SYMMETRIES, FIELDS, AND PARTICLES

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Lecture 1.

Thursday, October 4, 2018

Today we'll outline the content of this course and motivate it with a few examples. To begin with, symmetry as a principle has led physicists all the way to our current model of physics. This course's content will be almost exclusively mathematical, yet more pragmatic about introducing the necessary tools to apply symmetries to the physical systems we're interested in.

Resources

- Notes (online)
  - Nick Manton's notes (concise, more on geometry of Lie groups)
  - Hugh Osborn's notes (comprehensive, don't cover Cartan classification)
  - Jan Gutowski's notes (classification of Lie algebras). There is actually a second set of notes on an earlier version of the course which can be found here, but I believe the notes referred to in lecture are the first set.
- Books: "Symmetries, Lie Algebras and Representations", Fuchs & Schweigert Ch. 1-7.

Introduction

Definition 1.1. We define a *symmetry* as a transformation of dynamical variables that leaves the form of physical laws invariant.

Example 1.2. A rotation is a transformation, e.g. on $\mathbf{x} \in \mathbb{R}^3$ such that $\mathbf{x}' = M \cdot \mathbf{x} \in \mathbb{R}^3$. There are *orthogonal* matrices which satisfy $MM^T = 1_3$ and also *special* matrices which satisfy $\det M = 1$.

It's also useful for us to define the notion of a group (likely familiar from an intro course on abstract algebra or mathematical methods).

Definition 1.3. A *group* G is a set equipped with a multiplication law (binary operation) obeying

- Closure ($\forall g_1, g_2 \in G, g_1 g_2 \in G$)
- Identity ($\exists e \in G \text{s.t.} \forall g \in G, eg = ge = g$)
- Existence of inverses ($\forall g \in G, \exists g^{-1} \in G$ s.t. $g^{-1}g = gg^{-1} = e$)
- Associativity ($\forall g_1, g_2, g_3 \in G, (g_1 g_2) g_3) = g_1(g_2 g_3)$).

Exercise 1.4. For rotations $G = SO(3)$, the group of 3-dimensional special orthogonal matrices, check that the group axioms apply (SO(3) forms a group).[1]

We also remark that the set may be finite or infinite[2]. A group G is called *abelian* if the multiplication law is commutative ($\forall g_1, g_2 \in G, g_1 g_2 = g_2 g_1$). Otherwise, it is called non-abelian.

We notice that a rotation in $\mathbb{R}^3$ depends continuously on 3 parameters: $\hat{n} \in S^2, \theta \in [0, \pi]$ (with $\hat{n}$ the axis of rotation, θ the angle of rotation). This leads us to introduce the idea of a Lie group.

Definition 1.5. A *Lie group* G is a group which is also a smooth manifold. It's key that the group and manifold structures must be compatible, and so G is (almost) completely determined by the behavior "near" e, i.e. by infinitesimal transformations in a small neighborhood of the identity element e. These correspond to the *tangent vectors* to G at e.

The tangent vectors are local objects which span the tangent space to the manifold at some given point. It turns out that $\forall v_1, v_2 \in T_e(G)$ the tangent space of G, we can define a binary operation $[,] : T_e(G) \times T_e(G) \to T_e(G)$ such that $[,]$ is bilinear, antisymmetric, and obeys the Jacobi identity.

Definition 1.6. The tangent space at the identity equipped with the Lie bracket defines a *Lie algebra* $\mathcal{L}(G)$.

It's a remarkable fact that *all* finite-dimensional semi-simple Lie algebras (over $\mathbb{C}$) can be classified into four infinite families A_n, B_n, C_n, D_n with $n \in \mathbb{N}$, plus five *exceptional cases* E_6, E_7, E_8, G_2, F_4.[3] We call this the *Cartan classification.*

Symmetries in physics In classical physics, (continuous) symmetries give rise to conserved quantities. This is the conclusion of Noether's theorem.

Example 1.7. Rotations in $\mathbb{R}^3$ correspond to conservation of angular momentum, $\mathbf{L} = (L_1, L_2, L_3)$.

In quantum mechanics, we have

- states: vectors in Hilbert space $|\psi\rangle \in \mathcal{H}$
- observables: linear operators $\hat{O} : \mathcal{H} \to \mathcal{H}$ with (generally) non-commutative multiplication.

We recall from previous courses in QM that operators which commute with the Hamiltonian (e.g. $[\hat{H}, \hat{L}_i] = 0, i = 1, 2, 3$) give rise to "quantum conserved quantities."

In fact, we recall that the angular momentum operators are associated to a Lie bracket: $[\hat{L}_i, \hat{L}_j] = i\epsilon_{ijk}\hat{L}_k$. But this is exactly the $\mathcal{L}(SO(3))$ Lie algebra.

Our angular momentum operators often act on finite-dimensional vector spaces, e.g. *electron spin*.

$$|\uparrow\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |\downarrow\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

This corresponds to a two-dimensional *representation* of $\mathcal{L}(SO(3)$, i.e. a set of 2×2 matrices $\Sigma_i, i = 1, 2, 3$ satisfying the same Lie algebra,

$$[\Sigma_i, \Sigma_J] = i\varepsilon_{ijk}\Sigma_k,$$

which is provided by setting $\Sigma_i = \frac{1}{2}\sigma_i$, our old friends the Pauli matrices.

More generally, we should think of a representation as a map e from a Lie group to some space of transformations on a vector space which preserves the Lie bracket, $e([v_1, v_2]) = [e(v_1), e(v_2)]$.

Now suppose we have a rotational symmetry in a quantum system,

$$[\hat{H}, \hat{L}_i] = 0, i = 1, 2, 3.$$

Then the spin states obey $\hat{H}|\uparrow\rangle = E|\uparrow\rangle, \hat{H}|\downarrow\rangle = E'|\downarrow\rangle$, with $E = E'$. More generally, degeneracies in the energy spectrum of quantum systems correspond to irreducible representations of symmetries.

[1] We'll prove this more generally for $SO(n)$ in a few lectures. The answer is in the footnote to Exercise 3.4.

[2] For example, cyclic groups $\mathbb{Z}_n$ (i.e. addition in modular arithmetic) vs. most matrix groups like GL_n.

[3] The exceptional groups have not yet come up in physical phenomena, but they seem to have a mysterious connection to the absence of anomalies in string theory.

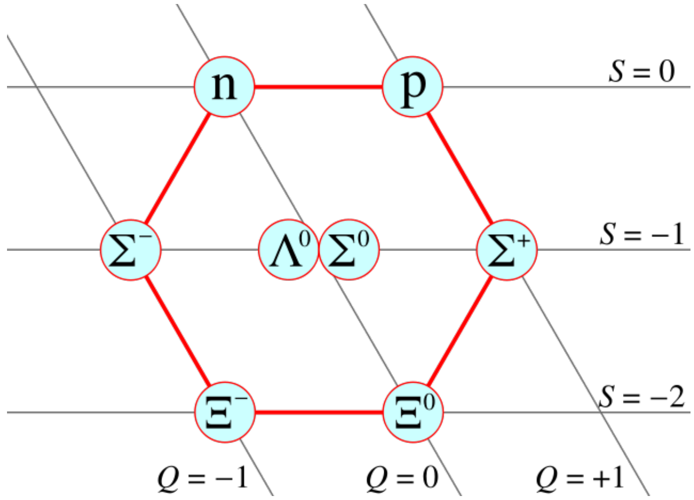

FIGURE 1. The baryon octet. Particles are arranged by their charge along the diagonals and by their strangeness on the horizontal lines.

Example 1.8. We have an approximate $SU(3)$ symmetry for the strong force, with

$$G = SU(3) \equiv \{3 \times 3 \text{ complex matrices } M \text{ with } MM^\dagger = I_3 \text{ and } \det M = 1.\}$$

The spectrum of mesons and baryons are thus defined by the representation of the Lie algebra $\mathcal{L}(SU(3))$. See also the "eightfold way," due to Murray Gell-Mann, who showed that plotting the various mesons and baryons with respect to certain quantum numbers (isospin and hypercharge) gives rise to a very nice picture corresponding to the 8-dimensional representation of the Lie algebra $\mathcal{L}(SU(3))$.

Lecture 2.

Saturday, October 6, 2018

So far, we have discussed global symmetries.

- Spacetime symmetries:
  - Rotation, $SO(3)$.
  - Lorentz transformations, $SO(3,1)$. (Rotations in $\mathbb{R}^3$ plus boosts.)
  - The Poincaré group (not a simple Lie group, so does not fit Cartan classifications)
  - Supersymmetry? (i.e. a symmetry between fermions and bosons, described by "super" Lie algebra)
- Internal symmetries:
  - Electric charge
  - Flavor, $SU(3)$ in hadrons
  - Baryon number

But we also have gauge symmetry.

Definition 2.1. A *gauge symmetry* is a redundancy in our mathematical description of physics. For instance, the phase of the wavefunction in quantum mechanics has no physical meaning:

$$\psi \to e^{i\delta}\psi \tag{2.2}$$

leaves all the physics unchanged ($\delta \in \mathbb{R}$).[4]

Example 2.3. Another gauge symmetry familiar to us is the gauge transformation in electrodynamics,

$$\mathbf{A}(\mathbf{x}) \to \mathbf{A}(\mathbf{x}) + \boldsymbol{\nabla}\chi(\mathbf{x}).$$

[4] However, differences in phase can have significant effects– see for instance the Aharanov-Bohm effect.

By adding the gradient of some scalar function χ of $\mathbf{x}$, this leaves $\mathbf{B} = \nabla \times \mathbf{A}$ unchanged (since $\nabla \times \nabla F = 0$) and so the fields corresponding to the vector potential produce the same physics. Gauge invariance turns out to be key to our ability to quantize the spin-1 field corresponding to the photon.

Example 2.4. Another example (maybe less familiar in the exact details) is the Standard Model of particle physics.[5] The Standard Model is a non-abelian gauge theory based on the Lie group

$$G_{SM} = SU(3) \times SU(2) \times U(1).$$

We started to describe Lie groups last time. Let us repeat the definition here: a Lie group G is a group which is also a (smooth) manifold. Informally, a manifold is a space which locally looks like $\mathbb{R}^n$– for every point on the manifold, there is a smooth map from an open set of $\mathbb{R}^n$ to the manifold (that patch "looks flat"), and these maps are compatible. For cute wordplay reasons, the collection of such maps is known as an atlas.

Sometimes it is useful to consider a manifold as embedded in an ambient space, e.g. S^2 embedded in $\mathbb{R}^3$: $\mathbf{x}(x,y,z) \in \mathbb{R}^3$ such that $x^2 + y^2 + z^2 = r^2, r > 0$.

More generally, we can take the set of all $\mathbf{x} = (x_1, x_2, \ldots, x_{n+m}) \in \mathbb{R}^{n+m}$ such that for a continuous, differentiable set of functions $F^\alpha(\mathbf{x}) : \mathbb{R}^{n+m} \to \mathbb{R}, \alpha = 1, \ldots, m$, a space M is defined by all such $\mathbf{x}$ satisfying $F^\alpha(\mathbf{x}) = 0, \alpha =, 1 \ldots, m$. That is,

$$M = \{\mathbf{x} \in \mathbb{R}^{n+m} : F^\alpha(\mathbf{x}) = 0, \alpha =, 1 \ldots, m\} \tag{2.5}$$

Then the following theorem holds.

Theorem 2.6. *M is a smooth manifold of dimension n if the Jacobian matrix J has rank m, with the Jacobian defined*

$$J_i^\alpha = \frac{\partial F^\alpha}{\partial x_i}.$$

In words, all this says is that M is a manifold if F^α imposes a nice independent set of m constraints on our $n + m$ variables, leaving us with a manifold of dimension n.

Example 2.7. For the sphere S^2, we have $m = 1, n = 2$ and we have the constraint $F^1(\mathbf{x}) = x^2 + y^2 + z^2 - r^2$ for some r. Then the Jacobian is simply

$$J = \left(\frac{\partial F^1}{\partial x}, \frac{\partial F^1}{\partial y}, \frac{\partial F^1}{\partial z}\right) = 2(x, y, z),$$

and this matrix indeed has rank 1 unless $x = y = z = 0$. Therefore we can represent S^2 as a manifold of dimension 2 embedded in $\mathbb{R}^3$.

Group operations (multiplication, inverses) define smooth maps on the manifold. The *dimension* of G, denoted $\dim(G)$, is the dimension of the group manifold $M(G)$. We may introduce coordinates $\{\theta^i\}, i = 1, \ldots, D = \dim(G)$ in some local coordinate patch P containing the identity $e \in G$. Then the group elements depend continuously on $\{\theta^i\}$, such that $g = g(\theta) \in G$ (the manifold structure is compatible with group elements). Set $g(0) = e$.

Thus if we choose two points θ, θ' on the manifold M, group multiplication,

$$g(\theta)g(\theta') = g(\phi) \in G,$$

corresponds to (induces) a smooth map $\phi : G \times G \to G$ which can be expressed in coordinates

$$\phi^i = \phi^i(\theta, \theta'), i = 1, \ldots, D$$

such that $g(0) = e \implies$

$$\phi^i(\theta, 0) = \theta^i, \phi^i(0, \theta') = \theta'^i.$$

We ought to be a little careful that our group multiplication doesn't take us out of the coordinate patch we've defined our coordinates on, but in practice this shouldn't cause us too many problems.

Similarly, group inversion defines a smooth map, $G \to G$. This map can be written as follows:

$$\forall g(\theta) \in G, \exists g^{-1}(\theta) = g(\tilde{\theta}) \in G$$

[5] We'll unpack the Standard Model more in next term's Standard Model class.

such that

$$g(\theta)g(\tilde{\theta}) = g(\tilde{\theta})g(\theta) = e.$$

In coordinates, the map

$$\tilde{\theta}^i = \tilde{\theta}^i(\theta), i = 1, \dots, D$$

is continuous and differentiable.

Example 2.8. Take the Lie group $G = (\mathbb{R}^D, +)$ (Euclidean D-dimensional space with addition as the group operation). Then the map defined by group multiplication is simply

$$\mathbf{x}'' = \mathbf{x} + \mathbf{x}' \forall \mathbf{x}, \mathbf{x}' \in \mathbb{R}^D$$

and similarly the map defined by group inversion is

$$\mathbf{x}^{-1} = -\mathbf{x} \forall \mathbf{x} \in \mathbb{R}^D.$$

This is a bit boring since the group multiplication law is commutative, so we'll next look at some important non-abelian groups– namely, the matrix groups.

Matrix groups Let $\text{Mat}_n(F)$ denote the set of $n \times n$ matrices with entries in a field $F = \mathbb{R}$ or $\mathbb{C}$. These satisfy some of the group axioms– matrix multiplication is closed and associative, and there is an obvious unit element, $e = I_n \in \text{Mat}_n(F)$ (with I_n the $n \times n$ unit matrix). However, $\text{Mat}_n(F)$ is not a (multiplicative) group because not all matrices are invertible (e.g. with $\det M = 0$). (Since it is not a group, it is also not a Lie group, though it does have a manifold structure, that of $\mathbb{R}^{n^2}$.) Thus, we define the *general linear groups.*

Definition 2.9. The general linear group $GL(n, F)$ is the set of matrices defined by

$$GL(n, F) \equiv \{M \in \text{Mat}_n(F) : \det M \neq 0\}. \tag{2.10}$$

Definition 2.11. We also define the *special linear groups* $SL(n, F)$ as follows:

$$SL(n, F) \equiv \{M \in GL(n, F) : \det M = 1.\} \tag{2.12}$$

Here, closure follows from the fact that determinants multiply nicely, $\forall M_1, M_2 \in GL(n, F), \det(M_1 M_2) = \det(M_1)\det(M_2) = 1$ for $SL(n, F)$ (is nonzero for $GL(n, F)$), and existence of inverses follows from the defining condition that $\det M \neq 0$.

It's less obvious that $GL(n, F)$ and $SL(n, F)$ are also Lie groups. In fact, our theorem (Thm. 2.6) applies here: the condition that $\det M = \pm 1$ corresponds to a nice $F(\mathbf{x}) = \det M - 1, \mathbf{x} \in \mathbb{R}^{n^2}$, which is sufficently nice as to define a manifold. The same is true for $SL(n, F)$, so these are indeed Lie groups. Note the dimensions of these sets are as follows.

$$\begin{array}{rr} \dim(GL(n, \mathbb{R})) = n^2 & \dim(GL(n, \mathbb{C})) = 2n^2 \\ \dim(GL(n, \mathbb{R})) = n^2 - 1 & \dim(SL(n, \mathbb{C})) = 2n^2 - 2 \end{array}$$

And now, a bit of extra detail on the dimensions and manifold properties of these Lie groups. In $\text{Mat}_n(F)$, we have our free choice of any numbers we like in F for the n^2 elements of our matrix. It turns out that imposing $\det M \neq 0$ is not too strong a constraint– it eliminates a set of zero measure from the space of possible $n \times n$ matrices, so we have our choice of n^2 real numbers in $GL(n, \mathbb{R})$ and n^2 complex numbers (so $2n^2$ real numbers) in $GL(n, \mathbb{C})$. Requiring that $\det M \neq 0$ means we can equivalently view $GL(n, \mathbb{R})$ as the preimage of an open set in $\mathbb{R}$ (since $\det M : \mathbb{R}^{n^2} \to \mathbb{R}$) under a continuous (and smooth!) map, which is therefore an open set in $\mathbb{R}^{n^2}$. It turns out that any open set in $\mathbb{R}^{n^2}$ is itself a manifold (really, any open subset of a manifold), so $GL(n, \mathbb{R})$ is indeed a manifold.

Note that the situation is easier in $SL(n, F)$, since our theorem then applies with $F = \det M - 1$. The corresponding Jacobian has rank 1 unless all the matrix elements vanish identically, so $SL(n, F)$ is a manifold Imposing the restriction that $\det M = 1$ is now a stronger algebraic condition– it reduces our choice of values by 1, since if we have picked $n^2 - 1$ values of the matrix, the last value is completely determined by $\det M = 1$. Thus the dimension of $SL(n, \mathbb{R})$ is $n^2 - 1$. Since we get to pick $n^2 - 1$ complex numbers in $SL(n, \mathbb{C})$ (equivalently there are two real constraints, one on the real components and one on the imaginary ones), that amounts to $2(n^2 - 1) = 2n^2 - 2$ real numbers. Hence, dimension $2n^2 - 2$.

Definition 2.13. A *subgroup* H of a group G is a subset ($H \subseteq G$) which is also a group. We write it as $H \leq G$. If H is also a smooth submanifold of G, we call H a *Lie subgroup* of G.

Lecture 3.

Tuesday, October 9, 2018

Having introduced the matrix groups, we'll next discuss some important subgroups of $GL(n, \mathbb{R})$. First, the *orthogonal groups.*

Definition 3.1. Orthogonal groups $O(n)$ are the matrix groups which preserve the Euclidean inner product,

$$O(n) = \{M \in GL(n, \mathbb{R}) : M^T M = I_N\}. \tag{3.2}$$

Their elements correspond to orthogonal transformations, so that for $\mathbf{v} \in \mathbb{R}^n$, an orthogonal matrix M acts on $\mathbf{v}$ by matrix multiplication,

$$\mathbf{v}' = M \cdot \mathbf{v}$$

and so in particular

$$|\mathbf{v}'|^2 = \mathbf{v}'^T \cdot \mathbf{v}' = \mathbf{v}^T \cdot M^T M \cdot \mathbf{v} = \mathbf{v}^T \cdot \mathbf{v} = |\mathbf{v}|^2.$$

It also follows that $\forall M \in O(n), \det(M^T M) = \det(M)^2 = \det(I_n) = 1 \implies \det M = \pm 1.$

The orthogonal group $O(n)$ has two connected components.

Definition 3.3. The *special orthogonal groups* $SO(n)$ are the subset of orthogonal groups which also preserve orientation (i.e. no reflections).

$$SO(n) \equiv \{M \in O(n) : \det M = 1\}.$$

That is, elements of $SO(n)$ preserve the sign of the volume element in $\mathbb{R}^n$,

$$\Omega = \epsilon^{i_1 i_2 \ldots i_n} v_1^{i_1} v_2^{i_2} \ldots v_n^{i_n}.$$

Exercise 3.4. Check the group axioms for $SO(n)$.[6] Show that $\dim(O(n)) = \dim(SO(n)) = \frac{1}{2}n(n-1)$.[7]

Orthogonal matrices have some nice properties. Let $M \in O(n)$ be an orthogonal matrix and suppose that $\mathbf{v}_\lambda$ is an eigenvector of M with eigenvalue λ. Then the following is true:

(a) If λ is an eigenvalue, then λ^* is also an eigenvalue (eigenvalues of M come in complex conjugate pairs).
(b) $|\lambda|^2 = 1$.

The proof is as follows:

(a) $M \cdot \mathbf{v}_\lambda = \lambda \mathbf{v}_\lambda \implies M \cdot \mathbf{v}_\lambda^* = \lambda^* \mathbf{v}_\lambda^*$ (since M is a real matrix).[8]
(b) For any complex vector $\mathbf{v}$, we have

$$(M \cdot \mathbf{v}^*)^T \cdot M \cdot \mathbf{v} = \mathbf{v}^\dagger \cdot M^T M \cdot \mathbf{v} = \mathbf{v}^\dagger \cdot \mathbf{v}.$$

Now if $\mathbf{v} = \mathbf{v}_\lambda$, then

$$(M \cdot \mathbf{v}_\lambda^*)^T \cdot M \cdot \mathbf{v}_\lambda = (\lambda^* \mathbf{v}_\lambda^*)^T \cdot (\lambda \mathbf{v}_\lambda) = |\lambda|^2 \mathbf{v}^\dagger \cdot \mathbf{v}.$$

By comparison to the earlier expression we see that $|\lambda|^2 = 1$. ☒

[6] As usual, we need to check closure and inverses. The identity matrix I satisfies $I^T I = I$ and $\det I = 1$, and associativity follows from standard matrix multiplication. Inverses: if $M \in SO(n)$, then M^{-1} is defined by $MM^{-1} = I$. But $\det(MM^{-1}) = \det(M)\det(M^{-1}) = (1)\det(M^{-1}) = \det I = 1$, so $\det(M^{-1}) = 1$. We also check that the inverse of an orthogonal matrix is also orthogonal: $MM^{-1} = I$, so $(M^{-1})^T(M^T) = (M^{-1})^T M^{-1} = I^T = I$. Closure: $\forall M, N \in SO(n), \det(MN) = \det(M)\det(N) = (1)(1) = 1$ and $(MN)^T(MN) = N^T M^T MN = I$, so $MN \in SO(n)$. ☒

[7] This can be seen by writing a matrix $M \in SO(n)$ as a row of n column vectors $(\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n})$. Then the condition that $M^T M = 1$ is equivalent to $\begin{pmatrix} \mathbf{x_1}\cdot\mathbf{x_1} & \mathbf{x_1}\cdot\mathbf{x_2} & \ldots & \mathbf{x_1}\cdot\mathbf{x_n} \\ \mathbf{x_2}\cdot\mathbf{x_1} & \mathbf{x_2}\cdot\mathbf{x_2} & \ldots & \mathbf{x_2}\cdot\mathbf{x_n} \\ \vdots & & & \\ \mathbf{x_n}\cdot\mathbf{x_1} & \ldots & \ldots & \mathbf{x_n}\cdot\mathbf{x_n} \end{pmatrix} = I_n$. We see that by the symmetry of the explicit form of $M^T M$, we get $1 + 2 + 3 + \ldots + n = n(n+1)/2$ independent constraints on the n^2 entries of M. Applying our theorem, we find that the resulting manifold has dimension $n^2 - n(n+1)/2 = n(n-1)/2$.

[8] This is generally true of real matrices with complex eigenvalues– it's not specific to orthogonal matrices.

Example 3.5. For the group $G = SO(2)$, $M \in SO(2) \implies M$ has eigenvalues

$$\lambda = e^{i\theta}, e^{-i\theta}$$

for some $\theta \in \mathbb{R}, \theta \sim \theta + 2\pi$ (identified up to a phase of 2π). A group element may be written explicitly as

$$M = M(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

which is uniquely specified by a rotation angle θ. Therefore the group manifold of $SO(2)$ is $M(SO(2) \cong S^1$, the circle.

Example 3.6. For the group $G = SO(3)$, we have instead $M \in SO(3) \implies M$ has eigenvalues

$$\lambda = e^{i\theta}, e^{-i\theta}, 1$$

for $\theta \in \mathbb{R}, \theta \sim \theta + 2\pi$. The normalized eigenvector for $\lambda = 1$, $\hat{\mathbf{n}} \in \mathbb{R}^3$, specifies the axis of rotation.