THE STANDARD MODE

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These notes were taken for the *The Standard Model* course taught by Christopher Thomas at the University of Cambridge as part of the Mathematical Tripos Part III in Lent Term 2019. I live-TEXed them using Overleaf, and as such there may be typos; please send questions, comments, complaints, and corrections to itel2@cam.ac.uk.
Many thanks to Arun Debray for the LATEX template for these lecture notes: as of the time of writing, you can find him at https://web.ma.utexas.edu/users/a.debray/.

Contents

Lecture 1.

Friday, January 18, 2019

Note. Here are some preliminary administrative notes on the course. Lecture notes and example sheets are available online at http://www.damtp.cam.ac.uk/user/cet34/teaching/. There are four example sheets and classes, plus a revision class in Easter Term. The instructor's email is c.e.thomas@damtp.cam.ac.uk. This course requires as prerequisites the Quantum Field Theory and Symmetries, Fields and Particles courses from Michaelmas term.

Some useful references are mentioned in the official course notes, including

- Peskin and Schroeder
- Aitchinson and Hey
- Halzen and Martin
- Donoghue, Golowich, and Holstein.

The sign conventions will be mostly in line with the Tong QFT notes, though note the sign of γ^5.

Quantum field theory was originally formulated to reconcile special relativity with quantum mechanics. The prototype for modern quantum field theories is quantum electrodynamics (QED), the quantum theory of light and charge. The *Standard Model* (SM) describes three fundamental forces (EM, weak, and strong) but does not include gravity. The model is an incredibly successful theory, having survived experimental tests up to the 1×10^8 GeV level. However, we know that because it does not include gravity, it must break down somewhere– perhaps at the Planck scale (1×10^{19} GeV).

In the SM, forces are mediated by gauge bosons (spin $= 1$).

- EM (QED): photon, γ (massless)
- Weak force: W boson and Z boson (massive)
- Strong force: gluon (massless)

Of course, our theory wouldn't be very good if we only had forces and no matter. In the SM, matter content is described by spin-1/2 fermions:

- neutrinos: ν_e, ν_μ, ν_τ (weak)
- charged leptons: e, μ, τ (weak and EM)

- quarks: $\begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} t \\ b \end{pmatrix}$.

We notice that there are three "generations" of matter particles where the properties of particles between generations are mostly the same, except the mass goes up in each generation.

Finally, we've got the Higgs boson, H (scalar, spin= 0). The Higgs is responsible for generating mass of the W and Z bosons as well as all the fermions. This was famously discovered at the Large Hadron Collider in 2012.[1]

Gauge bosons are manifestations of *local* symmetries (as opposed to global symmetries)– we discussed this towards the end of Symmetries last term. The Standard Model gauge group is

$$SU(3)_C \times SU(2)_L \times U(1)_Y.$$

Here, $SU(3)_C$ is the "colour" symmetry of the strong interaction, QCD. The $SU(2)_L$ symmetry is a chiral (handedness) symmetry. And $U(1)_Y$ corresponds to something called hypercharge. It's actually a combination of the $SU(2)_L \times U(1)_Y$ symmetries that gives rise to the $U(1)_{EM}$ gauge symmetry of QED– these two symmetries together govern the electroweak interactions.

Chiral and gauge symmetries As always, we will use natural units in which $\hbar = c = 1$. To discuss *chiral symmetries*, let us consider a spin-1/2 Dirac fermion with a spinor field ψ satisfying the Dirac equation,

$$(i\partial\!\!\!/ - m)\psi = 0. \tag{1.1}$$

We use the Feynman slash notation, such that

$$\partial\!\!\!/ = \partial_\mu \gamma^\mu.$$

The (Dirac) adjoint (bar notation) is defined $\bar{\psi} = \psi^\dagger \gamma^0$, and satisfies

$$\bar{\psi}(-i\partial\!\!\!/^{\leftarrow} - m) = 0, \tag{1.2}$$

where $\partial\!\!\!/^{\leftarrow}$ acts to the left. The Dirac matrices γ^μ are a set of 4×4 matrices which satisfy the Lorentz algebra,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I, \tag{1.3}$$

where we will take $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ (the Minkowski metric with the mostly minus convention) and curly braces denote anticommutators as usual. We also define the γ^5 matrix to be

$$\gamma^5 = +i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{1.4}$$

so that $(\gamma^5)^2 = I, \{\gamma^5, \gamma^\mu\} = 0$. In the *chiral/Weyl basis*, the gamma matrices take the form

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.5}$$

This basis is so named because γ^5 picks out the left- and right-handed components.

Consider the massless limit of the Dirac equation,

$$\partial\!\!\!/\psi = 0 \implies \partial\!\!\!/(\gamma^5\psi) = 0. \tag{1.6}$$

Then we can define the *projection operators*,

$$P_{R,L} = \frac{1}{2}(1 \pm \gamma^5).. \tag{1.7}$$

This allows us to describe the components of a Dirac spinor:

$$\psi_{R,L} \equiv P_{R,L}\psi \implies \gamma^5\psi_{R,L} = \pm\psi_{R,L}. \tag{1.8}$$

These are eigenstates of the chirality operator, and are called "right-handed" or "left-handed" depending on whether they change sign under application of γ^5.

These are only properly eigenstates in the massless limit– if the particles are massive, then right-handed and left-handed states can mix (e.g. under Lorentz boosts). In chiral bases, ψ_R (ψ_L) only contains lower (upper) 2-component spinor degrees of freedom.

[1] Strictly, a Higgs-like particle which we have since verified many of the other properties of.

The effect of the field after projection is that ψ_L (ψ_R) annihilates left-handed (right-handed) chiral particles. Note also that the Dirac adjoint is

$$\bar{\psi}_{R,L} = (P_{R,L}\psi)^\dagger \gamma^0 = \psi^\dagger \frac{1}{2}(1 \pm \gamma^5)\gamma^0 = \bar{\psi} P_{L,R}. \tag{1.9}$$

We now observe that a massless Dirac fermion has a *global* $U(1)_L \times U(1)_R$ chiral symmetry:

$$U(1)_{R,L} : \psi_{R,L} \to e^{i\alpha_{R,L}} \psi_{R,L},$$

as can be seen from the Dirac Lagrangian:

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi = \bar{\psi}_L i\not{\partial}\psi_L + \bar{\psi}_R i\not{\partial}\psi_R - m(\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R).$$

However, the mass term explicitly breaks this chiral symmetry (it couples the left- and right-handed eigenstates together). It changes our chiral symmetry to a vector symmetry where $\alpha_L = \alpha_R = \alpha$ so the the field as a whole transforms to

$$U(1)_L \times U(1)_R \to U(1)_V : \psi \to e^{i\alpha}\psi.$$

Lecture 2.

Monday, January 21, 2019

Today we will continue the review of the Dirac field (cf. http://www.damtp.cam.ac.uk/user/tong/qft.html).

Review of Dirac field Recall that we can write the Dirac field ψ as a sum over momenta and spin states,

$$\psi(x) = \sum_{p,s} \left[b^s(p) u^s(p) e^{-ip\cdot x} + d^{s\dagger} v^s(p) e^{+ip\cdot x} \right], \tag{2.1}$$

where $s = \pm 1/2$ and $\sum_p \equiv \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}}$. Momentum eigenstates are defined as

$$|p\rangle = b^\dagger(p) |0\rangle,$$

and the relativistic normalization of these momentum eigenstates is $\langle p | q \rangle = (2\pi)^3 (2E_{\mathbf{p}}) \delta^{(3)}(\mathbf{p} - \mathbf{q})$. The identity can be written as $I = \sum_p |p\rangle \langle p|$. Here, $b^\dagger, d^\dagger$ are creation operators for positive and negative frequency modes and u, v are our plane wave solutions to the Dirac equation.

That is, instead of writing a full four-component spinor we can write solutions

$$(\not{p} - m)u = 0,$$
$$(\not{p} + m)v = 0,$$

so that in the chiral basis, our plane wave solutions take the form

$$u^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\xi^s \\ \sqrt{p \cdot \bar{\sigma}}\xi^s \end{pmatrix}, \quad u^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\eta^s \\ -\sqrt{p \cdot \bar{\sigma}}\eta^s \end{pmatrix}. \tag{2.2}$$

Here, $\sigma^\mu = (I_2, \sigma^i)$ and $\bar{\sigma}^\mu = (I_2, -\sigma^i)$. (I write the 2×2 identity matrix as I_2 here to avoid confusion.)

Helicity is defined as the projection of angular momentum onto a linear momentum direction. That is, the helicity operator takes the form

$$h = \mathbf{J} \cdot \hat{\mathbf{p}} = \mathbf{s} \cdot \hat{\mathbf{p}} \tag{2.3}$$

where the angular momentum operator is

$$\mathbf{J} = \mathbf{r} \times \mathbf{p} + \mathbf{s} \tag{2.4}$$

with

$$s_i = \frac{i}{4}\epsilon_{ijk}\gamma^j\gamma^k = \frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \tag{2.5}$$

in the chiral basis.

A massless spinor u then satisfies $\not{p}u = 0$, which means that

$$hu(p) = \frac{\gamma^5}{2}u(p)$$
$$hu_{R,L} = \frac{\gamma^5}{2}u_{R,L} = \pm\frac{1}{2}u_{R,L}.$$

Thus u can be decomposed into a basis of eigenstates u_R, u_L of the chirality operator, where u_R has positive helicity and u_L, negative helicity.

A few notes on chirality:

- Chiral states are only eigenstates of the Dirac equation when $m = 0$ (i.e. they don't mix).
- Helicity is defined for $m = 0$ and $m \neq 0$, but it is not Lorentz invariant when $m \neq 0$. This is because for a massive spinor, we could always imagine Lorentz boosting into a frame where the particle appears to be going the other way (while the direction of its angular momentum is unchanged).
- There is only a 1-1 correspondence between helicity and chirality when $m = 0$.

Review of gauge symmetry (local symmetry) Recall that we had a global symmetry where $\psi \to e^{i\alpha}\psi$, with $\alpha \in \mathbb{C}$. Now suppose we promote α to a function of x, $\alpha = \alpha(x)$ and

$$\psi \to e^{i\alpha(x)}\psi. \tag{2.6}$$

Under this *local* transformation, the old kinetic term is no longer invariant, as it becomes

$$\bar{\psi}i\not{\partial}\psi \to \bar{\psi}i\not{\partial}\psi - (\bar{\psi}\gamma^\mu\psi)(\partial_\mu\alpha(x)). \tag{2.7}$$

The way around this is to introduce a *covariant derivative* D_μ such that

$$D_\mu\psi(x) \to \exp(i\alpha(x))D_\mu\psi(x). \tag{2.8}$$

That is, the derivative transforms like the field itself under a gauge transformation so that our kinetic terms are preserved.

To do this, let us introduce a gauge field $A_\mu(x)$ such that

$$D_\mu\psi = (\partial_\mu + igA_\mu)\psi \quad \text{where } A_\mu(x) \to A_\mu(x) - \frac{1}{g}\partial_\mu\alpha \tag{2.9}$$

so that $\bar{\psi}i\not{D}\psi$ *is* invariant.

We could also introduce a kinetic term for the gauge fields,

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.10}$$

Equivalently $F_{\mu\nu}$ can be defined by a condition on g,

$$igF_{\mu\nu} = [D_\mu, D_0]. \tag{2.11}$$

What other gauge theories can we discuss? The theory of QED has a $U(1)$ gauge symmetry that treats LH and RH fields equivalently ($\alpha_L(x) = \alpha_R(x)$). However, the weak gauge bosons only couple to LH fields, but $U(1)$ is actually not the appropriate symmetry– we will need $SU(2)$. This completes the review of abelian gauge symmetries. We will review non-abelian gauge symmetries a little later.

Types of symmetry Symmetries may manifest themselves in a variety of ways.

(1) We can have a symmetry that is *intact* (unbroken), e.g. the $U(1)_{EM}$ and $SU(3)_C$ gauge symmetries.
(2) The symmetry of $\mathcal{L}$ is broken by an *anomaly* (i.e. it holds classically but when we quantize, something breaks). Not a true symmetry. For example, the global axial $U(1)$ symmetry in the SM.
(3) A symmetry can hold for some terms in the Lagrangian but not others (i.e. the terms which break the symmetry might be small at some relevant energy scale, so we can treat them perturbatively). This is an *explicitly broken* symmetry, though it may be an approximate symmetry if the breaking terms are small. For example, the global *isospin* symmetry relating u and d quarks in QCD.
(4) We might have a *hidden symmetry* which is respected by the Lagrangian but not by the vacuum.
    (a) A *spontaneously broken symmetry* results in a vacuum expectation value (VEV) for one or more scalar fields (cf. Higgs mechanism). In the SM, the $SU(2)_L \times U(1)_\gamma \to U(1)_{EM}$ is a spontaneously broken symmetry.

(b) Even without scalar fields, we can have *dynamical breaking* from quantum effects, e.g. the $SU(2)_L \times SU(2)_R$ global symmetry in QCD (massless quarks).

Discrete symmetries Some discrete symmetries we should be familiar with include

- Parity (P): $(t, \mathbf{x}) \to (t, -\mathbf{x})$
- Time reversal (T): $(t, \mathbf{x}) \to (-t, \mathbf{x})$
- Charge conjugation (C): exchanges particles $\leftrightarrow$ antiparticles.

These first two are spacetime symmetries, while the last is a bit different.

Example 2.12. Let's look at some examples of these symmetries in the Standard Model.

- The $\bar{\psi}\gamma^\mu\psi$ couplings between gauge bosons and fermions, e.g. QED and QCD, are invariant under P and C separately.
- $\bar{\psi}\gamma^\mu(1-\gamma^5)\psi$ couplings to fermions, e.g. the weak interaction, are not.
- The weak interaction violates CP, which implies that T-symmetry is also violated from the CPT theorem (i.e. a system must be invariant under the combination of C, P, and T).

To understand these statements, it will be useful to investigate the consequences of these C, P, T symmetries individually and together.

Lecture 3.

Wednesday, January 23, 2019

Today we'll discuss the consequences of discrete symmetries (CPT).

Symmetry operators We will start by quoting a result proven by Wigner.

Theorem 3.1. *If physics is invariant under some transformation $\Psi \to \Psi'$ (with $\Psi, \Psi' \in$ some Hilbert space), then there is an operator W such that $\Psi' = W\Psi$ and where either W is linear and unitary, or* anti*linear and* anti-*unitary.*

That is, writing the inner product on the hilbert space as $(\cdot, \cdot)$, we have either

- W is unitary and linear,

$$(W\Phi, W\Psi) = (\Phi, \Psi) \text{ and } W(\alpha\Phi + \beta\Psi) = \alpha W\Phi + \beta W\Psi \; (\alpha, \beta \in \mathbb{C}) \tag{3.2}$$

- or W is antiunitary and antilinear,

$$(W\Phi, W\Psi) = (\Phi, \Psi)^* \text{ and } W(\alpha\Phi + \beta\Psi) = \alpha^* W\Phi + \beta^* W\Psi. \tag{3.3}$$

Note that W being antiunitary as an operator is not the same as W being an antiunitary matrix ($W^{-1} = -W^\dagger$).

Now, let us recall the Poincaré transformations, which take

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu. \tag{3.4}$$

In particular we have some improper Lorentz transformations (not of det $= +1$) which are of special importance. There's the parity transformation,

$$\Lambda^\mu{}_\nu = \mathbb{P}^\mu{}_\nu = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \tag{3.5}$$

and also time reversal,

$$\mathbb{T}^\mu{}_\nu = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}. \tag{3.6}$$

Consider an infinitesimal transformation

$$\Lambda^\mu{}_\nu + \delta^\mu{}_\nu + \omega^\mu{}_\nu, a_\mu = \epsilon_\mu. \tag{3.7}$$

Then the corresponding operator can be expanded as

$$W(\Lambda, a) = W(1+\omega, \epsilon) = 1 + \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} - i\epsilon_\mu P^\mu, \tag{3.8}$$

where $J^{\mu\nu}$ is the generator of boosts and rotations and P^μ is a four-momentum operator with $P^0 = H$ the Hamiltonian and $p^i =$ the three-momentum operator.

Thus we can write the parity and time reversal operators as

$$\hat{P} = W(\mathbb{P}, 0)$$
$$\hat{T} = W(\mathbb{T}, 0).$$

From the general composition rule, we can write

$$\hat{P}W(\Lambda, a)\hat{P}^{-1} = W(\mathbb{P}\Lambda\mathbb{P}^{-1}, \mathbb{P}a). \tag{3.9}$$

If we now insert the expansion of W 3.8 on both sides of the equation and compare the coefficients of ϵ_0, we find that

$$\hat{P}iH\hat{P}^{-1} = iH, \tag{3.10}$$

where we recall that $H = P^0$. Similarly,

$$\hat{T}W(\Lambda, a)\hat{T}^{-1} = W(\mathbb{T}\Lambda\mathbb{T}^{-1}, \mathbb{T}a), \tag{3.11}$$

which implies that

$$\hat{T}iH\hat{T}^{-1} = -iH. \tag{3.12}$$

We've been careful not to move the i through the operator $\hat{T}$, since we don't yet know whether the operator is unitary or anti-unitary.

Suppose now Ψ is an energy eigenstate,

$$(\Psi, iH\Psi) = iE.$$

If $\hat{P}$ and $\hat{T}$ are symmetries, then $\hat{P}\Psi$ and $\hat{T}\Psi$ should also be energy eigenstates with energy E.

Suppose $\hat{P}$ is linear. Then we have

$$(\hat{P}\Psi, iH\hat{P}\Psi) = (\hat{P}\Psi, \hat{P}iH\Psi) = (\hat{P}\Psi, \hat{P}iE\Psi) = iE(\hat{P}\Psi, \hat{P}\Psi) = iE, \tag{3.13}$$

by 3.10 and linearity. We could have also run this argument with unitarity instead.

Similarly, suppose $\hat{T}$ is linear. Then

$$(\hat{T}\Psi, iH\hat{T}\Psi) = -(\hat{T}\Psi, \hat{T}iH\Psi) = -iE \tag{3.14}$$

by an equivalent argument using 3.12. But this tells us that $\hat{T}$ has produced an energy eigenstate with energy $-iE$, which is wrong.

Therefore, suppose $\hat{T}$ is anti-linear. Then

$$(\hat{T}\Psi, iH\hat{T}\Psi) = -(\hat{T}\Psi, \hat{T}iH\Psi) = -(\hat{T}\Psi, \hat{T}iE\Psi) = +iE(\hat{T}\Psi, \hat{T}\Psi) = +iE. \tag{3.15}$$

Therefore $\hat{T}$ must be anti-linear and anti-unitary.

To sum up, the parity operator $\hat{P}$ is unitary and linear, while the time reversal operator $\hat{T}$ is antiunitary and antilinear.

Parity Now that we've defined some basic properties of these symmetries, let's consider what parity does to different fields.

For a complex scalar field,

$$\phi(x) = \sum_p \Big[a(p)e^{-ip\cdot x} + c^\dagger(p)e^{+ip\cdot x}\Big], \tag{3.16}$$

where the operator a annihilates a particle and $c^\dagger$ creates an antiparticle.

The operator $\hat{P}$ maps momentum eigenstates $|p\rangle \mapsto \eta^{a*}|p_P\rangle$ where

$$p_P = (p^0, -\mathbf{p}) \tag{3.17}$$
$$X_P = (x^0, -\mathbf{x}) \tag{3.18}$$

and η^{a*} is a complex phase.

Thus

$$\hat{P}a^\dagger(p)\,|0\rangle = \eta^{a*}a^\dagger(p_P)\,|0\rangle\,. \tag{3.19}$$

Since $\hat{P}\hat{P}^{-1} = I$ and assuming $\hat{P}\,|0\rangle = |0\rangle$ (the vacuum is invariant under $\hat{P}$), we conclude that

$$\hat{P}a^\dagger(p)\hat{P}^{-1} = \eta^{a*}a^\dagger(p_P). \tag{3.20}$$

To preserve the normalization, we must have

$$\hat{P}a(p)\hat{P}^{-1} = \eta^{a}a(p_P). \tag{3.21}$$

Similarly, we can work out that

$$\hat{P}c^\dagger(p)\hat{P}^{-1} = \eta^{c*}c^\dagger(p_P). \tag{3.22}$$

Now since $\hat{P}$ is linear and unitary, we can write $\hat{P}\phi(x)\hat{P}^{-1}$ as follows:

$$\begin{aligned}
\hat{P}\phi(x)\hat{P}^{-1} &= \sum_p \Big[\hat{P}a(p)\hat{P}^{-1}e^{-ip\cdot x} + \hat{P}c^\dagger(p)\hat{P}^{-1}e^{+ip\cdot x}\Big] \\
&= \sum_p \Big[\eta^a a(p_P)e^{-ip\cdot x} + \eta^{c*}c^\dagger(p_P)e^{+ip\cdot x}\Big] \\
&= \sum_{p_P} \Big[\eta^a a(p)e^{-ip_P\cdot x} + \eta^{c*}c^\dagger(p)e^{+ip_P\cdot x}\Big] \text{ relabeling } p \leftrightarrow p_P \\
&= \sum_{p_P} \Big[\eta^a a(p)e^{-ip\cdot x_P} + \eta^{c*}c^\dagger(p)e^{+ip\cdot x_P}\Big] \text{ using } p_P\cdot x = p\cdot x_P \\
&= \sum_{p} \Big[\eta^a a(p)e^{-ip\cdot x_P} + \eta^{c*}c^\dagger(p)e^{+ip\cdot x_P}\Big] \text{ relabeling } \sum_p = \sum_{p_P}.
\end{aligned}$$

Note that this does not "look like" $\phi(x_P)$ unless $\eta^a = \eta^{c*} \equiv \eta_p$ (if you like, we're matching the coefficients of Fourier modes). If you're not convinced by this, notice that we would not in general find the commutator $[\phi(x), \hat{P}\phi^\dagger(y)\hat{P}^{-1}]$ vanishes for spacelike $x-y$.

Lecture 4.

Friday, January 25, 2019

Last time, we argued that the parity transformation takes the form

$$\hat{P}\phi(x)\hat{P}^{-1} = \sum_p \Big[\eta_p a(p)e^{-ip\cdot x_p} + \eta_p c^\dagger(p)e^{+ip\cdot x_p}\Big], \tag{4.1}$$

with η_p the *intrinsic parity* of ϕ. In this notation, we found that

$$\hat{P}\phi(x)\hat{P}^{-1} = \eta_p\phi(x_p), \tag{4.2}$$

with $x_P = (x^0, -\mathbf{x})$.

Let us make some comments on the parity transformation.

- For a real scalar field, $a = c$ (the particle and antiparticle operators are the same) and so $\eta^a = \eta^{a*} = \eta_p$, which tells us that $\eta_p = \pm 1$. We say that $\eta_p = +1$ is a scalar and -1 a pseudoscalar.
- For a complex scalar field, η_p may not be real, but if there is a conserved charge then we can redefine the operator $\hat{P}$ so that $\eta_p = \pm 1$ (not obvious, but cf. Weinberg §3.3 and 2.2).
- For a vector field,

$$V^\mu(x) = \sum_{p,\lambda}\Big[\epsilon^\mu(\lambda,p)a^\lambda(p)e^{-ip\cdot x} + \epsilon^{\mu*}(\lambda,p)c^{\lambda\dagger}(p)e^{+ip\cdot x}\Big], \tag{4.3}$$

  where $\lambda = -1, 0, +1$ is the helicity (for a massive particle, or else we would not get the zero helicity state). The ϵs are polarization vectors, like for photons. If we use

$$\mathbb{P}^\mu{}_\nu\epsilon^\nu(\lambda,p) = -\epsilon^\mu(\lambda,p_P), \tag{4.4}$$

  then by a similar argument as above,

$$\hat{P}V^\mu(x)\hat{P}^{-1} = -\eta_p\mathbb{P}^\mu{}_\nu V^\nu(x_P). \tag{4.5}$$

Vectors have $\eta_p = -$ and axial vectors have $\eta_p = +1$.

The Dirac field For the Dirac field, creation and annihilation operators should behave like those for bosons. The three-momentum reverses direction, but the spin component is unchanged, so

$$\hat{P}b^s(p)\hat{P}^{-1} = \eta^b b^s(p_P) \tag{4.6}$$

and

$$\hat{P}d^{s\dagger}(p)\hat{P}^{-1} = \eta^{d*}d^{s\dagger}(p_P). \tag{4.7}$$

Recalling that

$$\psi(x) = \sum_{p,s}\Big[b^s(p)u^s(p)e^{-ip\cdot x} + d^{s\dagger}(p)v^s(p)e^{+ip\cdot x}\Big], \tag{4.8}$$

we notice that the spinors are just a set of four complex numbers, and four-vector inner products are unchanged by parity, so only the operators $b^s, d^{s\dagger}$ are hit by the parity operator, giving

$$\begin{aligned}\hat{P}\psi(x)\hat{P}^{-1} &= \sum_{p,s}\Big[\eta^b b^s(p_P)u^s(p)e^{-ip\cdot x} + \eta^{d*}d^{s\dagger}(p_P)v^s(p)e^{+ip\cdot x}\Big] \\ &= \sum_{p,s}\Big[\eta^b b^s(p)u^s(p_P)e^{-ip\cdot x_P} + \eta^{d*}d^{s\dagger}(p)v^s(p_P)e^{+ip\cdot x_P}\Big],\end{aligned}$$

We leave it as an exercise to check that $u^s(p_P) = \gamma^0 u^s(p), v^s(p_P) = -\gamma^0 v^s(p)$. Using these relations, it follows that

$$\sum_{p,s}\Big[\eta^b b^s(p)\gamma^0 u^s(p)e^{-ip\cdot x_P} - \eta^{d*}d^{s\dagger}(p)\gamma^0 v^s(p)e^{+ip\cdot x_P}\Big] \implies \eta^b = -\eta^{d*} \equiv \eta_p \tag{4.9}$$

so that

$$\psi^P(x) \equiv \hat{P}\psi(x)\hat{P}^{-1} = \eta_p\gamma^0\psi(x_P). \tag{4.10}$$

Thus

$$\bar{\psi}^P(x) \equiv \hat{P}\bar{\psi}(x)\hat{P}^{-1} = \eta_p^*\bar{\psi}(x_P)\gamma^0. \tag{4.11}$$

Thus for the Dirac field, parity sends left-handed fields to right-handed fields under

$$\hat{P}\psi_L(x)\hat{P}^{-1} = \eta_p\gamma^0\psi_R(x_P). \tag{4.12}$$

One should also check that $\psi^P(x)$ satisfies the Dirac equation if $\psi(x)$ does. Thus we can determine the transformation properties of fermion bilinears, e.g.

$$\bar{\psi}(x)\psi(x) \to \hat{P}\bar{\psi}(x)\hat{P}\hat{P}^{-1}\bar{\psi}(x)\hat{P}^{-1} = \bar{\psi}(x_P)\psi(x_P). \tag{4.13}$$

Since we don't pick up a sign flip, we call this a scalar fermion bilinear. Similarly a bit of direct computation yiels the pseudoscalar case:

$$\bar{\psi}(x)\gamma^5\psi(x) \to -\bar{\psi}(x_P)\gamma^5\psi(x_P), \tag{4.14}$$

the vector case:

$$\bar{\psi}(x)\gamma^\mu\psi(x) \to \mathbb{P}^\mu{}_\nu\bar{\psi}(x_P)\gamma^\nu\psi(x_P), \tag{4.15}$$

and the axial vector case:

$$\bar{\psi}(x)\gamma^\mu\gamma^5\psi(x) \to -\mathbb{P}^\mu{}_\nu\bar{\psi}(x_P)\gamma^\nu\gamma^5\psi(x_P), \tag{4.16}$$

Charge conjugation Having thoroughly discussed parity, let us now talk about charge conjugation, $\hat{C}$. The operator $\hat{C}$ is unitary and linear, and it sends particles to antiparticles. Note that Lorentz symmetry constrains the phases, so

$$\hat{C}a(p)\hat{C}^{-1} = \eta_c c(p), \quad \hat{C}c(p)\hat{C}^{-1} = \eta_c^* a(p). \tag{4.17}$$

Thus

$$\hat{C}\,|\text{particle}, p\rangle = \hat{C}a^\dagger(p)\,|0\rangle = \eta_c^* c^\dagger(p)\,|0\rangle = \eta_c^*\,|\text{antiparticle}, p\rangle\,. \tag{4.18}$$

From the decomposition of the field, we find that

$$\begin{aligned}\hat{C}\phi(x)\hat{C}^{-1} &= \eta_c\phi^\dagger(x) \\ \hat{C}\phi^\dagger(x)\hat{C}^{-1} &= \eta_c^*\phi(x).\end{aligned}$$

For a real scalar field, $\phi^\dagger = \phi$ and so $\eta_C = \pm 1$.

This has some important consequences. For instance, the photon field must obey $\hat{C} A_\mu(x) \hat{C}^{-1} = -A_\mu(x)$. Note that the π^0 meson can decay to 2γ, which tells us that $\eta_c^{\pi^0} = (-1)^2 = +1$.