# QUANTUM FIELD THEORY

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These notes were taken for the *Quantum Field Theory* course taught by Ben Allanach at the University of Cambridge as part of the Mathematical Tripos Part III in Michaelmas Term 2018. I live-TEXed them using Overleaf, and as such there may be typos; please send questions, comments, complaints, and corrections to itel2@cam.ac.uk.
Many thanks to Arun Debray for the LATEX template for these lecture notes: as of the time of writing, you can find him at https://web.ma.utexas.edu/users/a.debray/.

## Contents



Lecture 1.

## Thursday, October 4, 2018

> $2 = \pi = i = -1$ *in these lectures.* –a former lecturer of Prof. Allanach's.

To begin with, some logistic points. The notes (and I assume course material) will be based on David Tong's QFT notes plus some of Prof. Allanach's on cross-sections and decay rates. See http://www.damtp.cam.ac.uk/user/examples/indexP3.html and in particular http://www.damtp.cam.ac.uk/user/examples/3P1l.pdf for the notes on cross-sections.

After Tuesday's lecture, we'll be assigned one of four course tutors:

- Francesco Careschi, fc435cam.ac.uk
- Muntazir Abidi, sma74
- Khim Leong, lkw30
- Stefano Vergari, sv408

Also, the Saturday, November 24th lecture has been moved to 1 PM Monday 26 November, still in MR2. That's it for logistics for now.

**Definition 1.1.** A *quantum field theory* (QFT) is a field theory with an infinite number of degrees of freedom (d.o.f.). Recall that a field is a function defined at all points in space and time (e.g. air temperature is a scalar field in a room wherever there's air). The states in QFT are in general multi-particle states.

Special relativity tells us that energy can be converted into mass, and so particles are produced and destroyed in interactions (particle number not conserved). This reveals a conflict between SR and quantum mechanics, where particle number is fixed. Interaction forces in our theory then arise from structure in the theory, dependent on things like

- symmetry
- locality
- "renormalization group flow."

**Definition 1.2.** A *free QFT* is a QFT that has particles but no interactions. The classic free theory is a relativistic theory with infinitely many quantized harmonic oscillators.

Free theories are generally not realistic but they are important, as interacting theories can be built from these with perturbation theory. The fact we can do this means the particle interactions are somehow weak (weak coupling), but other theories have strong coupling and cannot be described with perturbation theory.

**Units in QFT** In QFT, we usually set $c = \hbar = 1$. Since $[c] = [L][T]^{-1}, [\hbar] = [L]^2[M][T]^{-1}$, we find that in natural units,

$$[L] = [T] = [M]^{-1} = [E]^{-1}$$

(where the last equality follows from $E = mc^2$ with $c = 1$). We often just pick one unit, e.g. an energy scale like eV, and describe everything else in terms of powers of that unit. To convert back to metres or seconds, just reinsert the relevant powers of $c$ and $\hbar$.

**Example 1.3.** The de Broglie wavelength of a particle is given by $\lambda = \hbar/(mc)$. An electron has mass $m_e \simeq 10^6\,\text{eV}$, so $\lambda_e = 2 \times 10^{-12}\,\text{m}$.

If a quantity $x$ has dimension $(mass)^d$, we write $[x] = d$, e.g.

$$G = \frac{\hbar c}{M_p^2} \implies [G] = -2.$$

$M_p \approx 10^{19}\,\text{GeV}$ corresponds to the Planck scale, $\lambda_p \sim 10^{-33}\,\text{cm}$, the length/energy scales where we expect quantum gravitational effects to become relevant. We note that the problems associated with relativising the Schrödinger equation are fixed in QFT by particle creation.

Before we do QFT, let's review classical field theory. In classical particle mechanics, we have a finite number of generalized coordinates $q_a(t)$ (where $a$ is a label telling you which coordinate you're talking about) and in general they are a function of time $t$. But in field theory, we instead have $\phi_a(x,t)$ where $a$ labels the field in question and $x$ is no longer a coordinate but a label like $a$.[1]

In our classical field theory, there are now an infinite number of d.o.f., at least one for each $x$, so position has been demoted from a dynamical variable to a mere label.

**Example 1.4.** The classical electromagnetic field is a vector field with components $E_i(x,t), B_i(x,t)$ such that $i, j, k \in \{1, 2, 3\}$ label spatial directions. In fact, these six fields are derived from four fields (or rather four field components), the four-potential $A_\mu(x,t) = (\phi, \mathbf{A})$ where $\mu \in \{0, 1, 2, 3\}$.

Then the classical fields are simply related to the four-potential by

$$E_i = \frac{\partial A_i}{\partial t} - \frac{\partial A_0}{\partial x_i} \text{ and } B_i = \frac{1}{2}\epsilon_{ijk}\frac{\partial A_k}{\partial x_j}$$

with $\epsilon_{ijk}$ the usual Levi-Civita symbol, and where we have used the Einstein summation convention (repeated indices are summed over).

The dynamics of a field are given by a *Lagrangian* $L$, which is simply a function of $\phi_a(x,t), \dot{\phi}_a(x,t)$, and $\boldsymbol{\nabla}\phi_a(x,t)$.

**Definition 1.5.** We write

$$L = \int d^3x \mathcal{L}(\phi_a, \partial_\mu \phi_a),$$

where we call $\mathcal{L}$ the *Lagrangian density*, or by a common abuse of terminology simply the Lagrangian.

**Definition 1.6.** We may then also define the *action*

$$S \equiv \int_{t_0}^{t_1} L dt = \int d^4x \mathcal{L}(\phi_a, \partial_\mu \phi_a)$$

Let us also note that in these units we have $[S] = 0$ (since it appears alone in an exponent, for instance, $e^{iS}$) and so since $[d^4x] = -4$ we have $[\mathcal{L}] = 4$.

The dynamical principle of classical field theory is that fields evolve s.t. $S$ is stationary with respect to variations of the field that don't affect the intiial or final values (boundary conditions). A general variation of the fields produces a variation in the action

$$\delta S = \sum_a \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a}\delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta(\partial_\mu \phi_a) \right\}.$$

[1] See for instance Anthony Zee's *QFT in a Nutshell* to see a more detailed description of how we go from discrete to continuous systems.

With an integration by parts we find that the variation is the action becomes

$$\delta S = \sum_a \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a \right) - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \delta\phi_a \right\}.$$

The integral of the total derivative term vanishes for any term that decays at spatial $\infty$ (i.e. $\mathcal{L}$ is reasonably well-behaved) and has $\delta\phi_a(x,t_1) = \delta\phi_a(x,t_0) = 0$. Therefore the boundary term goes away and we find that stationary action implies the *Euler-Lagrange equations*,

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0.$$

**Example 1.7.** Consider the Klein-Gordon field $\phi$, defined

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2.$$

Here $\eta^{\mu\nu}$ is the standard Minkowski metric[2].

To compute the Euler-Lagrange equation for this field theory, we see that

$$\frac{\partial L}{\partial \phi} = -m^2\phi \text{ and } \frac{\partial L}{\partial(\partial_\mu \phi)} = \partial^\mu \phi.$$

The Euler-Lagrange equations then tell us that

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0,$$

which we call the *Klein-Gordon equation*. It has wave-like solutions $\phi = e^{-ipx}$ with $(-p^2 + m^2)\phi = 0$ (so that $p^2 = m^2$, which is what we expect in units where $c = 1$).

Lecture 2.

## Saturday, October 6, 2018

Last time, we derived the Euler-Lagrange equations for Lagrangian densities:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0. \tag{2.1}$$

**Example 2.2.** Consider the Maxwell Lagrangian,

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A^\mu)^2. \tag{2.3}$$

Plugging into the E-L equations, we find that $\frac{\partial L}{\partial A_\nu} = 0$ and

$$\frac{\partial L}{\partial \partial_\mu A_\nu} = \partial^\mu A^\nu + \eta^{-\mu\nu}\partial_\rho A^\rho. \tag{2.4}$$

Thus E-L tells us that

$$0 = -\partial^2 A^\nu + \partial^\nu(\partial_\rho A^\rho) = -\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu). \tag{2.5}$$

Defining the field strength tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, we can write the E-L equation for Maxwell as the simple

$$0 = \partial_\mu F^{\mu\nu},$$

which written explicitly is equivalent to Maxwell's equations in vacuum (we'll revisit this when we do QED).

The Lagrangians we'll consider here and afterwards are all *local*– in other words, there are no couplings $\phi(\mathbf{x},t)\phi(\mathbf{y},t)$ with $\mathbf{x} \neq \mathbf{y}$. There's no reason a priori that our Lagrangians have to take this form, but all physical Lagrangians seem to do so.

[2] We use the mostly minus convention here, but honestly the sign conventions are all arbitrary and relativity often uses the other one where time gets the minus sign.

**Lorentz invariance** Consider the Lorentz transformation on a scalar field $\phi(x) \equiv (\phi(x^\mu)$. The coordinates $x$ transform as $x' = \Lambda^{-1}x$ with $\Lambda^\mu_\sigma \eta^{\sigma\tau}\Lambda^\nu_\tau = \eta^{\mu\nu}$. Under $\Lambda$, our field transforms as $\phi \to \phi'$ where $\phi'(x) = \phi(x')$. Recall that Lorentz transformations generically include boosts as well as rotations in $\mathbb{R}^3$. As we've discussed in Symmetries, Fields and Particles, Lorentz transformations form a Lie group ($O(3,1)$, or specifically the proper orthochronous Lorentz group) under matrix multipication. They have a representation given on the fields (i.e. a mapping to a set of transformations on the fields which respects the group multiplication law).

For a scalar field, this is $\phi(x) \to \phi(\Lambda^{-1}x)$ (an active transformation). We could have also used a passive transformation where we re-label spacetime points: $\phi(x) \to \phi(\Lambda x)$. It doesn't matter too much– since Lorentz transformations form a group, if $\Lambda$ is a Lorentz transformation, so is $\Lambda^{-1}$. In addition, most of our theories will be well-behaved and Lorentz invariant.

**Definition 2.6.** *Lorentz invariant* theories are ones where the action $S$ is unchanged by Lorentz transformations.

**Example 2.7.** Consider the action given by

$$S = \int d^4x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi)\right],$$

where $U(\phi)$ is some potential density. $U \to U'(x) \equiv U(\phi'(x)) = U(x')$ means that $U$ is a scalar field (check this!) and we see that

$$\partial_\mu\phi' = \frac{\partial}{\partial x^\mu}\phi(x') = \frac{\partial x'^\sigma}{\partial x^\mu}\partial'_\sigma\phi(x') = (\Lambda^{-1})^\sigma_\mu\partial'_\sigma\phi(x')$$

where $\partial'_\sigma \equiv \frac{\partial}{\partial x'^\sigma}$. Thus the kinetic term transforms as

$$L_{kin} \to L'_{kin} = \eta^{\mu\nu}\partial_\mu\phi'\partial_\nu\phi' = \eta^{\mu\nu}(\Lambda^{-1})^\sigma_\mu(\Lambda^{-1})^\tau_\nu\partial'_\sigma\phi(x')\partial'_\tau\phi(x') = \eta^{\sigma\tau}\partial'_\sigma\phi(x')\partial'_\tau\phi(x') = L_{kin}(x).$$

Thus we see that the action overall transforms as

$$S \to S' = \int d^4xL(x') = \int d^4xL(\Lambda^{-1}x).$$

Under a change of variables $u \equiv \Lambda^{-1}x$, we see that $\det(\Lambda^{-1}) = 1$ (from group theory) so the volume element is the same, $d^4y = d^4x$ and therefore

$$S' = \int d^4yL(y) = S.$$

We conclude that $S$ is invariant under Lorentz transformations.

We also remark that under a LT, a vector field $A_\mu$ transforms like $\partial_\mu\phi$, so

$$A'_\mu(x) = (\Lambda^{-1})^\sigma_\mu A_\sigma(\Lambda^{-1}x).$$

This is enough to attempt Q1 from example sheet 1.[3]

**Theorem 2.8.** *Every continuous symmetry of $L$ gives rise to a current $J^\mu$ which is conserved, $\partial_\mu j^\mu = 0$. Each $j^\mu$ has a conserved charge $Q = \int_{\mathbb{R}^3} j^0 d^3x$.*

*This conserved charge appears because $\frac{dQ}{dt} = \int_{\mathbb{R}^3}\partial_0 j^0 d^3x = -\int_{\mathbb{R}^3} d^3x \boldsymbol{\nabla}\cdot\mathbf{j} = 0$ by the divergence theorem, assuming $|\mathbf{j}| \to 0$ as $|\mathbf{x}| \to \infty$.*

Let us define an infinitesimal variation of a field $\phi$, $\phi(x) \to \phi'(x) = \phi(x) + \alpha\Delta\phi(x)$ with $\alpha$ an infinitesimal change. If $S$ is invariant, we call this a *symmetry* of the theory.

Since $S$ is invariant up to adding a total 4-divergence (a total derivative $\partial_\mu$) to the Lagrangian, our symmetry doesn't affect the Euler-Lagrange equations. $L$ transforms as

$$L(x) \to L(x) + \alpha\partial_\mu X^\mu(x), \tag{2.9}$$

[3]Copied here for quick reference: Show directly that if $\phi(x)$ satisfies the Klein-Gordon equation, then $\phi(\Lambda^{-1}x)$ also satisfies this equation for any Lorentz transformation $\Lambda$.

and expanding to leading order in $\alpha$ we have

$$L \to L(x) + \alpha \frac{\partial L}{\partial \phi} \Delta\phi + \alpha \frac{\partial L}{\partial(\partial_\mu \phi)} \partial_\mu(\Delta\phi) + O(\alpha^2). \tag{2.10}$$

We can rewrite this in terms of a total derivative $\partial_\mu \left( \frac{\partial L}{\partial(\partial_\mu \phi)} \Delta\phi \right)$ so that

$$L' = L(x) + \alpha\partial_\mu \left( \frac{\partial L}{\partial(\partial_\mu \phi)} \Delta\phi \right) + \alpha \left( \frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi)} \right) \Delta\phi. \tag{2.11}$$

By Euler-Lagrange, the second term in parentheses vanishes, so we identify the first term in parentheses as none other than $\alpha\partial_\mu X^\mu(x)$ from Eqn. 2.9 (in other words, $\frac{\partial L}{\partial(\partial_\mu \phi)} \Delta\phi = X^\mu$) and recognize

$$j^\mu \equiv \frac{\partial L}{\partial(\partial_\mu \phi)} \Delta\phi - X^\mu$$

as our conserved current.

**Example 2.12.** Take a complex scalar field $\psi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x))$. We can then trea $\psi, \psi^*$ as independent variables and write a Lagrangian

$$L = \partial_\mu \psi^* \partial^\mu \psi - V(|\psi|^2).$$

Then we observe that under $\psi \to e^{i\beta}\psi, \psi^* \to e^{-i\beta}\psi^*$, the Lagrangian is invariant. The differential changes are $\Delta\psi = i\psi$ (think of expanding $\psi \to e^{i\beta}\psi$ to leading order) and similarly $\Delta\psi^* = -i\psi^*$ (here, $X^\mu = 0$.

We add the currents from $\psi, \psi^*$ to find

$$j^\mu = i\{\psi\partial_\mu\psi^* - \psi^*\partial_\mu\psi\}.$$

This is enough to do questions 2 and 3 on the example sheet.

**Example 2.13.** Under infinitesimal translation $x^\mu \to x^\mu - \alpha\epsilon^\mu$, we have $\phi(x) \to \phi(x) + \alpha\epsilon^\mu\partial_\mu\phi(x)$ by Taylor expansion (similar for $\partial_\mu\phi$). If the Lagrangian doesn't depend explicitly on $x$, then $L(x) \to L(x) + \alpha\epsilon^\mu\partial_\mu L(x)$.

Rewriting to match the form $L + \alpha\partial_\mu X^\mu$, we see that our new Lagrangian takes the form $L(x) + \alpha\epsilon^\nu\partial_\mu(\delta^\mu_\nu L)$. We get one conserved current for each component of $\epsilon^\nu$, so that

$$(j^\mu)_\nu = \frac{\partial L}{\partial(\partial_\mu \phi)} \partial_\mu\phi - \delta^\mu_\nu L$$

with $\partial_\mu(j^\mu)_\nu = 0$. We write this as $j^\mu_\nu \equiv T^\mu_\nu$, the energy-momentum tensor. The conserved charges end up being the total energy $E = \int d^3x T^{00}$ and the total momentum $P^i = \int d^3x T^{0i}$.