BLACK HOLES

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These notes were taken for the *Black Holes* course taught by Jorge Santos at the University of Cambridge as part of the Mathematical Tripos Part III in Michaelmas Term 2018. I live-TEXed them using Overleaf, and as such there may be typos; please send questions, comments, complaints, and corrections to itel2@cam.ac.uk.
Many thanks to Arun Debray for the LATEX template for these lecture notes: as of the time of writing, you can find him at https://web.ma.utexas.edu/users/a.debray/.

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Lecture 1.

Friday, January 18, 2019

> *"The integral curves of the timelike Killing vector don't intersect, or else you could go back in time and kill your own grandmother... which would make you a WEIRDO." –Jorge Santos*

Note. Some very important admininistrative details for this course! Lectures will be Monday, Wednesday, *Thursday*, and Friday, with M/W/F lectures from 12:00-13:00 and Thursday lectures from 13:00-14:00. There will be no classes from 4th February to 15th February, due to Prof. Santos anticipating a baby.

Some useful readings include

- Harvey Reall's notes on black holes and general relativity
- Wald's "General Relativity"
- Witten's review, "Light Rays, Singularities and All That"

To begin with, some conventions. Naturally, we set $c = G = 1$. We use the $- + ++$ sign convention for the Minkowski metric. We shall use the abstract tensor notation where tensor expressions with Greek indices μ, ν, σ are only valid in a particular coordinate basis, while Latin indices a, b, c are valid in any basis, e.g. the Riemann scalar is defined to be $R = g^{ab} R_{ab}$, whereas the Christoffel connection takes the form $\Gamma^\mu_{\nu\rho} = \frac{g^{\mu\epsilon}}{2}(g_{\epsilon\nu,\rho} + g_{\epsilon\rho,\nu} - g_{\nu\rho,\epsilon})$. We also define $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$.

Stars Black holes are one possible endpoint of a star's life cycle. Let's start by assuming spherical stars. Now, stars radiate energy and burn out. However, even very cold stars can avoid total gravitational collapse because of *degeneracy pressure*. If you make a star out of fermions (e.g. electrons) then the Pauli exclusion principle says they can't be in the same state (or indeed get too close), and it might be that the degeneracy pressure is enough to balance the gravitational forces. When this happens, we call the star a *white dwarf*. It turns out this can only happen for stars up to $1.4M_\odot$ (solar masses). If a star is instead made of neutrons (naturally we call these *neutron stars*) then the pressure of the neutrons can prevent gravitational collapse in a mass range from $1.4M_\odot < 3M_\odot$. *Beyond $3M_\odot$, stars are doomed to collapse into black holes.* We'll spend some time understanding this limit.

Spherical symmetry A normal sphere is invariant under rotations in 3-space, $SO(3)$. The line element on the 2-sphere of unit radius is

$$d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2.$$

It is also invariant under reflections sending $\theta \to \pi - \theta$ (the full group $O(3)$), and perhaps some other symmetries.

Definition 1.1. A spacetime (M, g) is *spherically symmetric* if it possesses the same group of isometries as the round two-sphere $d\Omega_2^2$. That is, it has an $SO(3)$ symmetry where the orbits are S^2s (two-spheres). Important remark– there are spacetimes such as Taub-NUT spacetime which enjoy $SO(3)$ symmetry but are *not* spherically symmetric.

In a spherically symmetric spacetime, we shall define a “radius” $r : M \to \mathbb{R}^+$ defined by

$$r(p) = \sqrt{\frac{A(p)}{4\pi}},$$

where $A(p)$ is the area of the S^2 orbit from a point p. This only makes good sense to define under spherical symmetry, but the idea is that we invert the old relationship $A = 4\pi r^2$ to define a radius given an area.

Definition 1.2. A spacetime (M, g) is *stationary* if it admits a Killing vector field K^a which is everywhere timelike. That is,

$$K^a K^b g_{ab} < 0.$$

Using the assumptions of time independence and spherical symmetry, we’ll show some constraints on the resulting spacetime. Let us pick a hypersurface Σ which is nowhere tangent to the Killing vector K. We assign coordinates t, x^i where x^i is defined on the hypersurface, and t then describes a distance along the integral curves of K^a through each point on Σ. That is, we follow the curves such that $\frac{dx^a}{dt} = K^a$.

But in this coordinate system, K^a now takes the wonderfully simple form

$$K^a = \left(\frac{\partial}{\partial t}\right)^a$$

Since K^a is a Killing vector, the Lie derivative of the metric with respect to K vanishes, $\mathcal{L}_K g = 0$. (See Harvey Reall’s notes for the definition of a Lie derivative– it’s just a derivative, “covariant-ized.”) In this case, $K^c \partial_c g_{ab} + K^c{}_{,a} g_{cb} + K^c{}_{,b} g_{ac} = 0$.

With these assumptions, our metric takes the form

$$ds^2 = g_{tt}(x^k) dt^2 + 2g_{ti}(x^k) dt dx^i + g_{ij}(x^k) dx^i dx^j,$$

where $g_{tt}(x^k) < 0$ (stationarity).

We shall also consider *static spacetimes*. Take Σ to be a hypersurface defined by $f(x) = 0$ for some function $f : M \to \mathbb{R}, df \neq 0$. Then df is orthogonal to Σ. Let’s prove this.

Proof. Take Z^a to be tangent to Σ. Thus

$$(df)(Z) = Z(f) = Z^\mu \partial_\mu f = 0$$

on Σ. A useful example might be to compute this for the two-sphere.

Now take a general 1-form normal to Σ. This 1-form can be written as

$$m = g df + f m'. \tag{1.3}$$

That is, on the surface Σ the one-form m is precisely normal to Σ, but if we go off Σ a little bit then we can smoothly extend m off by a bit. We require that g is a smooth function and that m' is smooth but otherwise arbitrary.

Then the differential of m is

$$dm = dg \wedge df + df \wedge m' + f \wedge dm' \implies dm|_\Sigma = (dg - m') \wedge df.$$

We find that $m|_\Sigma = g\, df \implies (m \wedge dm)|_\Sigma = 0$– this follows since $df \wedge df$ is zero. So if Σ is a hypersurface with m orthogonal then $(m \wedge dm)_\Sigma = 0$. ☒

The converse is also true (a theorem due to Frobenius)– if m is a non-zero 1-form such that $m \wedge dm = 0$ everywhere, then there exists f, g such that $m = g\, df$.

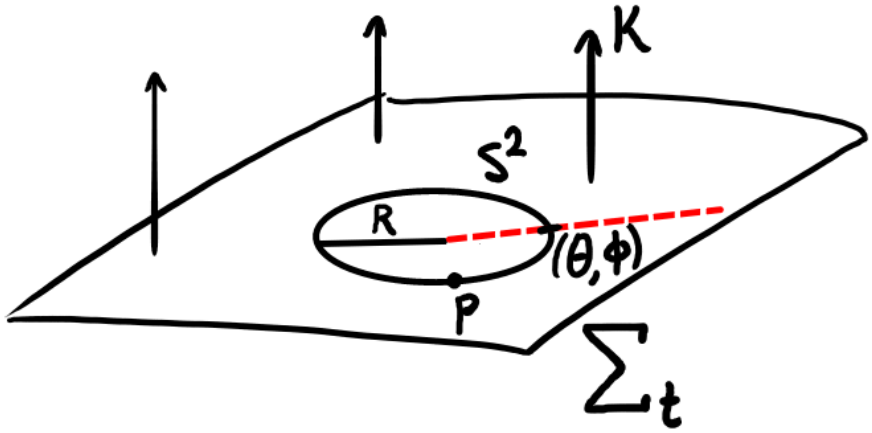

FIGURE 1. An illustration of our coordinates for static, spherically symmetric solutions. We can always choose a hypersurface Σ_t which is orthogonal to the timelike Killing vector K. On Σ_t, choose a point p and trace out its S^2 orbit (drawn here as a circle, S^1) under the action of the $SO(3)$ symmetry. On the S^2 orbit, we can define angular coordinates (θ, ϕ), and we can then extend these to the rest of Σ_t by defining θ, ϕ to be constant on spacelike geodesics normal to the S^2 orbit (red dashed line). The radial coordinate r is given by the area formula $r = \sqrt{A(p)/4\pi}$. This defines coordinates on Σ_t, which we can extend to the entire manifold by following the integral curves of K.

Definition 1.4. A spacetime is *static* if it admits a hypersurface-orthogonal timelike Killing vector field. In particular, static $\implies$ stationary.

In practice, for a static spacetime, we know that K^a is hypersurface orthogonal, so when defining coordinates we shall choose Σ to be orthogonal to K^a. Equivalently, this means that we can choose a hypersurface Σ to be the surface $t = 0$, which implies that $K_\mu \propto (1, 0, 0, 0)$. Of course, this means that $K_i = 0$. But recall that

$$K^a = \left(\frac{\partial}{\partial t}\right)^a \implies g_{ti} = K_i = 0.$$

So our generic metric simplifies considerably– the cross-terms g_{ti} go away and spherical symmetry will further constrain the spatial g_{ij} terms.

Lecture 2.

Monday, January 21, 2019

"Not only are we going to make our cow spherical, we're going to shoot it down so it doesn't move."
–Jorge Santos

So far, we discussed two key concepts. We discussed the condition of a spacetime being static, i.e. stationary and enjoying invariance under $t \leftrightarrow -t$, which forced the line element to take the form

$$ds^2 = g_{tt}(x^k)dt^2 + g_{ij}(x^k)dx^i dx^j. \tag{2.1}$$

We also required spherical symmetry, i.e. the $SO(3)$ orbits of points p in the manifold are S^2s.

Let us see what we get when we combine static and spherically symmetric solutions. We know from staticity that there is a timelike Killing vector $K = \left(\frac{\partial}{\partial t}\right)^a$. Suppose we take a hypersurface Σ_t which is normal to our timelike Killing vector K. Then take any point $p \in \Sigma_t$. By spherical symmetry, its $SO(3)$ orbit is a sphere S^2. Assign angular coordinates θ, ϕ on the S^2 orbit. Using spacelike geodesics normal to the sphere S^2, we can then extend θ, ϕ to the entire hypersurface, a process which is shown in Fig. 1.

Thus our metric on Σ_t takes the form

$$ds^2_{\Sigma_t} = e^{2\psi(r)}dr^2 + r^2 d\Omega_2^2, \tag{2.2}$$

where the coefficient of dr^2 must only depend on r by spherical symmetry, and r is given by our old area relation, $r : \mathcal{M} \to \mathbb{R}^+$ with $r(p) = \sqrt{\frac{A(p)}{4\pi}}$. Now using the property our spacetime is static, we can write down the full spacetime metric,

$$ds^2 = -e^{2\Phi(r)}dt^2 + ds^2_{\Sigma_t}. \tag{2.3}$$

So far we have two degrees of freedom, $(\psi(r), \Phi(r))$. For fluids, recall that the stress-energy tensor takes the form

$$T_{ab} = (\rho + p)U_a U_b + p g_{ab} \tag{2.4}$$

where U_a is a four-velocity, ρ is an energy density and p is a pressure. By spherical symmetry, ρ and p can only be functions of the radial coordinate r, so $\rho = \rho(r)$ and $p = p(r)$. We take

$$U^a U_a = -1 \implies U_t^2 g^{tt} = -1 \implies U^a = e^{-\Phi}\left(\frac{\partial}{\partial t}\right)^a \tag{2.5}$$

so that $p, \rho > 0$. This is an energy condition.

We want to describe spherical stars (with finite spatial extent), so outside the star both the pressure and energy density must vanish,

$$p = \rho = 0 \text{ for } r > R \tag{2.6}$$

with R the radius of the star. Now, we know that the defining property of stress-energy is that it is conserved– $\nabla^a T_{ab} = 0$. But the Einstein equation says that

$$R_{ab} - \frac{R}{2}g_{ab} = T_{ab}, \tag{2.7}$$

and by the contracted Bianchi identity we know that the divergence of the LHS always vanishes, so it suffices to look at the Einstein equation since it automatically implies the conservation equation for fluids. This is not generally true for other energy content since

Let's look at a specific example, the tt componetn of the Einstein equations.

$$G_{tt} = \frac{e^{2(\Phi-\psi)}}{r^2}[e^{2\psi} + 2r\psi' - 1], \tag{2.8}$$

where the prime indicates a $\frac{\partial}{\partial R}$.

Let us also define a function $m(r)$, given by

$$e^{2\psi} \equiv \left[1 - \frac{2m(r))}{r}\right]^{-1}. \tag{2.9}$$

From the various components we learn that

$$tt : m' = 4\pi r^2 \rho(r) \tag{2.10}$$

$$nn : \Phi' = \frac{m + 4\pi r^3 p}{r(r-2m)} \tag{2.11}$$

$$\theta\theta : p' = -(p+\rho)\frac{m + 4\pi n^3 p}{r[r - 2m(r)]}. \tag{2.12}$$

We call these the Tollman-Oppenheimer-Volkoff equations (TOV for short). We have three equations but four unknowns: m, Φ, p and ρ. We need one more bit of information– namely, an equation of state relating the pressure and energy density. Normally, p depends on ρ and also T the temperature. But for our purposes, we will assume cold stars so that p is only a function of the energy density ρ.

What can we figure out before imposing any sort of conditions on $p(\rho)$? Well, outside the star, $r > R$, we have $p = \rho = 0$. But we see immediately that $m' = 0 \implies m = M$ constant. This in turn implies that

$$\psi(r) = -\frac{1}{2}\log\left(1 - \frac{2M}{r}\right) = -\Phi(r). \tag{2.13}$$

We can now write down the line element,

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega_2^2. \tag{2.14}$$

This is the *Schwarzschild line element.* We associate the parameter M with the mass of the system.

There's a bit of physics to extract from this– for *stars*, we need $R > 2M$ to keep the signs correct in the metric. For the sun, we have $2M_\odot = 3\,\text{km}$ and $R \simeq 7 \times 10^5\,\text{km}$, so this is a (very loose) bound which is easily satisfied.

Inside the star, life is not so easy. The mass now depends on the radius, and it has a solution

$$m(r) = 4\pi \int_0^r \rho(\tilde{r})\tilde{r}^2 d\tilde{r} + m_*, \tag{2.15}$$

with m_* some integration constant. Fortunately, we note that by physical concerns, $m(r) \to 0$ as $r \to 0$ in order to preserve regularity (the metric should look flat), which tells us that this integration constant is zero, $m_* = 0$.

At the surface of the star ($r = R$), the metric is continuous. This tells us that

$$M = 4\pi \int_0^R \rho(r) r^2 dr, \tag{2.16}$$

so the mass M is related to an integration of the energy density. It is not however the total energy, which is given by

$$E = \int_V \rho r^2 \sin\theta e^\psi > M.$$

The total energy differs by a factor which corresponds to the gravitational binding energy.

Restoring units to our $R < 2M$ bound on the star radius, we write

$$\frac{GM}{c^2 R} < \frac{1}{2}. \tag{2.17}$$

This isn't hard to satisfy but it's a start, considering we haven't assumed anything about the equations of state.

Let's add some assumptions. For reasonable matter,

$$\frac{dp}{d\rho} > 0, \quad \frac{dp}{dr} \leq 0 \implies \frac{d\rho}{dr} \leq 0. \tag{2.18}$$

This first condition says that more stuff (density) means more pressure, and the second says that pressure decreases as we go towards the surface of the star. The $\theta\theta$ component then tells us that

$$\frac{m(r)}{r} < \frac{2}{9}\left[1 - 6\pi r^2 p + (1 + 6\pi r^2 p)^{1/2}\right], \tag{2.19}$$

which we will prove on Example Sheet 1. Knowing that the pressure vanishes at the surface of the star, $r = R$, we arrive at the Buchdahl bound,

$$R > \frac{9}{4}M. \tag{2.20}$$

This already improves on our naïve bound.

Now using the TOV equations, we could just consider the m' and p' equations. Recall that p is a function of ρ, so we can consider these as two first-order equations for p and m. Normally, each of these conditions would require a boundary condition. But recall we have one integration constant (our m_* from earlier) fixed to be zero, so really we just need to specify one boundary condition, $\rho(r = 0)$.

By the form of p', we see that the pressure decreases as we go towards the surface, so we just integrate outwards until p vanishes and we hit the surface of the star at some value R. This tells us that $M(\rho(0))$ and $R(\rho(0))$, so all the physical parameters of the star are fixed by just one number– the energy density at the center of the star, $\rho(0)$.

We could now introduce an equation of state, in principle. But let's try to be a bit more clever and deduce something independent of the equation of state of whatever this star is made of. This star could be super dense in its core, and maybe we don't know anything about physics in the interior, up to some radius r_0. But outside the core there's some envelope region $r_0 < r < R$ where we do know what's happening– see Fig. 2 for an illustration.

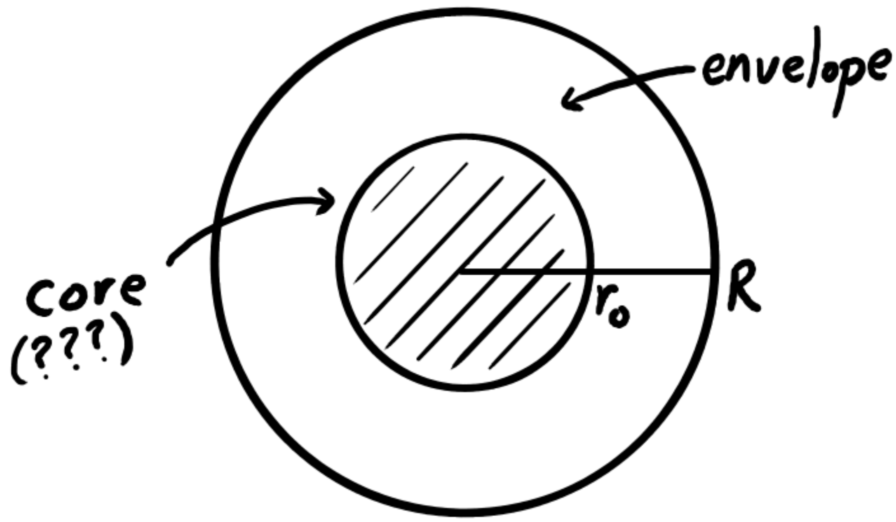

FIGURE 2. A schematic drawing of the interior + envelope model for a star. The interior region extends from $0 < r \leq r_0$ and the exterior region (envelope) goes from $r_0 < r < R$, to the surface of the star.

What could happen in the interior? If ρ takes on some value $\rho(r_0) = \rho_0$ on the surface of the core, then by integrating we can put the bound

$$m_0 \geq \frac{4\pi}{3} r_0^3 \rho_0 \tag{2.21}$$

on m_0 the mass contained in the core. That is, in the best case $\rho(r)$ is constant in the core region– the star certainly cannot be less dense in its core. But we have another inequality on $m(r)$, the Buchdahl limit 2.19, which we can see is a decreasing function of p. So we evaluate this condition at $r = r_0, m(r_0) = m_0$, noting that the most general bound we can put on m_0 in terms of r_0 occurs when $p = 0$. We find that

$$\frac{m_0}{r_0} < \frac{4}{9}. \tag{2.22}$$

These two inequalities in the space of core masses m_0 and core radii r_0 plus a value for the core density ρ_0 tell us that there is a limit on the total core mass– taking $\rho_0 = 5 \times 10^{14}\,\mathrm{g/cm^3}$, the density of nuclear matter, we find that $m_0 < 5M_\odot$. Strictly, these are only limits on the core mass, but it turns out that the envelope region is generally insignificant, so

$$M \approx m_0 < 5M_\odot. \tag{2.23}$$