

BLACK HOLES

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LAST UPDATED FEBRUARY 26, 2019

These notes were taken for the *Black Holes* course taught by Jorge Santos at the University of Cambridge as part of the Mathematical Tripos Part III in Michaelmas Term 2018. I live-TeXed them using Overleaf, and as such there may be typos; please send questions, comments, complaints, and corrections to itel2@cam.ac.uk.

Many thanks to Arun Debray for the L^AT_EX template for these lecture notes: as of the time of writing, you can find him at <https://web.ma.utexas.edu/users/a.debray/>.

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Lecture 1.

Friday, January 18, 2019

“The integral curves of the timelike Killing vector don’t intersect, or else you could go back in time and kill your own grandmother. . . which would make you a WEIRDO.” –Jorge Santos

Note. Some very important administrative details for this course! Lectures will be Monday, Wednesday, Thursday, and Friday, with M/W/F lectures from 12:00-13:00 and Thursday lectures from 13:00-14:00. There will be no classes from 4th February to 15th February, due to Prof. Santos anticipating a baby.

Some useful readings include

- Harvey Reall’s notes on black holes and general relativity
- Wald’s “General Relativity”
- Witten’s review, “Light Rays, Singularities and All That”

To begin with, some conventions. Naturally, we set $c = G = 1$. We use the $-+++$ sign convention for the Minkowski metric. We shall use the abstract tensor notation where tensor expressions with Greek indices μ, ν, σ are only valid in a particular coordinate basis, while Latin indices a, b, c are valid in any basis, e.g. the Riemann scalar is defined to be $R = g^{ab}R_{ab}$, whereas the Christoffel connection takes the form $\Gamma_{\nu\rho}^{\mu} = \frac{g^{\mu\epsilon}}{2}(g_{\epsilon\nu,\rho} + g_{\epsilon\rho,\nu} - g_{\nu\rho,\epsilon})$. We also define $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$.

Stars Black holes are one possible endpoint of a star's life cycle. Let's start by assuming spherical stars. Now, stars radiate energy and burn out. However, even very cold stars can avoid total gravitational collapse because of *degeneracy pressure*. If you make a star out of fermions (e.g. electrons) then the Pauli exclusion principle says they can't be in the same state (or indeed get too close), and it might be that the degeneracy pressure is enough to balance the gravitational forces. When this happens, we call the star a *white dwarf*. It turns out this can only happen for stars up to $1.4M_\odot$ (solar masses). If a star is instead made of neutrons (naturally we call these *neutron stars*) then the pressure of the neutrons can prevent gravitational collapse in a mass range from $1.4M_\odot < 3M_\odot$. *Beyond $3M_\odot$, stars are doomed to collapse into black holes.* We'll spend some time understanding this limit.

Spherical symmetry A normal sphere is invariant under rotations in 3-space, $SO(3)$. The line element on the 2-sphere of unit radius is

$$d\Omega_2^2 = d\theta^2 + \sin^2 \theta d\phi^2.$$

It is also invariant under reflections sending $\theta \rightarrow \pi - \theta$ (the full group $O(3)$), and perhaps some other symmetries.

Definition 1.1. A spacetime (M, g) is *spherically symmetric* if it possesses the same group of isometries as the round two-sphere $d\Omega_2^2$. That is, it has an $SO(3)$ symmetry where the orbits are S^2 s (two-spheres). Important remark—there are spacetimes such as Taub-NUT spacetime which enjoy $SO(3)$ symmetry but are *not* spherically symmetric.

In a spherically symmetric spacetime, we shall define a “radius” $r : M \rightarrow \mathbb{R}^+$ defined by

$$r(p) = \sqrt{\frac{A(p)}{4\pi}},$$

where $A(p)$ is the area of the S^2 orbit from a point p . This only makes good sense to define under spherical symmetry, but the idea is that we invert the old relationship $A = 4\pi r^2$ to define a radius given an area.

Definition 1.2. A spacetime (M, g) is *stationary* if it admits a Killing vector field K^a which is everywhere timelike. That is,

$$K^a K^b g_{ab} < 0.$$

Using the assumptions of time independence and spherical symmetry, we'll show some constraints on the resulting spacetime. Let us pick a hypersurface Σ which is nowhere tangent to the Killing vector K . We assign coordinates t, x^i where x^i is defined on the hypersurface, and t then describes a distance along the integral curves of K^a through each point on Σ . That is, we follow the curves such that $\frac{dx^a}{dt} = K^a$.

But in this coordinate system, K^a now takes the wonderfully simple form

$$K^a = \left(\frac{\partial}{\partial t} \right)^a$$

Since K^a is a Killing vector, the Lie derivative of the metric with respect to K vanishes, $\mathcal{L}_K g = 0$. (See Harvey Reall's notes for the definition of a Lie derivative—its just a derivative, “covariant-ized.”) In this case, $K^c \partial_c g_{ab} + K^c_{,a} g_{cb} + K^c_{,b} g_{ac} = 0$.

With these assumptions, our metric takes the form

$$ds^2 = g_{tt}(x^k) dt^2 + 2g_{ti}(x^k) dt dx^i + g_{ij}(x^k) dx^i dx^j,$$

where $g_{tt}(x^k) < 0$ (stationarity).

We shall also consider *static spacetimes*. Take Σ to be a hypersurface defined by $f(x) = 0$ for some function $f : M \rightarrow \mathbb{R}, df \neq 0$. Then df is orthogonal to Σ . Let's prove this.

Proof. Take Z^a to be tangent to Σ . Thus

$$(df)(Z) = Z(f) = Z^\mu \partial_\mu f = 0$$

on Σ . A useful example might be to compute this for the two-sphere.

Now take a general 1-form normal to Σ . This 1-form can be written as

$$m = gdf + fm'. \tag{1.3}$$

That is, on the surface Σ the one-form m is precisely normal to Σ , but if we go off Σ a little bit then we can smoothly extend m off by a bit. We require that g is a smooth function and that m' is smooth but otherwise arbitrary.

Then the differential of m is

$$dm = dg \wedge df + df \wedge m' + f \wedge dm' \implies dm|_{\Sigma} = (dg - m') \wedge df.$$

We find that $m|_{\Sigma} = g df \implies (m \wedge dm)|_{\Sigma} = 0$ —this follows since $df \wedge df$ is zero. So if Σ is a hypersurface with m orthogonal then $(m \wedge dm)_{\Sigma} = 0$. \square

The converse is also true (a theorem due to Frobenius)—if m is a non-zero 1-form such that $m \wedge dm = 0$ everywhere, then there exists f, g such that $m = g df$.

Definition 1.4. A spacetime is *static* if it admits a hypersurface-orthogonal timelike Killing vector field. In particular, static \implies stationary.

In practice, for a static spacetime, we know that K^a is hypersurface orthogonal, so when defining coordinates we shall choose Σ to be orthogonal to K^a . Equivalently, this means that we can choose a hypersurface Σ to be the surface $t = 0$, which implies that $K_{\mu} \propto (1, 0, 0, 0)$. Of course, this means that $K_i = 0$. But recall that

$$K^a = \left(\frac{\partial}{\partial t} \right)^a \implies g_{ti} = K_i = 0.$$

So our generic metric simplifies considerably—the cross-terms g_{ti} go away and spherical symmetry will further constrain the spatial g_{ij} terms.

Lecture 2.

Monday, January 21, 2019

“Not only are we going to make our cow spherical, we’re going to shoot it down so it doesn’t move.”
—Jorge Santos

So far, we discussed two key concepts. We discussed the condition of a spacetime being static, i.e. stationary and enjoying invariance under $t \leftrightarrow -t$, which forced the line element to take the form

$$ds^2 = g_{tt}(x^k)dt^2 + g_{ij}(x^k)dx^i dx^j. \quad (2.1)$$

We also required spherical symmetry, i.e. the $SO(3)$ orbits of points p in the manifold are S^2 s.

Let us see what we get when we combine static and spherically symmetric solutions. We know from staticity that there is a timelike Killing vector $K = \left(\frac{\partial}{\partial t} \right)^a$. Suppose we take a hypersurface Σ_t which is normal to our timelike Killing vector K . Then take any point $p \in \Sigma_t$. By spherical symmetry, its $SO(3)$ orbit is a sphere S^2 . Assign angular coordinates θ, ϕ on the S^2 orbit. Using spacelike geodesics normal to the sphere S^2 , we can then extend θ, ϕ to the entire hypersurface, a process which is shown in Fig. 1.

Thus our metric on Σ_t takes the form

$$ds_{\Sigma_t}^2 = e^{2\psi(r)} dr^2 + r^2 d\Omega_2^2, \quad (2.2)$$

where the coefficient of dr^2 must only depend on r by spherical symmetry, and r is given by our old area relation, $r : \mathcal{M} \rightarrow \mathbb{R}^+$ with $r(p) = \sqrt{\frac{A(p)}{4\pi}}$. Now using the property our spacetime is static, we can write down the full spacetime metric,

$$ds^2 = -e^{2\Phi(r)} dt^2 + ds_{\Sigma_t}^2. \quad (2.3)$$

So far we have two degrees of freedom, $(\psi(r), \Phi(r))$. For fluids, recall that the stress-energy tensor takes the form

$$T_{ab} = (\rho + p)U_a U_b + p g_{ab} \quad (2.4)$$

where U_a is a four-velocity, ρ is an energy density and p is a pressure. By spherical symmetry, ρ and p can only be functions of the radial coordinate r , so $\rho = \rho(r)$ and $p = p(r)$. We take

$$U^a U_a = -1 \implies U_t^2 g^{tt} = -1 \implies U^a = e^{-\Phi} \left(\frac{\partial}{\partial t} \right)^a \quad (2.5)$$



FIGURE 1. An illustration of our coordinates for static, spherically symmetric solutions. We can always choose a hypersurface Σ_t which is orthogonal to the timelike Killing vector K . On Σ_t , choose a point p and trace out its S^2 orbit (drawn here as a circle, S^1) under the action of the $SO(3)$ symmetry. On the S^2 orbit, we can define angular coordinates (θ, ϕ) , and we can then extend these to the rest of Σ_t by defining θ, ϕ to be constant on spacelike geodesics normal to the S^2 orbit (red dashed line). The radial coordinate r is given by the area formula $r = \sqrt{A(p)/4\pi}$. This defines coordinates on Σ_t , which we can extend to the entire manifold by following the integral curves of K .

so that $p, \rho > 0$. This is an energy condition.

We want to describe spherical stars (with finite spatial extent), so outside the star both the pressure and energy density must vanish,

$$p = \rho = 0 \text{ for } r > R \quad (2.6)$$

with R the radius of the star. Now, we know that the defining property of stress-energy is that it is conserved— $\nabla^a T_{ab} = 0$. But the Einstein equation says that

$$R_{ab} - \frac{R}{2}g_{ab} = T_{ab}, \quad (2.7)$$

and by the contracted Bianchi identity we know that the divergence of the LHS always vanishes, so it suffices to look at the Einstein equation since it automatically implies the conservation equation for fluids. This is not generally true for other energy content since

Let's look at a specific example, the tt component of the Einstein equations.

$$G_{tt} = \frac{e^{2(\Phi-\psi)}}{r^2} [e^{2\psi} + 2r\psi' - 1], \quad (2.8)$$

where the prime indicates a $\frac{\partial}{\partial R}$.

Let us also define a function $m(r)$, given by

$$e^{2\psi} \equiv \left[1 - \frac{2m(r)}{r} \right]^{-1}. \quad (2.9)$$

From the various components we learn that

$$tt : m' = 4\pi r^2 \rho(r) \quad (2.10)$$

$$nn : \Phi' = \frac{m + 4\pi r^3 p}{r(r - 2m)} \quad (2.11)$$

$$\theta\theta : p' = -(p + \rho) \frac{m + 4\pi r^3 p}{r[r - 2m(r)]}. \quad (2.12)$$

We call these the Tollman-Oppenheimer-Volkoff equations (TOV for short). We have three equations but four unknowns: m, Φ, p and ρ . We need one more bit of information—namely, an equation of state relating the pressure and energy density. Normally, p depends on ρ and also T the temperature. But for our purposes, we will assume cold stars so that p is only a function of the energy density ρ .

What can we figure out before imposing any sort of conditions on $p(\rho)$? Well, outside the star, $r > R$, we have $p = \rho = 0$. But we see immediately that $m' = 0 \implies m = M$ constant. This in turn implies that

$$\psi(r) = -\frac{1}{2} \log\left(1 - \frac{2M}{r}\right) = -\Phi(r). \quad (2.13)$$

We can now write down the line element,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega_2^2. \quad (2.14)$$

This is the *Schwarzschild line element*. We associate the parameter M with the mass of the system.

There's a bit of physics to extract from this—for stars, we need $R > 2M$ to keep the signs correct in the metric. For the sun, we have $2M_\odot = 3 \text{ km}$ and $R \simeq 7 \times 10^5 \text{ km}$, so this is a (very loose) bound which is easily satisfied.

Inside the star, life is not so easy. The mass now depends on the radius, and it has a solution

$$m(r) = 4\pi \int_0^r \rho(\tilde{r}) \tilde{r}^2 d\tilde{r} + m_*, \quad (2.15)$$

with m_* some integration constant. Fortunately, we note that by physical concerns, $m(r) \rightarrow 0$ as $r \rightarrow 0$ in order to preserve regularity (the metric should look flat), which tells us that this integration constant is zero, $m_* = 0$.

At the surface of the star ($r = R$), the metric is continuous. This tells us that

$$M = 4\pi \int_0^R \rho(r) r^2 dr, \quad (2.16)$$

so the mass M is related to an integration of the energy density. It is not however the total energy, which is given by

$$E = \int_V \rho r^2 \sin \theta e^\psi > M.$$

The total energy differs by a factor which corresponds to the gravitational binding energy.

Restoring units to our $R < 2M$ bound on the star radius, we write

$$\frac{GM}{c^2 R} < \frac{1}{2}. \quad (2.17)$$

This isn't hard to satisfy but it's a start, considering we haven't assumed anything about the equations of state.

Let's add some assumptions. For reasonable matter,

$$\frac{dp}{d\rho} > 0, \quad \frac{dp}{dr} \leq 0 \implies \frac{d\rho}{dr} \leq 0. \quad (2.18)$$

This first condition says that more stuff (density) means more pressure, and the second says that pressure decreases as we go towards the surface of the star. The $\theta\theta$ component then tells us that

$$\frac{m(r)}{r} < \frac{2}{9} \left[1 - 6\pi r^2 p + (1 + 6\pi r^2 p)^{1/2} \right], \quad (2.19)$$

which we will prove on Example Sheet 1. Knowing that the pressure vanishes at the surface of the star, $r = R$, we arrive at the Buchdahl bound,

$$R > \frac{9}{4} M. \quad (2.20)$$

This already improves on our naïve bound.

Now using the TOV equations, we could just consider the m' and p' equations. Recall that p is a function of ρ , so we can consider these as two first-order equations for p and m . Normally, each of these conditions would require a boundary condition. But recall we have one integration constant (our m_* from earlier) fixed to be zero, so really we just need to specify one boundary condition, $\rho(r = 0)$.

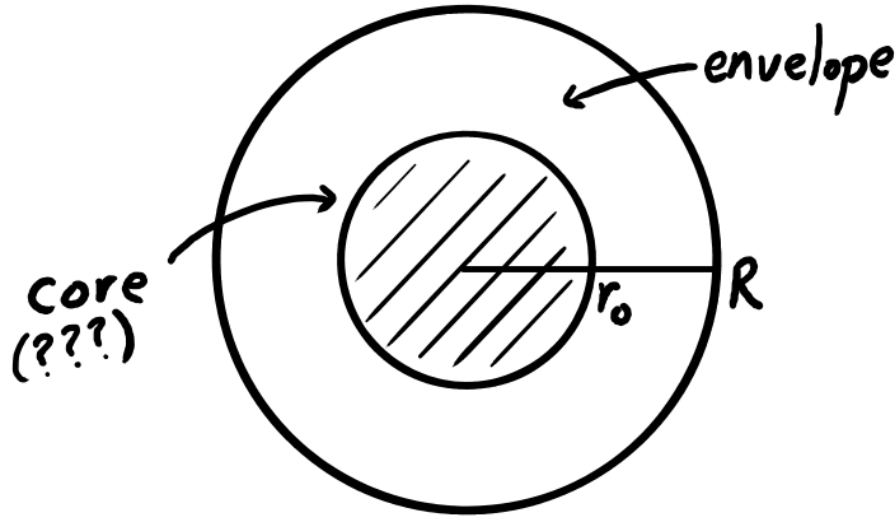


FIGURE 2. A schematic drawing of the interior + envelope model for a star. The interior region extends from $0 < r \leq r_0$ and the exterior region (envelope) goes from $r_0 < r < R$, to the surface of the star.

By the form of p' , we see that the pressure decreases as we go towards the surface, so we just integrate outwards until p vanishes and we hit the surface of the star at some value R . This tells us that $M(\rho(0))$ and $R(\rho(0))$, so all the physical parameters of the star are fixed by just one number– the energy density at the center of the star, $\rho(0)$.

We could now introduce an equation of state, in principle. But let's try to be a bit more clever and deduce something independent of the equation of state of whatever this star is made of. This star could be super dense in its core, and maybe we don't know anything about physics in the interior, up to some radius r_0 . But outside the core there's some envelope region $r_0 < r < R$ where we do know what's happening– see Fig. 2 for an illustration.

What could happen in the interior? If ρ takes on some value $\rho(r_0) = \rho_0$ on the surface of the core, then by integrating we can put the bound

$$m_0 \geq \frac{4\pi}{3} r_0^3 \rho_0 \quad (2.21)$$

on m_0 the mass contained in the core. That is, in the best case $\rho(r)$ is constant in the core region– the star certainly cannot be less dense in its core. But we have another inequality on $m(r)$, the Buchdahl limit 2.19, which we can see is a decreasing function of p . So we evaluate this condition at $r = r_0, m(r_0) = m_0$, noting that the most general bound we can put on m_0 in terms of r_0 occurs when $p = 0$. We find that

$$\frac{m_0}{r_0} < \frac{4}{9}. \quad (2.22)$$

These two inequalities in the space of core masses m_0 and core radii r_0 plus a value for the core density ρ_0 tell us that there is a limit on the total core mass– taking $\rho_0 = 5 \times 10^{14} \text{ g/cm}^3$, the density of nuclear matter, we find that $m_0 < 5M_\odot$. Strictly, these are only limits on the core mass, but it turns out that the envelope region is generally insignificant, so

$$M \approx m_0 < 5M_\odot. \quad (2.23)$$

Today is an exciting class! For the first time, we will look at the Schwarzschild line element. We will not actually define what a black hole is for another ten lectures, but we'll plow ahead and learn some thing about them anyway.

In Schwarzschild coordinates t, r, θ, ϕ , the line element of a Schwarzschild black hole takes the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\Omega_2^2. \quad (3.1)$$

For stars, we said this solution was only valid for $r > 2M$, but as you may already know, the $r = 2M$ singularity is just a coordinate singularity and can be defined away by a good choice of coordinates (as we'll do today). We take the parameter M to be a physical mass right now, and thus $M > 0$ (we will treat the case $M < 0$ later). The radius $r = 2M$ is known as the *Schwarzschild radius*.

Last time, we assumed our solutions were both spherically symmetric and stationary. What if we drop the assumption that the solution is stationary? The answer is the following theorem by Birkhoff.

Theorem 3.2 (Birkhoff's theorem). *Any spherically symmetric solution of the vacuum Einstein equations is isometric to Schwarzschild.*

Proof. What could the spherically symmetric line element look like? The angular bit must look like $r^2d\Omega_2^2$. The rest of it is unconstrained— we can have dt^2, dr^2 , and $drdt$ cross terms. Thus

$$ds^2 = -f(t, r)[dt + \chi dr]^2 + \frac{dr^2}{g} + r^2d\Omega_2^2. \quad (3.3)$$

Here, χ, f , and g can all depend on t, r . Let us begin by rescaling time t to set the factor $\chi = 0$. If you like, $dt + \chi dr = dt'$. We can still send $t \rightarrow p(t)$ a function of t . Now our line element reduces to

$$ds^2 = -f(t, r)dt^2 + \frac{dr^2}{g(t, r)} + r^2d\Omega_2^2. \quad (3.4)$$

From the tr component of the vacuum Einstein equations, we find that

$$0 = R_{tr} - g_{tr} \frac{R}{2} \implies \frac{\partial}{\partial t} g = 0 \implies g(r), \quad (3.5)$$

g does not depend on time. Looking at the tt component we get

$$1 - g - rg' = 0 \implies g(r) = 1 - \frac{2M}{r}, \quad (3.6)$$

where M is an integration parameter. From the rr component we get instead

$$1 - \frac{1}{g} + r \frac{f'}{f} = 0 \implies f(t, r) = C(t) \left[1 - \frac{2M}{r}\right]. \quad (3.7)$$

This is almost good— all we need to do is reparametrize t and we can set this function $C(t) = 1$. We find that with f, g defined in this way, our general spherically symmetric metric has been put into the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\Omega_2^2. \quad \square$$

Gravitational redshift Suppose we have two observers Alice and Bob. They sit at some constant coordinates (r_A, θ, ϕ) and (r_B, θ, ϕ) respectively. Now Alice sends some signals (light pulses) to Bob, separated by a coordinate time Δt . Because $\frac{\partial}{\partial t}$ is a Killing vector, our scenario has time translation symmetry— each pulse will be the same as the last, just translated in time, so Alice and Bob agree on the difference in coordinate time Δt .

Let us notice that Alice measures the photons as being separated by a proper time

$$\Delta\tau_A = \sqrt{1 - \frac{2M}{r_A}} \Delta t. \quad (3.8)$$

An equivalent expression is true for Bob at r_B . Eliminating Δt , we find that

$$\frac{\Delta\tau_B}{\Delta\tau_A} = \sqrt{\frac{1 - \frac{2M}{r_B}}{1 - \frac{2M}{r_A}}} > 1 \quad (3.9)$$

for $r_B > r_A$. This interval is a proxy for a perceived wavelength, $\lambda_B > \lambda_A$. If Bob is far away, $R_B \gg 2M$, then the measured redshift z is given by

$$1 + z \equiv \frac{\lambda_B}{\lambda_A} = \frac{1}{\sqrt{1 - \frac{2M}{r_A}}}. \quad (3.10)$$

For an observer outside a physical star, $R = 9M/4$, $Z = 2$ is the maximum redshift Bob can measure as Alice gets close to the surface of the star. But $R = 2M$ represents a surface of infinite redshift, so it seems to represent something very strange. This is the *event horizon*. We'll explore the consequences of this calculation more as we go on.

Geodesics of Schwarzschild We'll introduce some nice coordinates to let us cross the event horizon. Let $x^\mu(\lambda)$ be an affinely parametrized geodesic with tangent vector

$$U^\mu \equiv \frac{dx^\mu}{d\tau}.$$

Since we have two Killing vectors $K = \frac{\partial}{\partial t}$ and $m = \frac{\partial}{\partial \phi}$, we get two conserved charges,

$$E = -K \cdot U = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau} \quad (3.11)$$

$$h = m \cdot U = r^2 \sin^2 \theta \frac{d\phi}{d\tau}. \quad (3.12)$$

Note that $U^a \nabla_a U_b = 0$ defines an affinely parametrized geodesic, and Killing vectors obey $\nabla_a K_b \nabla_b K_a = 0$. These two facts are enough to prove the existence of a conserved charge:

$$U^c \nabla_c (U^a K_a) = (U^c \nabla_c U^a) K_a + U^c U^a \nabla_c K_a. \quad (3.13)$$

But this first term vanishes by the definition of an affinely parametrized geodesic, and the second can be symmetrized since U^c and U^a commute, so the second term vanishes by Killing's equation. Thus $\nabla_U (U^a K_a) = 0$.

For timelike particles, if τ is the proper time, then E has the interpretation of energy per unit mass, and h is the angular momentum per unit mass. For null geodesics, the quantity

$$b = \left| \frac{h}{E} \right|$$

is the physical impact factor.

Note that there is a third conserved charge— it is the value $U^a U_a = \pm 1, 0$ depending on if the geodesic is spacelike, timelike, or null.

We can write an action

$$S = \int d\tau \dot{x}^a \dot{x}^b g_{ab}. \quad (3.14)$$

which naturally gives us a Lagrangian that we can apply our conserved charges to and calculate Euler-Lagrange equations for. In Schwarzschild, the action takes the form

$$S = \int d\tau \left[g_{tt} \dot{t}^2 + g_{rr} \dot{r}^2 + \dot{\theta}^2 r^2 + \dot{\phi}^2 r^2 \sin^2 \theta \right]. \quad (3.15)$$

Note also that

$$\frac{d}{d\tau} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0 \implies r^2 \frac{d}{d\tau} (r^2 \dot{\theta}) - \frac{\cos \theta}{\sin^3 \theta} h^2 = 0. \quad (3.16)$$

WLOG we can always set $\theta(0) = \pi/2$ and $\dot{\theta}(0) = 0$, which means that $\dot{\theta} = 0 \implies \theta(\tau) = \pi/2$ for all τ .

Substituting in our conserved quantities, we have only one equation which we need to consider: \dot{r} . For a geodesic such that

$$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -\tilde{v}, \quad (3.17)$$

we have

$$\dot{r}^2/2 + V(r) = E^2/2, \text{ with } V(r) = \frac{1}{2} \left(\tilde{v} + \frac{h^2}{r^2} \right) \left(1 - \frac{2M}{r} \right). \quad (3.18)$$

What are the radial free-fall paths of Schwarzschild? For radial trajectories, $h = 0$, so that

$$E = 1 \implies \dot{t} = \frac{1}{1 - \frac{2M}{r}}, \dot{r} = \pm 1. \quad (3.19)$$

For the upper sign, $\dot{r} = +1$, we have $\dot{t}/\dot{r} > 0, r > 2M$, which represents outgoing trajectories. For the lower sign, $\dot{t}/\dot{r} < 0, r > 2M$ (ingoing). In any case, from $\dot{r} = -1$ we see that we can reach $r = 2M$ without a problem.¹

In Schwarzschild, we can then write

$$\frac{dt}{dr} = \pm \left(1 - \frac{2M}{r} \right)^{-1}. \quad (3.20)$$

We define Regge-Wheeler coordinates (AKA tortoise coordinates) as follows:

$$dr_* = \frac{dr}{1 - \frac{2M}{r}} \implies r_* = r + 2M \log \left| \frac{r}{2M} - 1 \right|. \quad (3.21)$$

We will show that in classical gravity, we can indeed cross the horizon in finite proper time.

Lecture 4.

Thursday, January 24, 2019

"This [tidal force divergence] is related to something that happens in AdS/CFT. ... don't tell anyone I mentioned that." –Jorge Santos

Quick announcement– office hours from this course will be held at 14:00 on Tuesdays. If you plan to attend the office hours, however, do send an email in advance.

Last time, we started looking at the geodesics of Schwarzschild. We found two particularly simple ones: the ingoing and outgoing radial null geodesics, with equation

$$\frac{dt}{dr} = \pm \left(1 - \frac{2M}{r} \right)^{-1}, r > 2M. \quad (4.1)$$

We also defined the tortoise (Regge-Wheeler) coordinate r_* such that

$$dr_* = \frac{dr}{1 - \frac{2M}{r}}. \quad (4.2)$$

Thus our null geodesic equation becomes

$$\frac{dt}{dr_*} = \pm 1. \quad (4.3)$$

In tortoise coordinates, null geodesics therefore obey

$$t = \pm r_* + \text{constant}. \quad (4.4)$$

This seems kind of trivial. But now we introduce a new coordinate, the *ingoing Eddington-Finkelstein coordinate*, defining

$$v \equiv t + r_*. \quad (4.5)$$

This is clearly constant on ingoing geodesics, just by looking at the previous equation. That is, $dv|_{\text{geodesic}} = dt + dr_* = 0$. Good. (We could have done the same for outgoing geodesics taking $u \equiv t - r_*$.) Now we eliminate the original coordinate time t from the line element using $dt = dv - \frac{dr}{1 - \frac{2M}{r}}$. We get the new line element

$$ds^2 = - \left(1 - \frac{2M}{r} \right) dv^2 + 2dvdr + r^2 d\Omega_2^2. \quad (4.6)$$

¹At least from a pure gravity perspective. Once we introduce quantum mechanical effects, we have to worry about firewalls and such, and all bets are off.

Let's write this in matrix notation. Nothing fancy.

$$g_{\mu\nu} = \begin{pmatrix} -1 + \frac{2M}{r} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}. \quad (4.7)$$

We haven't done anything too extreme—just made a change of coordinates. But we see something very nice—none of the metric components are singular at $r = 2M$. In fact, the determinant of the metric is still perfectly nice at $r = 2M$ —by an explicit computation, $\det g = -r^2 \sin^2 \theta$. This only vanishes at $\theta = 0$ (the regular coordinate badness of spherical coordinates) and at $r = 0$, which may (a priori, we don't know yet) be a real problem.

So we have found some coordinates which appear to extend r not just from $r > 2M$ but to all $r > 0$. Our metric is real and analytic so it is a nice analytic continuation of the old (bad) Schwarzschild coordinates. This is related to the problem of extendibility— are there other metrics which cover more of the spacetime manifold which are compatible with the solution that we've found?

However, something really bad does happen as $r \rightarrow 0$. The Kretschmann scalar $R^{abcd}R_{abcd} = \frac{48M^2}{r^6} \rightarrow \infty$ as $r \rightarrow 0$, and scalars by definition are invariant under coordinate transformations. So we cannot get rid of this by a simple redefinition of coordinates.

Moreover, let us observe that $\frac{\partial}{\partial t} = \frac{\partial}{\partial v}$. Thus our Killing vector field becomes

$$K = \frac{\partial}{\partial v}, \quad K^2 = -\left(1 - \frac{2M}{r}\right). \quad (4.8)$$

So our metric appears to be no longer static or stationary in these coordinates.

We now introduce the Finkelstein diagram. Recall that for $r = 2M$, on outgoing geodesics we have $t - r_* = \text{constant} \implies v = 2r + 4M \log\left|\frac{r}{2M} - 1\right| + \text{constant}$. Let us draw a plot in the $t_* \equiv v - r, r$ plane. What we see is that the ingoing geodesics follow 45° paths, while the outgoing geodesics follow some curved paths in these coordinates.

What we observe is that in these coordinates, the light cones “tip over” at $r = 2M$ —even the “outgoing” null geodesics are forced to proceed towards $r = 0$ for $r < 2M$. Now, this does not yet prove that this is a black hole. We've only examined radial geodesic trajectories, and it's not clear that adding even a little bit of angular momentum can't save us from a spaghetti death by black hole-like object.

Suppose that we² sit on the surface of a collapsing star. What we can show is that the time to singularity is $\Delta t = \mathbb{T}M$, where $M = M_\odot \implies \Delta t = 10 \times 10^{-5} \text{ s}$. It's a weird fact that we can indeed cross the horizon and hit the singularity in finite time, and yet because the event horizon is a surface of infinite redshift, a far-away observer will never see us actually cross the event horizon.

Black hole region How do we define a black hole? Before we can do that, we'll need some preliminaries.

Definition 4.9. A vector is *causal* if it is null or timelike and nonzero. A curve is causal if its tangent vector is everywhere causal.

Definition 4.10. A spacetime is *time-orientable* if it admits a time orientation, i.e. a causal vector field T^a . Another causal vector field x^a is then *future-directed* if it lies in the same light cone as T^a and is past-directed otherwise (i.e. $x^a T_a \leq 0$).

Note that the old coordinate time t in Schwarzschild was really a bad choice inside the event horizon. This is because $\frac{\partial}{\partial t}$ becomes spacelike inside the horizon, related to the change of sign of g_{tt} for $r < 2M$. So let us take $\pm \frac{\partial}{\partial r}$, and note that in our EF coordinates, $g_{rr} = 0$. Therefore $\frac{\partial}{\partial r} \frac{\partial}{\partial r} = 0$, meaning that we've found a vector which is null everywhere.

In fact, recall our timelike Killing vector outside, K . This gave us a good sense of time outside. We see now that

$$K \cdot \left(-\frac{\partial}{\partial r}\right) = -g_{vr} = -1,$$

²“Imagine that you, not me, are situated at the surface of this star. I don't want to go there.” – Jorge Santos

where $K \equiv \frac{\partial}{\partial v}$. Therefore we've found a timelike coordinate which is good everywhere and defines a good time orientation.

Proposition 4.11. *Let $x^\mu(\lambda)$ be any future-directed causal curve, i.e. one whose tangent vector is everywhere future-directed and causal. Assume $r(\lambda_0) \leq 2M$. Then $r(\lambda) \leq 2M$ for any $\lambda \geq \lambda_0$.*

Let us define the tangent vector $V^\mu = \frac{dx^\mu}{d\lambda}$. Since $\left(-\frac{\partial}{\partial r}\right)$ and V^a are both future-directed, we have

$$0 \geq \left(-\frac{\partial}{\partial r}\right)V = -g_{r\mu}V^\mu = -V^r = -\frac{dr}{d\lambda} \implies \frac{dr}{d\lambda} \geq 0. \quad (4.12)$$

Now

$$V^2 \equiv V^a V_a = -\left(1 - \frac{2M}{r}\right)\left(\frac{dr}{d\lambda}\right)^2 + 2\left(\frac{dv}{d\lambda}\right)\left(\frac{dr}{d\lambda}\right) + r^2\left(\frac{d\Omega}{d\lambda}\right)^2, \quad (4.13)$$

where $\left(\frac{d\Omega}{d\lambda}\right)^2 \equiv \left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2$. Then

$$-2\left(\frac{dv}{d\lambda}\right)\left(\frac{dr}{d\lambda}\right) = -V^2 + \left(\frac{2M}{r} - 1\right)\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\Omega}{d\lambda}\right)^2. \quad (4.14)$$

But if V is causal and $r \leq 2M$, then the right side of 4.14 is non-negative, so it follows that

$$\frac{dv}{d\lambda} \frac{dr}{d\lambda} \leq 0. \quad (4.15)$$

Let us assume that $\frac{dr}{d\lambda} > 0$ (our curve at any point is directed towards larger r). Then $\frac{dv}{d\lambda} = 0$, which means that by 4.14, $V^2 = 0$ and $\frac{d\Omega}{d\lambda} = 0$. But then the only nonvanishing component of V is $V^r = \frac{dr}{d\lambda} > 0 \implies V$ is a positive multiple of $\frac{\partial}{\partial r}$, and hence is past-directed. We have reached a contradiction.

Therefore $\frac{dr}{d\lambda} \leq 0$ if $r \leq 2M$. If $r < 2M$, the equality must be strict. If $\frac{dr}{d\lambda} = 0$ then by 4.14, $\frac{d\Omega}{d\lambda} = \frac{dv}{d\lambda} = 0 \implies V^\mu = 0$. Hence if $r(\lambda_0) < 2M$, then $r(\lambda)$ is monotonically decreasing for all $\lambda \geq \lambda_0$.

Lecture 5.

Friday, January 25, 2019

"[The white hole region is] like a norovirus. Nothing can stay in. Everything has to come out." -Jorge Santos

Last time, we made the proposition that the Schwarzschild spacetime really does feature a black hole region, i.e. a region where any future-directed causal curves with $r(\lambda) \leq 2M$ are forced to have $r(\lambda) \leq 2M \forall \lambda \geq \lambda_0$. We proved last time that if you are inside the horizon, $r(\lambda_0) < 2M$, the $r(\lambda)$ is monotonically decreasing for $\lambda \geq \lambda_0$.

Formally, we must consider the case $r(\lambda_0) = 2M$. Note that if $\frac{dr}{d\lambda} < 0|_{\lambda=\lambda_0}$, then we're done. In the next ϵ time later, we'll be inside the horizon and our proof holds. If $\frac{dr}{d\lambda} = 0$, then we sit at $r = 2M$ forever and we're also done.

So there's only one case we have to consider, $dr/d\lambda > 0$ at $\lambda = \lambda_0$.

We've seen that

$$-2\left(\frac{dv}{d\lambda}\right)\left(\frac{dr}{d\lambda}\right) = -V^2 + \left(\frac{2M}{r} - 1\right)\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\Omega}{d\lambda}\right)^2.$$

(4.14 from last time). At $\lambda = \lambda_0$, this vanishes, so $\frac{d\Omega}{d\lambda} = V^2 = 0$. This means that $\frac{dv}{d\lambda} \neq 0$, or else $V^\mu = 0$ (which is a contradiction). Then

$$\frac{dv}{d\lambda}|_{\lambda=\lambda_0} > 0,$$

since we proved last time that $\frac{dv}{d\lambda} \geq 0$.

Hence at least near $\lambda = \lambda_0$, we can use v instead of λ as a parameter along the curve with $r = 2M$ at $v = v_0 = v(\lambda_0)$. Dividing 4.14 by $\left(\frac{dv}{d\lambda}\right)^2$ gives

$$-2\frac{dr}{dv} \geq \frac{2M}{r} - 1 \implies 2\frac{dr}{dv} \leq 1 - \frac{2M}{r}. \quad (5.1)$$

Hence for v_2, v_1 greater than v_0 with $v_2 > v_1$, we have

$$2 \int_{r(v_1)}^{r(v_2)} \frac{dr}{1 - \frac{2M}{r}} \leq v_2 - v_1. \quad (5.2)$$

This completes the $r = 2M$ case. \square

Technically, this doesn't prove that we have a black hole because it is a local statement. To establish a global event horizon will take more work.

Detecting black holes Note the following facts.

- (a) There is no upper bound on the mass of a black hole.
- (b) Black holes are very small. For instance, a black hole with the mass of the Earth would have a radius of 0.9 cm.

We observe black holes by noticing their gravitational pull on stars, for instance, and considering their incredible compactness. Fun fact– there are massive and supermassive (billions of solar mass) black holes in the universe which we have detected astrophysically, and we have no idea where they came from. It seems like there isn't enough time in the age of the universe for them to have formed.

Orbits around a black hole Consider a timelike geodesic around a black hole. The turning points of the potential are given by points $\dot{r} = 0$ in the potential

$$V(r) = \frac{E^2}{2} = \frac{1}{2} \left(\tilde{v} + \frac{h^2}{r^2} \right) \left(1 - \frac{2M}{r} \right), \quad (5.3)$$

and where

$$V'(r) = 0 \implies r_{\pm} = \frac{h^2 \pm \sqrt{h^4 - 12h^2\tilde{v}M^2}}{2M\tilde{v}}, \tilde{v} = 1. \quad (5.4)$$

If $h^2 < 12m$, then we are in free-fall– there are no turning points. However, if $h^2 > 12M^2$, then there is a local minimum of the potential at r_+ , and a local maximum at r_- . One can show that

$$3M < r_- < 6M < r_+, \quad (5.5)$$

where $r = 6M$ is known as the ISCO (innermost stable circular orbit). There is no Newtonian analogue to this– $r = 6M$ lies well within the star.

The energy of the orbit is then

$$E_{\pm} = \frac{r_{\pm} - 2M}{r_{\pm}^{1/2}(r_{\pm} - 3M)^{1/2}}, \quad (5.6)$$

and for $r_+ \gg M$, we find that

$$E_+ \approx 1 - \frac{M}{2r_+} \rightarrow m - \frac{Mm}{(2r_+,)} \quad (5.7)$$

tells us that the energy at this orbit is the relativistic mass-energy $E = m$ minus a correction for the gravitational energy.

Let us approximate the accretion disk of the black hole as non-interacting so that particles basically travel along geodesics. Orbits will radiate off energy, decreasing towards $r \rightarrow 0$. When a particle hits $r = 6M$, it will suddenly turn towards the singularity, releasing a burst of energy (cf. brehmsstrahlung). It is these bursts of energy that we can detect in astrophysical systems.

White holes We looked at ingoing null geodesics in EF coordinates. What about outgoing (radial) null geodesics? We have the outgoing EF coordinate

$$u \equiv t - r_*, \quad (5.8)$$

which is constant along radial outgoing null geodesics. Here, $\frac{dt}{dr_*} = 1$. Then the metric takes the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right)du^2 - 2dudr + r^2 d\Omega_2^2. \quad (5.9)$$

Importantly, this is *not* the same region that we found before! We have a new $r < 2M$ region, but outgoing null geodesics are forced to leave this region. However, we just spent all that time proving that geodesics

inside the event horizon were trapped inside, so this cannot possibly be the same region as in the ingoing coordinates. For constant u (outgoing null radial geodesics), the metric tells us that

$$\frac{dr}{d\tau} = 1. \quad (5.10)$$

So these can propagate from the curvature singularity at $r = 0$ through the surface $r = 2M$. Weak cosmic censorship says that naked singularities are forbidden in nature, and to define a white hole we must have an initial singularity, so this seems to be unphysical. Another argument for this involves time reversal– it has been proved in some generality that black holes are stable to perturbations, so after perturbation they will settle down to another black hole state. If we take the time reversal of this statement, this suggests that white holes will be highly unstable to perturbation, so they are probably not physical.

The Kruskal extension Can we get both the black and white hole regions in a single set of coordinates? Yes! We start in the exterior region $r > 2M$, and define *Kruskal-Szekeres* coordinates (U, V, θ, ϕ) , defining

$$U = -e^{-u/4M}, \quad V = e^{v/4M} \quad (5.11)$$

in terms of the EF coordinates from before. So for $r > 2M$, $U < 0$ and $V > 0$.

Now we can directly compute

$$UV = -e^{\frac{+r}{2M}} \left(\frac{r}{2M} - 1 \right). \quad (5.12)$$

Let us observe that the RHS of this equation is a monotonic function of r , and hence determines $r(U, V)$ uniquely. We also have

$$\frac{V}{U} = -e^{t/2M}, \quad (5.13)$$

which determines $t(U, V)$.

Computing the differentials, we find that

$$dU = \frac{1}{4M} e^{-u/4M} du, \quad dV = \frac{1}{4M} e^{v/4M} dv. \quad (5.14)$$

Therefore

$$\begin{aligned} dUdV &= \frac{1}{16M^2} \exp\left(\frac{v-u}{4M}\right) dudv \\ &= \frac{1}{16M^2} \exp\left(\frac{r_*}{2M}\right) (dt^2 - dr_*^2) \\ &= \frac{1}{16M^2} \exp\left(\frac{r_*}{2M}\right) \left[dt^2 - \frac{dr^2}{\left(1 - \frac{2M}{r}\right)^2} \right]. \end{aligned}$$

This is almost what we want– now we just multiply by the appropriate factors to find that our new line element is

$$ds^2 = -32M^3 \frac{\exp\left(-\frac{r(U,V)}{2M}\right)}{r(U,V)} dUdV + r^2(U,V) d\Omega_2^2. \quad (5.15)$$

Lecture 6.

Monday, January 28, 2019

“What about the bifurcation sphere?” “It will be the throat of an Einstein-Rosen bridge. Those of you who watched Thor... this will be familiar.” –a student and Jorge Santos

Last time, we tried to extend the Schwarzschild spacetime. Our first attempt took us inside the black hole with the EF coordinates, and we then defined the Kruskal extension. Defining Kruskal coordinates U and V , we observe that

$$UV = -e^{\frac{+r}{2M}} \left(\frac{r}{2M} - 1 \right), \quad \frac{V}{U} = -e^{t/2M}. \quad (6.1)$$

Note that $U < 0$ and $V > 0$ for $r > 2M$ in Schwarzschild coordinates. But here’s another surprise– if we switch the signs of U and V , we can still sensibly define $r(U, V)$ for $U \geq 0, V \leq 0$ via 6.1.

This is an entirely new region of spacetime, which is isometric to our black hole exterior region with one notable caveat—time runs backwards. The surface $R = 2M$ is actually achieved by either taking $U = 0$ or $V = 0$. These lines intersect at $U = V = 0$, and note that at $r = 0$, we have $UV = +1$.

If we're interested in AdS/CFT, we should take all four regions seriously. However, if we're interested in astrophysical black holes, then most of the diagram is really the interior of the star as it proceeds to collapse along a geodesic.

Now recall that we have a Killing vector

$$K = \frac{\partial}{\partial t} = \frac{1}{4M} \left(V \frac{\partial}{\partial v} - U \frac{\partial}{\partial U} \right). \quad (6.2)$$

What happens at the point $U = V = 0$?

The Einstein-Rosen bridge Let us make the following coordinate transformation. Define the coordinate ρ by

$$r = \rho + M + \frac{M^2}{4\rho} \quad (6.3)$$

so that as $\rho \rightarrow +\infty, r \rightarrow +\infty$. Naturally this is quadratic(ish) in ρ so we'll get two solutions for each value of r . We choose $\rho > M/2$ in region I, the exterior region, and $0 < \rho < M/2$ in region IV. Then our metric becomes

$$ds^2 = - \left[\frac{1 - \frac{M}{2\rho}}{1 + \frac{M}{2\rho}} \right]^2 dt^2 + \left(1 + \frac{M}{2\rho} \right)^4 \underbrace{(d\rho^2 + \rho^2 d\Omega_2^2)}_{\mathbb{R}^3} \quad (6.4)$$

So we've put the metric in a form where the spatial part looks like \mathbb{R}^3 up to a scaling factor. It's an exercise to check that the transformation $\rho \rightarrow M^2/4\rho$ leaves this metric unchanged.

Now what does a line of constant t look like?

$$ds_{\Sigma_t}^2 = \left(1 + \frac{M}{2\rho} \right)^4 \underbrace{(d\rho^2 + \rho^2 d\Omega_2^2)}_{\mathbb{R}^3}, \quad (6.5)$$

As $\rho \rightarrow +\infty$, we get one asymptotically flat region, and as $\rho \rightarrow 0$ we get another asymptotically flat region. Connecting them we get a wormhole where there is an S^2 at $\rho = M/2$ with some minimum (nonzero) radius.

Now, note that from our Kruskal diagram, this wormhole is non-traversable. In order to get to region IV, we must enter the black hole region. And once we're in the black hole region (region II), we are bound to hit the singularity. Tough luck.

Extendability and singularities

Definition 6.6. A spacetime (\mathcal{M}, g) is *extendible* if it is isometric to a proper subset of another spacetime (\mathcal{M}', g) . The latter is called an *extension* of (\mathcal{M}, g) .

Thus the Kruskal spacetime is an extension of the Schwarzschild spacetime, and moreover it is the maximal analytic extension of Schwarzschild.

Let's talk a bit about singularities now. There are different sorts of singularities we might be interested in.

- Coordinate singularities: the metric (or determinant) is not smooth in some coordinate chart. Nothing physically bad has (necessarily) happened, but we just chose bad coordinates.
- Curvature singularities: a curvature scalar becomes singular. These are physically significant, since we cannot define them away by a coordinate choice.
- Conical singularities: consider the line element

$$g = dr^2 + \lambda^2 r^2 d\phi^2, \quad (6.7)$$

with $\lambda > 0$. We can always make this look like $dr^2 + r^2 d\tilde{\phi}^2$, which looks like \mathbb{R}^2 . But suppose $\lambda \neq 1$. Then the period of our new $\tilde{\phi} = \lambda\phi$ coordinate is not 2π . If we take a "circle" of radius ϵ , we see that $\frac{\text{circumference}}{\text{radius}} = \frac{2\pi\lambda\epsilon}{\epsilon} = 2\pi\lambda$, which does not go to 2π as $\epsilon \rightarrow 0$. What results is a conical singularity at the origin.

- There are certain metrics such that the components of the Riemann tensor are singular in every coordinate system, which means that the tidal forces become infinite as you approach the singularity. We will see an example of this on the examples sheet.

Definition 6.8. A *curve* is a smooth map from some interval of the real line to our manifold, $\gamma : (a, b) \rightarrow \mathcal{M}$, with $a, b \in \mathbb{R}$.

Definition 6.9. A point $p \in \mathcal{M}$ is a *future endpoint* of a future-directed causal curve if for any neighborhood \mathcal{O} of p , there exists t_0 such that $\gamma(t) \in \mathcal{O}$ for all $t > t_0$. That is, our curve gets trapped arbitrarily close to p at late times.

Definition 6.10. We say that γ is *future-inextendible* if it has no future endpoints. Equivalent notions of past endpoints and past-inextendible can be defined by replacing future with past, and we say a curve is *inextendible* if it is future-inextendible and past-inextendible.

Example 6.11. Let $\gamma : (-\infty, 0) \rightarrow \mathcal{M}$, with $\gamma(t) = (t, 0, 0, 0)$. Clearly, this has a future endpoint at $(0, 0, 0, 0)$. If we remove that point from the manifold, however, there is no future endpoint—no other point on our manifold will satisfy the future endpoint condition since we can always take γ at some time arbitrarily close to zero. So the notion of extendibility depends crucially on our choice of manifold.

Definition 6.12. A *geodesic* is complete if an affine parameter for the geodesic exists to $\pm\infty$. A spacetime is *geodesically complete* if all inextendible causal geodesics are complete. A spacetime is *singular* if it is inextendible and geodesically incomplete.

Example 6.13. The Kruskal spacetime is inextendible, and it has plenty of geodesics which hit the $r = 0$ singularity and are therefore incomplete. Therefore by our definition, the Kruskal spacetime is singular.³

Lecture 7.

Wednesday, January 30, 2019

“I could draw here AdS, but I know most of you here are allergic to that.” –Jorge Santos

Today, we will introduce the initial value problem in general relativity. In classical physics, it’s natural to say that we set initial conditions and (using some differential equations) evolve them to the future. But in GR, the question is more complicated because we are trying to evolve things to “the future,” yet the future is part of what we are solving for! It will take some care to define precisely what we mean.

Definition 7.1. Let (\mathcal{M}, g) be a time-orientable spacetime. A *partial Cauchy surface* Σ is a hypersurface for which no two points $p, p' \in \Sigma$ are connected by a causal curve in \mathcal{M} .

Definition 7.2. The *future domain of dependence* of Σ , denoted $D^+(\Sigma)$, is the set $p \in \mathcal{M}$ such that every past-inextendible causal curve through p intersects Σ . The past domain of dependence $D^-(\Sigma)$ is defined equivalently for future-inextendible causal curves, and the domain of dependence is then $D(\Sigma) = D^+(\Sigma) \cup D^-(\Sigma)$.

Example 7.3. Consider Minkowski space, with a partial Cauchy surface $\Sigma : t = 0, x > 0$. Then the domain of dependence $D(\Sigma)$ is the 45-degrees region with $x \geq |t|$ —see Fig. 3 for an illustration.

Definition 7.4. As far as we (as physicists) are concerned, a *hyperbolic partial differential equation* (second order) is defined to be a differential equation of the form

$$g^{ef} \nabla_e \nabla_f T^{ab...}_{cd...} = \tilde{G}^{ab...}_{cd...}, \quad (7.5)$$

where the tensor we’ve put on the RHS depends only on T and first derivatives of T in some smooth way. That is, the second derivative is the highest derivative, and it appears only linearly. These are sometimes called quasilinear equations.

³In fact, incomplete geodesics are a sufficient but not necessary condition for a spacetime to be singular. We should also in principle be concerned about accelerating observers falling off the edge of spacetime into a singularity—cf. Hawking and Ellis, and also work by Geroch. According to Hawking and Ellis, the more appropriate notion is not geodesic completeness but a more general idea of “b-completeness,” completeness of general inextendible causal curves in the spacetime.

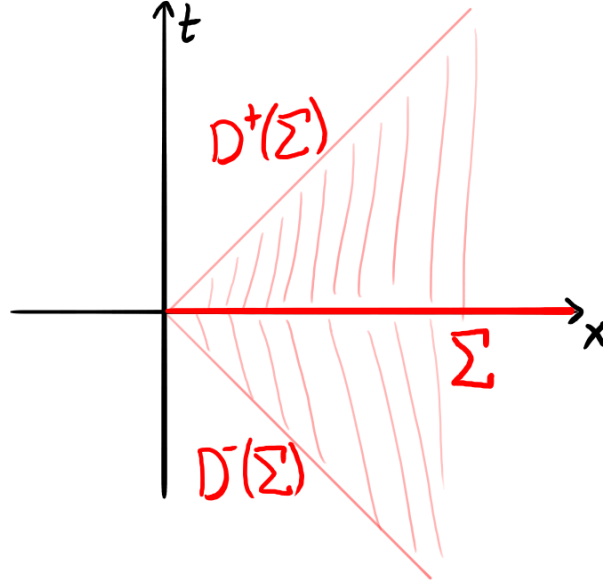


FIGURE 3. In Minkowski space, we select the partial Cauchy surface $\Sigma : t = 0, x > 0$ (in red). Given initial conditions on Σ , we can predict physics in the shaded region $D^+(\Sigma)$ and retrodict in the region $D^-(\Sigma)$. However, Σ is not a Cauchy surface because there are points in spacetime which lie outside the domain of dependence $D(\Sigma)$. Note also that the boundary of $D^+(\Sigma)$ (i.e. the Cauchy horizon) coincides with the boundary of the causal future of points outside Σ .

Definition 7.6. A *Cauchy surface* for a spacetime (\mathcal{M}, g) is a partial Cauchy surface Σ with $D(\Sigma) = \mathcal{M}$ the entire manifold. Spacetimes (\mathcal{M}, g) which admit Cauchy surfaces are called *globally hyperbolic*.

Note that if $D(\Sigma) \neq \mathcal{M}$, then the solution of hyperbolic equations will *not* be uniquely specified on $\mathcal{M} \setminus D(\Sigma)$ by data on Σ .

Example 7.7. A very simple example of a spacetime that is not globally hyperbolic is $\mathbb{R}^{1,1} \setminus 0$. There are points where a past-directed inextendible causal curve can drop off the manifold (i.e. hit the point we have removed).

Theorem 7.8 (Wald). *Let (\mathcal{M}, g) be a globally hyperbolic spacetime. Then*

- (i) *There exists a global-time function $t : \mathcal{M} \rightarrow \mathbb{R}$ such that $-(dt)^a$ (the normal to surfaces of constant t) is future-directed and timelike.*
- (ii) *Surfaces of constant t , Σ_t , are Cauchy surfaces and all have the same topology.*
- (iii) *The topology of \mathcal{M} is $\mathbb{R} \times \Sigma$.*

Remark: spacetimes which have singularities can still be globally hyperbolic. Consider the Kruskal spacetime— we can set initial conditions on the Cauchy surface $U + V = \text{constant}$.

Extrinsic curvature Let Σ denote a spacelike or timelike hypersurface with unit normal n_a (i.e. such that $n^a n_a = \pm 1$).

Lemma 7.9. *For any $p \in \Sigma$, let $h^a_b = \delta^a_b \mp n^a n_b$ so that $h^a_b n^b = 0$ (where upper/lower signs apply for spacelike/timelike respectively). Then*

- (a) $h^a_c h^c_b = h^a_b$
- (b) *Any vector x^a at p can be written as $x^a_{\parallel} + x^a_{\perp}$, where $x^a_{\parallel} = h^a_b x^b$ and $x^a_{\perp} = \pm n_b x^b n^a$.*
- (c) *If X^a, Y^a are tangent to Σ , then $h^{ab} X_a Y_b = g^{ab} X_a Y_b$. h is sometimes called the first fundamental form.*

That is, let N^a be a normal to Σ at p . If we parallel transport N^a on Σ along X^a (that is, $X^b \nabla_b N_a = 0$), will the parallel-transported N^a still be normal? Take Y tangent to Σ , so that $Y^a N_a = 0$ at the point p . Then

$$\nabla_X(Y^a N_a) = X^b \nabla_b(Y^a N_a) = N_a X^b \nabla_b Y^a. \quad (7.10)$$

Definition 7.11. Up to now, n_a has been defined only on Σ . First, let us extend it to a neighborhood of Σ in an arbitrary way. The *extrinsic curvature* (second fundamental form) is defined at $p \in \Sigma$ by

$$K(X, Y) = -n_a (\nabla_{X_{\parallel}} Y_{\parallel})^a. \quad (7.12)$$

Here, X and Y need not be tangent to Σ , but we are interested in their projections parallel to Σ .

Lemma 7.13. K_{ab} is independent of how n_a is extended off Σ , and in particular

$$K_{ab} = h_a^c h_b^d \nabla_c n_d. \quad (7.14)$$

Proof. The RHS of $K(X, Y)$ is

$$-n_d X_{\parallel}^c \nabla_c Y_{\parallel}^d = X_{\parallel}^c Y_{\parallel}^d \nabla_c n_d \quad (7.15)$$

by the Leibniz rule, since $n_d Y_{\parallel}^d = 0$. So

$$K(X, Y) = X_{\parallel}^c Y_{\parallel}^d \nabla_c n_d = X^a Y^b h_a^c h_b^d \nabla_c n_d. \quad (7.16)$$

Therefore

$$K_{ab} = h_a^c h_b^d \nabla_c n_d. \quad (7.17)$$

To demonstrate that K_{ab} is independent of how n_a is extended, consider a different extension n'_a , and let $m_a \equiv n'_a - n_a$. Note that on Σ , $m_a = 0$. Then on Σ ,

$$\begin{aligned} X^a Y^b (K'_{ab} - K_{ab}) &= X_{\parallel}^c Y_{\parallel}^d \nabla_c m_d \\ &= \nabla_{X_{\parallel}} (Y_{\parallel}^d m_d) = 0 \end{aligned}$$

since $Y_{\parallel}^d m_d = 0$ on Σ . □

We can use

$$n^b \nabla_c n_b = \frac{1}{2} \nabla_c (n_b n^b) = 0 \quad (7.18)$$

and conclude that

$$K_{ab} = h_a^c \nabla_c n_b. \quad (7.19)$$

At home, we should show that the extrinsic curvature is given by a Lie derivative,

$$K_{ab} = \frac{1}{2} \mathcal{L}_m (h_{ab}) \quad (7.20)$$

where h here is the first fundamental form.

Lecture 8.

Thursday, January 31, 2019

"I know that you all really hate my handwriting. And our equations have hundreds of indices. So today I will receive threatening emails by the end of the day." –Jorge Santos

Last time, we introduced the extrinsic curvature,

$$K_{ab} h_a^c h_b^d \nabla_c n_d, \quad (8.1)$$

where $n^a n_a = \pm 1$, with n^a the unit normal to the hypersurface Σ . We also define the projection

$$h_{ab} = g_{ab} \mp n_a n_b. \quad (8.2)$$

Lemma 8.3. $K_{ab} = K_{ba}$, so K is a symmetric 2-tensor.

Proof. Let $f : \mathcal{M} \rightarrow \mathbb{R}$ be constant on Σ with $df \neq 0$ on Σ . Let X^a be tangent to Σ . Thus

$$X(f) = X^a \nabla_a f = 0. \quad (8.4)$$

This implies that $(df)^a$ is normal to Σ . Thus on Σ we can write

$$n_a = \alpha (df)_a, \quad (8.5)$$

where α is chosen such that $n_a n^a = \pm 1$. It follows that

$$\nabla_c n_d = \alpha \nabla_c \nabla_d f + \frac{(\nabla_c \alpha)}{\alpha} n_d. \quad (8.6)$$

But this tells us that

$$K_{ab} = h_a^c h_b^d \nabla_c n_d = \alpha h_a^c h_b^d \nabla_c \nabla_d f, \quad (8.7)$$

and for all torsion-free spacetimes, covariant derivatives commute on scalars, so K_{ab} is symmetric. \square

As it turns out, the property that

$$K_{ab} = \frac{1}{2}(\mathcal{L}_m h)_{ab} \quad (8.8)$$

means that the extrinsic curvature has the right form for an initial condition in the initial value problem of general relativity.

The Gauss-Coducci equation A tensor at a point $p \in \Sigma$ is invariant under projection h_a^b if

$$T^{a_1 \dots a_n}_{b_1 \dots b_s} = h_{c_1}^{a_1} \dots h_{c_n}^{a_n} h_{b_1}^{d_1} \dots h_{b_s}^{d_s} T^{c_1 \dots c_n}_{d_1 \dots d_s}. \quad (8.9)$$

Proposition 8.10. A covariant derivative D on Σ can be identified by projection of the covariant derivative on \mathcal{M} . Thus

$$D_a \Gamma^{b_1 \dots b_n}_{c_1 \dots c_s} = h_a^d h_{e_1}^{b_1} \dots h_{e_n}^{b_n} h_{c_1}^{f_1} \dots h_{c_s}^{f_s} \nabla_d T^{e_1 \dots e_n}_{f_1 \dots f_s}. \quad (8.11)$$

Lemma 8.12. The covariant derivative D is precisely the Levi-Civita (metric) connection associated to the metric h_{ab} on the submanifold Σ :

$$D_a h_{bc} = 0, \quad (8.13)$$

and D is torsion-free.

Proof. The covariant derivative D acting on the metric h_{ab} is the projection

$$D_a h_{bc} = h_e^a h_f^b h_g^c \nabla_a h_{bc}. \quad (8.14)$$

Recalling that $h_{ab} = g_{ab} \mp n_a n_b$, let's expand the ordinary covariant derivative:

$$\nabla_a h_{bc} = \mp n_c \nabla_a n_b \mp n_b \nabla_a n_c, \quad (8.15)$$

where the $\nabla_a g_{bc}$ term is zero by metric compatibility. But recall that $h_a^c n_c = 0$ (i.e. the projection of a normal vector onto the submanifold is zero), so when we compute the projection of $\nabla_a h_{bc}$ onto Σ , we find that

$$D_a h_{bc} = h_e^a h_f^b h_g^c (\mp n_c \nabla_a n_b \mp n_b \nabla_a n_c) = 0, \quad (8.16)$$

which tells us that D is metric compatible with respect to h_{ab} .

To prove it is torsion-free, let $f : \Sigma \rightarrow \mathbb{R}$ be a scalar function on Σ , and extend it off Σ to a function $f : \mathcal{M} \rightarrow \mathbb{R}$ so that $\nabla_a f$ is well-defined. Then consider $D_a D_b f$. This is

$$\begin{aligned} D_a D_b f &= D_a h_b^e \nabla_e f \\ &= h_a^c h_b^d \nabla_c (h_d^e \nabla_e f) \\ &= h_a^c h_b^e \nabla_c \nabla_e f + h_a^c h_b^d (\nabla_c h_d^e) (\nabla_e f). \end{aligned}$$

This first term is already manifestly symmetric in a and b since the original connection was torsion free (i.e. covariant derivatives $\nabla_c \nabla_e$ commute on scalars), so let us rewrite the second term (neglecting the $\nabla_e f$ part) as follows:

$$\begin{aligned} h_a^c h_b^d \nabla_c h_d^e &= g^{ef} h_a^c h_b^d \nabla_c h_d^e \\ &= \mp g^{ef} h_a^c h_b^d (n_f \nabla_c n_d + n_d \nabla_c n_f) \\ &= \mp n^e K_{ab}, \end{aligned}$$

where we have grouped together $h_a^c h_b^d \nabla_c n_d = K_{ab}$ and gotten rid of the $n_d \nabla_c n_f$ term by projection with h_b^d . But K_{ab} is symmetric, so it also vanishes under antisymmetrization. We conclude that

$$D_{[a} D_{b]} f = 0. \quad (8.17)$$

\square

We'd like to relate the extrinsic curvature to our old-fashioned Riemann curvature. How do we do this?

Proposition 8.18. Denote the Riemann tensor associated with D_a on Σ by R'^a_{bcd} . This is given by Gauss's equation:

$$R'^a_{bcd} = h^a_e h_b^f h_c^g h_d^h R^e_{fgh} \pm 2K_{[c}^a K_{d]b}. \quad (8.19)$$

Proof. Let x^a be tangent to Σ . Then the Ricci identity for D is

$$R'^a{}_{bcd}X^b = 2D_{[c}D_{d]}X^a. \quad (8.20)$$

Let us compute the RHS of this equation.

$$D_c D_d X^a = h_c^e h_d^f h_g^a \nabla_e (D_f X^g) \quad (8.21)$$

$$= h_c^e h_d^f h_g^a \nabla_e (h_f^h h_i^g \nabla_h X^i) \quad (8.22)$$

$$= h_c^e h_d^f h_g^a \nabla_e \nabla_h X^i + h_c^e h_d^f h_g^a (\nabla_e h_f^h) \nabla_h X^i + h_c^e h_d^f h_g^a (\nabla_e h_i^g) \nabla_h X^i. \quad (8.23)$$

This first term is already looking good. Note that we have to project the indices everywhere, so in the first line we project c, d, a indices, and in the second line we project f, g indices, and in the final line we expand out the covariant derivative.⁴

We have seen that

$$h_c^e h_d^f h_g^a \nabla_e h_f^h = \mp n^e K_{ab}. \quad (8.24)$$

We can use this identity in the last two terms of 8.23 to get

$$D_c D_d X^a = h_c^e h_d^f h_g^a \nabla_e \nabla_h X^i \mp K_{cd} h_i^a n^h \nabla_h X^i \mp K_c^a n_a h_d^h \nabla_h X^i. \quad (8.25)$$

This second term already drops out since K_{cd} is symmetric and we're antisymmetrizing over c and d in the final expression. The last term can be recast as

$$K_c^a h_d^h \nabla_h (n_i X^i) \pm K_c^a X^i h_d^h \nabla_h n_i = \pm K_c^a K_{bd} X^b. \quad (8.26)$$

Antisymmetrizing, we find that

$$\begin{aligned} R'^a{}_{bcd}X^b &= 2h_{[c}^e h_{d]}^f h_g^a \nabla_e \nabla_f X^g \pm 2K_{[c}^a K_{d]b} X^b \\ &= h_c^e h_d^f h_g^a h^a{}_b R^g{}_{hef} X^b \pm 2K_{[c}^a K_{d]b} X^b. \end{aligned}$$

Since X^b is arbitrary, this holds as a tensor identity. □

Lemma 8.27. *The Ricci scalar of Σ is*

$$R' = R \mp 2R_{ab}n^a n^b \pm K^2 \mp K^{ab}K_{ab}, \quad (8.28)$$

where $K = g^{ab}K_{ab}$, the trace of K_{ab} .

Proposition 8.29 (Codacci's equation). $D_a K_{bc} - D_b K_{ac} = h_a^d h_b^e h_c^f n^a R_{defg}$.

The proof of Codacci's equation is homework but simple compared to Gauss's equation.

Lemma 8.30. $D_a K^a{}_b - D_b K = h_b^c R_{cd}n^d$. This is sometimes referred to as the Codacci equation, though we will call it the contracted Codacci equation (if we refer to it by name at all).

The constraint equations We now have several definitions and identities. What are they good for? Assume the hypersurface Σ is spacelike, with a timelike normal n^a . The Einstein equation is just

$$R_{ab} - \frac{R}{2}g_{ab} \equiv G_{ab} = 8\pi T_{ab}.$$

We're going to contract it with $n^a n^b$. Using the same notation as before, the Einstein equation takes the form

$$R' - K^{ab}K_{ab} + K^2 = 16\pi\rho. \quad (8.31)$$

A priori, we might have thought that we had two free choices for our initial conditions on Σ — we could have chosen a metric on Σ and also an extrinsic curvature (like a first derivative of the metric moving off of Σ). But in fact the choice is not entirely free.

⁴As a rule of thumb, all free indices (i.e. that are not contracted over) must be projected in *each* covariant derivative. In my (limited) experience, I have found it most useful to begin with the innermost covariant derivative and work my way out. So for instance we know that $D_d X^a$ will come with two projectors, one with d as the free index and another with a as the free index. Our final expression (say, $D_d X^a$) still has two free indices d, a but it also has two projectors, so that when we take the covariant derivative $D_c(D_d X^a)$, there are three free indices to project. Hence we get three projectors outside all the derivatives and two inside a first derivative, as in Eqn. 8.22.

The equation 8.31 is known as the *Hamiltonian constraint*. Contracting with n^a and projecting with h gives us instead

$$D_b K^b_a - D_a K = 8\pi h_a^b T_{bc} n^c, \quad (8.32)$$

which is sometimes called the momentum constraint.⁵

Theorem 8.33 (Choquet-Bruhat and Geroch (1969)). *Let (Σ, h, K) be initial data satisfying the vacuum Hamiltonian and momentum constraints ($T_{ab} = 0$). Then there exists a unique (up to diffeomorphism) spacetime (\mathcal{M}, g_{ab}) called the maximal Cauchy development of (Σ, h, K) such that*

- (i) (\mathcal{M}, g) satisfies the Einstein equation,
- (ii) (\mathcal{M}, g) is globally hyperbolic with Cauchy surface Σ ,
- (iii) the induced metric and the extrinsic curvature of Σ are h and K , respectively,
- (iv) and any other spacetime satisfying (i), (ii), and (iii) is isometric to a subset of (\mathcal{M}, g) .

Analogous conditions exist for matter that obeys reasonable energy conditions. However, it is possible that the maximal Cauchy development is extendible. In that case, the region of the manifold in the complement of the maximal Cauchy development is not unique. Physics is not predictable outside the maximal Cauchy development, and the boundary of this region is called a *Cauchy horizon*. Note that in order to avoid trivial Cauchy horizons from poor choices of Σ , we require the initial data to be inextendible.

A final note. Look at the Schwarzschild solution with negative M , such that the line element is

$$ds^2 = -\left(1 + \frac{2|M|}{r}\right) dt^2 + \frac{dr^2}{1 + \frac{2|M|}{r}} + r^2 d\Omega_2^2. \quad (8.34)$$

But if we take any $t = 0$ surface, it is impossible for us to avoid the $r = 0$ singularity. Our initial data should not be singular! Look at outgoing geodesics,

$$\frac{dt}{dr} = \frac{1}{1 + \frac{2|M|}{r}} \implies t \simeq t_0 + \frac{r^2}{4|M|}. \quad (8.35)$$

This tells us that a Cauchy horizon has appeared, and this one we cannot avoid by picking a different Cauchy surface.

Lecture 9.

Friday, February 1, 2019

“Look at this word, the most naughty word in mathematics. ‘Generically.’” –Jorge Santos

Last time, we formulated the initial value problem in general relativity. We said that some initial data is bad– we require that it is at a minimum inextendible and geodesically complete. However, something else bad can happen. See the figure: if our initial data is prescribed on an asymptotically null surface, then it could be our domain of dependence is cut off by a light cone. Therefore, we require that the initial data is also *asymptotically flat*. We haven’t yet defined what this means, though.

Definition 9.1. An initial data set (Σ, h, K) has an asymptotically flat end if

- (i) Σ is diffeomorphic to $\mathbb{R}^3 \setminus B$, where B is a closed ball centered on the origin in \mathbb{R}^3 ,
- (ii) If we pull back the \mathbb{R}^3 coordinates to define coordinates x^i on Σ , then $h_{ij} = \delta_{ij} + O(1/r)$ (the Euclidean metric plus $1/r$ terms) and $K_{ij} = O(1/r^2)$ as $r \rightarrow +\infty$, where $r = \sqrt{x^i x_i}$.
- (iii) Derivatives of the latter expression also hold, e.g.

$$\partial_k h_{ij} = O(1/r^2).$$

Definition 9.2. An initial data set is asymptotically flat with N ends if it is the union of a compact set with N asymptotically flat ends (e.g. the Kruskal spacetime has two asymptotically flat ends).

Having defined our wish list for an initial data set, it would be extremely disturbing if we made this data set as nice as we like, and yet ended up with an extendible spacetime as our maximal Cauchy development. That such spacetimes do not (generically) occur is the content of the strong cosmic censorship conjecture.

⁵It’s diffeomorphism invariance that gives us this constraint, analogous to the constraint on trying to solve QED in Coulomb gauge.

Theorem 9.3. *Let (Σ, h, K) be a geodesically complete, asymptotically flat, initial data set for the vacuum Einstein equations. Then generically the maximal Cauchy development of the initial data is inextendible.*

This has been proved (with much work) that strong cosmic censorship is true in asymptotically flat space (i.e. Minkowski space is stable to small perturbations). It has nearly been proved (by Dafermos et al) for the Kerr black hole. And violations are known for Reissner-Nordström solution in de Sitter space.

Singularity theorems Now, in Newtonian gravity the formation of singularities is generally avoidable. If we drop a bit of matter in a Newtonian potential, it will fall straight to $r = 0$. But if we add a bit of angular momentum, the angular momentum will prevent the matter from falling straight in, and so our chunk of stuff cannot actually hit the $r = 0$ singularity. This is *not* true in general relativity.

It is the focus of an amazing set of theorems originally due to Penrose, and later generalized in conjunction with Hawking,⁶ that the formation of singularities is generic in general relativity given a set of very reasonable conditions.⁷

Definition 9.4. A *null hypersurface* is a hypersurface whose normal is everywhere null. For example, take the metric

$$g^{\mu\nu} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 - \frac{2M}{r} & 0 & 0 \\ 0 & 0 & 1/r^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (9.5)$$

The 1-form $n = dr$ is normal to the surfaces of constant Ω . Thus note that

$$n^2 = g^{\mu\nu} n_\mu n_\nu = g^{rr} = \left(1 - \frac{2M}{r}\right). \quad (9.6)$$

So the surface $r = 2M$ is a null hypersurface.

Now let n_a be normal to a null hypersurface \mathcal{N} . Then any nonzero vector X^a tangent to the hypersurface obeys

$$X^a n_a = 0. \quad (9.7)$$

But in particular we could have $X^a = n^a$, which implies that either X^a is spacelike or X^a is parallel to n^a . In particular, note that n^a is tangent to the hypersurface. Hence on \mathcal{N} , the integral curves of n^a lie within \mathcal{N} .

Proposition 9.8. *The integral curves of n^a are null geodesics. These are called the generators of \mathcal{N} .*

Proof. Let the null hypersurface \mathcal{N} be given by an equation $f = \text{constant}$ for some function f with $df \neq 0$. df is everywhere normal to \mathcal{N} , so we must have $n = h df$. Define $N \equiv df$. Since \mathcal{N} is null, we must have

$$N_a N^a = 0 \text{ on } \mathcal{N}. \quad (9.9)$$

Hence the function $(N^a N_a)$ is constant on \mathcal{N} . This implies that the gradient of this function is normal to \mathcal{N} . Thus

$$\nabla_a (N_b N^b)|_{\mathcal{N}} = 2\alpha N_a \quad (9.10)$$

for some proportionality constant α . Now, we also have that

$$\nabla_a N_b = \nabla_a \nabla_b f = \nabla_b \nabla_a f = \nabla_b N_a,$$

where we have used the assumption that our spacetime is torsion-free, so that covariant derivatives commute on scalars. But then by the Leibniz rule, we conclude that

$$N^b \nabla_b N_a|_{\mathcal{N}} = \alpha N_a, \quad (9.11)$$

which tells us that N^a is the tangent vector of a non-affinely parametrized geodesic. We conclude that the integral curves of N are null geodesics. \square

⁶It's kind of great to learn about black holes from a guy who's on a first-name basis with Stephen Hawking.

⁷I gave a talk on this last term!

Geodesic deviation We first saw geodesic deviation last term in *General Relativity*– for “nearby” geodesics, we can define relative “velocities” and “accelerations” leading to tidal forces. We’ll make this notion more precise now.

Definition 9.12. A 1-parameter family of geodesics is a map $\gamma : I \times I' \rightarrow \mathcal{M}$, where I, I' are both open intervals in \mathbb{R} , such that

- (i) for fixed s , $\gamma(s, \lambda)$ is a geodesic with affine parameter λ ,
- (ii) the map $(s, \lambda) \mapsto \gamma(s, \lambda)$ is smooth and one-to-one with smooth inverse. Thus γ defines a surface in \mathcal{M} .

Let U^a be the tangent vector to the geodesics and S^a be the vector tangent to curves of constant λ (i.e. takes us between neighboring geodesics). In a chart x^μ , the geodesics are specified in coordinates $x^\mu(s, \lambda)$ where $S^\mu = \frac{\partial x^\mu}{\partial s}$. Hence

$$x^\mu(s + \delta s, \lambda) = x^\mu(s, \lambda) + \delta s S^\mu + O(\delta s^2),$$

where S^μ points towards the next geodesic at $s + \delta s$. For this reason, we call S^μ the *deviation vector*.

On the surface (i.e. the image of γ), we can use s and λ coordinates. This gives a coordinate chart in which

$$S^\mu = \left(\frac{\partial}{\partial s} \right)^\mu, \quad U^\mu = \left(\frac{\partial}{\partial \lambda} \right)^\mu \quad (9.13)$$

on the surface. But in this coordinate system, these are just partial derivatives, so they automatically commute, and the commutator is covariant, so S^a and U^a commute in general, independent of basis. Thus

$$0 = [S, U] \iff U^b \nabla_b S^a = S^b \nabla_b U^a. \quad (9.14)$$

This implies that

$$U^c \nabla_c (U^b \nabla_b S^a) = R^a_{bcd} U^b U^c S^d. \quad (9.15)$$

Solutions of 9.15 are called *Jacobi fields*.

Geodesic congruence We’ll introduce a final definition for today.

Definition 9.16. Let $U \subset \mathcal{M}$ be open. A *geodesic congruence* on U is a family of geodesics such that exactly one geodesic passes through each $p \in U$.

A geodesic congruence provides us with a notion of a set of geodesics covering a subset of the manifold without overlap.

Lecture 10.

Monday, February 18, 2019

“We’ll see that the shear in AdS tells us amazing things. Well, we won’t see the AdS. But we’ll see the amazing thing.” –Jorge Santos

It’s been 15 days since the last lecture. Where did we leave off? We introduced null surfaces, and suggested that null surfaces will be important to our understanding of singularity theorems.

Last time, we talked about geodesic congruences (families of geodesics which cover some submanifold of the spacetime). Consider a congruence where all geodesics are of the same type,

$$U^a U_a = \pm 1$$

in affine parametrization (for spacelike or timelike geodesics respectively) or $U^a U_a = 0$ for null congruences. Now consider a 1-parameter family of geodesics which form such a congruence. In terms of the geodesic tangent vector U and the deviation vector S , we have

$$[S, U] = 0 \iff U^b \nabla_b S^a = S^b \nabla_b U^a = S^b B^a_b \quad (10.1)$$

where we have defined

$$B^a_b \equiv \nabla_b U^a. \quad (10.2)$$

That is, B measures the failure of S^a to be parallel propagated along U^b . This tensor has some nice properties:

$$B^a_b U^b = 0, \quad U_a B^a_b = \frac{1}{2} \nabla_b (U^2) = 0. \quad (10.3)$$

The first follows from the geodesic equation and the latter from $U^2 = \text{constant}$.⁸ Consider the following expression:

$$U \cdot \nabla (U \cdot S) = (U \cdot \nabla U^a) S_a + U^a U \cdot S_{,a}, \quad (10.4)$$

where (\cdot) indicates contracted indices. But this first term is zero by the geodesic equation, and the second is zero by the property $U^a B_{ab} = 0$ from above. Thus $U \cdot S$ is constant along the integral curves of U .

The existence of some quantity along geodesics suggests to us there might be some gauge freedom we get to fix. In particular, we can make a very nice choice to fix the affine parameter along geodesics:

$$\lambda' = \lambda - a(s) \quad (10.5)$$

where a is some function of our choice. In particular this allows us to shift S :⁹

$$S'^a \equiv S^a + \frac{da}{ds} U^a \implies U \cdot S' = U \cdot S + \frac{da}{ds} U^2. \quad (10.6)$$

What's nice about this? $U^2 = \pm 1$ for timelike or spacelike coordinates, which means that WLOG we can set $U \cdot S' = 0$. But for the null case, $U^2 = 0$ means that the reparametrization cannot be fixed by a choice of a —we just get $U \cdot S' = U \cdot S$.

Gauge fixing, the easy way For null geodesic congruences, we have $U^2 = 0$ and thus

$$U \cdot S' = U \cdot S,$$

so a doesn't help us pick a nice reparametrization. Instead, we pick a spacelike hypersurface Σ which intersects each geodesic once— we can do this since we're looking at a congruence. Now let N^a be a vector field defined on Σ obeying $N^2 = 0, N \cdot U = -1$ on Σ .

Now extend N^a off Σ by parallel transport along the geodesic

$$U \cdot \nabla N^a = 0. \quad (10.7)$$

There's a nice discussion of equivalence classes and the full freedom we have to fix the gauge, but we'll leave it to Wald. For our purposes, notice that we have the three properties

$$N^2 = 0, \quad U \cdot N = -1, \quad U \cdot \nabla N^a = 0. \quad (10.8)$$

Take a deviation vector S^a and decompose it as follows:

$$S^a = \alpha U^a + \beta N^a + \hat{S}^a, \quad (10.9)$$

where

$$U \cdot \hat{S} = N \cdot \hat{S} = 0. \quad (10.10)$$

This is a bit like a Gram-Schmidt process— we can subtract off the bits parallel to the geodesic U^a and also the bit parallel to N^a . Note that $U \cdot S = -\beta$, where β is constant, as we found at the start of the calculation. It's also important to observe that any vector which is orthogonal to two null vectors is either spacelike or the zero vector.¹⁰

⁸Just one line of algebra. $U_a B^a_b = U_a \nabla_b U^a = \frac{1}{2} \nabla_b (U^2) = 0$.

⁹Recall that $S \equiv \frac{dx^\mu}{ds}$, and $U^\mu \equiv \frac{dx^\mu}{d\lambda}$. Thus for a geodesic parametrized by λ', s , we may write

$$\begin{aligned} x^\mu(\lambda', s) &= x^\mu(\lambda - a(s), s) \\ &= x^\mu(\lambda, s) + \frac{dx^\mu}{d\lambda} a(s) \\ &= x^\mu(\lambda, s) + U^\mu a(s), \end{aligned}$$

and taking a derivative with respect to s now yields

$$\frac{dx^\mu(\lambda', s)}{ds} = \frac{dx^\mu(\lambda, s)}{ds} + \frac{da}{ds} U^\mu.$$

¹⁰To see this, try the calculation in Minkowski space. I haven't worked out the details myself yet.

So we can write a deviation vector S^a as the sum of a part

$$\alpha U^a + \hat{S}^a$$

which is orthogonal to U^a and a part

$$\beta N^a$$

that is parallel transported along each geodesic. We are interested in a congruence containing the generators of a null hypersurface \mathcal{N} . In this case, if we pick a 1-parameter family of geodesics contained in \mathcal{N} , then the deviation vector S^a will be tangent to \mathcal{N} and hence $U \cdot S = 0$. Since U^a is normal to \mathcal{N} , we have $\beta = 0$.

We can write

$$\hat{S}^a = P^a_b S^b \quad (10.11)$$

under the projection

$$P^a_b = \delta^a_b + N^a U_b + U^a N_b, \quad (10.12)$$

where it's a quick exercise to check that $P^a_b P^b_e = P^a_e$. This projection projects the tangent space at p onto the spacelike T_\perp , the space perpendicular to both null vectors N^a and U^a . One can also check that

$$U \cdot \nabla P^a_b = 0, \quad (10.13)$$

so P is parallel-propagated trivially along geodesics U^a .

Proposition 10.14. *A deviation vector for which $U \cdot S = 0$ satisfies*

$$U \cdot \nabla \hat{S}^a = \hat{B}^a_b \hat{S}^b, \quad (10.15)$$

where \hat{B}^a_b is a projected version of B^a_b into the perpendicular space,

$$\hat{B}^a_b = P^a_c B^c_d P_b^d. \quad (10.16)$$

Proof.

$$\begin{aligned} U \cdot \nabla \hat{S}^a &= U \cdot \nabla (P^a_c S^c) \\ &= P^a_c U \cdot \nabla S^c \\ &= P^a_c B^c_d S^d \\ &= P^a_c B^c_d P^d_e S^e, \end{aligned}$$

where we used $U \cdot S = 0$ and $B^c_d U^d = 0$. Finally, we can use the fact that P is a projector so that $P^2 = P$. So we replace $P^d_e = P^d_f P^f_e$ and this gives us the projectors to write everything with hats. \square

Expansion, rotation, and shear We've proved some nice properties about this B^a_b matrix. But what exactly is it? As it turns out, \hat{B}^a_b can be regarded as a matrix that acts on the 2D space T_\perp . It's very natural for us to decompose such a matrix into its symmetric (trace-free) part, its antisymmetric part, and its trace part.

Definition 10.17. Let us define

$$\theta \equiv \hat{B}^a_a, \quad \hat{\sigma}_{ab} \equiv \hat{B}_{(ab)} - \frac{1}{2} P_{ab} \theta, \quad \hat{\omega}_{ab} = \hat{B}_{[ab]}. \quad (10.18)$$

Note that the factor of $1/2$ works for $3+1$ spacetime dimensions because the trace of P_{ab} is 2 here. This then implies that

$$\hat{B}^a_b = \frac{1}{2} \theta P^a_b + \hat{\sigma}^a_b + \hat{\omega}^a_b. \quad (10.19)$$

With this definition, notice that

$$\theta \equiv g^{ab} \hat{B}_{ab} = g^{ab} B_{ab} = \nabla_a U^a, \quad (10.20)$$

which does not depend on the choice of N . In fact, this is true for any scalar like the eigenvalues of the rotation matrix.

Proposition 10.21. *If the congruence contains the generators of a null hypersurface \mathcal{N} , then $\hat{\omega}_{ab} = 0$ on \mathcal{N} . Conversely, if $\hat{\omega}_{ab} = 0$ everywhere on \mathcal{N} , then U^a is everywhere hypersurface orthogonal.¹¹*

¹¹This is not quite true in some extensions of general relativity, but it is exact in pure GR.

Proof. The definition of \hat{B} and the fact that $U \cdot U = B \cdot U = 0$ implies that

$$\hat{B}^b_c = B^b_c + U^b N_d B^d_c + U_c B^b_d N^d + U^b U_c N_d B^d_e N^e. \quad (10.22)$$

Using this we have

$$U_{[a} \hat{\omega}_{bc]} = U_{[a} \nabla_c U_{b]} = -\frac{1}{6} (U \wedge dU)_{abc}, \quad (10.23)$$

where we have just rewritten the expression in form notation. If U^a is normal to \mathcal{N} , then $U \wedge dU = 0$ on \mathcal{N} , and hence on \mathcal{N} ,

$$0 = U_{[a} \hat{\omega}_{bc]} = \frac{1}{3} (U_a \hat{\omega}_{bc} + U_b \hat{\omega}_{ca} + U_c \hat{\omega}_{ab}), \quad (10.24)$$

where we've just opened up the antisymmetrization and used the fact that $\hat{\omega}_{bc}$ is already antisymmetric to absorb factors of 2.

Contracting with N^a gives

$$\hat{\omega}_{bc} = 0 \quad (10.25)$$

on \mathcal{N} , where $U \cdot N = -1$ and $\hat{\omega} \cdot N = 0$. Conversely if $\hat{\omega} = 0$ everywhere, then 10.23 implies that U is hypersurface orthogonal by the Frobenius theorem. \square

Lecture 11.

Wednesday, February 20, 2019

"Roger (Penrose) noticed something very interesting about the signs of these guys. ...well, I have no idea if that was actually the case, I'm just making this up as I go along." –Jorge Santos

We showed last time that we can characterize the behavior of null congruences by three objects– the expansion, the shear, and the rotation. Having showed that the rotation was zero under certain conditions, we now consider the consequences of these objects.

Gaussian null coordinates For a null hypersurface \mathcal{N} , take a 2D spacelike surface S with coordinates y_i on $\mathcal{N}(U)$. Take another null vector field V on \mathcal{N} satisfying

$$V \cdot \frac{\partial}{\partial y^i} = 0 \text{ and } V \cdot U = 1, V^2 = 0. \quad (11.1)$$

Now we assign coordinates (r, λ, y^i) to the point at affine parameter distance r along the null geodesic which starts at the point on \mathcal{N} with coordinate (λ, y^i) and has tangent vector V^a . Thus we follow U along \mathcal{N} ($r = 0$) for distance λ and then go off \mathcal{N} by following V for affine parameter r .

This defines a coordinate chart in a neighborhood of \mathcal{N} such that the \mathcal{N} is at $r = 0$, with $U = \frac{\partial}{\partial \lambda}$ on \mathcal{N} and $\frac{\partial}{\partial r}$ tangent to affinely parametrized null geodesics.

The latter condition implies that $g_{rr} = 0$ everywhere. It also follows from the geodesic equation for $\frac{\partial}{\partial r}$ that

$$g_{r\mu, r} = 0, \quad (11.2)$$

where μ is now a free index that runs over r, λ, y^i . But at $r = 0$, we have $g_{r\lambda} = U \cdot V = 1$, and $g_{ri} = V \cdot \frac{\partial}{\partial y^i} = 0$. So these are the first indications these are good coordinates. We also know that $g_{\lambda\lambda} = 0$ at $r = 0$, as U^a is null, and $g_{\lambda i} = 0$ at $r = 0$, as $\frac{\partial}{\partial y^i}$ is tangent to \mathcal{N} and hence orthogonal to U^a . Thus the metric components take the form

$$g_{\lambda\lambda} = rF, \quad g_{\lambda i} = rh_i, \quad (11.3)$$

and we find the metric in these coordinates takes the form

$$ds^2 = 2drd\lambda + rFd\lambda^2 + 2rh_id\lambda dy^i + h_{ij}dy^i dy^j. \quad (11.4)$$

Note that at $r = 0$, F must vanish. To see this, we use the fact that

$$\lambda \mapsto (0, \lambda, y^i) \quad (11.5)$$

are affinely parametrized null geodesics. For this, the only non-vanishing component of the geodesic equation is the r component,

$$\partial_r(rF) = 0. \quad (11.6)$$

Integrating this at $r = 0$ tells us that $F = 0$ at $r = 0$, so we can write $F = r\hat{F}$ for \hat{F} some smooth function. WLOG we can therefore write

$$ds^2 = 2drd\lambda + r^2\hat{F}d\lambda^2 + 2rh_id\lambda dy^i + h_{ij}dy^i dy^j. \quad (11.7)$$

What does this metric look like on our null hypersurface? On \mathcal{N} , the metric is simply

$$g|_{\mathcal{N}} = 2drd\lambda + h_{ij}dy^i dy^j. \quad (11.8)$$

Thus $U^\mu = (0, 1, 0, 0)$ on \mathcal{N} , which implies $U_\mu = (1, 0, 0, 0)$ (where we've lowered the index using $g|_{\mathcal{N}}$). Recall that

$$U \cdot B = B \cdot U = 0 \implies B^r_\mu = B^\mu_\lambda = 0. \quad (11.9)$$

We also know that $\theta = B^\mu_\mu$, but we've just shown that two of the trace components vanish, $B^r_r = B^\lambda_\lambda = 0$, so it must be the y^i components which contribute to the expansion. On \mathcal{N} ,

$$\theta = B^i_i = \nabla_i U^i = \partial_i U^i + \Gamma^i_{i\mu} U^\mu \quad (11.10)$$

$$= \Gamma^i_{i\lambda} = \frac{1}{2} h^{ij} (g_{ji,\lambda} + g_{j\lambda,i} - g_{i\lambda,j}) \quad (11.11)$$

$$= \frac{\partial_\lambda \sqrt{h}}{\sqrt{h}} \quad (11.12)$$

where $h = \det h_{ij}$. To make the simplifications in the first and second lines, we notice that $\partial_i U^i = 0$ and $g_{i\lambda,j} = 0$. Hence we have

$$\frac{d\sqrt{h}}{d\lambda} = \theta\sqrt{h}. \quad (11.13)$$

Trapped surfaces Consider a 2D spacelike surface S , i.e. a 2D submanifold for which all tangent vectors are spacelike. For any $p \in S$, there will be *precisely* two future-directed null vectors U^a_1, U^a_2 orthogonal to S (up to a freedom of rescaling).¹² This generalizes the notion of ingoing and outgoing trajectories.

Example 11.14. Let S be a 2-sphere with $U = U_0, V = V_0$ in the Kruskal spacetime. (That is, points in the Kruskal diagram represent 2-spheres.) By spherical symmetry, the generators of \mathcal{N}_i will be radial null geodesics. Hence, \mathcal{N}_i must be surfaces of constant U or constant V , with the generators tangent to dU and dV . Raising the index, we have

$$U^a_1 = re^{r/2M} \left(\frac{\partial}{\partial V} \right)^a, \quad U^a_2 = re^{r/2M} \left(\frac{\partial}{\partial U} \right)^a. \quad (11.15)$$

We have fixed signs such that both are future-directed. Now

$$\theta_1 = \nabla_a U^a_1 = \frac{1}{\sqrt{g}} \partial_a (\sqrt{-g} U^a_1) = -\frac{8M^2}{r} U, \quad (11.16)$$

$$\theta_2 = -\frac{8M^2}{r} V. \quad (11.17)$$

But note that U and V have different signs in the different quadrants of the Kruskal diagram. Something very interesting happens because of this.

For S in region I, we have

$$U < 0, V > 0 \implies \theta_1 > 0, \theta_2 < 0. \quad (11.18)$$

Region IV is similar. But region II (the black hole interior) is different. Here,

$$U > 0, V > 0 \implies \theta_1 < 0, \theta_2 < 0, \quad (11.19)$$

so now both the expansion coefficients are negative, and everything is shrinking as we flow along congruences.

Definition 11.20. A compact orientable 2D spacelike surface is *trapped* if both families of null geodesics orthogonal to S have negative expansion everywhere on S .

¹²"If you want to see if you really understand the class, go home and think about why the word precisely is here."

As it turns out, the existence of trapped surfaces is closely related with the presence of singularities—they are a condition in the famous Penrose singularity theorems.

The Raychaudhuri equation

Proposition 11.21. *With the expansion defined as before,*

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 + \hat{\sigma}^{ab}\hat{\sigma}_{ab} + \hat{\omega}^{ab}\hat{\omega}_{ab} - R_{ab}U^aU^b. \quad (11.22)$$

Proof. We start out with

$$\begin{aligned} \frac{d\theta}{d\lambda} &= U \cdot \nabla (B^a_b P^b_a) \\ &= P^a_b U \cdot \nabla^a_b \\ &= P^a_b U^c (\nabla_c \nabla_b U^a + R^a_{dc} U^d) \\ &= P^a_b [\nabla_b (U^c \nabla_c U^a) - (\nabla_b U^c) (\nabla_c U^a)] + P^a_b R^a_{dc} U^c U^d \\ &= -B^c_b P^b_a B^a_c - R_{cd} U^c U^d \\ &= -\hat{B}^c_a \hat{B}^a_c - R_{ab} U^a U^b. \end{aligned}$$

where we've used the Ricci identity to commute the derivatives and moved a U inside a derivative with Leibniz in order to get something that vanishes by the geodesic equation.

Instead of just B^2 , we've got \hat{B}^2 , which is even nicer, and from here we simply insert the explicit expression for \hat{B} in terms of $\theta, \hat{\sigma}$, and $\hat{\omega}$. \square

But notice that none of this proof depends on the Einstein equations! It can be formulated as a purely geometric statement. Where Einstein comes in is in recognizing that the Ricci tensor has appeared in the Raychaudhuri equation, and in particular the Einstein equations connect the Ricci tensor to the stress tensor. $R_{ab}U^aU^b$ is some contracted quantity, and what this starts to suggest to us is conditions on the matter that lives in spacetime, i.e. energy conditions.

Lecture 12.

Thursday, February 21, 2019

"You might not be familiar [with these topological concepts]. But you need to get familiar because we're going to squeeze physics out of this like there's no tomorrow." –Jorge Santos

We continue our discussion of focusing of geodesics. Last time, we wrote down Raychaudhuri's equation,

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 + \hat{\sigma}^{ab}\hat{\sigma}_{ab} + \hat{\omega}^{ab}\hat{\omega}_{ab} - R_{ab}U^aU^b,$$

where $U^a U_a = 0$ (i.e. we consider null geodesics). Today, we will set up the proof that singularities are generic in general relativity.

The first three terms on the RHS of the Raychaudhuri equation have definite signs (e.g. we showed that the third is zero along the integral curves of the generators of a null hypersurface). These were purely geometric statements. But to say anything useful about this last term with the Ricci tensor in it, we'll need some physics. Let u^a be the 4-velocity of an observer. We now define a current

$$J^a = -T^a_b u^b. \quad (12.1)$$

Definition 12.2. The *Dominant Energy Condition (DEC)* states that $-T^a_b V^b$ is a future-directed causal vector (or zero) for all future-directed timelike vectors V^a .

Definition 12.3. The *Weak Energy Condition (WEC)* states that $T_{ab} V^a V^b \geq 0$ for any causal vector V^a .

Definition 12.4. The *Null Energy Condition (NEC)* states that $T_{ab} V^a V^b \geq 0$ for any null vector V^a .

There is also a strong energy condition, but note that it *does not* imply the weak energy condition.

Definition 12.5. The *Strong Energy Condition (SEC)* states that

$$(T_{ab} - \frac{1}{2}g_{ab}T^c{}_c)V^aV^b \geq 0 \quad (12.6)$$

for all causal vectors V^a .

Thus the DEC \implies WEC \implies NEC and SEC \implies NEC, but there are no other implications. Fortunately, we will only need the weakest of these– the null energy condition.

Conjugate points

Lemma 12.7. In a spacetime satisfying Einstein's equation with matter satisfying the null energy condition,

$$\frac{d\theta}{d\lambda} \leq -\frac{\theta^2}{2}. \quad (12.8)$$

This follows since the NEC fixes the sign of the R_{ab} term in Raychaudhuri, and the $\hat{\sigma}$ term is projected onto a null hypersurface, so has only spacelike indices. The $\hat{\omega}$ contribution is zero, as argued before. Thus the lemma follows.

Corollary 12.9. If $\theta = \theta_0 < 0$ at a point p on a generator γ of a null hypersurface, then $\theta \rightarrow -\infty$ along γ within finite parameter distance $2/|\theta_0|$ provided that γ extends this far.

Proof. Let $\lambda = 0$ at p , and notice that our lemma can be written as

$$\frac{d(\theta^{-1})}{d\lambda} \geq \frac{1}{2} \implies \theta^{-1} - \theta_0^{-1} \geq \frac{\lambda}{2} \implies \theta \leq \frac{\theta_0}{1 + \lambda \frac{\theta_0}{2}}. \quad (12.10)$$

But now notice that this denominator becomes singular as $\lambda \rightarrow 2/|\theta_0|$, and since the numerator is positive, we see that $\theta \rightarrow -\infty$ as $\lambda \rightarrow 2/|\theta_0|$. \square

Definition 12.11. Points p, q on a geodesic γ are *conjugate* if there exists a Jacobi field along γ that vanishes at p and q but is not identically zero.

These give us a sense of focal points, so they are natural to think about in the context of singularity theorems.

Theorem 12.12. Consider a null geodesic congruence which includes all of the null geodesics through p .¹³ If $\theta \rightarrow -\infty$ at a point q on a null geodesic γ through p , then q is conjugate to p along γ .

Theorem 12.13. Let γ be a causal curve with $p, q \in \gamma$. Then there does not exist a smooth 1-parameter family of causal curves γ_s connecting p, q with $\gamma_0 = \gamma$ and γ_s timelike for $s > 0$ if and only if γ is a null geodesic with no point conjugate to p along γ between p and q .

Example 12.14. Take the Einstein static universe with metric

$$ds^2 = -dt^2 + d\Omega_2^2.$$

Let us observe that null geodesics emitted from the south pole at time $t = 0$ all reconverge at the north pole at time $t = \pi$. However, if we now look at a point q where we would have to travel farther than R along a great circle, then by deforming the great circle into a shorter path, one can travel from p to q with a velocity less than the speed of light. Thus there exists a timelike curve from p to q .¹⁴

Consider now a 2D spacelike surface S . We have seen that (in $3 + 1$ dimensions) we can introduce two future-directed null vector fields (ingoing/outgoing trajectories, if you like) U_1^a and U_2^a on S that are normal to S . Consider the null geodesics which have one of these vectors as their tangent on S . A point is *conjugate* to a surface if there is a Jacobi field which is tangent on the surface and vanishes at that point. The analogue of Thm. 12.12 is then as follows.

¹³This is not quite a congruence as we've defined it before. It is necessarily singular at p since p lies on more than one geodesic– by definition, it lies on all of them! However, we won't worry too much about the details. Just take the set of all null geodesics through p and work from there.

¹⁴If we like, we can say that timelike geodesics therefore only have the property of extremizing proper time until they hit their first focal (conjugate) point. As Witten writes, "A geodesic that is continued past a focal point and thus has gone more than half way around the sphere is no longer length minimizing, as it can be 'slipped off' the sphere."

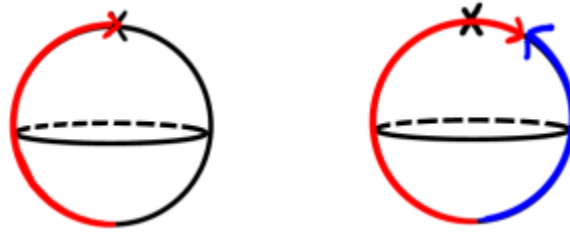


FIGURE 4. Left: a 2-sphere with a geodesic that travels from the south pole to the north pole (marked by an X). Right: a 2-sphere with a geodesic (red) that continues through the north pole. As this geodesic has passed through a focal point, it no longer minimizes the distance between its endpoints, and can be deformed into the geodesic in blue which is of shorter length.

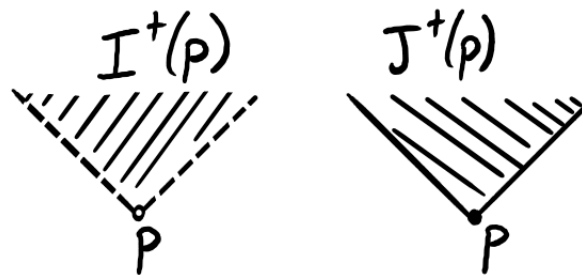


FIGURE 5. The chronological future I^+ and the causal future J^+ in Minkowski space. Notice that the chronological future $I^+(p)$ is strictly the interior of the light cone of the point p and does not include p itself, whereas the causal future includes the light cone and p itself.

Theorem 12.15. A point P is conjugate to a surface S if $\theta \rightarrow -\infty$ at p along one of the geodesics parametrized by U_1^a, U_2^a .

Definition 12.16. Let (\mathcal{M}, g) be a time-orientable spacetime $U \subset \mathcal{M}$. The *chronological future/past* of U , denoted $I^\pm(U)$, is the set of points of \mathcal{M} which can be reached by a future-/past-directed timelike curve starting on U (not including U).

Definition 12.17. The *causal future/past* of U , denoted $J^\pm(U)$, is the union of U with the set of points of \mathcal{M} that can be reached by a future-/past-directed *causal* curve starting on U .

Theorem 12.18. Let S be a two-dimensional orientable spacelike submanifold of a globally hyperbolic spacetime (\mathcal{M}, g) . Then every point $p \in \dot{J}^+(S)$ (the topological boundary of $J^+(S)$) lies on a future-directed null geodesic starting from S which is orthogonal to S and has no point conjugate to S between S and p . Furthermore, $\dot{J}^+(U)$ is a submanifold of \mathcal{M} and is achronal (no two points on $\dot{J}^+(U)$ are connected by a timelike curve), where $U \subset \mathcal{M}$.

Lecture 13.

Friday, February 22, 2019

"Is this clear? Now go home and prove it." –Jorge Santos, on the Penrose singularity theorem

We stated an important theorem last time, and we'll use it to prove something great.

Theorem 13.1 (Penrose singularity theorem (1965)). Let (\mathcal{M}, g) be globally hyperbolic with a non-compact Cauchy surface Σ . Assume the Einstein equations and the null energy condition hold, and that \mathcal{M} contains a trapped surface T . Let $\theta_0 < 0$ be the maximum value of θ on T for both sets of null geodesics orthogonal to T . Then at least one of these geodesics is future-directed inextendible and has a finite length no greater than $2/|\theta_0|$.

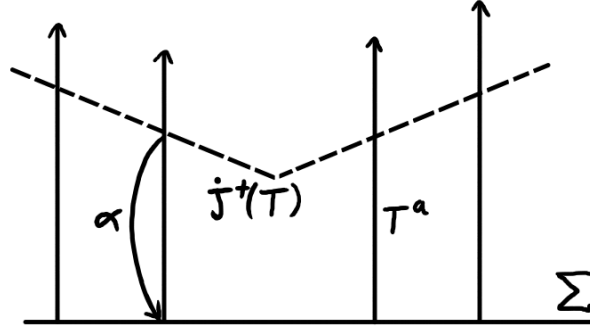


FIGURE 6. An illustration of the homeomorphism between $j^+(T)$ and Σ . By following the timelike curves which are the integral curves of T^a (the time-orientation of our spacetime) backwards from $j^+(T)$ to some subset of Σ , we get a continuous one-to-one map $\alpha : j^+(T) \rightarrow \Sigma$.

Proof. We will prove this by contradiction. Assume that all inextendible null geodesics orthogonal to T have affine length greater than $2/|\theta_0|$. By the Raychaudhuri equation, we saw that along any of these geodesics, we will have $\theta \rightarrow -\infty$. Hence a point conjugate to T within affine parameter distance no greater than $2/|\theta_0|$.

It's time to use our theorem from last time. Let $p \in j(T)$, $p \notin T$. From Thm. 12.18, we know that p lies on a future-directed null geodesic γ starting from T which is orthogonal to T and has no point conjugate to T between T and p . It follows that p cannot lie beyond the point on γ conjugate to T on γ .

Therefore $j^+(T)$, the topological boundary of the causal future of T , is a subset of a compact set consisting of the set of points along the null geodesics orthogonal to T with affine parameter less than or equal to $2/|\theta_0|$. But $j^+(T)$ is a boundary, hence closed, which implies that $j^+(T)$ is in fact compact.¹⁵

Now recall that $j^+(T)$ is a (sub)manifold, which implies that it cannot have a boundary.¹⁶ In addition, we said that Σ is a non-compact Cauchy surface. Therefore pick a timelike vector field T^a (we are free to do this since our manifolds are time-orientable). By global hyperbolicity, integral curves of T^a intersect Σ exactly once. Notice that they will intersect $j^+(T)$ at most once because $j^+(T)$ is achronal (otherwise, two points would be connected by a timelike curve). This defines a continuous one-to-one map $\alpha : j^+(T) \rightarrow \Sigma$, as illustrated in Fig. 6. In particular this is a homeomorphism between $j^+(T)$ and $\alpha(j^+(T)) \subset \Sigma$. Since the former is a closed set, so is the latter.

Now $j^+(T)$ is a 3D submanifold, and hence for any $p \in j^+(T)$, we can find a neighborhood V of p with $V \subset j^+(T)$. Then $\alpha(V)$ gives a neighborhood of $\alpha(p)$ in the image $\alpha(j^+(T))$. Hence the latter set is open in Σ .

Since $\alpha(j^+(T))$ is both open and closed in Σ , it must be the entire set,

$$\alpha(j^+(T)) = \Sigma.$$

But we showed that $\alpha(j^+(T))$ was compact, and by assumption Σ was noncompact. Therefore we have reached a contradiction. \square

It seems like we haven't quite gotten the result we were promised. Penrose requires a trapped surface, and maybe it's the case that trapped surfaces never form. However, due to an incredible result by Christodoulou, it turns out that trapped surfaces do form when you scatter enough gravitons, at least in Minkowski backgrounds. Moreover, there's the notion of Cauchy stability, i.e. given initial conditions that lead to a trapped surface, we can perturb them, solve the Einstein equations, and see how the new solution depends on the perturbation. It turns out that generically, perturbed spacetimes which had trapped surfaces before perturbation do still feature trapped surfaces, and we can do analytic calculations in spherical symmetry

¹⁵That is, a closed subset of a compact set is compact.

¹⁶Some people make a distinction between manifold with boundary and manifold without boundary. We won't bother here, and assume that a manifold just looks like \mathbb{R}^n everywhere.

which do have trapped surfaces in them. Numerically, trapped surfaces turn up all the time. Therefore life is not so bad. Trapped surfaces are basically generic and so singularities are generic. QED.

Asymptotic flatness We have discussed asymptotic flatness in the context of initial conditions. What does it mean for a spacetime to be asymptotically flat? To understand this, we'll need the notion of *conformal compactification*.

Given a spacetime (\mathcal{M}, g) , we can define a new metric

$$\bar{g} = \Omega^2 g. \quad (13.2)$$

In particular, we can choose Ω such that the resulting spacetime (\mathcal{M}, \bar{g}) is extendible into $(\bar{\mathcal{M}}, \bar{g})$. That is, M is a proper subset of $\bar{\mathcal{M}}$. Note that since Ω^2 is non-negative (generally, positive), the causal structure of the spacetime will be respected by this transformation. Spacelike separations in g will also be spacelike in \bar{g} , and so on.

Let's start with Minkowski space. Let (\mathcal{M}, g) be Minkowski in spherical coordinates,

$$g = -dt^2 + dr^2 + r^2 d\omega^2, \quad (13.3)$$

where we denote the round 2-sphere metric by ω to avoid confusion with the conformal factor Ω . Define advanced and retarded (light cone) coordinates

$$u = t - r, \quad v = t + r. \quad (13.4)$$

Under this change of coordinates, we get the metric

$$g = -dudv + \frac{1}{4}(u - v)^2 d\omega^2. \quad (13.5)$$

Now we'll make a somewhat stranger change of coordinates,

$$u = \tan p, \quad v = \tan q \quad (13.6)$$

where in the (p, q) coordinates, $-\pi/2 < p \leq q < \pi/2$. Substituting back into the metric, we get something that looks kind of weird. It's

$$g = \frac{1}{(2 \cos p \cos q)^2} \left[-4dpdq + \sin^2(q - p) d\omega^2 \right]. \quad (13.7)$$

Notice that this denominator diverges when $q \rightarrow \pi/2$, so this suggests to us a very natural choice of conformal factor Ω . It's just

$$\Omega = 2 \cos q \cos p, \quad (13.8)$$

which gives us a new metric

$$\bar{g} = -4dpdq + \sin^2(q - p) d\omega^2 = \Omega^2 g. \quad (13.9)$$

Finally, make the coordinate transformation

$$T = q + p \in (-\pi, \pi), \quad \chi = q - p \in [0, \pi]. \quad (13.10)$$

What we get is the metric

$$\bar{g} = -dT^2 + \underbrace{d\chi^2 + \sin^2 \chi d\omega^2}_{S^3}, \quad (13.11)$$

which looks very much like the Einstein static universe! Except there's a little caveat— the coordinate ranges mean that we don't get the whole of the cylinder. In fact, the range of χ also depends on T . An illustration of this appears in Fig. 7. Moreover, by projecting down into the T - χ plane (so that each point represents a 2-sphere), we get the Penrose diagram for Minkowski space, seen in Fig. 8. Penrose diagrams are very good for radial motion, but they come at the slight cost of hiding all angular motion behind our projection. In addition, note that under this projection, the points $(T = T_0, \chi = +\chi_0)$ and $(T = T_0, \chi = -\chi_0)$ are collapsed to the same point. I emphasize that there is nothing special about $\chi = 0$ in Minkowski space— it is utterly non-singular. This is just a quirk of our projection, taking advantage of the spherical symmetry of our spacetime.

It's also important to notice that we've introduced an unphysical, extendible metric \bar{g} . Minkowski space itself is inextendible, but after a conformal transformation we got a subset of the Einstein static universe (where this subset is very clearly extendible). Moreover, conformal transformations preserve the causal

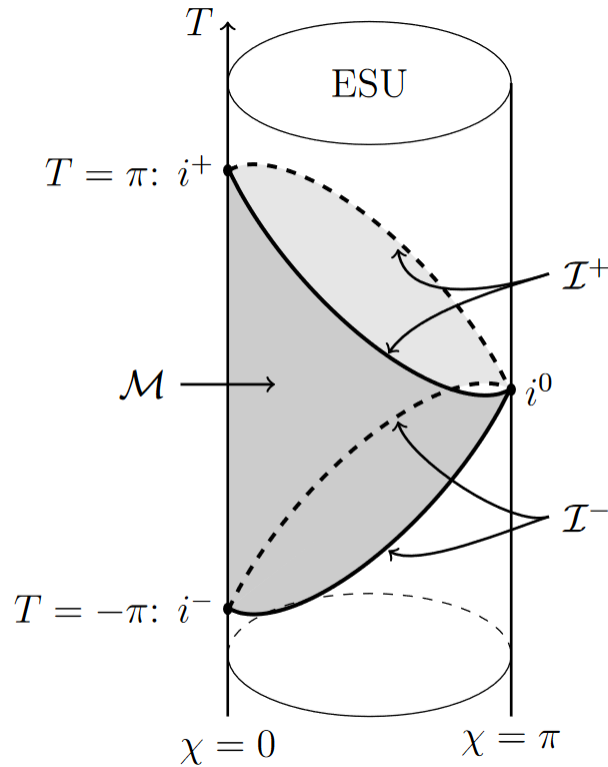


FIGURE 7. The patch of the Einstein static universe conformally equivalent to Minkowski space. The ESU has the topology of $\mathbb{R} \times S^3$, so we suppress two of the angular coordinates to draw this as a cylinder instead, $\mathbb{R} \times S^1$, where the T direction runs vertically and one of the angular coordinates (say, χ) runs around the circumference of the cylinder. Also indicated are future/past timelike infinity i^\pm , future/past null infinity \mathcal{I}^\pm , and spacelike infinity i^0 . Image credit to Prof. Reall's [Black Holes notes](#), §5.1.

structure of spacetime, so we've found a very nice way to illustrate the causal structure of a spacetime. In the next lecture, we'll look at the solutions to wave equations living in curved spacetime.

Lecture 14.

Monday, February 25, 2019

Last time, we saw that under a conformal compactification, we could bring infinity into a finite distance and draw the Penrose diagram to study its causal structure.

We could study perturbations to the metric itself (i.e. the graviton), but we'll do something more straightforward— a massless scalar field. Take the massless Klein-Gordon equation:

$$\nabla^a \nabla_a \psi = 0. \quad (14.1)$$

Suppose we study spherically symmetric solutions, $\psi(t, r)$. The most general solution is

$$\psi = \frac{1}{r} [f(t - r) + g(t + r)]. \quad (14.2)$$

It's easy to check this— if $\psi = \frac{\Pi}{r}$ for some other field, then Π satisfies a regular wave equation which has left- and right-moving modes. Now, regularity at $r = 0$ demands that

$$\psi(t, r) = \frac{1}{r} [f(u) - f(v)] = \frac{1}{r} [F(p) - F(q)], \quad (14.3)$$

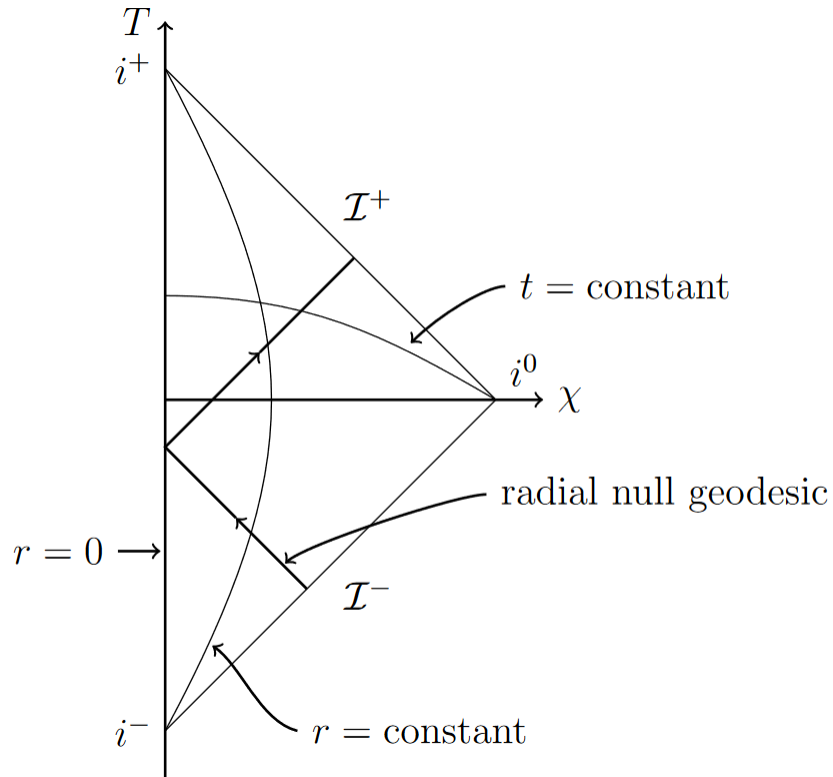


FIGURE 8. The Penrose diagram for Minkowski space. Note that the radial null geodesic drawn is not “reflecting” off anything; there is nothing special about $r = 0$ in Minkowski space. The geodesic simply passes through $r = 0$ and then proceeds back out to larger values of r as it heads towards future null infinity, \mathcal{I}^+ . Image credit to Prof. Reall’s [Black Holes](#) notes, §5.1.

where $F(x) := f(\tan x)$. Let $F_0(q)$ denote the limiting value of $r\psi$ on \mathcal{I}^- , at $p = -\pi/2$. Thus

$$F(-\pi/2) - F(q) = F_0(q). \quad (14.4)$$

But this is valid for any q , so it is valid for any p . We conclude that

$$\psi(t, r) = \frac{1}{r} [F_0(p) - F_0(q)]. \quad (14.5)$$

Thus it suffices to specify initial data on \mathcal{I}^- , past null infinity.

Penrose diagram of Kruskal spacetime It’s a bit fiddly to explicitly construct the Penrose diagram for the Kruskal spacetime, so instead we’ll use our knowledge of the asymptotics to draw the diagram. Take the Kruskal diagram, and notice that regions I and IV are asymptotically flat, so must look like Minkowski. Null rays already travel at 45 degrees, so this is nice. Finally, we can straighten out the singularity by some choice of Ω to get the Penrose diagram for the Schwarzschild black hole in asymptotically flat space. It’s pretty straightforward to see from the lines of constant r that i^+ is a singular point since it intersects $r = 0$, which is singular. But it’s also true (though much harder to prove) that spacelike null infinity i^0 is also singular.

We can also draw the Penrose diagram for spherical collapse. The surface of the star follows a timelike geodesic, collapsing to $r = 0$. When it hits $r = 0$, we get a singularity, and there’s also some horizon cutting off the black hole region.

Now, we’d like to discuss asymptotic flatness, but timelike and spacelike infinity are both singular. What’s left? Null infinity, \mathcal{I}^\pm .

Asymptotic flatness

Definition 14.6. A *manifold-with-boundary* is defined in the same way as a manifold, except charts now map $\phi : \mathcal{M} \rightarrow \mathbb{R}^n / 2 = \{(x^1, \dots, x^n); x^1 \leq 0\}$. The boundary $\partial\mathcal{M}$ of \mathcal{M} is defined to be the set of points which have $x^1 = 0$ in some chart.

Definition 14.7. A time-orientable spacetime (\mathcal{M}, g) is asymptotically flat at null infinity if there exists a spacetime $(\bar{\mathcal{M}}, \bar{g})$ such that

- (a) \exists a positive function Ω defined on \mathcal{M} such that $(\bar{\mathcal{M}}, \bar{g})$ is an extension of $(\mathcal{M}, \Omega^2 g)$.
- (b) Within $\bar{\mathcal{M}}$, \mathcal{M} can be extracted to obtain a manifold-with-boundary $\bar{\mathcal{M}} \cup \partial\bar{\mathcal{M}}$.
- (c) Ω can be extended to a function on $\bar{\mathcal{M}}$ such that $\Omega = 0$ and $d\Omega \neq 0$ on $\partial\bar{\mathcal{M}}$.
- (d) The boundary $\partial\bar{\mathcal{M}}$ is a disjoint union of two components of $\mathcal{I}^+, \mathcal{I}^-$, each diffeomorphic to $\mathbb{R} \times S^2$.
- (e) No past- (future-)directed causal curve starting in \mathcal{M} intersects $\mathcal{I}^+ (\mathcal{I}^-)$.
- (f) \mathcal{I}^\pm are ‘complete.’

To recap, conditions a-c are conditions on how to choose Ω , while the other conditions describe the spacetime, and we haven’t actually explained what ‘complete’ means yet.

Now, we’ve seen that

$$R_{ab} = \bar{R}_{ab} + 2\Omega^{-1}\bar{\nabla}_a\bar{\nabla}_b\Omega + \bar{g}_{ab}\bar{g}^{cd}\left(\Omega^{-1}\bar{\nabla}_c\bar{\nabla}_d\Omega - 3\frac{\bar{\nabla}_c\Omega\bar{\nabla}_d\Omega}{\Omega^2}\right). \quad (14.8)$$

In pure gravity, we take vacuum solutions of the Einstein equations and hence $R_{ab} = 0$. Note barred quantities are taken with respect to $(\mathcal{M}, \Omega^2 g)$. Multiplying by a single factor of Ω , we get

$$\Omega\bar{R}_{ab} + 2\bar{\nabla}_a\bar{\nabla}_b\Omega + \bar{g}_{ab}\bar{g}^{cd}\left(\bar{\nabla}_c\bar{\nabla}_d\Omega - 3\frac{\bar{\nabla}_c\Omega\bar{\nabla}_d\Omega}{\Omega}\right) = 0. \quad (14.9)$$

Regularity at $\Omega = 0$ implies that

$$\bar{g}^{cd}\bar{\nabla}_c\Omega\bar{\nabla}_d\Omega \propto \Omega. \quad (14.10)$$

So $d\Omega$ is null when $\Omega = 0$ (\mathcal{I}^+).

We deduce that \mathcal{I}^+ is a null hypersurface. There’s still a residual freedom in our choice of Ω , though—we can still choose $\Omega' = \omega^2\Omega$. And this means we can be more clever about our choice of Ω . Choose ω^2 such that

$$\Omega^{-1}\bar{g}^{cd}\bar{\nabla}_c\Omega\bar{\nabla}_d\Omega = 0 \text{ on } \mathcal{I}^+. \quad (14.11)$$

That is, $\bar{g}^{cd}\bar{\nabla}_c\Omega\bar{\nabla}_d\Omega$ goes as Ω^2 so that $\Omega^{-1}\bar{g}^{cd}\bar{\nabla}_c\Omega\bar{\nabla}_d\Omega \rightarrow 0$ as $\Omega \rightarrow 0$.

Using this, we find that

$$\bar{\nabla}_a\bar{\nabla}_b\Omega = 0 \text{ on } \mathcal{I}^+, \quad (14.12)$$

and hence

$$\bar{\nabla}_a n^b = 0 \text{ on } \mathcal{I}^+, \quad (14.13)$$

which tells us that

$$n^a\bar{\nabla}_a n^b = 0 \text{ on } \mathcal{I}^+. \quad (14.14)$$

So the geodesics of n^a are null and they foliate \mathcal{I}^+ . We introduce coordinates near \mathcal{I}^+ as follows: if we have some coordinates on \mathcal{I}^+ , we can follow n^a on \mathcal{I}^+ a affine parameter distance u to get (u, θ, ϕ) coordinates, and we follow Ω off of \mathcal{I}^+ to get to points at $(\Omega, u, \theta, \phi)$. But in fact this is nothing more than Gaussian null coordinates, which means we can immediately write down the metric:

$$\bar{g}_{\Omega=0} 2du d\Omega + d\theta^2 + \sin^2\theta d\phi^2. \quad (14.15)$$

For small $\Omega \neq 0$, the metric components will differ from the above by $O(\Omega)$ terms. (This is hard to prove and not particularly illuminating, so we won’t do it.)

If we now introduce $r = 1/\Omega$, then

$$g = -2durd + r^2(d\theta^2 + \sin^2\theta d\phi^2) + \dots \quad (14.16)$$

where \dots indicates corrections to higher order in Ω . The completeness condition on \mathcal{I}^+ says that $u \in (-\infty, +\infty)$.

Definition of a black hole

Definition 14.17. A *black hole interior* is the region

$$B = \mathcal{M} \setminus [\mathcal{M} \cap J^-(\mathcal{I}^+)]. \quad (14.18)$$

The black hole *future event horizon* is

$$\mathcal{H}^+ = \mathcal{M} \cap j^-(\mathcal{I}^+). \quad (14.19)$$

That is, take the causal past of future null infinity. Follow the light rays back everywhere you can get on your manifold, and take the complement. What remains is the black hole.