ADVANCED QUANTUM FIELD THEORY

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LAST UPDATED MARCH 8, 2019

These notes were taken for the *Advanced Quantum Field Theory* course taught by Matthew Wingate at the University of Cambridge as part of the Mathematical Tripos Part III in Michaelmas Term 2018. I live-TEXed them using Overleaf, and as such there may be typos; please send questions, comments, complaints, and corrections to itel2@cam.ac.uk. Many thanks to Arun Debray for the LATEX template for these lecture notes: as of the time of writing, you can find him at https://web.ma.utexas.edu/users/a.debray/.

Contents

Lecture 1.

Saturday, January 19, 2019

Note. There will not be official typed course notes, but there will be scanned handwritten notes (which I will link here as they become available). Previous lecturers' notes are currently online (Skinner, Osborn).

Today we introduce path integrals in a QFT context. There are some benefits to working with path integrals– some computations are simplified or more straightforward, and Lorentz invariance is manifest (unlike in the canonical formalism).

Path integrals in quantum mechanics Rather than trying to tackle the full machinery of QFT, we'll start with $0+1$ dimensional non-relativistic quantum mechanics (cf. Osborn § 1.2. We'll set $\hbar = 1$ for now, though we may restore it later in order to make arguments when $\hbar \ll 1$ in a classical limit. In these units,

$$[E][t] = [\hbar] = [p][x]$$

using uncertainty relations.

Let us consider a Hamiltonian in 1 spatial dimension,

$$\hat{H} = H(\hat{x}, \hat{p}) \quad \text{with } [\hat{x}, \hat{p}] = i.$$

We'll further assume for simplicity that the Hamiltonian has a kinetic term and a potential based only on position,

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}).$$

Now the Schrödinger equation takes the form

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle \tag{1.1}$$

which has formal solution

$$|\psi(t)\rangle = e^{-i\hat{H}t}|\psi(0)\rangle. \tag{1.2}$$

Let us consider some position eigenstates $|x, t\rangle$ such that

$$\hat{x}(t)|x, t\rangle = x|x, t\rangle, \quad x \in \mathbb{R},$$

where these states obey some normalization

$$\langle x', t|x, t\rangle = \delta(x' - x).$$

In the Schrödinger picture, states depend on time, while operators are constant. In terms of fixed (time-independent) eigenstates $\{|x\rangle\}$ of the position operator $\hat{x}$, we may write the wavefunction as

$$\psi(x, t) = \langle x|\psi(t)\rangle, \tag{1.3}$$

so that applying the Hamiltonian to the wavefunction $\psi(x, t)$ yields

$$\hat{H}\psi(x, t) = (-\frac{1}{2m}\frac{\partial^2}{\partial x^2} + V(x))\psi(x, t). \tag{1.4}$$

This is the traditional presentation of quantum mechanics and the wavefunction. In the path integral formalism, we'll consider a more particle-like treatment, where we express time evolution as a sum over all trajectories (meeting some boundary conditions) appropriately weighted (by an action).

Recall that our formal solution 1.2 tells us what $|\psi(t)\rangle$ is– we can therefore rewrite the wavefunction as

$$\psi(x, t)\langle x|e^{-i\hat{H}t}|\psi(0)\rangle. \tag{1.5}$$

By inserting a complete set of (position eigen)states, $1 = \int dx_0 |x_0\rangle\langle x_0|$, we get

$$\begin{aligned}\psi(x, t) &= \int dx_0 \langle x|e^{-i\hat{H}t}|x_0\rangle\langle x_0|\psi(0)\rangle \\ &= \int dx_0\, K(x, x_0; t)\psi(x_0, 0),\end{aligned}$$

where we have defined $K(x, x_0; t) \equiv \langle x|e^{-i\hat{H}t}|x_0\rangle$. Let us further consider time evolution in discrete steps, with $0 \equiv t_0 < t_1 < t_2 < \ldots < t_n < t_{n+1} \equiv T$ so that

$$e^{-i\hat{H}T} = e^{-i\hat{H}(t_{n+1}-t_n)} \ldots e^{-i\hat{H}(t_1-t_0)}.$$

As before, we insert complete sets of states, finding that our generic time evolution from any x_0 to an x of our choosing:

$$K(x, x_0; T) = \int \left[\prod_{r=1}^{n} dx_r \langle x_{r+1}|e^{-i\hat{H}(t_{r+1}-t_r)}|x_r\rangle\right] \langle x_1|e^{-i\hat{H}t_1}|x_0\rangle. \tag{1.6}$$

That is, we integrate over all intermediate positions x_r for each t_r. Naturally, dx_{n+1} must be x.

Let's look at the free theory first to understand what we've done, $V(x) = 0$. Now this weird K_0 object we've defined takes the form

$$K_0(x, x'; t) = \langle x| e^{-i\frac{\hat{p}^2}{2m}t} |x'\rangle. \tag{1.7}$$

We'll instead insert a complete set of momentum eigenstates $|p\rangle$ with the normalization

$$\int \frac{dp}{2\pi} |p\rangle\langle p| = 1,$$

recalling that $\langle x|p\rangle = e^{ipx}$ are simply plane waves. Then

$$K_0(x, x'; t) = \int \frac{dp}{2\pi} e^{-ip^2 t/2m} e^{ip(x-x')}.$$

We can compute this– completing the square with a change of variables to $p' = p - \frac{m(x-x')}{t}$, K_0 becomes a gaussian integral,

$$\begin{aligned} K_0(x, x'; t) &= e^{im(x-x')^2/2t} \int_{-\infty}^{\infty} \exp\left[-\frac{i(p')^2 t}{2m}\right] \\ &= e^{im(x-x')^2/2t} \sqrt{\frac{m}{2\pi i t}}. \end{aligned}$$

Note that as $t \to 0$,[1]

$$\lim_{t\to 0} K_0(x, x'; t) = \delta(x - x'),$$

which agrees with the fact that $\langle x'|x\rangle = \delta(x - x')$.

For $V(\hat{x}) \neq 0$, we still need small time steps but since operators generically do not compute, exponentials don't add in the usual way:

$$e^{\hat{A}} e^{\hat{B}} = \exp(\hat{A} + \hat{B} + \frac{1}{2}[\hat{A}, \hat{B}] + \ldots) \neq e^{\hat{A}+\hat{B}} \quad \text{when } [\hat{A}, \hat{B}] \neq 0.$$

This is the Baker-Campbell-Hausdorff (BCH) formula. However, for small ϵ we can write

$$e^{\epsilon\hat{A}} e^{\epsilon\hat{B}} = \exp(\epsilon\hat{A} + \epsilon\hat{B} + O(\epsilon^2)),$$

or equivalently

$$e^{\epsilon(\hat{A}+\hat{B})} = e^{\epsilon\hat{A}} e^{\epsilon\hat{B}} (1 + O(\epsilon^2)),$$

so we conclude that

$$e^{\hat{A}+\hat{B}} = \lim_{n\to\infty} \left(e^{\hat{A}/n} e^{\hat{B}/n}\right)^n.$$

Suppose now that we divide our time into n time steps so that $t_r - 1 - t_r = \delta t$, with $T = n\delta t$. Then one of the intermediate time evolution steps looks like

$$\begin{aligned} \langle x_{r+1}| e^{-i\hat{H}\delta t} |x_r\rangle &= e^{-iV(x_r)\delta t} \langle x_{r+1}| e^{-i\hat{p}^2\delta t/2m} |x_r\rangle \\ &= \sqrt{\frac{m}{2\pi i\delta t}} \exp\left[\frac{i}{2} m \left(\frac{x_{r+1} - x_r}{\delta t}\right)^2 \delta t - iV(x_r)\delta t\right]. \end{aligned}$$

Taking $T = n\delta t$, we find that the entire K becomes

$$K(x, x_0; T) = \int \left(\prod_{r=1}^{n} dx_r\right) \left(\frac{m}{2\pi i\delta t}\right)^{\frac{n+1}{2}} \exp\left(i \sum_{r=0}^{n} \left[\frac{m}{2}\left(\frac{x_{r+1} - x_r}{\delta t}\right)^2 - V(x_r)\right] \delta t\right). \tag{1.8}$$

Now we take the limit as $n \to \infty, \delta t \to 0$ with T fixed. Then the argument of the exponential becomes

$$\int_0^T \frac{m}{2}\dot{x}^2 - V(x)]dt = \int_0^t L dt, \tag{1.9}$$

where $L(x, \dot{x})$ is the classical Lagrangian and this integral is nothing more than the action. We conclude that

$$K(x, x_0; T) = \langle x| e^{-i\hat{H}t} |x_0\rangle = \int \mathcal{D}x\, e^{iS[x]}, \tag{1.10}$$

[1] This was more obvious from the original expression for K_0 where $K_0(x, x'; t = 0) = \int \frac{dp}{2\pi} e^{ip(x-x')}$.

where $S[x] = \int_0^T L(x, \dot{x}) dt$ is the classical action and the $\mathcal{D}$ conceals all our sins (the continuum limit) in a cute integration measure. Note that the action has units of energy $\times$ time, so if we restore $\hbar$, we see that this integral becomes

$$K(x, x_0; T) = \int \mathcal{D}x \, e^{iS/\hbar}, \tag{1.11}$$

and in the $\hbar \to 0$ limit (the classical limit), the integral is dominated by paths x which minimize the classical action, and we recognize this as Hamilton's principle from classical mechanics.

Lecture 2.

Tuesday, January 22, 2019

Last time, we introduced the path integral in quantum mechanics, and we said it took the form

$$\langle x | e^{-iHt/\hbar} | x_0 \rangle = \int \mathcal{D}x e^{iS[x]/\hbar}. \tag{2.1}$$

Let us consider now a "rotation" to imaginary time, $t \to -i\tau$ (Wick rotation). Then our path integral becomes

$$\langle x | e^{-H\tau/\hbar} | x_0 \rangle = \int \mathcal{D}x e^{-S[x]/\hbar}. \tag{2.2}$$

Working with a real exponent has some benefits– the convergence of the integral is more obvious, and in the $\hbar \to 0$ limit we expect the integral to be dominated by the classical path x which minimizes the action $S[x]$.

We can make the observation that 1D quantum mechanics is like a $0 + 1$D quantum field theory– the field is

$$x(t) : \mathbb{R} \to \mathbb{R}.$$

In fact, 3D quantum mechanics is also like a $0 + 1$D QFT, where the field is now

$$\mathbf{x}(t) : \mathbb{R} \to \mathbb{R}^3.$$

Given a single spacetime label t, a QM theory gives us a real scalar in $\mathbb{R}$ or a vector in $\mathbb{R}^3$– cf. Srednicki Ch. 1. There are different approaches to quantization, but in the second quantization formalism we demote position $\mathbf{x}$ from an operator to a label on a spacetime point $(\mathbf{x}, t)$. Therefore QFT in $3 + 1$ dimensions has e.g. a scalar field ϕ which is a map

$$\phi : \mathbb{R}^{1,3} \to \mathbb{R}.$$

Path integral methods Let's begin with the simplest possible case, QFT in zero dimensions.[2] All of spacetime is a single point p,[3] and our (real scalar) field ϕ is a map $\phi : \{p\} \to \mathbb{R}$.

Using our imaginary time (Euclidean signature) convention for the path integral, we write

$$Z = \int_{\mathbb{R}} d\phi \, e^{-S[\phi]/\hbar}. \tag{2.3}$$

We'll take our action $S[\phi]$ to be polynomial in ϕ, with highest power even.

As in statistical field theory, we are interested in correlation functions and expectation values. Given a function $f(\phi)$, we might like to compute the expectation value

$$\langle f \rangle = \frac{1}{Z} \int d\phi \, f(\phi) e^{-S[\phi]/\hbar}. \tag{2.4}$$

For this to have a chance of convergence, f should not grow too rapidly as $|\phi| \to \infty$. Usually the functions we are interested in are polynomial in ϕ.

[2] Cf. Skinner Ch. 2, Srednicki §8,9.

[3] If you're reading my SUSY notes, you should be getting déjà vu right about now.

Free field theory Suppose we have N real scalar fields $\phi_a, a = 1, \ldots, N$. We can compactly write this as a single field

$$\phi : \{p\} \to \mathbb{R}^N, \tag{2.5}$$

and we'd like to compute the integral

$$Z_0 = \int d^N \phi e^{-S[\phi]/\hbar}. \tag{2.6}$$

Now, a free theory simply means that the action is quadratic in our fields. A priori it could have included kinetic terms, but since we are in zero dimensions, there are no derivatives to take and therefore no kinetic terms in this model. Then we can write our action as

$$S(\phi) = \frac{1}{2}\mathcal{M}_{ab}\phi_a\phi_b = \frac{1}{2}\phi^T\mathcal{M}\phi, \tag{2.7}$$

where $\mathcal{M}$ is an $N \times N$ symmetric matrix with $\det M > 0$. So our action could include terms like $\frac{1}{2}\phi_1^2$ and $\frac{5}{2}\phi_1\phi_4$. Since $\mathcal{M}$ is symmetric, we can diagonalize it as

$$\mathcal{M} = P\Lambda P^T$$

for some orthogonal matrix P. But equivalently we could just redefine our fields to some new fields $\phi' = P^T\phi$ so that

$$S(\phi) = \frac{1}{2}\phi'^T\Lambda\phi' = \frac{1}{2}\sum_{i=1}^{N}\lambda_i(\phi_i')^2,$$

where λ_i are the eigenvalues of $\mathcal{M}$. Since P is orthogonal, $\det P = 1 \implies d^N\phi = (\det P) d^N\phi' = d^N\phi'$, so our path integral separates into N Gaussian integrals of the form

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{\lambda}{2\hbar}x^2} = \sqrt{\frac{2\pi\hbar}{\lambda}}. \tag{2.8}$$

Thus

$$Z_0 = \int d^N\phi\, e^{-\frac{1}{2\hbar}\phi^T\mathcal{M}\phi} = \prod_{i=1}^{N} d\phi_i\, e^{-\frac{1}{2\hbar}\lambda_i(\phi_i)^2} = \frac{(2\pi\hbar)^{N/2}}{\sqrt{\det\mathcal{M}}}. \tag{2.9}$$

We can now introduce a source term J, modifying our action to

$$S(\phi) = \frac{1}{2}\phi^T\mathcal{M}\phi + J\cdot\phi. \tag{2.10}$$

If we complete the square and make a change of variables $\tilde{\phi} = \phi + \mathcal{M}^{-1}J$, we find that the new path integral with a source is

$$\begin{aligned} Z_0[J] &= \int d^N\phi\, \exp\left[-\frac{1}{2\hbar}\phi^T\mathcal{M}\phi - \frac{1}{\hbar}J\cdot\phi\right] \\ &= \exp\left(\frac{1}{2\hbar}J^T\mathcal{M}^{-1}J\right)\int d^N\tilde{\phi}\, e^{-\frac{1}{2\hbar}\tilde{\phi}^T\mathcal{M}\tilde{\phi}} \\ &= Z_0 \exp\left(\frac{1}{2\hbar}J^T\mathcal{M}^{-1}J\right). \end{aligned}$$

We see that $\frac{\partial}{\partial J}$ derivatives will bring down ϕs, which will allow us to compute correlation functions just like we did in statistical physics with the partition function.

Example 2.11. What is the value of the correlation function $\langle \phi_a \phi_b \rangle$ in this theory? We can compute it directly:

$$\begin{aligned}
\langle \phi_a \phi_b \rangle &= \frac{1}{Z_0} \int d^N\phi\, \phi_a \phi_b \exp\left[-\frac{1}{2\hbar} \phi^T \mathcal{M} \phi - \frac{1}{\hbar} J \cdot \phi \right] \Bigg|_{J=0} \\
&= \frac{1}{Z_0} \int d^N\phi \left(-\hbar \frac{\partial}{\partial J_a} \right) \left(-\hbar \frac{\partial}{\partial J_b} \right) \exp\left[-\frac{1}{2\hbar} \phi^T \mathcal{M} \phi - \frac{1}{\hbar} J \cdot \phi \right] \Bigg|_{J=0} \\
&= (-\hbar)^2 \frac{\partial}{\partial J_a} \frac{\partial}{\partial J_b} \exp\left[\frac{1}{2\hbar} J^T \mathcal{M}^{-1} J \right] \Bigg|_{J=0} \\
&= \hbar (\mathcal{M}^{-1})_{ab}.
\end{aligned}$$

Note that the first J derivative brings down an $\mathcal{M}^{-1}J$ (so our expression is of the form $\mathcal{M}^{-1}J \exp(J^T \mathcal{M}^{-1} J)$), and when we take the second J derivative, we will get two terms, one of the form $\mathcal{M}^{-1} \exp(\dots)$ and another of the form $(\mathcal{M}^{-1}J)^2 \exp(\dots)$. The second term is zero when we set $J = 0$, and the exponential becomes 1 in both cases, so we are just left with $\mathcal{M}^{-1}$.

What we have calculated is a two-point function, otherwise known as a propagator (though it's a bit silly to call this a propagator when the spacetime is just a single point). We can associate a Feynman diagram to this process:

There is another method we can use to compute propagators (cf. Osborn §1.3):

$$\begin{aligned}
\mathcal{M}_{ca} \langle \phi_a \phi_b \rangle &= \frac{1}{Z_0} \int d^N\phi\, \mathcal{M}_{ca} \phi_a \phi_b \exp\left[-\frac{1}{2\hbar} \phi^T \mathcal{M} \phi \right] \\
&= -\frac{\hbar}{Z_0} \int d^N\phi\, \phi_b \frac{\partial}{\partial \phi_c} \exp\left[-\frac{1}{2\hbar} \phi^T \mathcal{M} \phi \right] \\
&= \frac{\hbar}{Z_0} \int d^N\phi\, \frac{\partial \phi_b}{\partial \phi_c} \exp\left[-\frac{1}{2\hbar} \phi^T \mathcal{M} \phi \right] \\
&= \hbar \delta_{bc} \implies \langle \phi_a \phi_b \rangle = \hbar (\mathcal{M}^{-1})_{ab}.
\end{aligned}$$

In going from the second to the third line, we have integrated by parts to move the $\frac{\partial}{\partial \phi_c}$ to ϕ_b, and then recognized the remaining integral as Z_0.

More generally, let $l(\phi) = l \cdot \phi = \sum_{a=1}^N l_a \phi_a (\neq 0)$ be a linear function of ϕ, with $l_a \in \mathbb{R}$. Then the expected value $\left\langle l_a(\phi) \dots l_p(\phi) \right\rangle$ is given by

$$\left\langle l_a(\phi) \dots l_p(\phi) \right\rangle = (-\hbar)^p \prod_{i=1}^p \left(l_i \frac{\partial}{\partial J_i} \right) \frac{Z_0[J]}{Z_0} \Bigg|_{J=0}.$$

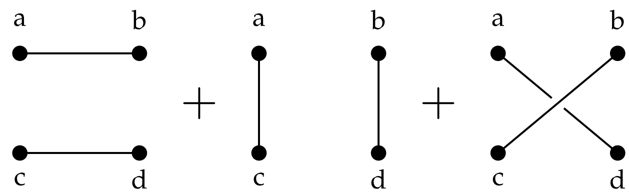
Notice that if we play this game for an odd number of J_i derivatives, all our terms will be of the form $J^p \exp(\dots)$ where p is odd. When we set $J = 0$, all these terms therefore vanish, which tells us that $\left\langle \phi_{a_1} \dots \phi_{a_p} \right\rangle = 0$ for n odd. If we compute it for $p = 2k, k \in \mathbb{N}$, the terms that survive setting $J = 0$ will have k factors of $\mathcal{M}^{-1}$.

Example 2.12. What is the value of the four-point function $\langle \phi_a \phi_b \phi_c \phi_d \rangle$ in free field theory? It is simply

$$\langle \phi_a \phi_b \phi_c \phi_d \rangle = \hbar^2 \Big[(\mathcal{M}^{-1})_{ab} (\mathcal{M}^{-1})_{cd} + (\mathcal{M}^{-1})_{ac} (\mathcal{M}^{-1})_{bd} + (\mathcal{M}^{-1})_{ad} (\mathcal{M}^{-1})_{bc} \Big].$$

Though we haven't said it, this is effectively a toy version of Wick's theorem– we are taking contractions of the fields using $(\mathcal{M}^{-1})$s as propagators.

We can depict these contractions as connecting some $2k$ dots pairwise with lines using a simplified Feynman diagram notation:



In general, the number of distinct ways we can pair $2k$ elements is

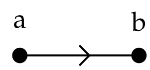
$$\frac{(2k)!}{2^k k!}.$$

The logic here is that we could take all $(2k)!$ permutations of the $2k$ elements, and then take neighboring pairs, e.g. if our elements are $\{a, b, c, d, e, f\}$, one set of pairs is

$$abdcfe \to ab|dc|fe.$$

The order of the 2 elements in each of the k pairs doesn't matter ($ab|dc = ba|dc$), so we've overcounted by a factor of 2^k, and the order of all the k pairs also doesn't matter ($ab|dc = dc|ab$), so we divide by another factor of $k!$ to get the final result.

Example 2.13. One last example– if our free fields are instead complex, $\phi : \{p\} \to \mathbb{C}$, then $\mathcal{M}$ is hermitian. Therefore $(\mathcal{M}^{-1})$ will in general not be symmetric, and so the order of the indices matters. That is, $\langle \phi_a \phi_b^* \rangle = \hbar (\mathcal{M}^{-1})_{ab}$. Then the associated Feynman diagram has an arrow to indicate direction:



Lecture 3.

Thursday, January 24, 2019

Today, we will continue our exploration of zero-dimensional path integrals in quantum field theory.

Interacting theory Let us consider a single real scalar field $\phi : \{\text{point}\} \to \mathbb{R}$. We choose the action

$$S(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4. \tag{3.1}$$

We'll take $\lambda > 0$ for stability and $m^2 > 0$ such that $\min(S)$ lies at $\phi = 0$ so that we can easily expand around the minimum of S.

The path integral is then

$$Z = \int d\phi \exp\left[-\frac{1}{\hbar} \left(\frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right) \right]. \tag{3.2}$$

This will be equivalent to expanding about $\hbar = 0$ (semi-classical limit). We can obviously open up the exponential and rewrite as a series in ϕ and $\hbar$,

$$\begin{aligned} Z &= \int d\phi e^{-\frac{m^2\phi^2}{2\hbar}} \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\lambda}{\hbar\, 4!} \right)^n \phi^{4n} \\ &= \frac{\sqrt{2\hbar}}{m} \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\hbar \lambda}{4! m^4} \right)^n \cdot 2^{2n} \int_0^\infty dx e^{-x} x^{2n + \frac{1}{2} - 1}, \end{aligned}$$

where we have performed a change of variables $x = \frac{m^2 \phi^2}{2\hbar}$. This integral is in fact just a gamma function,

$$\Gamma(2n + \frac{1}{2}) = \frac{(4n)! \sqrt{\pi}}{4^{2n} (2n)!}.$$

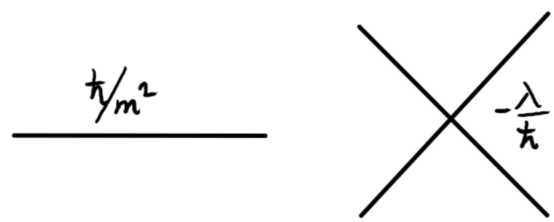
Thus our path integral computation using the gamma function is

$$Z = \frac{\sqrt{2\pi\hbar}}{m} \sum_{n=0}^{N} \left(-\frac{\lambda\hbar}{m^4} \right)^n \underbrace{\frac{1}{(4!)^n n!}}_{(1)} \underbrace{\frac{(4n)!}{2^{2n}(2n)!}}_{(2)}. \tag{3.3}$$

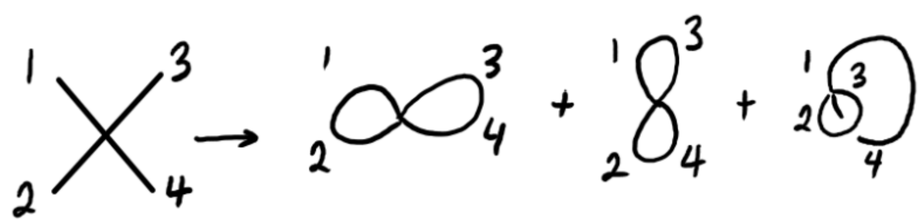
Note that from Stirling's approximation, $n! \approx e^{n\log n}$, Thus these two combinatorial-looking terms scale roughly as $e^{n\log n} \approx n!$. The factorial growth of the coefficients means that this path integral actually has zero radius of convergence. This is an asymptotic series– it looks like it is getting better and better, and then everything goes to hell.[4]

In practice the "true" function can differ from the truncated series by some transcendental function which might be small. Cf. Skinner Ch. 2 for more discussion of asymptotic series.

Note that term (1) in the path integral series expansion 3.3 comes from expanding the $\frac{\lambda]4!\hbar}{\phi}^4$ term in the exponent, while term (2) is the number of ways of joining $4n$ elements in distinct pairs (compare our discussion at the end of the previous lecture). We can associate some Feynman diagrams to this– a propagator and a four-point vertex.



Note also that Z has no ϕ dependence, meaning that the Feynman diagrams have no external legs. Let D_n be the set of *labelled* vacuum diagrams with n vertices, so that D_1 is the following set of diagrams, with $|D_1| = 3$.



Then let G_n be the group which permutes each of the 4 fields at each vertex ($(S_4)^n$) and also permutes the n vertices (S_n). The size of this group is

$$|G_n| = |S_4|^n |S_n| = (4!)^n n!.$$

We therefore recognize that

$$\begin{aligned} \frac{Z}{Z_0} &= \sum_{n=0}^{N} \left(-\frac{\lambda\hbar}{m^4}\right)^n \frac{|D_n|}{|G_n|} \\ &= 1 - \frac{\hbar\lambda}{8m^4} + \frac{35}{384}\frac{\hbar^2\lambda^2}{m^8} + \dots \end{aligned}$$

with $Z_0 = \frac{\sqrt{2\pi\hbar}}{m}$. Physically, we can consider $\frac{|D_n|}{|G_n|}$ to be the sum over topologically distinct graphs divided by a symmetry factor. Equivalently, we write

$$\frac{|D_n|}{|G_n|} = \sum_{\Gamma} \frac{1}{S_\Gamma} \tag{3.4}$$

where Γ is a distinct graph free from labels and S_Γ is the number of permutations of lines and vertices leaving Γ invariant. Some examples appear in Fig.

In dimensions > 0, loops correspond to integrals over internal momenta, so these diagrams may have different contributions aside from the symmetry factors.

[4] What we mean by an asymptotic series is that it converges not in the limit as the number of terms in the power series gets very large but rather as the expansion parameter gets very small.

If we introduce an external source, then our path integral has a generating function

$$Z(J) = \int d\phi \exp -\frac{1}{\hbar}\left(\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + J\phi\right) \tag{3.5}$$

and our correlation functions are modified as before, with $\langle\phi^2\rangle = \frac{(-\hbar)^2}{Z(0)} \frac{\partial^2}{\partial J^2} Z(J)\Big|_{J=0}$. Source terms correspond to lines terminating on vertices J, so that the expansion of $Z(J)$ involves not only $Z(0)$ vacuum diagrams but also diagrams that terminate with even numbers of source vertices.

Lecture 4.

Saturday, January 26, 2019

The official course notes from this class will be available from `www.damtp.cam.ac.uk/user/wingate/AQFT`.

Last time, we computed the ϕ^2 correlation function, $\langle\phi^2\rangle$. In principle this sum also includes disconnected diagrams[5] with "vacuum bubbles." As it turns out, the source-free partition function $Z(0)$ is exactly the sum of the vacuum bubble diagrams, so that when we compute the correlation function, it suffices to consider only connected diagrams.

Effective actions Let's introduce now the *Wilsonian effective action* (named for Ken Wilson of the renormalization group).

Definition 4.1. The Wilsonian effective action W is defined to be

$$Z = e^{-W/\hbar}. \tag{4.2}$$

Schematically,

$$\sum(\text{all vacuum diagrams}) = \exp\left(-\frac{1}{\hbar}\sum(\text{connected diagrams})\right). \tag{4.3}$$

To understand this, note that any diagram D is a product of connected diagrams C_I, such that

$$D = \frac{1}{S_D}\prod_I (C_I)^{n_I}, \tag{4.4}$$

where I indexes over connected diagrams, C_I includes its own internal symmetry factors, n_I is the number of C_Is in D, and S_D is the number of rearranging the identical $C_I s$ in D. That is,

$$S_D = \prod_I (n_I)!. \tag{4.5}$$

Therefore we have

$$\begin{aligned}
\frac{Z}{Z_0} &= \sum_{\{n_I\}} D \\
&= \sum_{\{n_I\}} \prod_I \frac{1}{n_I!}(C_I)^{n_I} \\
&= \prod_I \sum_{n_I} \frac{1}{n_I!}(C_I)^{n_I} \\
&= \exp\left(\sum_I C_I\right) \\
&= e^{-(W-W_0)/\hbar},
\end{aligned}$$

where $W = W_0 - \hbar\sum_I C_I$ is a sum over connected diagrams.

[5]Disconnected means that part of the diagram is not connected to any of the external legs. There are diagrams which look "disconnected" in the informal sense, but in which every line is still connected to an external line (real particle).

Why is W an "effective" action? Consider a theory with two real scalar fields ϕ, χ. Our theory has an action

$$S(\phi, \chi) = \frac{m^2}{2}\phi^2 + \frac{M^2}{2}\chi^2 + \frac{\lambda}{4}\phi^2\chi^2. \tag{4.6}$$

Note there's no factorial in the λ term because the fields are distinguishable.

We can associate some Feynman rules to the theory. Then there are some vacuum bubbles we can draw (see figure) associated to these rules to produce a sum

$$-\frac{W}{\hbar} = -\frac{\hbar\lambda}{m^2M^2}\left(\frac{1}{4}\right) + \left(\frac{\hbar\lambda}{m^2M^2}\right)^2\left(\frac{1}{16} + \frac{1}{16} + \frac{1}{8}\right) + O(\lambda^3). \tag{4.7}$$

Similarly for the connected loop diagrams, we have

$$\left\langle \phi^2 \right\rangle = \frac{\hbar}{m^2}\left(1 - \frac{\hbar\lambda}{m^2M^2}\frac{1}{2} + \left(\frac{\hbar\lambda}{m^2M^2}\right)^2\left[\frac{1}{4} + \frac{1}{4} + \frac{1}{2}\right] + O(\lambda^3)\right). \tag{4.8}$$

This is well and good. We can write down the Feynman rules for the full theory, draw the diagrams, and in principle compute any cross section we like. But now say we want to remove the explicit χ dependence from our theory. That is, maybe the χ particle is very massive, $M \gg m$, and so we are unlikely to see it in our collider. We say that we "integrate out" the heavy field.

For this toy theory, define W such that

$$e^{-W(\phi)/\hbar} = \int d\chi e^{-S(\phi,\chi)/\hbar}. \tag{4.9}$$

Returning to our action, we see that the $\phi^2\chi^2$ term acts like a source term for χ^2.

Correlation functions can then be expressed as

$$\langle f(\phi)\rangle = \frac{1}{Z}\int d\phi d\chi f(\phi) e^{-S(\phi,\chi)/\hbar} = \frac{1}{Z}\int d\phi f(\phi) e^{-W(\phi)/\hbar}, \tag{4.10}$$

with W our new effective action.

In our example, the integral can be done exactly.

$$\int d\chi e^{-S(\phi,\chi)/\hbar} = e^{-m^2\phi^2/2}\sqrt{\frac{2\pi\hbar}{M^2 + \frac{\lambda\phi^2}{2}}}, \tag{4.11}$$

and taking the log we find that

$$W(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\hbar}{2}\log\left(1 + \frac{\lambda}{2M^2}\phi^2\right) + \frac{\hbar}{2}\log\frac{M^2}{2\pi\hbar}. \tag{4.12}$$

For our purposes, this constant piece won't affect QFT correlation functions since it appears both in Z and Z_0. However, these constant energy shifts are important where gravity is concerned, and in principle they should contribute to the cosmological constant of the universe. It's an open problem why the observed Λ is so small compared to the quantum fluctuations that should be contributing to it.

Now in our effective action we can expand the logarithm to get

$$W(\phi) = \left(\frac{m^2}{2} + \frac{\hbar\lambda}{4M^2}\right)\phi^2 - \frac{\hbar\lambda^2}{16M^4}\phi^4 + \frac{\hbar\lambda^3}{48M^6}\phi^6 + \dots \tag{4.13}$$

$$= \frac{m_{\text{eff}}^2}{2}\phi^2 + \frac{\lambda_4}{4!}\phi^4 + \frac{\lambda_6}{6!}\phi^6 + \dots + \frac{\lambda_{2k}}{(2k)!}\phi^{2k} + \dots \tag{4.14}$$

where

$$m_{\text{eff}}^2 = m^2 + \frac{\hbar\lambda}{2M^2}$$

$$\lambda_{2k} = (-1)^{k+1}\hbar\frac{(2k)!}{2^{k+1}k}\frac{\lambda^k}{M^{2k}},$$

This tells us that all new terms are $\propto \hbar$, so these are quantum corrections, and they are also suppressed by $1/M^{2p}$. In a sense, this is very good for our ability to make predictions about the low-energy theory. We can treat these higher order corrections as small and do calculations in our effective theory. But conversely, it will be hard to probe the high energy theory because the corrections are suppressed.

Our toy model was very nice because it had an exact solution, but usually we must find $W(\phi)$ perturbatively. That is, we construct Feynman rules with $\frac{\lambda}{4}\phi^2\chi^2$ as a source term, so that our effective action goes as

$$W(\phi) \sim \frac{m^2\phi^2}{2} + \frac{1}{2}\frac{\hbar\lambda}{2M^2}\phi^2 - \frac{1}{4}\frac{\hbar\lambda^2}{4M^4}\phi^4 + \frac{1}{3!}\frac{\hbar\lambda^3}{8M^6}\phi^6 + \dots, \tag{4.15}$$

as before.

Either way, with our effective action we can then compute correlation functions for ϕ with our effective action, e.g.

$$\left\langle \phi^2 \right\rangle = \frac{1}{Z}\int d\phi\, \phi^2 e^{-W/\hbar} = \frac{\hbar}{m_{\text{eff}} - \frac{\lambda_4\hbar^2}{2m_{\text{eff}}^6}} + \dots, \tag{4.16}$$

as before.

Lecture 5.

Tuesday, January 29, 2019

Last time, we saw our first QFT example of an effective action. We introduced the Wilson effective action $W(J)$, where we averaged over the quantum fluctuations of some degrees of freedom (e.g. a heavy particle). We showed explicitly that we can construct an effective action for a two-particle theory by integrating out one of the fields and treating it as a source,

$$e^{-W(\phi)/\hbar} = \int d\chi e^{-S(\phi,\chi)/\hbar}.$$

Today, we'll show that we can take this further and construct a quantum effective action $\Gamma(\Phi)$ and average over all quantum fluctuations. This will lead us to defined an effective potential $V(\Phi)$. Effective actions of this form help us to determine the true vacuum of a theory and answer questions like "Do quantum effects induce spontaneous symmetry breaking?"

Let us define an average field in the presence of some source J,

$$\begin{aligned}\Phi \equiv \frac{\partial W}{\partial J} &= -\frac{\hbar}{Z(J)}\frac{\partial}{\partial J}\int d\phi e^{-(S[\phi]+J\phi)/\hbar} &(5.1)\\ &= \langle\phi\rangle_J\,, &(5.2)\end{aligned}$$

where W is the Wilson effective action and $J \neq 0$.

Thus $\Gamma(\Phi)$ is defined to be the Legendre transform of $W(J)$, i.e.

$$\Gamma(\Phi) = W(J) - \Phi J. \tag{5.3}$$

Note that

$$\begin{aligned}\frac{\partial\Gamma}{\partial\Phi} &= \frac{\partial W}{\partial\Phi} - J - \Phi\frac{\partial J}{\partial\Phi}\\ &= \underbrace{\frac{\partial W}{\partial J}}_{\Phi}\frac{\partial J}{\partial\Phi} - J - \Phi\frac{\partial J}{\partial\Phi}\\ &= -J,\end{aligned}$$

by applying the chain rule and the definition of Φ. We conclude that

$$J = -\frac{\partial\Gamma}{\partial\Phi}. \tag{5.4}$$

Note also that

$$\frac{\partial\Gamma}{\partial\Phi}|_{J=0} = 0,$$

i.e. in the absence of sources, $J = 0$, the average field $\Phi = \langle\phi\rangle_{J=0}$ corresponds to an extremum of $\Gamma(\Phi)$.

In higher dimensions, we write

$$\Gamma(\Phi) = \int d^d x \left[-V(\Phi) - \frac{1}{2}\partial^\mu \Phi \partial_\mu \Phi + \dots \right], \tag{5.5}$$

where the ... indicate higher derivatives and the first term $V(\Phi)$ is called the *effective potential*.

To make contact with statistical field theory, consider an Ising model, some spins $s(x)$ with an external magnetic field h and a Hamiltonian $\mathcal{H}$. The partition function is

$$Z(h) = e^{-\beta F(h)} = \int \mathcal{D}s \exp\left[-\beta \int d^d x (\mathcal{H}(s) - hs) \right]. \tag{5.6}$$

The magnetization is

$$M = -\frac{\partial F}{\partial h} = \int d^d x \left\langle s(x) \right\rangle, \tag{5.7}$$

and under a Legendre transform we have the Gibbs free energy

$$G = F + hM, \quad \frac{\partial G}{\partial M} = h. \tag{5.8}$$

When $h \to 0$, the equilibrium magnetization is given by the minimum of G.

Returning to QFT, let us try to perturbatively calculate $\Gamma(\Phi)$. We will treat Φ as we did ϕ, i.e. as a proper field. A quantum path integral over Φ then takes the form

$$e^{-W_\Gamma(J)/g} = \int d\Phi e^{-(\Gamma(\Phi)+J\Phi)/g}, \tag{5.9}$$

where g is some "fictional" new Planck constant.

Schematically, $W_\Gamma(J)$ is the sum of connected diagrams with Φ propagators and vertices. Expanding in g (i.e. in loops), we see that

$$W_\Gamma(J) = \sum_{l=0}^{\infty} g^l W_\Gamma^{(l)}(J) \tag{5.10}$$

where $W_\Gamma^{(l)}$ has all the l-loop diagrams.

Tree diagrams are those composing $W_\Gamma^{(0)}(J)$. In the $g \to 0$ (semi-classical?) limit, only tree-level diagrams contribute, so

$$W_\Gamma(J) \approx W_\Gamma^{(0)}(J) \tag{5.11}$$

as $g \to 0$. In addition, as $g \to 0$, our path integral 5.9 over Φ will be dominated by the minimum of the exponent (steepest descent), i.e. the average field Φ such that

$$\frac{\partial \Gamma}{\partial \Phi} = -J.$$

We learn that

$$W_\Gamma(J) = W_\Gamma^{(0)}(J) = \Gamma(\Phi) + J\Phi = W(J), \tag{5.12}$$

where the last equality follows from our earlier definition 5.3. Therefore the sum of connected diagrams $W(J)$ (with action $S(\phi) + J\phi$) can be obtained as the sum of tree diagrams $W_\Gamma^{(0)}(J)$ (with action $\Gamma(\Phi) + J\Phi$).

Definition 5.13. A line (edge) of a connected graph is a *bridge* if removing it would make the graph disconnected.

Definition 5.14. A connected graph is said to be one-particle irreducible (1PI) if it has no bridges.

The quantum effective action $\Gamma(\Phi)$ sums the 1PI graphs of the theory with action $S(\phi)$ yielding many vertices.[6] Then correlation functions can be found using tree graphs with vertices from $\Gamma(\Phi)$.

For example, an N-component field ϕ has a correlation function

$$\left\langle \phi_a \phi_b \right\rangle^{\text{conn}} = \left\langle \phi_a \phi_b \right\rangle - \left\langle \phi_a \right\rangle \left\langle \phi_b \right\rangle, \tag{5.15}$$

[6]??? I think this means we get modified Feynman rules for computing correlation functions.

where the correlation function over connected diagrams is

$$\begin{aligned} -\hbar\frac{\partial^2 W}{\partial J_a \partial J_b} &= \langle \phi_a \phi_b \rangle^{\text{conn}} \\ &= \hbar \left(\frac{\partial^2 \Gamma}{\partial \Phi_a \partial \Phi_b} \right)^{-1}, \end{aligned}$$

which is $\hbar$ times the inverse of the quadratic part of Γ.

Lecture 6.

Thursday, January 31, 2019

Today we'll finish our discussion of the zero-dimensional path integral by introducing fermions to our theory. To model fermions, we will introduce Grassmann variables,[7] i.e. a set of n elements $\{\theta_a\}_{a=1}^n$ obeying anticommutation relations,

$$\theta_a \theta_b = -\theta_b \theta_a. \tag{6.1}$$

Note also that for (complex) scalars $\phi_b \in \mathbb{C}$,

$$\theta_a \phi_b = \phi_b \theta_a, \tag{6.2}$$

i.e. scalars commute with Grassmann variables. In addition, $\theta_a^2 = 0$ by the anticommutation relations, which implies that any function of n Grassmann variables can be written in finite form. That is, polynomials in Grassmann variables are forced to terminate since at some point we run out of distinct Grassmann variables to multiply. A general function $F(\theta)$ can be written

$$F(\theta) = f + \rho_a \theta_a + \frac{1}{2!} g_{ab} \theta_a \theta_b + \ldots + \frac{1}{n!} h_{a_1 \ldots a_n} \theta_{a_1} \ldots \theta_{a_n}. \tag{6.3}$$

Note that the coefficients $\rho, g, \ldots, h$ are totally antisymmetric under interchange of indices.

We also want to define differentiation and integration of these guys. Differentiation anticommutes with the Grassmann variables, i.e.

$$\left(\frac{\partial}{\partial \theta_a} \theta_b + \theta_b \frac{\partial}{\partial \theta_a} \right) * = (\delta_{ab}) * \tag{6.4}$$

where the derivative in the first term acts on everything coming after. This leads us to a modified Leibniz rule.

To define integration, note that for a single Grassmann variable θ, a function takes the form

$$F(\theta) = f + \rho\theta, \tag{6.5}$$

so we just need to define $\int d\theta$ and $\int d\theta\, \theta$. If we require translational invariance, i.e.

$$\int d\theta (\theta + \eta) = \int d\theta\theta \implies \int d\theta = 0. \tag{6.6}$$

We can then choose the normalization so that $\int d\theta\, \theta = 1$. Note the similarity between differentiation and integration (i.e. an integral $\int d\theta\, \theta = \frac{\partial}{\partial\theta} \theta = 1$). This process is called *Berezin integration*. Using these rules, we also find that

$$\int d\theta \frac{\partial}{\partial \theta} F(\theta) = 0, \tag{6.7}$$

since the term linear in θ will go to a constant by the derivative and be killed by the integral, and any constant terms will be killed by the derivative. Either way the result is zero.

Suppose now we have n Grassmann variables. Then the only nonvanishing integrals involve exactly one power of each integration variable, e.g.

$$\int d^n\theta\, \theta_1 \theta_2 \ldots \theta_n = \int d\theta_n d\theta_{n-1} \ldots d\theta_1\, \theta_1 \theta_2 \ldots \theta_n = 1. \tag{6.8}$$

[7] We've seen these in *Supersymmetry* already.

In general we can just anticommute the Grassmann variables until they're in the right order, picking up a factor for the parity of the permutation. That is,

$$\int d^n\theta \theta_{a_1}\theta_{a_2}\dots\theta_{a_n} = \epsilon^{a_1 a_2 \dots a_n}, \tag{6.9}$$

where ϵ is the totally antisymmetric symbol with value $+1$ for even permutations of $1, 2, \dots, n$, -1 for odd permutations, and 0 if any indices are repeated.

What if we now make a change of variables $\theta'_a = A_{ab}\theta_b$? Then

$$\int d^n\theta \theta'_{a_1}\theta'_{a_2}\dots\theta'_{a_n} = A_{a_1 b_1}\dots A_{a_n b_n} \underbrace{\int d^n\theta\, \theta_{b_1}\dots\theta_{b_n}}_{\epsilon^{b_1\dots b_n}} \tag{6.10}$$

$$= \det A\, \epsilon^{a_1\dots a_n} \tag{6.11}$$

$$= \det A \int d^n\theta'\, \theta'_{a_1}\dots\theta'_{a_n} \tag{6.12}$$

We conclude that under a change of variables, the integration measures are related by

$$d^n\theta = \det A\, d^n\theta'. \tag{6.13}$$

Note that this is the opposite of the convention for scalars, where

$$\phi'_a = A_{ab}\phi_b \implies d^n\phi = \frac{1}{|\det A|} d^n\phi'. \tag{6.14}$$

Free fermion field theory Consider $d = 0$, with two fermion fields θ_1, θ_2. The action must be bosonic (scalar), so the only possible nonconstant action is

$$S(\theta) = \frac{1}{2}A\theta_1\theta_2, A \in \mathbb{R} \tag{6.15}$$

Then the path integral is

$$Z_0 = \int d^2\theta\, e^{-S(\theta)/\hbar} = \int d^2\theta \left(1 - \frac{A}{2\hbar}\theta_1\theta_2\right) = -\frac{A}{2\hbar}, \tag{6.16}$$

where the exponential has terminated thanks to our Grassmann variables.

Suppose now we have $n = 2m$ fermion fields θ_a. Then our action might be quadratic in the fields,

$$S = \frac{1}{2}A_{ab}\theta_a\theta_b \tag{6.17}$$

with A an antisymmetric matrix, and the path integral is then

$$\begin{aligned} Z_0 &= \int d^{2m}\theta\, e^{-S(\theta)/\hbar} = \int d^{2m}\theta \sum_{j=0}^{m} \frac{(-1)^j}{(2\hbar)^j j!}(A_{ab}\theta_a\theta_b)^j \\ &= \frac{(-1)^m}{(2\hbar)^m m!}\int d^{2m}\theta A_{a_1a_2}A_{a_3a_4}\dots A_{a_{2m-1}a_{2m}}\theta_{a_1}\theta_{a_2}\dots\theta_{2m} \\ &= \frac{(-1)^m}{(2\hbar)^m m!}\epsilon^{a_1a_2\dots a_{2m}}A_{a_1a_2}A_{a_3a_4}\dots A_{a_{2m-1}a_{2m}} \\ &= \frac{(-1)^m}{\hbar^m}\mathrm{Pf}(A), \end{aligned}$$

where $\mathrm{Pf}(A)$ is the *Pfaffian* of the matrix A, defined by

$$\mathrm{Pf}(A) \equiv \frac{1}{2^m}\epsilon^{a_1a_2\dots a_{2m}}A_{a_1a_2}A_{a_3a_4}\dots A_{a_{2m-1}a_{2m}}, \tag{6.18}$$

which we will show on the examples sheet is in fact $\pm\sqrt{\det A}$. Thus $\mathrm{Pf}\begin{pmatrix} 0 & -1 \\ a & 0 \end{pmatrix} = a$. Using this property, we find that for fermionic fields,

$$Z_0 = \pm\sqrt{\frac{\det A}{\hbar^n}} \tag{6.19}$$

with A antisymmetric, whereas for bosonic fields with some symmetric mass matrix M,[8] we have

$$Z_0 = \sqrt{\frac{(2\pi\hbar)^n}{\det M}}. \tag{6.20}$$

We can now introduce an external source function to our action, a Grassmann-values $\{\eta_a\}$, such that the new action is

$$S(\theta, \eta) = \frac{1}{2} A_{ab}\theta_a\theta_b + \eta_a\theta_b. \tag{6.21}$$

Taking care to respect the anticommutation relations and completing the square as before, we can rewrite the action as

$$S(\theta, \eta) = \frac{1}{2}(\theta_a + \eta_c(A^{-1})_{ca})A_{ab}(\theta_b + \theta_d(A^{-1})_{db}) + \frac{1}{2}\eta_a(A^{-1})_{ab}\eta_b. \tag{6.22}$$

We can make a change of variables using the translational invariance of θ_a and pull out the constant factor to find

$$Z_0(\eta) = \exp\left(-\frac{1}{2\hbar}\eta^T(A^{-1})\eta\right)Z_0(0). \tag{6.23}$$

This allows us to get propagators by taking derivatives with respect to the source η, as we are wont to do:

$$\langle\theta_a\theta_b\rangle = \frac{\hbar^2}{Z_0(0)}\frac{\partial^2 Z_0(\eta)}{\partial\eta_a\partial\eta_b}|_{\eta=0} = \hbar(A^{-1})_{ab}. \tag{6.24}$$

We see that the propagator is proportional to the inverse of the bilinear part of the action for Grassmann variables.

Lecture 7.

Saturday, February 2, 2019

Quick admin note: there are some typos on Example Sheet 1. The expression in problem 1 should read

$$\exp\left(\frac{im(x-x_0)^2}{2(t-t_0)}\right),$$

where the denominator is not squared, and in problem 2,

$$\exp\left(\frac{\ldots - 2xx_0}{\ldots}\right).$$

Today we shall return to the world of $3+1$ dimensions and set path integrals aside for a moment. Our main result today is the LSZ reduction formula, named for Lehmann-Symanzik-Zimmermann (cf. Srednicki §5). This result provides a direct relationship between scattering amplitudes. For example, consider the $2 \to 2$ scattering of real scalar particles. For a free scalar, we have the field written in terms of creation and annihlation operators,

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2E}\left[a(\mathbf{k})e^{-ik\cdot x} + a^\dagger(\mathbf{k})e^{ik\cdot x}\right] \tag{7.1}$$

where $k \cdot x = Et - \mathbf{k}\cdot\mathbf{x}$, using the mostly minus $(+---)$ signature.

Equivalently we can Fourier transform the field to find expressions for $a, a^\dagger$ in terms of the field ϕ:

$$\int d^3x\, e^{ik\cdot x}\phi(x) = \frac{1}{2E}a(\mathbf{k}) + \frac{1}{2E}e^{2iEt}a^\dagger(-\mathbf{k}),$$
$$\int d^3x\, e^{ik\cdot x}\partial_0\phi(x) = -\frac{i}{2}a(\mathbf{k}) + \frac{i}{2}e^{2iEt}a^\dagger(-\mathbf{k}),$$

which tells us that

$$a(\mathbf{k}) = \int d^3x e^{ik\cdot x}(i\partial_0\phi(x) + E\phi(x)) \tag{7.2}$$

$$a^\dagger(\mathbf{k}) = \int d^3x e^{-ik\cdot x}(-i\partial_0\phi(x) + E\phi(x)). \tag{7.3}$$

[8]That is, for an action $S = \frac{1}{2}M_{ab}\phi_a\phi_b$.

Now for the free theory, a one-particle state is given by

$$|k\rangle = a^\dagger(\mathbf{k})|0\rangle, \tag{7.4}$$

with $|0\rangle$ the normalized vacuum state such that $\langle 0|0\rangle = 1$ and $a(\mathbf{k})|0\rangle = 0 \forall \mathbf{k}$. We require that these momentum eigenstates are (relativistically) normalized such that

$$\langle k'|k\rangle = (2\pi)^3(2E)\delta^{(3)}(\mathbf{k}-\mathbf{k}'), \tag{7.5}$$

with $E = \sqrt{|\mathbf{k}|^2 + m^2}$. We can now introduce a Gaussian wavepacket in momentum space by

$$a_1^\dagger \equiv \int d^3k f_1(\mathbf{k}) a^\dagger(k) \tag{7.6}$$

where

$$f_1(\mathbf{k}) \propto \exp\left[-\frac{(\mathbf{k}-\mathbf{k}_1)^2}{4\sigma^2}\right] \tag{7.7}$$

for some $\mathbf{k}_1, \sigma$. We can define a second particle with $a_2^\dagger$ for some $f_2, \mathbf{k}_2$ such that $\mathbf{k}_2 \neq \mathbf{k}_1$.

Now if we evolve Gaussian wavepackets from the far distant past (or future), the overlap between the Gaussians in coordinate space should be small (the particles are far apart in the past and future). Thus their interaction is effectively limited in both space and time to some bounded interaction region.

We shall assume this works even when interactions are present. However, there is a complication– $a^\dagger(\mathbf{k})$ becomes time-dependent, e.g. their energies depend on their proximity to other particles, and therefore $a_1^\dagger(t), a_2^\dagger(t)$ are now functions of time. We therefore assume that as $t \to \pm\infty$, the wavepacket operators $a_1^\dagger, a_2^\dagger$ coincide with the free theory expressions.

Our initial and final (in/out) states are therefore

$$|i\rangle = \lim_{t\to-\infty} a_1^\dagger(t) a_2^\dagger(t)|0\rangle \tag{7.8}$$

$$|f\rangle = \lim_{t\to+\infty} a_{1'}^\dagger(t) a_{2'}^\dagger(t)|0\rangle \tag{7.9}$$

where initial and final states are normalized, $\langle i|i\rangle = \langle f|f\rangle = 1$, and $\mathbf{k}_1 \neq \mathbf{k}_2, \mathbf{k}_1' \neq \mathbf{k}_2'$. The scattering amplitude is then the overlap of the initial and final states, $\langle f|i\rangle$.

Note that

$$\begin{aligned}
a_1^\dagger(\infty) - a_1^\dagger(-\infty) &= \int_{-\infty}^{\infty} dt\, \partial_0 a_1^\dagger(t) \\
&= \int d^3k f_1(\mathbf{k}) \int d^4x \partial_0 \left[e^{-ik\cdot x}(-i\partial_0\phi E\phi)\right] \\
&= -i \int d^3k f_1(\mathbf{k}) \int d^4x e^{-ik\cdot x}(\partial_0^2 + E^2)\phi \\
&= -i \int d^3k f_1(\mathbf{k}) \int d^4k e^{-ik\cdot x}(\partial_0^2 + |\mathbf{k}|^2 + m^2)\phi.
\end{aligned}$$

In going from the second to third line, the cross terms from the ∂_0 derivative cancel. We also recognize that $|\mathbf{k}|^2 e^{-ik\cdot x}\phi(x) = -\nabla^2(e^{-ik\cdot x})\phi(x) = -e^{-ik\cdot x}\nabla^2\phi(x)$ by integrating by parts. Therefore this last line becomes

$$a_1^\dagger(\infty) - a_1^\dagger(-\infty) = -i \int d^3k f_1(\mathbf{k}) \int d^4x e^{-ik\cdot x}(\partial^2 + m^2)\phi. \tag{7.10}$$

Note that in a free theory, the Klein-Gordon equation tells us that $(\partial^2 + m^2)\phi = 0$, so that $a_1^\dagger(\infty) = a_1^\dagger(-\infty)$.

Now

$$\langle f|i\rangle = \langle 0|\mathcal{T} a_{1'}(\infty) a_{2'}(\infty) a_1^\dagger(-\infty) a_2^\dagger(-\infty)|0\rangle, \tag{7.11}$$

where $\mathcal{T}$ indicates time ordering. Of course, the expression is already time ordered, so we can insert it for free. We can then use equations like 7.10 to substitute

$$a_j^\dagger(-\infty) = a_j^\dagger(\infty) + i \int d^3k\, f_j(\mathbf{k}) \int d^4x\, e^{-ik\cdot x}(\partial^2 + m^2)\phi, \tag{7.12}$$

and something similar for $a_{j'}(\infty) = a_{j'}(-\infty) + \ldots$. Time ordering then moves $a_j^\dagger(\infty)$ to the left, annihilating $\langle 0|$ and $a_{j'}(-\infty)$ to the right, annihilating $|0\rangle$. What remains is the integral terms, which form the LSZ formula:

$$\langle f|i\rangle = (i)^4 \int d^4x_1 d^4x_2 d^4x_{1'} d^4x_{2'} e^{-ik_1\cdot x_1} e^{-ik_2\cdot x_2} e^{ik_{1'}\cdot x_{1'}} e^{ik_{2'}\cdot x_{2'}} \\ \times (\partial_1^2 + m^2)(\partial_{1'}^2 + m^2)(\partial_2^2 + m^2)(\partial_{2'}^2 + m^2)\langle 0|\mathcal{T}\phi(x_1)\phi(x_2)\phi(x_{1'})\phi(x_{2'})|0\rangle,$$

having taken the $\sigma \to 0$ limit in all the $f_j(\mathbf{k})$ so to get delta functions $\delta^{(3)}(\mathbf{k} - \mathbf{k}_j)$. It's this last term, the expectation value of the time-ordered fields, which contains all the physics.

We have the following assumptions in this formula (noting that the interacting ϕ is not exactly like the free ϕ field):

- We assume there is a unique ground state so that the first excited state is a single particle.
- We also want $\phi|0\rangle$ to be a single particle, i.e. $\langle 0|\phi|0\rangle = 0$. If instead $\langle 0|\phi|0\rangle = v \neq 0$, we simply redefine the field by a shift, $\tilde{\phi} = \phi - v$ such that $\langle 0|\tilde{\phi}|0\rangle = 0$.
- We want ϕ normalized such that $\langle k|\phi|0\rangle = e^{ik\cdot x}$ as in the free case. With interactions, we may need to instead rescale $\phi \to Z_\phi^{1/2}\phi$.

With these assumptions (and some careful thought about multi-particle states), the LSZ formula still applies. For instance,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \text{interactions} \\ \to \frac{1}{2}Z_\phi\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}Z_m m^2\phi^2 + \ldots$$

after renormalization.

Lecture 8.

Tuesday, February 5, 2019

Today we will begin our discussion of scalar field theory in the path integral formalism. Let us begin with a preliminary note that we can trivially shift time variables from $it \to \tau$ and thereby go from a Minkowski to Euclidean metric. Thus in Minkowski (with signature $+---$) we have a Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$$

(so the kinetic term has a $+$ sign) and in Euclidean signature $(++++)$ we have

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi).$$

For instance, we might have some potential like $V(\phi) = \frac{1}{2}m^2\phi^2 + sum_{n>2}\frac{1}{n!}V^{(n)}\phi^n$.

Our path integral is then

$$Z = \int \mathcal{D}\phi e^{i\int dx^0 d^3x \mathcal{L}} = \int \mathcal{D}\phi e^{-\int dx_4 d^3x \mathcal{L}}, \tag{8.1}$$

where we have defined $ix^0 = x_4$ and work in units with $\hbar = 1$.

The Minkowski propagator takes the form

$$\frac{i}{k^2 - m^2 + i\epsilon} = \frac{i}{(k^0)^2 - |\mathbf{k}|^2 - m^2 + i\epsilon}, \tag{8.2}$$

whereas in Euclidean signature we have instead

$$\frac{1}{k^2 + m^2}. \tag{8.3}$$

In Euclidean signature, we do not need to move the poles since they no longer lie on the real axis.

Generating functional We have written down a free field action with a source (cf. Srednicki §8):

$$S_0[\phi, J] = \int d^4x(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi = \frac{1}{2}m^2\phi^2 + J(x)\phi(x)). \tag{8.4}$$

Taking the Fourier transform of the field we have

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx}\tilde{\phi}(k). \tag{8.5}$$

In terms of the Fourier transformed field, we get an action

$$S_0[\tilde{\phi}, \tilde{J}] = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\left[\tilde{\phi}(-k)(k^2+m^2)\tilde{\phi}(k) + \tilde{J}(-k)\tilde{\phi}(k) + \tilde{J}(k)\tilde{\phi}(-k)\right]. \tag{8.6}$$

Our aim will be to construct a partition function $Z[J]$, integrating out ϕ. To do this, let us rewrite our action in terms of the shifted field

$$\tilde{\chi}(k) \equiv \tilde{\phi}(K) + \frac{\tilde{J}(k)}{k^2+m^2}, \tag{8.7}$$

completing the square. If we make this change of variables we get

$$S_0[\tilde{\phi}, \tilde{J}] = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\left[\tilde{\chi}(-k)(k^2+m^2)\tilde{\chi}(k) + \frac{\tilde{J}(-k)\tilde{J}(k)}{k^2+m^2}\right]. \tag{8.8}$$

The χ path integral is just over a Gaussian. If we assume normalization such that $Z_0[0] = 1$, we find that

$$Z_0[\tilde{J}] = \exp\left[-\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{\tilde{J}(-k)\tilde{J}(k)}{k^2+m^2}\right] \tag{8.9}$$

and Fourier transforming back, we have

$$Z_0[J] = \exp\left[-\frac{1}{2}\int d^4x d^4x' J(x)\Delta(x-x')J(x')\right], \tag{8.10}$$

where the Feynman propagator is

$$\Delta(x-x') \equiv \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot(x-x')}}{k^2+m^2}. \tag{8.11}$$

Recall that the Feynman propagator is a Green's function of the Klein-Gordon equation, such that

$$(\partial_x^2 + m^2)\Delta(x-x') = \delta^{(4)}(x-x'),$$

and (cf. Tong QFT §2.7.1) the Feynman propagator is also related to the time-ordered product

$$\Delta(x-x') = \langle 0|\mathcal{T}\phi(x)\phi(x')|0\rangle.$$

With these facts in mind, we observe that

$$\langle 0|\mathcal{T}\phi(x)\phi(x')|0\rangle = \left(-\frac{\delta}{\delta J(x)}\right)\left(-\frac{\delta}{\delta J(x')}\right)Z_0[J]|_{J=0}. \tag{8.12}$$

Here, we use the functional derivative notation that $\frac{\delta}{\delta f(x_1)}f(x_2) = \delta(x_1 - x_2)$. This is naturally the continuous generalization of $\frac{\partial}{\partial x_i}x_j = \delta_{ij}$.

Similarly, the four-point function (still in free theory) is the sum of the three unique Wick contractions of the four fields,

$$\langle 0|\mathcal{T}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle = [\Delta(x_1-x_2)\Delta(x_3-x_4) + \Delta(x_1-x_3)\Delta(x_2-x_4) + \Delta(x_1-x_4)\Delta(x_2-x_3)]. \tag{8.13}$$

The results of our 0-dimensional calculation apply, with the slight complication that the propagator $\Delta(x-x')$ is non-trivial. To complete the story, let us now turn on interactions and see what happens (cf. Srednicki §10). We write the full, exact propagator as

$$\mathbf{\Delta}(x_1 - x_2) \equiv \langle 0|\mathcal{T}\phi(x_1)\phi(x_2)|0\rangle. \tag{8.14}$$

Note that $|0\rangle$ is the interacting vacuum, not the free theory vacuum from before. Using the Wilsonian effective action $W[J] = -\log Z[J]$ and the notation that

$$\delta_i \equiv -\frac{\delta}{\delta J(x_i)}, \tag{8.15}$$

we see that the propagator now takes the form

$$\boldsymbol{\Delta}(x_1 - x_2) = \delta_1\delta_2 Z[J]|_{J=0} = -\delta_1\delta_2 W[J]|_{J=0} + (\delta_1 W[J])(\delta_2 W[J])|_{J=0}. \tag{8.16}$$

If we assume that $\langle 0|\phi(x_1)|0\rangle = -\delta_i W[J]|_{J=0} = 0$ (i.e. the field has no VEV), the result is therefore just the first term:

$$\boldsymbol{\Delta}(x_1 - x_2) = -\delta_1\delta_2 W[J]|_{J=0}. \tag{8.17}$$

If we consider the interacting theory four-point function, we find that

$$\begin{aligned}\langle 0|\mathcal{T}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle &= \delta_1\delta_2\delta_3\delta_4 Z[J]|_{J=0}\\ &= [-\delta_1\delta_2\delta_3\delta_4 W + (\delta_1\delta_2 W)(\delta_3\delta_4 W)\\ &\quad + (\delta_1\delta_3 W)(\delta_2\delta_4 W) + (\delta_1\delta_4 W)(\delta_2\delta_3 W)]_{J=0}.\end{aligned}$$

We now show that these last three terms are either zero or trivial (non-interacting). Consider the LSZ formula for $2 \to 2$ scattering:

$$\begin{aligned}\langle f|i\rangle = (i)^4 \int d^4x_1 d^4x_2 d^4x_{1'} d^4x_{2'} e^{-ik_1\cdot x_1} e^{-ik_2\cdot x_2} e^{ik_{1'}\cdot x_{1'}} e^{ik_{2'}\cdot x_{2'}}\\ \times (\partial_1^2 + m^2)(\partial_{1'}^2 + m^2)(\partial_2^2 + m^2)(\partial_{2'}^2 + m^2)\langle 0|\mathcal{T}\phi(x_1)\phi(x_2)\phi(x_{1'})\phi(x_{2'})|0\rangle,\end{aligned}$$

where we have Wick rotated back to Minkowski signature. Consider the term $(\delta_1\delta_3 W)(\delta_2\delta_4 W)$. This term can be rewritten as $\boldsymbol{\Delta}(x_1 - x_{1'})\boldsymbol{\Delta}(x_2 - x_{2'})$. We use the notation

$$F(x_{ij}) = (\partial_i^2 + m^2)(\partial_j^2 + m^2)\boldsymbol{\Delta}^{(m)}(x_{ij}),$$

where the superscript m indicates the propagator is being computed in Minkowski signature. We define $x_{ij'} = x_i - x_{j'}, \bar{k}_{ij} = \frac{1}{2}(k_i + k_{j'}$, and $\tilde{F}(k)$ indicates the Fourier transform of F. Thus the contribution of the $(13)(24)$ terms to $\langle f|i\rangle$ is

$$\int d^4x_1 d^4x_2 d^4x_{1'} d^4x_{2'} e^{(\dots)} F(x_{11'})F(x_{22'}) = (2\pi)^8 \delta^{(4)}(k_1 - k_{1'})\delta^{(4)}(k_2 - k_{2'})\tilde{F}(\bar{k}_{11'})\tilde{F}(\bar{k}_{22'})$$

But looking at these delta functions, we see that they set $k_1 = k_{1'}, k_2 = k_{2'} \implies$ there is no scattering. The other terms are similar. We conclude that the interesting bit is

$$\langle 0|\mathcal{T}\phi(x_1)\dots\phi(x_n)|0\rangle_C \equiv -\delta_1\dots\delta_n W[J]|_{J=0}, \tag{8.18}$$

where the C on the left indicates connected diagrams and the RHS is fully connected diagrams.

Lecture 9.

Thursday, February 7, 2019

Today we'll turn on interactions and try to understand path integrals/generating functionals in an interacting theory, cf. Osborn §2.2.

Feynman rules We start by stating the following identity: for functions F, G,

$$G(-\frac{\partial}{\partial J})F(-J) = F(\frac{\partial}{\partial \phi}G(\phi)e^{-J\phi}|_{\phi=0}. \tag{9.1}$$

Example 9.2. Here's an example. Let $F(J) = e^{\beta J}$ and $G(\phi) = e^{\alpha\phi}$. Evaluating the LHS of our identity, we have

$$\begin{aligned} G(-\frac{\partial}{\partial J})F(-J) &= e^{-\alpha\frac{\partial}{\partial J}}e^{-\beta J} \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}(-\alpha\frac{\partial}{\partial J})^n e^{-\beta J} \\ &= e^{\alpha\beta}e^{-\beta J} = F(\alpha - J). \end{aligned}$$

On the RHS we have instead

$$\begin{aligned} F(\frac{\partial}{\partial\phi})G(\phi)e^{-J\phi}|_{\phi=0} &= e^{\beta\frac{\partial}{\partial\phi}}e^{\alpha\phi - J\phi}|_{\phi=0} \\ &= e^{-\beta(\alpha-J)} = F(\alpha - J). \end{aligned}$$

Really, this is a notational abuse– we are using these functions both as maps on some values/fields ϕ, J and also on differential operators. But the result is valid[9] and for general F, G we may write these as Fourier series and proceed as above.

We will employ this identity in interacting scalar field theory in the form

$$e^{-\mathcal{L}_{int}(-\frac{\partial}{\partial J})}e^{-\frac{1}{2}J\Delta J} = e^{-\frac{1}{2}\frac{\partial}{\partial\phi}\Delta\frac{\partial}{\partial\phi}}e^{-\mathcal{L}_{int}(\phi)-J\phi}|_{\phi=0}, \tag{9.3}$$

where we will promote J, ϕ to fields.

In interacting scalar field theory, we can separate the Lagrangian into a free part and an interacting part,

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}, \quad \mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2. \tag{9.4}$$

Now the generating functional for this theory (possibly in the presence of a source J) takes the form

$$Z[J] = \int\mathcal{D}\phi\exp\left[-\int d^4x(\mathcal{L}_0 + \mathcal{L}_{int} + J\phi)\right] \tag{9.5}$$

$$= \exp\left\{-\int d^4y\mathcal{L}_{int}\left[-\frac{\partial}{\partial J}\right]\right\}\underbrace{\int\mathcal{D}\phi\exp\left[-\int d^4x(\mathcal{L}_0 + J\phi)\right]}_{Z_0[J]} \tag{9.6}$$

$$= \exp\left\{-\int d^4y\mathcal{L}_{int}\left[-\frac{\partial}{\partial J}\right]\right\}\exp\left[-\frac{1}{2}\int d^4xd^4x'J(x)\Delta(x-x')J(x')\right] \tag{9.7}$$

$$= \exp\left[-\frac{1}{2}\int d^4xd^4x'\frac{\delta}{\delta\phi(x)}\Delta(x-x')\frac{\delta}{\delta\phi(x')}\right]\exp\left[-\int d^4y(\mathcal{L}_{int}[\phi] + J(y)\phi(y)\right]|_{\phi=0}. \tag{9.8}$$

In line 9.6, we have used the fact that $\left(\frac{\delta}{\delta J(y)}\right)e^{-d^4xJ\phi} = \phi(y)e^{-\int d^4xJ\phi}$. In the next line, we used our free theory result for $Z_0[J]$. In the last line, we have used our identity, Eqn. 9.3.

The (position space) Feynman rules are then based on the series expansion of exponentials in $Z[J]$.

- Propagators come with factors of $\Delta(x - x')$.
- Vertices with n lines come from $\left(\frac{\delta}{\delta\phi(y)}\right)^n(-\mathcal{L}_{int}[\phi])|_{\phi=0} \equiv v^{(n)}$.
- Integrate over the positions of all internal vertices.
- Add symmetry factors as before.

Of course, it's usually more illuminating to do our calculations in momentum space instead. A Fourier transform will take us there. We can write down a momentum space propagator

$$\tilde{\Delta}(k) = \int d^4y\Delta(y)e^{-ik\cdot y} = \frac{1}{k^2 + m^2}. \tag{9.9}$$

Our integrals over position now become δ functions which conserve momentum at each vertex, and we will always get an overall factor $(2\pi)^4\delta^{(4)}(\sum_j p_j)$ where the sum is taken over external momenta. The momentum space Feynman rules are as follows:

[9] At least for sufficiently nice functions, I assume.

- Propagators get factors of $\frac{1}{k^2+m^2}$.
- Vertices get factors of $(2\pi)^4\delta^{(4)}(\sum p_i)$ where p_i is taken over momenta going into a vertex (or out, if you prefer)
- Integrate over all internal momenta with $\int \frac{d^4k}{(2\pi)^4}$.

For fully connected diagrams[10] we have a nice graph theory property due to Euler:

$$L = I - V + 1, \tag{9.10}$$

where L is the number of loops, I is the number of internal lines, and V is the number of vertices. We can use this to simplify some integrals by

$$\int \left[\prod_{i=1}^{I} \frac{d^4k_i}{(2\pi)^4}\right]\left[\prod_{v=1}^{V}(2\pi)^4\delta^{(4)}(\sum_j p_{j,v})\right]\dots \tag{9.11}$$

where ... indicates some integrand. We can therefore factor out the momentum-conserving delta function and do $V-1$ integrals over the rest of the δ functions, so we are left with L nontrivial integrals. The factors of 2π work out too: $\left(\frac{1}{(2\pi)^4}\right)^I(2\pi)^{4V} = \frac{1}{(2\pi)^{4(L-1)}}$.

We get the following simplified rules:

- External lines get $\frac{1}{p^2+m^2}$ factors
- Internal lines get $\frac{1}{k^2+m^2}$ factors
- n-point vertices get factors of $v^{(n)}$
- Impose momentum conservation at each vertex
- Integrate over each undetermined loop momentum (1 for each loop)
- Strip off the overall momentum conserving delta function $(2\pi)^4\delta^{(4)}(\sum_j p_j)$.

For example, if $\mathcal{L}_{int}$ contains a $\frac{\lambda}{4!}\phi^4$ term, then we get a one-loop diagram, resulting in

$$\frac{1}{2}\frac{1}{(p_1^2+m^2)(p_2^2+m^2)}(2\pi)^4\delta^{(4)}(p_1-p_2)(-\lambda)\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}. \tag{9.12}$$

Unfortunately, this is infinity. We'll see what to do with this a little later. If $\mathcal{L}_{int}$ instead contains $\frac{g}{3!}\phi^3$, we get a matrix element

$$\frac{1}{2}\frac{1}{(p_1^2+m^2)(p_2^2+m^2)}(2\pi)^4\delta^{(4)}(p_1-p_2)(-g)^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}\frac{1}{(k-p_1)^2+m^2} \tag{9.13}$$

Lecture 10.

Saturday, February 9, 2019

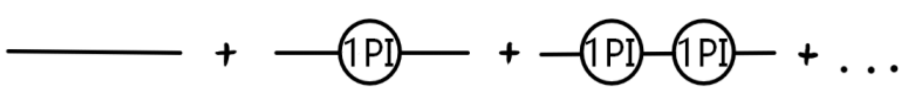
Today we'll begin our discussion of renormalization and why infinities might not be so scary after all (cf. Skinner §5.1). Let us consider ϕ^4 theory:

$$S[\phi] = \int d^4x\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right]. \tag{10.1}$$

In momentum space, the full propagator $\tilde{\mathbf{\Delta}}(p^2)$ takes the form

$$\tilde{\mathbf{\Delta}}(p^2) = \int d^4x e^{-ip\cdot x}\langle\phi(x)\phi(0)\rangle_{\text{connected}}. \tag{10.2}$$

Schematically, we can represent the propagator as the following sum of diagrams:

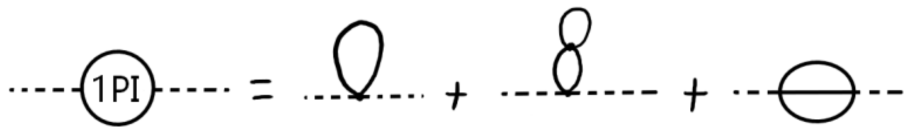


[10] In David Tong's notes, he refers to connected diagrams where every point is connected to an external line, and *fully connected diagrams*, where all points are connected to all other points. This distinction was previously missed in these lectures.

or equivalently the following geometric series:

$$\begin{aligned}\tilde{\mathbf{\Delta}}(p^2) &= \frac{1}{p^2+m^2} + \frac{1}{p^2+m^2}\Pi(p^2)\frac{1}{p^2+m^2} + \frac{1}{p^2+m^2}\Pi(p^2)\frac{1}{p^2+m^2}\Pi(p^2)\frac{1}{p^2+m^2} + \dots \\ &= \frac{1}{p^2+m^2-\Pi(p^2)}\end{aligned}$$

where $\Pi(p^2)$ is called the self-energy. Note that Π contributes to the quantum effective action Γ. Perturbatively, we get contributions from diagrams like the following:



Note that dashed lines are omitted from the computation of the 1PI factor $\Pi(p^2)$ since they are external propagators and already accounted for in our expansion.

One of the simplest diagrams we can draw is the one-loop diagram, and it corresponds to the amplitude

$$-\frac{\lambda}{2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}. \tag{10.3}$$

This is divergent, since the integral goes as d^4k/k^2. To see this explicitly, let us introduce an ultraviolet (UV) cutoff Λ so that we integrate over only momenta with $|k| < \Lambda$. Since the integral depends only on k^2, we can change to spherical coordinates and integrate:

$$-\frac{\lambda S_4}{2(2\pi)^4}\int_0^\Lambda \frac{k^3dk}{k^2+m^2} = -\frac{\lambda S_4 m^2}{4(2\pi)^4}\int_0^{\Lambda^2/m^2}\frac{udu}{1+u} = -\frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2\log\left(1+\frac{\Lambda^2}{m^2}\right)\right], \tag{10.4}$$

where $d^dk = S_d|k|^{d-1}d|k|$ with $S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ and we've made the substitution $u = k^2/m^2$ to perform the integral.[11] After performing the integral, we arrive at an amplitude that indeed diverges as $\Lambda \to \infty$.

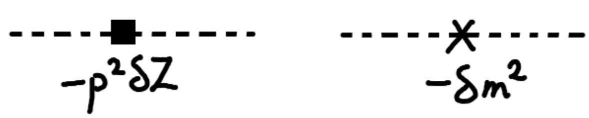
Suppose we allow the coupling to depend on Λ by adding "counterterms" to the action. That is,

$$S[\phi] \to S[\phi] + (\hbar)S^{CT}[\phi,\Lambda]. \tag{10.5}$$

For instance, we might define a set of counterterms as

$$S^{CT}[\phi,\Lambda] = \int d^4x\left[\frac{\delta Z(\Lambda)}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\delta m^2(\Lambda)\phi^2 + \frac{\delta\lambda(\Lambda)}{4!}\phi^4\right]. \tag{10.6}$$

Note that these correspond to some new vertices and thus new contributions to $\Pi(p^2)$. The δZ coupling will contribute to a new two-point vertex, as will the δm^2 coupling:

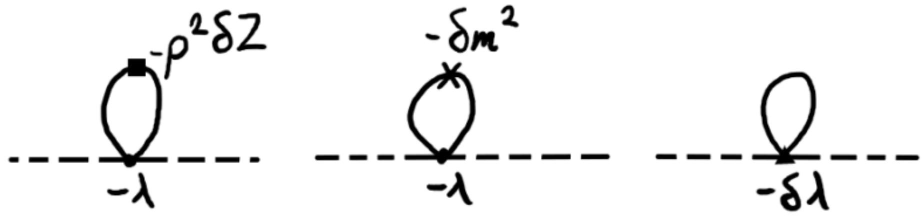


At 1 loop, the 1PI contribution becomes

$$\Pi^{\text{1-loop}}(p^2) = -p^2\delta Z - \delta m^2 - \frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2\log\left(1+\frac{\Lambda^2}{m^2}\right)\right]. \tag{10.7}$$

At two loops, the counterterm diagrams

[11] Quick note: that gamma function in the volume of the $d-1$-sphere will be nice in even dimension, and somewhat less nice in odd dimension. Here, we have $d = 4$, so that $\Gamma(d/2) = \Gamma(2) = 1! = 1$, and the numerator is just $2\pi^2$. In odd dimension, we evaluate the gamma function at a half-integer, and its value is given by a weird double factorial: $\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$, where $n!! = (n)(n-2)(n-4)\dots 3\cdot 1$ for odd n.



must also be included.[12] Now we can tune the parameters $\delta Z, \delta m^2, \delta\lambda$ in our counterterms to cancel the divergences. In other words, we *renormalize* ϕ, m^2, λ.

On-shell renormalization scheme The need to "regulate" the theory by cancelling divergences does not uniquely determine the counterterms, so we impose additional renormalization conditions, which we call a *scheme*. It will turn out that physical observables do not depend on our choice of scheme.

The on-shell scheme is as follows. We fix $\delta Z, \delta m^2, \delta\lambda$ by requiring that

1. $\tilde{\mathbf{\Delta}}(p^2)$ has a simple pole at some experimentally observable mass, i.e. $-p^2 = m^2_{\text{phys}}$ (in Euclidean signature), and
2. The residue of this pole is equal to 1.

Therefore since

$$\tilde{\mathbf{\Delta}}(p^2) = \frac{1}{p^2 + m^2 - \Pi(p^2)}, \tag{10.8}$$

our first condition says that

$$\Pi(-m^2_{\text{phys}}) = m^2 - m^2_{\text{phys}}, \tag{10.9}$$

and $m^2 - m^2_{\text{phys}} = 0$ if we want the mass in $\mathcal{L}$ to equal m_{phys} at this order of counterterm. Imposing the second condition tells us that

$$\frac{\partial \Pi}{\partial p^2}\Big|_{p^2 = -m^2_{\text{phys}}} = 0, \tag{10.10}$$

by L'Hôpital's rule.[13] If we now compare to the expression for the self-energy at one loop, Eqn. 10.7, we see that

$$2. \implies \delta Z = 0, \tag{10.11}$$

$$1. \implies \delta m^2 = -\frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2 \log\left(1 + \frac{\Lambda^2}{m^2}\right)\right]. \tag{10.12}$$

Note that $\delta Z = 0$ and $\Pi(p^2) = 0\,\forall p^2$ is due to the one-loop diagram not depending on p^2. If we instead tried to construct the counterterms to the two-loop diagram we would get $\delta Z \neq 0$ since the integral depends on p, i.e. the integral corresponding to

---

[12] Note that all these diagrams visually look like they only have one loop. What's going on? Basically, there's a factor of $\hbar$ that suppresses all of the counterterm couplings. What we're really doing is expanding in powers of $\hbar$ and/or λ, since it was the four-point λ coupling that gave rise to loops even before we introduced our counterterms. Thus the first two diagrams pick up a factor of λ from the original action and an $\hbar$ from the counterterm couplings $\delta Z, \delta m^2$. The third diagram corresponds to the $\delta\lambda$ coupling, which is part of the counterterms and so has a factor of $\hbar$ but is also a λ correction itself, and so is treated at two loop order.

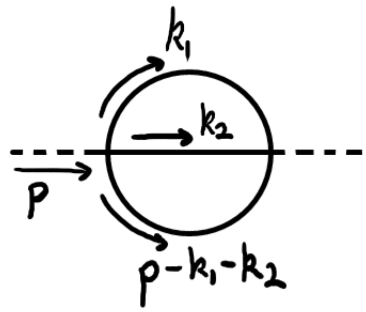
Put more simply (as Skinner says), each counterterm vertex counts as a loop.

[13] This is a little quick. Recall that for a function $f(z)$ with a simple pole at z_0, the residue is the value $\lim_{z\to\zeta_0} f(z)(z - z_0)$. Here, we want $\frac{1}{p^2+m^2+\Pi(p^2)}$ to have unit residue at the pole $p^2 = -m^2_{\text{phys}}$. So the residue is

$$\lim_{p^2\to -m^2_{\text{phys}}} \frac{p^2 + m^2_{\text{phys}}}{p^2 + m^2 + \Pi(p^2)} = \lim_{p^2\to -m^2_{\text{phys}}} \frac{1}{1 + \frac{\partial\Pi}{\partial p^2}} = \frac{1}{1 + \frac{\partial\Pi}{\partial p^2}\Big|_{p^2 = -m^2_{\text{phys}}}}.$$

Setting this expression equal to 1 gives us precisely the desired condition,

$$\frac{\partial\Pi}{\partial p^2}\Big|_{p^2 = -m^2_{\text{phys}}} = 0.$$



In general, UV divergences are not too hard to spot– we saw that

$$\int^{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2} \sim \Lambda^2, \tag{10.13}$$

and generically

$$\int^{\Lambda} \frac{d^nk}{k^m} \sim \begin{cases} \Lambda^{n-m}, & n \neq m \\ \log \Lambda, & n = m \end{cases}. \tag{10.14}$$

Non-lectured aside: renormalization It's worth taking a minute to think about what we're really doing here in renormalizing our theory. To recap, we wrote down a bunch of Lagrangians and actions last term. We derived their Feynman rules, and used the Feynman rules to make some computations of matrix elements, scattering amplitudes, and other physically important quantities.

But there was a trap hidden in our attempt to write down quantum field theories. The Feynman rules tell us that strictly, we should include loop diagrams in our calculations, though the corrections from those diagrams are in principle suppressed by additional factors of the coupling constant. A priori, this sounds like it's not too bad. But once we start writing down loop diagrams and integrating over loop momenta, what we discover is that these corrections we hoped were small are in fact infinite. The integrals over momentum generically diverge, and no factors of the coupling constant can suppress an infinity.

So what do we do? A pragmatic first step to remedying this problem is to say that our theory simply isn't valid to arbitrarily large momentum. As physicists, we know that at very high energies, different forces may unify and perhaps quantum gravitational effects are significant, so it's simply a question of admitting our ignorance and treating our theory as an effective theory at low energies.

But now that we have this cutoff, we would still like to make predictions with our theory that don't depend on our choice of cutoff. Renormalization allows us to do this. It tells us that our first attempt at writing down the actions of quantum field theories was wrong, since it had divergences baked into it from the very beginning. Instead, what we should really be working with is a renormalized effective theory, i.e. we add in counterterms to modify the original couplings to precisely cancel the divergences at some desired order.

This is a very practical thing to do, because once we make some measurements of actual scattering amplitudes, we can fix the values of the couplings in our effective theories and make predictions using for instance the physical mass of a particle (i.e. the effective mass), rather than some mysterious constant m which we can never actually measure. What would have been an honest mass in classical field theory becomes impossible to measure once we add in quantum corrections– the physically relevant thing is then m_{phys} (though we can sometimes set $m^2 = m^2_{\text{phys}}$ by a choice of scheme), and the same is true of other coupling constants.

Lecture 11.

Tuesday, February 12, 2019

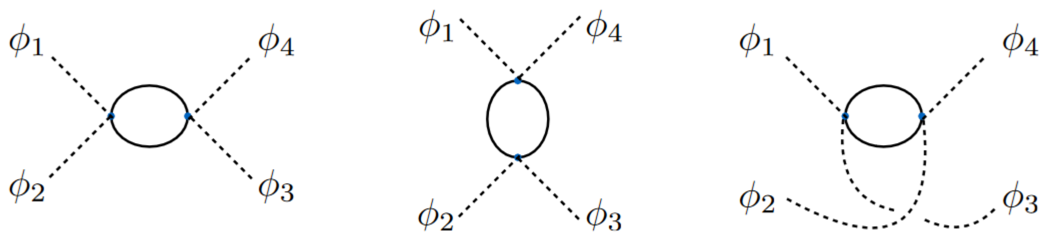
Last time, we wrote down a counterterm action to cancel UV divergences at one-loop order:

$$S^{CT}[\phi, \Lambda] = \int d^4x \left[\frac{\delta Z(\Lambda)}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \delta m^2(\Lambda) \phi^2 + \frac{\delta \lambda(\Lambda)}{4!} \phi^4 \right].$$

In our on-shell renormalization scheme, we set

$$\delta Z = 0,$$
$$\delta m^2 = -\frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2 \log\left(1 + \frac{\Lambda^2}{m^2}\right)\right].$$

To determine $\delta\lambda$, we must look at the 1-loop level correction to the quartic coupling, $\frac{\lambda}{4!}\phi^4$. Before considering any counterterms, there are three diagrams[14] which modify the quartic coupling:



which correspond to an amplitude

$$\frac{\lambda^2}{2}\int^{\Lambda} \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}\left[\frac{1}{(p_1+p_2+k)^2+m^2} + \frac{1}{(p_1+p_4+k)^2+m^2} + \frac{1}{(p_1+p_3+k)^2+m^2}\right]. \tag{11.1}$$

The overall factor of $1/2$ is a symmetry factor since the two internal lines are identical and can be exchanged, and the propagators can be read off by conservation of momentum at each vertex (taking all external momenta to be flowing in). We can evaluate this integral in terms of the external momenta p_i, but let's try to get a feel for the divergence first. We see that this integral goes as d^4k/k^4, so we expect a $\log\Lambda$ divergence. More precisely, the large k behavior (where we care about this divergence) will look like[15]

$$\begin{aligned}\frac{3\lambda^2}{2}\int^{\Lambda}\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+m^2)^2} &= \frac{3\lambda^2}{16\pi^3}\int_0^{\Lambda}\frac{k^3dk}{(k^2+m^2)^2} &(11.2)\\ &= \frac{3\lambda^2}{32\pi^2}\int_0^{\Lambda^2/m^2}\frac{u\,du}{(1+u)^2} &(11.3)\\ &= \frac{3\lambda^2}{32\pi^2}\left[\log\left(1+\frac{\Lambda^2}{m^2}\right) - \frac{\Lambda^2}{\Lambda^2+m^2}\right]. &(11.4)\end{aligned}$$

This value is the shift in the λ coupling before we introduce the $\delta\lambda$ counterterm. If we then choose

$$\delta\lambda = \frac{3\lambda^2}{32\pi^2}\left[\log\frac{\Lambda^2}{m^2} - 1\right], \tag{11.5}$$

we can then produce an effective coupling of

$$\lambda_{\text{eff}} = \lambda - \frac{3\lambda^2}{32\pi^2}\left[\log\left(1+\frac{m^2}{\Lambda^2}\right) + \frac{m^2}{m^2+\Lambda^2}\right], \tag{11.6}$$

---

[14] Diagram credit to Skinner, §5.1.2.

[15] This integral isn't totally immediate. To evaluate this, rewrite $k^3dk = \frac{1}{2}k^2d(k^2)$. Next, divide through in the numerator and denominator by m^4 to get

$$\frac{1}{2}\int_0^{\Lambda}\frac{(k^2/m^2)d(k^2/m^2)}{(k^2/m^2)+1)^2} = \frac{1}{2}\int_0^{\Lambda^2/m^2}\frac{u\,du}{(u+1)^2}.$$

Finally, to evaluate the u integral, just integrate by parts. Some similar integrals like $\int\frac{u\,du}{1+u}$ are amenable to a simple rewriting as $\frac{u}{1+u} = 1 - \frac{1}{1+u}$, but in general you'll want to integrate by parts:

$$\int_0^{\Lambda^2/m^2} u\frac{du}{(1+u)^2} = -\left.\frac{u}{1+u}\right|_0^{\Lambda^2/m^2} - \int\left(-\frac{1}{1+u}\right)du = -\left.\frac{u}{1+u}\right|_0^{\Lambda^2/m^2} + \log(1+u)|_0^{\Lambda^2/m^2} = \log\left(1+\frac{\Lambda^2}{m^2}\right) - \frac{\Lambda^2}{\Lambda^2+m^2}.$$

which is finite as $\Lambda \to \infty$.[16]

Having computed the log divergence one way, let us try to be a little more precise and account for the external momenta. For this next discussion, we'll need a trick due to Feynman:

$$\int_0^1 \frac{dx}{[xA+(1-x)B]^2} = \frac{1}{B-1}\left[\frac{1}{xA+(1-x)B}\right]_0^1 = \frac{1}{AB}. \tag{11.7}$$

We'll use this to rewrite products of denominators (i.e. propagators) as these sorts of integrals, i.e. from right to left. Note that the integral can be put in a more manifestly symmetric form as

$$\int_0^1 dx \int_0^1 dy \frac{\delta(x+y-1)}{(xA+yB)^2}.$$

To compute the loop integral for our first diagram, let $p_{12} \equiv p_1 + p_2$. Using Feynman's trick, we can rewrite the propagators as

$$\begin{aligned}
\frac{1}{(p_{12}+k)^2+m^2}\frac{1}{k^2+m^2} &= \int_0^1 \frac{dx}{[x((p_{12}+k)^2+m^2)+(1-x)(k^2+m^2)]^2} \\
&= \int_0^1 \frac{dx}{\left[(k+xp_{12})^2+m^2+x(1-x)p_{12}^2\right]^2} \\
&= \int_0^1 \frac{dx}{\left[l^2+m^2+x(1-x)p_{12}^2\right]^2}
\end{aligned}$$

where we have defined $l = k + xp_{12}$ and completed the square. In the $\Lambda \to \infty$ limit, the shifted integration range $|k| \le \Lambda \to |l| \le \Lambda$ vanishes, so we can turn our d^4k integral into a d^4l and write

$$\begin{aligned}
\int \frac{d^4l dx}{\left[l^2+m^2+x(1-x)p_{12}^2\right]^2} &= S_4 \int_0^1 dx \int_0^\Lambda \frac{l^3 dl}{\left[l^2+m^2+x(1-x)p_{12}^2\right]^2} \\
&= \pi \int_0^1 dx \left\{ \log\left[\frac{\Lambda^2+m^2+x(1-x)p_{12}^2}{m^2+m^2+x(1-x)p_{12}^2}\right] + \frac{m^2+x(1-x)p_{12}^2}{\Lambda^2+m^2+x(1-x)p_{12}^2} - 1 \right\}
\end{aligned}$$

after a change of variables and an integration by parts. Note that this term with the $1/\Lambda^2$ goes to zero as $\Lambda \to \infty$, so we will not need to worry about renormalizing it. We also notice that all three diagrams are related by the Mandelstam variables

$$s = -(p_1+p_2)^2, \quad t = -(p_1+p_4)^2, \quad u = -(p_1+p_3)^2, \tag{11.8}$$

so that the sum of our three diagrams (restoring prefactors) is then

$$\frac{\lambda^2}{32\pi^2}\int_0^1 dx \left\{ \log\left(\frac{\Lambda^2}{m^2-x(1-x)s}\right) + \log\left(\frac{\Lambda^2}{m^2-x(1-x)t}\right) + \log\left(\frac{\Lambda^2}{m^2-x(1-x)u}\right) - 3 \right\}. \tag{11.9}$$

The coefficient of $\tilde{\phi}^4$ in the effective action $\Gamma(\tilde{\phi})$ (i.e. $\tilde{\phi}$ in momentum space) is

$$\lambda + \delta\lambda - \frac{\lambda^2}{32\pi^2}\int d\{\ldots\} \tag{11.10}$$

[16] This is just λ plus the one-loop correction we computed to be 11.4 plus our choice of $\delta\lambda$ (which is itself treated as a one-loop correction). In fact, there's a relative minus sign between the original coupling and the one-loop correction. That is, the original coupling contributes $(-\lambda\delta^{(4)}(\ldots)$, while the one-loop diagrams contribute $(-\lambda)^2\delta^{(4)}(\ldots)$. Thus

$$\begin{aligned}
\lambda_{\text{eff}} &= \lambda - \frac{3\lambda^2}{32\pi^2}\left[\log\left(1+\frac{\Lambda^2}{m^2}\right) - \frac{\Lambda^2}{\Lambda^2+m^2}\right] + \frac{3\lambda^2}{32\pi^2}\left[\log\frac{\Lambda^2}{m^2} - 1\right] \\
&= \lambda - \frac{3\lambda^2}{32\pi^2}\left[\log\left(1+\frac{m^2}{\Lambda^2}\right) + \frac{m^2}{m^2+\Lambda^2}\right].
\end{aligned}$$

with $\delta\lambda$ from above, and replacing $(m^2, \lambda) \mapsto (m^2_{\text{phys}}, \lambda_{\text{eff}})$. We find that

$$\lambda_{\text{eff}} + \frac{\lambda_{\text{eff}}^2}{32\pi^2}\int_0^1 dx \left\{ \log\left[1 - \frac{x(1-x)s}{m^2_{\text{phys}}}\right] + \log\left[1 - \frac{x(1-x)t}{m^2_{\text{phys}}}\right] + \log\left[1 - \frac{x(1-x)u}{m^2_{\text{phys}}}\right] \right\} \tag{11.11}$$

is finite– no more counterterms are necessary after δm^2 and $\delta\lambda$. Our capacity to regulate these terms depends on the idea of operators being relevant, irrelevant, or marginal (depending on their mass dimension as compared to the dimension of spacetime).

Lecture 12.

Thursday, February 14, 2019

Dimensional regularization ("dim reg") We would like to understand the behavior of our QFT in different dimensions. Though it may seem like a strange construction, let us work in d dimensions where

$$d = 4 - \epsilon, \text{ with } 0 \le \epsilon < 1.^{17} \tag{12.1}$$

We take the action of ϕ^4 theory,

$$S = \int d^d x \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right]. \tag{12.2}$$

Working in $\hbar = c = 1$ units, let's analyze the mass dimension of the different quantities and couplings in this action. Given that the action must be dimensionless, $[S] = 0$, we get $[\partial] = [m] = [x^{-1}] = 1$. Recall that in 4 dimensions, λ was a dimensionless coupling and therefore "marginal." From the mass term we have

$$[m^2\phi^2] = 2[m] + 2[\phi] = d \implies [\phi]\frac{d}{2} - 1, \tag{12.3}$$

which means our λ term now has dimensions given by

$$[\lambda\phi^4] = d \implies [\lambda] = 4 - d = \epsilon. \tag{12.4}$$

Thus λ is no longer dimensionless. Let us now introduce an arbitrary mass scale μ and a new dimensionless coupling g such that

$$\lambda = \mu^\epsilon g(\mu), \tag{12.5}$$

If we return to our one-loop graph, we see that the self-energy $\Pi^{\text{1-loop}}$ is

$$\Pi^{\text{1-loop}} = -\frac{1}{2}g(\mu)\mu^\epsilon \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m^2} = -\frac{1}{2}g(\mu)\mu^\epsilon \frac{S_d}{2(2\pi)^d}\int_0^\infty \frac{k^{d-1}dk}{k^2+m^2}, \tag{12.6}$$

with S_d the surface area of the d-sphere. Note that we can actually define the surface area of the unit sphere in d dimensions even when d is not an integer:

$$(\sqrt{\pi})^d = \int \prod_{i=1}^d e^{-x_i^2}dx_i = S_d \int_0^\infty e^{-r^2}r^{d-2}dr = \frac{S_d}{2}\Gamma(d/2). \tag{12.7}$$

Thus

$$S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}, \tag{12.8}$$

which we take to be the definition of S_d in $d \in \mathbb{C}$ dimensions. We can also analytically continue the Γ function by establishing a recursion relation:

$$\begin{aligned}\Gamma(\alpha) &= \int_0^\infty dx\, x^{\alpha-1}e^{-x} \\ &= \frac{1}{\alpha}\left[x^{-\alpha}e^{-x}\right]_0^\infty + \frac{1}{\alpha}\int_0^\infty dx\, x^\alpha e^{-x} \\ &= 0 + \frac{1}{\alpha}\Gamma(\alpha+1)\end{aligned}$$

[17] Compare the epsilon expansion from David Tong's *Statistical Field Theory* notes.

where the first term is zero if $\text{Re}(\alpha) > 0$. We analytically continue and define $\Gamma(\alpha)$ for $\text{Re}(\alpha) > -1$ through

$$\Gamma(\alpha) = \frac{1}{\alpha}\Gamma(\alpha+1). \tag{12.9}$$

Note that there are poles when $\text{Re}(\alpha) \in \mathbb{Z}^- \cup \{0\}$.

Finally, note that there is an expansion of Γ for small α: it is

$$\log\Gamma(\alpha+1) = -\gamma\alpha - \sum_{k=2}^{\infty}(-1)^k\frac{1}{k}\zeta(k)\alpha^k \tag{12.10}$$

where $\gamma \approx 0.577216\ldots$ is the Euler-Mascheroni constant and ζ is the Riemann zeta function,

$$\zeta(k) = \sum_{n=1}^{\infty}\frac{1}{n^k}.$$

We will usually use this to write

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + O(\epsilon) \text{ for small } \epsilon. \tag{12.11}$$

It is also useful to note that there exists an Euler "beta function" given by

$$B(s,t) = \int_0^a du\, u^{s-1}(1-u)^{t-1} = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}. \tag{12.12}$$

Now let us return to our integral 12.6. Using these facts, it becomes

$$\begin{aligned}
\mu^\epsilon\int_0^\infty \frac{k^{d-1}dk}{k^2+m^2} &= \frac{\mu^\epsilon}{2}\int_0^\infty\frac{(k^2)^{d/2-1}dk^2}{k^2+m^2} \\
&= \frac{m^2}{2}\left(\frac{\mu}{m}\right)^\epsilon\int_0^1 du(1-u)^{d/2-1}u^{-d/2} \\
&= \frac{m^2}{2}\left(\frac{\mu}{m}\right)^\epsilon\frac{\Gamma(d/2)\Gamma(1-d/2)}{\Gamma(1)},
\end{aligned}$$

where we have used the substitution $u = \frac{m^2}{k^2+m^2}$. Therefore

$$\Pi^{\text{1-loop}} = -\frac{g(\mu)m^2}{2(4\pi)^{d/2}}\left(\frac{\mu}{m}\right)^\epsilon\Gamma(1-d/2). \tag{12.13}$$

Using the recursion relation, we have

$$\Gamma(1-d/2) = \Gamma(\epsilon/2-1) = -\frac{1}{(1-\epsilon/2)}\Gamma(\epsilon/2). \tag{12.14}$$

We can now unpack some of the factors as

$$\left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon/2} = 1 + \frac{\epsilon}{2}\log\left(\frac{4\pi\mu^2}{m^2}\right) + O(\epsilon^2). \tag{12.15}$$

Using these last two expressions, we have

$$\Pi^{\text{1-loop}} = -\frac{g(\mu)m^2}{32\pi^2}\left[\frac{2}{\epsilon} - \gamma + 1 + \log\left(\frac{4\pi\mu^2}{m^2}\right)\right] + O(\epsilon). \tag{12.16}$$

We see that the old $\Lambda \to \infty$ divergence appears here as a $1/\epsilon$ pole. In order to make this contribution converge, we must add a counterterm $\frac{\delta m^2}{2}\phi^2$. The first thing we might think to do is the *minimal subtraction* (MS) scheme,

$$\delta m^2 = -\frac{g(\mu)m^2}{16\pi^2\epsilon}, \tag{12.17}$$

by which we just get rid of the epsilon divergence directly. There's also the *modified minimal subtraction scheme* ($\overline{\text{MS}}$), where we also get rid of the extra constants hanging around,

$$\delta m^2 = -\frac{g(\mu)m^2}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma + \log 4\pi\right). \tag{12.18}$$

After this regularization, we have

$$\Pi^{\text{1-loop},\overline{MS}} = \frac{g(\mu)m^2}{32\pi^2}\left(\log\frac{\mu^2}{m^2} - 1\right). \tag{12.19}$$

We can also renormalize the quartic term $\frac{\lambda}{4!}\phi^4$ in an equivalent way by writing out the Feynman loop diagrams and calculating the self-energy from these guys. We'll get amplitudes like

$$\frac{\lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}\frac{1}{(p_1+p_2+k)^2+m^2} + \text{ two similar terms.} \tag{12.20}$$

The one-loop contribution to $g(\mu) = \lambda\mu^{-\epsilon}$ is

$$\frac{g^2\mu^\epsilon}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m^2}\frac{1}{(p_1+p_2+k)^2+m^2} + t, u\text{-channel diagrams.} \tag{12.21}$$

If we set $p_i = 0$ to find the pure ϕ^4 coupling in the effective action $\Gamma[\phi]$, we get an integral that looks like

$$\mu^\epsilon\int_0^\infty\frac{k^{d-1}dk}{(k^2+m^2)^2} = \frac{1}{2}\left(\frac{\mu}{m}\right)^\epsilon\frac{\Gamma(2-d/2)\Gamma(d/2)}{\Gamma(2)}. \tag{12.22}$$

and we get the result

$$\frac{3g^2S_4}{2(2\pi)^d}\frac{1}{2}\left(\frac{\mu}{m}\right)^\epsilon\frac{\Gamma(2-d/2)\Gamma(d/2)}{\Gamma(2)} = \frac{3g^2}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma + \log\frac{4\pi\mu^2}{m^2}\right) + O(\epsilon). \tag{12.23}$$

Introducing a counterterm $\delta g = \frac{3g^2}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma + \log 4\pi\right)$ (the $\overline{\text{MS}}$ scheme), we get an effective coupling

$$g_{\text{eff}}(\mu) = g(\mu) - \frac{3g^2(\mu)}{32\pi^2}\log\frac{\mu^2}{m^2} + \ldots \tag{12.24}$$

Lecture 13.

Saturday, February 16, 2019

Last time, we computed an effective coupling g_{eff} to be

$$g_{\text{eff}}(\mu) = g(\mu) - \frac{3g^2(\mu)}{32\pi^2}\log\frac{\mu^2}{m^2} + \ldots,$$

and in particular since g_{eff} is the coefficient of $\frac{1}{4!}\phi^4$ in the quantum effective action $\Gamma[\phi]$, it should be independent of μ. Thus

$$0 = \frac{dg_{\text{eff}}}{d\log\mu} \tag{13.1}$$

$$= \mu\frac{dg_{\text{eff}}}{d\mu} \tag{13.2}$$

$$= \mu\frac{d}{d\mu}\left[g(\mu) - \frac{3g^2(\mu)}{32\pi^2}\log\frac{\mu^2}{m^2}\right]. \tag{13.3}$$

This tells us the "running" of the renormalized coupling $g(\mu)$, which we refer to as the *beta function*,[18]

$$\beta(g) \equiv \mu \frac{dg}{d\mu} = \frac{3\hbar g^2}{16\pi^2} + O(\hbar^2), \tag{13.4}$$

restoring $\hbar$ which counts the loop order of the corrections. Note that $\beta(g) > 0$ for this coupling in this theory.

Integrating the ODE 13.4 for $\frac{dg}{d\mu}$ between μ, μ', we find that

$$\frac{1}{g(\mu')} = \frac{1}{g(\mu)} + \frac{3\hbar}{16\pi^2} \log \frac{\mu}{\mu'} \tag{13.5}$$

$$\implies g(\mu') = \frac{g(\mu)}{1 + \frac{3\hbar g(\mu)}{16\pi^2} \log \frac{\mu}{\mu'}} = g(\mu) - \frac{3\hbar g^2(\mu)}{16\pi^2} \log \frac{\mu}{\mu'} + O(\hbar^2). \tag{13.6}$$

For $\mu' > \mu$, we see that $g(\mu') > g(\mu)$. Note that there is a special scheme-dependent mass scale Λ_{ϕ^4} such that when $\mu' \to \Lambda_{\phi^4}, g(\mu') \to \infty$. For our scheme, this happens when

$$\frac{3\hbar g(\mu)}{16\pi^2} \log \frac{\mu}{\Lambda_{\phi^4}} = -1 \tag{13.7}$$

at one loop. Thus, knowing this scale allows us to write our coupling as

$$g(\mu) = \frac{16\pi^2}{3\hbar} \frac{1}{\log(\Lambda_{\phi^4}/\mu)}. \tag{13.8}$$

Exchanging our dimensionless coupling for a dimensionful scale (Λ_{ϕ^4}) is called *dimensionful transmutation.* All we're saying is that Λ_{ϕ^4} is the scale at which perturbation theory breaks down– perturbation theory works for $\mu \ll \Lambda_{\phi^4}$.

Quantum electrodynamics We'll begin our discussion of QED and the photon in the path integral formalism (cf. Skinner Ch. 5, Peskin & Schroeder). In Euclidean space, we have the classical action

$$S[\psi, \bar{\psi}, A] = \int d^4x \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi}(\not{D} + m)\psi \right] \tag{13.9}$$

where $\not{D} = \gamma^\mu(\partial_\mu - ieA_\mu)$ is a covariant derivative and $\psi, \bar{\psi}$ are four-spin-component Grassmann fields. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is just the electromagnetic field strength tensor.

The partition function for our theory is

$$Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A \, e^{-S[\psi,\bar{\psi},A]}. \tag{13.10}$$

Let us consider the first novel feature of our theory, the electromagnetic field. We write the Fourier transform

$$A_\mu(x) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x} \tilde{A}_\mu(k). \tag{13.11}$$

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[18] It takes a little algebra to get here. Explicitly, we have

$$\begin{aligned}
\beta(g) \equiv \mu \frac{dg}{d\mu} &= \mu \frac{d}{d\mu} \left[\frac{3g^2(\mu)}{32\pi^2} \log \frac{\mu^2}{m^2} \right] \\
&= \mu \left[\frac{3(2g'(\mu)g(\mu))}{32\pi^2} \log \frac{\mu^2}{m^2} + \frac{3g^2}{32\pi^2} \left(\frac{2}{\mu} \right) \right] \\
&= \frac{3g}{16\pi^2} \log \frac{\mu^2}{m^2} \left(\mu \frac{dg}{d\mu} \right) - \frac{3g^2}{16\pi^2},
\end{aligned}$$

so rearranging we find that

$$\mu \frac{dg}{d\mu} = \frac{3g^2}{16\pi^2} \left(1 - \frac{3g}{16\pi^2} \log \frac{\mu^2}{m^2} \right) = \frac{3\hbar g^2}{16\pi^2} + O(\hbar^2),$$

restoring $\hbar$.

A few steps of algebra[19] reveals that

$$\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\tilde{A}_\mu(-k)(k^2\delta^{\mu\nu} - k^\mu k^\nu)\tilde{A}_\nu(k), \tag{13.12}$$

where derivatives have brought down ks and the integral over d^4x has related the momenta in $\tilde{A}_\mu, \tilde{A}_\nu$.

Note that for a field $\tilde{A}_\mu(k) = k_\mu\tilde{\alpha}(k)$ with $\tilde{\alpha}$ a scalar function, this integral vanishes. This is bad– since $\int d^4x F_{\mu\nu}F^{\mu\nu}$ is in the action S, any path integral configuration of this form will pick up a weight of 1 from $e^{-S[\psi,\bar{\psi},A]} = e^0 = 1$, and there are infinitely many such configurations to integrate over in $\mathcal{D}A$, so Z diverges. In position space, this choice of $\tilde{A}_\mu$ corresponds to $A_\mu(x) = \partial_\mu\alpha(x)$. Recall that under gauge transformations,

$$A_\mu(x) \mapsto A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x). \tag{13.13}$$

Therefore these troublesome modes are all gauge-equivalent to $A_\mu(x) = 0$, and so the solution will come from a gauge fixing procedure.

Faddeev-Popov method for gauge-fixing Before we do the whole procedure, let's take a simple example. Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ that is rotationally invariant,

$$f(r;\rho) \text{ with } r^2 = x^2 + y^2, \rho \text{ another parameter.} \tag{13.14}$$

Consider the integral

$$\begin{aligned} Z(\rho) &= \int d^2x\, f(r;\rho) \\ &= \int_0^{2\pi} d\phi \int r dr f(r;\rho) = 2\pi \int dr\, r f(r;\rho), \end{aligned}$$

where we used a change of coordinates to do the integral. Easy enough to compute. We separated and integrated out a trivial part of the integral, the $d\phi$ part, leaving only the interesting r dependence.

But there's another way to think about this integration.[20] Consider an integration path given by the constraint $g(\mathbf{x}) = 0$ for some function of our choosing $g(\mathbf{x})$. We may (following Skinner) call this path a *gauge slice* and the function a *gauge-fixing function*. In particular, let's say we want to integrate only along the x-axis, i.e. $g(\mathbf{x}) = y = 0$. Consider then the related integral

$$\int d^2x\, \delta(g(\mathbf{x}))f(r;\rho). \tag{13.15}$$

With the delta function, this integral does what we wanted– it restricts the integration path precisely to $g(\mathbf{x}) = 0$. However, its value clearly depends on our choice of path, since rescaling $g \to cg$ for some constant c will rescale the entire integral by a factor $1/|c|$. This is because the δ function changes with our gauge fixing function g. However, we can account for this as follows. We introduce the factor

$$\Delta_g(\mathbf{x}) = \left.\frac{\partial}{\partial\theta}g(R_\theta\mathbf{x})\right|_{\theta=0} \tag{13.16}$$

where R_θ indicates that we rotate our coordinates by an angle θ before evaluating our gauge fixing function $g(\mathbf{x})$. This factor precisely captures how the delta function changes as we change g by an infinitesimal rotation, so that the integral

$$\int d^2x |\Delta_g(\mathbf{x})| \delta(g(\mathbf{x}))f(r;\rho) \tag{13.17}$$

is now independent of both reparametrization of g and rotations by θ. In fact, notice that so long as our gauge slice only hits each gauge orbit once, the integral is completely independent of the gauge slice. For our example, it is straightforward to compute $\Delta_g(\mathbf{x})$:

$$\Delta_g(\mathbf{x}) = \frac{\partial}{\partial\theta}(y\cos\theta - x\sin\theta)|_{\theta=0} = -y\sin\theta - x\cos\theta|_{\theta=0} = -x. \tag{13.18}$$

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20 Following Skinner's conventions (Ch. 8 of his notes), I've significantly rewritten this section from how it was presented in class in anticipation of later gauge fixing content.

To see how this factor emerges, consider again our delta function $\delta(g(\mathbf{x})) = \delta(y)$. Under a rotation by θ, $y \mapsto y_\theta = y\cos\theta - x\sin\theta$, so

$$\delta(y) \mapsto \delta(y_\theta) = \delta(y\cos\theta - x\sin\theta). \tag{13.19}$$

Let us now write this delta function in terms of θ using the composition rule of the δ function: for a function f with a single zero $f(x_0) = 0$, $\delta(f(x)) = \frac{\delta(x-x_0)}{|f'(x_0)|}$. Thus

$$\delta(y_\theta) = \frac{\delta(\theta - \tan^{-1}\frac{y}{x})}{|y\sin\theta + x\cos\theta|_{\theta=0}} = \frac{\delta(\theta - \tan^{-1}\frac{y}{x})}{|x|}. \tag{13.20}$$

By definition, when $\tan^{-1} y/x \in (0,\pi)$, the delta function satisfies

$$\begin{aligned} 1 &= \int_0^\pi d\theta\, \delta(\theta - \tan^{-1}\frac{y}{x}) \\ &= \int_0^\pi d\theta\, \delta(y\cos\theta - x\sin\theta)|x| \\ &= \int_0^\pi d\theta\, \delta(y_\theta)\left|\frac{\partial y_\theta}{\partial\theta}\right|. \end{aligned}$$

We recognize $\left|\frac{\partial y_\theta}{\partial\theta}\right| = \Delta_g(\mathbf{x})$. Then

$$Z(\rho) = \int d^2x \int_0^\pi d\theta\, \delta(y_\theta)\left|\frac{\partial y_\theta}{\partial\theta}\right| f(r;\rho). \tag{13.21}$$

The factor of $\left|\frac{\partial y_\theta}{\partial\theta}\right|$ is a simple example of a *Faddeev-Popov determinant*, which we have already met in *String Theory*.

We are now free to change integration variables $y \to y_\theta$ and relabel to y so that our integral becomes

$$Z(\rho) = \int d^2x \int_0^\pi d\theta\, \delta(y)\left|\frac{\partial y}{\partial\theta}\right|_{\theta=0} f(r;\rho), \tag{13.22}$$

where the integral is now θ-independent. In particular, notice that

$$\begin{aligned} \left|\frac{\partial y}{\partial\theta}\right|_{\theta=0} = |x| \implies Z(\rho) &= \pi\int_{\mathbb{R}^2} d^2x\, \delta(y)|x| f(r;\rho) \\ &= 2\pi\int_0^\infty dx\, x f(x;\rho) \end{aligned}$$

as before. If we had N variables and $N-1$ rotations with angles $\theta_{1i}, i = 2,\ldots,N$, then we would have instead the determinant

$$1 = \int d\theta_{1i}\delta(x^i_{\theta i})\left|\frac{dx^i_{\theta_{1i}}}{d\theta_{1i}}\right|. \tag{13.23}$$

This approach generalizes– in our gauge theory, we fix the gauge with some functional of $A_\mu(x)$, i.e. $G[A] = 0$. For example, $G[A] = \partial_\mu A^\mu$ in Lorenz [sic] gauge. Consider now gauge transformations

$$A^\alpha_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x). \tag{13.24}$$

We now use the identity

$$1 = \int \mathcal{D}\alpha(x)\delta(G[A])\det\left(\frac{\delta G[A]}{\delta\alpha}\right), \tag{13.25}$$

where this last factor is a functional determinant. We'll see how the gauge-fixing procedure modifies the photon propagator next time.

Non-lectured aside: on Fadeev-Popov determinants Let us briefly remark on what we've done. We had a theory over some potentially complicated space, but we recognized that there was a redundancy in our description. In the case of our rotationally invariant integral, we saw that on $r =$ constant orbits, the integrand was also constant. This led us to take a gauge slice, i.e. to fix a path through the space which only intersects each gauge orbit once, and then multiply by the "size" of a gauge orbit. More generally, the "size" of a gauge orbit will be infinite, but we can still use this method to choose a gauge slice in our configuration space, and we include the Faddeev-Popov determinant to ensure that the integral is independent of our choice of path.

The way it was presented in lecture is a bit backwards from how we use this method. Here is a quick recap of how we will use this.

(a) Identify a gauge symmetry of the theory.
(b) Identify the gauge orbits.
(c) Choose a gauge-fixing function such that $g = 0$ along the gauge slice (integration path).
(d) Calculate the Faddeev-Popov determinant, i.e. compute the variation of g as we go around a gauge orbit, evaluated at our gauge slice.
(e) Insert the delta function and the Faddeev-Popov determinant into the integral.
(f) Perform the integral using the delta function.

How does this connect to our example?

(a) We identified an $SO(2)$ gauge symmetry.
(b) We identified the gauge orbits as sets of constant r, and within each orbit there was a gauge freedom described by θ.
(c) We chose as our gauge-fixing function $g = y$.
(d) We calculated the Faddeev-Popov determinant by varying with respect to the parameter θ describing the gauge freedom: $\Delta_g(\mathbf{x}) = \frac{\partial}{\partial\theta} g(R_\theta \mathbf{x})\Big|_{\theta=0} = -x$.
(e) We insert the delta function $\delta(y)$ and the Faddeev-Popov determinant into our integral to get $\int d^2x\, \delta(y)|x| f(r;\rho) = \int_{-\infty}^{\infty} dx\, |x| f(x;\rho)$.

Lecture 14.

Tuesday, February 19, 2019

Last time, we wrote down an action for the electromagnetic field,

$$S_g[A] = \frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \tilde{A}_\mu(-k)(k^2\delta^{\mu\nu} - k^\mu k^\nu)\tilde{A}_\nu(k),$$

We introduced the Faddeev-Popov method for fixing the gauge. We write the identity in terms of a delta function and an (as yet unspecified) functional $G[A]$,

$$1 = \int \mathcal{D}\alpha(x)\delta(G[A]) \det\left(\frac{\delta G[A]}{\delta\alpha}\right).$$

For the $G[A]$ we will use, the determinant (Jacobian factor) will be independent of A, though this will not be true more generally in non-Abelian gauge theories. If we choose to work in Lorenz gauge, then

$$G[A^\alpha] = \partial_\mu A^\mu + \frac{1}{e}\partial^2\alpha, \tag{14.1}$$

so

$$\det\left(\frac{\delta G[A]}{\delta\alpha}\right) = \det\left(\frac{\partial^2}{e}\right). \tag{14.2}$$

Thus we rewrite the path integral with our delta function as

$$\int \mathcal{D}A e^{-S_g[A]} = \det\left(\frac{\delta G A^\alpha}{\delta\alpha}\right)\int \mathcal{D}\alpha \int \mathcal{D}A e^{-S_g[A]}\delta(G[A]) \tag{14.3}$$

$$= \det\left(\frac{\delta G A^\alpha}{\delta\alpha}\right)\left(\int \mathcal{D}\alpha\right)\int \mathcal{D}A e^{-S_g[A]}\delta(G[A]), \tag{14.4}$$

where we have changed variable $A \mapsto A^\alpha$ and S_g is invariant since this is a gauge transformation.

To fix the gauge, let's choose the functional

$$G[A] = \partial_\mu A^\mu(x) - \omega(x) \implies G[A^\alpha] = \partial_\mu A^\mu - \omega + \frac{1}{e}\partial^2\alpha. \tag{14.5}$$

We now integrate over ω with a Gaussian weight factor of mean 0 and variance ξ. Thus

$$\int \mathcal{D}\omega \exp\left[-\int d^4x \frac{\omega^2}{2\xi}\right] \det\left(\frac{\partial^2}{e}\right) \int \mathcal{D}\alpha \mathcal{D}A e^{-S[A]} \delta(\partial_\mu A^\mu - \omega), \tag{14.6}$$

which becomes (similar to before)

$$\det\left(\frac{\partial^2}{e}\right)\left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A e^{-S_g[A]} - \underbrace{\int d^4x \frac{1}{2\xi}(\partial_\mu A^\mu)^2}_{S_{gf}[A]}, \tag{14.7}$$

where S_{gf} indicates a gauge-fixing action. Thus

$$S_g[A] + S_{gf}[A] = \frac{1}{2}\int \frac{d^4k}{(2\pi)^k} \tilde{A}_\mu(-k)\left[k^2\delta^{\mu\nu} - (1-\frac{1}{\xi})k^\mu k^\nu\right]\tilde{A}_\nu(k), \tag{14.8}$$

where we've just taken a Fourier transform as usual. The propagator solves

$$(k^2\delta^{\mu\nu} - (1-\frac{1}{\xi})k^\mu k^\nu)\tilde{D}_{\nu\rho}(k) = \delta^\mu{}_\rho, \tag{14.9}$$

so the photon propagator takes the form

$$\tilde{D}^{\mu\nu}(k) = \frac{1}{k^2}\left(\delta^{\mu\nu} - (1-\xi)\frac{k^\nu k^\nu}{k^2}\right). \tag{14.10}$$

Here, $\xi = 0$ is known as Lorenz or Landau gauge depending on when the ξ condition is imposed, while $\xi = 1$ is known as Feynman gauge.

Free fermions (electrons) Let us consider an action in terms of fermions,

$$S[\psi, \bar{\psi}] = \int d^4x(-\bar{\psi}(\partial\!\!\!/ + m)\psi), \tag{14.11}$$

where we work in Euclidean signature, $\partial\!\!\!/ = \gamma_\mu\partial^\mu$, and the anticommutation relations hold, $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. Our gamma matrices are hermitian, $\gamma_\mu^\dagger = \gamma_\mu$, and our γ_5 is taken to be $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$. For example,

$$\gamma_j = \begin{pmatrix} 0 & -i\sigma_j \\ i\sigma_j & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{14.12}$$

We take the Fourier transform of our fermionic fields using

$$\psi(x) = \int_p e^{ip\cdot x}\tilde{\psi}(p), \tag{14.13}$$

$$\bar{\psi}(x) = \int_p e^{-ip\cdot x}\tilde{\bar{\psi}}(p) \tag{14.14}$$

where $\int_p = \int \frac{d^4p}{(2\pi)^4}$. Thus in Fourier space our action takes the form

$$S[\tilde{\psi}, \tilde{\bar{\psi}}] = \int_p \tilde{\bar{\psi}}(ip\!\!\!/ + m)\tilde{\psi}. \tag{14.15}$$

Adding sources $\tilde{\eta}, \tilde{\bar{\eta}}$, the generating functional is then

$$Z[\tilde{\eta}, \tilde{\bar{\eta}}] = \int \mathcal{D}\tilde{\psi}\mathcal{D}\tilde{\bar{\psi}} \exp - \int_p \left[\tilde{\bar{\psi}}(ip\!\!\!/ + m)\tilde{\psi} - \tilde{\bar{\eta}}\tilde{\psi} + \tilde{\bar{\psi}}\tilde{\eta}\right] \tag{14.16}$$

$$= Z[0,0]e^{-\int_p \tilde{\bar{\eta}}(ip\!\!\!/+m)^{-1}\tilde{\eta}}, \tag{14.17}$$

where we have completed the square as usual.

Feynman rules In addition to the propagators and vertices, fermion loops pick up minus signs. For instance, the position space propagator takes the form

$$S_F^{\alpha\beta}(x-y) = \left\langle \psi^\alpha(x)\bar{\psi}^\beta(y) \right\rangle = \int_p e^{ip\cdot(x-y)} \left(\frac{1}{1\not{p}+m} \right)^{\alpha\beta}, \tag{14.18}$$

where α, β are spin indices $1, \ldots, 4$. If we expand the action $e^{-S_{QED}}$ to second order in the electron-photon coupling, we find terms

$$(-ie)^2 \left\langle \left(\int d^4x \bar{\psi} \not{A} \psi \right) \left(\int d^4y \bar{\psi} \not{A} \psi \right) \right\rangle = (-ie)^2 \int d^4x d^4y \left\langle \not{A}^{\alpha\beta}(x) \not{A}^{\gamma\delta}(y) \bar{\psi}^\alpha(x) \psi^\beta(x) \bar{\psi}^\gamma(y) \psi^\delta(y) \right\rangle.$$

In general, we need to anticommute the ψs and $\bar{\psi}$s to form propagators. One term is

$$-(-ie)^2 \int d^4x d^4y \left(\not{A}^{\alpha\beta}(x) \not{A}^{\gamma\delta}(y) \underbrace{\psi^\beta(x)\bar{\psi}^\gamma(y)}_{S_F^{\beta\gamma}(x-y)} \underbrace{\psi^\delta(y)\bar{\psi}^\alpha(x)}_{S_F^{\delta\alpha}(y-x)} \right) \tag{14.19}$$

where the overall minus sign comes from anticommuting and we've recognized the pairs $\psi\bar{\psi}$ as propagators.

We get some Feynman rules for this theory:

(a) The fermion propagator is an oriented line, with $\tilde{S}_F(p) = \frac{1}{i\not{p}+m}$
(b) The photon propagator is a squiggly line, $\tilde{D}^{\mu\nu}(k) = \frac{1}{k^2}(\delta^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2})$
(c) The vertex gets a $-ie\gamma^\mu$.
(d) We pick up an overall factor of $(-1)^{l_F}$, where l_F is the number of fermion loops.

Lecture 15.

Thursday, February 21, 2019

Vacuum polarization Here, we want to compute the vacuum polarization (cf. Skinner §5.2.1), corrections to the photon propagator. Let's start by amputating the external legs of our diagram and calculating the amplitude corresponding to a single electron-positron loop– see diagram. In d dimensions, we write $e^2 = \mu^\epsilon g^2(\mu)$ in order to get g a dimensionsless coupling, with $\epsilon = 4-d$. Thus the value of this one-loop diagram is

$$\begin{aligned}
\Pi^{\mu\nu}_{\text{1-loop}}(q) &= -\mu^\epsilon (ig)^2 \int \frac{d^dp}{(2\pi)^d} \operatorname{tr}\left(\frac{1}{i\not{p}+m} \gamma^\mu \frac{1}{i(\not{p}-\not{q})+m} \gamma^\nu \right) \\
&= -\mu^\epsilon (ig)^2 \int_p \frac{\operatorname{tr}[(-i\not{p}+m)\gamma^\mu(-i(\not{p}-\not{q})+m)\gamma^\nu]}{(p^2+m^2)((p-q)^2+m^2)}.
\end{aligned}$$

The overall minus sign comes from the fermion loop. To get this in a form we can actually integrate, we use Feynman's trick:

$$\frac{1}{AB} = \int_0^1 dx \int_0^a dy \frac{\delta(x+y-1)}{[xA+yB]^2}. \tag{15.1}$$

Thus our integral becomes

$$\int_0^1 \frac{dx}{\{(p^2+m^2)(1-x) + [(p-q)^2+m^2]x\}^2} = \int_0^1 \frac{dx}{[(p-qx)^2+m^2+q^2x(1-x)]^2}. \tag{15.2}$$

Since we want to integrate over the internal momentum p, let $p' = p - qx$ and WLOG drop the prime. Thus

$$\Pi^{\mu\nu}_{\text{1-loop}}(q) = \mu^\epsilon g^2 \int_p \int_0^1 dx \frac{\operatorname{tr}\{[-i(\not{p}+\not{q}x)+m]\gamma^\mu[-i(\not{p}-\not{q}(1-x))+m]\gamma^\nu\}}{(p^2+\Delta)^2}, \tag{15.3}$$

where we've denoted

$$\Delta \equiv m^2 + q^2x(1-x). \tag{15.4}$$

To simplify the big trace in the numerator, we use the following spin traces:

$$\mathrm{tr}(\gamma^\mu\gamma^\nu) = 4\delta^{\mu\nu} \tag{15.5}$$

$$\mathrm{tr}(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma) = 4(\delta^{\mu\rho}\delta^{\nu\sigma} - \delta^{\mu\nu}\delta^{\rho\sigma} + \delta^{\mu\sigma}\delta^{\rho\nu}). \tag{15.6}$$

After some algebra we find that the numerator simplifies (sort of) to

$$\mathrm{Tr}\{\ldots\} = 4\{-(p+qx)^\mu[p-q(1-x)]^\nu + (p+qx)\cdot[p-q(1-x)]\delta^{\mu\nu} - (p+qx)^\nu[p-q(1-x)]^\mu + m2\delta^{\mu\nu}\}. \tag{15.7}$$

As $d \to 4$ notice that since the integral is taken over all p, integrals with odd powers of p_μ vanish, so we neglect them here. Also, only the diagonal parts of $p^\mu p^\nu$ have nonzero integrals. The nonvanishing terms can be obtained by replacing

$$p^\mu p^\nu \to \frac{1}{d}\delta^{\mu\nu}p^2 \tag{15.8}$$

$$p^\mu p^\nu p^\rho p^\sigma \to \frac{(p^2)^2}{d(d+2)}(\delta^{\mu\nu}\delta^{\rho\sigma} + \delta^{\mu\rho}\delta^{\nu\sigma} + \delta^{\mu\sigma}\delta^{\nu\rho}). \tag{15.9}$$

Now our integrand is rotationally invariant, so we can split it up as usual with

$$\int \frac{d^dp}{(2\pi)^d} \to S_d \int \frac{p^{d-1}dp}{(2\pi)^d} = \int_0^\infty \frac{(p^2)^{d/2-1}d(p^2)}{(4\pi)^{d/2}\Gamma(d/2)}. \tag{15.10}$$

With these substitutions, our vacuum polarization contribution becomes

$$\begin{aligned}\Pi^{\mu\nu}_{\text{1-loop}}(q) = \frac{4\mu^\epsilon g^2}{(4\pi)^{d/2}\Gamma(d/2)} \int_0^1 dx \int_0^\infty dp^2 (p^2)^{d/2-1}\frac{1}{p^2+\Delta} \\ \times \left[p^2(1-\frac{2}{d})\delta^{\mu\nu} + (2q^\mu q^\nu - q^1\delta^{\mu\nu})(1-x) + m^2\delta^{\mu\nu}\right].\end{aligned}$$

These are Euler beta functions– letting $u = \frac{\Delta}{p^2+\Delta}$, our integral takes the form

$$\int_0^\infty d(p^2)\frac{(p^2)^{d/2-1}}{p^2+\Delta} = \left(\frac{1}{\Delta}\right)^{2-d/2}\frac{\Gamma(2-d/2)\Gamma(d/2)}{\Gamma(2)}$$

$$\int_0^\infty d(p^2)\frac{(p^2)^{d/2}}{p^2+\Delta} = \left(\frac{1}{\Delta}\right)^{1-d/2}\frac{\Gamma(1+d/2)\Gamma(1-d/2)}{\Gamma(2)},$$

which means that

$$\Pi^{\mu\nu}_{\text{1-loop}}(q) = (q^2\delta^{\mu\nu} - q^\mu q^\nu)\pi_{\text{1-loop}}(q^2) \tag{15.11}$$

where

$$\pi_{\text{1-loop}}(q^2) = -\frac{8g^2\Gamma(\epsilon/2)}{(4\pi)^{d/2}}\int_0^1 dx\, x(1-x)\left(\frac{\mu^2}{\Delta}\right)^{\epsilon/2}, \tag{15.12}$$

where this μ is the mass scale in $e^2 = \mu^\epsilon g^2(\mu)$. Note that the Lorentz structure is the same as for the free propagator in Lorenz gauge, i.e.

$$q_\mu\pi^{\mu\nu}_{\text{1-loop}} = 0.$$

Ward identity

- The gauge invariance of our theory leads to a massless photon (n.b. a coupling $\frac{1}{2}m^2A^2$ breaks gauge invariance).
- There are only two polarizations for the photon:

$$\varepsilon^\mu(p) = c_1\varepsilon_1^\mu(p) + c_2\varepsilon_2^\mu(p) \tag{15.13}$$

  where $\varepsilon^\mu_{1,2}$ are basis polarization vectors.
- Under a Lorentz boost, the polarization vector transforms as

$$\varepsilon^\mu(p) \to c_1'\varepsilon_1^\mu(p') + c_2'\varepsilon_2^m u(p') + c_3p^{\mu\prime} \tag{15.14}$$

- Consider a scattering amplitude $\mathcal{M}$ with at least one photon in the initial and final state. This depends on ε^μ:
$$\mathcal{M} = \varepsilon^\mu \mathcal{M}_\mu.$$
- After a boost, the amplitude transforms to
$$\begin{aligned}\mathcal{M} &\to (c_1' \varepsilon_1^\mu(p') + c_2' \varepsilon_2^m u(p') + c_3 p^{\mu\prime})\mathcal{M}'_\mu \\ &= \varepsilon'^\mu \mathcal{M}'_\mu\end{aligned}$$
in this frame. But the photon in a boosted frame still only has 2 polarizations, so we conclude that
$$p^\mu \mathcal{M}_\mu = 0, \tag{15.15}$$
i.e. there is no longitudinal polarization.
- Note that with the tree-level propagator $\tilde{D}^{\mu\nu}(k)$,
$$k_\mu \tilde{D}^{\mu\nu}(k) = \frac{k_\mu}{k^2}(\delta^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}) = \xi \frac{k^\nu}{k^2}, \tag{15.16}$$
so the longitudinal term will not be renormalized by loop diagrams and the ξ-dependence will cancel out of gauge-invariant quantities.

Lecture 16.

Saturday, February 23, 2019

QED counterterms Last time, we calculated the vacuum polarization at one loop. There are other loop diagrams we might be interested in, like the electron self-energy and the one-loop correction to the electron-photon interaction (see diagram).

To renormalize our theory, we will add counterterms of the form
$$S^{CT}[\psi, \bar{\psi}, A, \epsilon] = \int d^d x \left[\frac{\delta Z_3}{4} F_{\mu\nu}F^{\mu\nu} + \delta Z_2 \bar{\psi} \not{D} \psi + \delta m \bar{\psi}\psi\right]. \tag{16.1}$$
It's this second (gauge invariant) term we'll need in the following calculation. As $d \to 4$ ($\epsilon \to 0$), we have
$$\pi_{\text{1-loop}}(q^2) = -\frac{g^2(\mu)}{2\pi^2}\int_0^1 dx\, x(1-x)\left(\frac{2}{\epsilon} - \gamma + \log\frac{4\pi\mu^2}{\Delta}\right) + O(\epsilon) \tag{16.2}$$
with Δ, μ as before and γ the Euler-Mascheroni constant. In the $\overline{\text{MS}}$ term,
$$\delta Z_3 = -\frac{g^2(\mu)}{12\pi^2}\left(\frac{2}{\epsilon} - \gamma + \log 4\pi\right). \tag{16.3}$$
After renormalization, the 1-loop self-energy takes the form
$$\pi^{\text{ren}}_{\text{1-loop}}(q^2) = \frac{g^2(\mu)}{2\pi^2}\int_0^1 dx\, x(1-x)\log\left(\frac{m^2 + x(1-x)q^2}{\mu^2}\right). \tag{16.4}$$
Thus we've killed off the $1/\epsilon$ divergence and the constants, and have substituted back in $\Delta = m^2 + q^2 x(1-x)$. Note the branch cut in this integral for $m^2 + x(1-x)q^2 \leq 0$, when the argument of the log becomes negative or zero. In fact, there's a physical interpretation for this. For $x \in [0,1]$ we see that $0 \leq x(1-x) \leq 1/4$. In Minkowski signature, $q_0 = iE$, the branch cut therefore corresponds to
$$m^2 \leq x(1-x)(E^2 - \mathbf{q}^2) \leq \frac{1}{4}(E^2 - \mathbf{q}^2) \tag{16.5}$$
so the smallest E that satisfies this is simply $E^2 = (2m)^2$. This is precisely the minimum energy required to create a real (on-shell) electron-positron pair.

QED β function The simplest way to obtain the beta function for the QED coupling is to rescale the field

$$A_\mu \to \frac{A_\mu}{g}$$

so that the action is

$$\frac{1}{4g_{\text{eff}}^2}\int d^4x F_{\mu\nu}F^{\mu\nu} = \frac{1-\pi^{\text{ren}}_{\text{1-loop}}(0)}{4g^2(\mu)}\int d^4x\, F_{\mu\nu}F^{\mu\nu} \tag{16.6}$$

$$= \frac{1}{4}\left[\frac{1}{g^2(\mu)} - \frac{\hbar}{12\pi^2}\log\frac{m^2}{\mu^2} + O(\epsilon)\right]\int d^4x\, F_{\mu\nu}F^{\mu\nu}. \tag{16.7}$$

Then the beta function is

$$\beta(g) = \frac{\hbar g^3(\mu)}{12\pi^2} + O(\hbar^2) > 0. \tag{16.8}$$

Integrating, we can see how g depends on μ: it is

$$\frac{1}{g^2(\mu')} = \frac{1}{g^2(\mu)} + \frac{\hbar}{6\pi^2}\log\frac{\mu}{\mu'} + O(\hbar^2). \tag{16.9}$$

Let Λ_{QED} be the scale at which the coupling diverges,

$$g^2(\mu) = \frac{g\pi^2}{\hbar}\frac{1}{\log\Lambda_{QED}/\mu}. \tag{16.10}$$

Given that $m_e = 0.511\,\text{MeV}$, we have a coupling constant $\alpha = \frac{g^2(m_e)}{4\pi} \approx \frac{1}{137}$ (the fine structure constant), which tells us that $\Lambda_{QED} \sim 10^{286}\,\text{GeV}$. Note that at scales $\mu = 100\,\text{GeV}$, the EM and weak forces merge into the electroweak force.

Renormalization group Let's begin our discussion of renormalization by remarking that a QFT is not fully defined by its Lagrangian $\mathcal{L}$ (or equivalently an action S). A full definition requires that we regularize the theory in order to tame its divergences, which introduces an associated (unphysical) mass scale. Thus we impose some renormalization conditions in order to uniquely set the parameters of our theory, which requires empirical input (i.e. experiments to set the effective couplings). Once this is done, we can use a QFT to make predictions.

However, the physical predictions of our theory should be independent of arbitrary choices made in defining the theory. That is, the predictions must be the same at low energies independent of regularization scheme. This leads us to a sense of *universality*, just as we saw in *Statistical Field Theory*. That is, no matter what is going on in the UV, our effective theory describes the same IR physics emerging from families of theories with different regularization schemes or scales.

The renormalization group is therefore the natural structure to study how theories with different short-distance (UV) details can give rise to the same long-distance (IR) physics.

To see an example of this, consider a real scalar theory in d dimensions with a momentum cutoff Λ_0.

$$S_{\Lambda_0}[\phi] = \int d^dx\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \sum_i \frac{1}{\Lambda_0^{d_i-d}} g_{i0}O_i(x)\right] \tag{16.11}$$

where O_i are local operators of mass dimension $d_i > 0$ made up of fields and their derivatives. For instance,

$$O_i = (\partial\phi)^{r_i}\phi^{s_i}. \tag{16.12}$$

We can count the total number of fields in O_i– call it n_i, where in this example $n_i = r_i + s_i$.

The partition function with a cutoff, denoted $\mathcal{Z}$, is now

$$\mathcal{Z}(g_{i0}) = \int^{\Lambda_0}\mathcal{D}\phi e^{-S_{\Lambda_0}[\phi]}. \tag{16.13}$$

This tells us to integrate over fields with $|p| \leq \Lambda_0$.

Lecture 17.

Tuesday, February 26, 2019

Renormalization group Let's work with the action

$$S_{\Lambda_0}[\phi] = \int d^d x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \sum_i \frac{1}{\Lambda_0^{d_i - d}} g_{i0} O_i(x) \right], \tag{17.1}$$

where we're temporarily disregarding the mass coupling. The partition function $\mathcal{Z}$ is then

$$\mathcal{Z}_{\Lambda_0}(g_{i0}) = \int^{\Lambda_0} \mathcal{D}\phi e^{-S_{\Lambda_0}[\phi]}. \tag{17.2}$$

That is, we integrate over field configurations such that $|k| \leq \Lambda_0$. We might write the fields in terms of their Fourier transforms, i.e.

$$\phi(x) = \int_{|p| \leq \Lambda_0} \frac{d^d p}{(2\pi)^d} e^{ipx} \tilde{\phi}(p). \tag{17.3}$$

Let us introduce $\Lambda < \Lambda_0$, a lower cutoff, and split the integral as

$$\begin{aligned} \phi(x) &= \int_{|p| \leq \Lambda} \frac{d^d p}{(2\pi)^d} e^{ipx} \tilde{\phi}(p) + \int_{\Lambda < |p| \leq \Lambda_0} \frac{d^d p}{(2\pi)^d} e^{ipx} \tilde{\phi}(p) \\ &= \phi^-(x) + \phi^+(x). \end{aligned}$$

These sets of modes are disjoint, so we can write $\mathcal{D}\phi = \mathcal{D}\phi^- \mathcal{D}\phi^+$. Integrating over ϕ^+ gives an effective action

$$S_\Lambda^{\text{eff}}[\phi] = -\log \int_\Lambda^{\Lambda_0} \mathcal{D}\phi^+ e^{-S_{\Lambda_0}[\phi^- + \phi^+]}. \tag{17.4}$$

This "RG equation" tells us how to map $S_{\Lambda_0} \to S_\Lambda^{\text{eff}}$ as UV modes are integrated out, and this process can be iterated.[21] We therefore write

$$S_{\Lambda_0}[\phi^- + \phi^+] = S^0[\phi^-] + S^0[\phi^+] + S_{\Lambda_0}^{\text{int}}[\phi^-, \phi^+] \tag{17.5}$$

with free actions

$$S^0[\phi] = \int d^d x \, \frac{1}{2} \left[(\partial \phi)^2 + m^2 \phi^2 \right]. \tag{17.6}$$

Note that since ϕ^-, ϕ^+ have disjoint support, there are no mixed $\phi^- \phi^+$ terms. Equivalently the Fourier transform of such a term would be $\tilde{\phi}^-(k) \tilde{\phi}^+(k') \delta(k + k')$, and since these modes are in different regions of momentum space, they will not mix. Note this will be different for higher-order couplings.

Performing our integration over UV modes, we get some effective interactions

$$S_\Lambda^{\text{int}}[\phi^-] = -\log \int \mathcal{D}\phi^+ \, \exp\left[-S^0[\phi^+] - S_{\Lambda_0}^{\text{int}}[\phi^-, \phi^+] \right]. \tag{17.7}$$

Running couplings Remember, our basic principle is that the physics at low energies must be independent of the cutoffs Λ, Λ_0. Therefore

$$\int^\Lambda \mathcal{D}\phi \, e^{-S_\Lambda^{\text{eff}}[\phi]} = \int^{\Lambda_0} \mathcal{D}\phi \, e^{-S_{\Lambda_0}[\phi]}. \tag{17.8}$$

It follows that after integration, we will have some new, modified couplings which depend on Λ. Thus

$$\mathcal{Z}_\Lambda(g_i(\Lambda)) = \mathcal{Z}_{\Lambda_0}(g_{i0}; \Lambda_0). \tag{17.9}$$

But since the RHS is independent of Λ, so is the LHS. This places some constraints on the "flow" of the coupling constants, which we call the *Callan-Symantzik equation*. It takes the form

$$0 = \Lambda \frac{d\mathcal{Z}_\Lambda(g)}{d\Lambda} = \left(\Lambda \frac{\partial}{\partial \Lambda}|_{g_i} + \Lambda \frac{\partial g_i}{\partial \Lambda} \frac{\partial}{\partial g_i}|_\Lambda \right) \mathcal{Z}_\Lambda(g). \tag{17.10}$$

[21] In *Statistical Field Theory*, we also had to rescale the momenta to match the original upper limit Λ_0 in order to study the "same" kind of theory. I'm not sure if we're just not interested in that here because we want an effective action, or if there's something else going on.

Now, S_{Λ_0} is completely general, so S^{eff}_{Λ} should have the same form. We write

$$S^{\text{eff}}_{\Lambda}[\phi] = \int d^d x \left[\frac{Z_\Lambda}{2}(\partial\phi)^2 + \sum_i \frac{Z_\Lambda^{n_i/2}}{\Lambda^{d_i-d}} g_i(\Lambda) O_i(x) \right]. \tag{17.11}$$

in terms of some new coefficients g_i. Integrating may give a $Z_\Lambda \neq 1$ factor. LSZ therefore implies we want a canonically normalized kinetic term. Let $\phi^r = Z_\Lambda^{1/2}\phi$ be the renormalized field. The remaining variations describing the Λ dependence are given by the $g_i(\Lambda)$.

We can associate some β-functions to this theory by[22]

$$\beta_i^{\text{cl}} = (d_i - d), \quad \beta_i^{\text{q}} = \Lambda \frac{\partial g_i}{\partial \Lambda}, \tag{17.12}$$

where the superscripts indicate classical and quantum contributions. For example, ϕ^4 theory in 4 dimensions with a cutoff Λ_0 has an action of the form

$$S + S^{CT} = \int d^4 x \left[\frac{1}{2}(1+\delta Z)(\partial\phi)^2 + \frac{1}{2}(m^2 + \delta m^2)\phi^2 + \frac{1}{4!}(\lambda + \delta\lambda)\phi^4 \right]. \tag{17.13}$$

At one loop, we used the on-shell scheme to fx $\delta Z = 0$ and choose δm^2 such that $m^2 = m^2_{\text{phys}}$. In the language of the renormalization group, we can write

$$g_2(\Lambda_0) = \frac{1}{\Lambda_0^2}(m^2 + \delta m^2) = g_{20} - \frac{\lambda}{32\pi^2}\left(1 - g_{20}\log\left(1 + \frac{1}{g_{20}}\right)\right), \tag{17.14}$$

which is Λ-independent. Similarly,

$$g_4(\Lambda_0) = \lambda + \delta\lambda = g_{40} + \frac{3g_{40}^2}{32\pi}\left(\log\frac{\Lambda_0^2}{m^2} - 1\right). \tag{17.15}$$

With our rescaled field $\phi^r = Z_\Lambda^{1/2}\phi$ (here $Z_\Lambda = 1$, but not generally) we set

$$g_{\text{eff}} = \left. \frac{\delta^4 \Gamma[\tilde{\phi}^r]}{\delta\tilde{\phi}^r(p_1)\delta\tilde{\phi}^r(p_2)\delta\tilde{\phi}^r(p_3)\delta\tilde{\phi}^r(p_4)} \right|_{p_i=0},$$

which we can write as our sum of one-loop diagrams as before, but with a factor of Z_Λ^{-2} to account for that these variations are taken with respect to the renormalized field. We find that

$$\frac{dg_{\text{eff}}}{d\Lambda} = 0. \tag{17.16}$$

Anomalous dimensions in general $\delta Z \neq 0$, which means that $Z_\Lambda = 1 + \delta Z \neq 1$ and therefore the kinetic term will transform nontrivially under renormalization. The anomalous dimension of the field ϕ is given by

$$\gamma_\phi \equiv -\frac{\Lambda}{2}\frac{\partial}{\partial\Lambda}\log Z_\Lambda, \tag{17.17}$$

which is a "β-function" for the kinetic term. In our last example this was identically zero.

For instance, look at the n-point correlation function

$$\langle \phi(x_1)\ldots\phi(x_n)\rangle = Z_\Lambda^{-n/2}\langle \phi^r(x_1)\ldots\phi^r(x_n)\rangle. \tag{17.18}$$

We focus on the 1PI n-point functions calculated by variations with respect to ϕ^r:

$$\Gamma_\Lambda^{(n)}(x_1,\ldots,x_n; g_i(\Lambda)) = \frac{\delta^n\Gamma}{\delta\phi^r(x_1)\ldots\delta\phi^r(x_n)}. \tag{17.19}$$

The fact that our predictions must be independent of Λ tell us that we get the same expectation values for $s\lambda$ as for Λ, with $0 < s < 1$. Thus

$$Z_{s\Lambda}^{-n/2}\Gamma_{s\Lambda}^{(n)}(x_1,\ldots,x_n; g(s\Lambda)) = Z_\Lambda^{-n/2}\Gamma_\Lambda^{(n)}(x_1,\ldots,x_n; g(\Lambda)). \tag{17.20}$$

[22] I'm not sure this is correct– compare Skinner's Eqn. 5.15 in `http://www.damtp.cam.ac.uk/user/dbs26/AQFT/Wilsonchap.pdf`. There, he gives $\beta_i^{\text{cl}} = (d_i - d)g(\Lambda)$ and he leaves β_i^{q} undetermined, calling the overall beta-function $\beta_i = \Lambda\frac{\partial g_i}{\partial\Lambda}$.

Under an infinitesimal $\delta s = 1 - s$, we get

$$0 = \Lambda \frac{d}{d\Lambda} \Gamma_\Lambda^{(n)}(x_1, \ldots, x_n; g(\Lambda)) = \left(\Lambda \frac{\partial}{\partial \Lambda} + \beta_i \frac{\partial}{\partial g_i} + n\gamma_\phi \right) \Gamma_\Lambda^{(n)}(x_1, \ldots, x_n; g_i(\Lambda)) \tag{17.21}$$

where $\beta_i = \Lambda \frac{\partial g_i}{\partial \Lambda}$ is the quantum β-function. This is called the *generalized Callan-Symanzik equation*.

Lecture 18.

Thursday, February 28, 2019

Note. We report the following erratum. Last time, we wrote an action as $S + S^{CT}$, and tried to connect it to the Wilsonian flow. However, this action has a definite cutoff, whereas the renormalization group interpretation requires us to integrate from Λ to Λ_0, so these aren't quite comparable.

Let us now return to our discussion of anomalous dimension.[23] Last time, we wrote down the anomalous dimension of a field ϕ, given by

$$\gamma_\phi \equiv -\frac{\Lambda}{2} \frac{\partial}{\partial \Lambda} \log Z_\Lambda, \tag{18.1}$$

and we derived the generalized Callan-Symanzik equation,

$$0 = \Lambda \frac{d}{d\Lambda} \Gamma_\Lambda^{(n)}(x_1, \ldots, x_n; g(\Lambda)) = \left(\Lambda \frac{\partial}{\partial \Lambda} + \beta_i \frac{\partial}{\partial g_i} + n\gamma_\phi \right) \Gamma_\Lambda^{(n)}(x_1, \ldots, x_n; g_i(\Lambda)). \tag{18.2}$$

Thus if we let $\Lambda' = s\Lambda$, the differentiating with respect to s we have

$$s \frac{\partial}{\partial s} Z_{s\Lambda}^{-n/2} = -\frac{n}{2} Z_{s\Lambda}^{-n/2} s \frac{\partial}{\partial s} \log Z_{s\Lambda} = n\gamma \tag{18.3}$$

using $s \frac{\partial}{\partial s} = \Lambda' \frac{\partial}{\partial \Lambda'}$.

Our RG process is then as follows.

(a) Integrate out modes with momenta in $(s\Lambda, \Lambda)$.

(b) Rescale coordinates $x^\mu \mapsto x'^\mu = sx^\mu$ in order to keep the kinetic term canonically normalized, $\frac{1}{2} \int d^d x \frac{1}{2} \partial_\mu \phi \partial^\mu \phi$, so that

$$\phi^r(sx) = s^{1-d/2} \phi^r(x). \tag{18.4}$$

The rest of the action is invariant if $\Lambda \to \Lambda/s$.

Then

$$\begin{aligned} \Gamma_\Lambda^{(n)}(x_1, \ldots, x_n; g_i(\Lambda)) &= \left(\frac{Z_\Lambda}{Z_{s\Lambda}} \right)^{n/2} \Gamma_{s\Lambda}^{(n)}(x_1, \ldots, x_n; g_i(s\Lambda)) \\ &= \left(s^{2-d} \frac{Z_\Lambda}{Z_{s\Lambda}} \right)^{n/2} \Gamma_\Lambda^{(n)}(sx_1, \ldots, sx_n; g_i(s\Lambda)). \end{aligned}$$

Note the values $g_i(s\Lambda)$ and $Z_{s\Lambda}$ do not change under rescaling. Now a relabeling $x_i \mapsto \frac{x_i}{s}$ yields

$$\Gamma_\Lambda^{(n)}(x_1/s, \ldots, x_n/s; g_i(\Lambda)) = \left(s^{2-d} \frac{Z_\Lambda}{Z_{s\Lambda}} \right)^{n/2} \Gamma_\Lambda^{(n)}(x_1, \ldots, x_n; g_i(s\Lambda)). \tag{18.5}$$

Thus we can think of a running coupling while integrating out high-momentum modes as equivalent to the same coupling under a scaling transformation. As $s \to 0$, we are integrating out more modes. On the LHS, we see the separation between points increasing $\frac{|x_i - x_j|}{s} \to \infty$ (flowing to the long-distance infrared behavior), whereas on the RHS the separation is fixed but the coupling is "running" to lower energy scales (becomes insensitive to UV phenomena).

For infinitesimal $\delta s = 1 - s$ we can expand to linear order,

$$\left(s^{2-d} \frac{Z_\Lambda}{Z_{s\Lambda}} \right)^{1/2} = 1 + \left(\frac{d-2}{2} + \gamma_\phi \right) \delta s + \ldots \tag{18.6}$$

[23] Cf. Tong *Statistical Field Theory* §3.2.

and we see that the fields scale with mass dimension

$$\frac{d-2}{2} + \gamma_\phi \equiv \Delta\phi, \tag{18.7}$$

where there is an "engineering dimension" that we always get (and could have read off from the kinetic term), plus an "anomalous dimension." These add up to make an overall scaling dimension $\Delta\phi$ which in general is not the engineering dimension.

RG flow The renormalization group process tells us how couplings run as we integrate out high momentum modes and flow to the IR. These trace out trajectories, lines in the space of coupling constants which are governed by β-functions. Remarkably, some theories flow to the same endpoints, and therefore share the same IR physics. This is known as universality.

Where do such theories end up? If they end somewhere, they must end at fixed points (critical points), i.e. points in the space of coupling constants $g_i = g_i^*$ such that

$$\beta_i|_{g_i=g_i^*} = 0 \forall i, \tag{18.8}$$

where we now mean the full β-function, including classical and quantum contributions.

In ϕ^4 theory, there's an easy fixed point to spot. This is the Gaussian fixed point, $g_j^* = 0 \forall j$, which is a massless free theory with no mass. If there are no couplings, there's nothing to flow and the theory stays at the fixed point. There are also nontrivial fixed points which require β^{cl} and β^{q} to cancel, such as the Wilson-Fisher fixed point.

Scale invariance at fixed points Let us note that at fixed points, the couplings g_i^* must be independent of scale, and dimensionless functions of g_i^* become constant, e.g. $\gamma_\phi(g_i^*) = \gamma_\phi^*$.

Consider Callan-Symanzik for $n = 2$ at a fixed point. We have

$$\Lambda \frac{\partial}{\partial \Lambda} \Gamma_\Lambda^{(2)}(x,y) = -2\gamma_\phi^* \Gamma^{(2)}(x,y). \tag{18.9}$$

Lorentz invariance tells us that $\Gamma^{(2)}$ must be a function of $|x-y|$ only. Like $\langle \phi(x)\phi(y)\rangle$, $\Gamma^{(2)}$ has mass dimensions of $d-2$, so we posit that

$$\Gamma_\Lambda^{(2)}(|x-y|; g_i^*) = \frac{\Lambda^{d-2}}{\Lambda^{2\Delta d}} \frac{c(g_i^*)}{|x-y|^{2\Delta d}}. \tag{18.10}$$

Lecture 19.

Saturday, March 2, 2019

Admin note: Example Sheet 3 is online, albeit with just three questions for now.

Let's finish up our discussion of RG today. Near a fixed point, we can linearize the RG equations. Let

$$\delta g_j = g_j - g_j^* \tag{19.1}$$

so that

$$\Lambda \left.\frac{\partial g_i}{\partial \Lambda}\right|_{g_j^*+\delta g_j} = B_{ij}\delta g_j + O(\delta g^2). \tag{19.2}$$

Then this matrix B_{ij} has some eigenvectors σ_i, which are Λ-dependent vectors in coupling constant space. Its eigenvalues are $\Delta_i - d$ with $\Delta_i =$ the scaling dimension of σ_i.

For instance, in the simplest case, the coupling constants all decouple and B_{ij} is just diagonal. The Λ dependence of σ_i is then obvious. More generally, the eigenvectors σ_i represent linear combinations of some operators $O_j(x)$ in the action, i.e.

$$\Lambda \frac{\partial \sigma_i}{\partial \Lambda} = (\Delta_i - d)\sigma_i \implies \sigma_i(\Lambda) = \left(\frac{\Lambda}{\Lambda_0}\right)^{\Delta_i - d} \sigma_i(\Lambda_0), \tag{19.3}$$

for an overall cutoff Λ_0. There are some cases to consider here:

- $\Delta_i > d \implies \sigma_i(\Lambda) < \sigma_i(\Lambda_0)$, so we flow back to the fixed point (an *irrelevant* direction in coupling constant space)
- $\Delta_i < d \implies \sigma_i(\Lambda) > \sigma_i(\Lambda_0)$, so we flow away from the fixed point (a *relevant* direction)

- $\Delta_i = d \implies$ the coupling is *marginal*, i.e. we can't tell (to this order) which way the coupling will flow.

The subspace of irrelevant couplings is called the *critical surface* C. Strictly, it is infinite-dimensional.[24] However, the codimension (i.e. the dimension of the perpendicular space) of C is finite and represents relevant directions. There is also a special trajectory off the critical surface, known as the "renormalized trajectory" (RT).

Continuum limit We used the RG to give us an effective action,

$$S_\Lambda^{\text{eff}}[\phi^-] = \log \int_\Lambda^{\Lambda_0} \mathcal{D}\phi^+ \exp\left(-S_{\Lambda_0}[\phi^- + \phi^+]\right), \tag{19.4}$$

which gave Λ-independent physics. What if we take $\Lambda_0 \to \infty$ (the continuum limit)? This is of course equivalent to taking a lattice spacing to zero in our theory.

Suppose we start at some point in theory space with a set of initial coupling constants $\{g_{i0}\}$ and a cutoff Λ_0. Now we flow to a new set of couplings $\{g_i^{\text{ref}}\}$. The distance we move along our trajectory in RG space then depends on the ratio μ/Λ_0, where μ is the scale at which $g_i(\mu) = g_i^{\text{ref}}$.

Keeping g_{i0} fixed but increasing Λ_0, we see that $g_i(\Lambda)$ is driven towards the fixed point in irrelevant directions but away from the fixed point in relevant directions. If we have only irrelevant couplings, i.e. if g_{i0} lies on C, then in the $\Lambda_0 \to \infty$ limit we flow into the fixed point. That is, $\lim_{\Lambda_0\to\infty} S_\Lambda^{\text{eff}}[\phi^-]$ exists and corresponds to a scale-invariant theory with couplings g_i^*.[25]

Now let us consider the continuum limit with relevant couplings (e.g. mass coupling, Yang-Mills coupling in 4D). Let g_i^{ref} be a point in coupling space "near" g_i^*. Let μ be the scale at which, for a given Λ_0, $g_i(\mu) = g_i^{\text{ref}}$. If we had started with some cutoff $\Lambda' = b\Lambda_0$ ($b > 1$) then we would have had to run to a scale $\mu' = b\mu$, i.e.

$$\frac{\mu'}{\Lambda'} = \frac{\mu}{\Lambda_0} \implies \mu = \Lambda_0 f(g_{i0}). \tag{19.5}$$

But we want μ finite, so we must change the initial action.

Counterterms, revisited We can change the initial action by adding counterterms:

$$S_{\Lambda_0}[\phi] = S_{\Lambda_0}[\phi] + S^{CT}[\phi, \Lambda_0]. \tag{19.6}$$

This modifies the couplings as $g_{i0} \mapsto \tilde{g}_{i0}$ so that the trajectory under RG flow is closer to C and the renormalization trajectory from the fixed point. Since the "flow" is "slower" nearer to the fixed point, the ratio of scales $\tilde{\mu}/\tilde{\Lambda}_0$ can be made smaller than the original μ/Λ_0.

Note the limit is taken after integrating out the high-momentum modes (Λ, Λ_0), i.e. $\lim_{\Lambda_0\to\infty} S_\Lambda^{\text{eff}}[\phi]$.

Note also that if the irrelevant operators are important for describing some physics, e.g. the 4-fermion operators describing low-energy β decay in the theory of the weak interaction, we cannot take the continuum limit. Such couplings will be suppressed under RG flow, so there is no way to keep these couplings nonzero as $\Lambda_0 \to \infty$. The theory is said to be nonrenormalizable. This usually indicates something new is going on– in the case of the weak interaction, this was the unification into the electroweak interaction at higher energies.

Lecture 20.

Tuesday, March 5, 2019

Nonabelian gauge theories Today, we begin our discussion of nonabelian gauge theories. For an external reference, see Peskin and Schroeder or Osborn.

Under a local $U(1)$ transformation of the fermion field

$$\psi(x) \mapsto e^{i\alpha(x)}\psi(x), \tag{20.1}$$

[24] See also Skinner's nice diagram on page 24 of the PDF. Or David Tong's for that matter.

[25] In lecture, it was stated that scale invariance implies this is a CFT. This is not generally true, or at least has not been proven. Polchinski completed a proof by Zamolodchikov that this holds in 2 dimensions but it's an open question for higher dimensions.

the term $\bar{\psi}\partial\!\!\!/\psi$ is not invariant. Consider the derivative in the direction of n^μ a unit vector, i.e.

$$n^\mu \partial_\mu \psi = \lim_{a\to 0} \frac{1}{a}[\psi(x+an) - \psi(x)]. \tag{20.2}$$

Definition 20.3. A *parallel transporter* (aka *Wilson line*) is an object $U(y,x)$ with the following ($U(1)$) gauge transformation:

$$U(y,x) \mapsto e^{i\alpha(y)} U(y,x) e^{-i\alpha(x)}. \tag{20.4}$$

If we also set $U(x,x) = 1$, then $U(y,x)$ can be written as a phase

$$U(y,x) = e^{i\phi(y,x)}, \tag{20.5}$$

and we moreover take $U(x,y) = (U(y,x))^*$.

With this Wilson line, we can define a covariant derivative for our theory,

$$n^\mu D_\mu \psi = \lim_{a\to 0} \frac{1}{a}(\psi(x+an) - U(x+an,x)\psi(x)) \tag{20.6}$$

such that

$$\bar{\psi} n^\mu D_\mu \psi = \lim_{a\to 0} \frac{1}{a}[\bar{\psi}_x \psi_{x+an} - \bar{\psi}_{x+an}\psi_x], \tag{20.7}$$

which is gauge-invariant. For small a, define

$$\begin{aligned} U(x+an,x) &= \exp\left[-iean^\mu A_\mu(x+\frac{a}{2}n) + O(a^3)\right] \\ &= 1 - iean^\mu A_\mu(x+\frac{a}{2}n) + O(a^2), \end{aligned}$$

so our covariant derivative takes the familiar form

$$D_\mu \psi(x) = \left[\partial_\mu + ieA_\mu(x)\right]\psi(x), \tag{20.8}$$

which is simply the minimal coupling of the gauge field $A_\mu(x)$.

Under gauge transformations,

$$A_\mu(x) \mapsto A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x) \tag{20.9}$$

$$D_\mu \psi(x) \mapsto e^{i\alpha(x)} D_\mu \psi(x), \tag{20.10}$$

which tells us that $D_\mu \psi$ transforms like ψ, as does $D_\nu(D_\mu\psi)$. We can consider how the commutator transforms under gauge transformations,

$$[D_\mu, D_\nu]\psi \mapsto e^{i\alpha(x)}[D_\mu, D_\nu]\psi, \tag{20.11}$$

where

$$[D_\mu, D_\nu] = ie(\partial_\mu A_\nu - \partial_\nu A_\mu) \equiv ieF_{\mu\nu}, \tag{20.12}$$

the gauge-invariant field strength tensor.

Generally, our Lagrangian can contain terms which are Lorentz invariant like

$$F_{\mu\nu}F^{\mu\nu} \text{ and } i\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}, \tag{20.13}$$

though the latter term here breaks P and T symmetry. In terms of the parallel transporters U, the $F_{\mu\nu}$ appears from gauge-invariant *Wilson loops*, i.e. closed Wilson lines. For instance, the plaquette, with overall value

$$P_{12}(x) = U(y_1,y_4)U(y_4,y_3)U(y_3,y_2)U(y_2,y_1). \tag{20.14}$$

We can expand this about small a to find

$$P_{12}(x) = 1 - iea^2 F_{12}(x) + O(a^3) \tag{20.15}$$

One can generalize this principle to a Lie group G (e.g. $SU(N)$). The Lie group acts on our fields by local transformations,

$$\psi(x) \mapsto V(x)\psi(x) \tag{20.16}$$

with $V(x) \in G$, and Wilson lines then take the form

$$U(y,x) \mapsto V(y)U(y.x)V^\dagger(x) \tag{20.17}$$

with $U(x,x) = 1$. If G has some (hermitian) generators t^a in the Lie algebra $L(G)$ corresponding to G, then

$$U(x+an, x) = 1 + igan^\mu A_\mu^a t^a + O(a^2), \tag{20.18}$$

where we take

$$[t^a, t^b] = if^{abc}t^c \tag{20.19}$$

for f^{abc} some structure constants which are totally antisymmetric in their indices. Note that the index a is summed over (and should not be confused with the expansion parameter a). As we learned in *Symmetries, Fields and Particles*, the Lie bracket obeys the Jacobi identity,

$$[t^a, [t^b, t^c]] + [t^b, [t^c, t^a]] + [t^c, [t^a, t^b]] = 0.$$

To find the transformation of the A^a gauge fields, we expand $V(x+an)$. Notice that $V(x)V^\dagger(x) = 1$ (where we take G to be unitary), so

$$\begin{aligned} V(x+an)V^\dagger(x) &= \Big[(1 + an^\mu \partial_\mu + O(a^2))V(x)\Big]V^\dagger(x) \\ &= 1 + an^\mu(\partial_\mu V)V^\dagger + \ldots \\ &= 1 - an^\mu V(\partial_\mu(V^\dagger) + \ldots, \end{aligned}$$

so we find that

$$A_\mu^a(x)t^a \mapsto V(x)\left[A_\mu^a(x)t^a + \frac{i}{g}\partial_\mu\right]V^\dagger(x). \tag{20.20}$$

Our covariant derivative is therefore

$$D_\mu = \partial_\mu - igA_\mu^a t^a, \tag{20.21}$$

and for infinitesimal transformations,

$$V(x) = 1 + i\alpha^a t^a + O(\alpha^2) \tag{20.22}$$

so that

$$\psi(x) \mapsto (1 + i\alpha^a(x)t^a)\psi(x) \tag{20.23}$$

$$A_\mu^a(x) \mapsto A_\mu^a(x) + \frac{1}{g}\partial_\mu\alpha^a(x) + f^{abc}A_\mu^b\alpha^c(x) = A_\mu^a + \frac{1}{g}D_\mu\alpha^a. \tag{20.24}$$

The field strength tensor is then defined

$$[D_\mu, D_\nu] = -igF_{\mu\nu}^a t^a \tag{20.25}$$

with

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c, \tag{20.26}$$

where this last term shows that some new structure comes in from the fact our Lie group is in general nonabelian. In electromagnetism, this last term vanished since the structure constants were all zero.

Under gauge transformations,

$$F_{\mu\nu}^a \mapsto F_{[}\mu\nu]^a - f^{abc}\alpha^b F_{\mu\nu}^c \tag{20.27}$$

alone is not gauge invariant, but

$$F_{\mu\nu}^a F^{a,\mu\nu} = \operatorname{Tr} F_{\mu\nu}F^{\mu\nu} \tag{20.28}$$

is gauge invariant (where the trace is taken over generators). Generically, the new term with the structure constants means that our theories will have self-interactions even at tree level.

Gauge fixing With our field strength tensor, we can write down a path integral

$$\int \mathcal{D}A \exp\left[-\frac{1}{4}\int d^4x \operatorname{Tr} F_{\mu\nu}F^{\mu\nu}\right] \tag{20.29}$$

To quantize, we need to avoid integrating over configurations which are pure gauge (i.e. gauge-equivalent to $A_\mu(x) = 0$). We do this through the Faddeev-Popov gauge fixing procedure, i.e. we set some $G[A] = 0$ at each point x such that

$$1 = \int \mathcal{D}\alpha(x)\delta(G[A^\alpha]) \det\left(\frac{\delta G[A^\alpha]}{\delta\alpha}\right) \tag{20.30}$$

where α is not an index but instead parametrizes the gauge transformation, i.e.

$$(A^\alpha)^a_\mu = A^a_\mu + \frac{1}{g}D_\mu\alpha^a. \tag{20.31}$$

Note that for $G[A]$ linear in A, the variation $\frac{\delta G[A^\alpha]}{\delta\alpha}$ will be independent of α. The gauge-fixing procedure is then as in QED:

(a) Intechange order of integration, $A \leftrightarrow \alpha$
(b) Change variables $A' = A^\alpha$, noting that $\mathcal{D}A' = \mathcal{D}A$
(c) Relabel (remove the $'$) and factor out the α integration (assuming linear $G[A]$).

We arrive at

$$\int \mathcal{D}Ae^{-S[A]} = \left(\int \mathcal{D}\alpha\right)\int \mathcal{D}Ae^{-S[A]}\delta(G[A])\det\left(\frac{\delta G[A^\alpha]}{\delta\alpha}\right). \tag{20.32}$$

Note that this final determinant can now depend on the gauge field A. We can calculate propagators similarly to QED, and we'll see that some new constraints emerge in our non-abelian theories.

Lecture 21.

Thursday, March 7, 2019

Gauge fixing, cont. We had a general gauge-fixing expression

$$\int \mathcal{D}Ae^{-S[A]} = \left(\int \mathcal{D}\alpha\right)\int \mathcal{D}Ae^{-S[A]}\delta(G[A])\det\left(\frac{\delta G[A^\alpha]}{\delta\alpha}\right),$$

and we can choose a particular gauge-fixing function $G[A] = \partial^\mu A^a_\mu(x) - \omega^a(x)$, just as in QED. We then integrate over an extra parameter $\omega^a(x)$ with Gaussian weight $1/2\xi$,

$$\int \mathcal{D}Ae^{-S[A]} \det\left(\frac{\delta G[A^\alpha]}{\alpha}\right) \exp\left[\int d^4x \frac{1}{2\xi}(\partial^\mu A_\mu)^2\right] \tag{21.1}$$

The gauge-transformed field is then

$$(A^\alpha)^a_\mu = A^a_\mu + \frac{1}{g}D_\mu\alpha^a \tag{21.2}$$

such that the variation is[26]

$$\frac{\delta G[A^\mu]}{\delta\alpha} = \frac{1}{g}\partial^\mu D_\mu. \tag{21.3}$$

The Faddeev-Popov method tells us to express the functional determinant as a path integral over new Grassmann fields $c, \bar{c}$ which transform in the adjoint representation of the gauge group. These new fields anticommute, but are otherwise Lorentz scalars (spin zero). These $c, \bar{c}$ fields do not represent physical particles (i.e. valid in/out states) and should be thought of as constraints. They are called *Faddeev-Popov ghosts.* One can then write a "ghost Lagrangian,"

$$\mathcal{L}_{\text{gh}} = \bar{c}^a(\partial^2\delta^{ac} + gf^{abc}\partial^\mu A^b_\mu)c^c. \tag{21.4}$$

We arrive at a ghost propagator,

$$\left\langle c^a(x)\bar{c}^b(y)\right\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{\delta^{ab}}{k^2}e^{-ik\cdot(x-y)} \tag{21.5}$$

[26] See Skinner to make sense of this expression. We've left off a delta-function.

so that a ghost propagator is associated with a δ^{ab}/k^2 and we also get a three-point vertex with amplitude $-gf^{abc}p^\mu$.

If these ghost fields do not represent physical particles, why do we need them in our theory?

- The Feynman rules for fermions and gauge bosons only (no ghosts) lead to unphysical gauge field polarizations, e.g. the following diagrams. At tree level we can neglect unphysical polarizations by focusing on physcial in/out states, but loops are a problem.
- Ghosts cancel the unphysical contributions, e.g. the sum is free from contributions of unphysical polarizations.
- It is possible to avoid the ghosts by using a Lorentz non-invariant gauge-fixing condition, e.g. axial gauge like $n \cdot A^a = 0 \forall a$ with n a unit vector. Thus $G[A^\alpha] = n \cdot (A^\alpha)^a$. The gauge transformation of such a field is then

$$n \cdot (A^\alpha)^a = n \cdot A^a + \frac{1}{g} n \cdot \partial \alpha^a + f^{abc} + f^{abc} n \cdot A^b \alpha^c = \frac{1}{g} n \cdot \partial \alpha^a, \tag{21.6}$$

which is independent of A as in QED. (We have simplified by the original gauge condition $n \cdot A = 0$.) Thus $\frac{\delta G[A^\alpha]}{\delta \alpha}$ is independent of A. However, the downside of doing this is that we get a more complicated propagator for the gauge field. We see that breaking Lorentz invariance has its costs.

BRST symmetry This symmetry is a constraint on physical states, named for Becchi-Rouet-Storz-Tyutin. We have a Lagrangian in Euclidean space,

$$\mathcal{L} = \frac{1}{4}(F^a_{\mu\nu})^2)^2 + \bar{\psi}(\not{D} + m)\psi + \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \bar{c}\partial^\mu D_\mu c. \tag{21.7}$$

This is just a theory with a term that looks like the $F_{\mu\nu}$ of QED, a fermion coupling, a kinetic term for the gauge field, and the ghost constraint. We may then introduce an auxiliary (i.e. non-dynamical) field B^a, modifying our Lagrangian to

$$\mathcal{L} = \frac{1}{4}(F^a_{\mu\nu})^2 + \bar{\psi}(\not{D} + m)\psi + \frac{1}{2\xi}(B^a)^2 - B^a \partial^\mu A^a_\mu + \bar{c}\partial^\mu D_\mu c, \tag{21.8}$$

where completing the square and integrating out B^a yields the original Lagrangian.

This new Lagrangian is then invariant under the global BRST transformation, written in infinitesimal form as

$$\delta A^a_\mu = \epsilon D^{ac}_\mu c^c \tag{21.9}$$

$$\delta \psi = i g \epsilon c^a t^a \psi \tag{21.10}$$

$$\delta c^a = -\frac{1}{2} g \epsilon f^{abc} c^b c^c \tag{21.11}$$

$$\delta \bar{c}^a = \epsilon B^a \tag{21.12}$$

$$\delta B^a = 0. \tag{21.13}$$

where ϵ must be an infinitesimal Grassmann (anticommuting) quantity. Here, the t^as are the generators of the gauge group as before and f^{abc} are the corresponding structure constants.

This transformation may seem a bit ad hoc, but in fact it will help us to make sense of the ghosts. Notice that the transformation of $\bar{c}^a$ is related to the auxiliary field B^a.

The transformation of the fields ψ, A_μ is a local gauge transformation with $\alpha^a(x) = g\epsilon c^a(x)$, so invariance of $(F^a_{\mu\nu})^2, \bar{\psi}(\not{D} + m)\psi$ is clear. The B^a term is also trivially invariant.

The transformation of $\bar{c}$ cancels the transformation of A_μ when combining the last two terms of $\mathcal{L}$, so what remains is the covariant derivative term.

$$\begin{aligned}\delta(D^{ac}_\mu c^c) &= D^{ac}_\mu \delta c^c + g f^{abc} \delta A^b_\mu c^c \\ &= -\frac{1}{2} g\epsilon \partial_\mu (f^{abc} c^b c^c) - \frac{1}{2} g^2 \epsilon f^{abc} f^{cde} A^b_\mu c^d c^e + g\epsilon f^{abc} (\partial_\mu c^b) c^c + g^2 \epsilon f^{abc} f^{bde} A^d_\mu c^c c^e.\end{aligned}$$

Looking at the $O(g)$ terms, we have

$$-\frac{f^{abc}}{2} \partial_\mu (c^b c^c) + f^{abc} (\partial_\mu c^b) c^c = -\frac{f^{abc}}{2} \left[(\partial_\mu c^b) c^c + c^b (\partial_\mu c^c) \right] + f^{abc} (\partial_- \mu c^b) c^c. \tag{21.14}$$

Using

$$f^{abc}c^b(\partial_\mu c^c) = -f^{abc}(\partial_\mu c^c)c^b = -f^{abc}(\partial_\mu c^b)c^c = +f^{abc}(\partial_\mu c^b)c^c, \tag{21.15}$$

we see that the whole $O(g)$ term is zero. Similar manipulations show that the $O(g^2)$ term is also zero, using the Jacobi identity:

$$f^{ade}f^{bcd} + f^{bde}f^{cad} + f^{cde}f^{abd} = 0. \tag{21.16}$$

We conclude that $\mathcal{L}$ is BRST-invariant.