac{1}{4}F_{ab}^2 + \psi_{\dot{\alpha}}^* \not{D}\psi_\alpha + \frac{1}{2}(D\phi_I)^2 + \frac{g^2}{4}[\phi_I, \phi_J]^2 + \frac{g}{2}\psi[\phi, \psi] + \frac{g}{2}\psi^*[\phi, \psi^*]\right). \tag{9.2}$$

What's interesting about such a theory is that it actually has so much supersymmetry that all the beta functions vanish: $\beta = 0$, which implies this is in fact a super-conformal field theory (SCFT) for all g.

At *strong coupling* ($g_s N \gg 1$), we have instead

$$I \sim \frac{1}{l_p^8}\int d^{10}x \sqrt{-g}\left(R - F_{(s)}^2\right), \tag{9.3}$$

which as an extremal geometry with AdS in the near-horizon region, i.e.

$$ds^2 = \frac{1}{\sqrt{1+\frac{L^4}{r^4}}}(-dt^2 + dx_1 62 + dx_2^2 + dx_3^2) + \sqrt{1+\frac{L^4}{r^4}}(dr^2 + r^2 d\Omega_5^2), \tag{9.4}$$

where $L^4 = 4\pi l_p^4 N$.

In the low-energy limit ($r \ll L$), the exterior and the deep throat region decouple, and hence the netric reduces to

$$ds^2 = \frac{r^2}{L^2}(-dt + dx_i^2) + \frac{L^2}{r^2}dr^2 + L^2 d\Omega_5^2. \tag{9.5}$$

If we define $z = L^2/r, L = R_{\mathrm{AdS}} = R_S$, then we simply get the geometry of AdS$_5$-Poincaré $\times S_5$, and we have type IIB supergravity in the limit $L \gg l_s$ (where $l_p = g_s^{1/4} l_s$).

Adjusting g from weak coupling to strong coupling, we should be able to translate between the Super Yang Mills with $SU(N)$ symmetry (from the weak coupling limit) and the AdS$_5$-Poincaré $\times S_5$ with IIB SUGRA (from the strong coupling limit). This is Maldacena's derivation of the AdS/CFT correspondence.

There are some units which will help us translate between the two:

- $N_{\mathrm{colors}} = N_{\mathrm{branes}} = N_{\mathrm{flux}}$
- $\frac{l_p}{L} = (4\pi N)^{-1/4}$
- $\frac{l_s}{L} = (4\pi g_s N)^{-1/4} = (g_{YM}^2 N)^{-1/4} = \lambda^{-1/4}$.

Both these length ratios must be small in order for us to recover classical supergravity. There are two expansion parameters we can use to go further– there's $1/N^2 \sim \hbar G \sim$ quantum corrections, and $1/\lambda^2 \sim \alpha' \sim$ stringy corrrections. At finite N, we can probe nonperturbative quantum gravity, and at finite λ we study the full string worldsheet.

There is an S-duality on both sides– on the string side, we can take $g_s \to 1/g_s$ (which switches around the branes), and this is actually equivalent to $g_{YM} \to 4\pi/g_{YM}$. If λ is too big, we just recover the original S-duality on the SUGRA side. Note that this tells us we must have large N in order to have an interesting limit of the duality, or else we just get the old S-duality back.

Lecture 10.

Friday, May 10, 2019

A brief follow-up from last time. We claimed that in AdS$_5 \times S_5$, we had $T = 0 \implies R = 0$ by the conformal symmetry. However, one might worry about the possibility of a trace anomaly in curved space. Normally, the trace of the stress tensor is proportional to $aE + \sum_i c_i C_i$, where E is some Euler number term and the C_is are constructed from many copies of the Riemannn tensor contracted with some ϵs. However, because AdS$_5 \times S_5$ is conformally flat, all the c_is vanish and moreover $E = 0$ for odd-D manifolds without boundary. Note also that $E(A \times B) = E(A) \times E(B)$. Hence $E = 0$ for $\mathrm{AdS}_5 \times S_5$, so indeed there is no trace anomaly.

Note that the Newton constant of our spacetime depends on the compact Kaluza-Klein dimensions. Hence for instance (up to order unity factors)

$$G_t \sim \frac{G_{10}}{\mathrm{Area}[S_5]/l_{p,10}^5}. \tag{10.1}$$

Hence gravity gets diluted in the presence of extra dimensions. In general $G \sim l_p^{D-2}$.

What evidence do we have that the AdS/CFT correspondence is true? In Maldacena's original derivation, he posited that the theory could be interpolated between weak and strong couplings so that the commuting square diagram really works.

- The symmetries match up. Hence the $SO(2,4) \times SO(6) + 32$ real supersymmetries line up. In $\mathcal{N} = 4$ SYM, we get Weyl spinors with 2 complex components. In the CFT, there are 16 supercharges Q_A^α and 16 superconformal $S_{\bar{A}}^\alpha = X_{A\bar{A}} Q_B^\alpha \epsilon^{AB}$.There are some anticommutation relations which allow us to construct the representations of the symmetries,

$$\begin{aligned}\{Q_A, \bar{Q}_{\bar{A}}\} &\sim P_{A\bar{A}}\\ \{S, \bar{S}\} &\sim K\\ \{Q, \bar{S}\} &\sim M + R + D,\end{aligned}$$

  where P are the $+1$ spin representations, Q the $+1/2$, R, M, D the 0 reps, S the $-1/2$ reps, and K the -1 reps.
- S-duality. $SL(2,\mathbb{Z})$: in SYM, we have $\tilde{g} = g + i\theta, \theta F \wedge F$ and on SUGRA, $g_5 : \Phi, C_0$
- Trace anomalies match (protected from weak$\to$strong coupling)
- BH entropy/holographic entanglement entropy (to be discussed more alter)
- "chiral" primaries of supergroup– irrep that has null states, $Q|\psi\rangle = 0$ for some Q, ψ, with $S|\psi\rangle = 0$ and $\{Q, S\}|\psi\rangle = 0$. Δ is determined by spin (M) and R-charge (R), hence the charge Q is protected in going to strong coupling.

In fact, all type IIB supermultiplets (KKK reduced to AdS_5 in regime of validity of SUGRA) is equivalent to the single trace chiral irreps which are built $\mathrm{tr}(\underbrace{\phi_{\{I}\phi_J\phi_{K\ldots\}}}_{n})$ where $\{\}$ indicates symmetric traceless. This single trace expression corresponds to the nth spherical harmonic on S_5. We then have the relations

$$\begin{aligned}[Q_A, \phi] &\sim \psi_A,\\ \{Q_A, \psi_B\} &\sim F_{AB} + [\phi^I, \phi^J]\epsilon_{AB}\\ \{Q_A, \psi_{\bar{B}}^*\} &\sim D_{A\bar{A}}\phi\\ [Q_A, A_{B\bar{B}}] &\sim \epsilon_{AB}\psi_{\bar{B}}^*.\end{aligned}$$

We can have traces of $2 \le n \le N$ ϕs– note that traces of more than N commuting $N \times N$ matrices are not independent (of traces of fewer than N matrices). This relation suggests that there is a shortest scale on the S_5, such that no new degrees of freedom emerge if we try to probe length scales smaller than the Planck scale.

Now, IIB has no additional fields, so it seems that the non-chiral irreps have $\Delta \to$large ($\Delta^2 \sim 1/l_s$) as $\lambda \to$large. In $n = 2$ we have $\mathrm{tr}(\phi_{\{I}\phi_{J\}})$ with $2 \le \Delta \le 4$ which includes all conserved CFT currents. $\Delta = 3 = d - 1$ is the R-charge J, $\Delta = 3 + 1/2$ gives J_{super}, and $\Delta = 4 = d$ gives Φ dual to the dilaton.

There are other versions of the duality which are useful.

- N M2's: $\mathrm{AdS}_4 \times S_6^{(m)} \longleftrightarrow d = 5, \mathcal{N} = 8$ ABJM model. $U(N) \times U(N)$ Chern-Simons gauge theory $\mathcal{L} = F \wedge A + \frac{2}{3} A \wedge A \wedge A$, also IR limit of $d = 3, \mathcal{N} = 8$ SYM (g =relevant) (D2)
- N M5s: $\mathrm{AdS}_2 \times S_4 \longleftrightarrow d = 6, \mathcal{N} = (2,0)$ max dimensional superconformal theory. Does not come from a Lagrangian but is known to exist from stringy arguments. Compactify on S_1, flows to $d = 5, \mathcal{N} = 4$ SYM (g =irrelevant) (D4)
- Q_1 D1 Q_5: $\mathrm{AdS}_3 \times S_3 \times T_4$ (IIB) (K3) $\longleftrightarrow$ nonlinear σ-model $d = 2$.

Lecture 11.

Monday, May 13, 2019

Erratum: in writing $\operatorname{tr}(\phi_I \phi^J)$, we should have had $\Delta^2 \sim m^2 L^2 \sim L^2/l_s^2$, where L was the AdS radius. See also the relation $\Delta(\Delta - 4) = m^2$.

Black holes We have seen that the Hilbert spaces of the theories on either side of the duality are equal,

$$\mathcal{H}_{\text{AdS}} = \mathcal{H}_{\text{CFT}}. \tag{11.1}$$

However, there is another way to phrase this correspondence, in terms of partition functions:

$$\mathcal{Z}_{\text{AdS}}^{[J_{bdy}]} = \mathcal{Z}_{\text{CFT}}^{[J_{CFT}]}. \tag{11.2}$$

Since the partition functions agree, this means that the n-point correlation functions will also agree, e.g.

$$\frac{\delta}{\delta J_1(x)} \frac{\delta}{\delta J_2(y)} \ldots \cdot \ln \mathcal{Z} = \langle \mathcal{O}_1(x) \mathcal{O}_2(y) \ldots \rangle_{\text{connected}} . \tag{11.3}$$

We can do this for the partition function on either side of the duality (initially proposed by Ed Witten), and we'll find that the correlation functions do work out.

Moreover, there's a simplification in the large N (and possibly strong coupling λ) limit– the bulk becomes *classical gravity*. In this case, we can do a saddle point approximation and expand the metric as

$$g_{ab} = g_{ab}^{\text{class}} + h_{ab}, \tag{11.4}$$

where g_{ab}^{class} solves the Einstein equations (possibly coupled to matter) and h_{ab} is some small quantum correction. This is known as a semi-classical approximation.

In this case, our saddle point approximation says that we get

$$\mathcal{Z}_{\text{AdS}} = \det(\ldots) e^{i I_{\text{grav}}[g_{ab}} \tag{11.5}$$

in Lorentzian signature, where I is the action evaluated at the original classical metric and the determinant factor depends on the quantum corrections. We might have e.g. $\det^{-1/2}(D)$ for D a wave equation operator. However, when we take the log, we get

$$\log \mathcal{Z}_{\text{AdS}} = I_{\text{grav}} + \text{subleading in } 1/N \text{ loop corrections}. \tag{11.6}$$

Sometimes we work in Euclidean signature and write $-I_{\text{grav}}$ instead to avoid issues of convergence in our saddle point approximation.

This gives us a new entry in the AdS/CFT dictionary. Suppose we have a Euclidean signature QFT. Then the log of the partition function is equal to the gravitational action with a least-action "instanton" solution to the equations of motion with a specified boundary metric γ_{ab}.

There are some caveats to this, though. For one, the Euclidean action of GR is not bounded below (i.e. if we go off-shell). In addition, we're making a saddle point approximation, and so we should check that we really can deform the contour.[4]

Let's see this in action. In Euclidean signature, we have an action

$$I_{\text{(grav)}}^{\text{Euc.}} = -\frac{1}{16\pi G} \int_M \sqrt{g} (R - 2\Lambda) d^D x. \tag{11.7}$$

Now we find a solution to the Einstein equation of motion in vacuum,

$$R_{ab} - \frac{1}{2} g_{ab} R + g_{ab} \Lambda = 0. \tag{11.8}$$

Tracing over, we have

$$\frac{D-2}{2} R = D\Lambda, \tag{11.9}$$

which we can substitute back into the action to get

$$I_{\text{grav}} \sim \int d^D x \sqrt{g} = \text{Vol}(M) = +\infty \tag{11.10}$$

[4] "Morally, we should do this. Practically, no one ever does this before it's too hard. You're allowed to [assume the calculation works], you just have to feel guilty for it." –Aron Wall

since our space is asymptotically AdS.

What's gone wrong? We neglected to treat the boundary conditions carefully. What we should really do is to impose a UV cutoff such that our integration is only over $z > \epsilon$. When we vary the action to compute the equations of motion, we usually discard boundary terms– we can't do that here. Generically, varying R gives us a term like $\delta K_{ij}\gamma^{ij}$, where we want $\gamma^{ij} \neq 0$. More specifically, we should have included

(a) $I_{\mathrm{GH}} = \frac{1}{8\pi G}\int_{\partial M}\sqrt{\gamma}K_{ij}\gamma^{ij}$ with $K_{ij} = \frac{1}{2}g_{ij,\hat{n}}$. This *Gibbons-Hawking* term tells us how to treat the boundary term in terms of the metric γ on the boundary and the corresponding extrinsic curvature.

(b) $I_{\mathrm{ct}} = \int_{\partial M}\sqrt{\gamma}H[\gamma_{ij}]$, local counterterms where H depends on up to $d/2$ derivatives of γ_{ij}. This is the usual sort of thing that happens in QFTs– we write down

$$\ln \mathcal{Z}_{\mathrm{phys}} = \ln \mathcal{Z}_{\mathrm{reg}}(\epsilon) - \text{local divergences, e.g. } \epsilon^{-n}, \ln\epsilon. \tag{11.11}$$

That is, we introduce a regulating parameter ϵ of a naively divergent partition function and subtract off divergences to get the physical behavior. Interestingly, the coefficients of the log divergences seem to represent universal quantities which do agree between the AdS and CFT sides.

Consider now the thermal partition function in terms of $\beta = 1/T$. That is,

$$\mathcal{Z}[\beta] = \mathrm{tr}_{\mathcal{H}}(e^{-\beta E}). \tag{11.12}$$

This is analogous to evolving through $\beta = i\Delta t$ imaginary time ($e^{-i\Delta t E}$, and it gives us a geometry $S_1 \times S_{d-1}$. Hence

$$\ln Z = -\beta F = S - \beta E \tag{11.13}$$

in terms of the free energy, and S is now the von Neumann entropy

$$S = -\mathrm{tr}(\rho\ln\rho) = (1-\beta\partial_\beta)\ln\mathcal{Z} \tag{11.14}$$

with the energy $E = -\partial_\beta \ln\mathcal{Z}$.

We get what's called a thermofield double state (TFD), which is

$$|TFD\rangle = \sum_i e^{-\beta E_i/2}|i\rangle_L|\bar{E}_i\rangle_R \tag{11.15}$$

corresponding to a pure state in $\mathcal{H}\otimes\bar{\mathcal{H}}$. If we restrict to one system, we get a thermal state.

There are two types of solution.

(a) If we pinch the S_{d-1} to a point, we get two copies of the CFT at $t = 0$. This gives thermal AdS.

(b) We could instead pinch off the S_1 to a point. This instead corresponds to a connected wormhole geometry. More precisely, if we continue to Lorentz signature, we find the geometry of an eternal black hole, AdS-Schwarzschild.

This second point is still somewhat mysterious. It seems that entanglement on the CFT side is equivalent to a wormhole on the gravity side. In our semi-classical approximation, the entropy of thermal AdS is $S = 0$ (there may be subleading in $1/N$ corrections, which have the interpretation of thermal matter entropy). On the other hand, for the BH solution, we find that the entropy of the CFT is

$$S_{\mathrm{CFT}} = \frac{\mathrm{Area}[H]}{4G\hbar} = S_{\mathrm{BH}}. \tag{11.16}$$

This is none other than the famous Bekenstein-Hawking formula for the black hole entropy. Again, there may be subleading corrections from quantum fluctuations. This result tells us that the microstates being counted by the Bekenstein-Hawking entropy are (in AdS/CFT) just the microstates of the dual CFT.

Lecture 12.

Tuesday, May 14, 2019

There was a question last time about what we meant by "pinching off" the geometry. The boundary of our space is $S_1 \times S_{d-2}$. The idea is that we could either replace the S_{d-1} with a ball B_d to get $S_1 \times B_d$, or we could replace the S_1 with a disc B_2 to get a $B_2 \times S_{d-1}$. The former case gives two copies of thermal AdS, while the second gives the eternal black hole.

In the latter case, we get the AdS-Schwarzschild geometry. In AdS_5, we have a metric

$$ds^2 = -f dt + \frac{dr^2}{f} + r^2 d\Omega^2 \tag{12.1}$$

where

$$f = 1 + \frac{r^2}{L} - \frac{\mu}{r^2} \tag{12.2}$$

where the cosmological constant is related to the AdS radius by $\Lambda = -6/L^2$. The proportionality constant depends on how many dimensions we're working in. Now the horizon radius is given by

$$r_H = \frac{L^2}{2}\left(\sqrt{1 + \frac{4\mu}{L^2}} - \right) \tag{12.3}$$

where the inverse (Hawking) temperature is then

$$\beta = \frac{2\pi L^2 r_H}{2r_H^2 + L^2}. \tag{12.4}$$

If we plot T as a function of r_H, we find that in the small r_H regime (a small black hole), we get $T \sim 1/r_H$, which is just like the Schwarzchild black hole in asymptotically flat space. It has negative specific heat. On the other hand, at large r_H, we get $T \sim r_H$ and in this limit of a large black hole, we get a positive specific heat. Thus such a black hole is permitted in the canonical ensemble.

This tells us there is a minimum temperature corresponding to a maximized inverse temperature. For

$$\beta > \beta_{HP} = \frac{2\pi L}{3}, \tag{12.5}$$

we have $I^{\text{AdS}} < I^{\text{large BH}}$. Conversely for $\beta < \beta_{HP}$ we instead have $I^{\text{large BH}} < I^{\text{AdS}}$.

Equivalently, at low energies there exists a "confining" phase with $S \sim O(1)$ at low temperature (where the particles form color singlets), whereas at high temperatures, we have $S \sim O(N^2)$ which gives a (super) gluon plasma. This is kind of a special confinement because we made space a sphere– since our theory was conformal, a priori there's no special length scale to confine to (unlike in QCD). If we put our theory on a plane ($\mathbb{R}^4$ instead of $S_3 \times \mathbb{R}$) this is like taking the high-T limit. We find that

$$S_{BH} = \frac{\text{Area}}{4G\hbar} = \frac{3}{4} S_{\text{free}} \sim VT^3, \tag{12.6}$$

which is another qualitative check of the duality since this gives the entropy of a free thermal gas. One might worry about the 3/4 factor, but in fact perturbative calculations suggest that we might be able to interpolate smoothly between the $\lambda = 0$ weak coupling limit (with the constant $= 1$) and the $\lambda =$ large strong coupling limit (with the factor 3/4).

In the microcanonical ensemble, small black holes in 5d are in fact permitted, even though their temperature tends to decrease– this is allowed if we keep track of the thermal radiation as the black hole evaporates. Below this temperature there are 10d black holes, stringy behavior, and at the lowest energy scales some field theory limit. The CFT duals to the large black holes are better understood.

BH from collapse Let us suppose we send in a spherically symmetrical pulse of massless ϕ field (AdS-Vaidya). As this pulse compresses, a horizon forms.

We can actually predict physics outside the horizon based on the data specified on the boundary, but in general it's harder to predict what happens inside the horizon. In the tin-can picture, we send in some radiation and form a black hole, and this black hole then evaporates. At some point, the black hole reaches the Planck scale and quantum gravity kicks in. We don't really understand what happens here. It's expected that quantum gravitational effects will change the topology and allow the black hole to evaporate entirely. It shouldn't leave any remnant, since this would be hard to reconcile with the CFT description.

However, this leads to the "information paradox." For semiclassical bulk physics, the Hawking radiation should be thermal radiation, i.e. in a mixed state. One can study the Hawking radiation as related to some modes inside the horizon by suggesting that near the horizon, an infalling observer should simply see the vacuum. This allows us to treat Hawking radiation as a special case of the Unruh effect. But it seems that the mixed state of the thermal radiation is not correlated with what fell in.

On the other hand, on the CFT side, we observe thermalization. There's some Hamiltonian describing time evolution, so time evolution is in particular unitary– pure states remain pure under unitary operations, though it may look effectively thermal. Information should be preserved.

Most of the community now believes that the CFT side (unitary evolution) is correct and that information is not lost in black hole evaporation. Hence the question becomes: what goes wrong with the Hawking calculation?

Work by Mathur and later AMPS (Polchinski et al) suggested that if we want the Hawking radiation to also be pure, then the black hole interior is actually nonphysical for late time black holes. This is the "firewall paradox." Schematically, the argument suggests that the late-time radiation is strongly entangled with both the early-time radiation and the infalling radiation beyond the horizon. This violates the monogamy of entanglement, a consequence of strong subadditivity. We have

$$S(AB) + S(BC) \geq S(A) + S(C) \tag{12.7}$$

as bounds on the von Neumann entropies. Suppose A is the early time radiation, B the late time radiation, and C the internal radation. Then $S(AB) \sim 0, S(BC) \sim 0$ since these states are entangled (and nearly pure states), but $S(A) \sim \ln 2, S(C) \sim \ln 2$. This is very strange and there's no clear resolution.

Lecture 13.

Thursday, May 16, 2019

Today we'll introduce the idea of holographic entanglement entropy. We'll begin with a discussion of entanglement entropy in the context of field theory, and next lecture we'll discuss its holographic description with respect to the bulk theory.

If we have a Hilbert space which can be decomposed into a tensor product as

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2, \tag{13.1}$$

then for a pure state which lives in the bipartite Hilbert space $|\psi\rangle_{12} \in \mathcal{H}$, we can write down reduced density matrices by tracing over the degrees of freedom corresponding to the subsystems,

$$\rho_1 = \mathrm{Tr}_2(|\psi\rangle\langle\psi|) \tag{13.2}$$

where $\mathrm{tr}(\rho) = 1$ is normalized. The von Neumann entropy (perhaps familiar from *Quantum Information Theory*) is a measure of how mixed a state is. It is given by

$$S(\rho) = -\operatorname{tr}(\rho \ln \rho), \tag{13.3}$$

and it is zero for a pure state and $\ln \dim \mathcal{H}$ for a maximally mixed state. The von Neumann entropy has some nice properties.

(a) It is positive, $S(\rho) \geq 0$.
(b) Invariant under unitaries, $S(U\rho U^\dagger) = S(\rho)$ and invariant under adding extra $p = 0$ states.
(c) Additive under tensor products, $S(\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B)$
(d) Triangle inequality: $S(A) + S(B) \geq S(AB) \geq |S(A) - S(B)|$ (Araki-Lieb)
(e) Continuous for finite-dimensional $\mathcal{H}$ (lower semicontinous for infinite-dimension)
(f) Concavity, $S(\sum_i \lambda_i \rho_i) \geq \sum_i \lambda_i S(\rho_i)$ where $\sum \lambda_i = 1$.
(g) For $\rho = \oplus \lambda_i \rho_i$ (i.e. a block diagonal density matrix), $S(\rho) = \langle S(\rho_i)\rangle_\lambda - \sum_i \lambda_i \ln \lambda_i$ (the last term is the Shannon entropy of $\{\lambda_i\}$).
(h) Strong subadditivity, $S(AB) + S(BC) \geq S(ABC) + S(B)$.

These many properties of the von Neumann entropy will lead to some nontrivial checks of the duality.

Entanglement entropy in field theory Now, the notion of entropy is a little different for quantum field theories because our space does not factorize into clean Hilbert spaces as in quantum mechanics. Let us take a $d-1$-dimensional Cauchy surface Σ and further define a region R on Σ bounded by a surface E of codimension 2. There should be some density matrix ρ_R describing the state of the fields in R.

Naively, we would say that the entanglement entropy of R is then

$$S(\rho_R) = -\operatorname{tr}(\rho_R \ln \rho_R). \tag{13.4}$$

But there's a difficulty. The value of this depends on

(a) the QFT itself
(b) the region R
(c) the state ψ
(d) the short-distance cutoff ϵ, i.e. how we regulate the theory

Strictly speaking, one says that ρ is a state in a type III von Neumann algebra, but as is usual these constructions are harder to work with. It doesn't really make sense to take the trace over $\rho \ln \rho$ because individual entries in the density matrix correspond to pure states which may have arbitrarily high energy and large entanglement with the exterior region. We can then define ρ by the expectation values of operators for all $\mathcal{O} \subset \mathcal{A}$ in some algebra.

We should expect that for an entire system that is in a pure state, the entropy of our region R is equal to the entropy of its complement,

$$S(R) = S(\bar{R}), \tag{13.5}$$

given that we apply the same cutoff on both sides of the boundary. For a spacetime foliated by some Cauchy slices R_1, R_2, we can also say that $S(R_1) = S(R_2)$ if $D[R_1] = D[R_2]$ (their domains of dependence are the same).

Divergences For a $d = 2$ CFT, we get an entropy of

$$S = \frac{c}{3} \ln\left(\frac{r}{\epsilon}\right) + \text{finite}, \tag{13.6}$$

where the finite bit is scheme-dependent but the $c/3$ scaling of the log divergence is universal, with c the central charge. For $d > 2$, we instead get an area law,

$$S = \#\frac{\text{Area}[E]}{\epsilon^{d-2}} + \text{subleading} \tag{13.7}$$

where the constant in the first term depends on ϵ. In even dimension, we get a log divergence $\# \ln(\epsilon) \int [R]^{d/2}$ with R the curvature, and where the multiplicative factor is related to the central charge in the CFT. In odd dimensions, we just get a finite contribution without the log divergence. That is, the subleading terms will end with $1/\epsilon + \text{finite}$.

Geometric entropy The method we'll discuss now is valid when ρ comes from a path integral with a $U(1)$ rotational symmetry. One may for instance discuss the Rindler wedge of Minkowski. In Minkowski, a moving observer sees thermal radiation with boost energy due to the Unruh effect,

$$K = \int_0^\infty T_{tt} x dx dy dz. \tag{13.8}$$

That is,

$$\rho = \frac{e^{-2\pi K}}{Z} \tag{13.9}$$

with $Z = \text{tr}(e^{-2\pi K})$.

Since we have

$$S(\beta') = (1 - \beta\partial_\beta) \ln Z|_{\beta=\rho'}, \tag{13.10}$$

for $\beta' \neq 2\pi$ we get a conical singularity. Interestingly, if we take

$$\ln Z = -I_{\text{grav}} = \frac{1}{16\pi G} \int R\sqrt{g} d^d x, \tag{13.11}$$

the Einstein-Hilbert action, one in fact recovers

$$S = \frac{A}{4G\hbar}, \tag{13.12}$$

the Bekenstein-Hawking formula for the black hole entropy.

What if we do not have the $U(1)$ symmetry? We use the "replica trick," where we calculate a modified partition function

$$Z_n = \text{tr}(\rho^n), \tag{13.13}$$

d	CFT	bulk	strong λ	weak λ
2	D1-D5	$\mathrm{AdS}_3 \times S_3 \times T_4$	c	c
3	ABTM	$\mathrm{AdS}_4 \times S_2$	$N^{3/2}$ (IR)	N^2 (UV)
4	$\mathcal{N} = 4$ SYM	$\mathrm{AdS}_5 \times S_5$	$\sim N^2$	$\sim N^2$
6	$(2,0)$ model	$\mathrm{AdS}_7 \times S_4$	N^3 (UV)	N^2 (IR)

TABLE 2. Caption

with n an integer. That is, we take n copies of ρ and glue them together along some surface, and attempt to analytically continue to non-integer n. Then our entropy is given by

$$S = (1 - n\partial_n Z_n)|_{n=1} \tag{13.14}$$

$$= \lim_{n\to 1} \underbrace{\frac{1}{n} \ln \mathrm{tr}(\rho^n)}_{S_n,\text{ Rényi entropy}} \quad . \tag{13.15}$$

Lecture 14.

Friday, May 17, 2019

Today we'll continue our discussion of entanglement entropy, specifically the holographic entanglement entropy. The central result in this area is the *Ryu-Takayanagi formula*.

Last time, we said that by picking a Cauchy slice and some bounded region R with a boundary $E = \partial R$, we wanted to define the von Neumann entropy of the region R such that $S = -\,\mathrm{tr}(\rho_R \log \rho_R)$. The Ryu-Takayanagi formula gives us a way to compute the leading-order piece of the entanglement entropy S. The formula also applies to static geometries or $t \to -t$ Cauchy surfaces. Some of the scalings are given in Table 2.

The prescription is actually very simple. Given $E = \partial R$ on the boundary, we need only to construct the surface γ in the bulk with boundary corresponding to E which minimizes the area. That is,

$$S = -\,\mathrm{tr}(\rho \log \rho) = \min_\gamma \frac{\mathrm{Area}[\gamma]}{4G\hbar}. \tag{14.1}$$

It's no coincidence this looks like the Bekenstein-Hawking entropy formula. There are some additional constraints– the surface γ must be anchored to ∂R, and it must be homologous (can be smoothly deformed through the bulk) to R.

For instance, in the eternal AdS-Schwarzschild black hole, the correct minimal area surface lies at the throat of the wormhole. If we take R to be the CFT on one side, then all of space at an instant is S_{d-1}, which is closed and has no boundary ($\partial R = \{\}$). Hence the anchoring condition is trivial and the throat of the wormhole satisfies the homology condition.

If we try to do the bulk area calculation naively, we get infinity, since there is a redshift factor $1/z^2$ in our metric ($ds^2 = \frac{1}{z^2}(\eta_{ij}) dx^i dx^j$). This isn't too surprising since we also had to impose a cutoff in directly computing the entanglement entropy of the boundary CFT. Here, we will also introduce a cutoff and simply integrate the area of the minimizing surface starting at some $z = \epsilon$. In fact, if we are careful, we can use theories with large amounts of supersymmetry to check e.g. in $d = 4$ and $d = 2$ that the log divergences from the CFT calculation agree with the log divergences in the bulk calculation.

If there is a black hole in our space, as we tune the boundary region we will get a phase transition depending on which side of the black hole our minimizing surface wraps around. But in fact we need not have a black hole to see a phase transition. If we take a sphere and two caps A, B, then we can use RT to compute the entropy $S(AB)$. For small caps (small θ) we get

$$S(AB) = S(A) + S(B), \tag{14.2}$$

where the mutual information ($I_{A,B} = S_A + S_B - S_{AB} \geq 0$) is $I_{A,B} = \mathcal{O}(1)$. But for large θ, we instead have $S_{AB} < S_A + S_B$, with $I_{A,B} = \mathcal{O}(N^2)$. By sketching this, we see that the entropy itself is continuous but its derivative $\frac{\partial}{\partial\theta} S$ is discontinuous.

It's a remarkable fact that although strong subadditivity is really hard to prove in quantum information theory, there is a very nice proof from holography. Diagram to be added. We can prove other sorts of quantum information inequalities, e.g. the monogamy of mutual information:

$$S(AB) + S(BC) + S(AC) \geq S(A) + S(B) + S(C) + S(ABC). \tag{14.3}$$

It is not satisfied by general ρ_{ABC} in QM, but it does hold holographically (cf. Hayden-Headrick-Maloney).

There is also a covariant generalization of the RT formula, the Hubeny-Rangamani-Takayanagi (HRT) prescription. This formula kicks in when the spacetime $\mathcal{M}$ is dynamical or if the boundary E is time-dependent. Notice that in a metric with Lorentzian signature, our old notion of a minimal area surface breaks down because we can generally increase the area of such a surface by introducing "wiggles" in the time direction. Instead, we should try to extremize the area, like in Euler-Lagrange. Instead of seeking a global minimum, the best we can do is to find a saddle point.[5] Hence

$$S = \min \operatorname{ext}_\gamma \frac{\text{Area}[\gamma]}{4G\hbar} \tag{14.4}$$

for surfaces γ that are still homologous to the region in the boundary theory.

There is another way about this, the "max-min" procedure. For each Cauchy slice Σ passing through E, identify the minimal surface $\min(\text{Area}[\gamma], \Sigma)$. Then vary Σ to maximize, $\max_\Sigma \min_{\gamma\subset\sigma} \frac{\text{Area}[\gamma]}{4G\hbar}$. This is equivalent to HRT if $T_{ab}k^a k^b \geq 0$ for k^a null (i.e. the null energy condition) and the spacetime is AdS-hyperbolic. This max-min proceedure is good for proofs (establishing properties of the HRT surface) and bad for calculations (since we have to extremize over an infinite-dimensional space of Cauchy slices).

Lecture 15.

Monday, May 20, 2019

HKLL reconstruction formula Recall that our correspondence says that fields in the bulk can be written as the limit of operators near the boundary,

$$\phi \to \lim_{z\to 0} z^{-\Delta} \mathcal{O}. \tag{15.1}$$

We would like to define $\phi_{\text{bulk}}(z, x)$ near the boundary but for some finite z. That is, we wish to reconstruct the bulk from the boundary. To do this, we can write

$$\phi_{\text{bulk}}(z, x) = \int d^d x' \, K(x'|z, x) \mathcal{O}(x') \tag{15.2}$$

in terms of some Green's function (kernel) $K(x'|z, x)$. We wish to construct the $\tilde{\phi} = z^{(d-1)/2}\phi$ which solves

$$[\partial_z^2 - \frac{\tilde{m}^2}{z^2} + \Box^{(d)}]\tilde{\phi} = 0. \tag{15.3}$$

Notice this is a *nonstandard* Cauchy problem. We have a hyperbolic wave equation but instead of solving based on initial data on a Cauchy (hyper)surface, we are given data on a timelike $z = 0$ surface.

We want ϕ_{bulk} to be uniquely determined by the $z = 0$ data, but while our equation looks hyperbolic in the t-z plane, it looks elliptical in the t-$\mathbf{x}$ plane. To solve this, we can use the x-translation symmetry of the problem. That is, we expand $\phi, \mathcal{O}$ in plane waves in x so that

$$\phi_p(z, t) = \int d\mathbf{x} \, e^{i\mathbf{p}\cdot\mathbf{x}} \phi(z, t, \mathbf{x}). \tag{15.4}$$

The question reduces to a $1 + 1$ Lorentzian problem, where it is now not so hard to switch the roles of space and time. We can evolve in the z-direction at the cost of flipping the effective sign of m^2, i.e. yielding a tachyonic mass.

Note that it is not generally true that in a Lagrangian with a tachyon, our theory loses predictability (i.e. information can travel faster than light). Instead, what happens is that instead of a nice harmonic oscillator

[5] Both mathematicians and physicists somewhat abuse the terminology here. Mathematicians will say that solutions of the E-L equations are minimal surfaces, whereas physicists call these extremal surfaces. Technically, the surfaces need not be minima or maxima (as is implied by extremal), but saddle points of the thing we're varying.

$+\phi^2$ potential, we get a $-\phi^2$ potential which makes our theory unstable. That is, small perturbations from the vacuum state grow exponentially.

From our plane wave expansion, switching time and space allows us to write

$$\phi_p(z,t) = \int dt'\, K_p(t'|z,t)\mathcal{O}_p(t') \tag{15.5}$$

in terms of a kernel, as promised. However, note that K_p blows up at large values of $\mathbf{p}$. That's because the $\Box$ term grows bigger at large values of the spacelike momentum. Taking the inverse Fourier transform would then in principle allow us to reconstruct the bulk from 15.2.

We can, if we wish, evolve the operator $\mathcal{O}$ back to a 1-bdy Cauchy slice,

$$\mathcal{O}(t,x) = e^{iHt}(\mathcal{O}(0,x)e^{-iHt}. \tag{15.6}$$

That is, we can evolve the operator back to some preferred moment in time corresponding to a single Cauchy slice on the boundary. Having behavior in the bulk determined by a codimension 1 surface is standard field theory. Having behavior in the bulk determined by a codimension 2 surface is surprising. That's the holographic principle. Initial data on the boundary not only predicts time evolution on the boundary but the entire interior of the bulk.

In the complete AdS (tin can) picture, in order to reconstruct the bulk field at some point, we need data from a cylindrical chunk of AdS corresponding to how long it takes for null rays to reach the boundary. We can do this, provided that we sum over spherical harmonics (it suffices to add over s-waves).

We could also look at the Rindler patch of AdS, in which case we could determine a bulk field just from the intersection of its "light cone" with the boundary in one direction.

If we were feeling ambitious, we could include perturbative interactions in a $1/N$ expansion using Witten diagrams. We could use Green's functions in the spacelike directions to describe propagation from a bulk point to the boundary, provided that we're a little careful about gauge symmetries in the bulk. For instance, diffeo symmetry in the bulk is allowed so long as they vanish on the boundary ∂M_{bulk}.

One way to do the gauge fixing is to use Fefferman-Graham coordinates, i.e. trace geodesics from bulk points to near-boundary ($z = \epsilon$) points. What we'd find if we did this was that conditions on the boundary lead to nonlocal behavior, e.g. an electron on the boundary induces a nonlocal gravitational and electric field in the bulk. This is related to the notion of Wilson lines.

Now what if our theory loses translation symmetry? We wish to reconstruct (part of) the bulk from a general CFT region. To do this, take a slice R on the boundary and find its domain of dependence $D[R]$. Then determine the intersection of its causal future and past *within the bulk*, i.e

$$C_W = I^-(D[R]) \cap I^+(D[R]). \tag{15.7}$$

This is known as the causal wedge. The property of *bulk causality* then says that using the local equations of motion in the bulk (like HKLL), we can reconstruct at most the causal wedge C_W. The future and past boundaries of the causal wedge then define the future and past causal boundaries $\mathcal{H}^+, \mathcal{H}^-$, which as the notation suggests are similar to black hole event horizons.

So we can reconstruct at most the causal wedge, but can we ever get the entire causal wedge? The answer is contained in *Holmgrem's uniqueness theorem*, which implies that if bulk "sources" are analytic, we can indeed reconstruct all of C_W. In practice it is often assumed that the causal wedge can always be reconstructed in its entirety regardless of the analyticity of sources in the bulk. There are a few known counterexamples where e.g. one could construct an artificial field obeying $\Box\phi = f(x,z)\phi$ which decays sufficiently quickly near the boundary, in which case a geometric optics approach shows that the causal wedge cannot be completely reconstructed. But this is thought to be nonphysical in the sense that this field behavior could not have come from an action principle. So whether the causal wedge can always be reconstructed is an open question in bulk reconstruction.

However, if we have nonlocal operators, this leads us to a bigger version of the bulk, the *entanglement wedge*. That is,

$$E_W = D[\Sigma_{R\to m(R)}], \tag{15.8}$$

where one considers the HRT surface. Bulk reconstruction leads us to an interesting paradox known as the *ADH paradox*. Suppose we divide the boundary into three regions, A, B, C. Let ϕ_{bulk} lie in the very center. Hence ϕ_{bulk}

- is in $\mathcal{A}[\mathcal{H}_{AB}]$

- ◦ is in $\mathcal{A}[\mathcal{H}_{BC}]$
- ◦ is not in $\mathcal{A}[\mathcal{H}_B]$.

These first two implications suggests that in terms of $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, this operator must be $I \otimes \mathcal{O}_B \otimes I$. But we said that it couldn't lie in $\mathcal{H}_B$ alone.

Fortunately, there is a nice resolution, to do with the phenomenon of quantum error correction. One version requires a three-qutrit system (a 27D Hilbert space). These three qutrits lead to one logical qutrit (a 3D Hilbert space), such that if we encode a state as one of the three computational basis states, if we then lose one of the qutrits, we can still reconstruct the state of the logical qutrit state from the other two. That is, we can reconstruct the full state if we restrict to the "code subspace." This suggests that only a subspace of states in the bulk will have a nice semi-classical limit like this "code subspace," so whenever this is true, there is no paradox.