GAUGE/GRAVITY DUALITY

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LAST UPDATED APRIL 30, 2019

These notes were taken for the *Gauge/Gravity Duality* course taught by Aron Wall at the University of Cambridge as part of the Mathematical Tripos Part III in Easter Term 2018. I live-TEXed them using Overleaf, and as such there may be typos; please send questions, comments, complaints, and corrections to itel2@cam.ac.uk.
Many thanks to Arun Debray for the LATEX template for these lecture notes: as of the time of writing, you can find him at https://web.ma.utexas.edu/users/a.debray/.

Contents

Lecture 1.

Thursday, April 25, 2019

Size matters... not? To motivate our course, let us start with a story from Galileo. The astronomer Galileo wrote a treatise entitled "Two New Sciences." One of these was the heliocentric model of the solar system, and the other was an early version of the atomic theory. Galileo's work recognized that because of area-volume laws, the laws of physics seem to have a scale dependence. Building a scale model of a cathedral is very different than building a full-sized cathedral because mass scales with volume, whereas the strength of objects (based on local atomic interactions) scales with area.

On the other hand, there are a class of theories which follow the precepts of another great philosopher, Yoda, who stated in The Empire Strikes Back that "Size matters not." These are the so-called *conformal field theories*. We could have some object and then scale it up, and it would behave exactly the same. In fact, we might go so far as to posit that size is an extra *dimension* of our system! We shall call it z.

But maybe we object to this idea on a few grounds.

- *Objection #1.* Real dimensions should have conjugate momenta. In fact, Noether's theorem tells us that under the symmetry $\ln z \to \ln z + C$, we get a conserved quantity p_z corresponding to the dilation symmetry.
- *Objection #2.* We can rotate objects. This would require the group of Poincaré symmetries and dilations to be augmented to some bigger group with d extra generators. Indeed, this happens and we get the special conformal group.
- *Objection #3.* The speed of light is constant. This seems to tell us that we could measure distances by measuring the time that light takes to travel over the same scaled-up object. But by taking a cue from Einstein, we can answer this objection by saying that clocks run slower for bigger objects. That is, there is a redshift factor $ds \sim ds/z$. Our metric would look like
$$ds^2 = \frac{dz^2 + {}^{(d)}\eta_{ij}dx_i dx_j}{z^2}, \tag{1.1}$$
and in fact this is the unique metric which satisfies the conformal symmetries. This is precisely the metric of Anti-de Sitter space (AdS).
- *Objection #4.* We can't put objects on top of each other without them interfering (e.g. if we scale some things up). Fermions are the obvious case, where we expect to run into trouble with the Pauli

exclusion principle. However, there is a loophole. The objects won't interact much *if* there are a large number N of particle species, especially if objects are required to be in singlets (e.g. a gauge theory $SU(N_c)$, with $N \sim N_c^2$).

- *Objection #5.* If N is finite, there will still be a small interaction over large Δz, which implies the existence of a long-range universal force. But this looks like gravity! So things are going pretty well.
- *Objection #6.* When the gauge theory is heated up (e.g. we cram a lot of energy into a small space), we get "deconfinement," leading to a hyperentropic object with huge $O(N)$ entropy. What we've got is none other than a black hole.

This is the motivation for the AdS/CFT duality (originally posited by Maldacena and elaborated by Witten and others).

$$\mathrm{CFT}_d \leftrightarrow \mathrm{AdS}_{d+1} \times F \tag{1.2}$$

where F is a compact fiber. On the left lives an ordinary quantum field theory without gravity, and on the right lives a full theory of quantum gravity (typically a string theory). The large N limit of the QFT corresponds to the classical limit (where the Planck length is much smaller than the curvature scale, $l_p \ll R_{\mathrm{AdS}}$). Moreover, the QFT must be strongly coupled in order to produce local (pointlike) fields on scales below the AdS scale ($l_s \ll R_{\mathrm{Ads}}$, with l_s the string length). Finally, QFT has a set of known axioms (although they are hard to study at strong coupling), whereas we don't know how to treat quantum gravity nonperturbatively. Hence we can either use this to learn about strongly coupled field theories by studying general relativity, or we can try to learn about quantum gravity from the axioms of QFT.

Let's try to elaborate this idea a little more. In a 4d Maxwell theory, we have an action

$$I = \frac{1}{4}\int d^4x\sqrt{-g}F_{ab}F_{cd}g^{ac}g^{bd}, \tag{1.3}$$

which is invariant under the Weyl transformations $g_{ab} \to \Omega^2(x)g_{ab}$. This is because the two factors of the inverse metric each scale as Ω^{-2}, and the determinant of the metric is like a volume. Since g is like a length squared, $\sqrt{-g} \sim \Omega^d$, so these factors will cancel when $d = 4$. If the conformal symmetry holds in a QFT, we call it a CFT.

$$\frac{\delta \ln Z}{\delta\Omega(x)} \sim \langle T\rangle \sim \text{curvature}, \tag{1.4}$$

i.e. variations of the partition function with respect to the factor Ω are proportional to the trace of the stress tensor, which scales with the curvature. For $d = 2, T \sim cR$, while for $d = 4, T \sim a(GB) + cC^2$ where GB is the Gauss-Bonnet term and C is the Weyl tensor.

Conformal symmetry requires that the beta functions of the theory vanish. That is, if our couplings generically depend on scale, $\lambda(z)$, we require that

$$\frac{d\lambda_i}{d\ln z} = \beta_i = 0. \tag{1.5}$$

These usually represent some special isolated points in the space of theories, except in situations where there is enough supersymmetry to produce e.g. a 1-parameter family of CFTs. Hence many of the theories of interest in AdS/CFT are supersymmetric.

Lecture 2.

Friday, April 26, 2019

Recall from e.g. *General Relativity* or *String Theory* that a Weyl transformation is a rescaling of the metric

$$g_{ab} \to \Omega^2(x)g_{ab}. \tag{2.1}$$

Note that this is slightly different from a conformal symmetry. A *conformal symmetry* is a diffeomorphism ξ^a that preserves g_{ab} up to a Weyl factor Ω.

To describe symmetries we shall need a Killing vector ξ^a, which by definition satisfies $\nabla_a\xi_b + \xi_b\xi_a = 0$. (That is, $\mathcal{L}_\xi g = 0$.) Transformations corresponding to Killing vectors are isometries leaving the metric unchanged. More generally, we might like a *conformal Killing vector*, which satisfies

$$\nabla_a\xi_b + \nabla_b\xi_a - \frac{2}{d}g_{ab}(\nabla\cdot\xi) = 0. \tag{2.2}$$

d translations	$x^a \to x^a + c^a$	$p_a \equiv -i\frac{\partial}{\partial x^a} = -i\partial_a$
$\frac{d(d-1)}{2}$ Lorentz	$x^a \to \Lambda^a{}_b x^b (\Lambda^\dagger \Lambda = 1)$	$M_{ab} = i(x_a \partial_b - x_b \partial_a)$
1 scaling/dilation	$x^a \to \Omega x^a$	$D = -i(x \cdot \partial)$
d special	$x^a \to \frac{x^a - x^2 b^a}{(x-x^2b)^2}, \Omega = \frac{x^2}{(x-x^2b)^2}$	$K_a = i(x^2\partial_a - 2x_a(x\cdot\partial))$.

TABLE 1. A list of the conformal transformations, how points x^a transform, and their generators written in differential operator form.

The factor of $2/d$ comes from the fact that the trace of the metric is $g_{ab}g^{ab} = d$, so that when we take the trace of this, we will get something that is always zero.

In Minkowski, $g_{ab} = \eta_{ab}$. Notice that we can rescale the null coordinates (e.g. the EF coordinates u, v go to $f(u), g(v)$ for some monotonic functions f, g). In $d = 1, 2$, there are infinitely many generators for Minkowski. But for $d > 2$, there are precisely $\frac{1}{2}(d+1)(d+2)$ generators.

For a general d dimensions we have d translations, $\frac{d(d-1)}{2}$ Lorentz transformations, 1 scaling/dilation transformation, and d "special" conformal transformations, as seen in Table 1. Here, we've used an adjoint notation, $\Lambda^\dagger \Lambda^c{}_d g_{ca} g^{db}$. Notice that an inversion is a transformation $x^a \to \frac{x^a}{x^2}, \Omega = 1/x^2$ (since $1/x^a = x^a/x^2$. Hence the special conformal transformations are equivalent to a translation, an inversion, and another translation.

Recall that translation symmetry gives us a conserved stress tensor, $\nabla_a T^{ab} = 0$. We claim the current

$$J^a = T^{ab}\xi_b \tag{2.3}$$

is conserved. This follows since

$$\nabla_a J^a = (\nabla_a Tab)\xi_b + T^{ab}\nabla_a \xi_b \tag{2.4}$$

$$= T^{ab}\nabla_{(a}\xi_{b)}, \tag{2.5}$$

where the first term was zero because of the regular stress-energy conservation. We've symmetrized the second term by the symmetry of the stress tensor, so the second term is just zero for a Killing vector. As the stress tensor is traceless, we can certainly subtract off its trace for free so that

$$T^{ab}\nabla_{(a}\xi_{b)} - \frac{1}{d}T(\nabla \cdot \xi) = 0 \tag{2.6}$$

for a conformal killing vector. We may compute some commutators to get all the relations between the generators:

- Usual Poincaré commutators
- $[D, P_a] = iP_a$
- $[D, K_a] = -iK_a$
- $[K_a, P_b] = 2i(\eta_{ab}D - M_{ab})$
- $[K_a, M_{bc}] = i(\eta_{ab}K_c - \eta - acK_b)$

with all others zero. This is equivalent to the symmetries of $SO(d, 2)$.

We've got to be a little careful about our inversion transformations, since an inversion brings infinity in to the origin and vice versa. So this may not be a real symmetry of Minkowski since it "pushes infinity around." Instead, we should look at the *maximal conformal extension*. For the Euclidean plane $\mathbb{R}^2$, this is a 2-sphere with a "point at infinity." We can see the isomorphism by putting the 2-sphere on the plane and stereographically projecting from the "point at infinity" on the north pole.

On the other hand, the Lorentzian case is more subtle. In Lorentz signature, $|x| = 0$ on the light cone. What happens is we must go to the maximal conformal extension of Minkowski, which is $S_{d-1} \times \mathbb{R}$. It is a cylinder. Our conformal transformations have the effect of flipping or shifting which patch of the cylinder we are looking at.

We might additionally be interested in representations of the conformal group– how do conformal transformations act on fields? We shall focus on unitary, positive-energy irreps, which come from fields (i.e.

local operators). In QFT, we construct states by acting on the vacuum with some operator smeared out in spacetime by a suitable test function,

$$|\psi\rangle = \int d^d x\, f(x)\mathcal{O}(x)|0\rangle. \tag{2.7}$$

The operators $\mathcal{O}$ are classified by $SO(d)$ spin and weight of a "primary field" $\mathcal{O}$. By primary field (as we saw in *String Theory*), we mean a field which transforms as $\phi \to \Omega^\Delta \phi$ under conformal transformations, where Δ is called the weight. The derivatives $\partial^n\phi$ are called descendants, and they transform with derivatives of Ω. (This is slightly different from the $2D$ version of primary, where we are additionally interested in Virasoro). A gauge-invariant operator $\mathcal{O}$ must also satisfy "unitarity bounds." The details depend on the dimension d, and some concrete examples are as follows:

$$\Delta_\phi \geq \frac{d-2}{2} \text{ (scalar) except identity,} \tag{2.8}$$

$$\Delta_\psi \geq \frac{d-1}{2} \text{ (spinor)} \tag{2.9}$$

$$\Delta_J \geq d-1 \text{ (vector)} \tag{2.10}$$

$$\Delta_T \geq d \text{ (symmetric traceless tensor).} \tag{2.11}$$

These bounds are saturated for $\Box\phi = 0, \not{\partial}\psi = 0, \nabla_a J^a = 0, \nabla_a T^{ab} = 0$. Notice once things start interacting, we get anomalous dimensions.

Lecture 3.

Monday, April 29, 2019

Today we shall continue our discussion of conformal symmetry. Note that if you were in *Black Holes*, you might recall that when doing QFT in curved spacetime, the vacuum depends on our choice of reference frame. However, when we do the maximal conformal extension of Minkowski, no such ambiguity is present. The vacuum on the cylinder is the same as the vacuum in the original Minkowski space.

The picture of the cylinder is also the origin of the "operator-state correspondence," in which we may rewrite the time coordinate with Euclidean signature by a Wick rotation $\tau = it$. Hence the cylinder is isomorphic to a punctured plane (e.g. $S_1 \times \mathbb{R} \cong \mathbb{R}^2 \setminus \{0\}$) and we can set up a correspondence between states on the cylinder and operators inserted as boundary conditions on the punctured plane, with time ordering corresponding to radial ordering on the plane.

Recall that last time, we said that scalars have weights obeying

$$\Delta_\phi \geq \frac{d-2}{2} \text{ (scalar)} \tag{3.1}$$

except the identity, which clearly has weight zero. This discussion of weight also requires that we set $\hbar = c = 1$, as is conventional, so that there is only one scale in the game. A free scalar field theory has terms in the action like $\int \partial_\mu\phi\partial^\mu\phi d^d x$. Since ∂_μ has mass dimension $+1$ and $d^d x$ has dimension $-d$, our scalars must have (naive) dimension $\frac{d-2}{2}$.

One may then write down the 2-point (correlation) function, which by dimensional analysis can only look like

$$\langle 0|\phi(y)\phi(x)|0\rangle = \frac{1}{|x-y|^{2\Delta}}. \tag{3.2}$$

That is, it must have dimensions of $d-2$, and it can only depend on the separation between x and y (under translation symmetry). In fact, this needs to be slightly modified to

$$\langle 0|\phi(y)\phi(x)|0\rangle = \frac{1}{|x-y+i\epsilon\hat{t}|^{2\Delta}}, \tag{3.3}$$

where $\hat{t}$ allows us to slightly shift a pole to imaginary time. We haven't fully explained this correction yet, but we'll see why this is the correct correlation function later. The upshot is that the form of the correlation function is scaled by the scaling symmetry.

The three-point function is also fixed by the full conformal symmetry. Note that the fields in the three point function could a priori have different weights $\Delta_1, \Delta_2, \Delta_3$:

$$\langle 0|\phi_3(z)\phi_2(y)\phi_1(x)|0\rangle = \frac{C}{|x-y|^{\Delta_1+\Delta_2-\Delta_3}|x-z|^{\Delta_1+\Delta_3-\Delta_2}|y-z|^{\Delta_2+\Delta_3-\Delta_1}}. \tag{3.4}$$

n.b. the four-point and higher functions cannot be derived directly from the conformal symmetry. Specifying the three-point function is sufficient to completely define the CFT. However, note that in general the problem is overdetermined– if we pick a three-point function at random, it probably won't correspond to a meaningful CFT. The program of trying to determine which three-point functions will give valid CFTs based on self-consistency conditions is known as the *conformal bootstrap*.

Spectral decomposition The spectral decomposition is a Fourier transform which allows us to work in a nicer basis. For a state

$$|\psi\rangle = \int f(x)\phi(x)d^d x, \tag{3.5}$$

we can define the momentum-space wavefunction $\phi(p)$ as

$$\phi(p) = \int e^{ip\cdot x}\phi(x)d^d x. \tag{3.6}$$

Suppose we want to evaluate

$$\langle 0|\phi(-q)\phi(p)|0\rangle \tag{3.7}$$

up to overall normalization. We can do this with dimensional analysis:

$$\langle 0|\phi(-q)\phi(p)|0\rangle \propto \delta^d(p-q)\theta(E)|p|^{2\Delta-d}, \tag{3.8}$$

with $\Delta > \frac{d-2}{2}$. This is because the Fourier transform takes us to a wavefunction which lives in inverse momentum space. Hence we get two contributions of d from the $d^d x$ integrals, one of which is absorbed by the momentum-conserving delta function.

Note that for $\Delta = \frac{d-2}{2}$, we might naively assume that the $|p|^{2\Delta-d}$ dependence turns into $|p|^{-2}$. However, this is wrong. In fact, we get a delta function instead, $\delta(p^2)$. We can think of this as coming from the fact that free fields (which saturate this bound) satisfy $\Box\phi = 0$, which in momentum space corresponds to the constraint $p^2 = 0$. States with $\Delta < \frac{d-2}{2}$ will not satisfy unitarity consistent with some positivity constraint.

Hence for the special case $\Delta = \frac{d-2}{2}$, we only get states on the light cone. Taking a metric

$$ds^2 = -du dv + dy_i^2 \tag{3.9}$$

with $d-2$ coordinates y_i and two null coordinates v, u, we have $\frac{1}{p^2} = \frac{1}{p_{y_i}^2 - 2p_u p_v}$. For states which satisfy and do not saturate the unitarity bound, $\Delta > \frac{d-2}{2}$, our states fill the interior of the future light cone. There are no normalizable states $|\psi\rangle$ with $E^2 = \mathbf{p}^2$ exactly, though states may come arbitrarily close to being null. Thus all the physical states do not obey a wave equation *unless it is in one higher dimension*.

AdS spacetimes Before focusing on AdS, we'll start more generally with maximally symmetric spacetimes in D dimensions (where eventually we shall set $D = d+1$). In Minkowski, we have at most $\frac{D(D+1)}{2}$ Killing vectors in a D-dimensional spacetime. However, we can also set this many constraints on our spacetime by specifying the curvature, $R_{ab} = g_{ab}\Lambda$ for Λ some cosmological constant. (The counting works out since R_{ab} is symmetric.)

There are two interesting cases. For $\Lambda > 0$, we get de Sitter space, which can be thought of as the unit hyperboloid in $D+1$ Minkowski space. However, if we take anti-de Sitter space, we instead have the unit hyperboloid in a signature $(D-1, 2)$ spacetime with two time coordinates.