QUANTUM FIELD THEORY

IAN LIM
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These notes were taken for the *Quantum Field Theory* course taught by Ben Allanach at the University of Cambridge as part of the Mathematical Tripos Part III in Michaelmas Term 2018. I live-TEXed them using Overleaf, and as such there may be typos; please send questions, comments, complaints, and corrections to itel2@cam.ac.uk.
Many thanks to Arun Debray for the LATEX template for these lecture notes: as of the time of writing, you can find him at https://web.ma.utexas.edu/users/a.debray/.

Contents

Lecture 1.

Thursday, October 4, 2018

$2 = \pi = i = -1$ *in these lectures.* –a former lecturer of Prof. Allanach's.

To begin with, some logistic points. The notes and much of the course material will be based on David Tong's QFT notes plus some of Prof. Allanach's on cross-sections and decay rates. See http://www.damtp.cam.ac.uk/user/examples/indexP3.html and in particular http://www.damtp.cam.ac.uk/user/examples/3P1l.pdf for the notes on cross-sections. In revising these notes, I'll be cross-referencing the Tong QFT notes as well as my copy of Anthony Zee's *Quantum Field Theory in a Nutshell*, which takes a different pedagogical order in starting from the path integral formalism and introducing second quantization (the approach described here) later. Any good education in QFT requires an understanding of both formalisms, and we'll see the path integral next term in *Advanced Quantum Field Theory*.[1]

After Tuesday's lecture, we'll be assigned one of four course tutors:

[1]Note that the path integral formulation from *Statistical Field Theory*, also taught this term, is precisely equivalent to the path integral that appears in QFT under the identification of one of the Euclidean dimensions of a statistical field theory with the imaginary time dimension of a QFT. This will be more obvious in hindsight.

- Francesco Careschi, fc435@cam.ac.uk
- Muntazir Abidi, sma74
- Khim Leong, lkw30
- Stefano Vergari, sv408

Also, the Saturday, November 24th lecture has been moved to 1 PM Monday 26 November, still in MR2. That's it for logistics for now.

Definition 1.1. A *quantum field theory* (QFT) is a field theory with an infinite number of degrees of freedom (d.o.f.). Recall that a field is a function defined at all points in space and time (e.g. air temperature is a scalar field in a room wherever there's air). The states in QFT are in general multi-particle states.

Special relativity tells us that energy can be converted into mass, and so particles are produced and destroyed in interactions (particle number is in general not conserved). This reveals a conflict between SR and quantum mechanics, where particle number is fixed. Interaction forces in our theory then come from additional structure in the theory, depending on things like

- symmetry
- locality
- "renormalization group flow."

Definition 1.2. A *free QFT* is a QFT that has particles but no interactions. The classic free theory is a relativistic theory with which treats particles as excitations of infinitely many quantized harmonic oscillators.

Free theories are generally not realistic but they are important, as interacting theories can be built from these with perturbation theory. The fact we can do this means the particle interactions are somehow weak (we say these theories have *weak coupling*), but other theories of interest (e.g. the strong force) have strong coupling and cannot be described with perturbation theory.

Units in QFT In QFT, we usually set $c = \hbar = 1$. Since $[c] = [L][T]^{-1}$ and $[\hbar] = [L]^2[M][T]^{-1}$, we find that in natural units,

$$[L] = [T] = [M]^{-1} = [E]^{-1}$$

(where the last equality follows from $E = mc^2$ with $c = 1$, for example). We often just pick one unit, e.g. an energy scale like eV, and describe everything else in terms of powers of that unit. To convert back to metres[2] or seconds, just reinsert the relevant powers of c and $\hbar$.

Example 1.3. The de Broglie wavelength of a particle is given by $\lambda = \hbar/(mc)$. An electron has mass $m_e \simeq 10^6\,\text{eV}$, so $\lambda_e = 2 \times 10^{-12}\,\text{m}$.

If a quantity x has dimension $(mass)^d$, we write $[x] = d$, e.g.

$$G = \frac{\hbar c}{M_p^2} \implies [G] = -2.$$

$M_p \approx 10^{19}\,\text{GeV}$ corresponds to the Planck scale, $\lambda_p \sim 10^{-33}\,\text{cm}$, the length/energy scales where we expect quantum gravitational effects to become relevant. We note that the problems associated with relativising the Schrödinger equation are fixed in QFT by particle creation and annihilation.

Classical field theory Before we do QFT, let's review classical field theory. In classical particle mechanics, we have a finite number of generalized coordinates $q_a(t)$ (where a is a label telling you which coordinate you're talking about), and in general they are a function of time t. But in field theory, we instead have continuous fields $\phi_a(\mathbf{x}, t)$, where a labels the field in question and $\mathbf{x}$ is no longer a coordinate but a label like a.[3]

In our classical field theory, there are now an infinite number of degrees of freedom, at least one for each position in space $\mathbf{x}$, so position has been demoted from a dynamical variable to a mere label.

[2] As a USAmerican, I am likely to be bewilderingly inconsistent with regards to using American versus British spellings. Please bear with me.

[3] See for instance Anthony Zee's *QFT in a Nutshell* to see a more detailed description of how we go from discrete to continuous systems.

Example 1.4. The classical electromagnetic field is a vector field with components $E_i(x,t), B_i(x,t)$ such that $i,j,k \in \{1,2,3\}$ label spatial directions. In fact, these six fields are derived from four fields (or rather four field components), the four-potential $A_\mu(x,t) = (\phi, \mathbf{A})$ where $\mu \in \{0,1,2,3\}$.

Then the classical fields are simply related to the four-potential by

$$E_i = \frac{\partial A_i}{\partial t} - \frac{\partial A_0}{\partial x_i} \text{ and } B_i = \frac{1}{2}\epsilon_{ijk}\frac{\partial A_k}{\partial x_j} \tag{1.5}$$

with ϵ_{ijk} the usual Levi-Civita symbol, and where we have used the Einstein summation convention (repeated indices are summed over).

The dynamics of a field are given by a *Lagrangian* L, which is simply a function of $\phi_a(x,t), \dot{\phi}_a(x,t)$, and $\boldsymbol{\nabla}\phi_a(x,t)$. This is in precise analogy to the Lagrangian of a discrete system, which is a function of the coordinates $q_a(t)$ and their derivatives $\dot{q}_a(t)$.

Definition 1.6. We write

$$L = \int d^3x \mathcal{L}(\phi_a, \partial_\mu \phi_a), \tag{1.7}$$

where we call $\mathcal{L}$ the *Lagrangian density*, or by a common abuse of terminology simply the Lagrangian.

Definition 1.8. We may then also define the *action*

$$S \equiv \int_{t_0}^{t_1} L dt = \int d^4x \mathcal{L}(\phi_a, \partial_\mu \phi_a) \tag{1.9}$$

Let us also note that in these units we take the action S to be dimensionless, $[S] = 0$ (since it appears alone in an exponent, for instance, e^{iS}), and so since $[d^4x] = -4$ we have $[\mathcal{L}] = 4$.

The dynamical principle of classical field theory is that fields evolve such that S is stationary with respect to variations of the field that don't affect the initial or final values (boundary conditions). That is, $\delta S = 0$. A general variation of the fields produces a variation in the action

$$\delta S = \sum_a \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a}\delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta(\partial_\mu \phi_a) \right\}.$$

Integrating the second term by parts, we find that the variation in the action becomes

$$\delta S = \sum_a \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a}\delta\phi_a + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta\phi_a \right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \delta\phi_a \right\}.$$

The integral of the total derivative term vanishes for any term that decays at spatial ∞ (i.e. $\mathcal{L}$ is reasonably well-behaved) and has $\delta\phi_a(x,t_1) = \delta\phi_a(x,t_0) = 0$, as guaranteed by our boundary conditions. Therefore the boundary term goes away and we find that stationary action, $\delta S = 0$, implies the *Euler-Lagrange equations*,

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0. \tag{1.10}$$

Example 1.11. Consider the Klein-Gordon field ϕ, defined as the real-valued field ϕ which has a Lagrangian

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2. \tag{1.12}$$

Here $\eta^{\mu\nu}$ is the standard Minkowski metric[4].

To compute the Euler-Lagrange equation for this field theory, we see that

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi \text{ and } \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi.$$

The Euler-Lagrange equations then tell us that ϕ obeys the equation of motion

$$\partial_\mu \partial^\mu \phi + m^2\phi = 0,$$

which we call the *Klein-Gordon equation*. It has wave-like solutions $\phi = e^{-ipx}$ with $(-p^2 + m^2)\phi = 0$ (so that $p^2 = m^2$, which is what we expect in units where $c = 1$).

[4] We use the mostly minus convention here, but honestly the sign conventions are all arbitrary and relativity often uses the other one where time gets the minus sign.

Non-lectured aside: on functional derivatives If you're like me, you get a little anxious about taking complicated functional derivatives. The easiest way to manage these is to rewrite the Lagrangian so that all terms precisely match the form of the quantity you are taking the derivative with respect to, and remember that matching indices produce delta functions.

Here's a quick example. To compute $\frac{\partial}{\partial(\partial_\alpha\phi)}\left[\partial_\mu\phi\partial^\mu\phi\right]$, rewrite the term in the brackets as $\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ (since we are deriving with respect to a function of the form $\partial_\alpha\phi$) and make sure to take the derivative with respect to a new index not already in the expression, e.g. $\partial_\alpha\phi$. Then

$$\begin{aligned}\frac{\partial}{\partial(\partial_\alpha\phi)}\left[\partial_\mu\phi\partial^\mu\phi\right] &= \frac{\partial}{\partial(\partial_\alpha\phi)}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \\ &= \eta^{\mu\nu}(\delta^\alpha_\mu)\partial_\nu\phi + \eta^{\mu\nu}\partial_\mu\phi(\delta^\alpha_\nu) \\ &= 2\partial^\alpha\phi,\end{aligned}$$

where we have raised the index with $\eta^{\mu\nu}$ and written the final expression in terms of α using the delta function. The functional derivative effectively finds all appearances of the denominator exactly as written, including indices up or down, and replaces them with delta functions so the actual indices match. This is especially important in computing the Euler-Lagrange equations for something like Maxwell theory, where one may have to derive by $\partial_\mu A_\nu$ and both those indices must match exactly to their corresponding appearances in the Lagrangian.

No one ever taught me exactly how to approach such variational problems, so I wanted to record my strategy here for posterity. It may take a little longer than just recognizing that $\frac{\partial}{\partial(\partial_\mu\phi)}\frac{1}{2}\partial_\nu\phi\partial^\nu\phi = \partial^\mu\phi$, but this approach always works and it has the benefit of helping avoid careless mistakes like forgetting the factor of 2 in the example above.

Lecture 2.

Saturday, October 6, 2018

Last time, we derived the Euler-Lagrange equations for Lagrangian densities:

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} - \frac{\partial\mathcal{L}}{\partial\phi_a} = 0. \tag{2.1}$$

Today, we'll look at some more simple Lagrangians. We'll introduce Noether's theorem as it applies to fields and also derive the energy-momentum tensor in a field theory context.

Example 2.2. Consider the Maxwell Lagrangian,

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A^\mu)^2. \tag{2.3}$$

Plugging into the E-L equations, we find that $\frac{\partial\mathcal{L}}{\partial A_\nu} = 0$ and

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = \partial^\mu A^\nu + \eta^{-\mu\nu}\partial_\rho A^\rho. \tag{2.4}$$

Thus E-L tells us that

$$0 = -\partial^2 A^\nu + \partial^\nu(\partial_\rho A^\rho) = -\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu). \tag{2.5}$$

Defining the field strength tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, we can write the E-L equation for Maxwell as the simple

$$0 = \partial_\mu F^{\mu\nu},$$

which written explicitly is equivalent to Maxwell's equations in vacuum (we'll revisit this when we do QED).

The Lagrangians we'll consider here and afterwards are all *local*– in other words, there are no couplings $\phi(\mathbf{x},t)\phi(\mathbf{y},t)$ with $\mathbf{x}\neq\mathbf{y}$. There's no reason a priori that our Lagrangians have to take this form, but all physical Lagrangians seem to do so.

Lorentz invariance Consider the Lorentz transformation on a scalar field $\phi(x) \equiv (\phi(x^\mu)$. The coordinates x transform as $x' = \Lambda^{-1}x$ with $\Lambda^\mu_\sigma \eta^{\sigma\tau}\Lambda^\nu_\tau = \eta^{\mu\nu}$. Under Λ, our field transforms as $\phi \to \phi'$ where $\phi'(x) = \phi(x')$. Recall that Lorentz transformations generically include boosts as well as rotations in $\mathbb{R}^3$. As we've discussed in Symmetries, Fields and Particles, Lorentz transformations form a Lie group ($O(3,1)$, or specifically the proper orthochronous Lorentz group) under matrix multipication. They have a representation given on the fields (i.e. a mapping to a set of transformations on the fields which respects the group multiplication law).

For a scalar field, this is $\phi(x) \to \phi(\Lambda^{-1}x)$ (an active transformation). We could have also used a passive transformation where we re-label spacetime points: $\phi(x) \to \phi(\Lambda x)$. It doesn't matter too much– since Lorentz transformations form a group, if Λ is a Lorentz transformation, so is Λ^{-1}. In addition, most of our theories will be well-behaved and Lorentz invariant.

Definition 2.6. *Lorentz invariant* theories are ones where the action S is unchanged by Lorentz transformations.

Example 2.7. Consider the action given by

$$S = \int d^4x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi)\right],$$

where $U(\phi)$ is some potential density. $U \to U'(x) \equiv U(\phi'(x)) = U(x')$ means that U is a scalar field (check this!) and we see that

$$\partial_\mu\phi' = \frac{\partial}{\partial x^\mu}\phi(x') = \frac{\partial x'^\sigma}{\partial x^\mu}\partial'_\sigma\phi(x') = (\Lambda^{-1})^\sigma_\mu\partial'_\sigma\phi(x')$$

where $\partial'_\sigma \equiv \frac{\partial}{\partial x'^\sigma}$. Thus the kinetic term transforms as

$$\mathcal{L}_{kin} \to \mathcal{L}'_{kin} = \eta^{\mu\nu}\partial_\mu\phi'\partial_\nu\phi' = \eta^{\mu\nu}(\Lambda^{-1})^\sigma_\mu(\Lambda^{-1})^\tau_\nu\partial'_\sigma\phi(x')\partial'_\tau\phi(x') = \eta^{\sigma\tau}\partial'_\sigma\phi(x')\partial'_\tau\phi(x') = L_{kin}(x).$$

Thus we see that the action overall transforms as

$$S \to S' = \int d^4x\mathcal{L}(x') = \int d^4x\mathcal{L}(\Lambda^{-1}x).$$

Under a change of variables $u \equiv \Lambda^{-1}x$, we see that $\det(\Lambda^{-1}) = 1$ (from group theory) so the volume element is the same, $d^4y = d^4x$ and therefore

$$S' = \int d^4y\mathcal{L}(y) = S.$$

We conclude that S is invariant under Lorentz transformations.

We also remark that under a LT, a vector field A_μ transforms like $\partial_\mu\phi$, so

$$A'_\mu(x) = (\Lambda^{-1})^\sigma_\mu A_\sigma(\Lambda^{-1}x).$$

This is enough to attempt Q1 from example sheet 1.[5]

Theorem 2.8. *Every continuous symmetry of $\mathcal{L}$ gives rise to a current J^μ which is conserved, $\partial_\mu j^\mu = 0$. Each j^μ has a conserved charge $Q = \int_{\mathbb{R}^3} j^0 d^3x$.*

This conserved charge appears because $\frac{dQ}{dt} = \int_{\mathbb{R}^3} d^3x\partial_0 j^0 = -\int_{\mathbb{R}^3} d^3x\boldsymbol{\nabla}\cdot\mathbf{j} = 0$ by the divergence theorem, assuming $|\mathbf{j}| \to 0$ as $|\mathbf{x}| \to \infty$.

Let us define an infinitesimal variation of a field ϕ, $\phi(x) \to \phi'(x) = \phi(x) + \alpha\Delta\phi(x)$ with α an infinitesimal change. If S is invariant, we call this a *symmetry* of the theory.

Since S is invariant up to adding a total 4-divergence (a total derivative ∂_μ) to the Lagrangian, our symmetry doesn't affect the Euler-Lagrange equations. L transforms as

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha\partial_\mu X^\mu(x), \tag{2.9}$$

[5]Copied here for quick reference: Show directly that if $\phi(x)$ satisfies the Klein-Gordon equation, then $\phi(\Lambda^{-1}x)$ also satisfies this equation for any Lorentz transformation Λ.

and expanding to leading order in α we have

$$\mathcal{L} \to \mathcal{L}(x) + \alpha \frac{\partial \mathcal{L}}{\partial \phi} \Delta\phi + \alpha \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\Delta\phi) + O(\alpha^2). \tag{2.10}$$

We can rewrite this in terms of a total derivative $\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi \right)$ so that

$$\mathcal{L}' = \mathcal{L}(x) + \alpha \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi \right) + \alpha \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \Delta\phi. \tag{2.11}$$

By Euler-Lagrange, the second term in parentheses vanishes, so we identify the first term in parentheses as none other than $\alpha \partial_\mu X^\mu(x)$ from Eqn. 2.9 (in other words, $\partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \phi)} \Delta\phi \right) = \partial_\mu X^\mu$) and recognize

$$j^\mu \equiv \frac{\partial L}{\partial(\partial_\mu \phi)} \Delta\phi - X^\mu \tag{2.12}$$

as our conserved current (that is, $\partial_\mu j^\mu = 0$).

Example 2.13. Take a complex scalar field

$$\psi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x)).$$

We can then treat ψ, ψ^* as independent variables and write a Lagrangian

$$L = \partial_\mu \psi^* \partial^\mu \psi - V(|\psi|^2).$$

Then we observe that under $\psi \to e^{i\beta}\psi, \psi^* \to e^{-i\beta}\psi^*$, the Lagrangian is invariant. The differential changes are $\Delta\psi = i\psi$ (think of expanding $\psi \to e^{i\beta}\psi$ to leading order) and similarly $\Delta\psi^* = -i\psi^*$ (here we find that $X^\mu = 0$).

We add the currents from ψ, ψ^* to find

$$j^\mu = i\{\psi \partial_\mu \psi^* - \psi^* \partial_\mu \psi\}.$$

This is enough to do questions 2 and 3 on the example sheet.

Example 2.14. Under infinitesimal translation $x^\mu \to x^\mu - \alpha\epsilon^\mu$, we have $\phi(x) \to \phi(x) + \alpha\epsilon^\mu \partial_\mu \phi(x)$ by Taylor expansion (similar for $\partial_\mu \phi$). If the Lagrangian doesn't depend explicitly on x, then $\mathcal{L}(x) \to \mathcal{L}(x) + \alpha\epsilon^\mu \partial_\mu \mathcal{L}(x)$.

Rewriting to match the form $\mathcal{L} + \alpha \partial_\mu X^\mu$, we see that our new Lagrangian takes the form $L(x) + \alpha\epsilon^\nu \partial_\mu(\delta^\mu_\nu L)$. We get one conserved current for each component of ϵ^ν, so that

$$(j^\mu)_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}$$

with $\partial_\mu (j^\mu)_\nu = 0$. We write this as $j^\mu_\nu \equiv T^\mu_\nu$, the energy-momentum tensor.

Definition 2.15. The *energy-momentum tensor* (sometimes *stress-energy tensor*)is the conserved current corresponding to translations in time and space. It takes the form

$$T^{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L},$$

where we have raised an index with the Minkowski metric as is conventional. The conserved charges from integrating $\int d^3x T^{0\nu}$ end up being the total energy $E = \int d^3x T^{00}$ and the three components of momentum $P^i = \int d^3x T^{0i}$.[6]

[6]The definition of the energy-momentum tensor here is slightly different from the one used in general relativity. Here, we have used time and space translations to derive $T^{\mu\nu}$, but in general relativity, we use variations of the metric $g^{\mu\nu}$ instead. The benefit of the GR definition is that the resulting tensor is always symmetric, whereas the $T^{\mu\nu}$ from spacetime translations is not guaranteed to be symmetric. We'll see an example of this in the example sheets, but the $T^{\mu\nu}$ defined by spacetime translations can always be *made* symmetric by defining the "Belinfante-Rosenfeld tensor." The construction isn't anything too special, but people working in relativity insist that variations of the action with respect to the metric is the correct way to define the energy-momentum tensor.

Lecture 3.

Tuesday, October 9, 2018

Last time, we used Noether's theorem to find the stress-energy tensor

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}. \tag{3.1}$$

To better understand this object, we might ask: what is $T^{\mu\nu}$ for free scalar field theory? Recall the Lagrangian for this theory is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2. \tag{3.2}$$

Then by explicit computation, the stress-energy tensor is

$$T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}.$$

The energy is given by

$$E = \int d^3x \left[\frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\boldsymbol{\nabla} \phi)^2 + \frac{1}{2} m^2 \phi^2 \right]$$

(from integrating the T^{00} component) and the conserved momentum components are (from T^{0i})

$$p^i = \int d^3x \dot{\phi} (\partial^i \phi).$$

Note that the original Lagrangian terms don't show up here, since $\eta^{\mu\nu}$ is diagonal.

We'll note that $T^{\mu\nu}$ for this theory is symmetric in μ, ν, but a priori it doesn't have to be. If $T^{\mu\nu}$ is not symmetric initially, we can massage it into a symmetric form by adding $\partial_\rho \Gamma^{\rho\mu\nu}$ where $\Gamma^{\mu\rho\nu} = -\Gamma^{\rho\mu\nu}$ (antisymmetric in the first two indices). Then $\partial_\mu \left(\partial_\rho \Gamma^{\rho\mu\nu} \right) = 0$, which means that adding this term will not affect the conservation of $T^{\mu\nu}$. This is sufficient to attempt questions 1-6 of the first examples sheet.

Canonical quantization Here, we'll follow Dirac's lead and attempt to quantize our field theories. Recall that the Hamiltonian formalism also accommodates field theories (as well as our garden-variety QM).

Definition 3.3. We define the *conjugate momentum*

$$\pi(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

and the *Hamiltonian density* corresponding to a Lagrangian $\mathcal{L}$ is then

$$\mathcal{H} = \pi(x) \dot{\phi}(x) - \mathcal{L}(x).$$

As in classical mechanics, we eliminate $\dot{\phi}$ in favor of π everywhere in $\mathcal{H}$.

Example 3.4. For $\mathcal{L} = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} (\boldsymbol{\nabla} \phi)^2 - V(\phi)$ (and writing in terms of $\pi(x) = \dot{\phi}(x)$) we get

$$\mathcal{H} = \frac{1}{2} \pi^2 + \frac{1}{2} (\boldsymbol{\nabla} \phi)^2 + V(\phi).$$

The Hamiltonian is just the integral of the Hamiltonian density: $H = \int d^3x \mathcal{H}$. Hamlton's equations then yield the equations of motion:

$$\dot{\phi} = \frac{\partial H}{\partial \pi}, \dot{\pi} = -\frac{\partial H}{\partial \phi}.$$

Working these out explicitly for the free theory will give us back the Klein-Gordon equation. Note that H agrees with the total field energy E that we computed above.

There's a slight snag in working in the Hamiltonian formalism– because t is special in our equations, the theory is not manifestly Lorentz invariant (compare to the ∂_μs and variations with respect to $\delta \partial_\mu \phi$ in the Lagrangian formalism). Our original theory was LI, so our new theory is still LI– it just doesn't look LI.

Now let's recall that in quantum mechanics, canonical quantization takes the coordinates q_a and momenta p_a and promotes them to operators. We also replace the Poisson bracket $\{,\}$ with commutators $[,]$. In QM, we had

$$[q_a, p^b] = i \delta_a^b,$$

working in units where $\hbar = 1$. We'll do the same for our fields ϕ_a and the conjugate momenta π_b.

Definition 3.5. A *quantum field* is an operator-valued function of space obeying the commutation relations

$$[\phi_a(\mathbf{x}), \phi_b(\mathbf{y})] = 0 \tag{3.6}$$
$$[\pi_a(\mathbf{x}), \pi_b(\mathbf{y})] = 0 \tag{3.7}$$
$$[\phi_a(\mathbf{x}), \pi^b(\mathbf{y})] = i\delta^3(\mathbf{x}-\mathbf{y})\delta_a^b. \tag{3.8}$$

It's no coincidence that these precisely replicate the commutation relations of the operators $\hat{x}$ and $\hat{p}$ in ordinary quantum mechanics, except that now we have an additional label $\mathbf{x}$ on the fields. Note that $\phi_a(x), \pi^b(x)$ don't depend on t, since we are in the Schrödinger picture. All the t dependence sits in the states which evolve by the usual time-dependent Schrödinger equation

$$i\frac{d}{dt}|\psi\rangle = H|\psi\rangle.$$

We have an infinite number of degrees of freedom, at least one for each x in space. For some theories (free theories), different solutions ϕ can be added together and will evolve independently– free field theories have L quadratic in ϕ_a (plus derivatives thereof), which implies linear equations of motion.

We saw that the simplest free theory leads to the classical Klein-Gordon equation for a real scalar field $\phi(\mathbf{x}, t)$, i.e. $\partial_\mu\partial^\mu\phi + m^2\phi = 0$. To see explicitly why this is a free theory, take the Fourier transform of $\phi(\mathbf{x}, t)$ to write the equations of motion in momentum space:

$$\phi(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}}\phi(\mathbf{p}, t).$$

Then we get the equation of motion

$$\left[\frac{\partial^2}{\partial t^2} + (|\mathbf{p}|^2 + m^2)\right]\phi(\mathbf{p}, t) = 0.$$

We see that the solution is a harmonic oscillator with frequency $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$, so the general solution is a superposition of simple harmonic oscillators each vibrating at different frequencies $\omega_{\mathbf{p}}$. To quantize our field $\phi(\mathbf{x}, t)$, we have to quantize these harmonic oscillators.

Review of 1D harmonic oscillators Recall that the Hamiltonian for the simple harmonic oscillator is

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2,$$

subject to the quantization condition

$$[q, p] = i,$$

where p and q are the momentum and position operators as usual. It's certainly possible to solve this system by the series method, but the algebraic method is much more elegant by far and will generalize better. Our approach is as follows– we'd like to factor the Hamiltonian (since if p and q were classical quantities we could just write it as $\frac{1}{2}(p + i\omega q)(p - i\omega q)$, for instance) but we know that this doesn't quite work because p and q do not commute. Therefore, we define the following operators:

- The creation or raising operator, $a^\dagger \equiv -\frac{i}{\sqrt{2\omega}}p + \sqrt{\frac{\omega}{2}}q$
- The annihilation or lowering operator, $a \equiv +\frac{i}{\sqrt{2\omega}}p + \sqrt{\frac{\omega}{2}}q$.

Note that we can equivalently solve for p and q in terms of a and $a^\dagger$: $q = \frac{1}{\sqrt{2\omega}}(a + a^\dagger)$ and $p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger)$. Substituting p and q into the quantization condition yields the commutator of $a, a^\dagger$,

$$[a, a^\dagger] = 1.$$

We'll then factorize the Hamiltonian into a and $a^\dagger$, picking up an extra term from the commutation relation of p and q– a little more algebra allows us to rewrite the Hamiltonian as

$$H = \frac{1}{2}\omega(aa^\dagger + a^\dagger a) = \omega(a^\dagger a + \frac{1}{2}).$$

Computing the commutators $[H, a]$ and $[H, a^\dagger]$ reveals that

$$[H, a^\dagger] = \omega a^\dagger, [H, a] = -\omega a,$$

which tells us that $a, a^\dagger$ take us between energy eigenstates.[7] More specifically, they take us up and down a ladder of equally spaced energy eigenstates so that if we have one eigenstate with energy E, then we can reach a whole set of eigenstates with energy $\ldots E+2\omega, E+\omega, E, E-\omega, E-2\omega, \ldots$.

If we further postulate that the energy is bounded from below, this implies the existence of a ground state $|0\rangle$ such that the lowering operator acting on $|0\rangle$ kills the state: $a|0\rangle = 0$. In our original Hamiltonian, this ground state has energy given by

$$H|0\rangle = \omega(a^\dagger a + \frac{1}{2})|0\rangle = \frac{\omega}{2}|0\rangle,$$

so the ground state energy (or *zero point energy*) of the system is $\omega/2$. For our quantum theory it's really differences in energy which matter more than their absolute values,[8] so we can just as easily write an equivalent Hamiltonian $H = \omega a^\dagger a$ and set the ground state energy to 0.

We only need one state to construct our full ladder of energy eigenstates, and we can do so by passing our equation back to q-space (real coordinates) and further writing $p = i\frac{\partial}{\partial q}$. If we plug these back into the Hamiltonian, having set $H|0\rangle = 0$, we can then solve for the ground state and find that it is a Gaussian in q with some appropriate variance and normalization. Then we simply need to apply $a^\dagger$ repeatedly to get all the other states, labeling them as $|n\rangle \equiv (a^\dagger)^n|0\rangle$ with $H|n\rangle = n\omega|n\rangle$. (Here we've disregarded normalization, but it's easy enough to add some scaling factor in the definition of $|n\rangle$ so that $\langle n|\ m\rangle = \delta_{nm}$.)

That's about all there is to the quantum harmonic oscillator! We have recovered the quantized energy levels and defined operators to move between them. Next time, we'll repeat the same procedure with quantum fields.

Lecture 4.

Thursday, October 11, 2018

Today, we'll introduce the second quantization procedure, which generalizes the quantum harmonic oscillator to our free scalar field. We'll find that the Hamiltonian for a free scalar field takes the form of an integral over momentum of infinitely many uncoupled harmonic oscillator Hamiltonians with some characteristic frequencies $\omega_{\mathbf{p}}$. From this Hamiltonian, we'll recover the particle interpretation of the excitations of these harmonic oscillators.

Recall that for a free scalar field

$$\phi(\mathbf{x}, t) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \phi(\mathbf{p}, t),$$

where

$$\left[\frac{\partial^2}{\partial t^2} + (\mathbf{p}^2 + m^2)\right]\phi(\mathbf{p}, t) = 0.$$

We also defined $\omega_{\mathbf{p}}^2 \equiv \mathbf{p}^2 + m^2$, and remarked that our theory has plane wave solutions. Let's apply the simple harmonic oscillator quantization process to free fields now, defining

$$\phi(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} (a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}).$$

We also have the related conjugate momentum to the field,

$$\pi(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} (-i)\sqrt{\frac{\omega_{\mathbf{p}}}{2}} (a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} - a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}).$$

In the *second quantization* process, we've written our infinite number of harmonic oscillators in momentum space. We want to impose the equivalent of

$$[a_{\mathbf{p}}, a_{\mathbf{q}}] = [a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] = 0$$

[7] Explicitly, consider an eigenstate $|E\rangle$ with energy E. Then $Ha^\dagger|E\rangle = (a^\dagger H + \omega a^\dagger)|E\rangle = (E+\omega)a^\dagger|E\rangle$, so $a^\dagger|E\rangle$ is an eigenstate with energy $E+\omega$. The computation for a is similar.

[8] Remark: gravity is different! Gravity couples directly to energy, not to differences in energy. But in a simple theory like the 1D harmonic oscillator, all we care about is the spacing of the energy levels.

and

$$[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}).$$

Thus in the field theory context we have instead

$$[\phi(\mathbf{x}), \phi(\mathbf{y})] = [\pi(\mathbf{x}), \pi(\mathbf{y})] = 0$$

and

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^3(\mathbf{x}-\mathbf{y}).$$

It's a good exercise to check this, but we can for instance check this one way: assume the $a, a^\dagger$ commutation relations:

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{q}}{(2\pi)^3}\frac{(-1)}{2}\sqrt{\frac{\omega_{\mathbf{q}}}{\omega_{\mathbf{p}}}}\{-[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}]e^{i\mathbf{p}\cdot\mathbf{x}-i\mathbf{q}\cdot\mathbf{y}} + [a^\dagger_{\mathbf{p}}, a_{\mathbf{q}}]e^{-i\mathbf{p}\cdot\mathbf{x}+i\mathbf{q}\cdot\mathbf{y}}\}.$$

Using these commutation relations, we can rewrite and do the integral over $\mathbf{q}$ to get a delta function setting $\mathbf{p} = \mathbf{q}$,

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3}\left(\frac{-i}{2}\right)\{-e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\} = i\delta^3(\mathbf{x}-\mathbf{y})$$

since $\delta^3(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}e^{i\mathbf{p}\cdot\mathbf{x}}$ and $\mathbf{p}$ is a dummy integration variable, so we can freely switch the sign in the exponent.

Now we compute H in terms of $a_{\mathbf{p}}a^\dagger_{\mathbf{p}}$ to find (after some work with δ functions which you should check) that

$$\begin{aligned}
H &= \frac{1}{2}\int d^3x\left(\pi^2 + (\boldsymbol{\nabla}\phi)^2 + m^2\phi^2\right) \\
&= \frac{1}{2}\int d^3x\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{q}}{(2\pi)^3}\left[\frac{-\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}{2}(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} - a^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}})(a_q e^{iq\cdot x} - a^\dagger_q e^{-iq\cdot x})\right. \\
&\quad \left. + \frac{1}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}(ipa_p e^{ip\cdot x} - ipa^\dagger_p e^{-ip\cdot x})\right]
\end{aligned}$$

There's a lot of algebraic manipulation here (details in David Tong's notes) but the net result is that

$$H = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}(a_{\mathbf{p}}a^\dagger_{\mathbf{p}} + a^\dagger_{\mathbf{p}}a_{\mathbf{p}}).$$

This is simply the Hamiltonian for an infinite number of uncoupled simple harmonic oscillators with frequency ω_p, just as expected.

Now we can define a vacuum state $|0\rangle$ as the state which is annihilated by all operators $a_{\mathbf{p}}$:

$$a_{\mathbf{p}}|0\rangle = 0 \forall \mathbf{p}.$$

Then computing the vacuum state energy $H|0\rangle$ yields

$$\begin{aligned}
H|0\rangle &= \int \frac{d^3p}{(2\pi)^3}\omega_p(a^\dagger_p a_p + \frac{1}{2}[a_p, a^\dagger_p])|0\rangle \\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\omega_p[a_p, a^\dagger_p]|0\rangle \\
&= \int d^3p\frac{\omega_p}{2}\delta^3(\mathbf{0})|0\rangle,
\end{aligned}$$

which is infinite. Oh no!

What's happened is that $\int d^3p\left(\frac{1}{2}\omega_p\right)$ is the sum of ground state energies for each harmonic oscillator, but $\omega_p = \sqrt{|\mathbf{p}|^2 + m^2} \to \infty$ as $|\mathbf{p}| \to \infty$, so we call this a high-frequency or *ultraviolet divergence.* That is, at very high frequencies/short distances, our theory breaks down and our theory should cut off at high

momentum.[9] Of course, there's another way to handle this divergence in our theory– just redefine the Hamiltonian to set the ground state energy to zero.[10]

Thus, we redefine the Hamiltonian for our free scalar field theory to be

$$H = \int \frac{d^3p}{(2\pi)^3}\omega_p a_p^\dagger a_p,$$

such that $H|0\rangle = 0$. Nice. Subtractin' infinities. Because we're physicists.

More formally, the difference between the old and new Hamiltonians can be seen as due to an ordering ambiguity in moving from the classical theory to the quantum one. We could have written the classical Hamiltonian as

$$H = \frac{1}{2}(\omega q - ip)(\omega q + ip)$$

which is classically the same as the original simple harmonic oscillator but becomes

$$\omega a^\dagger a$$

when we quantize.

Definition 4.1. We define a *normal ordered* string of operators $\phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2)\dots\phi_n(\mathbf{x}_n)$ as follows. We write colons around the operators to be normal ordered,

$$:\phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2)\dots\phi_n(\mathbf{x}_n):,$$

and simply move all annihilation operators to the righthand side of the expression (so all the creation operators are on the left). Note that we totally ignore commutation relations in normal ordering! Just move the operators around[11] Normal ordered strings of operators are nice to work with because they make it easy to see what initial particle states will be annihilated and what final particle states will be created.

Example 4.2. For our free scalar field Hamiltonian, the normal-ordered version looks like

$$\begin{aligned} :H: &= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\omega_p :(a_p a_p^\dagger + a_p^\dagger a_p): \\ &= \int \frac{d^3p}{(2\pi)^3}\omega_p a_p^\dagger a_p. \end{aligned}$$

We'd like to recover particles from this theory. Recall that $\forall p, a_p|0\rangle = 0$, so $H|0\rangle = 0$ (where now H means the normal-ordered version of the Hamiltonian). It's easy to verify (exercise) that

$$[H, a_p^\dagger] = \omega_p a_p^\dagger$$

and similarly

$$[H, a_p] = -\omega_p a_p.$$

Let $|p'\rangle = a_{p'}^\dagger|0\rangle$. Then

$$H|p'\rangle = \int \frac{d^3p}{(2\pi)^3}\omega_p a_p^\dagger [a_p, a_{p'}^\dagger]|0\rangle = \omega_{p'}|p'\rangle.$$

Therefore the energy is given by $\omega_{p'} = \sqrt{p'^2 + m^2}$, the relativistic dispersion relation for a particle of mass m and momentum p'. We may thus interpret $|p\rangle$ as a momentum eigenstate of a single particle of mass m and momentum p. Recognizing ω_p as the energy, we'll write E_p instead of ω_p.

We can also write the (single-particle) momentum operator P such that

$$\mathbf{P}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle.$$

[9] This sort of cutoff behavior becomes especially important in the *renormalization group*, a method of studying the relationships of different field theories under special scaling transformations. We'll see this in *Statistical Field Theory.*

[10] "We're not interested in gravity, only energy differences, so we can just subtract ∞." –Ben Allanach

[11] There are sign flip subtleties when we come to working with fermions because of the antisymmetrization property but we won't worry about them for now.

P is simply the quantized version of the momentum operator from the stress-energy tensor:

$$\mathbf{P} = -\int \pi(x)\boldsymbol{\nabla}\phi(\mathbf{x})d^3x = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\mathbf{p}a^\dagger_{\mathbf{p}}a_{\mathbf{p}}.$$

Lecture 5.

Saturday, October 13, 2018

We previously found that we could write the field momentum operator (not the conjugate momentum!) as

$$\mathbf{P} = -\int \pi(x)\boldsymbol{\nabla}\phi(\mathbf{x})d^3x = \int \frac{d^3p}{(2\pi)^3}\mathbf{p}a^\dagger_{\mathbf{p}}a_{\mathbf{p}}.$$

We could also act on our momentum eigenstates with the angular momentum operator J^i, and what we find is that

$$J^i\left|\mathbf{p}\right\rangle = 0,$$

so the scalar field theory represents a spin 0 (scalar) boson.

In general we could imagine cooking up the multi-particle state

$$\left|\mathbf{p}_1,\mathbf{p}_2,\ldots,\mathbf{p}_n\right\rangle = a^\dagger_{\mathbf{p}_1}a^\dagger_{\mathbf{p}_2}\ldots a^\dagger_{\mathbf{p}_n}\left|0\right\rangle.$$

But it follows that

$$\left|\mathbf{p},\mathbf{q}\right\rangle = \left|\mathbf{q},\mathbf{p}\right\rangle,$$

since the creation operators for different momenta commute. So our states are symmetric under interchange, which means these particles are bosons. The full Hilbert space is spanned by

$$\left|0\right\rangle, a^\dagger_{\mathbf{p}}\left|0\right\rangle, a^\dagger_{\mathbf{p}_1}a^\dagger_{\mathbf{p}_2}\left|0\right\rangle,\ldots$$

and this is called *Fock space*.

If we use the number operator

$$N \equiv \int \frac{d^3p}{(2\pi)^3}a^\dagger_{\mathbf{p}}a_{\mathbf{p}}$$

which counts the number of particles in a state, we find (exercise)

$$N\left|\mathbf{p}_1,\ldots,\mathbf{p}_n\right\rangle = n\left|\mathbf{p}_1,\ldots,\mathbf{p}_n\right\rangle.$$

But it's easy to check that (and you should check this using the commutation relations)

$$[N,H] = 0,$$

which means that the number of particles is conserved in the free theory (crucially, this is not true once we add interactions).

Let's also note that our momentum eigenstates are *not* localized in space. We can describe a spatially localized state by a Fourier transform,

$$\left|\mathbf{x}\right\rangle = \int \frac{d^3p}{(2\pi)^3}e^{-i\mathbf{p}\cdot\mathbf{x}}\left|\mathbf{p}\right\rangle.$$

More generally we describe a wavepacket partially localized in position and momentum space, e.g. by

$$\left|\psi\right\rangle = \int \frac{d^3p}{(2\pi)^3}e^{-i\mathbf{p}\cdot\mathbf{x}}\psi(\mathbf{p})\left|\mathbf{p}\right\rangle\left\langle\mathbf{p}\right|.$$

Note that neither $\left|\mathbf{x}\right\rangle$ nor $\left|\psi\right\rangle$ are eigenstates of the Hamiltonian like in QM.

We consider now relativistic normalization. We define the vacuum such that $\left\langle 0\right|\left.0\right\rangle = 1$, which certainly must be Lorentz invariant (1 is just a number). So in general

$$\left\langle\mathbf{p}\right|\left.\mathbf{q}\right\rangle = \left\langle 0\right|[a_{\mathbf{p}},a^\dagger_{\mathbf{q}}]\left|0\right\rangle = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}).$$

Is this Lorentz invariant? Under the Lorentz transformation, we have

$$p^\mu \to \Lambda^\mu_\nu p^\nu \equiv p'^\mu.$$

We want the two states to be related by a unitary transformation so that the inner product $\langle \mathbf{p}| \ \mathbf{q}\rangle$ is Lorentz invariant (i.e. $\langle \mathbf{p}| \ \mathbf{q}\rangle \to \langle \mathbf{p}'| \ \mathbf{q}'\rangle = \langle \mathbf{p}| \, U(\Lambda)^\dagger U(\Lambda) \, |\mathbf{q}\rangle) = \langle \mathbf{p}| \ \mathbf{q}\rangle$ by unitarity).

To figure this out, we'll need to look at a Lorentz invariant object, e.g. the identity operator on 1-particle states.

$$1 = \int \frac{d^3p}{(2\pi)^3} \, |\mathbf{p}\rangle \, \langle \mathbf{p}| \, .$$

Either half of this (the d^3p part and the $|\mathbf{p}\rangle \, \langle \mathbf{p}|$ part) is not LI, but somehow the whole thing is (since it's equal to 1).

How do we prove this? We start by claiming that

$$\int \frac{d^3p}{2E_{\mathbf{p}}}$$

is Lorentz invariant. This follows because $\int d^4p$ is LI, since $\Lambda \in SO(1,3)$ (i.e. $\det \Lambda = 1$) so the factor of $\det \Lambda$ we would normally pick up from doing the coordinate transformation is just 1. So the four-volume element is Lorentz invariant, $\int d^4p = \int d^4p'$. It's also true that $p_0^2 = \mathbf{p}^2 + m^2$ is Lorentz invariant (in particular, it expresses the length of a four-vector $p_\mu p^\mu = m^2$). The solutions for p_0 have two branches, positive and negative:

$$p_0 = \pm\sqrt{\mathbf{p}^2 + m^2}.$$

But our choice of branch is also Lorentz invariant (we can't go from the positive to negative solutions via Lorentz transformation). Therefore combining the last few facts, we get

$$\int d^4p \delta(p_0^2 - \mathbf{p}^2 - m^2)|_{p^0>0} = \int \frac{d^3p}{2p_0}\bigg|_{p_0=E_p},$$

where we have used the fact that

$$\delta(g(x)) = \sum_{x_i \text{ roots of g}} \frac{\delta(x - x_i)}{|g'(x_i)|}.$$

(To see why this is true, consider Taylor expanding $\delta(g(x))$ around its roots to leading order.)

We make the next claim: $2E_p\delta^3(\mathbf{p} - \mathbf{q})$ is the Lorentz invariant version of a δ-function. The proof is as follows:

$$\int \frac{d^3p}{2E_p} 2E_p\delta^3(\mathbf{p} - \mathbf{q}) = 1.$$

But we showed that $\int d^3p/2E_p$ was Lorentz invariant and 1 is certainly Lorentz invariant, so it follows that $2E_p\delta^3(\mathbf{p} - \mathbf{q})$ is also Lorentz invariant.

We therefore learn that the correctly normalized states are

$$|p\rangle \equiv \sqrt{2E_p} \, |\mathbf{p}\rangle = \sqrt{2E_p} a_{\mathbf{p}}^\dagger \, |0\rangle \, ,$$

(where p is now the four-vector p, not the three-vector $\mathbf{p}$) with the inner product

$$\langle p| \ q\rangle = (2\pi)^3 2\sqrt{E_pE_q}\delta^3(\mathbf{p} - \mathbf{q}).$$

We can then rewrite the 1-particle identity operator[12] as an integral over the normalized states,

$$1 = \int \frac{d^3p}{2E_p(2\pi)^3} \, |p\rangle \, \langle p| \, .$$

[12] To see this really is the identity, let's act on the normalized $|q\rangle$. It's basically a one-liner:

$$\int \frac{d^3p}{2E_p(2\pi)^3} \, |p\rangle \, \langle p| \, |q\rangle = \int \frac{d^3p}{2E_p(2\pi)^3} \, |p\rangle \left[(2\pi)^3 2\sqrt{E_pE_q}\delta^3(\mathbf{p} - \mathbf{q})\right] = |q\rangle \, ,$$

since the delta function makes the integral trivial by setting $\mathbf{p} = \mathbf{q}$.

Free $\mathbb{C}$ scalar field We could also look at the free complex scalar field ψ, with Lagrangian

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - \mu^2 \psi^* \psi.$$

We can compute the Euler-Lagrange equations varying ψ, ψ^* separately to find

$$\partial_\mu \partial^\mu \psi + \mu^2 \psi = 0 \text{ and } \partial_\mu \partial^\mu \psi^* + \mu^2 \psi^* = 0$$

(the second equation is simply the complex conjugate of the first). Now we ought to write our field as a sum of two different creation and annihilation operators:

$$\psi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (b_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + c^\dagger_{\mathbf{p}} e^{-i\mathbf{p}\cdot x})$$

and similarly

$$\psi^\dagger = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (b^\dagger_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} + c_{\mathbf{p}} e^{+i\mathbf{p}\cdot x})$$

so that

$$\pi(x) = \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{E_p}{2}} (b^\dagger_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} - c_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}}).$$

The conjugate momentum to $\psi^\dagger$ is equivalently $\pi^\dagger$. The commutation relations are then (exercise)

$$[\psi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^3(\mathbf{x} - \mathbf{y})$$

$$\implies [b_{\mathbf{p}}, b^\dagger_{\mathbf{q}}] = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}) = [c_{\mathbf{p}}, c^\dagger_{\mathbf{q}}].$$

The interpretation of these equations is that different types of particle are created by the $b^\dagger_{\mathbf{p}}$ and $c^\dagger_{\mathbf{p}}$ operators. They are both spin 0 and of mass μ, so we should interpret them as a particle-antiparticle pair. This doesn't work for electrons, which have spin $1/2$, but it would describe something like a charged pion.

Indeed, if we compute the conserved charges in this theory by applying Noether's theorem, we get a conserved charge of the form $Q = i \int d^3x \dot{\psi}^* \psi - \psi^* \dot{\psi}$ or equivalently in terms of the conjugate momentum (since $\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \dot{\psi}^*$)

$$Q = i \int d^3x [\pi\psi - \psi^\dagger \pi^\dagger].$$

After normal ordering (exercise) one can write

$$Q = \int \frac{d^3p}{(2\pi)^3} (c^\dagger_{\mathbf{p}} c_{\mathbf{p}} - b^\dagger_{\mathbf{p}} b_{\mathbf{p}}) = N_c - N_b,$$

which shows that our conserved quantity has the interpretation of particle number (counting antiparticles as -1).

Since there are two real scalar fields in this theory, the Hamiltonian for this theory takes the form

$$H = \int \frac{d^3p}{(2\pi)^3} E_p (b^\dagger_{\mathbf{p}} b_{\mathbf{p}} + c^\dagger_{\mathbf{p}} c_{\mathbf{p}}).$$

As an exercise one can check that $[Q, H] = 0$ using the commutation relations, and therefore Q is conserved. This is also true in the interacting theory. N_c, N_b are individually conserved in the free theory, but in the interacting theory they aren't– instead, they can be created and destroyed in particle-antiparticle pairs so that $N_c - N_b$ is constant.

Lecture 6.

Tuesday, October 16, 2018

We've been working in the Schrödinger picture where the states evolve in time, but now it will be useful to pass to the Heisenberg picture, where the states are fixed and the *operators* evolve in time.

In the Schrödinger picture, it's not obvious how our theory is Lorentz invariant. We seem to have picked out time as a special dimension when we write things down (even though we started with a Lorentz

invariant theory, so our final theory should still be Lorentz invariant). The operators $\phi(\mathbf{x})$ don't depend on t, but the states evolve as

$$i\frac{d\,|p\rangle}{dt} = H\,|p\rangle = E_p\,|p\rangle \implies |p(t)\rangle = e^{-iE_pt}\,|p(0)\rangle\,.$$

In the Heisenberg picture, things are a bit better– time dependence is moved into the operators. Denoting Heisenberg picture operators as O_H and Schrödinger picture operators as O_S, we have[13]

$$O_H(t) \equiv e^{iHt}O_Se^{-iHt}.$$

Taking the time derivative of each side, one finds that[14]

$$\frac{dO_H}{dt} = i[H, O_H].$$

This is the general time evolution of operators in the Heisenberg picture. It's clear that $O_H(t=0) = O_S$, so our operators agree at $t = 0$ (but in general nowhere else). The field commutators then become *equal time commutation relations*:

$$[\phi(\mathbf{x},t), \phi(\mathbf{y},t)] = [\pi(\mathbf{x},t), \pi(\mathbf{y},t)] = 0$$

and

$$[\phi(\mathbf{x},t), \pi(\mathbf{y},t)] = i\delta^3(\mathbf{x}-\mathbf{y}).$$

Exercise 6.1. One should check (exercise) that $\frac{d\phi}{dt} = i[H,\phi]$ now means that the Heisenberg picture operator ϕ_H satisfies the Klein-Gordon equation, $\partial_\mu\partial^\mu\phi + m^2\phi = 0$.

We now write the Fourier transform of $\phi(x)$ (where x is now a four-vector) by deriving

$$e^{iHt}a_{\mathbf{p}}e^{-iHt} = e^{-iE_pt}a_{\mathbf{p}}$$

and

$$e^{iHt}a^\dagger_{\mathbf{p}}e^{-iHt} = e^{+iE_pt}a^\dagger_{\mathbf{p}}.$$

You should also check this (exercise) using the commutation relation $[H, a_{\mathbf{p}}] = -E_pa_{\mathbf{p}}$.

Therefore we can now write

$$\phi(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\{a_{\mathbf{p}}e^{-ip\cdot x} + a^\dagger_{\mathbf{p}}e^{+ip\cdot x}\}$$

where x and p are now four-vectors and $p_0 = E_p$.

Causality We might be concerned about the causal structure of this theory, since ϕ and π satisfy equal-time commutation relations. In general a Lorentz transform might mix up events which in one frame take place at "equal times." So what about arbitrary space-time separations? It turns out that causality requires that the commutators of spacelike separated operators is zero, i.e. two events which are spacelike separated cannot impact one another.

$$[O_1(x), O_2(y)] = 0 \forall (x-y)^2 < 0.$$

Does this condition hold? Let's define

$$\Delta(x-y) \equiv [\phi(x), \phi(y)]$$

and expand in the Fourier basis.

$$\begin{aligned}\Delta(x-y) &= \int \frac{d^3p}{(2\pi)^6}\frac{d^3p'}{\sqrt{4E_pE_{p'}}}\left([a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}]e^{-i(p\cdot x - p'\cdot y)} + [a^\dagger_{\mathbf{p}}, a_{\mathbf{p}'}]e^{i(p\cdot x - p'\cdot y)}\right)\\ &= \int \frac{d^3p}{2E_p(2\pi)^3}\left(e^{-ip\cdot(x-y)} - e^{ip'\cdot(x-y)}\right)\end{aligned}$$

[13] Here, the exponential of an operator is simply defined in terms of the power expansion of e, e.g. $e^{iHt} = \sum_{n=0}^\infty \frac{(iHt)^n}{n!}$.

[14] Explicitly, $\frac{dO_H(t)}{dt} = iHe^{iHt}O_Se^{-iHt} + e^{iHt}O_S(-iH)e^{-iHt} = ie^{iHt}[H,O_S]e^{-iHt} = i[H,O_H]$ since $e^{iHt}He^{-iHt} = H$. We also see from this that it doesn't matter to the Hamiltonian itself what picture we're in, since $H_S = H_H$.

Remarkably, this is just a c-number– it's not an operator at all but a (classical) number.[15] It is Lorentz invariant since the integration measure $d^3p/(2E_p)$ is Lorentz invariant and the integrand is too (it depends only on $p \cdot (x-y)$, which is the product of two four-vectors and therefore an invariant scalar). Moreover, each term is separately Lorentz invariant. In addition, if $x-y$ is spacelike then $x-y$ can be Lorentz transformed to $y-x$ in the first term, giving 0. It does not vanish for timelike separations, e.g.

$$[\phi(\mathbf{x},0),\phi(\mathbf{x},t)] = \int \frac{d^3p}{(2\pi)^3 2E_p}(e^{-imt} - e^{+imt}) \neq 0.$$

And at equal times

$$[\phi(\mathbf{x},t),\phi(\mathbf{y},t)] = \int \frac{d^3p}{(2\pi)^3 2E_p}(e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}) = 0$$

(since we can send the integration variable $\mathbf{p} \to -\mathbf{p}$). One can also see in this way that the commutator for spacelike separated operators vanishes, since a general spacelike separation can always be transformed into a frame where the two events take place at equal times.

Definition 6.2. We can then introduce the idea of a *propagator*– if we initially prepare a particle at point y, what is the amplitude to find it at x? We can write this as

$$\begin{aligned}
\langle 0|\,\phi(x)\phi(y)\,|0\rangle &= \int \frac{d^3p d^3p'}{(2\pi)^6\sqrt{4E_pE_{p'}}}\,\langle 0|\,a_{\mathbf{p}}a^\dagger_{\mathbf{p}'}\,|0\rangle\, e^{-ip\cdot x + ip'\cdot y} \\
&= \int \frac{d^3p d^3p'}{(2\pi)^6\sqrt{4E_pE_{p'}}}\,\langle 0|\,[a_{\mathbf{p}}a^\dagger_{\mathbf{p}'}]\,|0\rangle\, e^{-ip\cdot x + ip'\cdot y} \\
&= \int \frac{d^3p}{(2\pi)^3 2E_p} e^{-ip\cdot(x-y)} \equiv D(x-y),
\end{aligned}$$

where we have used the fact that $a_{\mathbf{p}}$ kills the ground state (so we can freely subtract off $a^\dagger_{\mathbf{p}'}a_{\mathbf{p}}$ to get a commutator) and used the resulting delta function to integrate over d^3p'.

In fact, one can show[16] that for spacelike separations $(x-y)^2 < 0$, the propagator decays as $D(x-y) \sim e^{-m|\mathbf{x}-\mathbf{y}|}$. The quantum field seems to "leak" out of the light cone. But we also computed that

$$\Delta(x-y) = [\phi(x),\phi(y)] = D(x-y) - D(y-x) = 0$$

if $(x-y)^2 < 0$. We can interpret this to mean that there's no Lorentz invariant way to order the two events at x and y. A particle can travel as easily from $y \to x$ as $x \to y$, so in a quantum measurement these two amplitudes cancel. With a complex scalar field, the story is more interesting. We find instead that the amplitude for a particle to go from $x \to y$ is cancelled by the amplitude for an anti-particle to go from $y \to x$.[17] This is also the case for the real scalar field, except the particle is its own antiparticle.

Definition 6.3. We now introduce the *Feynman propagator* Δ_F, which is like a regular propagator but with time ordering baked in. That is,

$$\Delta_F = \begin{cases} \langle 0|\,\phi(x)\phi(y)\,|0\rangle & \text{for } x^0 > y^0 \\ \langle 0|\,\phi(y)\phi(x)\,|0\rangle & \text{for } y^0 > x^0. \end{cases}$$

We claim the Feynman propagator can also be written as

$$\Delta_F = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2}e^{-ip\cdot(x-y)}.$$

[15] Wikipedia says this terminology is due to Dirac, who coined it to contrast with q-numbers (quantum numbers), which are just operators.

[16] The easiest way to do this is to set $y=0$ and take x and y at equal times, $x^0 = y^0 = 0$. This gets rid of p^0, and from here you can switch to spherical coordinates, rewriting $\mathbf{p}\cdot(x)$ as $|p||x|\cos\theta$.

[17] See also Wheeler's "one-electron universe"– https://en.wikipedia.org/wiki/One-electron_universe.

Note that this is Lorentz invariant– the volume element is certainly Lorentz invariant, and everything else is scalars. But there's an issue– this integral has a pole whenver $p^2 = m^2$, or equivalently for each value of $\mathbf{p}$, $p^2 - m^2 = (p^0)^2 - \mathbf{p}^2 - m^2 = 0$ when $p^0 = \pm E_{\mathbf{p}} = \pm\sqrt{\mathbf{p}^2 + m^2}$. We would like to integrate over p^0 to recover the earlier form of the propagator, so we can either deform the contour or push the poles of the real p^0 axis with an *$i\epsilon$ prescription*.

We'll finish the proof next time, but by analytically continuing p^0 to the complex plane, making this $i\epsilon$ prescription, and closing the contour appropriately we can do the p^0 integral and find that what we get is exactly the Feynman propagator as defined earlier in terms of time ordering.

Proof of Exercise 6.1 Let's find the equation of motion for ϕ. Recall that $[\phi(\mathbf{x}), \phi(\mathbf{y})] = 0$. We can also show that $\boldsymbol{\nabla}\phi(y)$ and $\phi(x)$ commute:

$$\boldsymbol{\nabla}\phi(\mathbf{y})\phi(\mathbf{x}) = \boldsymbol{\nabla}_y(\phi(\mathbf{y})\phi(\mathbf{x})) = \boldsymbol{\nabla}_y(\phi(\mathbf{x})\phi(\mathbf{y})) = \phi(\mathbf{x})\boldsymbol{\nabla}\phi(\mathbf{y})$$

so the only term in the Hamiltonian we need to worry about is the π^2 term.

$$\begin{aligned}
\dot{\phi} &= i[H, \phi] \\
&= \frac{i}{2}\int d^3y \left[\pi^2(y) + (\nabla\phi(y))^2 + m^2\phi(y)^2, \phi(x)\right] \\
&= \frac{i}{2}\int d^3y(\pi^2(y)\phi(x) - \phi(x)\pi^2(y)) \\
&= \frac{i}{2}\int d^3y(\pi(y)(-[\phi(x), \pi(y)] + \phi(x)\pi(y)) - \phi(x)\pi^2(y)) \\
&= \frac{i}{2}\int d^3y(-i\delta^3(x-y)\pi(y) + \pi(y)\phi(x)\pi(y) - \phi(x)\pi^2(y) \\
&= \frac{i}{2}\int d^3y(-2i\delta^3(x-y)\pi(y)) \\
&= \pi(x).
\end{aligned}$$

We can also compute the time evolution for π. Here, we do have to worry about the $\boldsymbol{\nabla}\phi$ terms as well as the ϕ terms.

$$\begin{aligned}
\dot{\pi} &= i[H, \pi] \\
&= \frac{i}{2}\int d^3y \left[\pi^2(y) + (\nabla\phi(y))^2 + m^2\phi(y)^2, \pi(x)\right] \\
&= \frac{i}{2}\int d^3y \nabla\phi(y)\nabla_y(\phi(y)\pi(x)) - \nabla_y(\pi(x)\phi(y))\nabla\phi(y) + 2im^2\delta^3(x-y)\phi(y) \\
&= \frac{i}{2}\int d^3y \nabla\phi(y)\nabla_y([\phi(y), \pi(x)]) - \nabla_y(-[\phi(y), \pi(x)])\nabla\phi(y) + 2im^2\delta^3(x-y)\phi(y) \\
&= \frac{i}{2}\int d^3y \nabla\phi(y)\nabla_y(i\delta^3(y-x)) + \nabla_y(i\delta^3(y-x))\nabla\phi(y) + 2im^2\delta^3(x-y)\phi(y) \\
&= \frac{i}{2}\int d^3y \left(-2i\delta^3(x-y)\nabla^2\phi(y) + 2im^2\delta^3(x-y)\phi(y)\right) \\
&= \nabla^2\phi - m^2\phi.
\end{aligned}$$

(where we have integrated by parts to move the ∇ from the delta function to ϕ). Thus ϕ obeys the equation

$$\ddot{\phi} = \dot{\pi} = \nabla^2\phi - m^2\phi$$

or equivalently

$$\ddot{\phi} - \nabla^2\phi + m^2 = \partial_\mu\partial^\mu\phi + m^2 = 0.$$

Therefore ϕ satisfies the Klein-Gordon equation. (This is also in David Tong's notes.) ⊠

We'll also make note of a potentially useful identity which can be proved by induction: if $[a, b] = \alpha$, then $[a^n, b] = n\alpha a^{n-1}$.

Proof of Heisenberg picture $a_p, a_p^\dagger$ Here, we'll show that

$$e^{iHt} a_p e^{-iHt} = e^{-iE_p t} a_p$$

using the commutation relation $[H, a_p] = -E_p a_p$. First, I'll claim that

$$H^n a_p = a(-E_p + H)^n.$$

Let's prove it by induction: for the base case, $n = 1$ and

$$Ha_p = [H, a_p] + a_p H = -E_p a_p + a_p H = a_p(-E_p + H).$$

Now the inductive step: suppose the hypothesis holds for n. Then

$$H^{n+1} a_p = H(H^n a_p) = H a_p (-E_p + H)^n = a_p(-E_p + H)^{n+1}.$$

Therefore we can use this in the expansion of e^{iHt}.

$$\begin{aligned} e^{iHt} a_p e^{-iHt} &= \sum_{n=0}^{\infty} \frac{(iHt)^n}{n!} a_p e^{-iHt} \\ &= a_p \sum_{n=0}^{\infty} \frac{(it(-E_p + H))^n}{n!} e^{-iHt} \\ &= a_p e^{-iE_p t} e^{iHt} e^{-iHt} \\ &= a_p e^{-iE_p t}. \end{aligned}$$

Rather than repeating this whole calculation, we can simply take the hermitian conjugate of each side (since H is hermitian) to get

$$e^{iHt} a_p^\dagger e^{-iHt} = e^{+iE_p t} a_p^\dagger.$$

Note that the sign flip in the exponent of $e^{\pm iHt}$ and the reversing of order from taking the hermitian conjugate cancel out. So the operators $a, a^\dagger$ do evolve in a nice way that allows us to write ϕ in terms of a four-vector product in the exponent, $p \cdot x$, and in turn this helps us to see that our theory has a sensible causal structure under Lorentz transformations. ⊠

Lecture 7.

Thursday, October 18, 2018

Today, we'll complete our initial discussion of propagators and then introduce interacting fields.

Last time, we claimed the Feynman propagator could be written as an integral over d^4p, and reduces to the regular propagator $D(x-y)$ or $D(y-x)$ depending on the sign of $x^0 - y^0$. The propagator $D(x-y)$ was an integral over d^3p only, so we need to integrate over the p^0 component. To evaluate the p^0 integral, one can make an $i\epsilon$ prescription and modify the pole to

$$\Delta_F = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}$$

with $\epsilon > 0$ and small. This helps us to keep track of which pole is inside our contour, but we can also equivalently shift the contour (see picture). This shifts the pole at E_p to $E_p - i\epsilon$ and from $-E_p$ to $-E_p + i\epsilon$. This is a little quick, so I'll work it out more carefully in a footnote later.

Which way we close the contour depends on the sign of $x^0 - y^0$ since $(x^0 - y^0) > 0$ means that $e^{ip^0(x^0-y^0)} \to 0$ when $p^0 \to +i\infty$, and for $(x^0 - y^0) < 0$ it goes to 0 when $p^0 \to -i\infty$.

In any case, we can evaluate this with the Cauchy integral formula to find

$$\Delta_F(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x-y)} = D(x-y)$$

for $x^0 > y^0$ and

$$\Delta_F(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(y-x)} = D(y-x)$$

for $y^0 > x^0$, where the sign flip has come from which way we close the contour and therefore which pole we pick up in the integration.

We can now observe that Δ_F is the *Green's function* of the Klein-Gordon equation. A Green's function (perhaps familiar from a class on PDEs or electrodynamics) is simply the inverse of a differential operator; it is a function which yields a delta function when you hit it with a given differential operator. You might have seen the Green's function for Poisson's equation, for instance.[18] In this case,

$$\begin{aligned}(\partial_t^2 - \nabla^2 + m^2)\Delta_F(x-y) &= \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m^2 + i\epsilon}(-p^2+m^2)e^{-ip\cdot(x-y)} \\ &= -i\int \frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)} \\ &= -i\delta^4(x-y).\end{aligned}$$

It can be useful to choose other integration contours, e.g. for the retarded propagator which takes

$$\Delta_R(x-y) = \begin{cases}[\phi(x),\phi(y)] & : x^0 > y^0 \\ 0 & : y^0 > x^0\end{cases}$$

The advanced propagator is similarly defined but for $x^0 < y^0$. In any case, the Feynman propagator is the most applicable for our purposes.

Interacting fields Our free theories have made for nice, exactly solvable models. They have Lagrangians which are quadratic in the fields, which means that

- the equations of motions are linear
- we have exact quantization
- we can produce multi-particle states, but there is no scattering.

It's this third point which is not realistic– we know in general that particles should interact and scatter. Therefore, we guess that interactions must come from higher-order terms in the Lagrangian $\mathcal{L}$. For example, in a real scalar field ϕ we could more generally write

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \sum_{n=3}\frac{\lambda_n}{n!}\phi^n,$$

where the λ_ns are called coupling constants. Ideally, we'd like these corrections to be small so we can take a perturbative expansion about the free theory solutions, which already look like particles.

Naïvely, we might say that small perturbations means that $\lambda_n \ll 1$, but that only makes sense when λ_n is dimensionless. So let's do some dimensional analysis to figure out what the dimensions of λ_n are. Recall that the action S is dimensionless, $[S] = 0$. Since $S = \int d^4x\mathcal{L}$ and $[d^4x] = -4$, we find that $[\mathcal{L}] = 4$. From looking at the kinetic term $\partial_\mu\phi\partial^\mu\phi$ and using the fact that $[\partial_\mu] = +1$, we conclude that $[\phi] = 1$, $[m] = 1$, and

$$[\lambda_n] = 4 - n$$

(where this 4 comes from the fact we are working in $3+1$ spacetime dimensions).

What we discover is that there are three important cases here:

(a) $[\lambda_3] = 1$. The dimensionless parameter is λ_3/E, where E is the energy scale of the process of interest (e.g. the scattering energy, on the order of TeV at the LHC). If $\lambda_3/E \ll 1$, then $\lambda_3\phi^3/3!$ is a small perturbation at high energies. We call this a *relevant perturbation* because it is important at low energies. In a relativistic setting, $E > m$ so we can make the perturbation small by taking $\lambda_3 \ll m$. We call this class of theories with positive mass dimension coupling constants *renormalizable*, meaning that we can reasonably deal with the infinities which crop up from weak coupling.

(b) $[\lambda_4] = 0$. Here, $\lambda_4\phi^4/4!$ is small if $\lambda_4 \ll 1$. We call these *marginal* couplings, and these are also renormalizable.

[18] Green's functions are useful because they allow us to easily fit the boundary conditions. Consider the operator equation $\hat{O}\psi(x) = f(x)$ for some differential operator $\hat{O}$ and some given function $f(x)$. If we could just write down $\hat{O}^{-1}$, it would be easy enough to solve any equation of this form: $\psi(x) = \hat{O}^{-1}f(x)$. This is sort of what Green's functions let us do. If we know that $\hat{O}\Delta(x-y) = \delta(x-y)$, it follows that $\hat{O}\left[\int dy\Delta(x-y)f(y)\right] = \int dy\delta(x-y)f(y) = f(x)$ (where any derivatives in $\hat{O}$ are taken with respect to x), so $\int dy\Delta(x-y)f(y) = \psi(x)$ solves the differential equation.

(c) $[\lambda_n] = 4 - n$ for $n \geq 5$. These are called *irrelevant* couplings. The dimensionless parameter is $\lambda_n E^{n-4}$, and they are small at low energies but large at higher energies. These lead to non-renormalizable theories, where the infinities are bigger and scarier and we cannot sweep them under the rug by just subtracting off infinity.

On the one hand, the nature of irrelevant couplings means that we can describe (relatively) low-energy physics well by only looking at the first few terms in the perturbative expansion, but it also makes it very difficult to probe very high-energy physics (for instance, on the scale of quantum gravity).

Example 7.1. Let's consider ϕ^4 theory, with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda\phi^4}{4!}; \lambda \ll 1.$$

We can already guess at the effects of this final term– in particular, $[H, N] \neq 0 \implies$ particle number is no longer conserved.[19] Expanding the last term, we expect some big integrals which will have

$$\int \ldots ((a_{\mathbf{p}}^\dagger)^4 \ldots) + \int \ldots a_{\mathbf{p}}^{\dagger 3} a_{\mathbf{p}} + \ldots$$

which will destroy particles.

Example 7.2. We could also consider scalar Yukawa theory, $\psi \in \mathbb{C}, \phi \in \mathbb{R}$ with the Lagrangian

$$\mathcal{L} = \partial_\mu\psi^*\partial^\mu\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \mu^2\psi^*\psi - \frac{1}{2}m^2\phi^2 - g\psi^*\psi\phi.$$

In this theory, $[g] = 1$ and we take $g \ll m, g \ll \mu$. We get a Noether current from noticing that the Lagrangian is invariant under $\psi \to e^{i\theta}\psi$, and this current has the interpretation of charge conservation– the number of ψ particles $-$ the number of ψ anti-particles is conserved, but there is no such conservation law for the number of ϕs.

The interaction picture Previously, we saw the familiar Schrödinger picture where operators are time-independent and states evolve in time by the Schrödinger equation,

$$i\frac{d}{dt}|\psi\rangle_S = H|\psi\rangle_S.$$

We then introduced the Heisenberg picture, where we moved the explicit time dependence into the operators,

$$|\psi\rangle_H = e^{iHt}|\psi\rangle_S, O_H(t) = e^{iHt}O_S e^{-iHt}.$$

The interaction picture is a hybrid of the Heisenberg and Schrödinger pictures. It splits the Hamiltonian into a free theory part and an interaction part:

$$H = H_0 + H_{\text{int}}.$$

Example 7.3. In ϕ^4 theory, we have $\mathcal{L}_{int} = -\lambda\phi^4/4!$ with

$$H_{int} = -\int d^3x\mathcal{L}_{int} = +\lambda\int \phi^4/4!$$

and H_0 the standard free theory Hamiltonian $H_0 = \int d^3x \frac{1}{2}\pi^2 + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2$.

Non-lectured supplement: contour integration and the p^0 integral If you haven't seen contour integration before, it's basically an integration technique for certain real integrals which makes use of a theorem called the Cauchy residue theorem. I'll use some different notation here (ks instead of ps and ω_k instead of E_p), but all the physics is the same. I'm also setting $y = 0$ here since Δ_F only depends on the combination $|x - y|$.

[19] Those of you with some familiarity with Feynman diagrams can probably cook up a simple diagram which goes from one to three particles using the ϕ^4 interaction. The interaction has four lines so just put one on the left and three on the right (no need to worry about antiparticles since this is a scalar field).

Cauchy came up with a nice formula which says that if a function $f(z)$ is analytic[20] on and inside a simple[21] closed curve C, then the value of the following contour integral[22] along C is given by

$$\oint \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

for $z = a$ a point inside C.

Mathematicians usually write this as a formula for the value of $f(a)$ in terms of the contour integral, but for our purposes it is more useful as a formula for the integral. The proof is not complicated and fits on a page or two (see for instance Boas Mathematical Methods 585-586 or http://mathworld.wolfram.com/CauchyIntegralFormula.html) but I will not repeat it here.

What's the practical use of this formula? Essentially, we can use it to compute real integrals which might have poles (singular points) along the integration path. Consider our expression for the propagator, and suppose $\varepsilon = 0$. Then the denominator becomes

$$k^2 - m^2 = (k^0)^2 - \mathbf{k}^2 - m^2 = (k^0)^2 - \omega_k^2$$

and written this way, it is clear that the integrand is going to become singular at $k^0 = \pm\omega_k$. Therefore, we make an "$i\varepsilon$ prescription," meaning that we add $i\varepsilon$ ($\varepsilon > 0$ and small) to push the poles off the real line into the complex plane so we can do the integral, and hope nothing bad happens as we let ε go to zero.

We'll need one more trick to compute this integral. You might have noticed that our integral isn't a closed curve yet (as required by the Cauchy formula)– it is an integral $\int_{-\infty}^{\infty} dk^0$. Therefore, we must close the contour by adding a curve whose final contribution to the overall integral will be zero. To warm up, suppose we want to compute

$$\int_{-\infty}^{\infty} dz \frac{e^{iz}}{z - iz_0}$$

for z_0 possibly complex. We can close the contour by adding a curve in the upper half-plane, "out at $+i\infty$." See Figure 1 for an illustration.

How do we decide whether to close the contour in the upper or lower half-plane? Notice that in the upper half-plane, $z = x + iy$ for $y > 0$, so $e^{iz} = e^{i(x+iy)} = e^{-y}e^{ix}$ with $y > 0$. Therefore, e^{iz} is exponentially damped in the upper half-plane and contributes basically zero to the overall integral. So we can close the curve "for free" and write

$$\int_{-\infty}^{\infty} dz \frac{e^{iz}}{z - z_0} = \oint dz \frac{e^{iz}}{z - z_0} = 2\pi i e^{iz_0}$$

if z_0 has imaginary part > 0 (is inside the contour) and 0 otherwise. Thanks, Cauchy integral formula.

In fact, the formula lends itself to an even better generalization, the *Cauchy residue theorem*. It states that

$$\oint_C f(z)dz = 2\pi i \cdot \text{sum of the residues of } f(z) \text{ inside } C,$$

where the integral around C is in the counterclockwise direction, and a *residue* is basically the value at the function at the pole if it didn't have that pole. Quick example: for the function $f(z) = \frac{z}{(1+z)(3-z)}$, $f(z)$ has a pole at $z = 3$. The residue $R(3)$ of $f(z)$ at $z = 3$ is simply $R(5) = \frac{z}{1+z}|_{z=3} = \frac{3}{4}$.

So to summarize, close the contour based on what the integrand is doing at $\pm i\infty$. Check which poles are inside your contour, and plug up the singularities one at a time to compute the residues. Sum up the residues, multiply by $2\pi i$, and you've got the value of your integral.

Returning to the problem at hand, we wish to compute the integral

$$\int dk^0 \frac{e^{ik^0x^0}}{(k^0)^2 - (\omega_k^2 - i\varepsilon)}.$$

[20] not singular

[21] does not cross itself

[22] A fancy name for a line integral in the complex plane.

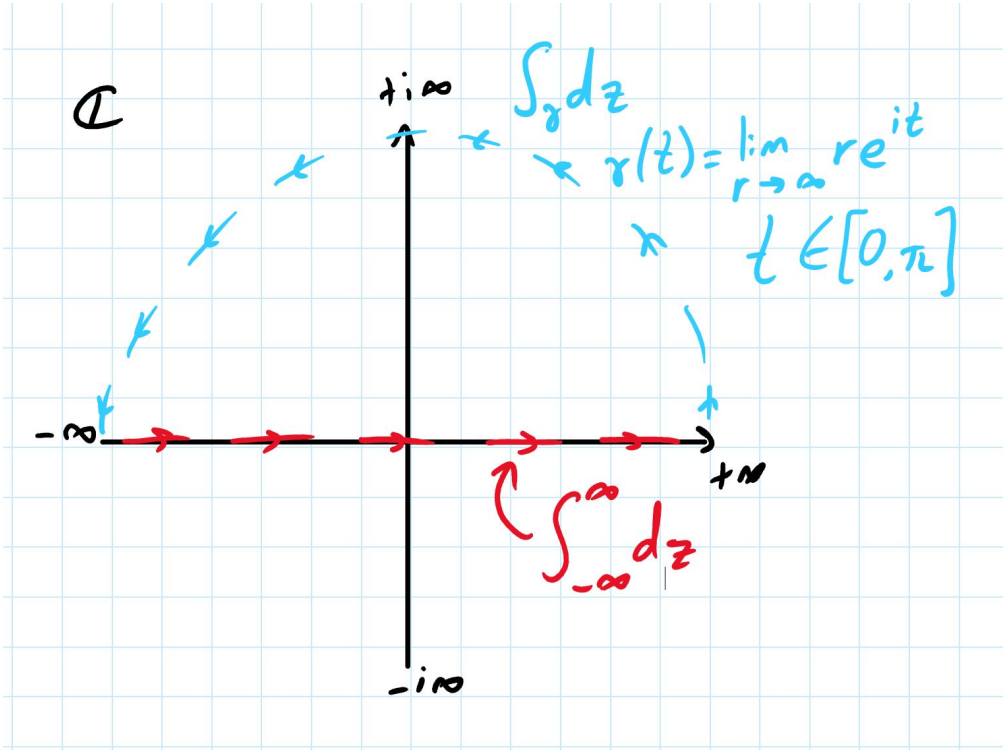

FIGURE 1. An illustration of closing the contour for a real integral (in red) so we can use the Cauchy integral formula to perform the integral. The integral along the light blue curve $\int_\gamma dz$ contributes nothing to the overall integration, so we can use it to close the contour out at $+i\infty$. Note the curve runs counterclockwise. If it ran clockwise, we would pick up a minus sign.

We have two poles at $\pm\sqrt{\omega_k^2 - i\varepsilon}$, which in the $\varepsilon \to 0$ limit become $+\omega_k - i\varepsilon$ and $-\omega_k + i\varepsilon$. Therefore, we rewrite as

$$\int_{-\infty}^{\infty} dk^0 \frac{e^{ik^0x^0}}{(k_0 - (\omega_k - i\varepsilon))(k_0 - (-\omega_k + i\varepsilon))}.$$

Since $e^{ik^0x^0}$ is exponentially damped in the upper half-plane, we close the contour at $+i\infty$, enclosing the pole at $-\omega_k + i\varepsilon$ (recall ω_k is real and ε is positive). Therefore, calculating the residue, this integral comes out to

$$2\pi i \frac{e^{-i\omega_k x^0}}{-2\omega_k}$$

(letting $\varepsilon \to 0$) and we conclude that for $x^0 > 0$,

$$\Delta_F(x) = -i \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-i(\omega_k t - \mathbf{k}\cdot\mathbf{x})}.$$

A similar calculation holds for $x^0 < 0$, so we recover our friend the Feynman propagator, which correctly accounts for the sign of x^0.

Lecture 8.

Saturday, October 20, 2018

Today, we'll take our first look at interacting theories in detail! Let's first complete our description of the interaction picture. Operators in the interaction picture evolve in time as

$$O_I(t) \equiv e^{iH_0 t} O_S e^{-iH_0 t},$$

with O_S the Schrödinger picture operator. In the context of quantum field theory,

$$\phi_I(x) = e^{iH_0t}\phi(\mathbf{x})e^{-iHt}$$

so that ϕ_I obeys the Klein-Gordon equation

$$(\partial^2 + m^2)\phi_I = 0,$$

with solution

$$\phi_I(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}(a_\mathbf{p}e^{-ip\cdot x} + a^\dagger_\mathbf{p}e^{ip\cdot x}.$$

Here, note that we're taking the four-vector inner product as in the Heisenberg picture, with $p^0 = E_p$ and $x^0 = t$. We also see that $\phi_I(t=0,\mathbf{x}) = \phi_S(\mathbf{x})$, so the fields at $t=0$ agree with the Schrödinger picture fields.

As before,

$$[a_\mathbf{p}, a^\dagger_\mathbf{p}] = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}'),$$

with other brackets vanishing. Note that $|0\rangle$ satisfies $a_\mathbf{p}|0\rangle = 0$ is the vacuum of the free theory, not the interacting theory. Interaction picture fields are related to the Heisenberg picture ones by

$$\phi_H(t,\mathbf{x}) = e^{iHt}e^{-iH_0t}\phi_I(x)e^{iH_0t}e^{-iHt},$$

where $e^{-iH_0t}\phi_I(x)e^{iH_0t} = \phi_S(\mathbf{x})$. We can also regroup the operators here to write

$$\phi_H(t,\mathbf{x}) \equiv U(t,0)^\dagger\phi_I(t,\mathbf{x})U(t,0)$$

where $U(t,t_0) \equiv e^{iH_0(t-t_0)}e^{-iH(t-t_0)}$ is a unitary time evolution operator such that

$$U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3)$$

and $U(t,t) = 1$. Equivalently,

$$|\psi(t)\rangle_I = U(t,t')\,|\psi(t')\rangle_I\,.$$

That is, U evolves interaction picture states in time.

By differentiating with respect to time, we see that

$$\begin{aligned} i\frac{dU(t,0)}{dt} &= i\left[iH_0e^{iH_0t}e^{-iHt} + e^{iH_0t}(-iH)e^{-iHt}\right] \\ &= e^{iH_0t}(H-H_0)e^{-iHt} \\ &= e^{iH_0t}(H_{int})_Se^{-iH_0t}e^{iH_0t}e^{-iHt} \\ &= (H_{int})_IU(t,0). \end{aligned}$$

If $(H_{int})_I = H_I$ were just a function, we could solve this by $U = \exp[-i\int_{t_0}^t H_i(t')dt']$, but because it is an operator we have ordering ambiguities,

$$[H_i(t'), H_I(t'')] \neq 0 \text{ for } t' \neq t''.$$

However, our differential equation for U tells us that $U(t,t_0)$ satisfies

$$U(t,t_0) = 1 + (-i)\int_{t_0}^t dt'H_I(t')U(t',t_0)$$

(you can check this explicitly in one line). Therefore we can substitute this back into itself to get the infinite series

$$U(t,t_0) = 1 + (-i)\int_{t_0}^t dt'H_I(t') + (-i)^2\int_{t_0}^t dt'\int_{t_0}^{t'} dt''H_I(t')H_I(t'') + \ldots.$$

From the ranges of integration, it's clear that the H_Is are automatically time-ordered– for instance, $H_I(t'')$ always takes place at $t'' \geq t'$. In general we can exploit symmetry to write the second term as

$$\frac{(-i)^2}{2!}\int_{t_0}^t dt'\int_{t_0}^t dt''T(H_I(t')H_I(t'')$$

(where we've picked up a 2 as a symmetry factor), and in general we get a symmetry factor of $n!$ to a term with n copies of $H_I(t')$.

Definition 8.1. We find that U can be written compactly as

$$U(t,t_0) = T\exp\{-i\int_{t_0}^{t} dt' H_I(t')\},$$

which we call *Dyson's formula.* (Note that $U(t,t_0 = T\exp\{+i\int_{t_0}^{t} dt' L_I(t')\}$, in terms of the Lagrangian.) This is a formal result, but we usually just expand to some finite order in terms of the coupling constants which live in the interacting Hamiltonian H_I.

This is the last bit of machinery we need to start computing scattering amplitudes in quantum field theory!

Definition 8.2. The time evolution used in scattering theory is called the *S-matrix* (S for scattering). The S matrix is defined to be

$$S = \lim_{t\to\infty, t_0\to-\infty} U(t,t_0).$$

We will consider interactions where the final state $|f\rangle$ and the initial state $|i\rangle$ are well-separated from each other and are far away from the interaction. Therefore, the initial and final states $|i\rangle, |f\rangle$ behave like free particles, i.e. they are eigenstates of H_0.[23]

This should seem at least plausible: at late/early times, the particles are well-separated and don't feel the effect of each other. As they approach, they may interact before going their separate ways. The scattering amplitude is then

$$\lim_{t\to\infty, t_0\to-\infty} \langle f| U(t,t_0) |i\rangle = \langle f| S |i\rangle .$$

Note that there are some cases that need to be treated differently, like bound states. For instance, a proton and an electron at low energies could interact, $p + e^- \to$ the bound state (H). Here, the assumption that the particles end up well-separated is violated. It turns out that such solutions appear as poles in the S-matrix, but this is a more advanced topic and we won't discuss it further here.

Let's return to scalar Yukawa theory. Now, we'll drop the I subscripts and assume uniformly that we are in the interaction picture. We have the Hamiltonian

$$\mathcal{H} = g\psi^*\psi\phi,$$

where ψ and ψ^* are (anti-)nucleons (e.g. a proton or neutron), and ϕ is a meson.

- ϕ has a and $a^\dagger$ terms which destroy and create mesons, respectively.
- ψ has b and $c^\dagger$ terms, where b destroys a nucleon and $c^\dagger$ creates an anti-nucleon.
- ψ^* has $b^\dagger$ and c terms, where $b^\dagger$ creates a nucleon and c destroys an anti-nucleon.

Looking at the possible terms in the Hamiltonian, we can already see interesting behavior– we'll have terms where nucleon-anti-nucleon pairs are created and destroyed, e.g. $b^\dagger c^\dagger a$ which destroys a meson and produces a nucleon-anti-nucleon pair. This contributes to meson decay, $\phi \to \psi\bar{\psi}$.[24] What we recover is the leading order in g term in $U(t,t_0)$.

At second order, we have more complicated terms like

$$g^2(b^\dagger c^\dagger a)(a^\dagger c b),$$

which describes nucleon-anti-nucleon scattering. We can draw a nice diagram for this process.

Returning to the case of meson decay, we have some ϕ meson going in with some defined momentum $\mathbf{p}$ as our initial state, and similarly we have $\psi, \bar{\psi}$ going out with some momenta $\mathbf{q}_1, \mathbf{q}_2$. We can write these states as

$$|i\rangle = \sqrt{2E_p} a^\dagger_{\mathbf{p}} |0\rangle$$

[23] This is a heuristically useful description but a little slippery in the details. A priori, there's no reason that eigenstates of the free Hamiltonian should be eigenstates of the interacting Hamiltonian. If you prefer, you can think of the scattering amplitude as the overlap (as measured by the inner product) between initial free particle states and final free particle states, with the possibility for some interaction in between. Even if we started with free particle eigenstates, our interaction is sure to evolve these states to some new ones, but we can look at the overlap between the time evolved versions of the free particle states $U(t,t_0)|i\rangle$ and the final free particle states we're interested in, $\langle f|$.

[24] When we talk about the fields, we use $*$, but when we denote antiparticles, we usually use the bar notation, e.g. an anti-ψ is a $\bar{\psi}$.

and

$$|f\rangle = \sqrt{4E_{q_1}E_{q_2}}b^\dagger_{\mathbf{q}_1}c^\dagger_{\mathbf{q}_2}|0\rangle\,.$$

To zeroth order there is no interaction and the scattering amplitude is zero. To first order, we have

$$\langle f|\,S\,|i\rangle = -ig\,\langle f|\int d^4x\psi^*(x)\psi(x)\phi(x)\,|i\rangle + O(g^2).$$

We'll compute this exactly next time and argue that the $O(g^2)$ corrections are relatively small, arriving at our first quantum field theory scattering amplitude.

Lecture 9.

Tuesday, October 23, 2018

Today, we'll introduce Wick's formula and contractions, calculate some more scattering amplitudes, and maybe see our first Feymman diagrams for calculating amplitudes in a more convenient way.

First, we complete the calculation of the order g scattering amplitude from last time. We were interested in meson decay, where we prepared initial and final states

$$|i\rangle = \sqrt{2E_p}a^\dagger_{\mathbf{p}}\,|0\rangle$$

and

$$|f\rangle = \sqrt{4E_{q_1}E_{q_2}}b^\dagger_{\mathbf{q}_1}c^\dagger_{\mathbf{q}_2}\,|0\rangle\,,$$

and we were interested in the scattering amplitude $\langle f|\,S\,|i\rangle$. To leading order, we found that

$$\langle f|\,S\,|i\rangle = -ig\,\langle f|\int d^4x\psi^*(x)\psi(x)\phi(x)\,|i\rangle + O(g^2),$$

and we'll see how to compute this.

We know how to expand each of these fields in terms of creation and annihilation operators, and we want to make sure that the initial state and final state are indeed proportional to each other so that this QFT amplitude will be reduced to a c-function of the four-momenta (i.e. it is just a number).

For instance, in ϕ we will have both $a^\dagger_{\mathbf{p}}$ and $a_{\mathbf{p}}$ terms, but our initial state already has an $a^\dagger$. So the $a^\dagger$ bit of ϕ acting on $|i\rangle$ will produce a two-meson state which the ψs won't touch, which means that its inner product with $\langle 0|$ will be zero. (Alternately you can think of the $a^\dagger$ from ϕ as acting on the $\langle 0|$ on the left, commuting through the ψs. That is, $a_{\mathbf{k}}\,|0\rangle = 0 \implies \langle 0|\,a^\dagger_{\mathbf{k}} = 0$.) So our matrix element takes the form

$$\langle f|\,S\,|i\rangle = -ig\,\langle f|\int d^4x\psi^*(x)\psi(x)\int\frac{d^3k}{(2\pi)^3}\frac{\sqrt{2E_p}}{\sqrt{2E_k}}(a_ka^\dagger_pe^{-ik\cdot x} + a^\dagger_ka^\dagger_pe^{ik\cdot x})\,|0\rangle\,,$$

but this second term is just zero and we can switch the $a_k, a^\dagger_p$ at the cost of a delta function $(2\pi)^3\delta^3(\mathbf{k}-\mathbf{p})$, which allows us to do the integral over d^3k.

We get

$$\begin{aligned}\langle f|\,S\,|i\rangle &= -ig\,\langle f|\int d^4x\psi^*\psi(x)e^{-ip\cdot x}\,|0\rangle\\ &= -ig\,\langle 0|\int\frac{d^4x}{(2\pi)^6}\frac{d^3k_1d^3k_2}{\sqrt{4E_{k_1}E_{k_2}}}\sqrt{4E_{q_1}E_{q_2}}c_{q_2}b_{q_1}(b^\dagger_{k_1}e^{ik_1\cdot x} + c_{k_1}e^{-ik_1\cdot x})(b_{k_1}e^{-ik_2\cdot x} + c^\dagger_{k_2}e^{ik_2\cdot x})e^{-ip\cdot x}\,|0\rangle\,.\end{aligned}$$

Only the $b^\dagger$ and $c^\dagger$ terms survive, and taking the appropriate commutators gives us delta functions over the momenta. That is,

$$\begin{aligned}\langle f|\,S\,|i\rangle &= -ig\,\langle 0|\int d^4xe^{i(q_1+q_2-q)\cdot x}\,|0\rangle\\ &= -ig[(2\pi)^4\delta^4(q_1+q_2-p)],\end{aligned}$$

where this delta function simply imposes overall 4-momentum conservation. Note that this is a matrix element, not a probability yet. To actually turn this into a measurable probability, we must take the mod squared and integrate over the possible outgoing momenta q_1, q_2– we'll defer this to later.

We'll now discuss *Wick's theorem* for a real scalar field. We wish to compute quantities like

$$\langle f|\, T\{H_I(x_1)\dots H_I(x_n)\}\,|i\rangle\,,$$

the amplitude of some time-ordered product– remember that Dyson says we ought to be evolving our states with time-ordered products. Our life would be easier if we worked in terms of normal-ordered products instead, where the as are on the RHS and the $a^\dagger$s are on the LHS. This would let us easily see which terms contribute to the final amplitude (e.g. which as cancel particles created by $a^\dagger$s on the RHS). Wick's theorem relates time-ordered products to normal-ordered products in a nice way.

Let's compute a simple example first: write

$$\phi(x) \equiv \phi^+(x) + \phi^-(x)$$

where

$$\phi^+(x) \equiv \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}} a_p e^{-ip\cdot x}$$

and

$$\phi^-(x) \equiv \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}} a_p^\dagger e^{+ip\cdot x}.$$

If we look at the case $x^0 > y^0$, the time-ordered product $T\{\phi(x)\phi(y)\}$ takes the form

$$\begin{aligned} T\{\phi(x)\phi(y)\} &= \phi(x)\phi(y) \\ &= (\phi^+(x)+\phi^-(x))(\phi^+(y)+\phi^-(y)) \\ &= \phi^+(x)\phi^+(y) + \phi^-(x)\phi^-(y) + \phi^-(y)\phi^+(x) + \phi^-(x)\phi^+(y) + [\phi^+(x),\phi^-(y)] \\ &=: \phi(x)\phi(y) : + D(x-y) \end{aligned}$$

where we've collected terms with all the ϕ^+ terms to the right of the ϕ^- terms into the normal-ordered product $: \phi(x)\phi(y) :$. If we had $y^0 > x^0$ instead, we would get

$$T\{\phi(x)\phi(y)\} =: \phi(x)\phi(y) : + D(y-x).$$

Putting these together, we see that

$$T\{\phi(x)\phi(y)\} =: \phi(x)\phi(y) : + \Delta_F(x-y),$$

where Δ_F is simply the Feynman propagator. It's important to note that while the time-ordered and normal-ordered products are both operators, their difference is Δ_F, a c-function.

Definition 9.1. We define a *contraction* of a pair of fields in a string in a string (denoted by a square bracket between the two fields) to mean replacing the two contracted fields by their Feynman propagator.

For instance, we saw that

$$T\{\phi(x)\phi(y)\} =: \phi(x)\phi(y) + \overbrace{\phi(x)\phi(y)}\,.$$

Wick's theorem states that

$$T\{\phi(x_1)\dots\phi(x_n)\} =: \phi(x_1)\dots\phi(x_n) : + : \text{all possible contractions} : .$$

Since all normal ordered terms kill the vacuum state, this allows us to immediately compute amplitudes like

$$\langle 0|\, T\{\phi_1\dots\phi_4\}\,|0\rangle = \Delta_F(x_1-x_2)\Delta_F(x_3-x_4) + \Delta_F(x_1-x_3)\Delta_F(x_2-x_4) + \Delta_F(x_1-x_4)\Delta_F(x_2-x_3).$$

The proof of Wick's theorem is by induction. Suppose it holds for $T\{\phi_2\dots\phi_n\}$. Then (see textbooks for detail)

$$T\{\phi_1\phi_2\dots\phi_n\} = (\phi_1^+ + \phi_1^-)[:\phi_2\dots\phi_n + : \text{all contractions of } \phi_2\dots\phi_n :].$$

The ϕ_1^- is okay where it is, while the ϕ_1^+ must be commuted to the RHS of the $\phi_2\dots\phi_n$ terms. Each commutator past the x_k term in $\phi_2\dots\phi_n$ gives us a $D(x_1-x_k)$, which is equivalent to a contraction between ϕ_1 and ϕ_k.

Wick's theorem has some immediate consequences. For instance,

$$\langle 0|\, T\phi_1\dots\phi_n\,|0\rangle = 0$$

if n is odd (since one ϕ is always left out of the contractions) and it is

$$\sum_{i_1,\ldots,i_n} \Delta_F(x_{i_1} - x_{i_2})\Delta_F(x_{i_3} - x_{i_4}) \ldots \Delta_F(x_{i_{n-1}} - x_{i_n})$$

where the sum is taken over symmetric permutations of $i_1, \ldots, i_n$.

Note that Wick's theorem also has a generalization to complex fields $\psi \in \mathbb{C}$, e.g.

$$T(\psi(x)\psi^*(y)) =: \psi(x)\psi^*(y) : +\Delta_F(x-y)$$

where the contraction of $\psi(x)\psi^*(y) \equiv \Delta_F(x-y)$ and the contractions of two ψs or two ψ^*s is zero.

Lecture 10.

Thursday, October 25, 2018

Last time, we began computing the amplitude for meson decay to first order in g. Now let's apply Wick's theorem to nucleon scattering, $\psi(p_1)\psi(p_2) \to \psi(p_1')\psi(p_2')$. Our initial state looks like

$$|i\rangle = \sqrt{4E_{p_1}E_{p_2}} b^\dagger_{\mathbf{p}_1} b_{\mathbf{p}_2} |0\rangle \equiv |p_1, p_2\rangle$$

and our final state is

$$|f\rangle = \sqrt{4E_{p_1'}E_{p_2'}} b^\dagger_{\mathbf{p}_1'} b_{\mathbf{p}_2'} |0\rangle \equiv |p_1', p_2'\rangle .$$

We aren't interested where there's no scattering (i.e. the zeroth order term where the nucleons just go on their way without any interaction). So what we're really interested in is the $O(g^2)$ term

$$\langle f| (S-1) |i\rangle .$$

(In the context of Feynman diagrams, it's pretty clear that any single interaction $O(g)$ would produce a meson we don't want in our final state. The leading order term will be $O(g^2)$.)

The amplitude takes the form

$$\frac{(-ig)^2}{2!} \int d^4x_1 d^4x_2 \langle p_1', p_2'| T\{\psi^*(x_1)\psi(x_1)\phi(x_1)\psi^*(x_2)\psi(x_2)\phi(x_2)\} |p_1, p_2\rangle .$$

Using Wick's theorem, we know there is a term of the form

$$: \psi^*(x_1)\psi(x_1)\psi^*(x_2)\psi(x_2) : \overbrace{\phi(x_1)\phi(x_2)}$$

in the time-ordered product. The contracted bit will make sure we have no issues with ϕ fields (since there are no ϕs in our initial and final states), while the normal-ordered part gives us ψs to annihilate the initial nucleons and ψ^*s to create the final nucleons. All other terms are zero.

We will ignore all terms involving $c, c^\dagger$ in the field expansions since they give zeroes (i.e. to this order, we don't need to worry about antiparticles). We ought to compute

$$\begin{aligned}\langle p_1', p_2'| : \psi^*(x_1)\psi(x_1)\psi^*(x_2)\psi(x_2) : |p_1, p_2\rangle = & \int \frac{d^3q_1 \ldots d^3q_4 \sqrt{16E_{p_1}\ldots E_{p_4}}}{(2\pi)^1 2\sqrt{2E_{q_1}\ldots 2E_{q_4}}} \\ & \times \langle 0| b_{\mathbf{p}_1'} b_{\mathbf{p}_2'} \underbrace{b^\dagger_{\mathbf{q}_1} b^\dagger_{\mathbf{q}_2}}_{\psi^* s} \underbrace{b_{\mathbf{q}_3} b_{\mathbf{q}_4}}_{\psi s} b^\dagger_{\mathbf{p}_1} b^\dagger_{\mathbf{p}_2} |0\rangle \\ & \times e^{i(q_1\cdot x_1 + q_2\cdot x_2 - q_3\cdot x_1 - q_2\cdot x_2)}.\end{aligned}$$

Using commutation relations one can check (in a few lines) that this big mess of creation and annihilation operators simplifies to a slightly more manageable mess of delta functions we can integrate over and get rid of. That is,

$$\begin{aligned}\langle 0| b_{\mathbf{p}_1'} b_{\mathbf{p}_2'} b^\dagger_{\mathbf{q}_1} b^\dagger_{\mathbf{q}_2} b_{\mathbf{q}_3} b_{\mathbf{q}_4} b^\dagger_{\mathbf{p}_1} b^\dagger_{\mathbf{p}_2} |0\rangle = & \left[\delta^3(\mathbf{p}_1' - \mathbf{q}_2)\delta^3(\mathbf{p}_2' - \mathbf{q}_1) + \delta^3(\mathbf{p}_2' - \mathbf{q}_2)\delta^3(\mathbf{p}_1' - \mathbf{q}_1)\right] \\ & \times \left[\delta^3(\mathbf{q}_4 - \mathbf{p}_1)\delta^3(\mathbf{q}_3 - \mathbf{p}_2) + \delta^3(\mathbf{q}_4 - \mathbf{p}_2)\delta^3(\mathbf{q}_3 - \mathbf{p}_1)\right].\end{aligned}$$

If we now integrate over this, our delta functions give us several exponential terms:

$$\langle p_1', p_2' | : \psi^*(x_1)\psi(x_1)\psi^*(x_2)\psi(x_2) : |p_1, p_2\rangle = \left[e^{i(p_1'\cdot x_2 + p_2'\cdot x_1)} + e^{i(p_2'\cdot x_2 + p_1'\cdot x_1)}\right] \times \left[e^{-i(p_1\cdot x_2 + p_2\cdot x_1)} + e^{-i(p_2\cdot x_2 + p_1\cdot x_1)}\right].$$

Writing this all out, one can perform the x_1, x_2 integrals to get (surprise) even more delta functions. We also integrate over the internal momentum k and find as our final result

$$(-ig)^2 \left\{ \frac{i}{(p_1 - p_1')^2 - m^2 + i\epsilon} + \frac{i}{(p_2' - p_1)^2 - m^2 + i\epsilon} \right\} (2\pi)^4 \delta^4(p_1 + p_2 - p_1' - p_2').$$

In fact, there are two terms here– one where the ψs exchange a meson and go on their ways, and one where the ψs exchange a meson and then cross over (so that what we thought was the first nucleon was actually the second). The meaning of this will be more obvious when we draw the Feynman diagrams. It should be clear that the delta function imposes conservation of overall momentum (i.e. the outgoing momentum is equal to the ingoing momentum, $p_1 + p_2 = p_1' + p_2'$).

Feynman diagrams This is basically the simplest interesting calculation we could have done, and using Wick's theorem to get there has given us a big mess. Surely there must be a better way, you say. And there is. We draw *Feynman diagrams* to keep track of the different possible Wick contractions, i.e. to represent the perturbative expansion of $\langle f | (S-1) | i \rangle$. We have a set of rules for how to draw the diagrams representing different processes, and can associate integrals to the diagrams.

Here are the rules.

- Draw an external line for each particle in the initial and final states $|i\rangle, |f\rangle$, assigning a four-momentum to each.
- For $\mathbb{C}$ fields we ought to add an arrow to label the flow of charge. Choose an in(out) going arrow for (anti-)particles in $|i\rangle$, and the opposite for $|f\rangle$.
- Join the lines together with vertices as prescribed by the Lagrangian, i.e. making sure that the interaction has a corresponding term and that charge is conserved in each vertex.
- Assign a momentum k to each internal line i.
- Add a delta function corresponding to each vertex for momentum conservation, $(-ig)(2\pi)^4\delta^4(\sum_i k_i)$, where $\sum_i k_i$ is the sum of all 4-momenta flowing into the vertex and g is the coupling constant in the Lagrangian.
- For each internal line with a 4-momentum k, write a factor of the propagator for that particle, e.g. in Yukawa theory,

$$\int \frac{d^4k}{(2\pi)^4} D(k^2) \text{ where } D(k^2) = \begin{cases} \frac{i}{k^2 - m^2 + i\epsilon} & \text{for } \phi \\ \frac{i}{k^2 - \mu^2 + i\epsilon} & \text{for } \psi \end{cases}$$

Using the Feynman rules, we can immediately write down the amplitude for our nucleon scattering process: it is

$$\begin{aligned} \langle f | (S-1) | i \rangle &= (-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} (2\pi)^8 \{\delta^4(p_1 - p_1' - k)\delta^4(p_2 - p_2' + k) \\ &\quad + \delta^4(p_1 - p_2' - k)\delta^4(p_2 + k - p_1')\} \\ &= i(-ig)^2 \left(\frac{1}{(p_1 - p_1')^2 - m^2 + i\epsilon} + \frac{1}{(p_1 - p_2') - m^2 + i\epsilon} \right) (2\pi)^4 \delta^4(p_1 + p_2 - p_1' - p_2'). \end{aligned}$$

The diagrams are suggestive of an analogous classical scattering process, like billiard balls colliding elastically. If we like, we can say that this is like the nucleons exchanging a meson of 4-momentum k. However, note that this meson doesn't necessarily satisfy the relativistic dispersion relation $k^2 = m^2$. If it doesn't, it's called "off-shell" or a virtual particle, and the impact of virtual particle interactions is a purely quantum effect.

Conversely, the external legs of our diagram are forced to be *on-shell*– because these are outgoing particles (that one could really observe and measure in a detector, for example), they had better satisfy the relativistic dispersion relation. It's also important to recognize that while internal momenta are fixed by momentum

conservation in "tree-level" diagrams, once we introduce loops into our Feynman diagrams all bets are off and we must integrate over all possible momenta for those virtual particles.

Lecture 11.

Saturday, October 27, 2018

Last time, we introduced the Feynman rules for drawing Feynman diagrams and computing scattering amplitudes, and it's good to check that these diagrams really do correspond to Wick contractions of our fields. Let's now make a canonical definition of the amplitude $\mathcal{A}_{fi}$, defined by

$$\langle f|\,(S-1)\,|i\rangle = i\mathcal{A}_{fi}\underbrace{(2\pi)^4\delta^4(\sum_{j\in f}p_j-\sum_{j\in i}p_j)}_{\text{from translational invariance}}$$

where the i is included by convention to match with non-relativistic QM.

We should then refine the Feynman rules to compute the amplitude (stripping away the overall momentum-conserving delta function, since we will always get it). Here are our revised rules:

- Draw all possible diagrams with appropriate external legs given by $|i\rangle$, $|f\rangle$.
- Impose 4-momentum conservation at each vertex.
- Write a factor of the coupling $(-ig)$ at each vertex.
- For each internal line, add a factor of the propagator.
- Integrate over internal momenta $\int \frac{d^4k}{(2\pi)^4}$. (This is trivial for tree-level diagrams since the momenta are all fixed by momentum conservation, but these will be real integrals for diagrams with internal loops.)

Example 11.1. Consider the scattering process $\psi+\bar{\psi}\to\phi+\phi$ in scalar Yukawa theory. There are two diagrams for this, and both are of order $(-ig)^2$. We can write down the amplitude almost by inspection:

$$i\mathcal{A}_{if} = (-ig)^2\left[\frac{i}{(p_1-p_1')^2-\mu^2}+\frac{i}{(p_1-p_2')^2-\mu^2}\right]$$

Note we've dropped the $i\epsilon$s here since the denominators don't vanish.

Example 11.2. We can now consider our first loop diagram, $\phi\phi\to\phi\phi$. It's a $O(g^4)$ diagram, so we write down the amplitude for this diagram as

$$i\mathcal{A}_{if} = (-ig)^4\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-\mu^2+i\epsilon}\frac{i}{(k-p_2')^2-\mu^2+i\epsilon}\frac{i}{(k+p_1'-p_1)^2-\mu^2+i\epsilon}\frac{i}{(k+p_1')^2-\mu^2+i\epsilon}.$$

We won't actually compute this integral, though we should note that at least it has a chance of converging since it goes as d^4k/k^8. These loop integrals can be tricky, and we'll revisit them in more detail next term in Advanced QFT. Sometimes the integrals won't converge, and we'll need the machinery of renormalization to sweep away the infinities and get actual numbers out of our integrals.

Let's now consider ϕ^4 theory, with $\mathcal{H}_{int}=\frac{\lambda}{4!}\phi^4$. Now we have a single interaction vertex– it's a 4-point vertex, where for each vertex we pay a cost of $-i\lambda$. The other Feynman rules are the same. Note that there's no $1/4!$ factor here. To see why, consider the simplest diagram for $\phi\phi\to\phi\phi$ scattering.

$$i\mathcal{A}_{fi}\sim-\frac{i\lambda}{4!}\langle p_1',p_2'|:\phi(x)\phi(x)\phi(x)\phi(x):|p_1,p_2\rangle.$$

Generically, this is

$$\langle 0|\,a_{\mathbf{p}_1'}a_{\mathbf{p}_2'}\ldots a^\dagger_{\mathbf{p}_a}a^\dagger_{\mathbf{p}_2}\,|0\rangle$$

and so any one of the fields ϕ can annihilate or create the external particles. Therefore there are 4! ways of matching up the operators and commuting them so that we start and end with $|0\rangle$. You can get other combinatoric factors like this (often 2 or 4). Having a term $\lambda_n\phi^n/n!$ is conventional, though.

Let's consider now

$$\langle 0|\,T\{\phi(x_1)\ldots\phi(x_m)S\}\,|0\rangle,$$

which we call a correlation function. This is analogous to the correlation functions we saw in Statistical Field Theory. It's a more elementary but less physical object than an S-matrix element. For brevity, denote

$$\phi_i \equiv \phi(x_i).$$

Now the nth term in the expansion for S gives

$$\frac{1}{n!}\left(\frac{-i\lambda}{4!}\right)^n \int d^4y_1 \dots d^4y_n \, \langle 0| \, T\{\phi_1 \dots \phi_m \phi^4(y_1) \dots \phi^4(y_n)\} \, |0\rangle \, .$$

Wick's theorem tells us to contract all pairs of fields in all possible ways. As an example, consider the case $n = 1, m = 4$. Then we have a term

$$-\frac{i\lambda}{4!} \int d^4x \, \langle 0| \, T\{\phi_1 \dots \phi_4 \phi^4(x)\} \, |0\rangle \, .$$

We're going to have to contract all the fields, since any uncontracted fields will kill the vacuum states. We could get a contraction where all the numbered ϕ fields contract with the xs, e.g.

$$-\frac{i\lambda}{4!} \int d^4x \, \overbrace{\phi_1\phi(x)}\overbrace{\phi_2\phi(x)}\overbrace{\phi_3\phi(x)}\overbrace{\phi_4\phi(x)}$$

and permutations of $\phi_1, \dots, \phi_4$. We could also contract two of the numbered ϕs,[25]

$$-\frac{i\lambda}{4!} \int d^4x \, \overbrace{\phi_1\phi_2}\overbrace{\phi_3\phi(x)}\overbrace{\phi_3\phi_4}\overbrace{\phi(x)\phi(x)}$$

and permutations of contracting 2 ϕ_is. Finally, we'll have contractions of all the ϕ_is together, which look like

$$-\frac{i\lambda}{4!} \int d^4x \, \overbrace{\phi_1\phi_2}\overbrace{\phi_3\phi_4}\overbrace{\phi(x)\phi(x)}\overbrace{\phi(x)\phi(x)} \, .$$

The first of these gives us 4! terms of Feynman propagators $\Delta_F(x_i - x)$ (4 choices for x_1, 3 for x_2, and so on). There are 12 ways of pairing ϕ_3, ϕ_4 with $\phi(x)$ (4 choices for ϕ_3 and 3 choices for ϕ_4). There are also 12 choices for which two fields ϕ_i, ϕ_j to contract with $\phi(x)$s. Finally, there are 3 ways of pairing only $\phi(x)$s (e.g. take ϕ_1. We get 3 choices of $\phi_{i\neq 1}$ to pair it with, and the other contraction is then fixed).

So the first term gets $-i\lambda$, the second gets $-i\lambda/2$, and the last gets $-i\lambda/8$. Note that $\Delta_F(x - x) = \Delta_F(0)$ diverges, so these "bubble" diagrams will diverge badly. Our theory turns out to be renormalizable, but again this isn't always the case.

Lecture 12.

Tuesday, October 30, 2018

Last time, we introduced the correlation functions

$$\langle 0| \, T\{\phi(x_1) \dots \phi(x_m) S\} \, |0\rangle \, .$$

Let's consider the term with $m = 4$ and $n = 2$ (four numbered fields ϕ_i and two four-point vertices ϕ^4). That term looks like

$$\frac{1}{2}\left(\frac{-i\lambda}{4!}\right)^2 \int d^4x d^4y \, \langle 0| \, T\{\phi_1\phi_2\phi_3\phi_4\phi^4(x)\phi^4(y)\} \, |0\rangle \, .$$

As before, we claim that the most important contributions are the completely connected ones, and anything not totally contracted will vanish in the time-ordered product. One such contraction is

$$\overbrace{\phi_1\phi(x)}\overbrace{\phi_2\phi(x)}\overbrace{\phi_3\phi(y)}\overbrace{\phi_4\phi(y)}\overbrace{\phi(x)\phi(y)}\overbrace{\phi(x)\phi(y)} \, .$$

But we could get some distinct diagrams depending on how we connect up the dots. The Feynman rules for the first diagram give

$$\frac{(-i\lambda)^2}{2} \int d^4x d^4y \Delta_F(x_1 - x)\Delta(x_2 - x)\Delta_F(x_3 - y)\Delta_F(x_4 - y)\Delta_F^2(x - y).$$

Let's work out the combinatoric factors: there are four choices for which $\phi(x)$ goes with x_1 and three choices for which $\phi(x)$ goes with x_2, for a factor of 12. The same is true for x_3, x_4 and y. We get a factor of

[25] This will give us a disconnected Feynman diagram.

2 for which of the remaining $\phi(y)$s the first $\phi(x)$ contracts with, and then the other is determined. We also get a factor 2! from interchange of x and y. The four $\phi(x)$s are identical, as are the four $\phi(y)$s, so we should add a factor of $(1/4!)^2$ to take care of that. Finally, we have $\binom{4}{2} = 1/2!$ choices of which ϕ_is to connect to $\phi(x)$s. Putting it all together we get

$$\frac{1}{2!} \times \left(\frac{1}{4!}\right)^2 \times \underbrace{12}_{x_1, x_2 \to x} \times \underbrace{12}_{x_3, x_4 \to y} \times \underbrace{2}_{x \to y} \times \underbrace{2!}_{x \leftrightarrow y} = \frac{1}{2}.$$

The Feynman rules for the correlation functions of ϕ^4 theory are then given by

$$\langle 0| \, T \left\{ \phi(x_1) \dots \phi(x_m) \exp\left(-\frac{i\lambda}{4!} \int d^4x \phi^4(x) \right) \right\} |0\rangle \,,$$

which is equal to the sum of all diagrams with m external points and any number of internal vertices connected by propagator lines. In perturbation theory, we categorize the diagrams based on the number of powers of λ, i.e. the number of vertices in the diagram. For each diagram, there is one integral containing

- Each propagator from y to z, $\Delta_F(y - z)$
- Each vertex at x, $-i\lambda \int d^4x$,

and we divide by a symmetry factor. Since the propagator is an integral over momentum space, it's easier to express the Feynman rules in momentum space. Rather than integrating over all space d^4x we can equivalently just integrate a momentum-conserving delta function. Let's work out the momentum space Feynman rules:

- To each propagator from x to y, assign $e^{ip \cdot y}$ to the y vertex (where the arrow is going out) and $e^{-ip \cdot x}$ to the vertex x with arrows in.
- Associate $\frac{i}{p^2 - m^2 + i\epsilon}$ to the line itself (for a particle with mass m) and an integral over all momentum $\int \frac{d^4p}{(2\pi)^4}$.
- Thus the integral at a vertex becomes

$$\int d^4x e^{-ip_1 \cdot x + ip_2 \cdot x - ip_3 \cdot x + ip_4 \cdot x} = (2\pi)^4 \delta^4(p_1 + p_3 - p_2 - p_4)$$

  where p_1, p_3 are flowing into the vertex, p_2, p_4 out. (There should also be a $-i\lambda$ for each vertex.)

However, as before the δ functions will make some of the momentum integrals trivial, and for each of these the $(2\pi)^4$ will cancel. We are left with the following momentum space rules:

- For each internal line associate a factor of $\frac{i}{p^2 - m^2 + i\epsilon}$.
- For each vertex associate a factor of $-i\lambda$.
- Impose four-momentum conservation at vertices, and overall.
- Integrate over undetermined momenta from internal lines, $\int \frac{d^4k}{(2\pi)^4}$.
- Divide by the appropriate symmetry factor.

Note that there isn't really a nice way to get the symmetry factors from looking at the Feynman diagrams–one must usually consider the Wick contraction to get these factors right.

Vacuum bubbles and connected diagrams What is the transition from the vacuum state to the vacuum state, $\langle 0| \, S \, |0\rangle$? In ϕ^4 theory, we get a sum of "vacuum bubbles," diagrams with *no external lines*. One should check (e.g. on the second example sheet) that the S-matrix element is simply the exponential of the various topologically distinct vacuum bubble diagrams. Weird!

In general we call the correlation function

$$\langle 0| \, T\{\phi(x_1) \dots \phi(x_m) S\} \, |0\rangle$$

an *m-point function*, and its value is the sum over diagrams with m external points. A typical diagram has some vacuum bubbles, e.g. at second order in ϕ^4 we have a disconnected diagram which looks like a line with a loop and the figure 8. Remarkably, the vacuum bubbles add to the same exponential as in the pure

vacuum case. We'll discuss this more in detail next term, but there is an apparently sensible way of treating the vacuum bubbles.[26] Therefore we may write

$$\langle 0|\, T\{\phi(x_1)\dots\phi(x_m)S\}\,|0\rangle = (\sum \text{connected diagrams}) \times \langle 0|\, S\,|0\rangle\,,$$

where connected means that every point in the diagram is connected to at least one external line.

Really, the issue here comes from the fact that the vacuum of the free theory is *not* the vacuum of the interacting theory.

Definition 12.1. Let $|\Omega\rangle$ be the vacuum of the *interacting* theory, normalized such that $H\,|\Omega\rangle = 0$ with $H = H_0 + H_{int}$ (n.b. $H_0\,|0\rangle = 0$) and $\langle\Omega|\,\Omega\rangle = 1$. Then we define

$$G^{(n)}(x_1\dots x_n) \equiv \langle\Omega|\, T\{\phi_H(x_1)\dots\phi_H(x_n)\}\,|\Omega\rangle\,.$$

We call these *Green's functions*.

We claim now that

$$\langle\Omega|\, T\{\phi_{1,H}\dots\phi_{m,H}\}\,|\Omega\rangle = \frac{\langle 0|\, T\{\phi_{1,I}\dots\phi_{m,I}S]\}}{\langle 0|\, s\,|0\rangle}.$$

What this means as that the Green's functions are precisely given by the sum of connected diagrams with m external points– we need not worry too much about the vacuum bubbles and disconnected diagrams because removing the vacuum bubbles gets the behavior relative to the *interacting* vacuum right (and S evolves our interaction picture fields to Heisenberg picture fields). We'll do the proof next time.

Lecture 13.

Thursday, November 1, 2018

A brief correction: the time evolution operator should be defined $U(t,t_0) = e^{iH_0}e^{iH(t-t_0)}e^{-iH_0t_0)}$ so that our time evolution relations all work out.

Now last time, we claimed that

$$\langle\Omega|\, T\{\phi_{1,H}\dots\phi_{m,H}\}\,|\Omega\rangle = \frac{\langle 0|\, T\{\phi_{1,I}\dots\phi_{m,I}S]\}\,|0\rangle}{\langle 0|\, S\,|0\rangle}.$$

That is, it suffices to consider only connected diagrams, since the vacuum bubbles add up to a multiplicative factor (namely, the vacuum energy) that can be factored out of the overall correlation function. Let us expand the numerator on the RHS as

$$\langle 0|\, U(\infty,t_1)\phi_{1,I}U(t_1,t_2)\phi_{2,I}\dots U(t_{n-1},t_n)\phi_{n,I}U(t_n,-\infty)\,|0\rangle\,,$$

and WLOG we label the fields to already be time-ordered, with $x_1^0 > x_2^0 > \dots > x_m^0$. That is, we've split up the overall time evolution operator $S = \lim_{t\to\infty,t_0\to-\infty} U(t,t_0)$ into intervals from $t_i = x_i^0$ to $t_{i+1} = x_{i+1}^0$. We can then break up the operators,

$$U(t_1,t_2) = U(t_1,0)U(0,t_2)$$

so that the numerator becomes

$$\langle 0|\, U(\infty,0)\underbrace{[U(0,t_1)\phi_{1,I}U(t_1,0)]}_{\phi_{1,H}}[U(0,t_2)\phi_{2,I}U(t_2,0)]\dots U(t_{n-1},0)[U(0,t_n)\phi_{n,I}U(t_n,0)]U(0,-\infty)\,|0\rangle\,,$$

which is a bunch of Heisenberg picture states sandwiched between the vacuum states and some Us. More compactly,

$$\underbrace{\langle 0|\, U(\infty,0)\phi_{I,H}\dots\phi_{m,H}}_{\langle\psi|}\,U(0,-\infty)\,|0\rangle$$

setting everything back to the Heisenberg picture at $t=0$. Consider now this same expression where instead of evolving the free theory ground state with the unitary operator $U(0,-\infty)$, we consider evolution from $t=t_0$ some finite time to $t=0$:

$$\langle\psi|\, U(0,t_0)\,|0\rangle = \langle\psi|\, e^{iHt_0}\,|0\rangle$$

[26] I believe this is related to renormalization.

since $H_0 |0\rangle = 0$. Insert a complete set of interacting states $|p_1, \dots, p_n\rangle$. Then

$$\lim_{t_0\to-\infty} \langle\psi| U(0,t_0) |0\rangle = \lim_{t_0\to-\infty} \langle\psi| e^{iHt_0} \left[|\Omega\rangle\langle\Omega| + \sum_{n=1}^{\infty} \int \prod_{j=1}^{n} \frac{d^3p_j}{2E_{p_j}(2\pi)^3} |p_1,\dots,p_n\rangle\langle p_1,\dots,p_n| \right] |0\rangle$$
$$= \langle\psi|\,\Omega\rangle\langle\Omega|\,0\rangle + \lim_{t_0\to-\infty} \sum_{n=1}^{\infty} \int \prod_{j=1}^{n} \frac{d^3p_j}{2E_{p_j}(2\pi)^3} e^{i\sum_{k=1}^{n} E_{p_k} t_0} \langle\psi|\, p_1,\dots,p_n\rangle\langle p_1,\dots,p_n|\,0\rangle .$$

Note that in the first term $\langle\psi| e^{iHt_0} |\Omega\rangle\langle\Omega|\,0\rangle$, all nonzero powers of H from the exponential will kill the vacuum state $|\Omega\rangle$ by definition, so the only thing that survives is the zeroth order term, $\langle\psi|\,\Omega\rangle\langle\Omega|\,0\rangle$. Luckily, the second term vanishes completely due to the Riemann-Lebesgue lemma: stated roughly, "for reasonable $f(x)$ (i.e. square-integrable), $\lim_{\mu\to\infty}\int_a^b f(x)e^{i\mu x}dx = 0$." Therefore we find that

$$\lim_{t_0\to-\infty} \langle\psi| U(0,t_0) |0\rangle = \langle 0| U(\infty,0)\phi_{1,H}\dots\phi_{m,H} |\Omega\rangle\langle\Omega|\,0\rangle$$

where we have put back in the definition of $\langle\psi|$. By the same token,

$$\lim_{t_0\to\infty} \langle 0| e^{-iHt_0} |\psi\rangle = \langle 0| U(\infty,0) |\psi\rangle ,$$

so our numerator becomes

$$\langle\Omega|\,\phi_{1,H}\dots\phi_{m,H} |\Omega\rangle\langle\Omega|\,0\rangle\langle 0|\,\Omega\rangle$$

and the denominator is just $\langle 0|\,\Omega\rangle\langle\Omega|\,0\rangle$.[27] Therefore

$$\frac{\langle 0|\,T\{\phi_{1,I}\dots\phi_{m,I}S]\}\,|0\rangle}{\langle 0|\,S\,|0\rangle} = \frac{\langle\Omega|\,\phi_{1,H}\dots\phi_{m,H}\,|\Omega\rangle\langle\Omega|\,0\rangle\langle 0|\,\Omega\rangle}{\langle\Omega|\,0\rangle\langle 0|\,\Omega\rangle} = \langle\Omega|\,T\{\phi_{1,H}\dots\phi_{m,H}\}\,|\Omega\rangle ,$$

as promised. Note we have put time ordering back in since we explicitly time-ordered the fields when we expanded out S. This completes the proof. ⊠

In words, this tells us that we can do our calculations relative to the vacuum of the interacting theory $|\Omega\rangle$ rather than the vacuum of the free theory $|0\rangle$, which means (in terms of our perturbative expansion) that we need not consider vacuum bubbles when we compute our correlation functions.

Going back to our previous example, we say that to describe scattering in the interacting theory, our external states, e.g. $|p_1, p_2\rangle$, should come from the interacting theory. This means that we exclude loops on the external lines (a process we call "amputation").

Mandelstam variables In two-particle scattering processes, the same combinations of p_1, p_2, p_1', p_2' (ingoing and outgoing four-momenta) often appear, so it's useful to introduce the *Mandelstam variables* s, t, and u, defined as

$$s = (p_1 + p_2)^2 = (p_1' + p_2')^2$$
$$t = (p_1 - p_1')^2 = (p_2 - p_2')^2$$
$$u = (p_1 - p_2')^2 = (p_2 - p_1')^2$$

where the squared here indicates a four-vector product (e.g. $(p_1 + p_2)^2 = (p_1^\mu + p_2^\mu)(p_{1\mu} + p_{2\mu})$.

Exercise 13.1. Show that the sum of the Mandelstam variables is

$$s + t + u = m_1^2 + m_2^2 + {m_1'}^2 + {m_2'}^2,$$

where m_1, m_2, m_1', m_2' are the masses of the initial and final particles, so the Mandelstam variables are not all independent.

WLOG, we can consider the initial particles in the center-of-mass frame, i.e. a frame in which the net 3-momentum is zero. Thus $\mathbf{p}_1 = -\mathbf{p}_2$. In this frame, s takes the simple form

$$s = (p_1 + p_2)^2 = (E_1 + E_2)^2.$$

Since s is a Lorentz scalar, it takes the same value in all frames. Therefore $\sqrt{s}$ is the center of mass energy, e.g. at the LHC we say that $\sqrt{s} = 13\,\text{TeV}$. In particular if $m_1 = m_2$, then by symmetry $E_1 = E_2 = \sqrt{s}/2$.

[27] It's literally the same calculation, but suppose there are no fields. Then the first factor is just $\langle\Omega|\,\Omega\rangle = 1$ by normalization of the interacting theory vacuum states.

Cross sections and decay rates So far, $|i\rangle$ and $|f\rangle$ have been states of definite momenta. What happens in a realistic situation where our ingoing states are now some distribution (a density function) smeared over momenta?

To understand this, suppose we have a collision with $2 \to n$ scattering, i.e. we have two particles ingoing with momenta p_1, p_2 and n outgoing particles with momenta $q_1, \ldots, q_n$. Then the scattering amplitude is proportional to

$$\langle q_1 q_2 \ldots q_n |\ p_1 p_2 \rangle\, (2\pi)^4 \delta^4(p_1 + p_2 - \sum_{i=1}^{n} q_i).$$

But probabilities are related to the amplitude squared, so it seems as if we've picked up an extra delta function in computing the physical probability of this interaction. The resolution is this– in reality, $|i\rangle, |f\rangle$ are very sharply peaked superpositions of momentum eigenstates. That is, our ingoing states take the form

$$|p_1 p_2\rangle_{in} = \int \frac{d^3\tilde{p}_1}{(2\pi)^3 2E_{\tilde{p}_1}} \frac{d^3\tilde{p}_2}{(2\pi)^3 2E_{\tilde{p}_2}} f_1(\tilde{p}_1) f_2(\tilde{p}_2)\, |\tilde{p}_1 \tilde{p}_2\rangle\,,$$

where $|\tilde{p}_1 \tilde{p}_2\rangle$ are the real four-momentum eigenstates.

If we suppose that the outgoing particles are also pure momentum eigenstates, then then our delta functions are soaked up by integrals when we try to compute the transition probability W. We then have

$$\begin{aligned} W &= (2\pi)^8) \int \frac{d^3\tilde{p}_1}{(2\pi)^3 2E_{\tilde{p}_1}} \frac{d^3\tilde{p}_2}{(2\pi)^3 2E_{\tilde{p}_2}} \frac{d^3p_1'}{(2\pi)^3 2E_{p_1'}} \frac{d^3p_2'}{(2\pi)^3 2E_{p_2'}} \\ &\quad \times \{|M|^2 f_1(\tilde{p}_1) f_1^*(p_1') f_2(\tilde{p}_2) f_2^*(p_2') \delta^4(\sum_i q_i - \tilde{p}_1 - \tilde{p}_2) \delta^4(\sum q_i - p_1' - p_2')\}. \end{aligned}$$

Note that what we have written as the square of the matrix element here is really

$$|M|^2 = \langle q_1 \ldots q_n |\ \tilde{p}_1 \tilde{p}_2 \rangle \left\langle p_1' p_2' \right|\ q_1 \ldots q_n \rangle\,.$$

We'll clean this up later to write everything in terms of the physical values p and q rather than dummy variables $p', \tilde{p}$.

This expression for W is the transition probability for $2 \to n$ scattering to states of definite momentum $q_1 \ldots q_n$. We can expand one of the delta functions in Fourier space to write

$$\begin{aligned} W &= \int d^4x \int \frac{d^3\tilde{p}_1}{(2\pi)^3 2E_{\tilde{p}_1}} f_1(\tilde{p}_1) e^{i\tilde{p}_1 \cdot x} \frac{d^3\tilde{p}_2}{(2\pi)^3 2E_{\tilde{p}_2}} f_2(\tilde{p}_2) e^{i\tilde{p}_2 \cdot x} \\ &\quad \times \frac{d^3p_1'}{(2\pi)^3 2E_{p_1'}} f_1^*(p_1') e^{ip_1' \cdot x} \frac{d^3p_2'}{(2\pi)^3 2E_{p_2'}} f_2^*(p_2') e^{ip_2' \cdot x} \\ &\quad \times \delta^4(\sum_i q_i - p_1' - p_2') \end{aligned}$$

Using the normalization we define the Fourier transform of the wavepacket,

$$|\psi_i\rangle \equiv \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_p}} f_i(p) e^{-ip \cdot x}\, |p\rangle\,.$$

What we've called the matrix element $|M|^2$ is still a function of $\tilde{p}_1, \tilde{p}_2, p_1', p_2', q_i$, but one can use the notion of sharp peaks (i.e. in our distributions f_i) to approximate $|M^2|$ by its value where $\tilde{p}_i = p_i' = p_i$. That is, our momentum distributions f_i are localized around some values p_i, so they behave similarly to delta functions and we can set all the dummy variables to the physical momenta p_1, p_2. Then the transition probability becomes

$$W = \int d^4x \frac{|\psi_1(x)|^2}{2E_1} \frac{|\psi_2(x)|^2}{2E_2} (2\pi)^4 \delta^4(\sum_i q_i - p_1 - p_2) |M|^2,$$

which means that the wavepacket in position space has some corresponding spread– like momentum, it is localized and not a single value. The total transition probability is a function of the spread in momentum $f_i(p)$ as well as the momenta themselves, $|M|^2$. Thus

$$\frac{dW}{d^4x} = \frac{|\psi_1(x)|^2}{2E_1} \frac{|\psi_2(x)|^2}{2E_2} (2\pi)^4 \delta^4(\sum_i q_i - p_1 - p_2) |M|^2,$$

where $|M|^2$ is now the actual matrix element $|\langle q_1 \dots q_n|\ p_1 p_2\rangle|^2$. We'll complete this discussion next time.

Lecture 14.

Saturday, November 3, 2018

Assorted remarks. There was another correction– time evolution is correctly given by $U(t,t_0) = e^{iH_0 t}e^{-iH(t-t_0)}e^{-iH_0 t_0}$, where the $-$ in the second exponential is key. After today's lecture, we'll be able to solve all the problems on Example Sheet 2 (in principle). Remark: on question 10b on sheet 2, the answer is incorrect. It should read *"Find $\frac{d\sigma}{dt}$ in terms of g, s, t, m, and M."* Note that the matrix element $\mathcal{M}$ and the amplitude A_{f_i} are the same thing (e.g. in Prof. Allanach's notes). Whew.

Okay, moving on. Last time we wrote down

$$\frac{dW}{d^4x} = \frac{|\psi_1(x)|^2}{2E_1}\frac{|\psi_2(x)|^2}{2E_2}(2\pi)^4\delta^4(\sum_i q_i - p_1 - p_2)|M|^2,$$

which is the *transition probability density per unit time*. It depends (perhaps very weakly) on x. Here's the picture we should imagine– we have a "beam" of particle 2, described in space by a wavefunction $|\psi_2(x)|^2$ and moving with velocity v. Thus the flux of particle 2 per unit area is $\phi = v|\psi_2(x)|^2$. In the rest frame of particle 1, we have a density of particle 1 $\rho = |\psi_1(x)|^2$, and it has some effective cross-sectional area $d\sigma$. Therefore we can rewrite this probability density as

$$\frac{dW}{d^4x} = d\sigma \cdot \phi \cdot \rho.$$

Equivalently we write the differential cross section as

$$d\sigma = \frac{(2\pi)^4\delta^4(p_1 + p_2 - \sum_i q_i)}{\mathcal{F}}|M|^2$$

where $\mathcal{F} = 4E_1E_2v$ is the "flux factor." Thus $d\sigma$ is the effective cross-sectional area to scatter into final states of momenta $\{q_i\}$. If we now boost to the rest frame of particle 2, in this frame the four-momenta take the form

$$p_2^\mu = (m_2, 0), \quad p_1^\mu = (\sqrt{m_1^2 + |\mathbf{p}_1|^2}, \mathbf{p}_1).$$

The relative velocity $v = |\mathbf{p}_1|/E_1$, so in this frame the flux factor takes the form

$$\mathcal{F} = 4E_1|\mathbf{p}_1| = 4m_2\sqrt{E_1^2 - m_1^2} = 4\sqrt{(p_1\cdot p_2)^2 - m_1^2m_2^2},$$

where we have used the fact that in this frame $p_1 \cdot p_2 = E_1 m_2$. This is the correct Lorentz invariant definition of the flux factor.

In the massless limit, $m_1, m_2 \ll E_1, E_2$. This is the case for high-energy colliders like the LHC ($\sqrt{s} =$ 13 TeV, while $m_p \sim 1$ GeV). In this limit, we therefore have

$$\mathcal{F} = 4\sqrt{(p_1\cdot p_2)^2 - m_1^2m_2^2} \approx 4(p_1\cdot p_2) \approx 2(m_1^2 + 2p_1\cdot p_2 + m_2^2) = 2(p_1+p_2)^2,$$

where we have added on and neglected mass terms rather freely in the limit where the masses are small compared to the $p_1 \cdot p_2$ term which is proportional to the energy E_1.

Then $\mathcal{F} \sim 2s$ where $s = (p_1+p_2)^2$. To compute the total cross-section, we then sum over the $\{q_i\}$ in the correct manner to get

$$\sigma = \int \prod_{i=1}^n \left(\frac{d^3q_i}{(2\pi)^3 2E_{q_i}}\right)\frac{|M|^2}{\mathcal{F}}(2\pi)^4\delta^4(p_1+p_2-\sum_{i=1}^n q_i).$$

We call the integrals over d^3q_i "phase space integrals."

$2 \to 2$ **scattering** Let us specialize in the case of 2 to 2 scattering. What is the behavior of the differential cross-section, e.g. in terms of the Mandelstam variables? Let's look at the variations of σ with respect to

$$t = (p_1 - q_1)^2 = m_1^2 + {m_1'}^2 - 2E_{p_1}E_{q_1} + 2\mathbf{p}_1 \cdot \mathbf{q}_1.$$

Notice that

$$\frac{dt}{d\cos\theta} = 2|\mathbf{p}_1||\mathbf{q}_1|,$$

where $\cos\theta$ is the angle between $\mathbf{p}_1$ and $\mathbf{q}_1$. But θ is a frame-dependent quantity, so we must be a little careful what frame we're working in. Let us instead write the integration measure

$$\frac{d^3q_2}{2E_{q_2}} = d^4q_2\delta(q_2^2 - {m_2'}^2)\theta(q_2^0)$$

with θ the step function. We proved this way back in Lecture 5, in a somewhat different form. What we wrote then was

$$\frac{d^3q_2}{2E_{q_2}} = d^4q_2\delta((q_2^0)^2 - \mathbf{q}_2^2 - {m_2'}^2)|_{q_2^0>0}.$$

But this is clearly equivalent– just turn the q_2^0 condition into a step function and rewrite $(q_2^0)^2 - \mathbf{q}_2^2$ in terms of the four-momentum q_2^2. We then rewrite the d^3q_1 integral in spherical coordinates for q_1:

$$\frac{d^3q_1}{2E_{q_1}} = \frac{|\mathbf{q}_1|^2 d|\mathbf{q}_1|}{2E_{q_1}} d\cos\theta d\phi.$$

Since $E_{q_1}^2 + m_1^2 = {m_1'}^2 + |\mathbf{q}_1|^2 \implies 2E_{q_1}dE_{q_1} = |\mathbf{q}_1|d|\mathbf{q}_1|$ allows us to rewrite our expression for $\frac{d^3q_1}{2E_{q_1}}$ (using the $dt/d\cos\theta$ expression) as

$$\frac{d^3q_1}{2E_{q_1}} = \frac{1}{4|\mathbf{p}_1|} dE_{q_1}d\phi dt.$$

If we explicitly substitute our expressions for $d^3q_1/2E_{q_1}$ and $d^3q_2/2E_{q_2}$ into the expression for σ, we get

$$\sigma = \int \frac{1}{(2\pi)^2}\left(\frac{1}{4|\mathbf{p}_1|}dE_{q_1}d\phi dt\right)\left(d^4q_2\delta(q_2^2 - {m_2'}^2)\theta(q_2^0)\right)\frac{|M|^2}{\mathcal{F}}\delta^4(p_1 + p_2 - (q_1 + q_2)).$$

The ϕ integral is trivial– it cancels a factor of 2π. The q_2 integral is also trivial by the last delta function– since it just sets $q_2 = q_1 - p_1 - p_2$. (All the step function tells us is that the energy of the final state is non-negative.) We now take the derivative d/dt of both sides to get an expression for $d\sigma/dt$:

$$\frac{d\sigma}{dt} = \frac{1}{8\pi\mathcal{F}|\mathbf{p}_1|}\int dE_{q_1}|M|^2\delta((q_1 - \sqrt{s})^2 - {m_2'}^2).$$

Expanding out the square we find that

$$(q_1 - \sqrt{s})^2 - {m_2'}^2 = q_1^2 - 2q_1 \cdot (p_1 + p_2) + s - {m_2'}^2,$$

so our final expression is

$$\frac{d\sigma}{dt} = \frac{1}{8\pi\mathcal{F}|\mathbf{p}_1|}\int dE_{q_1}|M|^2\delta(s - {m_2'}^2 + {m_1'}^2 - 2q_1 \cdot (p_1 + p_2)).$$

Boosting now to the center of mass frame where $p_1^\mu = (\sqrt{|\mathbf{p}_1|^2 + m_1^2}, \mathbf{p}_1)$ and $p_2^\mu = (\sqrt{|\mathbf{p}_1|^2 + m_2^2}, -\mathbf{p}_1)$, we note that s is some constant of the collision,

$$s = \left(\sqrt{|\mathbf{p}_1|^2 + m_1^2} + \sqrt{|\mathbf{p}_1|^2 + m_2^2}\right)^2.$$

We can solve for $|\mathbf{p}_1|$ as an exercise (see the end of this section) to find

$$|\mathbf{p}_1| = \frac{\lambda^{1/2}(s, m_1^2, m_2^2)}{2\sqrt{s}}$$

where

$$\lambda(x, y, z) \equiv x^2 + y^2 + z^2 - 2xy - 2xz - 2yz.$$

We therefore find that

$$\mathcal{F} = 2\lambda^{1/2}(s, m_1^2, m_2^2).$$

With our expressions for $|\mathbf{p}_1|$ and $\mathcal{F}$ firmly in hand, we can plug them back into our expression for $d\sigma/dt$, we get

$$\frac{d\sigma}{dt} = \frac{|M|^2}{16\pi\lambda(s, m_1^2, m_2^2)(1/2\sqrt{s})} \int dE_{q_1} \delta(s - m_2'^2 + m_1'^2 - 2q_1 \cdot (p_1 + p_2)).$$

Since we are in the center-of-mass frame, $p_1 + p_2 = (m_1 + m_2, 0, 0, 0) = (\sqrt{s}, 0, 0, 0)$, and so

$$\begin{aligned}\frac{d\sigma}{dt} &= \frac{|M|^2}{16\pi\lambda(s, m_1^2, m_2^2)(1/2\sqrt{s})} \int dE_{q_1} \delta(s - m_2'^2 + m_1'^2 - 2E_{q_1}\sqrt{s}) \\ &= \frac{|M|^2}{16\pi\lambda(s, m_1^2, m_2^2)(1/2\sqrt{s})} \int d\tilde{E}_{q_1} \frac{1}{2\sqrt{s}} \delta(s - m_2'^2 + m_1'^2 - \tilde{E}_{q_1}) \\ &= \frac{|M|^2}{16\pi\lambda(s, m_1^2, m_2^2)}.\end{aligned}$$

In the massless limit (a common approximation) we have $t = (p_1 - q_1)^2 - 2p_1 \cdot q_1 = -2|\mathbf{p}_1||\mathbf{q}_1|(1 - \cos\theta)$, and the total cross section is

$$\sigma_{tot} = \int_{-4|\mathbf{p}_1||\mathbf{q}_1|}^{0} dt \frac{d\sigma}{dt}.$$

In the center-of-mass frame $|\mathbf{p}_1| = |\mathbf{q}_1| = \sqrt{s}/2$ so $\frac{dt}{d\cos\theta} = \frac{s}{2}$. Defining the differential solid angle element $d\Omega$ by

$$d\Omega \equiv d\cos\theta d\phi$$

(a frame-dependent quantity) we find that

$$\frac{d\sigma}{d\Omega} = \frac{s}{4\pi}\frac{d\sigma}{dt} = \frac{|M|^2}{64\pi^2 s}$$

for particles with masses much less than the collision energy.[28]

We can also consider decay rates, which we treat much the same way. Take the initial state to be a sharply peaked superposition of momentum-space eigenstates. Our transition probability density is

$$\frac{dW}{d^4x} = \frac{|\psi(x)|^2}{2E_p}|M|^2(2\pi)^4\delta^4(p - \sum_i q_i),$$

where $\psi(x)$ is the space-time wavefunction of the decaying particle. dW/d^x is then the chance of finding the decaying particle per unit volume. We can equivalently define the differential decay rate $d\Gamma$ such that

$$\frac{dW}{d^4x} = |\psi(x)|^2 \times d\Gamma.$$

Thus

$$\Gamma = \frac{1}{2E_p} \int \prod_{i=1}^{n} \left(\frac{d^3q_i}{(2\pi)^3 2E_{q_i}} \right) |M|^2 (2\pi)^4 \delta^4(p - \sum_{i=1}^{n} q_i).$$

Note that Γ is *not* Lorentz invariant, as it goes as $1/E$ of the decaying particle. The standard convention is to define Γ in the rest frame of the decaying particle. The lifetime of a particle is given by

$$\tau = 6.58 \times 10^{-25} \text{ seconds} \times \frac{1\,\text{GeV}}{\Gamma}.$$

To link this back to our previous discussion of nucleon scattering, $\psi\psi \to \psi\psi$, we computed two diagrams for this process. We found that the matrix element was

$$iM = (-ig)^2 \left\{ \frac{1}{t - m^2} + \frac{1}{u - m^2} \right\},$$

with t and u the standard Mandelstam variables.

[28] Solid angle is the generalization of angles in the plane. A normal angle measured in radians corresponds to an arc length subtended by that angle on a circle of unit radius. In the same way, solid angle (measured in steradians) can be thought of as a surface area on a 2-sphere of unit radius, so that the total solid angle for a sphere is 4π.

Non-lectured aside– solving for $|\mathbf{p}_1|$ We have

$$s = \left(\sqrt{|\mathbf{p}_1|^2 + m_1^2} + \sqrt{|\mathbf{p}_1|^2 + m_2^2}\right)^2.$$

To solve for $|\mathbf{p}_1|$, let us expand out

$$s = (|\mathbf{p}_1|^2 + m_1^2) + (|\mathbf{p}_1|^2 + m_2^2) + 2\sqrt{(|\mathbf{p}_1|^2 + m_1^2)(|\mathbf{p}_1|^2 + m_2^2)}.$$

We move all the terms outside the square root to the LHS to get

$$\frac{s - 2|\mathbf{p}_1|^2 - m_1^2 - m_2^2}{2} = \sqrt{(|\mathbf{p}_1|^2 + m_1^2)(|\mathbf{p}_1|^2 + m_2^2)}$$

and square again to get rid of all the square roots. We can then expand the left side in a useful way, writing

$$\frac{(s - m_1^2 - m_2^2)^2}{4} + |\mathbf{p}_1|^4 - |\mathbf{p}_1|^2(s - m_1^2 - m_2^2) = (|\mathbf{p}_1|^2 + m_1^2)(|\mathbf{p}_1|^2 + m_2^2)$$

or equivalently

$$\frac{(s - m_1^2 - m_2^2)^2}{4} + |\mathbf{p}_1|^4 - |\mathbf{p}_1|^2(s - m_1^2 - m_2^2) = |\mathbf{p}_1|^4 + (m_1^2 + m_2^2)|\mathbf{p}_1|^2 + m_1^2 m_2^2.$$

The $|\mathbf{p}_1|^4$ terms cancel, as do the $(m_1^2 m_2^2)|\mathbf{p}_1|^2$s, so we are left with

$$s|\mathbf{p}_1|^2 = \frac{(s - m_1^2 - m_2^2)^2}{4} - m_1^2 m_2^2.$$

A little rearranging yields

$$|\mathbf{p}_1|^2 = \frac{(s - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2}{4s} = \frac{\lambda(s, m_1^2, m_2^2)}{4s}. \quad \boxtimes$$

Now we want to solve for $\mathcal{F}$. Note that

$$2p_1 \cdot p_2 = (p_1 + p_2)^2 - m_1^2 - m_2^2 = s - m_1^2 - m_2^2.$$

Then we can get $\mathcal{F}$ by writing

$$\begin{aligned} \mathcal{F} &= 4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} \\ &= 2\sqrt{(2p_1 \cdot p_2)^2 - 4m_1^2 m_2^2} \\ &= 2\sqrt{(s - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2} \\ &= 2\lambda^{1/2}, \end{aligned}$$

where we have recognized λ from the first calculation for $|\mathbf{p}_1|$. $\boxtimes$

Lecture 15.

Tuesday, November 6, 2018

Today, we'll introduce spinors, the mathematical framework describing the behavior of fermions! We'll start to show explicitly why spin 1/2 is different than spin 0.[29]

Now, so far we've only considered scalar fields ϕ. Under a Lorentz transformation, these transform as

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$$
$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x).$$

Most particles have an intrinsic angular momentum, which we call *spin*, and fields with spin have a non-trivial Lorentz transformation. For instance, spin 1 particles (i.e. *vector fields*) come with an index μ and transform as

$$A^\mu(x) \to A^{\mu\prime}(x) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x).$$

[29] It's pretty cool to learn about this in Cambridge, where Dirac actually discovered the behavior of spin 1/2 particles.

In general fields can transform as $\phi^a \to D^a{}_b(\Lambda)\phi^b(\Lambda^{-1}x)$, where we say the $D^a{}_b$ form a representation of the Lorentz group. These might be familiar from *Symmetries, Fields and Particles*, but to give a quick overview, a representation D of a group g is a map from that group to a space of linear transformations (usually taken to be matrices) which preserves the group multiplication. That is, it satisfies

$$D(\Lambda_1\Lambda_2) = D(\Lambda_1)D(\Lambda_2)$$
$$D(\Lambda^{-1}) = (D(\Lambda))^{-1}$$
$$D(I) = I.$$

To find the representations, we look at the Lorentz algebra by considering infinitesimal Lorentz transformations. If we write

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \epsilon\omega^\mu{}_\nu + O(\epsilon^2),$$

then the property that Λ preserves the inner product on four-vectors implies that $\omega_{\mu\nu}$ is a 4×4 antisymmetric matrix. In particular this means it has $\frac{4\times 3}{2} = 6$ independent components, corresponding to the three rotations and three Lorentz boosts.

We may introduce a basis of six 4×4 matrices, which we will label by four indices

$$(M^{\rho\sigma})^{\mu\nu} = \eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu},$$

where these matrices are antisymmetric in ρ,σ and in μ,ν. We take ρ,σ to specify which matrix we are looking at. Lowering the index ν, we take μ,ν to specify the row and column respectively. Therefore

$$(M^{\rho\sigma})^\mu_\nu = \eta^{\rho\mu}\delta^\sigma_\nu - \eta^{\sigma\mu}\delta^\rho_\nu.$$

Example 15.1. The basis vector $(M^{01})^\mu{}_\nu$ is given by

$$(M^{01})^\mu{}_\nu = \begin{pmatrix} 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This generates a boost in the x^1 direction (it mixes up x^1 and t).

Similarly, the basis vector $(M^{12})^\mu{}_\nu$ takes the form

$$(M^{12})^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This generates rotations in the $(x^1 - x^2)$ plane.

Note that when we lower ν in order to write the generators as matrices, the matrix may not explicitly look antisymmetric! We can now write

$$\omega^{\mu\nu} = \frac{1}{2}(\Omega_{\rho\sigma}M^{\rho\sigma})^\mu_\nu$$

where these Ms are the generators of the group of Lorentz transformations and Ω is some set of antisymmetric parameters.

Definition 15.2. The *Lorentz algebra* is defined by the bracket

$$[M^{\rho\sigma}, M^{\tau\nu}] = \eta^{\sigma\tau}M^{\rho\nu} - \eta^{\rho\tau}M^{\sigma\nu} + \eta^{\rho\nu}M^{\sigma\tau} - \eta^{\sigma\nu}M^{\rho\tau}.$$

The spinor representation means that we search for other matrices satisfying the Lorentz algebra.

Definition 15.3. We define the Clifford algebra (in any number of dimensions we like, though four is the most useful for our purposes) as a set of matrices γ^μ such that

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}I,$$

where we have defined the anticommutator $\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu$. In four dimensions, the γ^μ are a set of 4×4 matrices with $\mu = 0, 1, 2, 3$.

We need to find 4 matrices which anticommute, and such that $(\gamma^i)^2 = -I \forall i \in \{1,2,3\}$ and $(\gamma^0)^2 = I$. The simplest representation is in terms of 4×4 matrices. A common choice is the *chiral* or *Weyl representaion*, where we take

$$\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

where the σ^i are the usual 2×2 Pauli matrices. As a quick refresher, the Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They satisfy the commutation and anticommutation relations

$$[\sigma^i, \sigma^j] = 2ie^{ijk}\sigma^k \text{ and } \{\sigma^i, \sigma^j\} = 2\delta^{ij} I_2.$$

Note that the γ matrices under any similarity transformation $U\gamma^\mu U^{-1}$ (where U is an invertible constant matrix) also forms an equally good basis.

We now define

$$S^{\rho\sigma} \equiv \frac{1}{4}[\gamma^\rho, \gamma^\sigma] = \frac{1}{2}\gamma^\rho\gamma^\sigma - \frac{1}{2}\eta^{\rho\sigma}$$

by the Clifford algebra. We'll make the following claims: first,

$$[S^{\mu\nu}, \gamma^\rho] = \gamma^\mu \eta^{\nu\rho} - \gamma^\nu \eta^{\rho\mu}.$$

Second, using the previous claim and the definition of S, we can prove (e.g. on the example sheet) that S satisfies the commutation relation

$$[S^{\rho\sigma}, S^{\tau\nu}] = \eta^{\sigma\tau}S^{\rho\nu} - \eta^{\rho\tau}S^{\sigma\nu} + \eta^{\rho\nu}S^{\sigma\tau} - \eta^{\sigma\nu}S^{\rho\tau}.$$

But this is precisely the relations that the Lorentz group generators satisfy, and so S provides a representation of the Lorentz algebra.[30]

We now introduce a four-component *Dirac spinor* $\psi_\alpha(x), \alpha \in \{1,2,3,4\}$. The spinor then transforms under Lorentz transformations as

$$\psi^\alpha(x) \to S[\Lambda]^\alpha{}_\beta \psi^\beta(\Lambda^{-1}x).$$

Here,

$$S[\Lambda] = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right) \text{ and } \Lambda = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\right)$$

are both 4×4 matrices.

Is the spinor representation equivalent to the usual vector representation? No– one can look at specific Lorentz transformations to see this. For instance, the rotations $i, j \in \{1,2,3\}$ give

$$\begin{aligned} S^{ij} &= \frac{1}{4}\left[\gamma^i, \gamma^j\right] \\ &= \left[\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\right] \\ &= \frac{-i}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}. \end{aligned}$$

If we write $\Omega_{ij} = -\epsilon_{ijk}\phi^k$, where ϕ^k is a vector specifying a rotation axis, e.g. $\Omega_{12} = -\phi^3$. Then

$$S[\Lambda] = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right) = \begin{pmatrix} e^{i\boldsymbol{\phi}\cdot\boldsymbol{\sigma}/2} & 0 \\ 0 & e^{i\boldsymbol{\phi}\cdot\boldsymbol{\sigma}/2} \end{pmatrix}.$$

Therefore a rotation about the x^3 axis can be written as $\phi = (0,0,2\pi)$, and then

$$S[\Lambda] = \begin{pmatrix} e^{i\sigma^3\pi} & 0 \\ 0 & e^{i\sigma^3\pi} \end{pmatrix}. = -I_4.$$

[30] At this point, Professor Allanach made a slight digression to read from an interview with Dirac conducted by an USAmerican journalist. It's entertaining reading and can be found here: http://sites.math.rutgers.edu/~greenfie/mill_courses/math421/int.html

Therefore a rotation of 2π takes $\psi_\alpha(x) \to -\psi_\alpha(x)$. This is different from the vector representation, where

$$\Lambda = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\right) = \exp\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 2\pi & 0\\ 0 & -2\pi & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} = I_4,$$

as expected. So indeed spinors see the full $SU(2)$ rotational symmetry, and not just the $SO(3)$ symmetry of the ordinary Lorentz group.

What about boosts? Let us take

$$S^{0i} = \frac{1}{2}\begin{pmatrix}-\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix}$$

and write our boost parameter $\Omega_{0i} = -\Omega_{i0} \equiv \chi_i$. Then

$$S[\Lambda] = \begin{pmatrix}e^{-\boldsymbol{\chi}\cdot\boldsymbol{\sigma}/2} & 0\\ 0 & e^{-\boldsymbol{\chi}\cdot\boldsymbol{\sigma}/2}\end{pmatrix}.$$

For rotations, $S[\Lambda]$ is unitary since $S[\Lambda]S[\Lambda]^\dagger = I$, but for boosts this is *not* the case.

It turns out there are no finite-dimensional unitary representations of the Lorentz group: this is because the matrices

$$S[\Lambda] = \exp\left[\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right]$$

are only unitary if the $S^{\mu\nu}$ are anti-hermitian, $(S^{\mu\nu})^\dagger = -S^{\mu\nu}$. But

$$(S^{\mu\nu})^\dagger = -\frac{1}{4}[\gamma^{\mu\dagger}, \gamma^{\nu\dagger}]$$

can be anti-hermitian if all the γ^μs are either all hermitian or all anti-hermitian. However, this can't be arranged, since $\gamma^0)^2 = I \implies \gamma^0$ has real eigenvalues (and cannot be anti-hermitian), whereas $(\gamma^i)^2 = -I \implies \gamma^i$ has purely imaginary eigenvalues, and therefore cannot be hermitian.

Lecture 16.

Thursday, November 8, 2018

Today we will construct a LI action of spinor fields. Suppose we have a complex field ψ, with

$$\psi^\dagger(x) = (\psi^*)^T(x).$$

Is $\psi^\dagger(x)\psi(x)$ a Lorentz scalar? We'll check how it transforms. In general we have

$$\psi^\dagger(x)\psi * x) \to \psi^\dagger(\Lambda^{-1}x)\underbrace{S[\Lambda]^\dagger S[\Lambda]}_{\neq 1}\psi(\Lambda^{-1}x),$$

which is not quite what we want, since S is not unitary. Since $\gamma^0 = (\gamma^0)^\dagger$ is hermitian and $\gamma^i = -(\gamma^i)^\dagger$ is antihermitian in our representation, we have

$$\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger \implies (S^\mu\nu)^\dagger = -\frac{1}{4}[\gamma^{\mu\dagger}, \gamma^{\nu\dagger}] = -\gamma^0 S^{\mu\nu}\gamma^0.$$

(Note that Greek indices run from 0 to 3 here, while Latin indices are $1, 2, 3$. As they should be.)

Thus

$$S[\Lambda]^\dagger = \exp(\frac{1}{2}\Omega_{\mu\nu}(S^{\mu\nu})^\dagger) = \gamma^0 S[\Lambda]^{-1}\gamma^0,$$

which we get by using $(\gamma^0)^2 = 1$ repeatedly.

Definition 16.1. With this in mind, we define a *Dirac adjoint* of ψ:

$$\bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0.$$

We now claim that $\bar{\psi}(x)\psi(x)$ *is* a Lorentz scalar. Writing explicitly,

$$\begin{aligned}\bar{\psi}(x)\psi(x) &= \psi^\dagger(x)\gamma^0\psi(x)\\ &\to \psi^\dagger(\Lambda^{-1}x)S[\Lambda]^\dagger\gamma^0 S[\Lambda]\psi(\Lambda^{-1}(x)\\ &= \psi^\dagger(\Lambda^{-1}x)\gamma^0\psi(\Lambda^{-1}(x)\\ &= \bar{\psi}(\Lambda^{-1}x)\psi(\Lambda^{-1}x). \quad \boxtimes\end{aligned}$$

Moreover, we claim that $\bar{\psi}(x)\gamma^\mu\psi(x)$ is a Lorentz vector. Under a Lorentz transformation, it transforms as

$$\bar{\psi}(\Lambda^{-1}x)S[\Lambda]^\dagger\gamma^\mu S[\Lambda]\psi(\Lambda^{-1}(x).$$

If this is to be a Lorentz vector, we must have

$$S[\Lambda]^{-1}\gamma^\mu S[\Lambda] = \Lambda^\mu_\nu\gamma^\nu.$$

Now we know that

$$\Lambda^\mu_\nu = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\right)^\mu_\nu$$

and

$$S[\Lambda] = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right),$$

so infinitesimally we have

$$(M^{\rho\sigma})^\mu_\nu\gamma^\nu = -[S^{\rho\sigma},\gamma^\mu].$$

But from the definition of M, we have on the LHS

$$(\eta^{\rho\mu}\delta^\sigma_\nu - \eta^{\sigma\mu}\delta^\rho_\nu)\gamma^\nu = \eta^{\rho\mu}\gamma^\sigma - \gamma^\rho\eta^{\sigma\mu} = -[S^{\rho\sigma},\gamma^\mu],$$

which we proved previously.

Now we'll claim that

$$S = \int d^4x \underbrace{\bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x)}_{\mathcal{L}_D}$$

is a LI action, where $\mathcal{L}_D$ is the *Dirac Lagrangian*. This action describes a free spinor field, and it has some strange properties. If we look at the mass dimension of the field with $[m] = 1$, we find that $[\psi] = [\bar{\psi}] = \frac{3}{2}$. We can now vary $\psi, \bar{\psi}$ independently to get the equations of motion. Varying ψ, we find that

$$(i\gamma^\mu\partial_\mu - m)\psi = 0,$$

which is known as the *Dirac equation.* Note that this equation is only first-order in ∂_μ, whereas the scalar field yielded a second-order equation in ∂_μ. One arrives at a similar equation of $\bar{\psi}$ after an integration by parts:

$$i\partial_\mu\gamma^\mu\bar{\psi} + m\bar{\psi} = 0.$$

Let us now introduce the *slash notation*:

$$A_\mu\gamma^\mu = \gamma_\mu A^\mu = \not{A}.$$

Hence the Dirac equation is written

$$(i\not{\partial} - m)\psi = 0.$$

Note that the Dirac equation mixes up different components of ψ, but each individual component solves the Klein-Gordon equation:

$$\begin{aligned}(i\not{\partial} + m)(i\not{\partial} - m)\psi = 0 &\implies -(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2)\psi = 0\\ &\iff -(\frac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\mu\partial_\nu + m^2)\psi = 0\\ &\iff -(\partial_\mu\partial^\mu + m^2)\psi = 0.\end{aligned}$$

Remember, we should think of the spinor as secretly four components with a non-trivial transformation under rotations.

Now in our representation (the chiral representation), $S[\Lambda]$ is block diagonal. It takes the form

$$S[\Lambda] = \begin{cases} \begin{pmatrix} e^{i\boldsymbol{\phi}\cdot\boldsymbol{\sigma}/2} & 0 \\ 0 & e^{i\boldsymbol{\phi}\cdot\boldsymbol{\sigma}/2} \end{pmatrix} & \text{for rotations,} \\ \begin{pmatrix} e^{-\boldsymbol{\chi}\cdot\boldsymbol{\sigma}/2} & 0 \\ 0 & e^{-\boldsymbol{\chi}\cdot\boldsymbol{\sigma}/2} \end{pmatrix} & \text{for boosts.} \end{cases}$$

From *Symmetries*, we might recall that since the representation takes a block diagonal form, it is *reducible*, i.e. it decomposes into two *irreducible* representations acting on U_L, U_R, where we now write

$$\psi = \begin{pmatrix} U_L \\ U_R \end{pmatrix}$$

with U_L, U_R some 2-component $\mathbb{C}$ objects. We call U_L and U_R (where L, R stand for left and right) *Weyl* or *chiral spinors.* They transform identically under rotations,

$$U_{L,R} \to e^{i\boldsymbol{\phi}\cdot\boldsymbol{\sigma}/2} U_{L,R}$$

but oppositely under boosts,

$$U_L \to e^{-\boldsymbol{\chi}\cdot\boldsymbol{\sigma}/2} U_L$$
$$U_R \to e^{+\boldsymbol{\chi}\cdot\boldsymbol{\sigma}/2} U_R.$$

In group theory language, we say that U_L is in the $(1/2, 0)$ representation of the Lorentz group, while U_R is in the $(0, 1/2)$ representation (where the Lorentz group $SO(1,3) \simeq SU(2) \times SU(2)$). A general spinor is in the direct product space,

$$\psi = (1/2, 0) \oplus (0, 1/2).$$

The Weyl equation Let us now decompose the Dirac Lagrangian $\mathcal{L}_D$ in terms of Weyl spinors. Thus

$$\mathcal{L}_D = \bar{\psi}(i\partial\!\!\!/ - m)\psi = iU_L^\dagger \sigma^\mu \partial_\mu U_L + iU_R^\dagger \bar{\sigma}^\mu \partial_\mu U_R - m(U_L^\dagger U_R + U_R^\dagger U_L),$$

where $\sigma^\mu \equiv (I, \boldsymbol{\sigma}), \bar{\sigma}^\mu \equiv (I, -\boldsymbol{\sigma})$. We observe that the kinetic terms separate entirely– it is only the mass term which mixes U_L and U_R. A massive spinor requires both U_L and U_R in general, but a massless fermion only requires a single one (e.g. U_L). This leads us to write

$$i\sigma^\mu \partial_\mu U_L = 0,$$
$$i\bar{\sigma}^\mu \partial_\mu U_R = 0,$$

which are known as Weyl's equations.

Naïvely, we expect that since U_L and U_R each have two complex components, our count of the real degrees of freedom should come out to $2 \times 2 \times 2 = 8$. But it turns out this is not quite right. In classical mechanics, the number of degrees of freedom are typically given by

$$\#\text{ d.o.f.} = \frac{1}{2} \times (\text{dimensionality of phase space.}).$$

In field theory, we discuss instead the d.o.f. per spacetime point. For a real scalar ϕ, the conjugate momentum is $\Pi_\phi = \dot{\phi} \implies \#\text{ d.o.f.} = \frac{1}{2} \times (2) = 1$. However, for a spinor we have $\Pi_\psi = \psi^\dagger$, not $\dot{\psi}$. Therefore we get 4 complex components $= 8$ real degrees of freedom is ψ, but no extra in $\psi^\dagger$. The upshot is that for spinors,

$$\#\text{ d.o.f.} = \frac{1}{2}(8) = 4.$$

We can choose spin $\uparrow$ or spin $\downarrow$, and consider particles or antiparticles, so $2 \times 2 = 4$. We'll explore what happens to the extra degrees of freedom next time.