STRING THEORY

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These notes were taken for the *String Theory* course taught by R.A. Reid-Edwards at the University of Cambridge as part of the Mathematical Tripos Part III in Lent Term 2019. I live-TEXed them using Overleaf, and as such there may be typos; please send questions, comments, complaints, and corrections to itel2@cam.ac.uk.
Many thanks to Arun Debray for the LATEX template for these lecture notes: as of the time of writing, you can find him at https://web.ma.utexas.edu/users/a.debray/.

Contents

Lecture 1.

Friday, January 18, 2019

Note. This is a 24 lecture course with lectures at 11 AM M/W/F. There will be PDF notes available online somehow (TBD), and also $3+1$ problem sets plus a revision in Easter. The instructor can be reached at rar31@cam.ac.uk. Some recommended course readings[1] include "easier" texts:

[1] Most of these are published by Cambridge University Press. Conspiracy– string theory was invented by CUP to sell textbooks?

- Schomerus[2]
- (Becker)^2 and Schwarz[3]

and "harder" texts:

- Polchinski, Vol 1.[4]
- Lüst and Theisen[5]
- Green, Schwarz, and Witten.[6]

Introduction Here are some of the major topics we'll be covering in this course.

- Classical theory and canonical quantization
- Path integral quantization
- Conformal field theory (CFT) and BRST quantization
- Scattering amplitudes
- Advanced topics (more on this later).

Historically, string theory emerged from ideas in QCD, the theory of the strong force. However, it really took hold as a theory of quantum gravity in the quest to reconcile quantum mechanics with general relativity. A bit of expectation management, first. Some of the motivating ideas which string theory attempts to address are as follows:

- What sets the parameters of the Standard Model?
- What sets the cosmological constant?
- Failure of perturbative GR (problems in the UV– gravity is non-renormalizable)
- The black hole information paradox (quantum information in gravitational systems)
- How do you quantize a theory in the absence of an existing causal structure? (Most of the causal structure of spacetime is encoded in the metric. But what if it's the metric itself you're trying to quantize?)

There are alternatives to string theory– for instance, one can do QFT in curved spacetime to learn about some limit of quantum gravity. There's also loop quantum gravity and causal set theory, among others, but we won't really discuss those in this course.

What is string theory? We just don't know.

In some sense, string theory is a set of rules which, given a 10-dimensional classical spacetime vacuum, allows us to do quantum perturbation theory around this vacuum. By doing perturbation theory, we seem to arrive at a unique quantum theory (details of this to be discussed more later).

In the popular science conception of string theory, we imagine replacing particles with strings, and the harmonics of these strings correspond to different particles, including the graviton. How do we reconcile this with the idea that gravity is just a function of the curvature of space time? Answer: we assume that we are close to some well-understood solution with metric $\eta_{\mu\nu}$ and take the new metric to be a perturbation,

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x).$$

Now that we have some spacetime structure, we can start to talk about interactions. We might have a propagator for strings, and also interaction vertices with some rules. We might think that an equivalent of Feynman diagrams emerges to tell us how strings can mingle and talk to each other.

In QFT, we were given some Lagrangian and from that Lagrangian, we derived interactions and Feynman rules. But in string theory, the situation is a bit backwards. It's as though we've been given some Feynman rules which do seem to reduce to the particle interactions in some limit, but we don't in some sense know the underlying theory where these rules come from.[7]

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[2]Available here for users with access to Cambridge University Press online: `https://doi.org/10.1017/9781316672631`

[3]Ditto: `https://doi.org/10.1017/CBO9780511816086`

[4]Here: `https://doi.org/10.1017/CBO9780511816079`

[5]Possibly available through Springer Link but not a CUP publication. `https://link.springer.com/book/10.1007/BFb0113507`

[6]Here: rlhttps://doi.org/10.1017/CBO9781139248563

[7]"There are many reasons to study string theory. I suppose for you lot, you've got nothing better to do between the hours of 11 to 12." –R.A. Reid-Edwards

Classical theory In quantum mechanics, we have time t as a parameter and position $\hat{\mathbf{x}}$ as an operator. Of course, when we started learning quantum field theory, we were motivated to take our quantum fields $\hat{\phi}(\mathbf{x}, t)$ as operators and demote $\mathbf{x}$ to a simple label, so that $(\mathbf{x}, t)$ are both parameters. Space and time are on equal footing. This is the "second quantization" approach.

However, this isn't the only way we could do it. We could look for a formalism in which $\hat{x}^\mu = (\hat{\mathbf{x}}, \hat{t})$ are operators.

Example 1.1. Consider the *worldline formalism*. Imagine we have a massive particle propagating on a flat spacetime with metric $\eta_{\mu\nu}$. A suitable action for this theory might be

$$S[x] = -m \int_{s_1}^{s_2} ds, \tag{1.2}$$

where we use natural units of $\hbar = c = 1$ and the m is some mass due to dimensional concerns. This has a sort of geodesic interpretation for some integration measure ds. We can parametrize the worldline (e.g. in terms of proper time) such that

$$S[x] = -m \int_{\tau_1}^{\tau_2} d\tau \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}. \tag{1.3}$$

Here, dots indicate derivatives with respect to proper time. The conjugate momentum is then

$$P_\mu(\tau) = -\frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}}, \tag{1.4}$$

which obeys $P^2 + m^2 = 0$, so this is an "on-shell" formalism. We could then vary $S[x]$ with respect to trajectories $x^\mu(\tau)$ to find the equations of motion. We could imagine doing the same for an extended object and tracing out a "worldsheet" instead.

However, before we do that, let us revisit our action 1.2. In particular, we shall rewrite it as

$$S[x, e] = \frac{1}{2} \int d\tau \left(e^{-1} \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - em^2 \right). \tag{1.5}$$

This new action has a sensible massless limit, unlike the previous action. For our new action, the $x^\mu(\tau)$ equation of motion is then

$$\frac{d}{d\tau}(e^{-1}\dot{x}^\mu) = 0 \tag{1.6}$$

and the $e(\tau)$ equation of motion gives

$$\dot{x}^2 + e^2 m^2 = 0. \tag{1.7}$$

Now $e(\tau)$ appears algebraically, so we can substitute it back into the action to recover our previous formulation 1.2.[8]

Our theory also has some symmetry. If we shift the proper time by a function $\tau \to \tau + \zeta(\tau)$, then x and e change as

$$\delta x^\mu = \zeta \dot{x}^\mu$$
$$\delta e = \frac{d}{d\tau}(\zeta e).$$

We can use the one arbitrary degree of freedom to gauge fix $e(\tau)$ to a convenient value.

There's also a *rigid symmetry* which takes

$$x^\mu(\tau) \to \Lambda^\mu{}_\nu x^\nu(\tau) + a^\mu,$$

[8] Explicitly, we see that

$$\begin{aligned} S[x,e] &= \frac{1}{2}\int d\tau (e^{-1}\dot{x}^2 - em^2) \\ &= \frac{1}{2}\int d\tau (e^{-1}(-e^2m^2) - em^2) \\ &= \int d\tau (-em^2) \end{aligned}$$

and by setting $e = 1/m$ we recover 1.2.

which we may recognize as Poincaré invariance in the background spacetime.[9]

Non-lectured aside: reparameterization invariance Here, we'll explicitly show that the action 1.5 is invariant under the transformation

$$\tau \to \tau + \xi(\tau). \tag{1.8}$$

For some reason, this is not spelled out in either David Tong's notes or the standard textbooks I've consulted so far.

We make the assumption as in lecture that x and e change as

$$\delta x^\mu = \zeta \dot{x}^\mu$$
$$\delta e = \frac{d}{d\tau}(\zeta e).$$

If so, then note that

$$\delta(\dot{x}^\mu) = \frac{d}{d\tau}(\delta x^\mu) = \frac{d}{d\tau}(\lambda \dot{x}^\mu) \tag{1.9}$$

and

$$\frac{1}{e+\delta(e)} \sim \frac{1}{e} - \frac{1}{e^2}\delta(e) \implies \delta(e^{-1}) = -\frac{1}{e^2}\delta(e). \tag{1.10}$$

To perform this calculation, we'll also need the equations of motion from lecture, 1.6 and 1.7, reproduced here:

$$\frac{d}{d\tau}(e^{-1}\dot{x}^\mu) = 0$$

and

$$\dot{x}^2 + e^2 m^2 = 0.$$

Let's vary the action!

$$\begin{aligned}
\delta S[x,e] &= \frac{1}{2}\int d\tau \left[\delta(e^{-1})\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + e^{-1}\eta_{\mu\nu}\delta(\dot{x}^\mu)\dot{x}^\nu + e^{-1}\eta_{\mu\nu}\dot{x}^\mu\delta(\dot{x}^\nu) - \delta(e)m^2\right] \\
&= \frac{1}{2}\int d\tau \left[-\frac{1}{e^2}\delta(e)\dot{x}^2 + 2e^{-1}\eta_{\mu\nu}\frac{d}{d\tau}(\lambda\dot{x}^\mu)\dot{x}^\nu - \delta(e)m^2\right] \\
&= \frac{1}{2}\int d\tau \left[-\frac{1}{e^2}\delta(e)(\dot{x}^2 + m^2e^2) + 2(e^{-1}\dot{x}^\nu)\eta_{\mu\nu}\frac{d}{d\tau}(\lambda\dot{x}^\mu)\right] \\
&= \frac{1}{2}\int d\tau \frac{d}{d\tau}(\lambda e^{-1}\dot{x}^2) \\
&= 0.
\end{aligned}$$

In going from the first to the second line, we have explicitly substituted the variations for e^{-1} and for $\dot{x}^\mu$. In going from the second to the third, we simply regrouped terms into $\dot{x}^2 + m^2e^2$, which is zero by the equations of motion, and into $e^{-1}\dot{x}^\nu$, which is constant by the other equation of motion and therefore can be moved inside the total time derivative $\frac{d}{d\tau}$.

We see that after variation, what remains is simply an integral $\int d\tau$ of a total derivative, which is zero when evaluated at the endpoints of the action integral by the boundary conditions. Therefore the action is indeed invariant under reparametrization. ☒

Lecture 2.

Monday, January 21, 2019

Last time, we introduced a *worldline action* with an einbein e (auxiliary field).

$$S[x,e] = \frac{1}{2}\int d\tau \left(e^{-1}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - em^2\right).$$

[9] We can see that the action respects this symmetry, since it only depends on $\dot{x}^\mu$ and not x^μ (so translational symmetry is preserved) and $\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \to \eta_{\mu\nu}\Lambda^\mu{}_\sigma\dot{x}^\sigma\Lambda^\nu{}_\tau\dot{x}^\tau = \eta_{\sigma\tau}\dot{x}^\sigma\dot{x}^\tau$, so $\dot{x}^2$ is also preserved under Lorentz transformations as it should be.

In the massless limit, this reduces to

$$S[X,e] = \frac{1}{2} d\tau e^{-1} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu, \tag{2.1}$$

where we have replaced the Minkowski metric with some generic metric. The classical equations of motion for $X^\mu(\tau)$ then give the geodesic equation,

$$\ddot{X}^\mu + \Gamma^\mu_{\nu\lambda} \dot{X}^\nu \dot{X}^\lambda = 0. \tag{2.2}$$

The $e(\tau)$ equations of motion would give some constraints. However, if we attempted to quantize this theory, we would find that the background metric $g_{\mu\nu}$ is not actually deformed in the solutions. Rather than being dynamic as in general relativity, it's sort of a thing that is given to us and sits in the background, unchanging, which is why for a particle this is not a theory of quantum gravity. As we'll see, this is *not* the case for strings.

Strings As a string moves through some flat spacetime $\mathcal{M}$ with metric $\eta_{\mu\nu}$, it sweeps out a worldsheet Σ. Assume that the string is closed, so it has a coordinate σ (along the length of the string, if you like):

$$\sigma \sim \sigma + 2n\pi, n \in \mathbb{Z}.$$

And it moves through time as parametrized by a proper time τ, so the embedding of the worldsheet is given by $X^\mu(\sigma, \tau)$. That is, σ and τ provide good coordinates for the worldsheet in $\mathcal{M}$.

Definition 2.3. We call these X^μ embedding fields. They are maps $X : \Sigma \to \mathcal{M}$ from the worldsheet to the background spacetime manifold.

We also say that the area of the worldsheet Σ is given by

$$\text{area} = \int d\tau d\sigma \sqrt{-\det(\eta_{\mu\nu} \partial_a X^\mu \partial_0 X^\nu)} \tag{2.4}$$

where $\sigma^a = (\tau, \sigma)$ so that $\partial_a = \frac{\partial}{\partial \sigma^a}$. In fact, we shall introduce an extra factor know (for historical reasons) as α' and write

$$S[X] = -\frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{-\det(\eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu)}, \tag{2.5}$$

where α' is a free parameter. We often refer to the *string length*,

$$l_s \equiv 2\pi\sqrt{\alpha'} \tag{2.6}$$

or the *tension*

$$T \equiv \frac{1}{2\pi\alpha'}. \tag{2.7}$$

Definition 2.8. The object

$$G_{ab} \equiv \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \tag{2.9}$$

is an induced metric on Σ, and the action 2.5 is called the *Nambu-Goto action*.

Having just defined this, we won't really do anything with it for the rest of the course. Bummer. However, to make up for it, let's write down a new and improved action, the *Polyakov action*.

Definition 2.10. Consider the action

$$S[X,h] = -\frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{-h}\, h^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu. \tag{2.11}$$

This should remind us of what we did with the einbein last lecture, where we introduced e into our action.

This *Polyakov action* is classically equivalent to the Nambu-Goto action, since this auxiliary h which we have introduced will turn out to be non-dynamical.

The h_{ab} equations of motion are given by a weird variation of the action,

$$-\frac{2\pi}{\sqrt{-h}} \frac{\delta S}{\delta h^{ab}} = 0. \tag{2.12}$$

These equations of motion give the vanishing of the stress tensor, $T_{ab} = 0$, where

$$T_{ab} = -\frac{1}{\alpha'} \left(\partial_a X^\mu \partial_b X_\mu - \frac{1}{2} h_{ab} \partial_c X^\mu \partial_d X_\mu h^{cd} \right). \tag{2.13}$$

Note that in two dimensions, $T_{ab}h^{ab} = 0$, i.e. T_{ab} is traceless. This is our first indication that something is different about two dimensions.

The X^μ equations of motion are

$$\frac{1}{\sqrt{-h}}(\partial_a \sqrt{-h}\, h^{ab} \partial_b X^\mu) = 0, \quad \Box X^\mu = 0. \tag{2.14}$$

Now we could imagine adding a cosmological constant (which would cause the trace of the stress tensor to change) or perhaps some sort of Einstein-Hilbert term to our metric h_{ab}. But we'll see why this might be more complicated than it initially seems.

Symmetries The Polyakov action 2.11 has the following symmetries:

- Rigid (global) symmetry, $X^\mu(\sigma,\tau) \to \Lambda^\mu{}_\nu X^\nu(\sigma,\tau) + a^\mu$ (Poincaré invariance).
- Local symmetries– the physics should be invariant under reparametrizations of the coordinates of the worldsheet, so under transformations $\sigma^a \to \sigma'^a(\sigma,\tau)$. The fields themselves transform as

$$\begin{aligned} X'^\mu(\sigma',\tau') &= X^\mu(\sigma,\tau) \\ h_{ab}(\sigma,\tau) &= \frac{\partial \sigma'^c}{\partial \sigma^a}\frac{\partial \sigma'^d}{\partial \sigma^b} h'_{cd}(\sigma',\tau'). \end{aligned}$$

  Infinitesimallly, this means that $\sigma^a \to \sigma^a - \xi^a(\sigma,\tau)$, which gives us the variations

$$\begin{aligned} \delta X^\mu &= \xi^a \partial_a X^\mu \\ \delta h_{ab} &= \xi^c \partial_c h_{ab} + \partial_a \xi^c h_{cb} + \partial_b \xi^c h_{ca} = \nabla_a \xi_b + \nabla_b \xi_a \\ \delta\sqrt{-h} &= \partial_a(\xi^a \sqrt{-h}). \end{aligned}$$

  Note this second variation, δh_{ab}, can be written in terms of some covariant derivatives for an appropriate connection, but we won't usually bother.
- Weyl transformations– we send

$$\begin{aligned} X'^\mu(\sigma,\tau) &= X^\mu(\sigma,\tau) \\ h'_{ab}(\sigma,\tau) &= e^{2\Lambda(\sigma,\tau)} h_{ab}(\sigma,\tau). \end{aligned}$$

  Thus $\delta X^\mu = 0$ and $\delta h_{ab} = 2\Lambda h_{ab}$. Under such transformations, we have three arbitrary degrees of freedom in (ξ^a, Λ) (two from the two components of ξ plus one from Λ), and we can use them to fix the three degrees of freedom in h_{ab} (there are three, since h is symmetric and 2×2).

Classical solutions Let us now use reparametrization invariance to fix

$$h_{ab} = e^{2\phi}\eta_{ab}, \quad \eta_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.15}$$

The Polyakov action then becomes

$$S[X] = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma(-\dot{X}^2 + X'^2), \tag{2.16}$$

where

$$\dot{X}^\mu \equiv \frac{\partial X^\mu}{\partial \tau}, \quad X'^\mu \equiv \frac{\partial X^\mu}{\partial \sigma} \tag{2.17}$$

and squares are taken by contracting with the metric h_{ab}. In that case, the $X^\mu(\sigma,\tau)$ equation of motion becomes the wave equation in 2D, so solutions are of the form

$$X^\mu(\sigma,\tau) = X^\mu_R(\tau - \sigma) + X^\mu_L(\tau+\sigma). \tag{2.18}$$

Moreover, since we have a wave equation it is useful to introduce modes $(\alpha^\mu_n, \bar{\alpha}^\mu_n)$ where

$$X^\mu_R(\tau-\sigma) = \frac{1}{2}x^\mu + \frac{\alpha'}{2}p^\mu(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-in(\tau-\sigma)}, \tag{2.19}$$

where x^μ, p^μ are some constants in (τ,σ) and similarly the left-going modes are

$$X_L^\mu(\tau+\sigma) = \frac{1}{2}x^\mu + \frac{\alpha'}{2}p^\mu(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\bar{\alpha}_n^\mu e^{-in(\tau+\sigma)}. \tag{2.20}$$

It's sometimes useful to define a zero-mode,

$$\alpha_0^\mu = \bar{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu. \tag{2.21}$$

Lecture 3.

Wednesday, January 23, 2019

Two announcements. First, the official course notes will be released this weekend (I'll link them here soon). Second, today's colloquium is being given by Johanna Erdmenger, a Part III alumna working on AdS/CFT (gauge-gravity duality). The ideas in AdS/CFT were motivated by stringy concepts, and so should be relevant to our course.

Last time, we introduced the Polyakov action,

$$S[X,h] = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{-h}\,h^{ab}\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu. \tag{3.1}$$

Note that $h = \det(h_{ab})$ with h_{ab} considered as a 2×2 matrix. The equations of motion for h_{ab} gave the requirement that the stress tensor vanishes, $T_{ab} = 0$, with

$$T_{ab} = \partial_a X^\mu \partial_b X_\mu - \frac{1}{2}h_{ab}\partial^c X^\mu \partial_c X_\mu. \tag{3.2}$$

Here, a,b indices are raised and lowered with the appropriate metric h_{ab} and $\mu\nu$ indices are raised and lowered with $\eta_{\mu\nu}$.

Now, how does the Polyakov action relate to the Nambu-Goto action? Let us define the quantity

$$G_{ab} \equiv \partial_a X^\mu \partial_b X_\mu. \tag{3.3}$$

If $T_{ab} = 0$, then by 3.2,

$$G_{ab} = \frac{1}{2}h_{ab}(h^{cd}G_{cd}). \tag{3.4}$$

Taking determinants of both sides yields

$$\det(G_{ab}) = \left(\frac{1}{2}h^{cd}G_{cd}\right)^2 \det(h_{ab}) = \frac{1}{4}\left(h^{cd}G_{cd}\right)^2 h. \tag{3.5}$$

Therefore

$$2\sqrt{-\det(G_{ab})} = (h^{ab}G_{ab})\sqrt{-h} = \sqrt{-h}h^{ab}\partial_a X^\mu \partial_b X_\mu. \tag{3.6}$$

Substituting this back into the Polyakov action now gives us

$$S[X] = -\frac{1}{2\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{-\det G_{ab}},$$

the Nambu-Goto action. However, the Polyakov action is nicer to work with since it does not involve square roots of the coordinates X.

The stress tensor Recall that the conjugate momentum to X^μ is

$$P_\mu = \frac{1}{2\pi\alpha'}\dot{X}_\mu, \tag{3.7}$$

where a dot is a derivative with respect to proper time τ. We can define a Hamiltonian density $\mathcal{H}$ as

$$\mathcal{H} = P_\mu \dot{X}^\mu - \mathcal{L} = \frac{1}{4\pi\alpha'}(\dot{X}^2 + X'^2). \tag{3.8}$$

Definition 3.9. For our Hamiltonian formalism, we'll also need some *Poisson brackets* which we denote $\{\, , \,\}_{PB}$ (to contrast with another use of brackets later in the quantum theory). Given F, G defined on the phase space, we have

$$\{F, G\}_{PB} \equiv \int_0^{2\pi} d\sigma \left(\frac{\delta F}{\delta X^\mu(\sigma)} \frac{\delta G}{\delta P_\mu(\sigma)} - \frac{\delta F}{\delta P_\mu(\sigma)} \frac{\delta G}{\delta X^\mu(\sigma)} \right). \tag{3.10}$$

In particular, $\{X^\mu(\tau, \sigma), P_\nu(\tau, \sigma')\}_{PB} = \delta^\mu_\nu \delta(\sigma - \sigma')$.

Last time, we introduced a mode expansion

$$X^\mu(\sigma, \tau) = X^\mu_R(\tau - \sigma) + X^\mu_L(\tau + \sigma), \tag{3.11}$$

writing e.g. the right-going mode in terms of modes α^μ_n,

$$X^\mu_R(\tau - \sigma) = \frac{1}{2} x^\mu + \frac{\alpha'}{2} p^\mu(\tau - \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha^\mu_n e^{-in(\tau - \sigma)}, \tag{3.12}$$

and something similar holds for X^μ_L using the modes $\bar{\alpha}^\mu_n$.

Let's try to work in terms of modes rather than the embedding fields X^μ. We assert that the Poisson brackets acting on the modes $\alpha^\mu_n, \bar{\alpha}^\mu_n$ are

$$\{\alpha^\mu_m, \alpha^\nu_n\}_{PB} = -im\delta_{m,-n}\eta^{\mu\nu} \tag{3.13}$$

$$\{\alpha^\mu_m, \bar{\alpha}^\nu_n\}_{PB} = 0 \tag{3.14}$$

$$\{\bar{\alpha}^\mu_m, \bar{\alpha}^\nu_n\}_{PB} = -im\delta_{m,-n}\eta^{\mu\nu} \tag{3.15}$$

for $n \neq 0, m \neq 0$. If we define $\alpha^\mu_0 = \bar{\alpha}^\nu_0 = \sqrt{\frac{\alpha'}{2}} p^\mu$, we see that $\{x^\mu, p_\nu\}_{PB} = \delta^\mu_\nu$.

Let's see why this might be reasonable. We will set $\tau = 0$ so that

$$X^\mu(\sigma) = x^\mu + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \left(\alpha^\mu_n e^{in\sigma} + \bar{\alpha}^\mu_n e^{-in\sigma} \right),$$

$$P^\nu(\sigma') = \frac{p^\nu}{2\pi} + \frac{1}{2\pi} \sqrt{\frac{1}{2\alpha'}} \sum_{m \neq 0} \left(\alpha^\nu_m e^{im\sigma'} + \bar{\alpha}^\nu_m e^{-im\sigma'} \right).$$

Recall that we get P^ν by deriving $X^\mu(\tau, \sigma)$ with respect to τ and dividing by a factor of 2π. (Check this expression for $P^\nu(\sigma, \tau = 0)$!)

Now we can compute the Poisson bracket: it is

$$\{X^\mu(\sigma), P_\nu(\sigma')\}_{PB} = \frac{1}{2\pi}\{x^\mu, p^\nu\} - \frac{1}{4\pi} \sum_{n,m \neq 0} \frac{1}{2m} \left(\{\alpha^\mu_m, \alpha^\nu_n\} e^{i(m\sigma + n\sigma')} + \{\bar{\alpha}^\mu_m, \bar{\alpha}^\nu_n\} e^{-i(m\sigma + n\sigma')} \right). \tag{3.16}$$

Using the Poisson bracket relations on the modes and the "periodic delta function"

$$\delta(\sigma - \sigma') = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\sigma - \sigma')}, \tag{3.17}$$

one can show that

$$\{X^\mu(0, \tau), P^\nu(0, \sigma')\}_{PB} = \eta^{\mu\nu} \delta(\sigma - \sigma'). \tag{3.18}$$

The Wit algebra We'll quickly introduce the following concept. On our worldsheet, it will be useful to use light-cone (null) coordinates

$$\sigma^\pm = \tau \pm \sigma. \tag{3.19}$$

Thus the metric becomes

$$ds^2 = -d\tau^2 + d\sigma^2 = (d\sigma^+, d\sigma^-) \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} d\sigma^+ \\ d\sigma^- \end{pmatrix}. \tag{3.20}$$

Derivatives become

$$\partial_\pm \equiv \frac{\partial}{\partial \sigma^\pm} = \frac{1}{2}(\partial_\tau \pm \partial_\sigma). \tag{3.21}$$

In these new coordinates, the action and equations of motion become

$$S[X] = -\frac{1}{2\pi\alpha'}\int_\Sigma d\sigma^+ d\sigma^-\, \partial_+ X^\mu \partial_- X_\mu, \quad \partial_+\partial_- X^\mu = 0. \tag{3.22}$$

The stress tensor becomes

$$T_{++} = -\frac{1}{\alpha'}\partial_+ X^\mu \partial_+ X_\mu, \quad T_{--} = -\frac{1}{\alpha'}\partial_- X^\mu \partial_- X_\mu, \tag{3.23}$$

with $T_{+-} = 0$ since this is nothing more than the trace of T_{ab}.

We can introduce modes $l_m, \bar{l}_m$ for the stress tensor, writing

$$\begin{aligned} l_n &= -\frac{1}{2\pi}\int_0^{2\pi} d\sigma T_{--} e^{-in\sigma} \\ \bar{l}_n &= -\frac{1}{2\pi}\int_0^{2\pi} d\sigma T_{++} e^{+in\sigma}. \end{aligned}$$

Again, our goal is to work with modes rather than the entire solutions.

For instance,

$$\partial_- X^\mu = \sqrt{\frac{\alpha'}{2}}\sum_n \alpha_n^\mu e^{-in\sigma}, \text{ where } \alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu.$$

Lecture 4.

Friday, January 25, 2019

Last time we introduced the light cone coordinates on Σ, defined as $\sigma^\pm = \tau \pm \sigma$. Recall also that we want to work with modes rather than embedding fields, and for $\tau = 0$, the modes are given by

$$\begin{aligned} l_n &= -\frac{1}{2\pi}\int_0^{2\pi} d\sigma T_{--}(\sigma) e^{-in\sigma} \\ \bar{l}_n &= -\frac{1}{2\pi}\int_0^{2\pi} d\sigma T_{++}(\sigma) e^{+in\sigma}, \end{aligned}$$

with $T_{+-} = 0$.

We shall see that $l_m, \bar{l}_m$ are conserved quantities on the space $T_{ab} = 0$. Using

$$\partial_- X^\mu(\sigma) = \sqrt{\frac{\alpha'}{2}}\sum_n \alpha_n^\mu e^{-in\sigma}, \text{ where } \alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu,$$

we would like to get expressions for the stress tensor modes l_n in terms of the string modes α_m^μ. We postulated some Poisson brackets on the modes, which will hopefully help us out in this calculation.

For instance,

$$\begin{aligned} l_n &= \frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma \partial_- X \cdot \partial_- X e^{in\sigma} \\ &= \frac{1}{4\pi}\sum_{m,p} \alpha_m \cdot \alpha_p \int_0^{2\pi} d\sigma e^{i(m+p-n)\sigma} \\ &= \frac{1}{4\pi}\sum_{m,p} \alpha_m \cdot \alpha_p (2\pi\delta_{m+p,n}) \\ \implies l_n &= \frac{1}{2}\sum \alpha_{n-m}\cdot\alpha_m, \quad \bar{l}_n = \frac{1}{2}\sum_m \bar{\alpha}_{n-m}\cdot\bar{\alpha}_m. \end{aligned}$$

Using these expressions and the PB relations for the αs, one can (and should) show that the l_n satisfy the following Poisson brackets:

$$\begin{aligned} \{l_m, l_n\}_{PB} &= (m-n) l_{m+n} \\ \{\bar{l}_m, \bar{l}_n\}_{PB} &= (m-n)\bar{l}_{m+n} \\ \{l_m, \bar{l}_n\}_{PB} &= 0.l_{m+n} \end{aligned}$$

This is often called the *Wit algebra*, and it is related to the Virasoro algebra in the quantum theory. n.b. the stress tensor modes $l_0, l_{\pm 1}, \bar{l}_0, \bar{l}_{\pm 1}$ generate the Lie algebra of $SL(2, \mathbb{C})$.

Now, the Hamiltonian may be written as

$$H = \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma \Big((\partial_+ X)^2 + (\partial_- X)^2 \Big) \tag{4.1}$$

$$= \frac{1}{2} \sum_n (\alpha_{-n} \cdot \alpha_n + \bar{\alpha}_{-n} \cdot \bar{\alpha}_n) \tag{4.2}$$

$$= l_0 + \bar{l}_0. \tag{4.3}$$

Anticipating the quantum case, we will call these l modes *Virasoro generators*.

On the constraint surface $l_n \approx 0$, one can show that $\{H, l_n\} \approx 0$, since

$$\frac{dl_n}{d\tau} = \{H, l_n\}_{PB} = -n l_n. \tag{4.4}$$

Canonical quantization We have been working entirely with the classical string so far, and our main approach will be the path integral formalism. However, it may be enlightening for us to consider how to canonically quantize the string.

In the classical theory, we have $\{X^\mu, P_\nu\}_{PB}$ the Poisson bracket, with $T_{ab} = 0$. In going to a quantum theory, we could *impose* $T_{ab} = 0$ and promote variables to operators, $\{q^\mu, \pi_\nu\}_{PB}$, and then promote the Poisson bracket to a commutator of quantum operators, $i[q^\mu, \pi_\nu]$. That is, we first constrain the phase space and then quantize. This gives us a Hilbert space $\mathcal{H}_{l.c.}$ on the light cone.

On the other hand, our approach will be a little different. We can quantize first, $\{\cdot, \cdot\}_{PB} \to i[\cdot, \cdot]$, giving us commutators $[X^\mu, P_\nu]$, and *then* impose $T_{ab} = 0$, where T_{ab} is now an operator and the constraint is $T_{ab}|\psi\rangle = 0 \forall |\psi\rangle$. This will yield another Hilbert space $\mathcal{H}_Q$, which we hope (and could prove, although it is non-trivial) is equivalent to the light cone Hilbert space.

Thus in our approach, we start by replacing *fundamental* Poisson bracket relations with canonical commutation relations,

$$\{X^\mu, P_\nu\} \to -i[X^\mu, P_\nu], \tag{4.5}$$

and can do something equivalent for the $\alpha_n^\mu, \bar{\alpha}_n^\mu$ modes.

We now introduce the *Virasoro operators*

$$L_n = \frac{1}{2} \sum_m \alpha_{n-m} \cdot \alpha_m, n \neq 0, \tag{4.6}$$

where we distinguish the L_ns from the classical l_n since the quantum Ls do not quite satisfy the Wit algebra. $\bar{L}_n$ is defined equivalently.

We also introduce a vacuum state $|0\rangle$, which we will define as the state annihilated by all α modes,

$$\alpha_n^\mu |0\rangle = 0 \text{ for } n \geq 0. \tag{4.7}$$

We think of $\alpha_n^\mu, n > 0$ as annihilation operators analogous to those of the harmonic oscillator, and $n < 0$ as creation operators.[10] What are these operators creating and annihilating? Harmonics of the string, essentially.

We now notice an ambiguity in the definition of L_0 and $\bar{L}_0$. We have

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n>0} \alpha_{-n} \cdot \alpha_n, \tag{4.8}$$

but note that the $\alpha_{-n} \cdot \alpha_n$ terms have an ordering ambiguity.

To resolve this, we define normal ordering (denoted by ::) in the usual way, moving all creation operators to the left and all annihilation operators to the right. We then define composite operators using this ordering, e.g.

$$T_{--}(\sigma^-) = -\frac{1}{\alpha'} : \partial_- X^\mu \partial_- X_\mu : . \tag{4.9}$$

[10] α_0 is a little special and has to do with the center of mass of the string, though it does annihilate the vacuum.

Physical state conditions We define the number operators $N_n, \bar{N}_n$ by

$$nN_n = \alpha_{-n} \cdot \alpha_n, \quad n\bar{N}_n = \bar{\alpha}_{-n} \cdot \bar{\alpha}_n, \tag{4.10}$$

and the total number operators as

$$N = \sum_n nN_n, \quad \bar{N} = \sum_n n\bar{N}_n. \tag{4.11}$$

The $L_0, \bar{L}_0$ may be written as

$$L_0 = \frac{\alpha'}{4}p^2 + N, \quad L_0 = \frac{\alpha'}{4}p^2 + \bar{N}. \tag{4.12}$$

Next time, we will impose the conditions

$$L_n|\phi\rangle = 0, n > 0 \text{ and } (L_0 - a)|\phi\rangle = 0 \tag{4.13}$$

for $|\phi\rangle$ to be a physical state, with $a \in \mathbb{R}$.

Lecture 5.

Monday, January 28, 2019

Last time, we began discussing the quantization of the string. We said that our approach would be to quantize the unconstrained first and then apply the quantum-ized constraint $T_{ab} = 0$ on all physical states in the Hilbert space. We do this by imposing the conditions

$$L_n|\phi\rangle = 0, \quad n > 0 \tag{5.1}$$

for $|\phi\rangle$ to be physical. Note that $\bar{L}_n|\phi\rangle = 0$ as well– for most of our theory, we'll get an exact copy of the behavior of the right-handed modes L_n in the left-handed modes $\bar{L}_n$.

We also observed that our definition of L_0 was ambiguous in the quantum theory. In the other operators, we always had products of modes α_n with different harmonics n, but for L_0 there is an ordering ambiguity. We therefor impose the physical condition that

$$(L_0 - a)|\phi\rangle = 0, \quad (\bar{L}_0 - a)|\phi\rangle = 0 \tag{5.2}$$

where $a \in \mathbb{R}$ quantifies this ordering ambiguity. We will see later (cf. BRST invariance) that the theory is consistent only if $D = 26, a = 1$. From now on we shall assume $a = 1$.

It will be useful to define

$$L_0^{\pm} = L_0 \pm \bar{L}_0, \tag{5.3}$$

so that we have

$$(L_0^+ - 2)|\psi\rangle = 0, \quad L_0^-|\psi\rangle = 0, \quad L_n|\psi\rangle = \bar{L}_n|\psi\rangle = 0, n > 0. \tag{5.4}$$

These three conditions characterize physical states. Recall that $L_0 = \frac{\alpha'}{4}p^2 + N, \bar{L}_0 = \frac{\alpha'}{4}p^2 + \bar{N}$.

The spectrum We'll start by looking at the lowest-lying modes of the theory. We haven't yet discussed the creation or destruction of strings, so the following discussion will, if you like, be centered on free propagators.

We begin by remarking that in our version of the theory, there are problems in the infrared which have to do with *tachyons*. These problems can be addressed in superstring theory, which is beyond the scope of this course.

The simplest state we can write down is the momentum eigenstate,

$$|k\rangle = e^{ik\cdot x}|0\rangle, \tag{5.5}$$

with k_μ some four-vector of our choice and x the center of mass coordinate for the string (i.e. the x such that $X^\mu(\sigma, \tau) = x^\mu + p^\mu\tau +$ oscillations). The action of the center of mass momentum p_μ is then

$$p_\mu|k\rangle = k_\mu|k\rangle. \tag{5.6}$$

We could define a general state by a weighted sum of these momentum eigenstates,

$$|T\rangle = \int d^Dk\, T(k)|k\rangle, \tag{5.7}$$

where $T(k)$ is a function of our choosing and we are working in D dimensions. Now the $L_0^-|\phi\rangle = 0$ condition imposes $N = \bar{N}$. This is called the "level-matching" condition. It turns out to be the only condition that relates the left-going and right-going modes– otherwise, they are totally uncoupled.

If we look at L_0^+, we get the condition

$$(L_0^+ - 2)T(k)|k\rangle = \left(\frac{\alpha'}{2}p^2 + N + \bar{N} - 2\right)T(k)|k\rangle = 0, \tag{5.8}$$

which tells us that $N = \bar{N} = 0$. Therefore

$$(L_0^+ - 2)T(k)|k\rangle = \left(\frac{\alpha'}{2}p^2 - 2\right)T(k)|k\rangle = 0, \tag{5.9}$$

which we can rewrite as a mass-shell condtion on the momentum space field $T(k)$:

$$(k^2 + M^2)T(k) = 0 \quad \text{where } M^2 = -\frac{4}{\alpha'}. \tag{5.10}$$

We notice that the field $T(k)$ is tachyonic, i.e. its mass squared is negative. (We use the mostly + sign convention for the Minkowski metric.) Note that

$$L_n|T\rangle = 0 = \bar{L}_n|T\rangle \text{ for } n > 0 \tag{5.11}$$

is satisfied trivially. A priori, tachyons need not sink our theory. It could be that we're just working relative to the wrong vacuum. This is an open question, though there are other reasons the bosonic string might not be quite the right model for our universe's physics. Having declared that superstring theory does provide some solution to this problem, we will pay it no more thought and move on.

Massless states Next, we consider states of the form

$$|\epsilon\rangle = \epsilon_{\mu\nu}(k)\alpha^\mu_{-1}\bar{\alpha}^\nu_{-1}|k\rangle, \tag{5.12}$$

where we have included both α and $\bar{\alpha}$ to satisfy level-matching, and we have thrown in an ϵ in order to kill the free indices.

The condition $(L_0^+ - 2)|\epsilon\rangle = 0$ gives $M^2 = 0$ since $N = \bar{N} = 1$. Note that $L_n|\epsilon\rangle = 0$ is satisfied trivially for $n > 1$ (and so is $\bar{L}_n|\epsilon\rangle = 0$).

What about $L_1|\epsilon\rangle = 0$? We have

$$\begin{aligned}
L_a|\epsilon\rangle &= \frac{1}{2}\sum_n \alpha_{1-n}\cdot\alpha_n\epsilon_{\mu\nu}\alpha^\mu_{-1}\bar{\alpha}^\nu_{-1}|k\rangle \\
&= \epsilon_{\mu\nu}(k)\alpha_0\cdot\alpha_1\alpha^\mu_{-1}\bar{\alpha}^\nu_{-1}|k\rangle \\
&= \sqrt{\frac{2}{\alpha'}}\epsilon_{\mu\nu}(k)k_\lambda\alpha^\lambda_1\alpha^\mu_{-1}\bar{\alpha}^\nu_{-1}|k\rangle \\
&= \sqrt{\frac{2}{\alpha'}}\epsilon_{\mu\nu}(k)k_\lambda\left([\alpha^\lambda_1, \alpha^\lambda_{-1}] + \alpha^\mu_{-1}\alpha^\lambda_1\right)\bar{\alpha}^\nu_{-1}|k\rangle.
\end{aligned}$$

We conclude that

$$\epsilon_{\mu\nu}(k)k^\mu = 0, \tag{5.13}$$

so two states related by

$$\epsilon_{\mu\nu}(k) \to \epsilon_{\mu\nu}(k) + k_\mu\xi_\nu \tag{5.14}$$

are physically equivalent since $k^2 = 0$, with ξ arbitrary. Similarly,

$$\bar{L}_1|\epsilon\rangle = 0 \implies k^\nu\epsilon_{\mu\nu}(k) = 0. \tag{5.15}$$

It is useful to decompose $\epsilon_{\mu\nu}(k)$ as follows:

$$\epsilon_{\mu\nu}(k) = \tilde{g}_{\mu\nu}(k) + \tilde{B}_{\mu\nu}(k) + \eta_{\mu\nu}\tilde{\phi}(k), \tag{5.16}$$

where $\tilde{g}_{\mu\nu}$ is traceless symmetric and $\tilde{B}_{\mu\nu}$ is antisymmetric. Now $\tilde{g}_{\mu\nu}(k)$ has the interpretation of a momentum space metric perturbation,

$$\tilde{g}_{\mu\nu}(k) \sim \tilde{g}_{\mu\nu}(k) + k_\mu\xi_\nu + \xi_\mu k_\nu, \tag{5.17}$$

which is simply (linearized) diffeomorphism invariance. What about this antisymmetric guy? We get a "B-field" which corresponds to a momentum spacetime field $\tilde{B}_{\mu\nu} = -\tilde{B}_{\nu\mu}$, where

$$\tilde{B}_{\mu\nu}(k) \sim \tilde{B}_{\mu\nu}(k) + k_\mu \lambda_\nu - k_\nu \lambda_\mu. \tag{5.18}$$

In spacetime this is a gauge invariance, where $B_{\mu\nu} \sim B_{\mu\nu} + \partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu$. Some older textbooks call this the notoph (which is nearly "photon" backwards).

Lecture 6.

Wednesday, January 30, 2019

Last time, we discovered the mildly disturbing fact that our bosonic string theory has tachyons. Having made note of this, we decided to take it on faith that superstring theory has a reasonable solution to this problem, and proceeded to define massless modes of the string by

$$|g\rangle = h_{\mu\nu}\alpha_{-1}^{(\mu}\bar{\alpha}_{-1}^{\nu)}|k\rangle \tag{6.1}$$

$$|B\rangle = B_{\mu\nu}\alpha_{-1}^{[\mu}\bar{\alpha}_{-1}^{\nu]}|k\rangle \tag{6.2}$$

$$|\phi\rangle = \phi\alpha_{-1}^{\mu}\bar{\alpha}_{-1\mu}|k\rangle. \tag{6.3}$$

These correspond sort of to a graviton ($g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$), a B-field (since $B_{\mu\nu} = -B_{\nu\mu}$), and a so-called dilaton ϕ (scalar field). One can show that these fields arise as a linear approximation to the theory described by the following spacetime action:

$$S = -\frac{1}{2K^2}\int d^D x\sqrt{-g}e^{-2\phi}\left(R - 4\partial_\mu\phi\partial^\mu\phi + \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right), \tag{6.4}$$

where $H_{\mu\nu\lambda} = \partial_{[\mu}B_{\nu\lambda]}$ and K is a coupling constant which will be related to Newton's gravitational constant in D dimensions.

This suggests to us that the fields and modes on our worldsheet have in fact told us something about how to deform the background (until now Minkowski) metric, so we could consider the more general starting point

$$S_1[X,h] = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{-h}h^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X). \tag{6.5}$$

Moreover, it turns out that the quantum theory has Weyl symmetry if $g_{\mu\nu}(x)$ satisfies

$$R_{\mu\nu} = 0$$

to first order in α' (the only parameter in our theory, really), which are simply the Einstein equations in vacuum. That is, imposing the symmetry of the quantum theory on the worldsheet results in a condition on the background metric in all of spacetime. Higher orders will give corrections to this result – to next order in α',

$$R_{\mu\nu} + \frac{\alpha'}{2}R_{\mu\rho\lambda\sigma}R_\nu^{\rho\lambda\sigma} = 0. \tag{6.6}$$

Our theory therefore suggests that there are higher order corrections to the Einstein equations.

We could also add a term like

$$S_2 = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{-h}\epsilon^{ab}\partial_a X^\mu\partial_b X^\nu B_{\mu\nu}, \tag{6.7}$$

with ϵ^{ab} the completely antisymmetric rank two tensor. This links the action to the stress-energy tensor of our B-field.

Finally, we could add a coupling to the dilaton,

$$S_3 = \frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{-h}\phi(X)R_\Sigma, \tag{6.8}$$

with R_Σ the worldsheet Ricci scalar.

The condition that the action

$$S = S_1 + S_2 + S_3$$

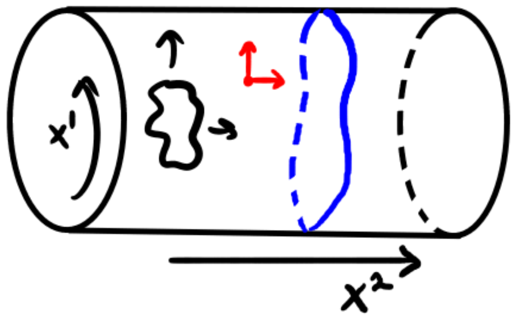

FIGURE 1. A spacetime with the topology $\mathbb{R}^2 \times S^1$. A particle (red) can move along the length of the cylinder and around its circumference. A string on the surface (black) can do the same. But this spacetime also admits the string configuration (blue) which wraps around the circumference.

gives a Weyl-invariant quantum theory results in what we might call equations of motion in spacetime for $g_{\mu\nu}$, $B_{\mu\nu}$, and ϕ. To leading order in α', these equations of motion may be derived from the action

$$S = -\frac{1}{2K^2}\int d^D x\sqrt{-g}e^{-2\phi}\left(R - 4\partial_\mu\phi\partial^\mu\phi + \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right) + O(\alpha').$$

If you like, the worldsheet theory couples to the metric of the background spacetime. Now, we could have just written down this action to start with. But deriving it from the worldsheet allows us to argue that any higher order terms are suppressed by the length scale of α'.

What happens if spacetime has some weird topology? Consider a theory where spacetime has the topology of $\mathbb{R}^2 \times S^1$, as in Fig. 1 Then a string can move around the spacetime just like a particle, but it can also wrap around the compact S^1 direction and probe the topology of the spacetime. Therefore something else interesting is happening which the modes we've currently defined seem totally insensitive to.

Path integral quantization Some of the details of path integral quantization are covered in *Advanced Quantum Field Theory*, and also in Polchinski (appendix in vol. 1), as well as in Ryder on QFT and Feynman and Hibbs (though this last one is broadly maligned for having errors in other sections).

Path integrals give us a conceptually different way to think about calculating amplitudes in QM and more generally in QFT. Morally speaking, a path integral is a weighted sum of paths satisfying some boundary conditions,

$$\langle x_f, t_t | x_i, t_i\rangle = \int_{x(t_i)}^{x(t_f)} \mathcal{D}x e^{iS[x]} \tag{6.9}$$

for some action $S[x] = \int_{t_i}^{t_f} dt L(x,\dot{x})$. We will be interested in the apath integral quantization of the Polyakov action.

That is, given some initial and final string states $\Psi_{i,f}$, the path integral is

$$\langle \Psi_f | \Psi_i\rangle = \int_i^f \mathcal{D}x\mathcal{D}h e^{iS[h,x]}, \tag{6.10}$$

with $S[h,x]$ the Polyakov action. But now by analogy with QFT we will have to deal with strings splitting and merging in our path integral, as shown in Fig. 2. There will be new complications when we try to compute the path integral.

Lecture 7.

Friday, February 1, 2019

Let us now continue our discussion of path integral quantization. Heuristically, we'll `import` the details of path integral quantization and see what works out. We want to understand how to make sense of

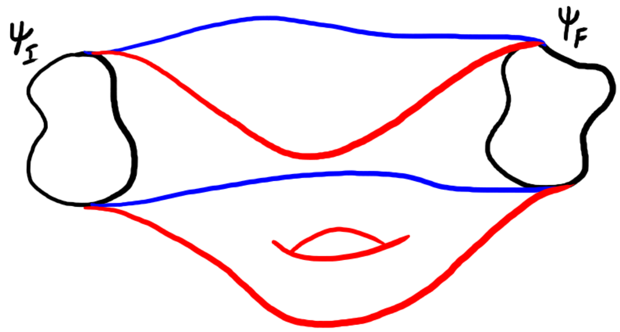

FIGURE 2. Two worldsheet configurations we might need to sum over in the path integral from Ψ_i (left) to Ψ_f (right). One worldsheet (blue) has the string propagating directly from Ψ_i to Ψ_f, while the other (red) has the string pinching off and splitting into two before merging back (the equivalent of a scattering process in QFT).

expressions like

$$\int \mathcal{D}h\mathcal{D}X\, e^{iS[h,X]} \tag{7.1}$$

where we are integrating over the space of metrics h_{ab} and embedding fields X^μs. When we do this calculation, we have to be careful not to overcount– there is a huge diffeomorphism symmetry and a Weyl symmetry in our theory relating physically equivalent states. If this path integral is to give us anything physically meaningful, we need to "quotient out" by the space of diffeomorphisms and Weyl transformations.

We would like to split the integral over all h_{ab} into integrals over physically inequivalent h_{ab} and those related by gauge transformations. Schematically,

$$\mathcal{D}h = \mathcal{D}h_{\text{phys}} \times \mathcal{J}\mathcal{D}h_{\text{Diff}\times\text{Weyl}}, \tag{7.2}$$

where $\mathcal{J}$ is a Jacobian factor whose importance we'll see in the following example.

Example 7.3. As a toy example, consider the following integral:

$$\int dxdy\, e^{-(x^2+y^2)}.$$

This isn't too hard to do– it separates into two Gaussian integrals readily. But notice that $x^2 + y^2$ is invariant under rotations about the origin. When we pass to polar coordinates, the θ angular integral becomes trivial, so we might really be interested in this integral modulo rotations. Thus our integral can be rewritten

$$\int d\theta \int dr\, re^{-r^2}.$$

This $\int d\theta$ will always give us a factor of 2π (the "volume" of an orbit of the rotation group)– our real interest is in the dr integral.

In this example, we needed the Jacobian of the coordinate transformation: $dxdy = rdrd\theta$. The same is true of our path integral. Formally, we will take

$$\frac{1}{|\text{Diff}| \times |\text{Weyl}|} \int \mathcal{D}h\mathcal{D}X = \int \mathcal{D}h_{\text{phys}}\mathcal{D}S_{\text{phys}}\, \mathcal{J}, \tag{7.4}$$

where $\mathcal{J}$ is now a functional determinant and $|\text{Diff}|, |\text{Weyl}|$ represents the orbits of diffeomorphisms and Weyl transformations. In the same way we could write

$$\sqrt{\frac{\pi}{\det M}} = \int_V dx\, e^{-(x,Mx)}, \tag{7.5}$$

we will write $\mathcal{J}$ as a functional integral,

$$\mathcal{J} = \int \mathcal{D}b\mathcal{D}c\, e^{-S[b,c]}. \tag{7.6}$$

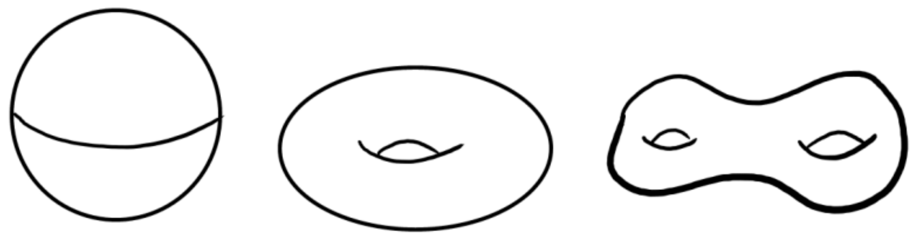

FIGURE 3. Three surfaces of genus $0, 1$, and 2, respectively. The first is the sphere S^2, the second is the torus T^2, and the final is a "handlebody" of genus two.

Global properties of the worldsheet We need to know more about what type of worldsheets appear in the path integral. This will take us on a crash course through Riemann surfaces.

We have looked at 2-dimensional Riemannian manifolds (Σ, h) modulo Weyl transformations. The set of Riemannian manifolds modulo Weyl transformations is known as *Riemann surfaces*. Quotienting out by diffeomorphisms is assumed. Note that worldsheets are Riemann surfaces.

We'll state a number of results without proof, though some of them are not too hard to prove– for more detail, see Farkas and Kra, and also Donaldson.

The first idea we'll consider is the *worldsheet genus*. For Riemann surfaces without boundary (i.e. a closed string, neglecting the initial and final string states), the relevant topological data is encoded in the *Euler characteristic*,

$$\chi = \frac{1}{4\pi}\int_\Sigma d^2\sigma\,\sqrt{h}R(h). \tag{7.7}$$

Here, $R(h)$ is the Ricci scalar with respect to the worldsheet metric h. The Euler characteristic captures the idea that while we can locally make the metric look however we want, in general there will be obstructions to globally bringing the metric to a required form. The *genus* g is given by

$$\chi = 2 - 2g, \tag{7.8}$$

and informally counts the "number of holes in Σ," as shown in Fig. 3. Why we care is because the genus is a topological invariant– we can't change the number of holes in a Riemann surface under smooth maps.

Moduli space of Riemann surfaces For a given genus g, the space of metrics on Σ_g modulo Weyl and diffeomorphisms is a finite-dimensional space called the *moduli space*. Schematically,

$$\mathcal{M}_g = \frac{\{\text{metrics } h_{ab}\}}{\{\text{Diff}\}\times\{\text{Weyl}\}}.$$

Both the numerator and denominator here are infinite dimensional, but our saving grace will be the following fact– the integral itself is finite-dimensional.

A useful result is the following: let s be the real dimension of the moduli space $\mathcal{M}_g$. Then

$$s = \dim\mathcal{M}_g = \begin{cases} 0, & g = 0 \\ 2, & g = 1 \\ 6g-6, & g \geq 2. \end{cases} \tag{7.9}$$

Example 7.10. Given a metric $\hat{h}_{ab}$ on a $g = 0$ surface, we can bring any metric to the form $e^{2w}\hat{h}_{ab}$. This is not the case for a torus ($g = 1$). We can build a torus by imposing identifications on $\mathbb{C}$, i.e. under the equivalence relation

$$z \sim z + n\lambda_1 + m\lambda_2, \tag{7.11}$$

where $n, m \in \mathbb{Z}$ and λ_1, λ_2 specify the "dimensions" of the torus.

One can show that the ratio $\tau \equiv \lambda_1/\lambda_2$ is Diff and Weyl invariant. However, we can always choose λ_1, λ_2 such that $\text{Im}(\tau) \geq 0$. We also get a metric

$$ds^2 = |dz + \tau d\bar{z}|^2. \tag{7.12}$$

If we transform $\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} \to U\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}$ for some matrix U, then we can undo that change by also changing the equivalence relation numbers $(n,m) \to (n,m)U^{-1}$. For n,m to be integers under any such transformation, we require the entries of U to all be integers, i.e. $U \in SL(2,\mathbb{Z})$.

Our moduli space is

$$\mathcal{M}_1 = \frac{UHP}{SL(2,\mathbb{Z})}, \tag{7.13}$$

with UHP the upper half-plane, $\tau, \operatorname{Im}\tau \geq 0$.

Lecture 8.

Monday, February 4, 2019

We've started our lightning tour of the theory of Riemann surfaces. Soon, we'll see the emergence of our first scattering amplitudes.

Conformal Killing vectors Recall from *General Relativity* that Killing vectors are very special objects which represent symmetries of the metric. In the language of Lie derivatives, a vector K is a Killing vector if the Lie derivative of the metric with respect to K is trivial, $\mathcal{L}_K g = 0$.[11] *Conformal Killing vectors* (CKV) generalize this idea. A conformal Killing vector generates diffeomorphisms that preserve the metric up to Weyl transformations.

Our gauge transformations are

$$\delta_V h_{ab} = \nabla_a V_b + \nabla_b V_a \tag{8.1}$$

$$\delta_\omega h_{ab} = 2\omega h_{ab}. \tag{8.2}$$

We are interested in V^a such that

$$\delta_{CK} h_{ab} = \nabla_a V_b + \nabla_b V_a + 2\omega h_{ab} = 0. \tag{8.3}$$

Note the covariant derivatives are taken with respect to the metric h_{ab}. Taking the trace, we have equivalently

$$2(\nabla_a V^a) + 4\omega = 0 \implies \omega = -\frac{1}{2}(\nabla_a V^a), \tag{8.4}$$

so V^a is a conformal Killing vector if

$$\delta h_{ab} = \nabla_a V_b + \nabla_b V_a - h_{ab}(\nabla_c V^c) = 0. \tag{8.5}$$

We define

$$(Pv)_{ab} \equiv \nabla_a V_b + \nabla_b V_a - h_{ab}(\nabla_c V^c) \tag{8.6}$$

so that V^a is a conformal Killing vector if $V^a \in \operatorname{Ker} P$.

Why have we introduced these? For closed Riemann surfaces of genus g, the (real) dimension of the conformal Killing group (CKG), i.e. the subgroup of diffeomorphisms generated by the conformal Killing vectors, is known: it is

$$\kappa = |\text{CKG}| = \begin{cases} 6, & g = 0 \\ 2, & g = 1 \\ 0, & g \geq 2. \end{cases} \tag{8.7}$$

On the sphere (think of this as $\mathbb{C}$ with the point at ∞), the CKVs generate the transformations

$$z \to \frac{az+b}{cz+d} \tag{8.8}$$

and similarly for $\bar{z}$, where $a,b,c,d \in \mathbb{C}$ and $ad - bc = 1$. This is in fact the Möbius group from complex analysis. We have four parameters and one algebraic constraint on complex values (hence two real constraints). Therefore we shall fix the conformal Killing symmetry by requiring that the V^a vanish at three distinct points on Σ (i.e. imposing six real constraints, since each point on Σ comes with two coordinates).

We'll need one more mathematical preliminary before moving forward. This is the *modular group.* First, observe that the diffeomorphism group on the Riemann surface Σ_g is in general not connected. Let us

[11] In terms of covariant derivatives, $\nabla_a K_b + \nabla_b K_a = 0$.

therefore define something useful– call the connected set of diffeomorphisms that includes the identity Diff_0. The modular group $\mathcal{M}_g$ is then

$$\mathcal{M}_g = \frac{\text{Diff}}{\text{Diff}_0}. \tag{8.9}$$

For example, for the torus we have $\mathcal{M}_1 = \text{SL}(2:\mathbb{Z})$.

Then the moduli space M_g can be written schematically as

$$M_g = \frac{\{\text{metrics}\}}{\{\text{Diff}\} \times \{\text{Weyl}\}} = \frac{\{\text{metrics}\}}{\{\text{Diff}_0\} \times \{\text{Weyl}\}} / \mathcal{M}_g. \tag{8.10}$$

We often call the space

$$\mathcal{T}_g = \frac{\{\text{metrics}\}}{\{\text{Diff}_0\} \times \{\text{Weyl}\}} \tag{8.11}$$

the Teichmüller space. In this notation, $M_g = \mathcal{T}_g / \mathcal{M}_g$.

The Faddeev-Popov determinant When we do path integrals, it's usually desirable to check our answer by other means, since path integrals have a way of hiding divergences which we as self-respecting physicists ought to care about. Happily, this will be possible for the following quantity we are about to define.

The idea is to choose a "gauge slice" through the space of metrics on Σ_g. That is, we choose a gauge such that the metric on the worldsheet h_{ab} takes some nice form, $h_{ab} = \hat{h}_{ab}$ (often diagonal), such that $\text{Diff}_0 \times \text{Weyl}$ orbits then take us everywhere else in our space of metrics. We formally define the *Fadeev-Popov determinant* as

$$1 = \Delta_{FG}(\hat{h}) \int_{\text{Diff}_0 \times \text{Weyl}} \mathcal{D}(\delta h) \delta[h - \hat{h}] \prod_i \delta(v(\hat{\sigma}_i)), \tag{8.12}$$

where $\delta[h - \hat{h}]$ can be thought of as a "delta functional" and σ_i indicates points on our worldsheet Σ_g where the CKVs vanish (in order to fix the CKG). We can think of this determinant in analogy to how $\delta(f(x)) \sim \frac{\delta x}{|f'(x_i)|}$ where $f(x_i) = 0$.

In more detail, we may write

$$1 = \Delta_{FG}(\hat{h}) \int_{\mathcal{T}} d^s t \int \mathcal{D}\omega \mathcal{D}v \delta[h_{ab} - \hat{h}_{ab}] \prod_i \delta(v(\hat{\sigma}_i)), \tag{8.13}$$

where the $d^s t$ integral is taken in Teichmüller space and our path integral is now written explicitly over the space of variations of h.

We will now write the delta functions and delta functions as integrals and functional integrals. Let us introduce numbers ζ^i_a and fields $\beta^{ab}(\sigma, \tau)$ such that

$$1 = \Delta_{FG}(\hat{h}) \int_{\mathcal{T}} d^s t \int \mathcal{D}\omega \mathcal{D}v \Big(d^\kappa \zeta^i_a \mathcal{D}\beta \exp(i(\beta | h - \hat{h}) + i\zeta^i_a v^a(\hat{\sigma}_i) \Big), \tag{8.14}$$

where the inner product $(\beta | h - \hat{h})$ is defined to be

$$(\beta | h - \hat{h}) = \int_\Sigma d^2\sigma \sqrt{|h|} \beta^{ab} (h_{ab} - \hat{h}_{ab}) \tag{8.15}$$

We can write $h_{ab} - \hat{h}_{ab} = \delta_{ab}$ as

$$\begin{aligned}
\delta h_{ab} &= \underbrace{\nabla_a v_b + \nabla_b v_a}_{\text{Diffeos}} + \underbrace{2\omega h_{ab}}_{\text{Weyl}} + \underbrace{t^I \partial_I h_{ab}}_{\text{moduli}} \\
&= (Pv)_{ab} + 2(\omega + \nabla_c v^c) h_{ab} + t^I \partial_I h_{ab} \\
&= (Pv)_{ab} + 2\bar{\omega} h_{ab} + t^I \mu_{Iab}
\end{aligned}$$

where $(Pv)_{ab}$ is as defined before, $\mu_{Iab} = \partial_I h_{ab} - \text{trace}$, and $\bar{\omega}$ contains the residual trace terms.

Lecture 9.

Wednesday, February 6, 2019

The official course notes and the first example sheet are online now. Note that David Tong's notes may also supplement the notes for this course. In addition, note that problems 4 and 5 are eligible for marking, while problem 6 has a typo and therefore the instructor asks that we ignore problem 6 entirely.

Last time, we introduced the Faddeev-Popov determinant. We found that

$$\Delta_{FP}^{-1}(\hat{h}) = \int_{\mathcal{T}} d^s t \int \mathcal{D}\omega \mathcal{D}v \left(d^\kappa \zeta_a^i \mathcal{D}\beta \exp(i(\beta|Pv + 2\bar{\omega}h + t^I \mu_I)) + i \sum_{i=1}^{k} \zeta_a^i v^a(\hat{\sigma}_i) \right) \tag{9.1}$$

Grassmann quantities If you're keeping up with my *AQFT* and *Supersymmetry* notes, this will be your third time seeing Grassmann quantities/variables. These are a set of quantities θ such that any two of them anticommute,

$$\theta_1\theta_2 = -\theta_2\theta_1.$$

Equivalently their anticommutator vanishes,

$$\{\theta_1, \theta_2\} = 0.$$

Objects (such as wavefunctions) that obey Fermi statistics can naturally be described by Grassmann numbers. One bit of motivation for this is the fact that for any θ, we have $\theta^2 = -\theta^2 = 0$, which is reminiscent of the Pauli exclusion principle. This anticommuting property also holds for integration measures,

$$d\theta_1 d\theta_2 = -d\theta_2 d\theta_1.$$

One can show (e.g. by considering $(\int d\theta)^2$) that

$$\int d\theta = 0,$$

and we can consistently define

$$\int d\theta\, \theta = 1.$$

The Dirac delta function for Grassman quantities is then $\delta(\theta) = \theta$, which leads to the somewhat unusual conclusion that integration and differentiation of Grassmann variables are essentially the same process.

Note that Taylor expansions are very easy for Grassman variables, since we cannot have anything of higher degree than 1 because $\theta^2 = 0$. Thus we can write some function $f(x, \theta)$ as

$$f(x, \theta) = f_0(x) + \theta f_1(x),$$

and so an integral can be written

$$\int d\theta\, f(x, \theta) = f_1(x) = \frac{\partial f(x, \theta)}{\partial \theta}. \tag{9.2}$$

Example 9.3. Let $\theta^a = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$ and $\bar{\theta}^a = (\bar{\theta}_1, \bar{\theta}_2)$. Consider the integral

$$\int d^2\theta d^2\bar{\theta} \exp\left(-\bar{\theta}^a M_{ab} \theta^b\right), \tag{9.4}$$

where M_{ab} is some normal 2×2 matrix. This exponential has a few terms but not too many. The first term is just 1, while the last term has 4 thetas and two Ms. Recalling that integration is like differentiation, we write the integral as

$$\frac{\partial^4}{\partial\theta_1 \partial\theta_2 \partial\bar{\theta}_1 \partial\bar{\theta}_2} \{(\bar{\theta}_1 M_{11} \theta_1)(\bar{\theta}_2 M_{22} \theta_2) + (\bar{\theta}_1 M_{12} \theta_2)(\bar{\theta}_2 M_{21} \theta_1\}, \tag{9.5}$$

noting that the only nonzero term must have all four of θ_1, θ_2, and their barred versions.

Equivalently this integral is

$$\int d^2\theta d^2\bar{\theta} \exp\left(-\bar{\theta}^a M_{ab} \theta^b\right) = (M_{11}M_{22} - M_{12}M_{21}) = \det(M_{ab}). \tag{9.6}$$

This result generalizes– the equivalent of a Gaussian for Grassmann variables is

$$\int d^n\theta d^n\bar{\theta}\exp(-\bar{\theta}^a M_{ab}\theta^b) = \det(M_{ab}). \tag{9.7}$$

Note that this is a bit different from the result for $z, \bar{z}$ real, where

$$\int d^2z d^2\bar{z}\exp(-\bar{z}Mz) = \frac{1}{\det(M_{ab})}. \tag{9.8}$$

This effect of inverting the determinant when we replace commuting (bosonic) variables with Grassmann (fermionic) variables carries over to the functional case, which we will just state but not prove.

With our "new" Grassmann variables in hand, we will now rewrite the Faddeev-Popov determinant in terms of Grassmann quantities to perform these crazy path integrals. That is, promote

$$v^a \to c^a, \beta^{ab} \to b^{ab}, t^I \to \xi^I, \zeta^i_a \to \eta^i_a, \tag{9.9}$$

where c^a and $b^{ab} = b^{ba}$ are Grassmann fields on Σ.

Note also that we can apparently get rid of the $\mathcal{D}\omega$ integral by writing

$$\begin{aligned}\Delta_{FP}^{-1} &\sim \int \mathcal{D}\omega\exp[i(\beta|2\bar{\omega}h)]\\ &\sim \int \mathcal{D}\bar{\omega}\exp\left[i\int_\Sigma d^2\sigma\sqrt{h}\beta^{ab}2\bar{\omega}h_{ab}\right].\end{aligned}$$

But we can do this $\bar{\omega}$ integral– it looks like a delta function, and fixes $\beta^{ab}h_{ab} = 0$. Thus β^{ab} is traceless.

Thus we rewrite the Fadeev-Popov determinant in terms of our shiny new Grassmann variables as

$$\Delta_{FP}(\hat{h}) = \int d^s\xi\int \mathcal{D}c\mathcal{D}b d^k\eta\exp(i(b|Pc+\xi^I\mu_I) + i\sum_{i=1}^k \eta^i_a c^a(\hat{\sigma}_i)), \tag{9.10}$$

having done the ω integral as above. Note that this is really just the Faddeev-Popov determinant and not its inverse, since we have promoted everything to Grassmann variables. We can also do the η^i_a and ξ^I integrals to get

$$\begin{aligned}\Delta_{FP}(\hat{h}) &= \int \mathcal{D}c\mathcal{D}b\, e^{i(b|Pc)}\prod_{I=1}^S \delta[(b|\mu_I)]\prod_{i=1}^K \delta(c^a(\hat{\sigma}_i))\\ &= \int \mathcal{D}c\mathcal{D}b\, e^{i(b|Pc)}\prod(b|\mu_I)\prod_{i=1}^k c^a(\hat{\sigma}_i).\end{aligned}$$

After all this computation, we therefore have

$$\Delta_{FP}(\hat{h}) = \int \mathcal{D}c\mathcal{D}b\, e^{iS[b,c]}\prod(b|\mu_I)\prod_{i=1,a=1,2}^k c^a(\hat{\sigma}_i) \tag{9.11}$$

where we have something that looks like an action,

$$S[b,c] = \int_\Sigma d^2\sigma\sqrt{h}b^{ab}(Pc)_{ab} = 2\int_\Sigma d^2\sigma\sqrt{h}b^{ab}(\nabla_a c_b). \tag{9.12}$$

Note that c^a, b_{ab} are Grassmann fields and therefore obey Fermi statistics. However, it turns out they also have integer "spin" (for some notion of spin we have not defined precisely yet). Fortunately, this is allowed because these quantities are not observables. We should think of them a bit like constraints on the observable variables of our theory, and we call them *Faddeev-Popov ghosts*.

Lecture 10.

Friday, February 8, 2019

Today, we'll wrap up our discsussion of global physics on the worldsheet. Let us return to the schematic path integral expression

$$Z = \frac{1}{|\text{Diff}|\times|\text{Weyl}|}\int \mathcal{D}X\mathcal{D}h\, e^{iS[h,X]}. \tag{10.1}$$

We will insert a factor of 1 using our expression for the Faddeev-Popov determinant:

$$1 = \Delta_{FG}(\hat{h}) \int_{\mathcal{T}_g} d^s t \int \mathcal{D}\bar{\omega}\mathcal{D}v\delta[h-\hat{h}] \prod_{i,a} \delta(v^a(\hat{\sigma}_i)). \tag{10.2}$$

The delta functional will do the $\mathcal{D}h$ integral for us, at the cost of introducing some other integrals into the picture. We rewrite

$$Z = \frac{1}{|\text{Diff}| \times |\text{Weyl}|} \int \mathcal{D}X e^{iS[X,\hat{h}]} \int_{\mathcal{T}_g} d^s t \int \mathcal{D}\bar{\omega}\mathcal{D}v \prod_{i,a} \delta(v^a(\hat{\sigma}))\Delta_{FP}(\hat{h}). \tag{10.3}$$

But now notice that

$$|\text{Weyl}| \times \frac{|\text{Diff}_0|}{|\text{CKG}|} = \int \mathcal{D}\bar{\omega} \int \mathcal{D}v \prod \pi_{i,a}\delta(v^a(\hat{\sigma}_i)).$$

That is, the delta functions are equivalent to quotienting out by the symmetries of the conformal Killing vectors, and these other integrals are taken over diffeomorphisms connected to the identity and related by Weyl transformations. This is still extremely schematic but we can "cancel" the Weyl groups and recognize $|\text{Diff}_0|/|\text{Diff}| = 1/|\mathcal{M}_g|$ so that

$$\frac{1}{|\text{Diff}| \times |\text{Weyl}|} \times |\text{Weyl}| \times \frac{|\text{Diff}_0|}{|\text{CKG}|} = \frac{1}{|\mathcal{M}_g| \times |\text{CKG}|}. \tag{10.4}$$

With this notation,

$$Z = \frac{1}{|\mathcal{M}_g||\text{CKG}|} \int_{\mathcal{T}_g} d^s t \int \mathcal{D}X e^{iS[x,\hat{h}]} \Delta_{FP}(\hat{h}). \tag{10.5}$$

We take this to mean an integral over the Teichmüller space quotiented by the modular group, i.e. over the moduli space M_g. Thus

$$\frac{1}{|\mathcal{M}_g|} \int_{\mathcal{T}_g} d^s t \equiv \int_{\mathcal{T}_g/\mathcal{M}_g} d^s t = \int_{M_g} d^s t,$$

and our full path integral is now an integral over the moduli space and the Grassmann fields b, c (substituting in our expression for Δ_{FG} explicitly):

$$Z = \frac{1}{|\text{CKG}|} \int_{M_g} d^s t \int \mathcal{D}X\mathcal{D}b\mathcal{D}c\, e^{iS[\hat{h},X,b,c]} \prod_{I=i}^{s} (b|\mu_I) \prod_{i,a} c^a(\hat{\sigma} i). \tag{10.6}$$

As before, our inner product is given by $(b|\mu_I) = \int_\Sigma d^\sigma \sqrt{|h|} b^{ab} \mu_{Iab}$ with $\mu_{Iab} = \partial_I h_{ab} -$ trace. We shall choose to define b, c such that the action takes the form

$$S[\hat{h}, X, b, c] = -\frac{1}{4\pi\alpha'} \int_\Sigma d^s\sigma \sqrt{|\hat{h}|} \hat{h}^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} + \frac{1}{2\pi} \int_\Sigma d^2\sigma \sqrt{|\hat{h}|} b^{ab} \nabla_a c_b. \tag{10.7}$$

It may be useful to consider the ghosts (bs and cs) as an integral part of the theory, rather than a hack we've added to make sense of these infinite-dimensional spaces of metrics. As we've said, these ghosts will represent important constraints, particularly when we try to figure out the dimensionality of the bigger spacetime in which our worldsheet lives.

Introduction to conformal field theory Conformal field theories (CFTs) are among the best-understood quantum field theories we have. Outside of string theory, they also have applications in condensed matter physics and other areas, and we'll see that our action as given above defines a CFT in two dimensions, which turns out to be a very special case.

We are interested in theories that are invariant under Weyl transformations. We can ask the following question: what is the natural generalization of the Poincaré group that preserves a metric up to Weyl transformations? In a general dimension $d > 1$, we are interested in transformations such that

$$\eta_{\rho\sigma} \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} = \Lambda(x)\eta_{\mu\nu}, \tag{10.8}$$

where infinitesimally, $x^\mu \to x'^\mu = x^\mu + V^\mu(x) + \ldots$. Morally, we are combining Lorentz boosts and rotations with local scale transformations.

We find that if $\Lambda(x) = e^{\omega(x)}$, then $\omega(x)$ and $v^\mu(x)$ are related by

$$\omega(x) = \frac{2}{d}\partial_\mu v^\mu(x), \tag{10.9}$$

so $v^\mu(x)$ satisfies

$$\partial_\mu v_\nu + \partial_\nu v_\mu = \frac{2}{d}\eta_{\mu\nu}\partial_\lambda v^\lambda(x). \tag{10.10}$$

We say that $V^\mu(x)$ satisfying this condition generates conformal transformations.[12]

Two dimensional CFTs Let us take

$$h_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

a metric up to a conformal factor (Wick rotation) where we have sent $t \to i\tau$ if you like. That is, we've switched from Lorentzian signature to Euclidean signature. Not a problem. We have some coordinates on the manifold given by

$$x^\mu \to \sigma^a = (\tau, \sigma). \tag{10.11}$$

The condition 10.10 now becomes

$$2\partial_\tau v_\tau = \partial_\tau v_\tau + \partial_\sigma v_\sigma \implies \partial_\tau v_\tau = \partial_\sigma v_\sigma, \tag{10.12}$$

in the case where $\mu = \nu$, and

$$\partial_\sigma v_\tau + \partial_\tau v_\sigma = 0 \tag{10.13}$$

for $\mu \neq \nu$. We write these as

$$\frac{\partial v_\tau}{\partial \tau} = \frac{\partial v_\sigma}{\partial \sigma}, \quad \frac{\partial v_\tau}{\partial \sigma} = -\frac{\partial v_\sigma}{\partial \tau}. \tag{10.14}$$

But these are just the Cauchy-Riemann equations for a complex function $v = v^\tau + iv^\sigma$, i.e. the requirement that v is holomorphic.

We conclude that in $d = 2$, the condition on $v = v^\tau + iv^\sigma$ given by 10.10 is that v is holomorphic,

$$\frac{\partial}{\partial \bar{z}} v = 0 = \bar{\partial} v \tag{10.15}$$

where $z = \tau + i\sigma, \bar{z} = \tau - i\sigma$. This tells us that it's natural to work not in worldsheet coordinates τ, σ but in the variables $z, \bar{z}$. However, we can do better– we also want variables which vary in some natural way under conformal transformations. Since all holomorphic transformations preserve our metric up to Weyl transformations, a better choice is

$$z = e^{\tau + i\sigma}, \quad \bar{z} = e^{\tau - i\sigma}. \tag{10.16}$$

In these variables, the worldsheet is mapped to the complex plane, with the infinite future mapped to the point at infinity. We can think of the worldsheet Σ as the Riemann sphere with two points removed.

In these new coordinates $(z, \bar{z})$, we find that the Polyakov action (remember that?) takes the form

$$S = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma \partial_a X^\mu \partial^a X^\nu \eta_{\mu\nu} = \frac{i}{2\pi\alpha'}\int_\Sigma d^2 z \partial X^\mu \bar{\partial} X^\mu \eta_{\mu\nu}, \tag{10.17}$$

where we have denoted $\partial \equiv \frac{\partial}{\partial z}, \bar{\partial} = \frac{\partial}{\partial \bar{z}}$. The stress tensor T_{ab} now has two non-trivial components:

$$T_{zz} \equiv T = -\frac{1}{\alpha'}\partial X^\mu \partial X^\nu \eta_{\mu\nu}, \tag{10.18}$$

$$T_{\bar{z}\bar{z}} \equiv \bar{T} = -\frac{1}{\alpha'}\bar{\partial} X^\mu \bar{\partial} X^\nu \eta_{\mu\nu}, \tag{10.19}$$

and $T_{z\bar{z}} = 0$ identically.

Finally, a quick note. In QFT we had a notion of time-ordering. For our theory, we see almost trivially that time ordering will be replaced by a "radial" ordering, i.e. curves at larger "time" τ correspond to larger radii in the complex plane.

[12] Note that v^μ looks a lot like the conformal Killing vectors we defined earlier.

Lecture 11.

Monday, February 11, 2019

Last time, we introduced conformal field theory. We found that for our two-dimensional worldsheet, we can construct a map

$$z = e^{\tau + i\sigma}, \quad \bar{z} = e^{\tau - i\sigma}.$$

We saw that the coordinates $z \to z' = f(z)$ were more generally holomorphic, i.e. we get some functions which satisfy the Cauchy-Riemann equations.

Conformal fields Here are some definitions common in the literature for conformal field theory.

Definition 11.1. A *chiral field* is a field Φ that depends on z only, i.e. $\Phi = \Phi(z)$. Similarly, an *anti-chiral field* is a field that depends only on $\bar{z}$.

Definition 11.2. The *conformal dimension* refers to how a field transforms under scalings $z \to z' = \lambda z, \bar{z} \to \bar{z}' = \bar{\lambda}\bar{z}$ ($\lambda \in \mathbb{C}$).

$$\Phi(z, \bar{z}) \to \Phi'(z', \bar{z}') = \lambda^h \bar{\lambda}^{\bar{h}} \Phi(\lambda z, \bar{\lambda}\bar{z}). \tag{11.3}$$

We shall call h and $\bar{h}$ the dimension of $\Phi(z, \bar{z})$.[13] Sometimes $h + \bar{h}$ is referred to as the dimension and $h - \bar{h}$ as the "conformal spin."

Definition 11.4. Under the conformal transformation $z \to z' = f(z)$, a *primary field* with dimension $(h, \bar{h})$ transforms as

$$\Phi(z, \bar{z}) \to \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \Phi(f(z), \bar{f}(\bar{z})). \tag{11.5}$$

That is, a primary field transforms like a tensor (with the appropriate exponents of $h, \bar{h}$).

Example 11.6. Consider an infinitesimal transformation

$$z \to z' = z + v(z) + \ldots = f(z). \tag{11.7}$$

Thus

$$\left(\frac{\partial f}{\partial z}\right)^h = (1 + \partial v)^h \tag{11.8}$$

$$\phi(f(z)) = \phi(z) + v(z)\partial\phi(z) + \ldots. \tag{11.9}$$

So for a field with $(h, \bar{h}) = (h, 0)$ we get

$$\delta\Phi(z) = (h\partial v(z) + v(z)\partial)\Phi(z) \tag{11.10}$$

where we have taken only the term to leading order in h.

Symmetries and the stress tensor For our classical theory, let us start with the action

$$S[X] = -\frac{1}{4\pi\alpha'} \int_\Sigma d^2 z \partial_a X^\mu \partial^a X^\nu \eta_{\mu\nu}. \tag{11.11}$$

Let us note that in going from τ, σ coordinates to $z, \bar{z}$, we pick up an i as the Jacobian factor, meaning that $e^{iS} \to_{(z,\bar{z})} e^{-S}$.

Consider the (conformal) transformation

$$\delta_v X^\mu = v^a \partial_a X^\mu. \tag{11.12}$$

The variation of the action is now

$$\delta_v S[X] = -\frac{1}{2\pi\alpha'} \int_\Sigma d^2 z \Big((\partial_a v^b) \partial_b X^\mu \partial^a X_\mu + v^b \partial_a (\partial_b X^\mu) \partial^a X_\mu \Big) \tag{11.13}$$

where all indices are raised and lowered with the Minkowski metric. After an integration by parts, this transformation becomes

$$\delta_v S[X] = \frac{1}{2\pi} \int_\Sigma d^2 z (\partial^a v^b) T_{ab}, \tag{11.14}$$

[13]This is a lot like what we did in *Statistical Field Theory*. In looking at the RG flows of different fields and couplings, we saw that they scaled in different ways with some scaling dimension.

with T_{ab} our old buddy the stress tensor. This tells us that $\delta S[X] = 0$ requires that

$$\partial^a T_{ab} = 0, \tag{11.15}$$

which is just Noether's theorem. That is, if the action is invariant under conformal transformations, then the stress tensor is conserved.

We could define a conserved charge

$$Q = Q_+ + Q_- \tag{11.16}$$

where

$$Q_\pm = \frac{1}{2\pi}\int_0^{2\pi} d\sigma T_{\pm\pm}(\sigma) \tag{11.17}$$

at $\tau = 0$. Classically, the symmetry transformations are generated by the charge Q:

$$\delta X^\mu = \{Q, X^\mu\}_{PB}. \tag{11.18}$$

What's the analogue of this in the quantum theory? Let's find out.

Conformal transformations and Ward identities For the following discussion, we stay in $d = 2$ but rather than focusing on our embedding fields X, we will work with more general fields $\phi(z, \bar{z})$. We shall be interested in the quantum analogue of Noether's theorem.

Consider a transformation

$$\phi \to \phi' = \phi + \delta\phi, \quad S[\phi'] = S[\phi] + \delta S[\phi]. \tag{11.19}$$

In the classical picture, we would say that if $\delta S = 0$, we've got a symmetry and that gives us some conserved quantity. But a classical action doesn't always uniquely specify a quantum action, and conversely there are some quantum actions we don't know the classical versions of. However, what we can say is that a symmetry of a quantum theory should preserve important features of that theory, and in particular it must preserve *correlation functions*.

Let us consider the correlation function

$$\langle \phi_1(z_1)\ldots\phi_n(z_n)\rangle \equiv \langle \phi_1 \ldots \phi_n\rangle .$$

Here, the $\bar{z}_i$ dependence is implicit. Under a transformation, our correlation functions become

$$\begin{aligned}
\langle \phi_1\ldots\phi_n\rangle \to \langle \phi_1'\ldots\phi_n'\rangle &= \int \mathcal{D}\phi' e^{-S[\phi']}\phi_1'\ldots\phi_n' \\
&= \int \mathcal{D}\phi e^{-S[\phi]}(1 - \delta S[\phi] + \ldots)(\phi_1 + \delta\phi_1 + \ldots)\ldots(\phi_n + \delta\phi_n + \ldots) \\
&= \langle \phi_1\ldots\phi_n\rangle - \int \mathcal{D}\phi e^{-S[\phi]}\delta S[\phi]\phi_1\ldots\phi_n + \sum_{k=1}^n \int \mathcal{D}\phi e^{-S[\phi]}\phi_1\ldots\delta\phi_k\ldots\phi_n.
\end{aligned}$$

where we have assumed that $\mathcal{D}\phi' = \mathcal{D}\phi$, i.e. the transformations are such that the integration measure is unchanged. If we require that the new correlations are the same as the old, i.e. $\langle \phi_1\ldots\phi_n\rangle = \langle \phi_1'\ldots\phi_n'\rangle$, then

$$\langle \delta S[\phi]\phi_1\ldots\phi_n\rangle = \sum_{k=1}^n \langle \phi_1\ldots\delta\phi_k\ldots\phi_n\rangle . \tag{11.20}$$

We would like to draw an analogue to the classical current, so we write $\delta S[\phi]$ as

$$\delta S[\phi] = \frac{1}{2\pi i}\int_\Sigma d^2z(\partial_a v(z))j^a(z), \tag{11.21}$$

where v is the parameter of the transformation and j is the classical Noether current. Remember, our aim here is to see how classical symmetries can be promoted to quantum conservation laws. Thus

$$\frac{1}{2\pi i}\int_\Sigma d^2z\partial_a v(z)\langle j^a(z)\phi_1\ldots\phi_n\rangle = \sum_{k=1}^n \langle \phi_1\ldots\delta\phi_k\ldots\phi_n\rangle . \tag{11.22}$$

We also choose Σ and $v(z)$ to isolate a particular $\delta\phi_k$. We define $\omega = z_k$ (thus $\phi_k(z_k) = \phi(\omega)$ and two curves C_1, C_2 such that $\partial\Sigma = C_1 \cup C_2$.We choose $v(z)$ to be constant within C_1, zero outside of C_2, and arbitrary on Σ. We also require C_1, C_2 to encircle $\omega = z_k$ only so that $v = 0$ at all other points $z_{j\neq k}$, which implies that all the other $\delta\phi_j, j \neq k$ vanish. In this way, we have

$$\frac{1}{2\pi i}\int_\Sigma d^2z\partial_a v(z)\langle j^a(z)\phi_1\ldots\phi_n\rangle = \langle\phi_1\ldots\phi(\omega)\ldots\phi_n\rangle. \tag{11.23}$$

Lecture 12.

Wednesday, February 13, 2019

Last time, we started looking at correlation functions in trying to understand how classical symmetries are promoted to quantum symmetries. We showed quite generally that a symmetry of the quantum theory means that the correlation functions are left invariant,

$$\langle\delta S[\phi]\phi_1\ldots\phi_n\rangle = \sum_{k=1}^n\langle\phi_1\ldots\delta\phi_k\ldots\phi_n\rangle,$$

and we saw that under conformal transformations,

$$\delta S[\phi] = \frac{1}{2\pi}\int_\Sigma d^2\sigma(\partial^a v^b)T_{ab} \tag{12.1}$$

with T_{ab} the stress tensor.

Substituting this into our expression relating correlation functions, we have

$$\frac{1}{2\pi}\int_\Sigma d^2\sigma(\partial^a v^b)\langle T_{ab}\phi_1\ldots\phi_n\rangle = \sum_{k=1}^n\langle\phi_1\ldots\delta\phi_k\ldots\phi_n\rangle. \tag{12.2}$$

We choose our Σ to select a single $\delta_V\phi_k$ on the RHS, i.e. define two curves C_1, C_2 with $\omega = z_k$ inside C_1, $v^a = 0$ outside and on C_2, and $v^a = (v^z(z,\bar z), v^{\bar z}(z,\bar z))$ inside and on C_1. Thus with this choice of Σ,

$$\frac{1}{2\pi}\int_\Sigma d^2\sigma(\partial^a v^b)\langle T_{ab}\phi_1\ldots\phi_n\rangle = \langle\phi_1\ldots\delta_v\phi(\omega,\bar\omega)\ldots\phi_n\rangle. \tag{12.3}$$

We denote $v^z(z,\bar z) = v(z)$ and $v^{\bar z}(z,\bar z) = \bar v(\bar z)$, though this notation is a little misleading since $\bar v$ is not necessarily the conjugate of v. It is just the part of v^a that depends only on $\bar z$.

Integrating by parts and applying Stokes's theorem we get

$$\begin{aligned}\frac{1}{2\pi}\int_\Sigma d^2\sigma(\partial^a v^b)\langle T_{ab}\phi_1\ldots\phi_n\rangle &= \frac{1}{2\pi}\int_\Sigma d^2\sigma\partial^a(v^b\langle T_{ab}\phi_1\ldots\phi_n\rangle) - \frac{1}{2\pi}\int_\Sigma d^2\sigma v^b\partial^a\langle T_{ab}\phi_1\ldots\phi_n\rangle\\ &= \frac{1}{2\pi i}\oint_{\partial\Sigma=C_1} dz\, v(z)\langle T(z)\phi_1\ldots\phi_n\rangle - \frac{1}{2\pi i}\oint_{\partial\Sigma=C_2} d\bar z\,\bar v(\bar z)\langle\bar T(\bar z)\phi_1\ldots\phi_n\rangle\\ &\quad - \frac{1}{2\pi}\int_\Sigma d^2\sigma\, v^b\partial^a\langle T_{ab}\phi_1\ldots\phi_n\rangle,\end{aligned}$$

where we've denoted $T_{zz}(z,\bar z) \equiv T(z,\bar z = T(z)$ and $T_{\bar z\bar z}(z,\bar z) \equiv \bar T(\bar z)$. We see that the boundary term can be rewritten as two contour integrals over the boundary of our region Σ, and moreover the integral over C_2 vanishes since $v^a = 0$ outside and on C_2.

We see that

$$\partial^a\langle T_{ab}\phi_1\ldots\phi_n\rangle = 0, \tag{12.4}$$

leaving

$$\langle\phi_1\ldots\delta_v\phi(\omega,\bar\omega)\ldots\phi_n\rangle = \oint_{C_1}\frac{dz}{2\pi i}v(z)\langle T(z)\phi_1\ldots\phi(\omega,\bar\omega)\ldots\phi_n\rangle - \oint_{C_2}\frac{d\bar z}{2\pi i}\bar v(\bar z)\langle\bar T(\bar z)\phi_1\ldots\phi(\omega,\bar\omega)\ldots\phi_n\rangle. \tag{12.5}$$

Abstractly, we have the variation

$$\delta_v\phi(\omega,\bar\omega) = \oint_{C(\omega)}\frac{dz}{2\pi i}v(z)T(z)\phi(\omega,\bar\omega) - \oint_{C(\omega)}\frac{d\bar z}{2\pi i}\bar v(\bar z)\bar T(\bar z)\phi(\omega,\bar\omega), \tag{12.6}$$

which we always think of as being inserted into a correlation function.

There are a few subtle points here. We need to take care to define the ordering of operators in this expression, since T, ϕ are operators. In addition, we can see that $T(z)$ ($\bar T(\bar z)$) generates holomorphic (resp.

anti-holomorphic) conformal transformations. Moreover, these are contour integrals, so our calculation reveals that it's the pole structure of $\lim_{z\to\omega} T(z)\phi(\omega,\bar{\omega})$ which governs the conformal transformations.

If we are interested in multiple variations $\langle\delta\phi_1\delta\phi_2\delta\phi_3\phi_4,\ldots\phi_n\rangle$, then we could choose some complicated region encircling just the corresponding points z_1, z_2, z_3.

Radial ordering Recall that we can map our worldsheet coordinates into

$$z = e^{\tau+i\sigma}, \tag{12.7}$$

where e^τ is the radial part of z, such that "time ordering" on the cylinder corresponds to radial ordering on $\mathbb{C}$. Thus $\tau_1 > \tau_2 \iff |z_1| > |z_2|$.

Last term in *Quantum Field Theory*, we computed expectation values of time-ordered objects, e.g. time-ordered correlation functions. Here, we will be interested in radially-ordered correlation functions. We define radial ordering as

$$\mathcal{R}(A(z), B(\omega)) \equiv \begin{cases} A(z)B(\omega) & |z| > |\omega| \\ B(\omega)A(z) & |\omega| > |z|. \end{cases} \tag{12.8}$$

But how should we radially order when we are integrating over some weird contour in the complex plane? For example,

$$\oint_{C(\omega)} R(a(z)b(\omega)$$

with the contour as shown in the image.

The answer is as follows. We can compute the answer in two regions where the ordering is clear, around a circle of some radius $R > |z-\omega|$ where $|z| > |\omega|$ and another circle oriented in the opposite direction with radius $R' < |z-\omega|$ where $|z| < |\omega|$. Thus we have the radial ordering

$$\oint_{C(\omega)} dz\, R(a(z)b(\omega)) = \oint_{C_1} dz\, R(a(z)b(\omega)) - \oint_{C_2} dz\, R(a(z)b(\omega)) = \oint_{C_1} dz\, a(z)b(\omega) - \oint_{C_2} dz\, b(\omega)a(z). \tag{12.9}$$

So our expression for $\delta_v\phi(\omega,\bar{\omega})$ is (once we include radial ordering)

$$\delta_v\phi(\omega) = \oint_{|\omega|<|z|} \frac{dz}{2\pi i} v(z)T(z)\phi(\omega) - \oint_{|\omega|>|z|} \frac{dz}{2\pi i}\phi(\omega)v(z)T(z) \tag{12.10}$$

(for a chiral field) where we only look at the ω dependence.

If we define

$$Q = \oint_{C(\omega)} \frac{dz}{2\pi i} v(z)T(z), \tag{12.11}$$

then we could define a bracket $[\cdot,\cdot]$ as

$$\delta_v\phi(\omega) = [Q, \phi(\omega)]. \tag{12.12}$$

Lecture 13.

Friday, February 15, 2019

Mode expansions Recall we had the expansion in σ, τ coordinates

$$X^\mu(\sigma^+,\sigma^-) = x^\mu + p^\mu\alpha'\tau + i\sqrt{\frac{\alpha}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^\mu e^{-in\sigma^-} + \bar{\alpha}_n^\mu e^{-in\sigma^+}\right), \tag{13.1}$$

and taking a derivative with respect to σ^- gives us

$$\partial_- X^\mu(\sigma^-) = \sqrt{\frac{\alpha'}{2}}\sum_n \alpha_n^\mu e^{-in\sigma^-}, \tag{13.2}$$

where α_0^μ is defined as before in terms of p^μ. We could look at the same object for a worldsheet with Euclidean signature, i.e. $\omega = \tau + i\sigma$, so that

$$\partial_\omega X^\mu(\omega) = -i\sqrt{\frac{\alpha'}{2}}\sum_n \alpha_n^\mu e^{-n\omega}. \tag{13.3}$$

But what we really want to consider is the theory on $\mathbb{C} \cup \{\infty\}$ with coordinates

$$z = e^{\omega} = e^{\tau+\sigma}. \tag{13.4}$$

Consider a chiral primary $\phi_{cyl}(\omega)$ of weight $(h, \bar{h}) = (h, 0)$ deifned on the cylinder. We expand

$$\phi_{cyl}(\omega) = \sum_n \phi_n e^{-n\omega}. \tag{13.5}$$

On the plane, we use the primary transformation law to get

$$\phi(z) = \left(\frac{\partial z}{\partial \omega}\right)^h \phi_{cyl}(\omega) = z^{-h}\phi_{cyl}(\omega) = z^{-h}\sum_n \phi_n z^{-n}. \tag{13.6}$$

Thus a natural mode expansion for $\phi(z)$ is

$$\phi(z) = \sum_n \phi_n z^{-n-h}. \tag{13.7}$$

More generally, a (primary) field of weight $(h, \bar{h})$ takes the form

$$\phi(z, \bar{z}) = \sum_{m,n} \phi_{mn} z^{-m-h} \bar{z}^{-n-\bar{h}}. \tag{13.8}$$

For instance, $T(z)$ and $\bar{T}(\bar{z})$ (which are holomorphic and antiholomorphic) have $(h, \bar{h})$ of $(2, 0)$ and $(0, 2)$ respectively, so

$$T(z) = \sum_n L_n z^{-n-2}, \quad \bar{T}(\bar{z}) = \sum_n \bar{L}_n \bar{z}^{-n-2}. \tag{13.9}$$

Note also that

$$\partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n^\mu z^{-n-1}, \tag{13.10}$$

where ∂ indicates a derivative with respect to z. Note also that

$$X^\mu(z, \bar{z}) = x^\mu - i\frac{\alpha'}{2} p^\mu \ln |z|^2 + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n}(\alpha_n^\mu z^{-n} + \bar{\alpha}_n^\mu \bar{z}^{-n}). \tag{13.11}$$

States and operators For a given physical operator $\Phi(z)$, there is a physical state $|\Phi\rangle$ given by

$$\lim_{z\to 0} \Phi(z)|0\rangle = |\Phi\rangle. \tag{13.12}$$

We shall take this as a definition for now. In the complex plane, we could imagine "inserting an operator" at the origin to produce some string state. With $\partial X^\mu(z)$ as before, consider

$$i\sqrt{\frac{2}{\alpha'}} \partial X^\mu(z)|0\rangle = \left(\ldots + \alpha_{-2}^\mu z + \alpha_{-1}^\mu + \frac{\alpha_0{}^\mu}{z} + \frac{\alpha_1}{z^2} + \ldots\right)|0\rangle. \tag{13.13}$$

For this limit to make sense, we see that some of these α_n^μs must annihilate the vacuum as we postulated earlier,

$$\alpha_n^\mu |0\rangle = 0, n \geq 0. \tag{13.14}$$

Then

$$\lim_{z=\to 0} i\sqrt{\frac{2}{\alpha'}} \partial X^\mu(z)|0\rangle = \alpha_{-1}^\mu |0\rangle. \tag{13.15}$$

For a more interesting example, we could look at

$$\lim_{z\to 0, \bar{z}\to 0} -\left(\frac{2}{\alpha'}\right) h_{\mu\nu} \partial X^\mu(z) \bar{\partial} X^\nu(\bar{z}) e^{ik\cdot X(z,\bar{z})}|0\rangle \tag{13.16}$$

whre k_μ is a momentum vector in spacetime and $h_{\mu\nu} = h_{\nu\mu}$ is a spacetime tensor. In this limit we have

$$h_{\mu\nu} \alpha_{-1}^\mu \bar{\alpha}_{-1}^\nu |k\rangle \tag{13.17}$$

our graviton state. Note that for a field of weight $(h, \bar{h})$ we require that

$$\phi_n |0\rangle = 0 \text{ for } n > -h. \tag{13.18}$$

Normal ordering and radial ordering We shall focus on the chiral field, which we shall call

$$j^\mu(z) \equiv \partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}}\sum_n \alpha_n^\mu z^{-n-1}. \tag{13.19}$$

Let us now split $j^\mu(z)$ into creation and annihilation parts. We won't be too careful about the zero mode, since it will drop out in the end. Thus we define

$$j_+^\mu = -i\sqrt{\frac{\alpha'}{2}}\sum_{n>0} \alpha_n^\mu z^{-n-1}, \tag{13.20}$$

$$j_-^\mu = -i\sqrt{\frac{\alpha'}{2}}\sum_{n\geq 0} \alpha_{-n}^\mu z^{n-1} \tag{13.21}$$

so that $j^\mu = j_+^\mu + j_-^\mu$. Remember that normal ordering is denoted by pairs of colons, $:(\ldots):$, as in QFT. For our chiral field, normal ordering is defined in an analogous way,

$$:j^\mu(z)j^\nu(\omega): = j_+^\mu(z)j_+^\nu(\omega) + j_-^\mu(z)j_+^\nu(\omega) + j_-^\nu(\omega)j_+^\mu(z) + j_-^\mu(z)j_-^\nu(\omega) \tag{13.22}$$

$$= j^\mu(z)j^\nu(\omega) + [j_-^\nu(\omega), j_+^\mu(z)]. \tag{13.23}$$

We can evaluate the commutator (as on the first examples sheet) to find

$$[j_-^\nu(\omega, j_+^\mu(z)] = -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-\omega)^2}. \tag{13.24}$$

However, in order to evaluate this commutator, we needed to sum a series, and that series only converged for $|z| > |\omega|$. Thus we see that normal ordering comes with a radial ordering requirement for the commutator to make sense. We find that

$$R(j^\mu(z)j^\nu(\omega) =: j^\mu(z)j^\nu(\omega): -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-\omega)^2}. \tag{13.25}$$

As in QFT, it is useful to introduce the "contraction" notation

$$\overbrace{j^\mu(z)j^\nu(\omega)} = -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-\omega)^2}. \tag{13.26}$$

If you like, this is a Green's function on Σ. On the examples sheet, we computed

$$\partial_z X^\mu(z)\partial_\omega X^\nu(\omega) = -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-\omega)^2}. \tag{13.27}$$

Up to arbitrary functions of $z, \bar{z}$ we integrate to find

$$\overbrace{X^\mu(z)X^\nu(\omega)} = -\frac{\alpha'}{2}\ln(z-\omega)\eta^{\mu\nu} \tag{13.28}$$

Splitting X into its holomorphic and antiholomorphic parts,

$$X^\mu(z,\bar{z}) = X^\mu(z) + \bar{X}^\mu(\bar{z}),$$

we can also show that

$$\overbrace{\bar{X}^\mu(\bar{z})\bar{X}^\nu(\bar{\omega})} = -\frac{\alpha'}{2}\ln(\bar{z}-\bar{\omega})\eta^{\mu\nu}, \tag{13.29}$$

$$\overbrace{X^\mu(z)\bar{X}^\nu(\bar{\omega})} = 0. \tag{13.30}$$

In total, we find that

$$\begin{aligned}\overbrace{X^\mu(z,\bar{z})X^\nu(\omega,\bar{\omega})} &= (X^\nu(z) + \overbrace{\bar{X}^\mu(\bar{z}))(X^\nu(\omega) + \bar{X}^\nu(\bar{\omega})}) \\ &= \left(-\frac{\alpha'}{2}\ln(z-\omega) - \frac{\alpha'}{2}\ln(\bar{z}-\bar{\omega})\right)\eta^{\mu\nu}\end{aligned}$$

where the $X^\mu(z), X^\nu(\omega)$ are also contracted over. We therefore learn that

$$\overbrace{X^\mu(z,\bar{z})X^\nu(\omega,\bar{\omega})} = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-\omega|^2, \tag{13.31}$$

which tells us the Green's function immediately:

$$\langle R(X^\mu(z,\bar{z})X^\nu(\omega,\bar{\omega})\rangle = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-\omega|^2. \tag{13.32}$$

We can use 13.31 to build contractions of more complicated operators constructed from X^μ via Wick's theorem. Notice that this Green's function diverges as $z \to \omega$, however, and its divergence also depends on this string parameter α'. This tells us that some interesting physics is captured in the particle limit as the string tension becomes infinite, $T \to \infty$, and $\alpha' \to 0$ since $T = -\frac{1}{2\pi\alpha'}$

Lecture 14.

Monday, February 18, 2019

We saw last time that a lot of the interest in our theory lies in its pole structure, i.e. the divergences that crop up when we bring two operators close together. From last term's *Quantum Field Theory*, we're familiar with Wick's theorem, which links time-ordered expressions to normal-ordered expressions with contractions. Here, we have radial ordering, so that

$$\begin{aligned} R(\phi_1(z_1)\dots\phi_n(z_n)) = {} & :\phi_1(z_1)\dots\phi_n(z_n) + \sum_{(i,j)} :\phi(z_1)\dots\overbrace{\phi_i(z_i)\dots\phi_j(z_j)}\dots\phi_n(z_n): \\ & + \sum_{(i,j),(k,l)} :\phi(z_1)\dots\overbrace{\phi_i(z_i)\dots\phi_j(z_j)}\overbrace{\phi_k(z_k)\dots\phi_l(z_l)}\dots\phi_n(z_n): + \dots \end{aligned}$$

where these sums are taken over all internal (pairwise) contractions. The contractions replace operator pairs with Green's functions, which means that there may be a lot of interest in the pole structure of this object.

We can use Wick's theorem and our knowledge of contractions to define composite operators, e.g. we found that

$$\overbrace{\partial X^\mu(z)\partial X^\nu(\omega)} = -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-\omega)^2}. \tag{14.1}$$

This gives us a natural definition for our stress tensor:

$$T(z) = \lim_{\omega\to z} -\frac{1}{\alpha'}\left(\partial X^\mu(z)\partial X_\mu(\omega) + \frac{\alpha'}{2}\frac{\eta^\mu{}_\mu}{(z-\omega)^2}\right). \tag{14.2}$$

Operator Product Expansions (OPEs) OPEs encode what happens when we bring two operators close together. Given a set of operators $\{O_i\}$, we write

$$O_i(\omega)O_j(z) = \sum_k f^k_{ij}(z-\omega)O_k(z) \tag{14.3}$$

as $\omega \to z$. Here, there's some sense of completeness in the set of operators $\{O_i\}$.

OPEs and conformal transformations For instance, let us consider the OPE $T(z)X^\mu(\omega)$ and conformal transformations. We are interested in

$$T(z)X^\mu(\omega) \text{ as } \omega \to z. \tag{14.4}$$

We have

$$T(z)X^\mu(\omega) = \frac{1}{\alpha'}:\partial X^\nu(z)\partial X_\nu(z): X^\mu(\omega), \tag{14.5}$$

and by integrating 14.1 we get

$$\partial X^\mu(z)X^\nu(\omega) = -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{z-\omega} + \dots \tag{14.6}$$

where the ... indicate terms that are finite as $z \to \omega$.

It follows that

$$\begin{aligned} T(z)X^\mu(\omega) &= -\frac{2}{\alpha'} : \overbrace{\partial X^\nu(z)\partial X_\nu(z) : X^\mu(\omega)} + \dots \\ &= -\frac{2}{\alpha'}\partial X_\nu(z)\left(-\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{z-\omega}\right) + \dots \\ &= \frac{\partial X^\mu(z)}{z-\omega} + \dots \end{aligned}$$

We then expand $\partial X^\mu(z)$ around $z = \omega$ to find

$$\partial X^\mu(z) = \partial X^\mu(\omega) + O(z-\omega),$$

so

$$T(z)X^\mu(\omega) = \frac{\partial X^\mu(\omega)}{z-\omega} + \dots \tag{14.7}$$

where the ... terms remain finite.

Recall that the conformal transformation of $X^\mu(\omega)$ may be given by

$$\delta_v X^\mu(\omega) = \oint_{z=\omega} \frac{dz}{2\pi i} R(v(z)T(z)X^\mu(\omega)), \tag{14.8}$$

where $v(z)$ is holomorphic and parametrizes our transformation.

We now substitute our OPE into $\delta_v X^\mu(\omega)$ to find

$$\delta_v X^\mu(\omega) = \oint_{z=\omega} \frac{dz}{2\pi i} v(z)\left(\frac{\partial X^\mu(\omega)}{z-\omega} + \dots\right) = v(\omega)\partial X^\mu(\omega). \tag{14.9}$$

where the contour is taken in a little loop around $z = \omega$.

Transformations of primary fields Consider a chiral primary $\phi(z)$ (where $\bar{h} = 0$). We know that

$$\delta_v \phi(z) = \oint_{C(z)} \frac{d\omega}{2\pi i} R(v(\omega)T(\omega)\phi(z)), \tag{14.10}$$

where we've swapped the z and ω in the integral to emphasize our primary field depends only on z. We want to retain the idea that a primary field transforms as a conformal tensor of weight $(h, \bar{h})$. Therefore we'll require that for $\phi(z)$ to be a chiral primary field, the OPE with $T(\omega)$ is such that

$$\delta_v \phi(z) = v(z)\partial\phi(z) + h\partial v(z)\phi(z). \tag{14.11}$$

Using the residue theorem in the following form,

$$\frac{1}{(n-1)!}\partial_z^{n-1} f(z) = \oint \frac{d\omega}{2\pi i}\frac{f(\omega)}{(\omega-z)^n}, \tag{14.12}$$

we find that the R part of the OPE can be rewritten as follows:

$$R(T(\omega)\phi(z)) = \frac{h}{(z-\omega)^2}\phi(\omega) + \frac{1}{z-\omega}\partial\phi(\omega) + \dots \tag{14.13}$$

in order to match the form of 14.11.

We could take this OPE with the stress tensor to define what we mean by a chiral primary of weight h. Thus by writing the radial ordering for some general ϕ we can read off the weight immediately.

A non-trivial OPE Consider now the OPE

$$T(z) : e^{ik\cdot X(\omega)} : \tag{14.14}$$

where $k \cdot X(\omega) = k_\mu X^\mu(\omega)$, with k_μ some constant spacetime vector. We think of this normal-ordered term in terms of its series expansion, i.e.

$$\sum_{n\geq 0} \frac{i^n}{n!} k_{\mu_1} \dots k_{\mu_n} : X^{\mu_1}(\omega) \dots X^{\mu_n}(\omega) : \tag{14.15}$$

We might wonder what the weight of $:e^{ik\cdot X}:$ is, but there's some non-trivial behavior going on in the normal ordering. Let's tack on $T(z)$ now:

$$-\frac{1}{\alpha'}:\partial X^{\nu}(z)\partial X_{\nu}(z):\sum_{n\geq 0}\frac{i^n}{n!}k_{\mu_1}\dots k_{\mu_n}:X^{\mu_1}(\omega)\dots X^{\mu_n}(\omega): \tag{14.16}$$

Single contractions contribute to this expression:

$$\sum\frac{i^n}{n!}n(k\cdot X(\omega)^{n-1}k_{\nu}\frac{1}{z-\omega}\partial X^{\nu}(\omega), \tag{14.17}$$

where we've contracted one of the ∂Xs with one of the X^{μ_i}s in the sum. Shifting the index we have

$$\sum_{m\geq 0}\frac{i^m}{m!}(k\cdot X(\omega))^m\frac{k_{\nu}\partial X^{\nu}(\omega)}{z-\omega}=\frac{1}{z-\omega}\partial_{\omega}(e^{ik\cdot X(\omega)}). \tag{14.18}$$

This already looks like the $\partial\phi(\omega)$ term in our expansion– we'll see how the double contractions give the other term on Wednesday.

Lecture 15.

Wednesday, February 20, 2019

Last time, we stopped mid-calculation. We were looking at the OPE for

$$T(Z)e^{ik\cdot X(\omega)}, \tag{15.1}$$

where $T(z)$ is the holomorphic part of the stress tensor, given by

$$T(z)=-\frac{1}{\alpha'}:\partial X^{\mu}(z)\partial X_{\mu}(z): \tag{15.2}$$

and the exponential is treated as a formal power series of the operator X. We found that single contractions gave us a term

$$\frac{1}{z-\omega}\partial_{\omega}(e^{ik\cdot X(\omega)}. \tag{15.3}$$

What about double contractions?

Double contributions contribute

$$-\frac{1}{\alpha'}:\partial X^{\mu}(z)\partial X_{\mu}(z):\sum_{(i,j)}\sum_{n\geq 0}\frac{(i)^n}{n!}k_{\mu_1}\dots k_{\mu_i}\dots k_{\mu_j}\dots k_{\mu_n}:X^{\mu_1}\dots X^{\mu_i}\dots X^{\mu_j}\dots X^{\mu_n}(\omega):, \tag{15.4}$$

where we must now perform contractions over the $\partial X^{\mu}(z)$s with the X^{μ_i}s on the right. There are no triple contractions since there are only two derivatives of Xs outside the sum and the normal ordering has already taken care of contractions in the X^{μ_i}s.

We can make this more precise. There are $n(n-1)$ options for which X^{μ_i}s to contract with, so we get an overall contribution

$$-\frac{1}{\alpha'}\sum_{n\geq 2}k_{\mu_2}\dots k_{\mu_n}\frac{(i)^n}{n!}n(n-1)\left(-\frac{\alpha'}{2}\right)^2\frac{k^2}{(z-\omega)^2}, \tag{15.5}$$

where two of the ks have been contracted since the contraction of $\partial X_{\mu}(z)\ X^{\mu_i}(\omega)$ comes with an $\eta_{\mu}^{\mu_i}$. Cleaning up a bit more, we have

$$=-\frac{\alpha'}{4}\frac{k^2}{(z-\omega)^2}\sum_{n\geq 2}:(k\cdot X(\omega))^{n-2}:i^2 i^{n-2}:X^{\mu_2}\dots X^{\mu_n}:\frac{n!}{n!(n-2)!} \tag{15.6}$$

$$=\frac{\alpha'}{4}\frac{k^2}{(z-\omega)^2}:e^{ik\cdot X(\omega)}: \tag{15.7}$$

In total, we have

$$T(z):e^{ik\cdot X(\omega)}:=\left(\frac{\alpha'}{4}\frac{k^2}{(z-\omega)^2}+\frac{\partial_{\omega}}{z-\omega}\right):e^{ik\cdot X(\omega)}+\dots \tag{15.8}$$

and we see that $:e^{ik\cdot X(\omega)}:$ has conformal weight

$$h = \frac{\alpha' k^2}{4}. \tag{15.9}$$

More generally $:e^{ik\cdot X(\omega,\bar{\omega})}:$ has weight

$$(h, \bar{h}) = \left(\frac{\alpha' k^2}{4}, \frac{\alpha' k^2}{4}\right). \tag{15.10}$$

Note that factors of the string tension α' go with factors of $\hbar$, which we've previously set to 1, so the relative factor of α' between the two terms in the expansion of $T(z):e^{ik\cdot X(\omega)}:$ tells us that there's a quantum correction going on here so that $:e^{ik\cdot X(\omega)}:$ doesn't just transform trivially as a scalar under conformal transformations.

It is now useful to separate the notion of a primary field from the definitions of h and $\bar{h}$.

Definition 15.11. A primary field is a field $\phi(\omega)$ with an OPE with $T(z)$ of the form

$$T(z)\phi(\omega) = \frac{h}{(z-\omega)^2}\phi(\omega) + \frac{1}{z-\omega}\partial\phi(\omega). \tag{15.12}$$

However, the $(z-\omega)^{-2}\phi(\omega)$ coefficient will still be called the *weight*, regardless of the presence of higher-order poles.

OPE of $T(z)T(\omega)$ and the Virasoro Algebra Recall that

$$T(z) = -\frac{1}{\alpha'}:\partial X^\mu(z)\partial X)\mu(z):, \tag{15.13}$$

and we have the contraction

$$\overbrace{\partial X^\mu(z)\partial X^\nu(\omega)} = -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-\omega)^2}. \tag{15.14}$$

We'll just go for it, then.

$$T(z)T(\omega) = \left(-\frac{1}{\alpha'}\right)^2 :\partial X^\mu(z)\partial X_\mu(z)::\partial X^\nu(\omega)\partial X_\nu(\omega): \tag{15.15}$$

where we need to take single contractions (e.g. $\overbrace{\partial X_\mu(z)\partial X^\nu(\omega)}$) and also double contractions

$$\overbrace{\partial X_\mu(z)\partial X^\nu(\omega)}\,\overbrace{\partial X^\mu(z)\partial X_\nu(\omega)}.$$

There will be four single contractions and two double contractions. Writing it all out, we find that

$$T(z)T(\omega) = -\frac{2}{\alpha'}\frac{\eta_{\mu\nu}}{(z-\omega)^2}:\partial X^\mu(z)\partial X^\nu(\omega): + \frac{1}{2}\frac{\delta^\mu{}_\nu}{(z-\omega)^2}\frac{\delta_\mu{}^\nu}{(z-\omega)^2} + \dots$$

We now expand $\partial X^\mu(z)$ about $z = \omega$:

$$\partial X^\mu(z) = \partial X^\mu(\omega) + (z-\omega)\partial^2 X^\mu(\omega) + \dots \tag{15.16}$$

We also recall that $\delta^\mu{}_\nu\delta^\nu{}_\mu = D$ the dimension of spacetime. Thus

$$T(z)T(\omega) = \frac{D/2}{(z-\omega)^4} - \frac{2}{\alpha'}\frac{1}{(z-\omega)^2}:\partial X^\nu(\omega)\partial X_\nu(\omega): - \frac{2}{\alpha'}\frac{1}{(z-\omega)}:\partial^2 X^\mu(\omega)\partial X_\mu(\omega): + \dots \tag{15.17}$$

using the expansion of $\partial X^\mu(z)$, 15.16. We arrive at

$$T(z)T(\omega) = \frac{D/2}{(z-\omega)^4} - \frac{2}{\alpha'}\frac{1}{(z-\omega)^2}T(\omega) - \frac{2}{\alpha'}\frac{1}{(z-\omega)}\partial T(\omega) + \dots \tag{15.18}$$

Clearly, this has weight $h = 2$, so $T(z)$ is of weight $(2,0)$. However, it is only a primary if $D = 0$. And of course there's no way to embed a nontrivial worldsheet in $D = 0$, so it seems like something very bad has happened.

The Virasoro algebra We've just show that $T(z)$ has $h=2$, so we expand it in modes as

$$T(z) = \sum_n L_n z^{-n-2}, \quad \bar{T}(z) = \sum_n \bar{L}_n \bar{z}^{-n-2}. \tag{15.19}$$

We can invert these expressions to find

$$L_m = \oint_{z=0} \frac{dz}{2\pi i} z^{m+1} T(z). \tag{15.20}$$

Let's now consider the commutator of two L_ns– to wit(t),

$$[L_m, L_n] = \oint_{\omega=0} \frac{d\omega}{2\pi i} \omega^{n+1} \oint_{z=0} \frac{dz}{2\pi i} z^{m+1} [T(z), T(\omega)]. \tag{15.21}$$

What do we mean by this commutator of operators? Let's just look at the dz integral first. In our discussion of radial ordering, we split up the contour integral as

$$\oint_{z=0} \frac{dz}{2\pi i} z^{m+1} [T(z), T(\omega)] := \oint_{|z|>|\omega|} z^{m+1} T(z) T(\omega) - \oint_{|z|<|\omega|} \frac{dz}{2\pi i} z^{m+1} T(\omega) T(z) \tag{15.22}$$

$$= \oint_{z=\omega} \frac{dz}{2\pi i} R(T(z)T(\omega)) z^{m+1}. \tag{15.23}$$

Using our $T(z)T(\omega)$ OPE, we have

$$[L_m, L_n] = \oint_{\omega=0} \frac{d\omega}{2\pi i} \omega^{n+1} \oint_{z=\omega} \frac{dz}{2\pi i} z^{m+1} \left(\frac{D/2}{(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial T(\omega)}{z-\omega} \right) \tag{15.24}$$

$$= \oint_{\omega=0} \frac{d\omega}{2\pi i} \omega^{n+1} \left(\frac{D/2}{3!} \frac{\partial^3}{\partial z^3} z^{m+1} + 2T(\omega) \frac{\partial}{\partial z} z^{m+1} + z^{m+1} \partial T(\omega) \right)_{z=\omega} \tag{15.25}$$

where we have dropped the ... from our OPE, as the contour integral will be evaluated by the residue theorem, and the residue theorem only cares about the pole structure of the thing we are integrating. Taking the derivatives, we get

$$\begin{aligned}
[L_m, L_n] &= \oint_{\omega=0} \frac{d\omega}{2\pi i} \left(\frac{D}{12}(m^3-m)\omega^{n+m-1} + 2(m+1)\omega^{m+n+1} T(\omega) - \omega^{m+n+2} \partial T(\omega) \right) \\
&= \oint_{\omega=0} \frac{d\omega}{2\pi i} \left(\frac{D}{12}(m^3-m)\omega^{n+m-1} + (m-n)\omega^{m+n+1} T(\omega) \right) \\
&= \frac{D}{12}(m^3-m)\delta_{m+n,0} + (m-n) L_{m+n}.
\end{aligned}$$

where in the second line we have integrated the final term by parts and simplified. This final result is something we were given on the first examples sheet. It looks almost like the Witt algebra, except there is an anomaly, the $D/12$ term. We call it the *Virasoro algebra*, and it is a consequence of the conformal symmetry of our quantum theory. It's sometimes called a central extension of the Witt algebra.

In fact, there's a complication that we've missed. Our theory isn't just defined by the Xs– there are also the ghosts, and as constraints, we expect that those ghosts will act like "negative degrees of freedom." They will contribute to this commutator and show that our theory can be consistent in $D \neq 0$

Lecture 16.

Friday, February 22, 2019

Last time, we looked at the OPE for the stress tensor. In our original calculation, we apparently learned that the stress tensor doesn't actually transform as a conformal tensor, thanks to a weird $\frac{D/2}{(z-\omega)^4}$ pole in the OPE, and this divergence was reflected in our calculation of an anomaly in the Virasoro algebra.

But we've only been paying attention to the Xs, and have neglected the b, c ghosts in our theory. To make sense of the path integral over h_{ab}, we introduced ghotst (b, c) via the Faddeev-Popov method. Thus our action included a ghost term

$$S[b,c] = \frac{i}{2\pi} \int_\Sigma d^2\sigma \sqrt{-\hat{h}} \hat{h}^{ab} b_{ab} \nabla_c c^b. \tag{16.1}$$

We could take $\hat{h}_{ab}$ as arbitrary and reimpose $\delta[\hat{h}-h]$ on the path integral. Now the stress tensor for the ghosts is derived by varying with respect to the metric. It is (naturally) symmetric but will otherwise be a mess, as we'll see now.

$$T^{gh}_{ab} = -i\left(\frac{1}{2}c^c\nabla_{(a}b_{b)c} + (\nabla_{(a}c^c)b_{b)c} - h_{ab}\text{ trace}\right). \tag{16.2}$$

This is pretty horrible. Let's return to the action. We will now work with a flat Euclidean metric and use our favorite worldsheet coordinates $(z,\bar{z})$ on $\mathbb{C}$. If we do that, the action becomes

$$S[b,c] = \frac{1}{2\pi}\int_\Sigma d^2z\left(b_{zz}\partial_{\bar{z}}c^z + b_{\bar{z}\bar{z}}\partial_z c^{\bar{z}}\right), \tag{16.3}$$

where $b_{z\bar{z}} = 0$ since b_{ab} is traceless.

In keeping with our (unfortunately conventional) notation of writing the dependence of quantities on $\bar{z}$ with bars themselves (the antiholomorphic bits), we will write

$$\begin{aligned} b_{zz} &\equiv b, \quad b_{\bar{z}\bar{z}} \equiv \bar{b} \\ c^z &\equiv c, \quad c^{\bar{z}} \equiv \bar{c}. \end{aligned}$$

In this notation, we now have the full action

$$S = \frac{1}{2\pi}\int_\Sigma d^2z(b\bar{\partial}c + \bar{b}\partial\bar{c}) - \frac{1}{2\pi\alpha'}\int_\Sigma d^2z\partial X^\mu\bar{\partial}X^\nu\eta_{\mu\nu}. \tag{16.4}$$

Remember, the ghosts don't have a physical embedding into the space, but they are still critical constraints which allow our theory to be (more) consistent and should be treated as a real part of the theory. The total stress tensor (the holomorphic part, anyway) decomposes by linearity into an X part and a ghost part:

$$T(z) = T_X(z) + T_{gh}(z) \tag{16.5}$$

where

$$T_X(z) = -\frac{1}{\alpha'} : \partial X^\mu(z)\partial X_\mu(z) : \tag{16.6}$$

$$T_{gh}(z) =: \partial b(z)c(z) : -2\partial(: b(z)c(z) :). \tag{16.7}$$

Ghost OPEs The ghosts are free, so Wick's theorem gives

$$R(b(z)c(\omega) =: b(z)c(\omega) + \overbrace{b(z)c(\omega)}. \tag{16.8}$$

We could have done this with a mode expansion like we did for X, but note that since this is a free theory, the classical Green's function for $\bar{\partial}$ gives $\overbrace{b(z)c(\omega)}$ exactly. Thus using the result

$$\frac{\partial}{\partial\bar{z}}\left(\frac{1}{z-\omega}\right) = 2\pi\delta^2(z-\omega), \tag{16.9}$$

so we then have

$$\overbrace{b(z)c(\omega)} = \frac{1}{z-\omega}. \tag{16.10}$$

Thus the OPE is

$$b(z)c(\omega) = \frac{1}{z-\omega} + \ldots (= c(z)b(\omega)). \tag{16.11}$$

We can use this to remove poles from composite operators. Thus

$$T_{gh}(z) = \lim_{\omega\to z}\left(-2b(\omega)\partial c(z) - \partial b(\omega)c(z) + \frac{1}{(z-\omega)^2}\right) \tag{16.12}$$

where the squared in the last term is because of the derivatives in the first two terms.

Conformal transformations of ghosts Consider the ghost stress tensor with the b ghosts:

$$T_{gh}(z)b(\omega) =: \partial_z b(z)\overbrace{c(z)b(\omega)} : -2 : \partial_z(b(z)\overbrace{c(z))b(\omega)} : + \dots \tag{16.13}$$

$$= \frac{\partial b(z)}{z-\omega} - 2\partial_z\left(\frac{b(z)}{z-\omega}\right) + \dots \tag{16.14}$$

$$= \frac{2}{(z-\omega)^2}b(z) - \frac{1}{z-\omega}\partial b(z). \tag{16.15}$$

where we assume that the other contractions (b with b, c with c) give regular things that do not contribute to the pole structure of the OPE. We can expand $b(z)$ about $z=\omega$ as

$$b(z) = b(\omega) + (z-\omega)\partial_\omega b(\omega) + \dots$$

so that in the limit our expression becomes

$$\frac{2b(\omega)}{(z-\omega)^2} + \frac{2}{z-\omega}\partial b(\omega) - \frac{1}{z-\omega}\partial b(\omega) + \dots,$$

and we conclude that

$$T_{gh}(z)b(\omega) = \frac{2}{(z-\omega)^2}b(\omega) + \frac{1}{z-\omega}\partial b(\omega) + \dots \tag{16.16}$$

where we see this is a primary field of weight $(2,0)$. A similar calculation for the c ghost gives

$$T_{gh}(z)c(\omega) = \frac{-1}{(z-\omega)^2}c(\omega) + \frac{1}{z-\omega}\partial c(\omega) + \dots, \tag{16.17}$$

i.e. c has weight $(-1,0)$.

We can now compute the full OPE of $T_{gh}(z)T_{gh}(\omega)$. It's a good exercise to reproduce

$$T_{gh}(z)T_{gh}(\omega) = \frac{-26/2}{(z-\omega)^4} + \frac{2}{(z-\omega)^2}T_{gh}(\omega) + \frac{1}{z-\omega}\partial T_{gh}(\omega) + \dots \tag{16.18}$$

which again looks almost like a primary field, except with this weird $1/(z-\omega)^4$ term. But this is the same dependence we saw in the X part of the stress tensor, and the mixing of the OPEs of the Xs and the ghosts is trivial, so when we write down the full stress tensor $T(z) = T_X(z) + T_{gh}(z)$, we now find that

$$T(z)T(\omega) = \frac{(D-26)/2}{(z-\omega)^4} + \frac{2}{(z-\omega)^2}T(\omega) + \frac{1}{z-\omega}\partial T(\omega) + \dots \tag{16.19}$$

And this provides us with a possible resolution: if $D = 26$, then there is no conformal anomaly, i.e. $T(z)$ transforms like an honest primary field under conformal transformations. We could, with some patience, introduce modes for this corrected stress tensor $T(z)$ (accounting for the ghosts) as

$$T(z) = \sum_n \mathcal{L}_n z^{-n-2}, \tag{16.20}$$

and as it turns out, these $\mathcal{L}$s would satisfy not the Virasoro algebra but the Witt algebra in $D = 26$, i.e.

$$[\mathcal{L}_n, \mathcal{L}_m] = (n-m)\mathcal{L}_{n+m}. \tag{16.21}$$

From now on, we will assume that we are working in 26 dimensions in order to have a quantum consistent theory (the tachyon aside). As it turns out, if we look at a general curved spacetime rather than our flat background, the requirement that our theory be anomaly-free will impose some nontrivial conditions on what kind of background spacetime our theory can live in. In fact, this condition will tell us that the background metric must satisfy the Einstein field equations to lowest order.

Mode expansions As we showed, b has weight $(2,0)$ and c has weight $(-1,0)$, so we can write mode expansions as

$$b(z) = \sum_n b_n z^{-n-2}, \quad c(z) \sum_n c_n z^{-n+1}. \tag{16.22}$$

What is the anticommutator of these modes $\{b_m, c_n\}$? We can invert the mode expansions to get

$$b_m = \oint_{z=0} \frac{dz}{2\pi i} z^{m+1} b(z), \quad c_n = \oint_{z=)} \frac{dz}{2\pi i} z^{n-2} c(z). \tag{16.23}$$

Writing out the anticommutator we have

$$\begin{aligned}
\{b_m, c_n\} &= \oint_{z=0} \frac{dz}{2\pi i} z^{m+1} \oint_{\omega=0} \frac{d\omega}{2\pi i} \omega^{n-2} \{b(z), c(\omega)\} \\
&= \oint_{z=0} \frac{dz}{2\pi i} \oint_{\omega=0} \frac{d\omega}{2\pi i} R(b(z)c(\omega)) z^{m+1} \omega^{n-2} \\
&= \oint_{z=0} \frac{dz}{2\pi i} \oint_{\omega=0} \frac{d\omega}{2\pi i} z^{m+1} \omega^{n-2} \left(\frac{1}{z-\omega} + \ldots \right) \\
&= \oint_{\omega=0} \frac{d\omega}{2\pi i} \omega^{m+n-1} = \delta_{\mathrm{m}+n,0},
\end{aligned}$$

so that

$$\{b_m, c_n\} = \delta_{m+n,0}. \tag{16.24}$$

In principle, the OPEs give us all the equations we need to understand the structure of the quantum theory, though they may not always correspond to a sensible classical limit.

Lecture 17.

Monday, February 25, 2019

We're tantalizingly close to actually calculating an observable of our theory (in a formal sense, anyway). But we'll need one more tool.

BRST symmetry As it turns out, there's an additional global symmetry left over even after we do gauge fixing. Recall our path integral,

$$\mathcal{Z} = \frac{1}{|\mathrm{CKG}|} \int_{M_g} d^s t \int \mathcal{D}X \mathcal{D}b \mathcal{D}c\, e^{iS[X,b,c]} \prod_{I=1}^{s} (b|\mu_I) \prod_{i,a} c^a(\hat{\sigma}_i). \tag{17.1}$$

The way to see the residual symmetry is to explicitly reintroduce the path integral over the metric, writing

$$\mathcal{Z} = \frac{1}{|\mathrm{CKG}|} \int_{M_g} d^s t \int \mathcal{D}X \mathcal{D}b \mathcal{D}c \mathcal{D}h\, \delta[h - \hat{h}] e^{iS[X,b,c]} \prod_{I=1}^{s} (b|\mu_I) \prod_{i,a} c^a(\hat{\sigma}_i). \tag{17.2}$$

We can write this delta functional as a functional integral over auxiliary fields B_{ab} and add to the action the following:

$$S_{gf}[B,h] = \frac{1}{4\pi} \int_\Sigma d^2\sigma \sqrt{-h} B^{ab} (\hat{h}_{ab} - h_{ab}), \tag{17.3}$$

which is simply the functional analogue of writing a delta function in terms of a Fourier transform. The full action is now

$$\begin{aligned}
S[X,h,b,c,B] = &-\frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} + \frac{i}{2\pi} \int_\Sigma d^2\sigma \sqrt{-h} b_{ab} \nabla^a c^b \\
&+ \frac{1}{4\pi} \int_\Sigma d^2\sigma \sqrt{-h} B^{ab} (\hat{h}_{ab} - h_{ab}).
\end{aligned} \tag{17.4}$$

So far, this construction seems pretty ad hoc. But there is a rigid symmetry of this action, given by

$$\delta_Q X^\mu(z) = i\epsilon c(z) \partial X^\mu(z). \tag{17.5}$$

This is just a diffeomorphism with $v(z) = \epsilon c(z)$, where ϵ is just some constant (Grassmann) parameter. We also need to change the metric:

$$\delta_Q h_{ab}(z) = \epsilon (Pc)_{ab} \tag{17.6}$$

and the ghosts:

$$\delta_Q c^a(z) = i\epsilon c^b \partial_b c^a \tag{17.7}$$

$$\delta_Q b_{ab} = i\epsilon B_{ab}. \tag{17.8}$$

And our new field is invariant,

$$\delta_Q B_{ab} = 0. \tag{17.9}$$

It seems plausible that the first term in the action 17.4 will be invariant under this symmetry, as some sort of diffeomorphism. To see that the remaining terms are also invariant, we introduce the "gauge-fixing fermion." The name is historical– our theory is still bosonic.

$$\Psi[b,h] = -\frac{i}{4\pi}\int_\sigma d^2\sigma\, b^{ab}(\hat{h}_{ab} - h_{ab}). \tag{17.10}$$

This is a Grassmann quantity, and under a BRST transformation, Ψ generates the second and third terms in 17.4. That is, we can write the action as

$$S[X,b,c,B,h] = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\,\sqrt{-h}h^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} + \delta_Q \Psi[b,h]. \tag{17.11}$$

In fact, one can show by direct calculation that $\delta_Q^2 = 0$ on any field. We'll try to show it in a better way, though. For the moment assume this holds. Then since the first term of the action is invariant under δ_Q,

$$S_Q X[\dots] = 0 + \delta_Q^2 \Psi[b,h] = 0, \tag{17.12}$$

so the entire action will be invariant. We now integrate out the auxiliary field B_{ab}. We have now fixed the metric, and the action (with $h_{ab} = \hat{h}_{ab}$ and B_{ab} integrated out) is invariant under the transformations

$$\delta_Q X^\mu(z) = i\epsilon c(z)\partial X^\mu(z),$$

$$\delta_Q b_{ab} = i\epsilon T_{ab},$$

$$\delta_Q c^a(z) = i\epsilon c^b \partial_b c^a,$$

where T_{ab} is the total stress tensor $T_X + T_{gh}$. Invariance under this set of variations is known as *BRST symmetry*.

BRST cohomology and physical states Let us introduce the BRST charge Q_B. We will argue (loosely) that physical states $|\phi\rangle$ are in the kernel of Q as an operator (i.e. $Q_B|\phi\rangle = 0$) but not in its image ($\nexists|\psi\rangle$ such that $|\phi\rangle = Q_B|\psi\rangle$). We call

$$\ker(Q_B)/\mathrm{Im}(Q_B) \simeq \mathrm{Cohom}(Q_B) \tag{17.13}$$

the cohomology. Along the way, we'll prove that δ_Q is nilpotent ($\delta_Q^2 = 0$).

Why are physical states in the kernel of Q_B? It's because any observables of our theory cannot depend on our choice of gauge. Consider the observable

$$\langle\phi_f|\phi_i\rangle = \int \mathcal{D}\phi\, T(\phi_i\phi_f)e^{iS[\phi]}$$

where ϕ_i, ϕ_f are some initial/final states and $S[\phi]$ is of the form $S_0[\phi] + \delta_Q\Psi = S_0[\phi] + \{Q_B, \psi\}$. T indicates time ordering.

Let us change our gauge choice by changing $\Psi \to \Psi + \delta\Psi$ ($\delta\Psi$ is not related to our Q_B transformation). Thus

$$\delta\langle\phi_f|\phi_i\rangle = \int \mathcal{D}\phi\,\phi_i\phi_f e^{iS[\phi]+i\{Q,\delta\psi\}} - \int \mathcal{D}\phi\,\phi_i\phi_f e^{iS[\phi]}. \tag{17.14}$$

To leading order, this variation is

$$\begin{aligned}\delta\langle\phi_f|\phi_i\rangle &= \int \mathcal{D}\phi\,\phi_i\phi_f i\{Q,\delta\Psi\}e^{iS[\phi]}\\ &= \langle\phi_f|\{Q_B,\delta\Psi\}|\phi_i\rangle = 0,\end{aligned}$$

where we require that this variation vanishes since our gauge freedom is a redundancy of our theory and cannot affect observables. For this to be true for any $\delta\Psi$, we require that

$$Q_B|\phi\rangle = 0, \tag{17.15}$$

where going forward we assume that $Q_B = Q_B^\dagger$. Thus $|\phi\rangle \in \ker(Q_B)$.

Next, we argue that $Q_B^2 = 0$. We want Q_B to be conserved, which means it commutes with the Hamiltonian. Under a change of gauge, $\psi \to \psi + \delta\psi$, we want Q to still be conserved, and we can ensure this by requiring that

$$[Q_B, \delta_Q(\delta\Psi)] = 0. \tag{17.16}$$

To sum up, $\delta\Psi$ is the change from the gauge transforamtion, $\delta_Q(\delta\Psi)$ the effect on the action, and $[Q_B, \delta_Q(\delta\psi)]$ the requirement that Q_B is still conserved. Thus

$$\begin{aligned} 0 &= [Q_B, \{Q_B, \delta\Psi\}] \\ &= -[\delta\Psi, \{Q_B, Q_B\}] - [Q_B, \{\delta\psi, Q_B\}] \implies \{Q_B, Q_B\} = 0. \end{aligned}$$

We conclude that $Q_B^2 = \frac{1}{2}\{Q_B, Q_B\} = 0$.

Lecture 18.

Wednesday, February 27, 2019

Today we'll continue our discussion of BRST symmetry. Last time, we argued that the BRST symmetry was related to physical states in a special way. We showed that the BRST charge $\mathcal{Q}_B$ must satisfy

$$\mathcal{Q}_B^2 = \frac{1}{2}\{\mathcal{Q}_B, \mathcal{Q}_B\} = 0 \tag{18.1}$$

for $\mathcal{Q}_B$ to be conserved. We also required that physical states must satisfy $\mathcal{Q}_B|\phi\rangle = 0$, i.e. they are $\mathcal{Q}_B$-closed.

Consider a state $|\zeta\rangle = \mathcal{Q}_B|\Lambda\rangle$ for some state $|\Lambda\rangle$ (i.e. a $\mathcal{Q}_B$-exact state). Clearly, such a state is $\mathcal{Q}_B$-closed, since

$$\mathcal{Q}_B|\zeta\rangle = \mathcal{Q}_B^2|\Lambda\rangle = 0. \tag{18.2}$$

However, notice that

$$\langle\zeta|\zeta\rangle = \langle\Lambda|\mathcal{Q}_B^2|\Lambda\rangle = 0, \tag{18.3}$$

so such states have zero norm. More generally, if $|\phi\rangle$ is a physical state (not necessarily $\mathcal{Q}_B$-exact), then

$$\langle\phi|\zeta\rangle = \langle\phi|\mathcal{Q}_B|\zeta\rangle = 0, \tag{18.4}$$

In fact, though we haven't proved it, such $\mathcal{Q}_B$-exact states decouple from the theory. That is, any correlation functions with $\mathcal{Q}_B$-exact states included will vanish.

Therefore the physical states we are interested in are in the kernel of $\mathcal{Q}_B$ ($\mathcal{Q}_B$-closed) but not in its image ($\mathcal{Q}_B$-exact). This is precisely the notion of the *cohomology* of $\mathcal{Q}_B$: all physical states $|\phi\rangle$ must satisfy

$$|\phi\rangle \in \ker(\mathcal{Q}_B)/\mathrm{Im}(\mathcal{Q}_B) \simeq \mathrm{Cohom}(\mathcal{Q}_B). \tag{18.5}$$

We might wonder whether there are still ghosts in the theory, but one can prove that the physical spectrum is actually in one-to-one correspondence with $\mathrm{Cohom}(\mathcal{Q}_B)$ (the no-ghost theorem).

BRST charge for bosonic string theory Having discussed heuristically why we might be interested in such a charge, let us try to construct it for the bosonic string. That is, we shall look for an operator $\mathcal{Q}_B$ such that $\mathcal{Q}_B^2 = 0$ which generates our BRST transformations. Recall that our theory comes as two copies, a holomorphic and antiholomorphic sector. We shall decompose the charge into its action on the holomorphic and antiholomorphic sectors,

$$\mathcal{Q}_B = Q_B + \bar{Q}_B, \tag{18.6}$$

and require that

$$\mathcal{Q}_B^2 = 0, \tag{18.7}$$

$$\{Q_B, Q_B\} = 0, \quad \{\bar{Q}_B, \bar{Q}_B\} = 0, \quad \{Q_B, \bar{Q}_B\} = 0. \tag{18.8}$$

What's our strategy to construct the charge? We require the embedding fields to vary as

$$\delta_Q X^\mu(\omega) = \epsilon c(\omega)\partial X^\mu(\omega), \tag{18.9}$$

where ϵ is some Grassmann parameter and c is the c-ghost. Thus we get the anticommutator

$$[Q_B, X^\mu(\omega)] = c(\omega)\partial X^\mu(\omega), \tag{18.10}$$

which looks like a conformal transformation! We can recover this from the charge

$$Q_B = \oint_{z=0} \frac{dz}{2\pi i} c(z) T_X(z).$$

In fact, that's not quite the whole story– we must also couple the charge to the ghosts, writing the charge

$$Q_B = \oint_{z=0} \frac{dz}{2\pi i} c(z) \left(T_X(z) + \frac{1}{2} T_{\text{gh}}(z) \right). \tag{18.11}$$

Recall that once the B_{ab} auxiliary field is integrated out, the gauge-fixed action was invariant under

$$[Q_B, X^\mu(\omega)] = c(\omega)\partial X^\mu(\omega) \tag{18.12}$$
$$\{Q_B, c(\omega)\} = c(\omega)\partial c(\omega) \tag{18.13}$$
$$\{Q_B, b(\omega)\} = T_X(\omega) + T_{\text{gh}}(\omega). \tag{18.14}$$

For example,

$$\begin{aligned}
\{Q_B, b(\omega)\} &= \oint_{z=0} \frac{dz}{2\pi i} \{(z)(T_X(z) + \frac{1}{2} T_{\text{gh}}(z)), b(\omega)\} \\
&= \oint_{z=\omega} \frac{dz}{2\pi i} \left(\overbrace{c(z)(T_X(z) + \frac{1}{2} T_{\text{gh}}(z)) b(\omega)} + \frac{1}{2} c(z) \overbrace{T_{\text{gh}} b(\omega)} \right) \\
&= \oint_{z=\omega} \frac{dz}{2\pi i} \left((T_X(z) + \frac{1}{2} T_{\text{gh}}(z)) \frac{1}{z-\omega} + \frac{1}{2} c(z) \left(\frac{2}{(z-\omega)^2} b(\omega) + \frac{1}{z-\omega} \partial b(\omega) \right) \right)
\end{aligned}$$

where we've used the OPEs to do the contraction. Expanding in powers of $z - \omega \ll 1$, $c(z) = c(\omega) + (z - \omega)\partial c(\omega) + O((z-\omega)^2)$, we have

$$\oint_{z=\omega} \frac{dz}{2\pi i} \left((T_X(\omega) + \frac{1}{2} T_{\text{gh}}(\omega)) \frac{1}{z-\omega} + \frac{1}{2} \left(\frac{2c(\omega)b(\omega)}{(z-\omega)^2} + 2\frac{\partial c(\omega) b(\omega)}{z-\omega} + \frac{c(\omega)\partial b(\omega)}{z-\omega} \right) \right). \tag{18.15}$$

But we now see that the $(z-\omega)^2$ pole does not depend on z in its numerator, and therefore does not contribute to the contour integral. The last two terms in the parentheses give a copy of $T_{\text{gh}}(\omega)$, leaving

$$\{\mathcal{Q}_B, b(\omega)\} = \{Q_B, b(\omega)\} = \oint_{z=\omega} \frac{dz}{2\pi i} (T_X(\omega) + T_{\text{gh}}) \frac{1}{z-\omega} = T_{\text{tot}}(\omega). \tag{18.16}$$

With (perhaps a lot of) work, one can show that

$$[Q_B, T_{\text{tot}}] = \frac{D-26}{12} \partial^3 c(\omega), \tag{18.17}$$

which suggests that our charge will be anomalous in any $D \neq 26$. Thus $Q_B^2 = 0$ if $D = 26$, the same anomaly we saw in the Virasoro algebra.

The BRST current and anomaly It is useful to define the BRST current

$$\mathcal{Q}_B = Q_B + \bar{Q}_B = \oint_{z=0} \frac{dz}{2\pi i} j_B(z) - \oint_{\bar{z}=0} \frac{d\bar{z}}{2\pi i} \bar{j}_B(\bar{z}). \tag{18.18}$$

Our BRST current takes the form

$$j_B(Z) = c(z) \left(T_X(z) + \frac{1}{2} T_{\text{gh}}(z) \right) + \frac{3}{2} \partial^2 c(z),$$

where this last term drops out in the contour integral. The rest is what we could have read off. As it turns out, the OPE of $j_B(z)$ with itself is

$$j_B(z) j_B(\omega) = -\frac{D-18}{2(z-\omega)^3} c(\omega)\partial c(\omega) - \frac{D-18}{4(z-\omega)^2} c(\omega)\partial^2 c(\omega) - \frac{(D-26)}{12(z-\omega)} c(\omega)\partial^3 c(\omega) + \ldots \tag{18.19}$$

and this seems pretty strange. Does this $D - 18$ factor mean that D is somehow both 18 and also 26?

No. When we check $Q_B^2 = 0$, we see that

$$\begin{aligned}\{Q_B, Q_B\} &= \oint_{z=0} \frac{dz}{2\pi i} \oint_{\omega=0} \frac{d\omega}{2\pi i} \{j_B(z), j_B(\omega)\} \\ &= \oint_{z=0} \frac{dz}{2\pi i} \oint_{\omega=0} \frac{d\omega}{2\pi i} (j_B(z) j_B(\omega) \text{ OPE}) \\ &= \oint_{z=0} \frac{dz}{2\pi i} \oint_{\omega=0} \frac{d\omega}{2\pi i} \left(-\frac{(D-26)}{12(z-\omega)} c(\omega) \partial^3 c(\omega) \right).\end{aligned}$$

In fact, the $D-18$ terms can be integrated by parts, and the two terms we have written turn out to make equal and opposite contributions to the contour integral. Performing the $d\omega$ integral, what remains is

$$\{Q_B, Q_B\} = -\frac{(D-26)}{12} \oint_{z=0} \frac{dz}{2\pi i} c(z) \partial^3(z). \tag{18.20}$$

As we've presented it, the BRST symmetry was something that emerged from our idea of physical states. But there's another viewpoint– the BRST operator is actually the fundamental object which tells us the structure of our theory. It tells us something deep about the constraints as we wee them through ghosts, and the requirement that states are not in the image of the BRST transformation is the statement that we are not interested in states which are pure gauge.

Lecture 19.

Friday, March 1, 2019

Today we'll begin our discussion of scattering amplitudes in string theory, i.e. the S-matrix.

The big idea Recall that we found it useful to conformally map our worldsheet cylinder onto the complex plane, or equivalently the Riemann sphere ($\mathbb{C} \cup \{\infty\}$) with two punctures, using the map $z = e^{\tau + i\sigma}$. What if there are some initial and final states $|\phi_i\rangle, |\phi_f\rangle$ in the picture?

We can encode the initial and final states in the Riemann sphere picture by inserting operators V_i, V_f at the punctures, where e.g.

$$|\phi_i\rangle = \lim_{z \to 0} V_i(z) |0\rangle. \tag{19.1}$$

But what if we want to discuss scattering? We could think of interactions between strings as described by worldsheets with many boundaries. In the same way, we will assume that $\exists$ a (conformal) map between e.g. a state with two initial strings merging and separating (see diagram) to a sphere with four punctures. We will also have loop diagrams, and these can be mapped to tori with punctures.[14]

Note that there have always been two theories in the game– the physics on our worldsheet, and the bigger spacetime it was embedded in. In order to properly discuss scattering amplitudes, we will need a few preliminaries.

Scattering preliminaries What constraints on the ghosts are required for the limit

$$|\phi\rangle = \lim_{z \to 0} \phi(z) |0\rangle \tag{19.2}$$

to exist? Suppose $\phi(z)$ is of weight $(h, 0)$ (a chiral field), such that

$$\phi(z) = \sum_n \phi_n z^{-n-h}. \tag{19.3}$$

Then the limit we want to evaluate is

$$\lim_{z \to 0} \sum_n \phi z^{-n-h} |0\rangle. \tag{19.4}$$

Notice that for $-n-h > 0$, the terms go to zero as $z \to 0$, but for $-n-h < 0$ these terms will generically blow up as $z \to 0$.

We can get a sensible limit if we require that

$$\phi_n |0\rangle = 0, \quad n > -h, \tag{19.5}$$

[14] We can definitely construct this from the "tree-level" interactions by "gluing" the punctured Riemann spheres together at the punctures.

and then

$$|\phi\rangle = \phi_{-h}|0\rangle, \tag{19.6}$$

the only mode that doesn't vanish.

For the ghosts, b has weight $h = 2$ and c has weight $h = -1$. This means that

$$c_0|0\rangle \neq 0, c_1|0\rangle \neq 0 \tag{19.7}$$

and

$$\langle 0|c_{-1}c_0c_{+1}|0\rangle \neq 0. \tag{19.8}$$

There's some freedom in how we choose to normalize the ghost vacuum. We can choose

$$\langle 0|c_{-1}c_0c_{+1}|0\rangle = 1 \tag{19.9}$$

One can then show as an exercise that the expectation of the c-ghosts at three points (e.g. by a mode expansion) that

$$\langle 0|c(z_1)c(z_2)c(z_3)|0\rangle = (z_1 - z_2)(z_2 - z_3)(z_3 - z_1). \tag{19.10}$$

The dilaton and the string coupling There's also an interesting point to be made about the dilaton, the scalar that popped out of our theory early on. We could consider the string as propagating in a spacetime with a background of $g_{\mu\nu}(X), B_{\mu\nu}(X), \Phi(X)$. That is, there's other stuff like background curvature and EM fields in the ambient spacetime. In that case, our worldsheet theory ought to be sensitive to this stuff.

In particular, the worldsheet metric would be modified to

$$S = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{-h}h^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X). \tag{19.11}$$

This is hard to solve. We already needed perturbation theory just to discuss the string in a flat Minkowski background– now we have some additional structure to perturb about. We might also pick up a factor

$$S = -\frac{i}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{-h}\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu B_{\mu\nu}(X) \tag{19.12}$$

with $\epsilon^{ab} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. If we like, the B-field is just a two-form, and its contribution to the action is simply the pullback of this two-form to the worldsheet.

There's one last thing we could do– we could couple to the dilaton.

$$S_\Phi = \frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{-h}\Phi(X)R_\Sigma, \tag{19.13}$$

where R_Σ is the Ricci scalar on Σ. There are a few strange features of this– this isn't at order $1/\alpha'$ but at order 1. The structure of this coupling also looks different than the other two, as it is diff invariant but not Weyl invariant. Moreover, if $\Phi(X)$ has a vacuum expectation value $\langle\Phi(X)\rangle = \Phi_0$, then our action picks up a contribution

$$S_\Phi = \frac{1}{4\pi}\Phi_0\int_\Sigma d^2\sigma\sqrt{-h}R_\Sigma = \Phi_0\chi = \Phi_0(2g-2) \tag{19.14}$$

where χ is the Euler characteristic of Σ and g is the genus of the worldsheet Σ.

So in the path integral, the sum over genus is weighted by a factor of e^{Φ_0}. Our path integral has the form

$$Z = \sum_{g=0}^\infty \frac{e^{\Phi_0(2g-2)}}{|\mathrm{CKG}|}\int_{\mathcal{M}_g} d^s t \int \mathcal{D}b\mathcal{D}c\mathcal{D}\bar{b}\mathcal{D}\bar{c}\mathcal{D}X \prod_{I=1}^s (\mu_I|b)(\bar{\mu}_I|\bar{b})e^{-S[X,b,c]}. \tag{19.15}$$

That is, in addition to the path integrals over the fields and ghosts, we must sum over all topologies with a weight given by the dilaton VEV, where we call

$$g_c \equiv e^{\Phi_0} \tag{19.16}$$

the closed string coupling constant.

For suppose we have a Riemann surface of genus g, weighted by g_c^{2g-2}. Now if we have a closed string state (e.g. a graviton) being emitted and then absorbed by Σ_g, this adds one more handle to Σ_g and hence increases the genus by 1. Thus $g \to g+1$ adds another factor of g_c^2 to our expression, so we associate a factor of g_c with each additional "vertex," i.e. one for the emission and one for the absorption in this process.

For now, we'll assume the background metric is flat Minkowski and there's no B-field. But there may be a nonzero dilaton field.

Vertex operators How can we build operators that live in the BRST cohomology? Suppose we have an operator $\Phi(z, \bar{z})$ which satisfies

$$[Q_B, \phi(z, \bar{z})] = \partial(c\phi), \quad [\bar{Q}_B, \phi(z, \bar{z})] = \bar{\partial}(\bar{c}\phi). \tag{19.17}$$

This is not chiral; it knows about both the holomorphic and antiholomorphic sectors of the theory. Then

$$V_\phi = \int_\Sigma d^2z\, \phi(z, \bar{z}) \tag{19.18}$$

is BRST-closed. Given one such solution, we'll get some others for free. Consider

$$U(z, \bar{z}) = c(z)\bar{c}(z)\phi(z, \bar{z}). \tag{19.19}$$

By linearity,

$$\begin{aligned}
[\mathcal{Q}_B, U] = [Q_B + \bar{Q}_B, U] &= c\partial c\bar{c}\phi + c\bar{c}\partial(c\phi) + \text{barred expressions} \\
&= c\partial c\bar{c}\phi + c\bar{c}\partial c\phi + \ldots \\
&= 0.
\end{aligned}$$

Thus U is BRST-closed.[15] We have therefore constructed two BRST-closed objects, one local and one non-local.

Lecture 20.

Monday, March 4, 2019

Let's continue our discussion of scattering amplitudes. We argued that if we have an operator $\phi(z, \bar{z})$ that transforms as

$$[Q_B, \phi] = \partial(c\phi) \text{ and } [\bar{Q}_B, \phi] = \bar{\partial}(\bar{c}\phi), \tag{20.1}$$

then we can construct two BRST-closed objects,

$$U(z, \bar{z}) = c(z, \bar{c}(z)\phi(z, \bar{z}), \quad V = \int_\Sigma d^2z\, \phi(z, \bar{z}). \tag{20.2}$$

Since this U we have constructed is local, it would be nice if it also transformed properly under conformal transformations. What sort of ϕs will satisfy this property? Assume that ϕ has weight $(h, \bar{h})$. Under conformal transformations, we have

$$\delta_v\phi = h\partial v\phi + v\partial\phi + \bar{h}\bar{\partial}\bar{v}\phi + \bar{v}\bar{\partial}\phi. \tag{20.3}$$

Therefore under BRST, we have

$$\begin{aligned}
[Q_B, \phi] &= h(\partial c)\phi + c\partial\phi \\
&= (h-1)(\partial c)\phi + \partial(c\phi).
\end{aligned}$$

Notice that ϕ transforms in the right way if $h = 1$. A similar argument for the antiholomorphic sector tells us we also require the $\bar{h} = 1$. Thus U and V are BRST-invariant if $(h, \bar{h}) = (1, 1)$.

[15] The $\partial\phi$ term goes away since it has a c^2, and the two terms we've written cancel once we anticommute $\bar{c}$ and ∂c.

The tachyon Imagine mapping a scattering process from our worldsheet to the Riemann sphere with some punctures. We might expect the tachyon vertex operator to tell us where the puncture is,

$$\delta^{26}[x^\mu - x^\mu(z,\bar{z})] \tag{20.4}$$

In momentum space, this becomes (after a Fourier transform)

$$\int d^{26}x\, \delta^{26}(x^\mu - X^\mu(z,\bar{z}))e^{ik_\mu x^\mu} = e^{ik_\mu X^\mu(z,\bar{z})}. \tag{20.5}$$

We might therefore propose that

$$\phi(z,\bar{z}) = e^{ik_\mu X^\mu(z,\bar{z})}. \tag{20.6}$$

Happily, this agrees with the state-operator correspondence as a momentum eigenstate.

Given this new operator ϕ, will U and V be BRST-invariant? We have shown that $e^{ik\cdot X(z,\bar{z})}$ has weight

$$(h,\bar{h}) = \left(\frac{\alpha' k^2}{4}, \frac{\alpha' k^2}{4}\right). \tag{20.7}$$

For the tachyon, $k^2 = 4/\alpha'$, so U and V will have weight $(1,1)$. The tachyon vertex operators are then

$$U_T(z,\bar{z}) = g_c c(z)\bar{c}(z) : e^{ik\cdot X(z,\bar{z})} :, \quad V_T = g_c \int_\Sigma d^2z\, : e^{ik\cdot X(z,\bar{z})} : . \tag{20.8}$$

Massless states Imagine a worldsheet embedding into a spacetime with metric

$$g_{\mu\nu}(X) = \eta_{\mu\nu} + \epsilon_{\mu\nu} e^{ik\cdot X(z,\bar{z})}, \tag{20.9}$$

almost Minkowski but with a little plane wave ripple in it. The action is

$$S = -\frac{1}{2\pi\alpha'} \int d^2z \Big(\eta_{\mu\nu} + \epsilon e^{ik\cdot X(z,\bar{z})}\Big)\partial X^\mu \bar{\partial} X^\nu. \tag{20.10}$$

Since we treat $\epsilon_{\mu\nu}$ as a small (symmetric) perturbation, we may as well expand in powers of that perturbation. Thus

$$\int \mathcal{D}X\, e^{-S[X]} \approx \int \mathcal{D}X e^{-S_0[X]} \left(1 + \frac{1}{4\pi\alpha'} \int_\Sigma d^2z\, \epsilon_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X} + \ldots\right) \tag{20.11}$$

where $S_0[X]$ is the unperturbed action with $g_{\mu\nu} = \eta_{\mu\nu}$. These operator insertions tell us how to deform our flat Minkowski spacetime into a slightly curved spacetime. That is, the insertion of an operator

$$\int_\Sigma d^2z\, \epsilon_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X} \tag{20.12}$$

results in an infinitesimal perturbation in $g_{\mu\nu}$. This suggests that

$$\phi(z,\bar{z}) = \epsilon_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X} \tag{20.13}$$

can be used to build graviton vertex operators. We checked the conformal weight of this object on the last examples sheet– the weight of $\phi(z,\bar{z})$ is

$$\left(1 + \frac{\alpha' k^2}{4}, 1 + \frac{\alpha' k^2}{4}\right). \tag{20.14}$$

So if $k^2 = 0$ then the ϕ operator has weight $(1,1)$, which tells us that our graviton candidate is massless. The corresponding graviton vertex operators are

$$U_g(z,\bar{z}) = g_c c\bar{c}\epsilon_{\mu\nu} : \partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X(z,\bar{z})} :, \quad V_g = g_c \epsilon_{\mu\nu} \int_\Sigma d^2z : \partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X} : . \tag{20.15}$$

As it turns out, if we add a small mass to the graviton then its vertex operators are no longer BRST-invariant. There are also massive states in our theory, but the constraints on these modes are more subtle– renormalization comes into play. We won't really discuss these except to note they exist.

The S-matrix The S-matrix entry describing the scattering of n states using the vertex operators $V_1, \ldots, V_n$ is

$$\begin{aligned} A_n = &\sum_{g=0}^{\infty} g_c^{2g-2} \frac{1}{|\text{CKG}|} \int_{\mathcal{M}} d^s t \int \mathcal{D}b\mathcal{D}c\mathcal{D}\bar{b}\mathcal{D}\bar{c}\mathcal{D}X \prod_{I=1}^{s} (\mu_I|b)(\bar{\mu}_I|\bar{b}) \\ &\times e^{-S[X,b,c,\bar{b},\bar{c}]} \prod_{i,a} c^a(\hat{\sigma}_i) V_1 \ldots V_n. \end{aligned}$$

Here, we have a sum over genus g, an integral over moduli space $\mathcal{M}$, a path integral over $b, c, \bar{b}, \bar{c}$ fields, the path integral weight e^{-S}, some Killing vector-fixing factors $c^a(\hat{\sigma}_i)$, and of course the vertex operators themselves.

At tree-level, we consider the $g = 0$ contributions (i.e. spheres). The CKG is $SL(2;\mathbb{C})$, and the moduli space is zero-dimensional (all spheres are conformally equivalent). Thus the factors $(\mu_I|b)$ drop out of the integral. We can fix the freedom in the CKG by choosing the $\hat{\sigma}_i^a$ to coincide with the first three punctures. In particular, it might be nice to select these to coincide with the operators V_1, V_2, V_3. Our amplitude becomes

$$\begin{aligned} A_n^{(g=0)} &= \frac{g_c^{-2}}{|\text{CKG}|} \left(\int \mathcal{D}b\mathcal{D}c\mathcal{D}\bar{b}\mathcal{D}\bar{c} e^{-S_{gh}[b,c]} \prod_{i=1}^{3} c(z_i)\bar{c}(\bar{z}_i) \right) \\ &\quad \times \int \mathcal{D}X e^{-S[X]} V_1 \ldots V_n \\ &= \frac{g_c^{n-2}}{|CKG|} \int d^2z_1 \ldots d^2z_n \left\langle \prod_{i=1}^{3} c(z_i)\bar{c}(\bar{z}_i) \right\rangle_{\text{gh}} \langle \phi_1(z_1, \bar{z}_1) \ldots \phi_n(z_n, \bar{z}_n) \rangle \\ &= \frac{g_c^{n-2}}{|\text{CKG}|} \int d^2z_1 d^2z_2 d^2z_3 \, \langle U_1(z_1, \bar{z}_1) U_2 U_3 V_4 \ldots V_n \rangle \,. \end{aligned}$$

Lecture 21.

Wednesday, March 6, 2019

Last time, we wrote down an amplitude for an n-state scattering process at tree level,

$$A_n = \frac{1}{|SL(2;\mathbb{C})|} g_c^{n-2} \int d^2z_1 \ldots d^2z_n \left\langle \prod_{i=1}^{3} c(z_i)\bar{c}(\bar{z}_i) \right\rangle_{bc} \langle \phi_1(z_1, \bar{z}_2) \ldots \phi_n(z_n, \bar{z}_n) \rangle \,. \tag{21.1}$$

One can show that

$$\left\langle \prod_{i=1}^{3} c(z_i)\bar{c}(\bar{z}_i) \right\rangle_{bc} = |(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)|^2 \underbrace{\langle 0 | c_{-1} c_0 c_1 \bar{c}_{-1} \bar{c}_0 \bar{c}_1 | 0 \rangle}_{=1} . \tag{21.2}$$

What about this $SL(2;\mathbb{C})$ volume? Remember, this corresponds to some gauge fixing in which we must pick three points on the Riemann sphere to fix the $SL(2;\mathbb{C})$ symmetry. It is natural for us to choose three of the punctures as the points to fix the symmetry. Notice that under an infinitesimal $SL(2;\mathbb{C})$ transformation,

$$z_i \to a_1 + a_2 z_i + a_3 z_i^2. \tag{21.3}$$

Here, a_i are parameters defining the transformation. We can relate an integral over the space of a_i ($i = 1, 2, 3$) to an integral over the locations of three punctures z_i as

$$|J|^2 d^2a_1 d^2a_2 d^2a_3 = d^2z_1 d^2z_2 d^2z_3, \tag{21.4}$$

where J is some Jacobian factor. In particular, it is

$$J = \det \left| \frac{\partial z_i}{\partial a_j} \right| + \begin{vmatrix} 1 & z_1 & z_1^2 \\ 1 & z_2 & z_2^2 \\ 1 & z_3 & z_3^2 \end{vmatrix} = (z_1 - z_2)(z_2 - z_3)(z_3 - z_1), \tag{21.5}$$

so we see that

$$\frac{1}{d^2a_1d^2a_2d^2a_3} = \frac{1}{d|SL(2;\mathbb{C})|} = \frac{|(z_1-z_2)(z_2-z_3)(z_3-z_1)|^2}{d^2z_1d^2z_2d^2z_3} = \frac{\left\langle \prod_{i=1}^3 c(z_i)\bar{c}(\bar{z}_i)\right\rangle_{bc}}{d^2z_1d^2z_2d^2z_3}. \tag{21.6}$$

What we see is that the Faddeev-Popov determinant is correctly capturing the Jacobian factor in going from d^2z_is to d^2a_is. So we interpret the $\frac{1}{|SL(2;\mathbb{C})|}$ factor as allowing us to fix the symmetry with the first three punctures, so we can write the amplitude as

$$A_n = g_c^{n-2}\int d^2z_4\dots d^3z_n \left\langle \prod_{i=1}^3 c(z_i)\bar{c}(\bar{z}_i)\right\rangle_{bc} \langle \phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)\rangle_X, \tag{21.7}$$

i.e. we integrate over $n-3$ of the punctures. Note that if $g=1$ (Σ is a torus), we integrate oer $n-1$ punctures, and if $g>1$ we integrate over all n punctures.

We can then compactly write the expression for A_n in terms of the U_i, V_i vertex operators, where recalling that

$$U_i = g_c c(z_i)\bar{c}(\bar{z}_i)\phi_i(z_i,\bar{z}_i), \quad V_i = g_c\int_\Sigma d^2z_i\phi(z_i,\bar{z}_i), \tag{21.8}$$

we have the amplitude

$$A_n = g_c^{-2}\left\langle \prod_{i=1}^3 U_i(z_i,\bar{z}_i)\prod_{j=4}^n V_j\right\rangle. \tag{21.9}$$

Tree-level scattering with path integrals Consider the correlation function

$$\langle\phi_1(z_1)\dots\phi_n(z_n)\rangle_X = \int \mathcal{D}Xe^{-S[X]}\phi_1(z_1)\dots\phi_n(z_n), \tag{21.10}$$

and introduce a source term to the action,

$$S_J[X] = \int_\Sigma d^2zJ_\mu X^\mu. \tag{21.11}$$

Then

$$\begin{aligned} S[X]+S_g[X] &= -\frac{1}{2\pi\alpha'}\int_\Sigma d^2x\partial X^\mu\bar{\partial}X_\mu + \int_\Sigma d^2zJ_\mu X^\mu \\ &= \frac{1}{2\pi\alpha'}\int_\Sigma d^2xX^\mu\Box X_\mu + \int_\Sigma d^2zJ_\mu X^\mu \\ &= \frac{1}{2\pi\alpha'}\int_\Sigma d^2zY^\mu\Box Y_\mu + \frac{1}{2}\int_{\Sigma\times\Sigma} d^2zd^2\omega J^\mu(z)G(z,\omega)J_\mu(\omega) + x^\mu\int_\Sigma d^2zJ_\mu(z), \end{aligned}$$

where we have integrated by parts and denote $\Box = \partial\bar{\partial}$. In this last step, we have separated off the constant part of X^μ,

$$X^\mu(z,\bar{z}) = x^\mu + \tilde{X}^\mu(z,\bar{z}), \tag{21.12}$$

and noticed that the derivatives in the first term kill the constant x^μ, leaving us with $\tilde{X}$s. We then denote

$$Y^\mu(z,\bar{z}) = \tilde{X}^\mu(z,\bar{z}) - \int_\Sigma d^2\omega G(z,\omega)J^\mu(\omega,\bar{\omega}) \tag{21.13}$$

where $G(z,\omega)$ is the Green's function

$$G(z,\omega) = -\frac{\alpha'}{2}\ln|z-\omega|^2 \tag{21.14}$$

satisfying

$$-\frac{1}{\pi\alpha'}\Box_z G(z,\omega) = \delta^2(z-\omega). \tag{21.15}$$

In principle, this is just completing the square in order to decouple the Y^μ integral. If we define

$$Z[J] = \int \mathcal{D}Xe^{-S[X]-S_J[X]}, \tag{21.16}$$

notice that up to zero modes which may be absorbed into the normalization of $Z[J]$, we have

$$Z[0] \sim \int \mathcal{D}Y \exp\left(-\frac{1}{2\pi\alpha'}\int_\Sigma d^2\partial Y^\mu \bar{\partial} Y_\mu\right), \tag{21.17}$$

and so

$$Z[J] = Z[0]\exp\left(\frac{1}{2}\int_{\Sigma\times\Sigma} d^2z d^2\omega J^\mu(z)G(z,\omega)J_\mu(\omega)\right)\int d^{26}x \exp\left(x^\mu \int d^2z J_\mu(z)\right) \tag{21.18}$$

There is a slight caveat, which is that $\mathcal{D}X = d^{26}x\mathcal{D}\tilde{X} = d^{26}x\mathcal{D}Y$, so we cannot discard the integral over zero modes, though it still separates out.

In a quantum field theory, we would write down Feynman rules by taking derivatives of $Z[J]$ with respect to J, bringing down factors of the propagator. However, that's not what we're going to do here.

Tachyon scattering The amplitude for n tachyon scattering includes

$$\left\langle e^{ik_1\cdot X(z_1)}\dots e^{ik_n\cdot X(z_n)}\right\rangle = \int \mathcal{D}X e^{-S[X]}\prod_{i=1}^n e^{ik\cdot X(z_i)}. \tag{21.19}$$

If we wanted to, we could write this as

$$\begin{aligned}\left\langle e^{ik_1\cdot X(z_1)}\dots e^{ik_n\cdot X(z_n)}\right\rangle &= \int cDX \exp\left(-S[X] + i\sum_{i=1}^n k_i\cdot X(z_i)\right)\\ &= \int \mathcal{D}X\exp\left(-S[X] - \int_\Sigma d^2z J^\mu(z)X_\mu(z)\right)\end{aligned}$$

where $J^\mu(z,\bar{z}) = -i\sum_{i=1}^n k_i^\mu \delta^2(z-z_i)$, so that we've constructed a "source term" and this amplitude looks a lot like $Z[J]$. Substituting this J^μ into 21.18 then requires us to compute

$$\int_\Sigma d^2z J_\mu(z) = -i\sum_{j=1}^n\int_\Sigma d^2z\delta^2(z-z_j)k_{\mu j} = -i\sum_{j=1}^n k_{\mu j}. \tag{21.20}$$

Hence

$$\int d^{26}x\exp(x^\mu\int_\Sigma d^2zJ_\mu) = \int d^{26}x\exp(ix^\mu\sum_{j=1}^n k_{\mu j}) = (2\pi)^{26}\delta^{26}\left(\sum_{j=1}^n k_j^\mu\right), \tag{21.21}$$

where this integral has turned out to simply enforce momentum conservation.

What about the other integral with the Green's function? Substituting in J^μ gives us

$$\begin{aligned}\frac{1}{2}_{\Sigma\times\Sigma} d^2zd^2\omega J^\mu(z)G(z,\omega)J_\mu(\omega) &= -\frac{1}{2}\int_{\Sigma\times\Sigma} d^2zd^2\omega\sum_{i\neq j}k_i^\mu\delta^2(z-z_i)G(z,\omega)k_{j\mu}\delta^2(z_j-\omega)\\ &= -\frac{1}{2}\sum_{i\neq j}k_i\cdot k_j\left(-\frac{\alpha'}{2}\ln|z_i-z_j|^2\right).\end{aligned}$$

This has cleaned up nicely, and so

$$\begin{aligned}\exp\frac{1}{2}\int_{\Sigma\times\Sigma} d^2zd^2\omega J^\mu(z)G(z,\omega)J_\mu(\omega) &= \prod_{i\neq j}|z_i-z_j|^{\alpha' k_i\cdot k_j/2}\\ &= \prod_{i<j}|z_i-z_j|^{\alpha' k_i\cdot k_j}.\end{aligned}$$

We find that

$$\left\langle\prod_{i=1}^n e^{ik_i\cdot X(z_i,\bar{z}_i)}\right\rangle_X = (2\pi)^{26}\delta^{26}\left(\sum_{i=1}^n k_{i\mu}\right)\prod_{i<j}|z_i-z_j|^{\alpha' k_i\cdot k_j}. \tag{21.22}$$

To sum up, we found that the tachyon amplitude could be written as an integral with an action plus a source term, which means that we can rewrite it (up to an overall normalization factor) in terms of our factorized result 21.18 and compute the path integral explicitly.

Lecture 22.

Friday, March 8, 2019

Last time, we went through most of the steps for evaluating our first string theory amplitude, the n-tachyon scattering amplitude. We found that the integral over the zero-modes for the Xs gave us overall momentum conservation, while the nontrivial bit gave us

$$\langle \phi_1 \dots \phi_n \rangle_X = (2\pi)^{26} \delta^{26} \left(\sum_{i=1}^n k_{i\mu} \right) \prod_{i<j} |z_i - z_j|^{\alpha' k_i \cdot k_j}, \tag{22.1}$$

which sort of measures the distance between the punctures on the Riemann sphere. Let's get more specific and set $n = 3$. We notice that

$$\alpha' k_1 \cdot k_2 = \frac{\alpha'}{2} \left[(k_1 + k_2)^2 - k_1^2 - k_2^2 \right], \tag{22.2}$$

and using momentum conservation we know that $k_1^\mu + k_2^\mu = -k_3^\mu$. Thus we can use the delta function to write

$$\alpha' k_1 \cdot k_2 = \frac{\alpha'}{2} (k_3^2 - k_1^2 - k_2^2), \tag{22.3}$$

and since tachyons have

$$k^2 = -m^2 = 4/\alpha', \tag{22.4}$$

we can write this explicitly as

$$\alpha' k_1 \cdot k_2 = -\frac{\alpha'}{2} \frac{4}{\alpha'} = -2, \tag{22.5}$$

and similar expressions hold for the other $\alpha' k_i \cdot k_j, i \neq j$. Hence we have[16]

$$\langle \phi_1 \phi_2 \phi_3 \rangle_X = (2\pi)^{26} \delta^{(26} \left(\sum_{i=1}^3 k_i^\mu \right) \frac{1}{|z_1 - z_2|^2 |z_2 - z_3|^2 |z_3 - z_1|^2}. \tag{22.6}$$

However, this isn't the whole story. The ghost contribution gives a factor of

$$|z_1 - z_2|^2 |z_2 - z_3|^2 |z_3 - z_1|^2, \tag{22.7}$$

and therefore we find that the amplitude is independent of the z_i. Thus

$$A_3 = g_c (2\pi)^{26} \delta^{26} \left(\sum_{i=1}^3 k_i^\mu \right). \tag{22.8}$$

This isn't too surprising– this is like the scattering of three scalar particles, and there isn't that much we could have written down that would be Lorentz invariant. We see that a "three-point vertex" in string theory is associated to a single factor of the closed string coupling g_c.

We can kick it up a notch with $n = 4$. The $n = 3$ case was very simple because the ghosts cancel the first three punctures– what if we have another one? Let us choose

$$z_1 = 0, z_2 = 1, z_3 = \lambda \to \infty \tag{22.9}$$

and take $z_4 = z$ to be integrated over. The amplitude for four-tachyon scattering will be

$$A_4 \sim \left\langle U_(z_1) U(z_2) U(z_3) V_4 \right\rangle \tag{22.10}$$

where we now have an integral to perform over $z_4 = z$. The amplitude includes a factor

$$\left\langle \prod_{l=1}^3 c_l \bar{c}_l \right\rangle \prod_{i<j=1}^4 |z_i - z_j|^{\alpha' k_i \cdot k_j} = |z|^{\alpha' k_1 \cdot k_4} |1 - z|^{\alpha' k_2 \cdot k_4}, \tag{22.11}$$

where we can derive this last expression using momentum conservation.

We can now introduce Mandelstam variables,

$$t = -(k_1 + k_3)^2, \quad u = -(k_1 + k_4)^2, \tag{22.12}$$

[16] This almost looks like a Feynman propagator.

so that e.g. $k_2 \cdot k_4$ can be written in terms of k_1, k_3 and therefore in terms of t. That is, we can write

$$\alpha' k_1 \cdot k_4 = -\frac{\alpha' u}{2} - 4, \quad \alpha' k_2 \cdot k_4 = -\frac{\alpha' t}{2} - 4. \tag{22.13}$$

This is sometimes useful when comparing to field theory calculations.

We find that

$$A_4 = g_c^2 (2\pi)^{26} \delta^{26} \left(\sum_{i=1}^{4} k_i^\mu \right) \int d^2 z |z|^{-\alpha/u/2-4} |1-z|^{-\alpha' t/2-4}. \tag{22.14}$$

Introducing the gamma function

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy, \tag{22.15}$$

the amplitude A_4 may be written as

$$A_4 = g_c^2 (2\pi)^{26} \delta^{26} \left(\sum_{i=1}^{4} k_{i\mu} \right) \frac{2\pi \Gamma(\alpha(s))\Gamma(\alpha(t))\Gamma(\alpha(u))}{\Gamma(\alpha(t)+\alpha(u))\Gamma(\alpha(s)+\alpha(u))\Gamma(\alpha(s)+\alpha(t))} \tag{22.16}$$

where $\alpha(s) = -1 - \frac{\alpha' s}{4}$ with

$$s = -(k_1+k_2)^2, \quad t = -(k_1+k_3)^2, \quad u = -(k_1+k_4)^2. \tag{22.17}$$

Notice that A_4 is completely symmetric in the Mandelstam variables s, t, u. This is what we might call a "duality" or "dual models."[17] String theory is a little different– since we can continuously deform our scattering processes, we actually get the other Feynman diagrams for free. Hence a single scattering amplitude at tree level contains the s, t, and u channels (up to an integral over moduli space).

Massless scattering For the tachyon, we argued that the vertex operator could be turned into a sort of source term, which allowed us to exactly calculate the amplitude for tachyon scattering. One might wonder if this technique generalizes for massless states, and ideed it does. Massless states have vertex operators of the form

$$V = \int d^2 z \epsilon_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X}. \tag{22.18}$$

It's a little more complicated than the tachyon operator. Introducing the dummy variables ρ and $\bar{\rho}$, we can write

$$\frac{\partial}{\partial \rho_{\mu j}} \left\{ \exp\left[i \int_\Sigma d^2 z (k_{\mu j} + \rho_{\mu j} \frac{\partial}{\partial z}) X^\mu(z,\bar{z}) \delta^2(z-z_j) \right] \right\}_{\rho_j = 0} = i\partial X^\mu(z_j) e^{ik_j \cdot X(z_j)}. \tag{22.19}$$

So this lets us write V as a pure exponential. Thus

$$\epsilon_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X} = -\epsilon_{\mu\nu} \frac{\partial^2}{\partial \rho_{\mu j} \partial \bar{\rho}_{\nu j}} \exp\left\{ i \int_\Sigma d^2 z \delta^2(z-z_j)(k_{\mu j} + \rho_{j\mu} \frac{\partial}{\partial z} + \bar{\rho}_{j\mu} \frac{\partial}{\partial \bar{z}}) X^\mu(z,\bar{z}) \right\}_{\rho=0,\bar{\rho}=0}. \tag{22.20}$$

This is a lot like what we did in field theory, introducing a source, taking derivatives, and setting the source to zero. Introducing now

$$J_\mu(z,\bar{z}) = -i \sum_{j=1}^{n} \delta^2(z-z_j)(k_{\mu_j} + \rho_{\mu_j} \frac{\partial}{\partial z} + \bar{\rho}_{j\mu} \frac{\partial}{\partial \bar{z}}), \tag{22.21}$$

the n-point amplitude for massless scattering may be written as

$$A_n = (-1)^n g_c^{n-2} \prod_{j=1}^{n} \left(\epsilon_{\mu_j \nu_j} \frac{\partial^2}{\partial \rho_{\mu_j} \partial \bar{\rho}_{\nu_j}} \right) \exp\left(\frac{1}{2} \int_{\Sigma\times\Sigma} d^2 z d^2 \omega \, J(z) J(\omega) G(z,\omega) \right)_{\rho=0,\bar{\rho}=0} \times (2\pi)^{26} \delta^{26} \left(\sum_{j=1}^{n} k_{\mu j} \right), \tag{22.22}$$

with $G(z,\omega)$ a Green's function.

It is useful to split

$$X^\mu(z,\bar{z}) = x^\mu + \tilde{X}^\mu(z) + \bar{X}^\mu(\bar{z}), \tag{22.23}$$

[17] In field theory we sometimes call these crossing symmetries.

i.e. into a center of mass bit, a holomorphic part, and an antiholomorphic part. Similarly we write[18]

$$G(z,\omega) = -\frac{\alpha'}{2}\ln|z-\omega|^2 = -\frac{\alpha'}{2}\ln(z-\omega) - \frac{\alpha'}{2}\ln(\bar{z}-\bar{\omega}). \tag{22.24}$$

Lecture 23.

Monday, March 11, 2019

Today we'll continue our discussion of the scattering of massless states. Recall that we had a trick of writing vertex operators as "source terms" in the path integral, which we could differentiate and then set sources to zero in, giving us the desired scattering amplitude.

Recall that we could write the vertex operator

$$i\partial X^\mu(z_j)e^{ik_j\cdot X(z_j)} = \frac{\partial}{\partial\rho_{\mu j}}\left\{\exp\left[i\int_\Sigma d^2z(k_{\mu j}+\rho_{\mu j}\frac{\partial}{\partial z})X^\mu(z,\bar{z})\delta^2(z-z_j)\right]\right\}_{\rho_j=0}. \tag{23.1}$$

It is useful to split X into a center-of-mass part and its holomorphic and antiholomorphic parts,

$$X^\mu(z,\bar{z}) = x^\mu + \tilde{X}^\mu(z) + \bar{X}^\mu(\bar{z}),$$

and similarly we split

$$J^\mu(z,\bar{z}) = j^\mu(z) + \bar{j}^\mu(\bar{z}). \tag{23.2}$$

The Green's function also splits:

$$G(z,\omega) = -\frac{\alpha'}{2}\ln|z-\omega|^2 = -\frac{\alpha'}{2}\ln(z-\omega) - \frac{\alpha'}{2}\ln(\bar{z}-\bar{\omega}). \tag{23.3}$$

Explicitly, our source terms splits as

$$j_\mu(z) = i\sum_{j=1}^n \delta^2(z-z_j)\left(\frac{1}{2}k_{j\mu}+\rho_{j\mu}\frac{\partial}{\partial z}\right). \tag{23.4}$$

As usual, our theory comes with two basically decoupled sectors. The amplitude may be written as

$$A_n = g_c^{n-2}|z_1-z_2|^2|z_2-z_3|^2|z_3-z_1|^2\delta^{26}\left(\sum_{j=1}^n k_{\mu j}\right)\int d^2z_4\ldots d^2z_n\,\epsilon^{(1)}_{\mu_1\nu_1}\ldots\epsilon^{(n)}_{\mu_n\nu_n}$$
$$\times\left\langle\prod_{j=1}^n\partial\tilde{X}^{\mu_j}(z_j)e^{ik_j\cdot\tilde{X}(z_j)}\right\rangle\left\langle\prod_{j=1}^n\bar{\partial}\bar{X}^{\mu_j}(\bar{z}_j)e^{ik_j\cdot\bar{X}(\bar{z}_j)}\right\rangle$$

where

$$\left\langle\prod_{j=1}^n\partial\tilde{X}^{\mu_j}(z_j)e^{ik_j\cdot X(z_j)}\right\rangle = \frac{1}{i^n}\frac{\partial^n}{\partial\rho_{1\mu_1}\ldots\partial\rho_{n\mu_n}}W[j]|_{\rho_1=\rho_2=\ldots=0} \tag{23.5}$$

and

$$W[j] = \exp\left(\frac{1}{2}\int_{\Sigma\times\Sigma}d^2zd^2\omega\,j(z)j(\omega)G(z,\omega)\right) \tag{23.6}$$

is an action with sources. We can think of W as a generating functional, just like in QFT. Here, $G(z,\omega) = -\frac{\alpha'}{2}\ln(z-\omega)$. If we do the integral, we find that

$$W[j] = \prod_{i<j}|z_i-z_j|^{\alpha'k_i\cdot k_j/2}\times\exp\left(\frac{\alpha'}{2}\sum_{i<j}\frac{\rho_i\cdot\rho_j}{(z_i-z_j)^2}+\frac{\alpha'}{2}\sum_{i\neq j}\frac{k_i\cdot\rho_j}{z_i-z_j}\right). \tag{23.7}$$

Actually taking the derivatives is a bit of a pain for many-particle scattering amplitudes, but in principle it is tractable. We can also compute amplitudes using a Wick contraction method– it's a matter of taste which way is preferable.

[18]"We find that... we find that we probably should not start this now." –R.A. Reid-Edwards

Example 23.8. Let's consider three-point graviton scattering. Here, the polarization vectors $\epsilon_{\mu\nu}$ are symmetric and traceless to represent gravitons. Recall that three-point interactions are nice since momentum conservation and the ghosts simplify our problem a bit. For three gravitons,

$$\alpha' k_1 \cdot k_2 = \frac{\alpha'}{2}(k_1 + k_2)^2 = \frac{\alpha'}{2}k_3^2 = 0. \tag{23.9}$$

We therefore have

$$|z_1 - z_j|^{\alpha' k_1 \cdot} = 1, \tag{23.10}$$

and so

$$\left\langle \prod_{j=1}^{3} \partial \tilde{X}^{\mu_j}(z - J) e^{ik_j \cdot X(z_j)} \right\rangle = \left(\frac{\alpha'}{2}\right)^2 \frac{T^{\mu_1\mu_2\mu_3}}{(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)} \tag{23.11}$$

with

$$T^{\mu_1\mu_2\mu_3} = \eta^{\mu_1\mu_2}k_2^{\mu_3} + \eta^{\mu_2\mu_3}k_3^{\mu_1} + \eta^{\mu_3\mu_1}k_1^{\mu_2} + \frac{\alpha'}{2}k_3^{\mu_1}k_1^{\mu_2}k_2^{\mu_2}, \tag{23.12}$$

where we used identities like $k^{\mu_1}\epsilon^{(1)}_{\mu_1\nu_1} = 0$ to simplify. Thus the three-point graviton scattering amplitude comes out to

$$A_3 = g_c(2\pi)^{26}\delta^{26}\left(\sum_{j=1}^{3} k_{\mu j}\right)\epsilon_{\mu_1\nu_1}\epsilon_{\mu_2\nu_2}\epsilon_{\mu_3\nu_3}T^{\mu_1\mu_2\mu_3}T^{\nu_1\nu_2\nu_3}. \tag{23.13}$$

One can show that this agrees with (tree-level) perturbation theory in general relativity, where there's some identification between g_c and the gravitational constant G_N in 26 dimensions. However, there might also be interactions at tree level in string theory, higher derivative corrections to the Einstein equations.

One loop This is non-examinable material (no exam questions on loop corrections) but it's certainly the next natural step after discussing tree-level amplitudes. We integrate over the moduli space $\mathcal{M}_1$, i.e. the moduli space of the torus (genus 1 Riemann surfaces). Thus

$$\mathcal{M}_1 = \{\tau = \tau_1 + i\tau_2 | \tau_2 > 0; -\frac{1}{2} \leq \tau_1 \leq \frac{1}{2}, |\tau| \geq 1\} \tag{23.14}$$

where $z \sim z + \tau$. At one loop, the amplitude is given by

$$A_n = \frac{1}{|U(1) \times U(1)|}\int_{\mathcal{M}_1}\frac{d^2\tau}{\tau_2}\int \mathcal{D}b\mathcal{D}\bar{b}\mathcal{D}c\mathcal{D}\bar{c}(\mu_\tau|b)(\bar{\mu}_\tau|\bar{b})e^{-S_{\text{gh}}[b,c]}\langle V_1 \ldots V_n\rangle_X\, c(z_1)\bar{c}(\bar{z}_1). \tag{23.15}$$

That is, this prefactor is equivalent to dividing out by the surface area of the torus. The weird inner products are then

$$(\mu_\tau|b) = \frac{1}{2\pi}\int_\Sigma d^2z\left(\frac{\partial h_{\bar{z}\bar{z}}}{\partial\tau}\right)b_{zz}. \tag{23.16}$$

We can compute this explicitly. Consider the worldsheet metric

$$h_{ab} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \tag{23.17}$$

under the deformation $h_{\bar{z}\bar{z}} = 0 \to \epsilon$. We find that

$$ds^2 = dzd\bar{z} \to (1 + \epsilon + \bar{\epsilon})dz'd\bar{z}' + O(\epsilon^2), \tag{23.18}$$

with $z' = z + \epsilon(\bar{z} - \bar{z}) + O(\epsilon^2)$. Looking at $z' \sim z' + \tau'$, we see that

$$\tau' = \tau + \epsilon(\bar{\tau}_\tau) + O(\epsilon^2). \tag{23.19}$$

We know $\frac{\partial h_{\bar{z}\bar{z}}}{\partial\epsilon}$ and $\frac{\partial\tau}{\partial\epsilon}$ and we have

$$\partial_\tau h_{\bar{z}\bar{z}} = \frac{i}{2\tau_2}. \tag{23.20}$$

Compared with tree level, we have b-ghost insertions in the ghost path integral.

However, observe that the Green's function also changes now that the topology of our Riemann surface has changed. $G(z,\omega)$ is more complicated due to periodicity requirements. Such requirements will lead us to introduce "theta functions" from the study of partial differential equations on manifolds of interesting topology.

We may ask whether our theory has divergences after we introduce loops. It turns out that it does, but these aren't UV divergences– they are associated to a tachyon, which is not present in the superstring theory. It seems that this theory is perturbatively finite to all orders.

Lecture 24.

Wednesday, March 13, 2019

Today we'll resolve some lingering questions about string theory in curved spacetime, starting with remarks about $\alpha' \sim l_s^2$. We'll think about α' corrections to general relativity, for instance.

Consider modifying the Minkowski metric to a general metric,

$$\eta_{\mu\nu} \to g_{\mu\nu}(X). \tag{24.1}$$

In general there could be some other fields in our theory like a B-field $B_{\mu\nu}(X)$ and a dilaton $\phi(X)$. Hence the action of our theory gets contributions from all of these,

$$S[X,h] = S_p[X,h] + S_B[X,h] + S_d[X,h] \tag{24.2}$$

where p indicates the Polyakov action,

$$S_p = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{h}h^{ab}g_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu. \tag{24.3}$$

Since the components $g_{\mu\nu}$ now depend explicitly on X, this action and theory are in principle highly nonlinear.

To have a chance at solving this, we expand $X^\mu = X_0^\mu + \eta^\mu$, where X_0^μ is a classical solution and η^μ is a quantum correction. For instance,

$$g_{\mu\nu}(X) = G_{\mu\nu}(X_0^\mu) + \frac{1}{3}R_{\mu\lambda\sigma\nu}(X_0)\eta^\lambda\eta^\sigma + \dots \tag{24.4}$$

where one could show this e.g. using Riemann normal coordinates. Hence the Polyakov action takes the form

$$S_p[X+\eta] = S_p[X_0] - \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{h}h^{ab}\delta_{ij}\nabla_a\eta^i\nabla_b\eta^j - \frac{1}{4\pi\alpha'}\int_\Sigma d^n\sigma\sqrt{h}h^{ab}R_{\mu i j\nu}(X_0)\partial_a X_0^\mu\partial^a X_0^\nu\eta^i\eta^j + \dots \tag{24.5}$$

with lots of other corrections to higher powers in η. This first term looks rather like a propagator, while the second is some sort of four-point vertex.

Exact solutions are rare, unless we are lucky or clever.[19] Hence we must resort to perturbation theory– worldsheet perturbation theory.

In flat spacetime, we had some nice properties of the stress tensor:

$$T_{ab} = 0, \quad T_{++} = T_{--} = 0, \quad T_{+-} = 0 = \mathrm{Tr}(T_{ab}). \tag{24.6}$$

Is it still true that $\langle T_{+-}\rangle = 0$ when we do perturbation theory? Let us start with the conservation law,

$$\nabla_a\langle T_{ab}\rangle = 0, \tag{24.7}$$

so that

$$\nabla^+\langle T_{++}\rangle + \nabla^-\langle T_{-+}\rangle = 0. \tag{24.8}$$

Switching to momentum space on Σ with the momenta q_-, q_+, we have

$$q_-\langle T_{++}\rangle + q_+\langle T_{-+}\rangle = 0. \tag{24.9}$$

The computations are not too illuminating, but we'll outline the proof here. We're interested in

$$\langle T_{++}\rangle = \int \mathcal{D}\eta\, T_{++}e^{-S_p[\eta]}. \tag{24.10}$$

To perform perturbation theory, we'll need to expand in something dimensionless, namely α' divided by some length scale squared. The relevant length scale will be given by the curvature of the background

[19]"In this business, we're more often lucky than clever. Maybe that will be different in the future– that's up to you lot." –R.A. Reid-Edwards

spacetime. That is, we expect these corrections to be valid when α' is small compared to the curvature of the background.

Contributions to $\langle T_{++}\rangle$ include some loop diagrams like

$$\left\langle \partial_+\eta^i\partial_+\eta_i \int_\Sigma d^2\sigma' R_{\mu ij\nu}\partial_a X_0^\mu(\sigma')\partial^a X_0^\nu(\sigma')\eta^i(\sigma')\eta^j(\sigma')\right\rangle. \tag{24.11}$$

We can rewrite this as

$$\langle T_{++}\rangle = -\frac{1}{4}\frac{q_+}{q_-}R_{\mu ij\nu}\partial_a X_0^\mu \partial^a X_0^\nu \eta^{ij} \tag{24.12}$$

where η^{ij} is just the Minkowski metric (unrelated to our perturbations). Similarly,

$$\langle T_{-+}\rangle = \frac{1}{4}R_{\mu\nu}\partial_a X_0^\mu \partial_b X^\nu h^{ab}. \tag{24.13}$$

Critically, this is perturbation theory on the worldsheet, not in terms of string scattering. According to this calculation, there's now no guarantee that the off-diagonal elements vanish, though this is okay if $R_{\mu\nu}(X_0) = 0$.

Moreover, we could get contributions from the B-field and the dilaton:

$$S_B = -\frac{i}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{h}\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu B_{\mu\nu}(X) \tag{24.14}$$

$$S_\phi = -\frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{h}R_\Sigma \phi(X), \tag{24.15}$$

where ϵ^{ab} is the totally antisymmetric rank two tensor. Taking the entire action to be the sum of a Polyakov action and the B-field and dilaton contributions, we get more Feynman rules and more corrections. If we calculate these corrections, we find that

$$\begin{aligned}\langle T_{-+}\rangle &= \frac{1}{4}\left(R_{\mu\nu} - \frac{1}{4}H^2_{\mu\nu} + 2\nabla_\mu\nabla_n u\phi\right)\partial_a X_0^\mu\partial_b X_0^\nu h^{ab}\\ &+ \frac{1}{4}\left(\nabla^\lambda H_{\lambda\ muv} - 2\nabla^\lambda\phi H_{\lambda\mu\nu}\right)\partial_a X_0^\mu \partial_b X_0^\nu \epsilon^{ab}\\ &+ \frac{1}{\alpha'}\left(\frac{D}{2} + \frac{\alpha'}{2}\left(-R + \frac{H^2}{12} + 4(\nabla\phi)^2 - 4\nabla^2\phi\right)\right)\partial_a X_0^\mu \partial_b X_{0\mu} h^{ab} + O(\alpha'^2).\end{aligned}$$

These lines represent contributions from the metric, the B-field, and the dilaton respectively. We haven't included the ghosts– as it turns out, they are largely indifferent to whether we're working in flat or curved spacetime, and will just modify the $D/2$ term in the dilaton contribution (the third line) with a $D-26$. Also, note that we have to work to two-loop order in the dilaton because it does not come with a $1/\alpha'$ like the other actions.

For these to be classically compatible, we may think of the vanishing of the trace of the stress tensor as giving us three equations of motion for $g_{\mu\nu}(X_0), B_{\mu\nu}(X_0), \phi(X_0)$. These can be derived from the action

$$S = \int d^{26}x\sqrt{g}e^{-2\phi}\left(R + 4(\nabla\phi)^2 - \frac{1}{12}H^2\right). \tag{24.16}$$

In fact, the equations of motion which emerge are precisely the equations of general relativity. This is remarkable. From a quantum consistency condition, we have derived Einstein's equations.

Can we go further? It takes more loop corrections, but it can be done. By performing the two-loop calculations for $S_p[X]$, we find that

$$R_{\mu\nu} + \frac{\alpha'}{2}R_{\mu k\lambda\sigma}R_\nu{}^{k\lambda\sigma} + \ldots = 0, \tag{24.17}$$

which represent higher-order "corrections" to general relativity.

The problem is then to find a background metric, B-field, and dilaton that solve the equations of motion to *all orders* in loop calculations, and then do perturbation theory. This is really hard– it requires us to perform high-order loop calculations just to write down the equations of motion. We don't have a general scheme for solving these equations, so we will have to be clever, lucky, or both.

T-duality There is however one trick we have to solve the equations. Consider a spacetime

$$\mathcal{M}_{26} = \mathbb{R}_{1,24} \times S^1, \tag{24.18}$$

exact to all orders in α' with B, ϕ constant. Let the 25th (i.e. the periodic S^1) coordinate satisfy

$$X^{25}(\sigma + 2\pi\tau) = X^{26}(\sigma, \tau) + 2\pi R m, m \in \mathbb{Z}. \tag{24.19}$$

Thus

$$X^{25}(\sigma, \tau) = x^{25} + \alpha' p^{25}\tau + mR\sigma + \sum_{n\neq 0}(\text{oscillations}). \tag{24.20}$$

Also, we'll take $p^{25} = n/R, n \in \mathbb{Z}$, so that the momentum is quantized by periodicity. The mass spectrum of such a theory is

$$\alpha' M^2 = \alpha' \frac{n^2}{R^2} + \frac{1}{\alpha'} m^2 R^2 + (N + \bar{N} - 2). \tag{24.21}$$

Hence there is energy in the oscillations (the last term, where $N, \bar{N}$ are oscillation number operators) and also the winding number m.

We notice there's a symmetry of this spectrum. If we interchange

$$m \leftrightarrow n,$$
$$\frac{\alpha'}{R^2} \leftrightarrow \frac{R^2}{\alpha'},$$

we see that the spectrum is left unchanged. In fact, it goes further– this is actually a symmetry of the entire theory. That is, by exchanging the momentum number and the winding number, and then inverting the radius, we get an equivalent description.

These statements generalize to other spacetimes with interesting geometries and topologies– not just tori and handlebodies but Calabi-Yau manifolds. This tells us that the string sees a structure not present in standard Riemannian geometry. In some sense, this is our first hint of a structure of spacetime beyond Einstein's geometric description, and it has led to much interesting research into what a (or the) theory of quantum gravity might look like.