ADVANCED QUANTUM FIELD THEORY

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These notes were taken for the *Advanced Quantum Field Theory* course taught by Matthew Wingate at the University of Cambridge as part of the Mathematical Tripos Part III in Michaelmas Term 2018. I live-TEXed them using Overleaf, and as such there may be typos; please send questions, comments, complaints, and corrections to itel2@cam.ac.uk. Many thanks to Arun Debray for the LATEX template for these lecture notes: as of the time of writing, you can find him at https://web.ma.utexas.edu/users/a.debray/.

Contents

Lecture 1.

Saturday, January 19, 2019

Note. There will not be official typed course notes, but there will be scanned handwritten notes (which I will link here as they become available). Previous lecturers' notes are currently online (Skinner, Osborn).

Today we introduce path integrals in a QFT context. There are some benefits to working with path integrals– some computations are simplified or more straightforward, and Lorentz invariance is manifest (unlike in the canonical formalism).

Path integrals in quantum mechanics Rather than trying to tackle the full machinery of QFT, we'll start with $0+1$ dimensional non-relativistic quantum mechanics (cf. Osborn § 1.2. We'll set $\hbar = 1$ for now, though we may restore it later in order to make arguments when $\hbar \ll 1$ in a classical limit. In these units,

$$[E][t] = [\hbar] = [p][x]$$

using uncertainty relations.

Let us consider a Hamiltonian in 1 spatial dimension,

$$\hat{H} = H(\hat{x}, \hat{p}) \quad \text{with } [\hat{x}, \hat{p}] = i.$$

We'll further assume for simplicity that the Hamiltonian has a kinetic term and a potential based only on position,

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}).$$

Now the Schrödinger equation takes the form

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle \tag{1.1}$$

which has formal solution

$$|\psi(t)\rangle = e^{-i\hat{H}t}|\psi(0)\rangle . \tag{1.2}$$

Let us consider some position eigenstates $|x,t\rangle$ such that

$$\hat{x}(t)|x,t\rangle = x|x,t\rangle , \quad x \in \mathbb{R},$$

where these states obey some normalization

$$\langle x',t|\, x,t\rangle = \delta(x'-x).$$

In the Schrödinger picture, states depend on time, while operators are constant. In terms of fixed (time-independent) eigenstates $\{|x\rangle\}$ of the position operator $\hat{x}$, we may write the wavefunction as

$$\psi(x,t) = \langle x|\, \psi(t)\rangle , \tag{1.3}$$

so that applying the Hamiltonian to the wavefunction $\psi(x,t)$ yields

$$\hat{H}\psi(x,t) = (-\frac{1}{2m}\frac{\partial^2}{\partial x^2} + V(x))\psi(x,t). \tag{1.4}$$

This is the traditional presentation of quantum mechanics and the wavefunction. In the path integral formalism, we'll consider a more particle-like treatment, where we express time evolution as a sum over all trajectories (meeting some boundary conditions) appropriately weighted (by an action).

Recall that our formal solution 1.2 tells us what $|\psi(t)\rangle$ is– we can therefore rewrite the wavefunction as

$$\psi(x,t)\, \langle x|\, e^{-i\hat{H}t}|\psi(0)\rangle . \tag{1.5}$$

By inserting a complete set of (position eigen)states, $1 = \int dx_0 |x_0\rangle \langle x_0|$, we get

$$\begin{aligned}\psi(x,t) &= \int dx_0 \langle x|\, e^{-i\hat{H}t}|x_0\rangle \langle x_0|\, \psi(0)\rangle \\ &= \int dx_0\, K(x,x_0;t)\psi(x_0,0),\end{aligned}$$

where we have defined $K(x,x_0;t) \equiv \langle x|\, e^{-i\hat{H}t}|x_0\rangle$. Let us further consider time evolution in discrete steps, with $0 \equiv t_0 < t_1 < t_2 < \ldots < t_n < t_{n+1} \equiv T$ so that

$$e^{-i\hat{H}T} = e^{-i\hat{H}(t_{n+1}-t_n)} \ldots e^{-i\hat{H}(t_1-t_0)}.$$

As before, we insert complete sets of states, finding that our generic time evolution from any x_0 to an x of our choosing:

$$K(x,x_0;T) = \int \left[\prod_{r=1}^{n} dx_r \langle x_{r+1}|\, e^{-i\hat{H}(t_{r+1}-t_r)}|x_r\rangle\right] \langle x_1|\, e^{-i\hat{H}t_1}|x_0\rangle . \tag{1.6}$$

That is, we integrate over all intermediate positions x_r for each t_r. Naturally, dx_{n+1} must be x.

Let's look at the free theory first to understand what we've done, $V(x) = 0$. Now this weird K_0 object we've defined takes the form

$$K_0(x,x';t) = \langle x|\, e^{-i\frac{\hat{p}^2}{2m}t}|x'\rangle . \tag{1.7}$$

We'll instead insert a complete set of momentum eigenstates $|p\rangle$ with the normalization

$$\int \frac{dp}{2\pi}|p\rangle \langle p| = 1,$$

recalling that $\langle x|\, p\rangle = e^{ipx}$ are simply plane waves. Then

$$K_0(x,x';t) = \int \frac{dp}{2\pi} e^{-ip^2t/2m} e^{ip(x-x')}.$$

We can compute this– completing the square with a change of variables to $p' = p - \frac{m(x-x')}{t}$, K_0 becomes a gaussian integral,

$$K_0(x, x'; t) = e^{im(x-x')^2/2t} \int_{-\infty}^{\infty} \exp\left[-\frac{i(p')^2 t}{2m}\right]$$
$$= e^{im(x-x')^2/2t} \sqrt{\frac{m}{2\pi i t}}.$$

Note that as $t \to 0$,[1]

$$\lim_{t\to 0} K_0(x, x'; t) = \delta(x - x'),$$

which agrees with the fact that $\langle x' | x \rangle = \delta(x - x')$.

For $V(\hat{x}) \neq 0$, we still need small time steps but since operators generically do not compute, exponentials don't add in the usual way:

$$e^{\hat{A}} e^{\hat{B}} = \exp(\hat{A} + \hat{B} + \frac{1}{2}[\hat{A}, \hat{B}] + \ldots) \neq e^{\hat{A}+\hat{B}} \quad \text{when } [\hat{A}, \hat{B}] \neq 0.$$

This is the Baker-Campbell-Hausdorff (BCH) formula. However, for small ϵ we can write

$$e^{\epsilon \hat{A}} e^{\epsilon \hat{B}} = \exp(\epsilon \hat{A} + \epsilon \hat{B} + O(\epsilon^2)),$$

or equivalently

$$e^{\epsilon(\hat{A}+\hat{B})} = e^{\epsilon \hat{A}} e^{\epsilon \hat{B}} (1 + O(\epsilon^2)),$$

so we conclude that

$$e^{\hat{A}+\hat{B}} = \lim_{n\to\infty} \left(e^{\hat{A}/n} e^{\hat{B}/n}\right)^n.$$

Suppose now that we divide our time into n time steps so that $t_r - 1 - t_r = \delta t$, with $T = n\delta t$. Then one of the intermediate time evolution steps looks like

$$\langle x_{r+1} | e^{-i\hat{H}\delta t} | x_r \rangle = e^{-iV(x_r)\delta t} \langle x_{r+1} | e^{-i\hat{p}^2 \delta t/2m} | x_r \rangle$$
$$= \sqrt{\frac{m}{2\pi i \delta t}} \exp\left[\frac{i}{2} m \left(\frac{x_{r+1} - x_r}{\delta t}\right)^2 \delta t - iV(x_r)\delta t\right].$$

Taking $T = n\delta t$, we find that the entire K becomes

$$K(x, x_0; T) = \int \left(\prod_{r=1}^{n} dx_r\right) \left(\frac{m}{2\pi i \delta t}\right)^{\frac{n+1}{2}} \exp\left(i \sum_{r=0}^{n} \left[\frac{m}{2}\left(\frac{x_{r+1} - x_r}{\delta t}\right)^2 - V(x_r)\right] \delta t\right). \tag{1.8}$$

Now we take the limit as $n \to \infty, \delta t \to 0$ with T fixed. Then the argument of the exponential becomes

$$\int_0^T \frac{m}{2} \dot{x}^2 - V(x)]dt = \int_0^t L dt, \tag{1.9}$$

where $L(x, \dot{x})$ is the classical Lagrangian and this integral is nothing more than the action. We conclude that

$$K(x, x_0; T) = \langle x | e^{-i\hat{H}t} | x_0 \rangle = \int \mathcal{D}x \, e^{iS[x]}, \tag{1.10}$$

where $S[x] = \int_0^T L(x, \dot{x}) dt$ is the classical action and the $\mathcal{D}$ conceals all our sins (the continuum limit) in a cute integration measure. Note that the action has units of energy $\times$ time, so if we restore $\hbar$, we see that this integral becomes

$$K(x, x_0; T) = \int \mathcal{D}x \, e^{iS/\hbar}, \tag{1.11}$$

and in the $\hbar \to 0$ limit (the classical limit), the integral is dominated by paths x which minimize the classical action, and we recognize this as Hamilton's principle from classical mechanics.

[1] This was more obvious from the original expression for K_0 where $K_0(x, x'; t = 0) = \int \frac{dp}{2\pi} e^{ip(x-x')}$.

Lecture 2.

Tuesday, January 22, 2019

Last time, we introduced the path integral in quantum mechanics, and we said it took the form

$$\langle x| e^{-iHt/\hbar} |x_0\rangle = \int \mathcal{D}x e^{iS[x]/\hbar}. \tag{2.1}$$

Let us consider now a "rotation" to imaginary time, $t \to -i\tau$ (Wick rotation). Then our path integral becomes

$$\langle x| e^{-H\tau/\hbar} |x_0\rangle = \int \mathcal{D}x e^{-S[x]/\hbar}. \tag{2.2}$$

Working with a real exponent has some benefits– the convergence of the integral is more obvious, and in the $\hbar \to 0$ limit we expect the integral to be dominated by the classical path x which minimizes the action $S[x]$.

We can make the observation that 1D quantum mechanics is like a $0+1$D quantum field theory– the field is

$$x(t) : \mathbb{R} \to \mathbb{R}.$$

In fact, 3D quantum mechanics is also like a $0+1$D QFT, where the field is now

$$\mathbf{x}(t) : \mathbb{R} \to \mathbb{R}^3.$$

Given a single spacetime label t, a QM theory gives us a real scalar in $\mathbb{R}$ or a vector in $\mathbb{R}^3$– cf. Srednicki Ch. 1. There are different approaches to quantization, but in the second quantization formalism we demote position $\mathbf{x}$ from an operator to a label on a spacetime point $(\mathbf{x}, t)$. Therefore QFT in $3+1$ dimensions has e.g. a scalar field ϕ which is a map

$$\phi : \mathbb{R}^{1,3} \to \mathbb{R}.$$

Path integral methods Let's begin with the simplest possible case, QFT in zero dimensions.[2] All of spacetime is a single point p,[3] and our (real scalar) field ϕ is a map $\phi : \{p\} \to \mathbb{R}$.

Using our imaginary time (Euclidean signature) convention for the path integral, we write

$$Z = \int_{\mathbb{R}} d\phi\, e^{-S[\phi]/\hbar}. \tag{2.3}$$

We'll take our action $S[\phi]$ to be polynomial in ϕ, with highest power even.

As in statistical field theory, we are interested in correlation functions and expectation values. Given a function $f(\phi)$, we might like to compute the expectation value

$$\langle f \rangle = \frac{1}{Z} \int d\phi\, f(\phi) e^{-S[\phi]/\hbar}. \tag{2.4}$$

For this to have a chance of convergence, f should not grow too rapidly as $|\phi| \to \infty$. Usually the functions we are interested in are polynomial in ϕ.

Free field theory Suppose we have N real scalar fields $\phi_a, a = 1, \ldots, N$. We can compactly write this as a single field

$$\phi : \{p\} \to \mathbb{R}^N, \tag{2.5}$$

and we'd like to compute the integral

$$Z_0 = \int d^N \phi e^{-S[\phi]/\hbar}. \tag{2.6}$$

Now, a free theory simply means that the action is quadratic in our fields. A priori it could have included kinetic terms, but since we are in zero dimensions, there are no derivatives to take and therefore no kinetic terms in this model. Then we can write our action as

$$S(\phi) = \frac{1}{2}\mathcal{M}_{ab}\phi_a\phi_b = \frac{1}{2}\phi^T \mathcal{M}\phi, \tag{2.7}$$

[2]Cf. Skinner Ch. 2, Srednicki §8,9.

[3]If you're reading my SUSY notes, you should be getting déjà vu right about now.

where $\mathcal{M}$ is an $N \times N$ symmetric matrix with $\det M > 0$. So our action could include terms like $\frac{1}{2}\phi_1^2$ and $\frac{5}{2}\phi_1\phi_4$. Since $\mathcal{M}$ is symmetric, we can diagonalize it as

$$\mathcal{M} = P\Lambda P^T$$

for some orthogonal matrix P. But equivalently we could just redefine our fields to some new fields $\phi' = P^T\phi$ so that

$$S(\phi) = \frac{1}{2}\phi'^T\Lambda\phi' = \frac{1}{2}\sum_{i=1}^N \lambda_i(\phi_i')^2,$$

where λ_i are the eigenvalues of $\mathcal{M}$. Since P is orthogonal, $\det P = 1 \implies d^N\phi = (\det P)d^N\phi' = d^N\phi'$, so our path integral separates into N Gaussian integrals of the form

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{\lambda}{2\hbar}x^2} = \sqrt{\frac{2\pi\hbar}{\lambda}}. \tag{2.8}$$

Thus

$$Z_0 = \int d^N\phi\, e^{-\frac{1}{2\hbar}\phi^T\mathcal{M}\phi} = \prod_{i=1}^N d\phi_i\, e^{-\frac{1}{2\hbar}\lambda_i(\phi_i)^2} = \frac{(2\pi\hbar)^{N/2}}{\sqrt{\det\mathcal{M}}}. \tag{2.9}$$

We can now introduce a source term J, modifying our action to

$$S(\phi) = \frac{1}{2}\phi^T\mathcal{M}\phi + J\cdot\phi. \tag{2.10}$$

If we complete the square and make a change of variables $\tilde\phi = \phi + \mathcal{M}^{-1}J$, we find that the new path integral with a source is

$$\begin{aligned}
Z_0[J] &= \int d^N\phi\, \exp\left[-\frac{1}{2\hbar}\phi^T\mathcal{M}\phi - \frac{1}{\hbar}J\cdot\phi\right] \\
&= \exp\left(\frac{1}{2\hbar}J^T\mathcal{M}^{-1}J\right)\int d^N\tilde\phi\, e^{-\frac{1}{2\hbar}\tilde\phi^T\mathcal{M}\tilde\phi} \\
&= Z_0\exp\left(\frac{1}{2\hbar}J^T\mathcal{M}^{-1}J\right).
\end{aligned}$$

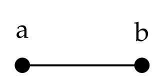
We see that $\frac{\partial}{\partial J}$ derivatives will bring down ϕs, which will allow us to compute correlation functions just like we did in statistical physics with the partition function.

Example 2.11. What is the value of the correlation function $\langle\phi_a\phi_b\rangle$ in this theory? We can compute it directly:

$$\begin{aligned}
\langle\phi_a\phi_b\rangle &= \frac{1}{Z_0}\int d^N\phi\,\phi_a\phi_b\exp\left[-\frac{1}{2\hbar}\phi^T\mathcal{M}\phi - \frac{1}{\hbar}J\cdot\phi\right]\Bigg|_{J=0} \\
&= \frac{1}{Z_0}\int d^N\phi\left(-\hbar\frac{\partial}{\partial J_a}\right)\left(-\hbar\frac{\partial}{\partial J_b}\right)\exp\left[-\frac{1}{2\hbar}\phi^T\mathcal{M}\phi - \frac{1}{\hbar}J\cdot\phi\right]\Bigg|_{J=0} \\
&= (-\hbar)^2\frac{\partial}{\partial J_a}\frac{\partial}{\partial J_b}\exp\left[\frac{1}{2\hbar}J^T\mathcal{M}^{-1}J\right]\Bigg|_{J=0} \\
&= \hbar(\mathcal{M}^{-1})_{ab}.
\end{aligned}$$

Note that the first J derivative brings down an $\mathcal{M}^{-1}J$ (so our expression is of the form $\mathcal{M}^{-1}J\exp(J^T\mathcal{M}^{-1}J)$), and when we take the second J derivative, we will get two terms, one of the form $\mathcal{M}^{-1}\exp(\ldots)$ and another of the form $(\mathcal{M}^{-1}J)^2\exp(\ldots)$. The second term is zero when we set $J = 0$, and the exponential becomes 1 in both cases, so we are just left with $\mathcal{M}^{-1}$.

What we have calculated is a two-point function, otherwise known as a propagator (though it's a bit silly to call this a propagator when the spacetime is just a single point). We can associate a Feynman diagram to this process:



There is another method we can use to compute propagators (cf. Osborn §1.3):

$$\begin{aligned}\mathcal{M}_{ca}\langle\phi_a\phi_b\rangle &= \frac{1}{Z_0}\int d^N\phi\,\mathcal{M}_{ca}\phi_a\phi_b\exp\left[-\frac{1}{2\hbar}\phi^T\mathcal{M}\phi\right]\\ &= -\frac{\hbar}{Z_0}\int d^N\phi\,\phi_b\frac{\partial}{\partial\phi_c}\exp\left[-\frac{1}{2\hbar}\phi^T\mathcal{M}\phi\right]\\ &= \frac{\hbar}{Z_0}\int d^N\phi\,\frac{\partial\phi_b}{\partial\phi_c}\exp\left[-\frac{1}{2\hbar}\phi^T\mathcal{M}\phi\right]\\ &= \hbar\delta_{bc}\implies\langle\phi_a\phi_b\rangle=\hbar(\mathcal{M}^{-1})_{ab}.\end{aligned}$$

In going from the second to the third line, we have integrated by parts to move the $\frac{\partial}{\partial\phi_c}$ to ϕ_b, and then recognized the remaining integral as Z_0.

More generally, let $l(\phi)=l\cdot\phi=\sum_{a=1}^N l_a\phi_a(\neq 0)$ be a linear function of ϕ, with $l_a\in\mathbb{R}$. Then the expected value $\langle l_a(\phi)\dots l_p(\phi)\rangle$ is given by

$$\langle l_a(\phi)\dots l_p(\phi)\rangle=(-\hbar)^p\prod_{i=1}^p\left(l_i\frac{\partial}{\partial J_i}\right)\frac{Z_0[J]}{Z_0}\bigg|_{J=0}.$$

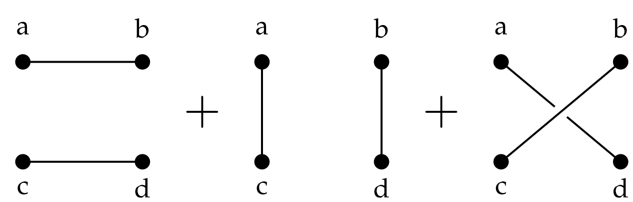
Notice that if we play this game for an odd number of J_i derivatives, all our terms will be of the form $J^p\exp(\dots)$ where p is odd. When we set $J=0$, all these terms therefore vanish, which tells us that $\left\langle\phi_{a_1}\dots\phi_{a_p}\right\rangle=0$ for n odd. If we compute it for $p=2k, k\in\mathbb{N}$, the terms that survive setting $J=0$ will have k factors of $\mathcal{M}^{-1}$.

Example 2.12. What is the value of the four-point function $\langle\phi_a\phi_b\phi_c\phi_d\rangle$ in free field theory? It is simply

$$\langle\phi_a\phi_b\phi_c\phi_d\rangle=\hbar^2\left[(\mathcal{M}^{-1})_{ab}(\mathcal{M}^{-1})_{cd}+(\mathcal{M}^{-1})_{ac}(\mathcal{M}^{-1})_{bd}+(\mathcal{M}^{-1})_{ad}(\mathcal{M}^{-1})_{bc}\right].$$

Though we haven't said it, this is effectively a toy version of Wick's theorem– we are taking contractions of the fields using $(\mathcal{M}^{-1})$s as propagators.

We can depict these contractions as connecting some $2k$ dots pairwise with lines using a simplified Feynman diagram notation:



In general, the number of distinct ways we can pair $2k$ elements is

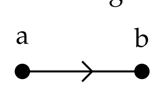
$$\frac{(2k)!}{2^k k!}.$$

The logic here is that we could take all $(2k)!$ permutations of the $2k$ elements, and then take neighboring pairs, e.g. if our elements are $\{a,b,c,d,e,f\}$, one set of pairs is

$$abdcfe\to ab|dc|fe.$$

The order of the 2 elements in each of the k pairs doesn't matter ($ab|dc=ba|dc$), so we've overcounted by a factor of 2^k, and the order of all the k pairs also doesn't matter ($ab|dc=dc|ab$), so we divide by another factor of $k!$ to get the final result.

Example 2.13. One last example– if our free fields are instead complex, $\phi:\{p\}\to\mathbb{C}$, then $\mathcal{M}$ is hermitian. Therefore $(\mathcal{M}^{-1})$ will in general not be symmetric, and so the order of the indices matters. That is, $\langle\phi_a\phi_b^*\rangle=\hbar(\mathcal{M}^{-1})_{ab}$. Then the associated Feynman diagram has an arrow to indicate direction:



Lecture 3.

Thursday, January 24, 2019

Today, we will continue our exploration of zero-dimensional path integrals in quantum field theory.

Interacting theory Let us consider a single real scalar field $\phi : \{\text{point}\} \to \mathbb{R}$. We choose the action

$$S(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4. \tag{3.1}$$

We'll take $\lambda > 0$ for stability and $m^2 > 0$ such that $\min(S)$ lies at $\phi = 0$ so that we can easily expand around the minimum of S.

The path integral is then

$$Z = \int d\phi \exp\left[-\frac{1}{\hbar}\left(\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right)\right]. \tag{3.2}$$

This will be equivalent to expanding about $\hbar = 0$ (semi-classical limit). We can obviously open up the exponential and rewrite as a series in ϕ and $\hbar$,

$$\begin{aligned} Z &= \int d\phi e^{-\frac{m^2\phi^2}{2\hbar}} \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{\lambda}{\hbar\, 4!}\right)^n \phi^{4n} \\ &= \frac{\sqrt{2\hbar}}{m} \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{\hbar\lambda}{4!m^4}\right)^n \cdot 2^{2n} \int_0^\infty dx e^{-x} x^{2n+\frac{1}{2}-1}, \end{aligned}$$

where we have performed a change of variables $x = \frac{m^2\phi^2}{2\hbar}$. This integral is in fact just a gamma function,

$$\Gamma(2n + \frac{1}{2}) = \frac{(4n)!\sqrt{\pi}}{4^{2n}(2n)!}.$$

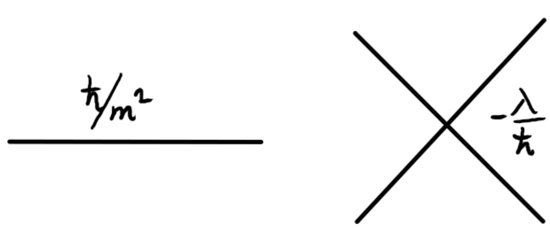
Thus our path integral computation using the gamma function is

$$Z = \frac{\sqrt{2\pi\hbar}}{m} \sum_{n=0}^{N} \left(-\frac{\lambda\hbar}{m^4}\right)^n \underbrace{\frac{1}{(4!)^n n!}}_{(1)} \underbrace{\frac{(4n)!}{2^{2n}(2n)!}}_{(2)}. \tag{3.3}$$

Note that from Stirling's approximation, $n! \approx e^{n\log n}$, Thus these two combinatorial-looking terms scale roughly as $e^{n\log n} \approx n!$. The factorial growth of the coefficients means that this path integral actually has zero radius of convergence. This is an asymptotic series– it looks like it is getting better and better, and then everything goes to hell.[4]

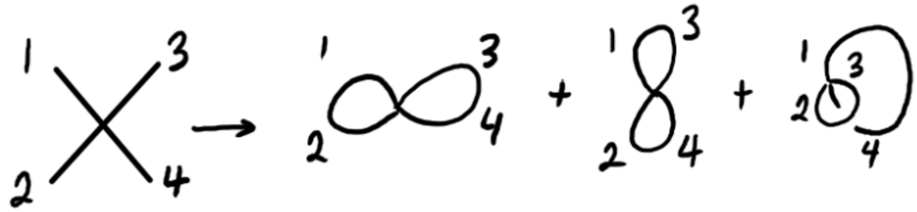
In practice the "true" function can differ from the truncated series by some transcendental function which might be small. Cf. Skinner Ch. 2 for more discussion of asymptotic series.

Note that term (1) in the path integral series expansion 3.3 comes from expanding the $\frac{\lambda]4!\hbar}{\phi}^4$ term in the exponent, while term (2) is the number of ways of joining $4n$ elements in distinct pairs (compare our discussion at the end of the previous lecture). We can associate some Feynman diagrams to this– a propagator and a four-point vertex.



[4] What we mean by an asymptotic series is that it converges not in the limit as the number of terms in the power series gets very large but rather as the expansion parameter gets very small.

Note also that Z has no ϕ dependence, meaning that the Feynman diagrams have no external legs. Let D_n be the set of *labelled* vacuum diagrams with n vertices, so that D_1 is the following set of diagrams, with $|D_1| = 3$.



Then let G_n be the group which permutes each of the 4 fields at each vertex ($(S_4)^n$) and also permutes the n vertices (S_n). The size of this group is

$$|G_n| = |S_4|^n |S_n| = (4!)^n n!.$$

We therefore recognize that

$$\begin{aligned} \frac{Z}{Z_0} &= \sum_{n=0}^{N} \left(-\frac{\lambda\hbar}{m^4}\right)^n \frac{|D_n|}{|G_n|} \\ &= 1 - \frac{\hbar\lambda}{8m^4} + \frac{35}{384}\frac{\hbar^2\lambda^2}{m^8} + \dots \end{aligned}$$

with $Z_0 = \frac{\sqrt{2\pi\hbar}}{m}$. Physically, we can consider $\frac{|D_n|}{|G_n|}$ to be the sum over topologically distinct graphs divided by a symmetry factor. Equivalently, we write

$$\frac{|D_n|}{|G_n|} = \sum_\Gamma \frac{1}{S_\Gamma} \tag{3.4}$$

where Γ is a distinct graph free from labels and S_Γ is the number of permutations of lines and vertices leaving Γ invariant. Some examples appear in Fig.

In dimensions > 0, loops correspond to integrals over internal momenta, so these diagrams may have different contributions aside from the symmetry factors.

If we introduce an external source, then our path integral has a generating function

$$Z(J) = \int d\phi \exp -\frac{1}{\hbar}\left(\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + J\phi\right) \tag{3.5}$$

and our correlation functions are modified as before, with $\langle \phi^2 \rangle = \frac{(-\hbar)^2}{Z(0)} \frac{\partial^2}{\partial J^2} Z(J)\Big|_{J=0}$. Source terms correspond to lines terminating on vertices J, so that the expansion of $Z(J)$ involves not only $Z(0)$ vacuum diagrams but also diagrams that terminate with even numbers of source vertices.

Lecture 4.

Saturday, January 26, 2019

The official course notes from this class will be available from `www.damtp.cam.ac.uk/user/wingate/AQFT`.

Last time, we computed the ϕ^2 correlation function, $\langle \phi^2 \rangle$. In principle this sum also includes disconnected diagrams[5] with "vacuum bubbles." As it turns out, the source-free partition function $Z(0)$ is exactly the sum of the vacuum bubble diagrams, so that when we compute the correlation function, it suffices to consider only connected diagrams.

[5]Disconnected means that part of the diagram is not connected to any of the external legs. There are diagrams which look "disconnected" in the informal sense, but in which every line is still connected to an external line (real particle).

Effective actions Let's introduce now the *Wilsonian effective action* (named for Ken Wilson of the renormalization group).

Definition 4.1. The Wilsonian effective action W is defined to be

$$Z = e^{-W/\hbar}. \tag{4.2}$$

Schematically,

$$\sum(\text{all vacuum diagrams}) = \exp\left(-\frac{1}{\hbar}\sum(\text{connected diagrams})\right). \tag{4.3}$$

To understand this, note that any diagram D is a product of connected diagrams C_I, such that

$$D = \frac{1}{S_D}\prod_I (C_I)^{n_I}, \tag{4.4}$$

where I indexes over connected diagrams, C_I includes its own internal symmetry factors, n_I is the number of C_Is in D, and S_D is the number of rearranging the identical $C_I s$ in D. That is,

$$S_D = \prod_I (n_I)!. \tag{4.5}$$

Therefore we have

$$\begin{aligned}
\frac{Z}{Z_0} &= \sum_{\{n_I\}} D \\
&= \sum_{\{n_I\}} \prod_I \frac{1}{n_I!}(C_I)^{n_I} \\
&= \prod_I \sum_{n_I} \frac{1}{n_I!}(C_I)^{n_I} \\
&= \exp\left(\sum_I C_I\right) \\
&= e^{-(W-W_0)/\hbar},
\end{aligned}$$

where $W = W_0 - \hbar\sum_I C_I$ is a sum over connected diagrams.

Why is W an "effective" action? Consider a theory with two real scalar fields ϕ, χ. Our theory has an action

$$S(\phi,\chi) = \frac{m^2}{2}\phi^2 + \frac{M^2}{2}\chi^2 + \frac{\lambda}{4}\phi^2\chi^2. \tag{4.6}$$

Note there's no factorial in the λ term because the fields are distinguishable.

We can associate some Feynman rules to the theory. Then there are some vacuum bubbles we can draw (see figure) associated to these rules to produce a sum

$$-\frac{W}{\hbar} = -\frac{\hbar\lambda}{m^2M^2}\left(\frac{1}{4}\right) + \left(\frac{\hbar\lambda}{m^2M^2}\right)^2\left(\frac{1}{16}+\frac{1}{16}+\frac{1}{8}\right) + O(\lambda^3). \tag{4.7}$$

Similarly for the connected loop diagrams, we have

$$\left\langle\phi^2\right\rangle = \frac{\hbar}{m^2}\left(1 - \frac{\hbar\lambda}{m^2M^2}\frac{1}{2} + \left(\frac{\hbar\lambda}{m^2M^2}\right)^2\left[\frac{1}{4}+\frac{1}{4}+\frac{1}{2}\right] + O(\lambda^3)\right). \tag{4.8}$$

This is well and good. We can write down the Feynman rules for the full theory, draw the diagrams, and in principle compute any cross section we like. But now say we want to remove the explicit χ dependence from our theory. That is, maybe the χ particle is very massive, $M \gg m$, and so we are unlikely to see it in our collider. We say that we "integrate out" the heavy field.

For this toy theory, define W such that

$$e^{-W(\phi)/\hbar} = \int d\chi\, e^{-S(\phi,\chi)/\hbar}. \tag{4.9}$$

Returning to our action, we see that the $\phi^2\chi^2$ term acts like a source term for χ^2.

Correlation functions can then be expressed as

$$\langle f(\phi)\rangle = \frac{1}{Z}\int d\phi d\chi f(\phi) e^{-S(\phi,\chi)/\hbar} = \frac{1}{Z}\int d\phi f(\phi) e^{-W(\phi)/\hbar}, \tag{4.10}$$

with W our new effective action.

In our example, the integral can be done exactly.

$$\int d\chi e^{-S(\phi,\chi)/\hbar} = e^{-m^2\phi^2/2}\sqrt{\frac{2\pi\hbar}{M^2+\frac{\lambda\phi^2}{2}}}, \tag{4.11}$$

and taking the log we find that

$$W(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\hbar}{2}\log\left(1+\frac{\lambda}{2M^2}\phi^2\right) + \frac{\hbar}{2}\log\frac{M^2}{2\pi\hbar}. \tag{4.12}$$

For our purposes, this constant piece won't affect QFT correlation functions since it appears both in Z and Z_0. However, these constant energy shifts are important where gravity is concerned, and in principle they should contribute to the cosmological constant of the universe. It's an open problem why the observed Λ is so small compared to the quantum fluctuations that should be contributing to it.

Now in our effective action we can expand the logarithm to get

$$W(\phi) = \left(\frac{m^2}{2}+\frac{\hbar\lambda}{4M^2}\right)\phi^2 - \frac{\hbar\lambda^2}{16M^4}\phi^4 + \frac{\hbar\lambda^3}{48M^6}\phi^6 + \ldots \tag{4.13}$$

$$= \frac{m_{\text{eff}}^2}{2}\phi^2 + \frac{\lambda_4}{4!}\phi^4 + \frac{\lambda_6}{6!}\phi^6 + \ldots + \frac{\lambda_{2k}}{(2k)!}\phi^{2k} + \ldots \tag{4.14}$$

where

$$m_{\text{eff}}^2 = m^2 + \frac{\hbar\lambda}{2M^2}$$
$$\lambda_{2k} = (-1)^{k+1}\hbar\frac{(2k)!}{2^{k+1}k}\frac{\lambda^k}{M^{2k}},$$

This tells us that all new terms are $\propto \hbar$, so these are quantum corrections, and they are also suppressed by $1/M^{2p}$. In a sense, this is very good for our ability to make predictions about the low-energy theory. We can treat these higher order corrections as small and do calculations in our effective theory. But conversely, it will be hard to probe the high energy theory because the corrections are suppressed.

Our toy model was very nice because it had an exact solution, but usually we must find $W(\phi)$ perturbatively. That is, we construct Feynman rules with $\frac{\lambda}{4}\phi^2\chi^2$ as a source term, so that our effective action goes as

$$W(\phi) \sim \frac{m^2\phi^2}{2} + \frac{1}{2}\frac{\hbar\lambda}{2M^2}\phi^2 - \frac{1}{4}\frac{\hbar\lambda^2}{4M^4}\phi^4 + \frac{1}{3!}\frac{\hbar\lambda^3}{8M^6}\phi^6 + \ldots, \tag{4.15}$$

as before.

Either way, with our effective action we can then compute correlation functions for ϕ with our effective action, e.g.

$$\left\langle \phi^2 \right\rangle = \frac{1}{Z}\int d\phi\, \phi^2 e^{-W/\hbar} = \frac{\hbar}{m_{\text{eff}} - \frac{\lambda_4\hbar^2}{2m_{\text{eff}}^6}} + \ldots, \tag{4.16}$$

as before.

Lecture 5.

Tuesday, January 29, 2019

Last time, we saw our first QFT example of an effective action. We introduced the Wilson effective action $W(J)$, where we averaged over the quantum fluctuations of some degrees of freedom (e.g. a heavy particle). We showed explicitly that we can construct an effective action for a two-particle theory by integrating out one of the fields and treating it as a source,

$$e^{-W(\phi)/\hbar} = \int d\chi e^{-S(\phi,\chi)/\hbar}.$$

Today, we'll show that we can take this further and construct a quantum effective action $\Gamma(\Phi)$ and average over all quantum fluctuations. This will lead us to defined an effective potential $V(\Phi)$. Effective actions of this form help us to determine the true vacuum of a theory and answer questions like "Do quantum effects induce spontaneous symmetry breaking?"

Let us define an average field in the presence of some source J,

$$\begin{aligned}\Phi \equiv \frac{\partial W}{\partial J} &= -\frac{\hbar}{Z(J)}\frac{\partial}{\partial J}\int d\phi e^{-(S[\phi]+J\phi)/\hbar} && (5.1)\\ &= \langle\phi\rangle_J\,, && (5.2)\end{aligned}$$

where W is the Wilson effective action and $J \neq 0$.

Thus $\Gamma(\Phi)$ is defined to be the Legendre transform of $W(J)$, i.e.

$$\Gamma(\Phi) = W(J) - \Phi J. \tag{5.3}$$

Note that

$$\begin{aligned}\frac{\partial\Gamma}{\partial\Phi} &= \frac{\partial W}{\partial\Phi} - J - \Phi\frac{\partial J}{\partial\Phi}\\ &= \underbrace{\frac{\partial W}{\partial J}}_{\Phi}\frac{\partial J}{\partial\Phi} - J - \Phi\frac{\partial J}{\partial\Phi}\\ &= -J,\end{aligned}$$

by applying the chain rule and the definition of Φ. We conclude that

$$J = -\frac{\partial\Gamma}{\partial\Phi}. \tag{5.4}$$

Note also that

$$\frac{\partial\Gamma}{\partial\Phi}|_{J=0} = 0,$$

i.e. in the absence of sources, $J = 0$, the average field $\Phi = \langle\phi\rangle_{J=0}$ corresponds to an extremum of $\Gamma(\Phi)$.

In higher dimensions, we write

$$\Gamma(\Phi) = \int d^d x\left[-V(\Phi) - \frac{1}{2}\partial^\mu\Phi\partial_\mu\Phi + \ldots\right], \tag{5.5}$$

where the $\ldots$ indicate higher derivatives and the first term $V(\Phi)$ is called the *effective potential*.

To make contact with statistical field theory, consider an Ising model, some spins $s(x)$ with an external magnetic field h and a Hamiltonian $\mathcal{H}$. The partition function is

$$Z(h) = e^{-\beta F(h)} = \int \mathcal{D}s \exp\left[-\beta\int d^d x(\mathcal{H}(s) - hs)\right]. \tag{5.6}$$

The magnetization is

$$M = -\frac{\partial F}{\partial h} = \int d^d x\,\langle s(x)\rangle\,, \tag{5.7}$$

and under a Legendre transform we have the Gibbs free energy

$$G = F + hM, \quad \frac{\partial G}{\partial M} = h. \tag{5.8}$$

When $h \to 0$, the equilibrium magnetization is given by the minimum of G.

Returning to QFT, let us try to perturbatively calculate $\Gamma(\Phi)$. We will treat Φ as we did ϕ, i.e. as a proper field. A quantum path integral over Φ then takes the form

$$e^{-W_\Gamma(J)/g} = \int d\Phi e^{-(\Gamma(\Phi)+J\Phi)/g}, \tag{5.9}$$

where g is some "fictional" new Planck constant.

Schematically, $W_\Gamma(J)$ is the sum of connected diagrams with Φ propagators and vertices. Expanding in g (i.e. in loops), we see that

$$W_\Gamma(J) = \sum_{l=0}^{\infty} g^l W_\Gamma^{(l)}(J) \tag{5.10}$$

where $W_\Gamma^{(l)}$ has all the l-loop diagrams.

Tree diagrams are those composing $W_\Gamma^{(0)}(J)$. In the $g \to 0$ (semi-classical?) limit, only tree-level diagrams contribute, so

$$W_\Gamma(J) \approx W_\Gamma^{(0)}(J) \tag{5.11}$$

as $g \to 0$. In addition, as $g \to 0$, our path integral 5.9 over Φ will be dominated by the minimum of the exponent (steepest descent), i.e. the average field Φ such that

$$\frac{\partial \Gamma}{\partial \Phi} = -J.$$

We learn that

$$W_\Gamma(J) = W_\Gamma^{(0)}(J) = \Gamma(\Phi) + J\Phi = W(J), \tag{5.12}$$

where the last equality follows from our earlier definition 5.3. Therefore the sum of connected diagrams $W(J)$ (with action $S(\phi) + J\phi$) can be obtained as the sum of tree diagrams $W_\Gamma^{(0)}(J)$ (with action $\Gamma(\Phi) + J\Phi$).

Definition 5.13. A line (edge) of a connected graph is a *bridge* if removing it would make the graph disconnected.

Definition 5.14. A connected graph is said to be one-particle irreducible (1PI) if it has no bridges.

The quantum effective action $\Gamma(\Phi)$ sums the 1PI graphs of the theory with action $S(\phi)$ yielding many vertices.[6] Then correlation functions can be found using tree graphs with vertices from $\Gamma(\Phi)$.

For example, an N-component field ϕ has a correlation function

$$\langle \phi_a \phi_b \rangle^{\text{conn}} = \langle \phi_a \phi_b \rangle - \langle \phi_a \rangle \langle \phi_b \rangle\,, \tag{5.15}$$

where the correlation function over connected diagrams is

$$\begin{aligned} -\hbar \frac{\partial^2 W}{\partial J_a \partial J_b} &= \langle \phi_a \phi_b \rangle^{\text{conn}} \\ &= \hbar \left(\frac{\partial^2 \Gamma}{\partial \Phi_a \partial \Phi_b} \right)^{-1}, \end{aligned}$$

which is $\hbar$ times the inverse of the quadratic part of Γ.

Lecture 6.

Thursday, January 31, 2019

Today we'll finish our discussion of the zero-dimensional path integral by introducing fermions to our theory. To model fermions, we will introduce Grassmann variables,[7] i.e. a set of n elements $\{\theta_a\}_{a=1}^n$ obeying anticommutation relations,

$$\theta_a \theta_b = -\theta_b \theta_a. \tag{6.1}$$

Note also that for (complex) scalars $\phi_b \in \mathbb{C}$,

$$\theta_a \phi_b = \phi_b \theta_a, \tag{6.2}$$

i.e. scalars commute with Grassmann variables. In addition, $\theta_a^2 = 0$ by the anticommutation relations, which implies that any function of n Grassmann variables can be written in finite form. That is, polynomials

[6] ??? I think this means we get modified Feynman rules for computing correlation functions.

[7] We've seen these in *Supersymmetry* already.

in Grassmann variables are forced to terminate since at some point we run out of distinct Grassmann variables to multiply. A general function $F(\theta)$ can be written

$$F(\theta) = f + \rho_a \theta_a + \frac{1}{2!} g_{ab} \theta_a \theta_b + \ldots + \frac{1}{n!} h_{a_1 \ldots a_n} \theta_{a_1} \ldots \theta_{a_n}. \tag{6.3}$$

Note that the coefficients $\rho, g, \ldots, h$ are totally antisymmetric under interchange of indices.

We also want to define differentiation and integration of these guys. Differentiation anticommutes with the Grassmann variables, i.e.

$$\left(\frac{\partial}{\partial \theta_a} \theta_b + \theta_b \frac{\partial}{\partial \theta_a} \right) * = (\delta_{ab}) * \tag{6.4}$$

where the derivative in the first term acts on everything coming after. This leads us to a modified Leibniz rule.

To define integration, note that for a single Grassmann variable θ, a function takes the form

$$F(\theta) = f + \rho \theta, \tag{6.5}$$

so we just need to define $\int d\theta$ and $\int d\theta\, \theta$. If we require translational invariance, i.e.

$$\int d\theta (\theta + \eta) = \int d\theta \theta \implies \int d\theta = 0. \tag{6.6}$$

We can then choose the normalization so that $\int d\theta\, \theta = 1$. Note the similarity between differentiation and integration (i.e. an integral $\int d\theta\, \theta = \frac{\partial}{\partial \theta} \theta = 1$). This process is called *Berezin integration*. Using these rules, we also find that

$$\int d\theta \frac{\partial}{\partial \theta} F(\theta) = 0, \tag{6.7}$$

since the term linear in θ will go to a constant by the derivative and be killed by the integral, and any constant terms will be killed by the derivative. Either way the result is zero.

Suppose now we have n Grassmann variables. Then the only nonvanishing integrals involve exactly one power of each integration variable, e.g.

$$\int d^n \theta\, \theta_1 \theta_2 \ldots \theta_n = \int d\theta_n d\theta_{n-1} \ldots d\theta_1\, \theta_1 \theta_2 \ldots \theta_n = 1. \tag{6.8}$$

In general we can just anticommute the Grassmann variables until they're in the right order, picking up a factor for the parity of the permutation. That is,

$$\int d^n \theta \theta_{a_1} \theta_{a_2} \ldots \theta_{a_n} = \epsilon^{a_1 a_2 \ldots a_n}, \tag{6.9}$$

where ϵ is the totally antisymmetric symbol with value $+1$ for even permutations of $1, 2, \ldots, n$, -1 for odd permutations, and 0 if any indices are repeated.

What if we now make a change of variables $\theta'_a = A_{ab} \theta_b$? Then

$$\int d^n \theta \theta'_{a_1} \theta'_{a_2} \ldots \theta'_{a_n} = A_{a_1 b_1} \ldots A_{a_n b_n} \underbrace{\int d^n \theta\, \theta_{b_1} \ldots \theta_{b_n}}_{\epsilon^{b_1 \ldots b_n}} \tag{6.10}$$

$$= \det A\, \epsilon^{a_1 \ldots a_n} \tag{6.11}$$

$$= \det A \int d^n \theta'\, \theta'_{a_1} \ldots \theta'_{a_n} \tag{6.12}$$

We conclude that under a change of variables, the integration measures are related by

$$d^n \theta = \det A\, d^n \theta'. \tag{6.13}$$

Note that this is the opposite of the convention for scalars, where

$$\phi'_a = A_{ab} \phi_b \implies d^n \phi = \frac{1}{|\det A|} d^n \phi'. \tag{6.14}$$

Free fermion field theory Consider $d = 0$, with two fermion fields θ_1, θ_2. The action must be bosonic (scalar), so the only possible nonconstant action is

$$S(\theta) = \frac{1}{2} A\theta_a\theta_2, A \in \mathbb{R} \tag{6.15}$$

Then the path integral is

$$Z_0 = \int d^2\theta\, e^{-S(\theta)/\hbar} = \int d^2\theta \left(1 - \frac{A}{2\hbar}\theta_1\theta_2\right) = -\frac{A}{2\hbar}, \tag{6.16}$$

where the exponential has terminated thanks to our Grassmann variables.

Suppose now we have $n = 2m$ fermion fields θ_a. Then our action might be quadratic in the fields,

$$S = \frac{1}{2} A_{ab}\theta_a\theta_b \tag{6.17}$$

with A an antisymmetric matrix, and the path integral is then

$$\begin{aligned} Z_0 &= \int d^{2m}\theta\, e^{-S(\theta)/\hbar} = \int d^{2m}\theta \sum j = 0^m \frac{(-1)^j}{(2\hbar)^j j!}(A_{ab}\theta_a\theta_b)^j \\ &= \frac{(-1)^m}{(2\hbar)^m m!}\int d^{2m}\theta A_{a_1a_2}A_{a_3a_4}\dots A_{a_{2m-1}a_{2m}}\theta_{a_1}\theta_{a_2}\dots\theta_{2m} \\ &= \frac{(-1)^m}{(2\hbar)^m m!}\epsilon^{a_1a_2\dots a_{2m}}A_{a_1a_2}A_{a_3a_4}\dots A_{a_{2m-1}a_{2m}} \\ &= \frac{(-1)^m}{\hbar^m}\mathrm{Pf}(A), \end{aligned}$$

where $\mathrm{Pf}(A)$ is the *Pfaffian* of the matrix A, defined by

$$\mathrm{Pf}(A) \equiv \frac{1}{2^m}\epsilon^{a_1a_2\dots a_{2m}}A_{a_1a_2}A_{a_3a_4}\dots A_{a_{2m-1}a_{2m}}, \tag{6.18}$$

which we will show on the examples sheet is in fact $\pm\sqrt{\det A}$. Thus $\mathrm{Pf}\begin{pmatrix} 0 & -1 \\ a & 0 \end{pmatrix} = a$. Using this property, we find that for fermionic fields,

$$Z_0 = \pm\sqrt{\frac{\det A}{\hbar^n}} \tag{6.19}$$

with A antisymmetric, whereas for bosonic fields with some symmetric mass matrix M,[8] we have

$$Z_0 = \sqrt{\frac{(2\pi\hbar)^n}{\det M}}. \tag{6.20}$$

We can now introduce an external source function to our action, a Grassmann-values $\{\eta_a\}$, such that the new action is

$$S(\theta, \eta) = \frac{1}{2}A_{ab}\theta_a\theta_b + \eta_a\theta_b. \tag{6.21}$$

Taking care to respect the anticommutation relations and completing the square as before, we can rewrite the action as

$$S(\theta,\eta) = \frac{1}{2}(\theta_a + \eta_c(A^{-1})_{ca})A_{ab}(\theta_b + \theta_d(A^{-1})_{db}) + \frac{1}{2}\eta_a(A^{-1})_{ab}\eta_b. \tag{6.22}$$

We can make a change of variables using the translational invariance of θ_a and pull out the constant factor to find

$$Z_0(\eta) = \exp\left(-\frac{1}{2\hbar}\eta^T(A^{-1})\eta\right)Z_0(0). \tag{6.23}$$

This allows us to get propagators by taking derivatives with respect to the source η, as we are wont to do:

$$\langle\theta_a\theta_b\rangle = \frac{\hbar^2}{Z_0(0)}\frac{\partial^2 Z_0(\eta)}{\partial\eta_a\partial\eta_b}|_{\eta=0} = \hbar(A^{-1})_{ab}. \tag{6.24}$$

[8] That is, for an action $S = \frac{1}{2}M_{ab}\phi_a\phi_b$.

We see that the propagator is proportional to the inverse of the bilinear part of the action for Grassmann variables.

Lecture 7.

Saturday, February 2, 2019

Quick admin note: there are some typos on Example Sheet 1. The expression in problem 1 should read

$$\exp\left(\frac{im(x-x_0)^2}{2(t-t_0)}\right),$$

where the denominator is not squared, and in problem 2,

$$\exp\left(\frac{\ldots-2xx_0}{\ldots}\right).$$

Today we shall return to the world of $3+1$ dimensions and set path integrals aside for a moment. Our main result today is the LSZ reduction formula, named for Lehmann-Symanzik-Zimmermann (cf. Srednicki §5). This result provides a direct relationship between scattering amplitudes. For example, consider the $2 \to 2$ scattering of real scalar particles. For a free scalar, we have the field written in terms of creation and annihlation operators,

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2E}\left[a(\mathbf{k})e^{-ik\cdot x} + a^\dagger(\mathbf{k})e^{ik\cdot x}\right] \tag{7.1}$$

where $k \cdot x = Et - \mathbf{k}\cdot\mathbf{x}$, using the mostly minus $(+---)$ signature.

Equivalently we can Fourier transform the field to find expressions for $a, a^\dagger$ in terms of the field ϕ:

$$\int d^3x\, e^{ik\cdot x}\phi(x) = \frac{1}{2E}a(\mathbf{k}) + \frac{1}{2E}e^{2iEt}a^\dagger(-\mathbf{k}),$$
$$\int d^3x\, e^{ik\cdot x}\partial_0\phi(x) = -\frac{i}{2}a(\mathbf{k}) + \frac{i}{2}e^{2iEt}a^\dagger(-\mathbf{k}),$$

which tells us that

$$a(\mathbf{k}) = \int d^3x e^{ik\cdot x}(i\partial_0\phi(x) + E\phi(x)) \tag{7.2}$$

$$a^\dagger(\mathbf{k}) = \int d^3x e^{-ik\cdot x}(-i\partial_0\phi(x) + E\phi(x)). \tag{7.3}$$

Now for the free theory, a one-particle state is given by

$$|k\rangle = a^\dagger(\mathbf{k})\,|0\rangle\,, \tag{7.4}$$

with $|0\rangle$ the normalized vacuum state such that $\langle 0|\,0\rangle = 1$ and $a(\mathbf{k})\,|0\rangle = 0 \forall \mathbf{k}$. We require that these momentum eigenstates are (relativistically) normalized such that

$$\langle k'|\,k\rangle = (2\pi)^3(2E)\delta^{(3)}(\mathbf{k}-\mathbf{k}'), \tag{7.5}$$

with $E = \sqrt{|\mathbf{k}|^2+m^2}$. We can now introduce a Gaussian wavepacket in momentum space by

$$a_1^\dagger \equiv \int d^3k f_1(\mathbf{k})a^\dagger(k) \tag{7.6}$$

where

$$f_1(\mathbf{k}) \propto \exp\left[-\frac{(\mathbf{k}-\mathbf{k}_1)^2}{4\sigma^2}\right] \tag{7.7}$$

for some $\mathbf{k}_1, \sigma$. We can define a second particle with $a_2^\dagger$ for some $f_2, \mathbf{k}_2$ such that $\mathbf{k}_2 \neq \mathbf{k}_1$.

Now if we evolve Gaussian wavepackets from the far distant past (or future), the overlap between the Gaussians in coordinate space should be small (the particles are far apart in the past and future). Thus their interaction is effectively limited in both space and time to some bounded interaction region.

We shall assume this works even when interactions are present. However, there is a complication– $a^\dagger(\mathbf{k})$ becomes time-dependent, e.g. their energies depend on their proximity to other particles, and therefore $a_1^\dagger(t), a_2^\dagger(t)$ are now functions of time. We therefore assume that as $t \to \pm\infty$, the wavepacket operators $a_1^\dagger, a_2^\dagger$ coincide with the free theory expressions.

Our initial and final (in/out) states are therefore

$$|i\rangle = \lim_{t\to-\infty} a_1^\dagger(t)a_2^\dagger(t)\,|0\rangle \tag{7.8}$$

$$|f\rangle = \lim_{t\to+\infty} a_{1'}^\dagger(t)a_{2'}^\dagger(t)\,|0\rangle \tag{7.9}$$

where initial and final states are normalized, $\langle i|\, i\rangle = \langle f|\, f\rangle = 1$, and $\mathbf{k}_1 \neq \mathbf{k}_2, \mathbf{k}_1' \neq \mathbf{k}_2'$. The scattering amplitude is then the overlap of the initial and final states, $\langle f|\, i\rangle$.

Note that

$$\begin{aligned}
a_1^\dagger(\infty) - a_1^\dagger(-\infty) &= \int_{-\infty}^{\infty} dt\, \partial_0 a_1^\dagger(t) \\
&= \int d^3k f_1(\mathbf{k}) \int d^4x \partial_0 \left[e^{-ik\cdot x}(-i\partial_0\phi E\phi)\right] \\
&= -i\int d^3k f_1(\mathbf{k}) \int d^4x e^{-ik\cdot x}(\partial_0^2 + E^2)\phi \\
&= -i\int d^3k f_1(\mathbf{k}) \int d^4k e^{-ik\cdot x}(\partial_0^2 + |\mathbf{k}|^2 + m^2)\phi.
\end{aligned}$$

In going from the second to third line, the cross terms from the ∂_0 derivative cancel. We also recognize that $|\mathbf{k}|^2 e^{-ik\cdot x}\phi(x) = -\nabla^2(e^{-ik\cdot x})\phi(x) = -e^{-ik\cdot x}\nabla^2\phi(x)$ by integrating by parts. Therefore this last line becomes

$$a_1^\dagger(\infty) - a_1^\dagger(-\infty) = -i\int d^3k f_1(\mathbf{k}) \int d^4x e^{-ik\cdot x}(\partial^2 + m^2)\phi. \tag{7.10}$$

Note that in a free theory, the Klein-Gordon equation tells us that $(\partial^2 + m^2)\phi = 0$, so that $a_1^\dagger(\infty) = a_1^\dagger(-\infty)$.

Now

$$\langle f|\, i\rangle = \langle 0|\, \mathcal{T} a_{1'}(\infty)a_{2'}(\infty)a_1^\dagger(-\infty)a_2^\dagger(-\infty)\,|0\rangle\,, \tag{7.11}$$

where $\mathcal{T}$ indicates time ordering. Of course, the expression is already time ordered, so we can insert it for free. We can then use equations like 7.10 to substitute

$$a_j^\dagger(-\infty) = a_j^\dagger(\infty) + i\int d^3k\, f_j(\mathbf{k}) \int d^4x\, e^{-ik\cdot x}(\partial^2 + m^2)\phi, \tag{7.12}$$

and something similar for $a_{j'}(\infty) = a_{j'}(-\infty) + \ldots$. Time ordering then moves $a_j^\dagger(\infty)$ to the left, annihilating $\langle 0|$ and $a_{j'}(-\infty)$ to the right, annihilating $|0\rangle$. What remains is the integral terms, which form the LSZ formula:

$$\begin{aligned}
\langle f|\, i\rangle = (i)^4 \int & d^4x_1 d^4x_2 d^4x_{1'} d^4x_{2'} e^{-ik_1\cdot x_1} e^{-ik_2\cdot x_2} e^{ik_{1'}\cdot x_{1'}} e^{ik_{2'}\cdot x_{2'}} \\
&\times (\partial_1^2 + m^2)(\partial_{1'}^2 + m^2)(\partial_2^2 + m^2)(\partial_{2'}^2 + m^2)\, \langle 0|\, \mathcal{T}\phi(x_1)\phi(x_2)\phi(x_{1'})\phi(x_{2'})\,|0\rangle\,,
\end{aligned}$$

having taken the $\sigma \to 0$ limit in all the $f_j(\mathbf{k})$ so to get delta functions $\delta^{(3)}(\mathbf{k} - \mathbf{k}_j)$. It's this last term, the expectation value of the time-ordered fields, which contains all the physics.

We have the following assumptions in this formula (noting that the interacting ϕ is not exactly like the free ϕ field):

- We assume there is a unique ground state so that the first excited state is a single particle.
- We also want $\phi\,|0\rangle$ to be a single particle, i.e. $\langle 0|\, \phi\,|0\rangle = 0$. If instead $\langle 0|\, \phi\,|0\rangle = v \neq 0$, we simply redefine the field by a shift, $\tilde{\phi} = \phi - v$ such that $\langle 0|\, \tilde{\phi}\,|0\rangle = 0$.
- We want ϕ normalized such that $\langle k|\, \phi\,|0\rangle = e^{ik\cdot x}$ as in the free case. With interactions, we may need to instead rescale $\phi \to Z_\phi^{1/2}\phi$.

With these assumptions (and some careful thought about multi-particle states), the LSZ formula still applies. For instance,

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \text{interactions} \\
&\to \frac{1}{2}Z_\phi\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}Z_m m^2\phi^2 + \ldots
\end{aligned}$$

after renormalization.

Lecture 8.

Tuesday, February 5, 2019

Today we will begin our discussion of scalar field theory in the path integral formalism. Let us begin with a preliminary note that we can trivially shift time variables from $it \to \tau$ and thereby go from a Minkowski to Euclidean metric. Thus in Minkowski (with signature $+---$) we have a Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$$

(so the kinetic term has a $+$ sign) and in Euclidean signature $(++++)$ we have

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi).$$

For instance, we might have some potential like $V(\phi) = \frac{1}{2}m^2\phi^2 + sum_{n>2}\frac{1}{n!}V^{(n)}\phi^n$.

Our path integral is then

$$Z = \int \mathcal{D}\phi e^{i\int dx^0 d^3x\mathcal{L}} = \int \mathcal{D}\phi e^{-\int dx_4 d^3x\mathcal{L}}, \tag{8.1}$$

where we have defined $ix^0 = x_4$ and work in units with $\hbar = 1$.

The Minkowski propagator takes the form

$$\frac{i}{k^2 - m^2 + i\epsilon} = \frac{i}{(k^0)^2 - |\mathbf{k}|^2 - m^2 + i\epsilon}, \tag{8.2}$$

whereas in Euclidean signature we have instead

$$\frac{1}{k^2 + m^2}. \tag{8.3}$$

In Euclidean signature, we do not need to move the poles since they no longer lie on the real axis.

Generating functional We have written down a free field action with a source (cf. Srednicki §8):

$$S_0[\phi, J] = \int d^4x(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi = \frac{1}{2}m^2\phi^2 + J(x)\phi(x)). \tag{8.4}$$

Taking the Fourier transform of the field we have

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4}e^{ikx}\tilde{\phi}(k). \tag{8.5}$$

In terms of the Fourier transformed field, we get an action

$$S_0[\tilde{\phi}, \tilde{J}] = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\Big[\tilde{\phi}(-k)(k^2 + m^2)\tilde{\phi}(k) + \tilde{J}(-k)\tilde{\phi}(k) + \tilde{J}(k)\tilde{\phi}(-k)\Big]. \tag{8.6}$$

Our aim will be to construct a partition function $Z[J]$, integrating out ϕ. To do this, let us rewrite our action in terms of the shifted field

$$\tilde{\chi}(k) \equiv \tilde{\phi}(K) + \frac{\tilde{J}(k)}{k^2 + m^2}, \tag{8.7}$$

completing the square. If we make this change of variables we get

$$S_0[\tilde{\phi}, \tilde{J}] = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\left[\tilde{\chi}(-k)(k^2 + m^2)\tilde{\chi}(k) + \frac{\tilde{J}(-k)\tilde{J}(k)}{k^2 + m^2}\right]. \tag{8.8}$$

The χ path integral is just over a Gaussian. If we assume normalization such that $Z_0[0] = 1$, we find that

$$Z_0[\tilde{J}] = \exp\left[-\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{\tilde{J}(-k)\tilde{J}(k)}{k^2 + m^2}\right] \tag{8.9}$$

and Fourier transforming back, we have

$$Z_0[J] = \exp\left[-\frac{1}{2}\int d^4x d^4x' J(x)\Delta(x - x')J(x')\right], \tag{8.10}$$

where the Feynman propagator is

$$\Delta(x-x') \equiv \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot(x-x')}}{k^2+m^2}. \tag{8.11}$$

Recall that the Feynman propagator is a Green's function of the Klein-Gordon equation, such that

$$(\partial_x^2 + m^2)\Delta(x-x') = \delta^{(4)}(x-x'),$$

and (cf. Tong QFT §2.7.1) the Feynman propagator is also related to the time-ordered product

$$\Delta(x-x') = \langle 0| \mathcal{T}\phi(x)\phi(x') |0\rangle .$$

With these facts in mind, we observe that

$$\langle 0| \mathcal{T}\phi(x)\phi(x') |0\rangle = \left(-\frac{\delta}{\delta J(x)}\right)\left(-\frac{\delta}{\delta J(x')}\right) Z_0[J]|_{J=0}. \tag{8.12}$$

Here, we use the functional derivative notation that $\frac{\delta}{\delta f(x_1)} f(x_2) = \delta(x_1 - x_2)$. This is naturally the continuous generalization of $\frac{\partial}{\partial x_i} x_j = \delta_{ij}$.

Similarly, the four-point function (still in free theory) is the sum of the three unique Wick contractions of the four fields,

$$\langle 0| \mathcal{T}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) |0\rangle = [\Delta(x_1-x_2)\Delta(x_3-x_4) + \Delta(x_1-x_3)\Delta(x_2-x_4) + \Delta(x_1-x_4)\Delta(x_2-x_3)]. \tag{8.13}$$

The results of our 0-dimensional calculation apply, with the slight complication that the propagator $\Delta(x-x')$ is non-trivial. To complete the story, let us now turn on interactions and see what happens (cf. Srednicki §10). We write the full, exact propagator as

$$\mathbf{\Delta}(x_1 - x_2) \equiv \langle 0| \mathcal{T}\phi(x_1)\phi(x_2) |0\rangle . \tag{8.14}$$

Note that $|0\rangle$ is the interacting vacuum, not the free theory vacuum from before. Using the Wilsonian effective action $W[J] = -\log Z[J]$ and the notation that

$$\delta_i \equiv -\frac{\delta}{\delta J(x_i)}, \tag{8.15}$$

we see that the propagator now takes the form

$$\mathbf{\Delta}(x_1-x_2) = \delta_1\delta_2 Z[J]|_{J=0} = -\delta_1\delta_2 W[J]|_{J=0} + (\delta_1 W[J])(\delta_2 W[J])|_{J=0}. \tag{8.16}$$

If we assume that $\langle 0| \phi(x_1) |0\rangle = -\delta_i W[J]|_{J=0} = 0$ (i.e. the field has no VEV), the result is therefore just the first term:

$$\mathbf{\Delta}(x_1-x_2) = -\delta_1\delta_2 W[J]|_{J=0}. \tag{8.17}$$

If we consider the interacting theory four-point function, we find that

$$\begin{aligned}\langle 0| \mathcal{T}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) |0\rangle &= \delta_1\delta_2\delta_3\delta_4 Z[J]|_{J=0} \\ &= [-\delta_1\delta_2\delta_3\delta_4 W + (\delta_1\delta_2 W)(\delta_3\delta_4 W) \\ &\quad + (\delta_1\delta_3 W)(\delta_2\delta_4 W) + (\delta_1\delta_4 W)(\delta_2\delta_3 W)]_{J=0}.\end{aligned}$$

We now show that these last three terms are either zero or trivial (non-interacting). Consider the LSZ formula for $2 \to 2$ scattering:

$$\begin{aligned}\langle f| i\rangle = (i)^4 \int d^4x_1 d^4x_2 d^4x_{1'} d^4x_{2'} e^{-ik_1\cdot x_1} e^{-ik_2\cdot x_2} e^{ik_{1'}\cdot x_{1'}} e^{ik_{2'}\cdot x_{2'}} \\ \times (\partial_1^2+m^2)(\partial_{1'}^2+m^2)(\partial_2^2+m^2)(\partial_{2'}^2+m^2) \langle 0| \mathcal{T}\phi(x_1)\phi(x_2)\phi(x_{1'})\phi(x_{2'}) |0\rangle ,\end{aligned}$$

where we have Wick rotated back to Minkowski signature. Consider the term $(\delta_1\delta_3 W)(\delta_2\delta_4 W)$. This term can be rewritten as $\mathbf{\Delta}(x_1 - x_{1'})\mathbf{\Delta}(x_2 - x_{2'})$. We use the notation

$$F(x_{ij}) = (\partial_i^2 + m^2)(\partial_j^2+m^2)\mathbf{\Delta}^{(m)}(x_{ij}),$$

where the superscript m indicates the propagator is being computed in Minkowski signature. We define $x_{ij'} = x_i - x_{j'}, \bar{k}_{ij} = \frac{1}{2}(k_i + k_{j'}$, and $\tilde{F}(k)$ indicates the Fourier transform of F. Thus the contribution of the $(13)(24)$ terms to $\langle f|\, i\rangle$ is

$$\int d^4x_1 d^4x_2 d^4x_{1'} d^4x_{2'} e^{(\ldots)} F(x_{11'})F(x_{22'}) = (2\pi)^8 \delta^{(4)}(k_1 - k_{1'})\delta^{(4)}(k_2 - k_{2'})\tilde{F}(\bar{k}_{11'})\tilde{F}(\bar{k}_{22'})$$

But looking at these delta functions, we see that they set $k_1 = k_{1'}, k_2 = k_{2'} \implies$ there is no scattering. The other terms are similar. We conclude that the interesting bit is

$$\langle 0|\, \mathcal{T}\phi(x_1)\ldots\phi(x_n)\, |0\rangle_C \equiv -\delta_1 \ldots \delta_n W[J]|_{J=0}, \tag{8.18}$$

where the C on the left indicates connected diagrams and the RHS is fully connected diagrams.

Lecture 9.

Thursday, February 7, 2019

Today we'll turn on interactions and try to understand path integrals/generating functionals in an interacting theory, cf. Osborn §2.2.

Feynman rules We start by stating the following identity: for functions F, G,

$$G(-\frac{\partial}{\partial J})F(-J) = F(\frac{\partial}{\partial \phi}G(\phi)e^{-J\phi}|_{\phi=0}. \tag{9.1}$$

Example 9.2. Here's an example. Let $F(J) = e^{\beta J}$ and $G(\phi) = e^{\alpha\phi}$. Evaluating the LHS of our identity, we have

$$\begin{aligned} G(-\frac{\partial}{\partial J})F(-J) &= e^{-\alpha\frac{\partial}{\partial J}}e^{-\beta J} \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}(-\alpha\frac{\partial}{\partial J})^n e^{-\beta J} \\ &= e^{\alpha\beta}e^{-\beta J} = F(\alpha - J). \end{aligned}$$

On the RHS we have instead

$$\begin{aligned} F(\frac{\partial}{\partial\phi})G(\phi)e^{-J\phi}|_{\phi=0} &= e^{\beta\frac{\partial}{\partial\phi}}e^{\alpha\phi - J\phi}|_{\phi=0} \\ &= e^{-\beta(\alpha-J)} = F(\alpha - J). \end{aligned}$$

Really, this is a notational abuse– we are using these functions both as maps on some values/fields ϕ, J and also on differential operators. But the result is valid[9] and for general F, G we may write these as Fourier series and proceed as above.

We will employ this identity in interacting scalar field theory in the form

$$e^{-\mathcal{L}_{int}(-\frac{\partial}{\partial J})}e^{-\frac{1}{2}J\Delta J} = e^{-\frac{1}{2}\frac{\partial}{\partial\phi}\Delta\frac{\partial}{\partial\phi}}e^{-\mathcal{L}_{int}(\phi) - J\phi}|_{\phi=0}, \tag{9.3}$$

where we will promote J, ϕ to fields.

In interacting scalar field theory, we can separate the Lagrangian into a free part and an interacting part,

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}, \quad \mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2. \tag{9.4}$$

[9] At least for sufficiently nice functions, I assume.

Now the generating functional for this theory (possibly in the presence of a source J) takes the form

$$Z[J] = \int \mathcal{D}\phi \exp\left[-\int d^4x(\mathcal{L}_0 + \mathcal{L}_{int} + J\phi)\right] \tag{9.5}$$

$$= \exp\left\{-\int d^4y\mathcal{L}_{int}\left[-\frac{\partial}{\partial J}\right]\right\} \underbrace{\int \mathcal{D}\phi \exp\left[-\int d^4x(\mathcal{L}_0 + J\phi)\right]}_{Z_0[J]} \tag{9.6}$$

$$= \exp\left\{-\int d^4y\mathcal{L}_{int}\left[-\frac{\partial}{\partial J}\right]\right\} \exp\left[-\frac{1}{2}\int d^4xd^4x' J(x)\Delta(x-x')J(x')\right] \tag{9.7}$$

$$= \exp\left[-\frac{1}{2}\int d^4xd^4x' \frac{\delta}{\delta\phi(x)}\Delta(x-x')\frac{\delta}{\delta\phi(x')}\right] \exp\left[-\int d^4y(\mathcal{L}_{int}[\phi] + J(y)\phi(y))\right]|_{\phi=0}. \tag{9.8}$$

In line 9.6, we have used the fact that $\left(\frac{\delta}{\delta J(y)}\right)e^{-d^4xJ\phi} = \phi(y)e^{-\int d^4xJ\phi}$. In the next line, we used our free theory result for $Z_0[J]$. In the last line, we have used our identity, Eqn. 9.3.

The (position space) Feynman rules are then based on the series expansion of exponentials in $Z[J]$.

- Propagators come with factors of $\Delta(x-x')$.
- Vertices with n lines come from $\left(\frac{\delta}{\delta\phi(y)}\right)^n(-\mathcal{L}_{int}[\phi])|_{\phi=0} \equiv v^{(n)}$.
- Integrate over the positions of all internal vertices.
- Add symmetry factors as before.

Of course, it's usually more illuminating to do our calculations in momentum space instead. A Fourier transform will take us there. We can write down a momentum space propagator

$$\tilde{\Delta}(k) = \int d^4y\Delta(y)e^{-ik\cdot y} = \frac{1}{k^2+m^2}. \tag{9.9}$$

Our integrals over position now become δ functions which conserve momentum at each vertex, and we will always get an overall factor $(2\pi)^4\delta^{(4)}(\sum_j p_j)$ where the sum is taken over external momenta. The momentum space Feynman rules are as follows:

- Propagators get factors of $\frac{1}{k^2+m^2}$.
- Vertices get factors of $(2\pi)^4\delta^{(4)}(\sum p_i)$ where p_i is taken over momenta going into a vertex (or out, if you prefer)
- Integrate over all internal momenta with $\int \frac{d^4k}{(2\pi)^4}$.

For fully connected diagrams[10] we have a nice graph theory property due to Euler:

$$L = I - V + 1, \tag{9.10}$$

where L is the number of loops, I is the number of internal lines, and V is the number of vertices. We can use this to simplify some integrals by

$$\int\left[\prod_{i=1}^{I}\frac{d^4k_i}{(2\pi)^4}\right]\left[\prod_{v=1}^{V}(2\pi)^4\delta^{(4)}(\sum_j p_{j,v})\right]\ldots \tag{9.11}$$

where ... indicates some integrand. We can therefore factor out the momentum-conserving delta function and do $V-1$ integrals over the rest of the δ functions, so we are left with L nontrivial integrals. The factors of 2π work out too: $\left(\frac{1}{(2\pi)^4}\right)^I(2\pi)^{4V} = \frac{1}{(2\pi)^{4(L-1)}}$.

We get the following simplified rules:

- External lines get $\frac{1}{p^2+m^2}$ factors
- Internal lines get $\frac{1}{k^2+m^2}$ factors
- n-point vertices get factors of $v^{(n)}$
- Impose momentum conservation at each vertex

[10] In David Tong's notes, he refers to connected diagrams where every point is connected to an external line, and *fully connected diagrams*, where all points are connected to all other points. This distinction was previously missed in these lectures.

- Integrate over each undetermined loop momentum (1 for each loop)
- Strip off the overall momentum conserving delta function $(2\pi)^4\delta^{(4)}(\sum_j p_j)$.

For example, if $\mathcal{L}_{int}$ contains a $\frac{\lambda}{4!}\phi^4$ term, then we get a one-loop diagram, resulting in

$$\frac{1}{2}\frac{1}{(p_1^2+m^2)(p_2^2+m^2)}(2\pi)^4\delta^{(4)}(p_1-p_2)(-\lambda)\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}. \tag{9.12}$$

Unfortunately, this is infinity. We'll see what to do with this a little later. If $\mathcal{L}_{int}$ instead contains $\frac{g}{3!}\phi^3$, we get a matrix element

$$\frac{1}{2}\frac{1}{(p_1^2+m^2)(p_2^2+m^2)}(2\pi)^4\delta^{(4)}(p_1-p_2)(-g)^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}\frac{1}{(k-p_1)^2+m^2} \tag{9.13}$$

Lecture 10.

Saturday, Feburary 9, 2019

Today we'll begin our discussion of renormalization and why infinities might not be so scary after all (cf. Skinner §5.1). Let us consider ϕ^4 theory:

$$S[\phi]=\int d^4x\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi+\frac{1}{2}m^2\phi^2+\frac{\lambda}{4!}\phi^4\right]. \tag{10.1}$$

In momentum space, the full propagator $\tilde{\mathbf{\Delta}}(p^2)$ takes the form

$$\tilde{\mathbf{\Delta}}(p^2)=\int d^4xe^{-ip\cdot x}\left\langle\phi(x)\phi(0)\right\rangle_{\text{connected}}. \tag{10.2}$$

Schematically, we can represent the propagator as the following sum of diagrams: (diagram to be inserted) or equivalently the following geometric series:

$$\begin{aligned}\tilde{\mathbf{\Delta}}(p^2)&=\frac{1}{p^2+m^2}+\frac{1}{p^2+m^2}\Pi(p^2)\frac{1}{p^2+m^2}+\frac{1}{p^2+m^2}\Pi(p^2)\frac{1}{p^2+m^2}\Pi(p^2)\frac{1}{p^2+m^2}+\dots\\&=\frac{1}{p^2+^2-\Pi(p)^2}\end{aligned}$$

where $\Pi(p^2)$ is called the self-energy. Note that Π contributes to the quantum effective action Γ. Perturbatively, we get contributions from diagrams like the following: (diagram here) Note that dashed lines are omitted from the computation of the 1PI factor $\Pi(p^2)$ since they are external propagators.

One of the simplest diagrams we can draw is the one-loop diagram, and it corresponds to the amplitude

$$-\frac{\lambda}{2}\int^\Lambda\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}. \tag{10.3}$$

This is divergent. To see this, let us introduce an ultraviolet (UV) cutoff so that we integrate over only modes with $|k|<\Lambda$. Since the integral depends only on k^2, we get

$$-\frac{\lambda S_4}{2(2\pi)^4}\int_0^\Lambda\frac{k^3dk}{k^2+m^2}=-\frac{\lambda S_4m^2}{4(2\pi)^4}\int_0^{\Lambda^2/m^2}\frac{udu}{1+u}=-\frac{\lambda}{32\pi^2}\left[\Lambda^2-m^2\log\left(1+\frac{\Lambda^2}{m^2}\right)\right], \tag{10.4}$$

where $d^dk=S_d|k|^{d-1}d|k|$ with $S_d=\frac{2\pi^{d/2}}{\Gamma(d/2)}$ and we've made the substitution $u=k^2/m^2$ to perform the integral. Here, $S_4=2\pi^2$. With all these substitutions, we reach an amplitude that diverges as $\Lambda\to\infty$.

Suppose we allow the coupling to depend on Λ by adding "counterterms" to the action. That is,

$$S[\phi]\to S[\phi]+(\hbar)S^{CT}[\phi,\Lambda]. \tag{10.5}$$

For instance, we might define a set of counterterms as

$$S^{CG}[\phi,\Lambda]=\int d^4x\left[\frac{\delta Z(\Lambda)}{2}\partial_\mu\phi\partial^\mu\phi+\frac{1}{2}\delta m^2(\Lambda)\phi^2+\frac{\delta\lambda(\Lambda)}{4!}\phi^4\right]. \tag{10.6}$$

Note that These correspond to some new vertices and thus new contributions to $\Pi(p^2)$: see diagram.

At 1 loop, the 1PI contribtuion becomes

$$\Pi^{\text{1-loop}}(p^2) = -p^2\delta Z - \delta m^2 - \frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2\log\left(1+\frac{\Lambda^2}{m^2}\right)\right]. \tag{10.7}$$

At two loops, the counterterm diagrams must also be included. Now we can tune the parameters $\delta Z, \delta m^2, \delta\lambda$ in our counterterms to cancel the divergences. In other words, we *renormalize* ϕ, m^2, λ.

On-shell renormalization scheme The need to "regulate" the theory by cancelling divergences does not uniquely determine the counterterms, so we impose additional renormalization conditions, which we call a *scheme*. It will turn out that physical observables do not depend on our choice of scheme.

The on-shell scheme is as follows. We fix $\delta Z, \delta m^2, \delta\lambda$ by requiring that

1. $\tilde{\boldsymbol{\Delta}}(p^2)$ has a simple pole at some experimentally observable mass, i.e. $-p^2 = m^2_{\text{phys}}$ (in Euclidean signature)
2. The residue of this pole is equal to 1.

Therefore since

$$\tilde{\boldsymbol{\Delta}}(p^2) = \frac{1}{p^2 + m^2 - \Pi(p^2)}, \tag{10.8}$$

our conditions say that

1. $\Pi(-m^2_{\text{phys}} = m^2 - m^2_{\text{phys}}$, which is zero if we want the mass in $\mathcal{L}$ to equal m_{phys} at this order of counterterm.
2. $\frac{\partial\Pi}{\partial p^2}|_{p^2} = -m^2_{\text{phys}} = 0$ (by L'Hôpital's rule).

These tell us that

$$2. \implies \delta Z = 0, \tag{10.9}$$

$$1. \implies \delta m^2 = -\frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2\log\left(1+\frac{\Lambda^2}{m^2}\right)\right]. \tag{10.10}$$

Note that $\delta Z = 0$ and $\pi(p^2) = 0 \forall p^2$ is due to the one-loop diagram not depending on p^2. If we instead tried to construct the counterterms to the two-loop diagram we would get $\delta Z \neq 0$ since the integral depends on p.

UV divergences are not too hard to spot– we saw that

$$\int^{\Lambda} \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2} \sim \Lambda^2, \tag{10.11}$$

and generically

$$\int^{\Lambda} \frac{d^nk}{k^m} \sim \begin{cases} \Lambda^{n-m}, & n \neq m \\ \log\Lambda, & n = m \end{cases}. \tag{10.12}$$