QUANTUM FIELD THEORY

IAN LIM
MICHAELMAS 2018

These notes were taken for the *Quantum Field Theory* course taught by Ben Allanach at the University of Cambridge as part of the Mathematical Tripos Part III in Michaelmas Term 2018. I live-TEXed them using Overleaf, and as such there may be typos; please send questions, comments, complaints, and corrections to itel2@cam.ac.uk.
Many thanks to Arun Debray for the LATEX template for these lecture notes: as of the time of writing, you can find him at https://web.ma.utexas.edu/users/a.debray/.

Contents

Lecture 1.

Thursday, October 4, 2018

$2 = \pi = i = -1$ *in these lectures.* –a former lecturer of Prof. Allanach's.

To begin with, some logistic points. The notes (and I assume course material) will be based on David Tong's QFT notes plus some of Prof. Allanach's on cross-sections and decay rates. See http://www.damtp.cam.ac.uk/user/examples/indexP3.html and in particular http://www.damtp.cam.ac.uk/user/examples/3P1l.pdf for the notes on cross-sections.

After Tuesday's lecture, we'll be assigned one of four course tutors:

- Francesco Careschi, fc435cam.ac.uk
- Muntazir Abidi, sma74
- Khim Leong, lkw30
- Stefano Vergari, sv408

Also, the Saturday, November 24th lecture has been moved to 1 PM Monday 26 November, still in MR2. That's it for logistics for now.

Definition 1.1. A *quantum field theory* (QFT) is a field theory with an infinite number of degrees of freedom (d.o.f.). Recall that a field is a function defined at all points in space and time (e.g. air temperature is a scalar field in a room wherever there's air). The states in QFT are in general multi-particle states.

Special relativity tells us that energy can be converted into mass, and so particles are produced and destroyed in interactions (particle number not conserved). This reveals a conflict between SR and quantum mechanics, where particle number is fixed. Interaction forces in our theory then arise from structure in the theory, dependent on things like

- symmetry
- locality
- "renormalization group flow."

Definition 1.2. A *free QFT* is a QFT that has particles but no interactions. The classic free theory is a relativistic theory with infinitely many quantized harmonic oscillators.

Free theories are generally not realistic but they are important, as interacting theories can be built from these with perturbation theory. The fact we can do this means the particle interactions are somehow weak (weak coupling), but other theories have strong coupling and cannot be described with perturbation theory.

Units in QFT In QFT, we usually set $c = \hbar = 1$. Since $[c] = [L][T]^{-1}, [\hbar] = [L]^2[M][T]^{-1}$, we find that in natural units,

$$[L] = [T] = [M]^{-1} = [E]^{-1}$$

(where the last equality follows from $E = mc^2$ with $c = 1$). We often just pick one unit, e.g. an energy scale like eV, and describe everything else in terms of powers of that unit. To convert back to metres or seconds, just reinsert the relevant powers of c and $\hbar$.

Example 1.3. The de Broglie wavelength of a particle is given by $\lambda = \hbar/(mc)$. An electron has mass $m_e \simeq 10^6$ eV, so $\lambda_e = 2 \times 10^{-12}$ m.

If a quantity x has dimension $(mass)^d$, we write $[x] = d$, e.g.

$$G = \frac{\hbar c}{M_p^2} \implies [G] = -2.$$

$M_p \approx 10^{19}$ GeV corresponds to the Planck scale, $\lambda_p \sim 10^{-33}$ cm, the length/energy scales where we expect quantum gravitational effects to become relevant. We note that the problems associated with relativising the Schrödinger equation are fixed in QFT by particle creation.

Before we do QFT, let's review classical field theory. In classical particle mechanics, we have a finite number of generalized coordinates $q_a(t)$ (where a is a label telling you which coordinate you're talking about) and in general they are a function of time t. But in field theory, we instead have $\phi_a(x, t)$ where a labels the field in question and x is no longer a coordinate but a label like a.[1]

In our classical field theory, there are now an infinite number of d.o.f., at least one for each x, so position has been demoted from a dynamical variable to a mere label.

Example 1.4. The classical electromagnetic field is a vector field with components $E_i(x, t), B_i(x, t)$ such that $i, j, k \in \{1, 2, 3\}$ label spatial directions. In fact, these six fields are derived from four fields (or rather four field components), the four-potential $A_\mu(x, t) = (\phi, \mathbf{A})$ where $\mu \in \{0, 1, 2, 3\}$.

Then the classical fields are simply related to the four-potential by

$$E_i = \frac{\partial A_i}{\partial t} - \frac{\partial A_0}{\partial x_i} \text{ and } B_i = \frac{1}{2}\epsilon_{ijk}\frac{\partial A_k}{\partial x_j}$$

with ϵ_{ijk} the usual Levi-Civita symbol, and where we have used the Einstein summation convention (repeated indices are summed over).

The dynamics of a field are given by a *Lagrangian* L, which is simply a function of $\phi_a(x, t), \dot{\phi}_a(x, t)$, and $\boldsymbol{\nabla}\phi_a(x, t)$.

Definition 1.5. We write

$$L = \int d^3x \mathcal{L}(\phi_a, \partial_\mu \phi_a),$$

where we call $\mathcal{L}$ the *Lagrangian density*, or by a common abuse of terminology simply the Lagrangian.

Definition 1.6. We may then also define the *action*

$$S \equiv \int_{t_0}^{t_1} L dt = \int d^4x \mathcal{L}(\phi_a, \partial_\mu \phi_a)$$

Let us also note that in these units we have $[S] = 0$ (since it appears alone in an exponent, for instance, e^{iS}) and so since $[d^4x] = -4$ we have $[\mathcal{L}] = 4$.

[1] See for instance Anthony Zee's *QFT in a Nutshell* to see a more detailed description of how we go from discrete to continuous systems.

The dynamical principle of classical field theory is that fields evolve s.t. S is stationary with respect to variations of the field that don't affect the intiial or final values (boundary conditions). A general variation of the fields produces a variation in the action

$$\delta S = \sum_a \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta(\partial_\mu \phi_a) \right\}.$$

With an integration by parts we find that the variation is the action becomes

$$\delta S = \sum_a \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a \right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \delta\phi_a \right\}.$$

The integral of the total derivative term vanishes for any term that decays at spatial ∞ (i.e. $\mathcal{L}$ is reasonably well-behaved) and has $\delta\phi_a(x, t_1) = \delta\phi_a(x, t_0) = 0$. Therefore the boundary term goes away and we find that stationary action implies the *Euler-Lagrange equations*,

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0.$$

Example 1.7. Consider the Klein-Gordon field ϕ, defined

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2.$$

Here $\eta^{\mu\nu}$ is the standard Minkowski metric[2].

To compute the Euler-Lagrange equation for this field theory, we see that

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi \text{ and } \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi.$$

The Euler-Lagrange equations then tell us that

$$\partial_\mu\partial^\mu\phi + m^2\phi = 0,$$

which we call the *Klein-Gordon equation*. It has wave-like solutions $\phi = e^{-ipx}$ with $(-p^2 + m^2)\phi = 0$ (so that $p^2 = m^2$, which is what we expect in units where $c = 1$).

A non-lectured aside on functional derivatives If you're like me, you get a little anxious about taking complicated functional derivatives. The easiest way to manage these is to rewrite the Lagrangian so that all terms precisely match the form of the quantity you are taking the derivative with respect to, and remember that matching indices produce delta functions.

Here's a quick example. To compute $\frac{\partial}{\partial(\partial_\alpha\phi)}\left[\partial_\mu\phi\partial^\mu\phi\right]$, rewrite the term in the brackets as $\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ (since we are deriving with respect to a function of the form $\partial_\alpha\phi$) and make sure to take the derivative with respect to a new index not already in the expression, e.g. $\partial_\alpha\phi$. Then

$$\begin{aligned}
\frac{\partial}{\partial(\partial_\alpha\phi)}\left[\partial_\mu\phi\partial^\mu\phi\right] &= \frac{\partial}{\partial(\partial_\alpha\phi)}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \\
&= \eta^{\mu\nu}(\delta^\alpha_\mu)\partial_\nu\phi + \eta^{\mu\nu}\partial_\mu\phi(\delta^\alpha_\nu) \\
&= 2\partial^\alpha\phi,
\end{aligned}$$

where we have raised the index with $\eta^{\mu\nu}$ and written the final expression in terms of α using the delta function. The functional derivative effectively finds all appearances of the denominator exactly as written, including indices up or down, and replaces them with delta functions so the actual indices match. This is especially important in computing the Euler-Lagrange equations for something like Maxwell theory, where one may have to derive by $\partial_\mu A_\nu$ and both those indices must match exactly to their corresponding appearances in the Lagrangian.

No one ever taught me exactly how to approach such variational problems, so I wanted to record my strategy here for posterity. It may take a little longer than just recognizing that $\frac{\partial}{\partial(\partial_\mu\phi)}\frac{1}{2}\partial_\nu\phi\partial^\nu\phi = \partial^\mu\phi$, but

[2] We use the mostly minus convention here, but honestly the sign conventions are all arbitrary and relativity often uses the other one where time gets the minus sign.

this approach always works and it has the benefit of helping avoid careless mistakes like forgetting the factor of 2 in the example above.

Lecture 2.

Saturday, October 6, 2018

Last time, we derived the Euler-Lagrange equations for Lagrangian densities:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0. \tag{2.1}$$

Example 2.2. Consider the Maxwell Lagrangian,

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A^\mu)^2. \tag{2.3}$$

Plugging into the E-L equations, we find that $\frac{\partial \mathcal{L}}{\partial A_\nu} = 0$ and

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \partial^\mu A^\nu + \eta^{-\mu\nu}\partial_\rho A^\rho. \tag{2.4}$$

Thus E-L tells us that

$$0 = -\partial^2 A^\nu + \partial^\nu(\partial_\rho A^\rho) = -\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu). \tag{2.5}$$

Defining the field strength tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, we can write the E-L equation for Maxwell as the simple

$$0 = \partial_\mu F^{\mu\nu},$$

which written explicitly is equivalent to Maxwell's equations in vacuum (we'll revisit this when we do QED).

The Lagrangians we'll consider here and afterwards are all *local*– in other words, there are no couplings $\phi(\mathbf{x}, t)\phi(\mathbf{y}, t)$ with $\mathbf{x} \neq \mathbf{y}$. There's no reason a priori that our Lagrangians have to take this form, but all physical Lagrangians seem to do so.

Lorentz invariance Consider the Lorentz transformation on a scalar field $\phi(x) \equiv (\phi(x^\mu)$. The coordinates x transform as $x' = \Lambda^{-1}x$ with $\Lambda^\mu_\sigma \eta^{\sigma\tau}\Lambda^\nu_\tau = \eta^{\mu\nu}$. Under Λ, our field transforms as $\phi \to \phi'$ where $\phi'(x) = \phi(x')$. Recall that Lorentz transformations generically include boosts as well as rotations in $\mathbb{R}^3$. As we've discussed in Symmetries, Fields and Particles, Lorentz transformations form a Lie group ($O(3,1)$, or specifically the proper orthochronous Lorentz group) under matrix multipication. They have a representation given on the fields (i.e. a mapping to a set of transformations on the fields which respects the group multiplication law).

For a scalar field, this is $\phi(x) \to \phi(\Lambda^{-1}x)$ (an active transformation). We could have also used a passive transformation where we re-label spacetime points: $\phi(x) \to \phi(\Lambda x)$. It doesn't matter too much– since Lorentz transformations form a group, if Λ is a Lorentz transformation, so is Λ^{-1}. In addition, most of our theories will be well-behaved and Lorentz invariant.

Definition 2.6. *Lorentz invariant* theories are ones where the action S is unchanged by Lorentz transformations.

Example 2.7. Consider the action given by

$$S = \int d^4x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi)\right],$$

where $U(\phi)$ is some potential density. $U \to U'(x) \equiv U(\phi'(x)) = U(x')$ means that U is a scalar field (check this!) and we see that

$$\partial_\mu\phi' = \frac{\partial}{\partial x^\mu}\phi(x') = \frac{\partial x'^\sigma}{\partial x^\mu}\partial'_\sigma\phi(x') = (\Lambda^{-1})^\sigma_\mu \partial'_\sigma\phi(x')$$

where $\partial'_\sigma \equiv \frac{\partial}{\partial x'^\sigma}$. Thus the kinetic term transforms as

$$\mathcal{L}_{kin} \to \mathcal{L}'_{kin} = \eta^{\mu\nu}\partial_\mu\phi'\partial_\nu\phi' = \eta^{\mu\nu}(\Lambda^{-1})^\sigma_\mu(\Lambda^{-1})^\tau_\nu\partial'_\sigma\phi(x')\partial'_\tau\phi(x') = \eta^{\sigma\tau}\partial'_\sigma\phi(x')\partial'_\tau\phi(x') = L_{kin}(x).$$

Thus we see that the action overall transforms as

$$S \to S' = \int d^4x \mathcal{L}(x') = \int d^4x \mathcal{L}(\Lambda^{-1}x).$$

Under a change of variables $u \equiv \Lambda^{-1}x$, we see that $\det(\Lambda^{-1}) = 1$ (from group theory) so the volume element is the same, $d^4y = d^4x$ and therefore

$$S' = \int d^4y \mathcal{L}(y) = S.$$

We conclude that S is invariant under Lorentz transformations.

We also remark that under a LT, a vector field A_μ transforms like $\partial_\mu \phi$, so

$$A'_\mu(x) = (\Lambda^{-1})^\sigma_\mu A_\sigma(\Lambda^{-1}x).$$

This is enough to attempt Q1 from example sheet 1.[3]

Theorem 2.8. *Every continuous symmetry of $\mathcal{L}$ gives rise to a current J^μ which is conserved, $\partial_\mu j^\mu = 0$. Each j^μ has a conserved charge $Q = \int_{\mathbb{R}^3} j^0 d^3x$.*

This conserved charge appears because $\frac{dQ}{dt} = \int_{\mathbb{R}^3} d^3x \partial_0 j^0 = -\int_{\mathbb{R}^3} d^3x \nabla \cdot \mathbf{j} = 0$ by the divergence theorem, assuming $|\mathbf{j}| \to 0$ as $|\mathbf{x}| \to \infty$.

Let us define an infinitesimal variation of a field ϕ, $\phi(x) \to \phi'(x) = \phi(x) + \alpha\Delta\phi(x)$ with α an infinitesimal change. If S is invariant, we call this a *symmetry* of the theory.

Since S is invariant up to adding a total 4-divergence (a total derivative ∂_μ) to the Lagrangian, our symmetry doesn't affect the Euler-Lagrange equations. L transforms as

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha\partial_\mu X^\mu(x), \tag{2.9}$$

and expanding to leading order in α we have

$$\mathcal{L} \to \mathcal{L}(x) + \alpha\frac{\partial\mathcal{L}}{\partial\phi}\Delta\phi + \alpha\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\Delta\phi) + O(\alpha^2). \tag{2.10}$$

We can rewrite this in terms of a total derivative $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right)$ so that

$$\mathcal{L}' = \mathcal{L}(x) + \alpha\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) + \alpha\left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\Delta\phi. \tag{2.11}$$

By Euler-Lagrange, the second term in parentheses vanishes, so we identify the first term in parentheses as none other than $\alpha\partial_\mu X^\mu(x)$ from Eqn. 2.9 (in other words, $\partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu\phi)}\Delta\phi\right) = \partial_\mu X^\mu$) and recognize

$$j^\mu \equiv \frac{\partial L}{\partial(\partial_\mu\phi)}\Delta\phi - X^\mu \tag{2.12}$$

as our conserved current (that is, $\partial_\mu j^\mu = 0$).

Example 2.13. Take a complex scalar field $\psi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x))$. We can then trea ψ, ψ^* as independent variables and write a Lagrangian

$$L = \partial_\mu\psi^*\partial^\mu\psi - V(|\psi|^2).$$

Then we observe that under $\psi \to e^{i\beta}\psi, \psi^* \to e^{-i\beta}\psi^*$, the Lagrangian is invariant. The differential changes are $\Delta\psi = i\psi$ (think of expanding $\psi \to e^{i\beta}\psi$ to leading order) and similarly $\Delta\psi^* = -i\psi^*$ (here we find that $X^\mu = 0$).

We add the currents from ψ, ψ^* to find

$$j^\mu = i\{\psi\partial_\mu\psi^* - \psi^*\partial_\mu\psi\}.$$

This is enough to do questions 2 and 3 on the example sheet.

[3]Copied here for quick reference: Show directly that if $\phi(x)$ satisfies the Klein-Gordon equation, then $\phi(\Lambda^{-1}x)$ also satisfies this equation for any Lorentz transformation Λ.

Example 2.14. Under infinitesimal translation $x^\mu \to x^\mu - \alpha\epsilon^\mu$, we have $\phi(x) \to \phi(x) + \alpha\epsilon^\mu\partial_\mu\phi(x)$ by Taylor expansion (similar for $\partial_\mu\phi$). If the Lagrangian doesn't depend explicitly on x, then $\mathcal{L}(x) \to \mathcal{L}(x) + \alpha\epsilon^\mu\partial_\mu\mathcal{L}(x)$.

Rewriting to match the form $\mathcal{L} + \alpha\partial_\mu X^\mu$, we see that our new Lagrangian takes the form $L(x) + \alpha\epsilon^\nu\partial_\mu(\delta^\mu_\nu L)$. We get one conserved current for each component of ϵ^ν, so that

$$(j^\mu)_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}$$

with $\partial_\mu(j^\mu)_\nu = 0$. We write this as $j^\mu_\nu \equiv T^\mu_\nu$, the energy-momentum tensor.

Definition 2.15. The *energy-momentum tensor* (sometimes *stress-energy tensor*)is the conserved current corresponding to translations in time and space. It takes the form

$$T^{\mu\nu} \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L},$$

where we have raised an index with the Minkowski metric as is conventional. The conserved charges from integrating $\int d^3xT^{0\nu}$ end up being the total energy $E = \int d^3xT^{00}$ and the three components of momentum $P^i = \int d^3xT^{0i}$.

Lecture 3.

Tuesday, October 9, 2018

Last time, we used Noether's theorem to find the stress-energy tensor

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}.$$

To better understand this object, we might ask: what is $T^{\mu\nu}$ for free scalar field theory? Recall the Lagrangian for this theory is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2.$$

Then by explicit computation, the stress-energy tensor is

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}.$$

The energy is given by

$$E = \int d^3x\left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2\right]$$

(from integrating the T^{00} component) and the conserved momentum components are (from T^{0i})

$$p^i = \int d^3x\dot{\phi}(\partial^i\phi).$$

Note that the original Lagrangian terms don't show up here, since $\eta^{\mu\nu}$ is diagonal.

We'll note that $T^{\mu\nu}$ for this theory is symmetric in μ, ν, but a priori it doesn't have to be. If $T^{\mu\nu}$ is not symmetric initially, we can massage it into a symmetric form by adding $\partial_\rho\Gamma^{\rho\mu\nu}$ where $\Gamma^{\mu\rho\nu} = -\Gamma^{\rho\mu\nu}$ (antisymmetric in the first two indices). Then $\partial_\mu\left(\partial_\rho\Gamma^{\rho\mu\nu}\right) = 0$, which means that adding this term will not affect the conservation of $T^{\mu\nu}$. This is sufficient to attempt questions 1-6 of the first examples sheet.

Canonical quantization Here, we'll follow Dirac's lead and attempt to quantize our field theories. Recall that the Hamiltonian formalism also accommodates field theories (as well as our garden-variety QM).

Definition 3.1. We define the *conjugate momentum*

$$\pi(x) \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}}$$

and the *Hamiltonian density* corresponding to a Lagrangian $\mathcal{L}$ is then

$$\mathcal{H} = \pi(x)\dot{\phi}(x) - \mathcal{L}(x).$$

As in classical mechanics, we eliminate $\dot{\phi}$ in favor of π everywhere in $\mathcal{H}$.

Example 3.2. For $\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi)$ (and writing in terms of $\pi(x) = \dot{\phi}(x)$) we get

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi).$$

The Hamiltonian is just the integral of the Hamiltonian density: $H = \int d^3x\mathcal{H}$. Hamlton's equations then yield the equations of motion:

$$\dot{\phi} = \frac{\partial H}{\partial \pi}, \dot{\pi} = -\frac{\partial H}{\partial \phi}.$$

Working these out explicitly for the free theory will give us back the Klein-Gordon equation. Note that H agrees with the total field energy E that we computed above.

There's a slight snag in working in the Hamiltonian formalism– because t is special in our equations, the theory is not manifestly Lorentz invariant (compare to the ∂_μs and variations with respect to $\delta\partial_\mu\phi$ in the Lagrangian formalism). Our original theory was LI, so our new theory is still LI– it just doesn't look LI.

Now let's recall that in quantum mechanics, canonical quantization takes the coordinates q_a and momenta p_a and promotes them to operators. We also replace the Poisson bracket $\{,\}$ with commutators $[,]$. In QM, we had

$$[q_a, p^b] = i\delta_a^b,$$

working in units where $\hbar = 1$. We'll do the same for our fields ϕ_a and the conjugate momenta π_b.

Definition 3.3. A *quantum field* is an operator-valued function of space obeying the commutation relations

$$[\phi_a(\mathbf{x}), \phi_b(\mathbf{y})] = 0 \tag{3.4}$$
$$[\pi_a(\mathbf{x}), \pi_b(\mathbf{y})] = 0 \tag{3.5}$$
$$[\phi_a(\mathbf{x}), \pi^b(\mathbf{y})] = i\delta^3(\mathbf{x}-\mathbf{y})\delta_a^b. \tag{3.6}$$

Note that $\phi_a(x), \pi^b(x)$ don't depend on t, since we are in the Schrödinger picture. All the t dependence sits in the states which evolve by the usual time-dependent Schrödinger equation

$$i\frac{d}{dt}|\psi\rangle = H|\psi\rangle.$$

We have an infinite number of degrees of freedom, at least one for each x in space. For some theories (free theories), the coordinates evolve independently. Free field theories have L quadratic in ϕ_a (plus derivatives thereof), which implies linear equations of motion.

We saw that the simplest free theory leads to the classical Klein-Gordon equation for a real scalar field $\phi(\mathbf{x}, t)$, i.e. $\partial_\mu\partial^\mu\phi + m^2\phi = 0$. To see why this is free, take the Fourier tranform

$$\phi(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}}\phi(\mathbf{p}, t).$$

Then we get the equation of motion

$$\left[\frac{\partial^2}{\partial t^2} + (|\mathbf{p}|^2 + m^2)\right]\phi(\mathbf{p}, t) = 0.$$

We see that the solution is a harmonic oscillator with frequency $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$, so the general solution is a superposition of simple harmonic oscillators each vibrating at different frequencies $\omega_{\mathbf{p}}$. To quantize our field $\phi(\mathbf{x}, t)$, we have to quantize these harmonic oscillators.

Review of 1D harmonic oscillators Recall that the Hamiltonian for the simple harmonic oscillator is

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2q^2,$$

subject to the quantization condition

$$[q, p] = i.$$

It's certainly possible to solve this system by the series method, but the algebraic method is much more elegant by far. Our approach is as follows– we'd like to factor the Hamiltonian, but we know that it doesn't quite work because p and q do not commute. Therefore, we define the following operators:

- The creation or raising operator, $a^\dagger \equiv -\frac{i}{\sqrt{2\omega}} p + \sqrt{\frac{\omega}{2}} q$
- The annihilation or lowering operator, $a \equiv +\frac{i}{\sqrt{2\omega}} p + \sqrt{\frac{\omega}{2}} q$.

Note that we can equivalently solve for p and q in terms of a and $a^\dagger$: $q = \frac{1}{\sqrt{2\omega}}(a + a^\dagger)$ and $p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger)$. Substituting p and q into the quantization condition yields the commutator of $a, a^\dagger$,

$$[a, a^\dagger] = 1.$$

A little more algebra allows us to rewrite the Hamiltonian as

$$H = \frac{1}{2}\omega(aa^\dagger + a^\dagger a) = \omega(a^\dagger a + \frac{1}{2}).$$

Computing the commutators $[H, a]$ and $[H, a^\dagger]$ reveals that

$$[H, a^\dagger] = \omega a^\dagger, [H, a] = -\omega a,$$

which tells us that $a, a^\dagger$ take us between energy eigenstates.[4] More specifically, they take us up and down a ladder of equally spaced energy eigenstates so that if we have one eigenstate with energy E, then we can reach a whole set of eigenstates with energy $\ldots E + 2\omega, E + \omega, E, E - \omega, E - 2\omega, \ldots$.

If we further postulate that the energy is bounded from below, this implies the existence of a ground state $|0\rangle$ such that the lowering operator acting on $|0\rangle$ kills the state: $a|0\rangle = 0$. In our original Hamiltonian, this ground state has energy given by

$$H|0\rangle = \omega(a^\dagger a + \frac{1}{2})|0\rangle = \frac{\omega}{2}|0\rangle,$$

so the ground state energy (or *zero point energy*) of the system is $\omega/2$. For our quantum theory it's really differences in energy which matter more than their absolute values,[5] so we can just as easily write an equivalent Hamiltonian $H = \omega a^\dagger a$ and set the ground state energy to 0.

We only need one state to construct our full ladder of energy eigenstates, and we can do so by passing our equation back to q-space (real coordinates) and further writing $p = i\frac{\partial}{\partial q}$. If we plug these back into the Hamiltonian, knowing that $H|0\rangle = 0$ now, we can solve for the ground state, finding that it is a Gaussian in q with some appropriate variance and normalization. Then we simply need to apply $a^\dagger$ to get all the other states, labeling them as $|n\rangle \equiv (a^\dagger)^n|0\rangle$ with $H|n\rangle = n\omega|n\rangle$. (Here we've disregarded normalization, but it's easy enough to add some scaling factor so that $\langle n|\, m\rangle = \delta_{nm}$.)

That's about all there is to the quantum harmonic oscillator! We have recovered the quantized energy levels and defined operators to move between them. Next time, we'll repeat the same procedure with quantum fields.

[4] Explicitly, consider an eigenstate $|E\rangle$ with energy E. Then $Ha^\dagger|E\rangle = (a^\dagger H + \omega a^\dagger)|E\rangle = (E + \omega)a^\dagger|E\rangle$, so $a^\dagger|E\rangle$ is an eigenstate with energy $E + \omega$. The computation for a is similar.

[5] Remark: gravity is different! Gravity couples directly to energy, not to differences in energy. But in a simple theory like the 1D harmonic oscillator, all we care about is the spacing of the energy levels.