QUANTUM FIELD THEORY

IAN LIM
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These notes were taken for the *Quantum Field Theory* course taught by Ben Allanach at the University of Cambridge as part of the Mathematical Tripos Part III in Michaelmas Term 2018. I live-TEXed them using Overleaf, and as such there may be typos; please send questions, comments, complaints, and corrections to itel2@cam.ac.uk.
Many thanks to Arun Debray for the LATEX template for these lecture notes: as of the time of writing, you can find him at https://web.ma.utexas.edu/users/a.debray/.

Contents

Lecture 1.

Thursday, October 4, 2018

$2 = \pi = i = -1$ *in these lectures.* –a former lecturer of Prof. Allanach's.

To begin with, some logistic points. The notes (and I assume course material) will be based on David Tong's QFT notes plus some of Prof. Allanach's on cross-sections and decay rates. See http://www.damtp.cam.ac.uk/user/examples/indexP3.html and in particular http://www.damtp.cam.ac.uk/user/examples/3P11.pdf for the notes on cross-sections.

After Tuesday's lecture, we'll be assigned one of four course tutors:

- Francesco Careschi, fc435cam.ac.uk
- Muntazir Abidi, sma74
- Khim Leong, lkw30
- Stefano Vergari, sv408

Also, the Saturday, November 24th lecture has been moved to 1 PM Monday 26 November, still in MR2. That's it for logistics for now.

Definition 1.1. A *quantum field theory* (QFT) is a field theory with an infinite number of degrees of freedom (d.o.f.). Recall that a field is a function defined at all points in space and time (e.g. air temperature is a scalar field in a room wherever there's air). The states in QFT are in general multi-particle states.

Special relativity tells us that energy can be converted into mass, and so particles are produced and destroyed in interactions (particle number not conserved). This reveals a conflict between SR and quantum mechanics, where particle number is fixed. Interaction forces in our theory then arise from structure in the theory, dependent on things like

- symmetry
- locality
- "renormalization group flow."

Definition 1.2. A *free QFT* is a QFT that has particles but no interactions. The classic free theory is a relativistic theory with infinitely many quantized harmonic oscillators.

Free theories are generally not realistic but they are important, as interacting theories can be built from these with perturbation theory. The fact we can do this means the particle interactions are somehow weak (weak coupling), but other theories have strong coupling and cannot be described with perturbation theory.

Units in QFT In QFT, we usually set $c = \hbar = 1$. Since $[c] = [L][T]^{-1}, [\hbar] = [L]^2[M][T]^{-1}$, we find that in natural units,

$$[L] = [T] = [M]^{-1} = [E]^{-1}$$

(where the last equality follows from $E = mc^2$ with $c = 1$). We often just pick one unit, e.g. an energy scale like eV, and describe everything else in terms of powers of that unit. To convert back to metres or seconds, just reinsert the relevant powers of c and $\hbar$.

Example 1.3. The de Broglie wavelength of a particle is given by $\lambda = \hbar/(mc)$. An electron has mass $m_e \simeq 10^6\,\text{eV}$, so $\lambda_e = 2 \times 10^{-12}\,\text{m}$.

If a quantity x has dimension $(mass)^d$, we write $[x] = d$, e.g.

$$G = \frac{\hbar c}{M_p^2} \implies [G] = -2.$$

$M_p \approx 10^{19}\,\text{GeV}$ corresponds to the Planck scale, $\lambda_p \sim 10^{-33}\,\text{cm}$, the length/energy scales where we expect quantum gravitational effects to become relevant. We note that the problems associated with relativising the Schrödinger equation are fixed in QFT by particle creation.

Before we do QFT, let's review classical field theory. In classical particle mechanics, we have a finite number of generalized coordinates $q_a(t)$ (where a is a label telling you which coordinate you're talking about) and in general they are a function of time t. But in field theory, we instead have $\phi_a(x,t)$ where a labels the field in question and x is no longer a coordinate but a label like a.[1]

In our classical field theory, there are now an infinite number of d.o.f., at least one for each x, so position has been demoted from a dynamical variable to a mere label.

Example 1.4. The classical electromagnetic field is a vector field with components $E_i(x,t), B_i(x,t)$ such that $i, j, k \in \{1, 2, 3\}$ label spatial directions. In fact, these six fields are derived from four fields (or rather four field components), the four-potential $A_\mu(x,t) = (\phi, \mathbf{A})$ where $\mu \in \{0, 1, 2, 3\}$.

Then the classical fields are simply related to the four-potential by

$$E_i = \frac{\partial A_i}{\partial t} - \frac{\partial A_0}{\partial x_i} \text{ and } B_i = \frac{1}{2}\epsilon_{ijk}\frac{\partial A_k}{\partial x_j}$$

with ϵ_{ijk} the usual Levi-Civita symbol, and where we have used the Einstein summation convention (repeated indices are summed over).

The dynamics of a field are given by a *Lagrangian* L, which is simply a function of $\phi_a(x,t), \dot{\phi}_a(x,t)$, and $\boldsymbol{\nabla}\phi_a(x,t)$.

Definition 1.5. We write

$$L = \int d^3x \mathcal{L}(\phi_a, \partial_\mu \phi_a),$$

where we call $\mathcal{L}$ the *Lagrangian density*, or by a common abuse of terminology simply the Lagrangian.

Definition 1.6. We may then also define the *action*

$$S \equiv \int_{t_0}^{t_1} L dt = \int d^4x \mathcal{L}(\phi_a, \partial_\mu \phi_a)$$

[1] See for instance Anthony Zee's *QFT in a Nutshell* to see a more detailed description of how we go from discrete to continuous systems.

Let us also note that in these units we have $[S] = 0$ (since it appears alone in an exponent, for instance, e^{iS}) and so since $[d^4x] = -4$ we have $[\mathcal{L}] = 4$.

The dynamical principle of classical field theory is that fields evolve s.t. S is stationary with respect to variations of the field that don't affect the intiial or final values (boundary conditions). A general variation of the fields produces a variation in the action

$$\delta S = \sum_a \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta(\partial_\mu \phi_a) \right\}.$$

With an integration by parts we find that the variation is the action becomes

$$\delta S = \sum_a \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a \right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \delta\phi_a \right\}.$$

The integral of the total derivative term vanishes for any term that decays at spatial ∞ (i.e. $\mathcal{L}$ is reasonably well-behaved) and has $\delta\phi_a(x, t_1) = \delta\phi_a(x, t_0) = 0$. Therefore the boundary term goes away and we find that stationary action implies the *Euler-Lagrange equations*,

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0.$$

Example 1.7. Consider the Klein-Gordon field ϕ, defined

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2.$$

Here $\eta^{\mu\nu}$ is the standard Minkowski metric[2].

To compute the Euler-Lagrange equation for this field theory, we see that

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi \text{ and } \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu\phi.$$

The Euler-Lagrange equations then tell us that

$$\partial_\mu\partial^\mu\phi + m^2\phi = 0,$$

which we call the *Klein-Gordon equation*. It has wave-like solutions $\phi = e^{-ipx}$ with $(-p^2 + m^2)\phi = 0$ (so that $p^2 = m^2$, which is what we expect in units where $c = 1$).

A non-lectured aside on functional derivatives If you're like me, you get a little anxious about taking complicated functional derivatives. The easiest way to manage these is to rewrite the Lagrangian so that all terms precisely match the form of the quantity you are taking the derivative with respect to, and remember that matching indices produce delta functions.

Here's a quick example. To compute $\frac{\partial}{\partial(\partial_\alpha\phi)}\left[\partial_\mu\phi\partial^\mu\phi\right]$, rewrite the term in the brackets as $\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ (since we are deriving with respect to a function of the form $\partial_\alpha\phi$) and make sure to take the derivative with respect to a new index not already in the expression, e.g. $\partial_\alpha\phi$. Then

$$\begin{aligned}
\frac{\partial}{\partial(\partial_\alpha\phi)}\left[\partial_\mu\phi\partial^\mu\phi\right] &= \frac{\partial}{\partial(\partial_\alpha\phi)}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \\
&= \eta^{\mu\nu}(\delta^\alpha_\mu)\partial_\nu\phi + \eta^{\mu\nu}\partial_\mu\phi(\delta^\alpha_\nu) \\
&= 2\partial^\alpha\phi,
\end{aligned}$$

where we have raised the index with $\eta^{\mu\nu}$ and written the final expression in terms of α using the delta function. The functional derivative effectively finds all appearances of the denominator exactly as written, including indices up or down, and replaces them with delta functions so the actual indices match. This is especially important in computing the Euler-Lagrange equations for something like Maxwell theory, where one may have to derive by $\partial_\mu A_\nu$ and both those indices must match exactly to their corresponding appearances in the Lagrangian.

[2]We use the mostly minus convention here, but honestly the sign conventions are all arbitrary and relativity often uses the other one where time gets the minus sign.

No one ever taught me exactly how to approach such variational problems, so I wanted to record my strategy here for posterity. It may take a little longer than just recognizing that $\frac{\partial}{\partial(\partial_\mu\phi)}\frac{1}{2}\partial_\nu\phi\partial^\nu\phi = \partial^\mu\phi$, but this approach always works and it has the benefit of helping avoid careless mistakes like forgetting the factor of 2 in the example above.

Lecture 2.

Saturday, October 6, 2018

Last time, we derived the Euler-Lagrange equations for Lagrangian densities:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0. \tag{2.1}$$

Example 2.2. Consider the Maxwell Lagrangian,

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A^\mu)^2. \tag{2.3}$$

Plugging into the E-L equations, we find that $\frac{\partial \mathcal{L}}{\partial A_\nu} = 0$ and

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \partial^\mu A^\nu + \eta^{-\mu\nu}\partial_\rho A^\rho. \tag{2.4}$$

Thus E-L tells us that

$$0 = -\partial^2 A^\nu + \partial^\nu(\partial_\rho A^\rho) = -\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu). \tag{2.5}$$

Defining the field strength tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, we can write the E-L equation for Maxwell as the simple

$$0 = \partial_\mu F^{\mu\nu},$$

which written explicitly is equivalent to Maxwell's equations in vacuum (we'll revisit this when we do QED).

The Lagrangians we'll consider here and afterwards are all *local*– in other words, there are no couplings $\phi(\mathbf{x}, t)\phi(\mathbf{y}, t)$ with $\mathbf{x} \neq \mathbf{y}$. There's no reason a priori that our Lagrangians have to take this form, but all physical Lagrangians seem to do so.

Lorentz invariance Consider the Lorentz transformation on a scalar field $\phi(x) \equiv (\phi(x^\mu)$. The coordinates x transform as $x' = \Lambda^{-1}x$ with $\Lambda^\mu_\sigma \eta^{\sigma\tau}\Lambda^\nu_\tau = \eta^{\mu\nu}$. Under Λ, our field transforms as $\phi \to \phi'$ where $\phi'(x) = \phi(x')$. Recall that Lorentz transformations generically include boosts as well as rotations in $\mathbb{R}^3$. As we've discussed in Symmetries, Fields and Particles, Lorentz transformations form a Lie group ($O(3,1)$, or specifically the proper orthochronous Lorentz group) under matrix multipication. They have a representation given on the fields (i.e. a mapping to a set of transformations on the fields which respects the group multiplication law).

For a scalar field, this is $\phi(x) \to \phi(\Lambda^{-1}x)$ (an active transformation). We could have also used a passive transformation where we re-label spacetime points: $\phi(x) \to \phi(\Lambda x)$. It doesn't matter too much– since Lorentz transformations form a group, if Λ is a Lorentz transformation, so is Λ^{-1}. In addition, most of our theories will be well-behaved and Lorentz invariant.

Definition 2.6. *Lorentz invariant* theories are ones where the action S is unchanged by Lorentz transformations.

Example 2.7. Consider the action given by

$$S = \int d^4x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi)\right],$$

where $U(\phi)$ is some potential density. $U \to U'(x) \equiv U(\phi'(x)) = U(x')$ means that U is a scalar field (check this!) and we see that

$$\partial_\mu\phi' = \frac{\partial}{\partial x^\mu}\phi(x') = \frac{\partial x'^\sigma}{\partial x^\mu}\partial'_\sigma\phi(x') = (\Lambda^{-1})^\sigma_\mu\partial'_\sigma\phi(x')$$

where $\partial'_\sigma \equiv \frac{\partial}{\partial x'^\sigma}$. Thus the kinetic term transforms as

$$\mathcal{L}_{kin} \to \mathcal{L}'_{kin} = \eta^{\mu\nu}\partial_\mu\phi'\partial_\nu\phi' = \eta^{\mu\nu}(\Lambda^{-1})^\sigma_\mu(\Lambda^{-1})^\tau_\nu\partial'_\sigma\phi(x')\partial'_\tau\phi(x') = \eta^{\sigma\tau}\partial'_\sigma\phi(x')\partial'_\tau\phi(x') = L_{kin}(x).$$

Thus we see that the action overall transforms as

$$S \to S' = \int d^4x\mathcal{L}(x') = \int d^4x\mathcal{L}(\Lambda^{-1}x).$$

Under a change of variables $u \equiv \Lambda^{-1}x$, we see that $\det(\Lambda^{-1}) = 1$ (from group theory) so the volume element is the same, $d^4y = d^4x$ and therefore

$$S' = \int d^4y\mathcal{L}(y) = S.$$

We conclude that S is invariant under Lorentz transformations.

We also remark that under a LT, a vector field A_μ transforms like $\partial_\mu\phi$, so

$$A'_\mu(x) = (\Lambda^{-1})^\sigma_\mu A_\sigma(\Lambda^{-1}x).$$

This is enough to attempt Q1 from example sheet 1.[3]

Theorem 2.8. *Every continuous symmetry of $\mathcal{L}$ gives rise to a current J^μ which is conserved, $\partial_\mu j^\mu = 0$. Each j^μ has a conserved charge $Q = \int_{\mathbb{R}^3} j^0 d^3x$.*

This conserved charge appears because $\frac{dQ}{dt} = \int_{\mathbb{R}^3} d^3x\partial_0 j^0 = -\int_{\mathbb{R}^3} d^3x\boldsymbol{\nabla}\cdot\mathbf{j} = 0$ by the divergence theorem, assuming $|\mathbf{j}| \to 0$ as $|\mathbf{x}| \to \infty$.

Let us define an infinitesimal variation of a field ϕ, $\phi(x) \to \phi'(x) = \phi(x) + \alpha\Delta\phi(x)$ with α an infinitesimal change. If S is invariant, we call this a *symmetry* of the theory.

Since S is invariant up to adding a total 4-divergence (a total derivative ∂_μ) to the Lagrangian, our symmetry doesn't affect the Euler-Lagrange equations. L transforms as

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha\partial_\mu X^\mu(x), \tag{2.9}$$

and expanding to leading order in α we have

$$\mathcal{L} \to \mathcal{L}(x) + \alpha\frac{\partial\mathcal{L}}{\partial\phi}\Delta\phi + \alpha\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\Delta\phi) + O(\alpha^2). \tag{2.10}$$

We can rewrite this in terms of a total derivative $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right)$ so that

$$\mathcal{L}' = \mathcal{L}(x) + \alpha\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) + \alpha\left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\Delta\phi. \tag{2.11}$$

By Euler-Lagrange, the second term in parentheses vanishes, so we identify the first term in parentheses as none other than $\alpha\partial_\mu X^\mu(x)$ from Eqn. 2.9 (in other words, $\partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu\phi)}\Delta\phi\right) = \partial_\mu X^\mu$) and recognize

$$j^\mu \equiv \frac{\partial L}{\partial(\partial_\mu\phi)}\Delta\phi - X^\mu \tag{2.12}$$

as our conserved current (that is, $\partial_\mu j^\mu = 0$).

Example 2.13. Take a complex scalar field $\psi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x))$. We can then trea ψ, ψ^* as independent variables and write a Lagrangian

$$L = \partial_\mu\psi^*\partial^\mu\psi - V(|\psi|^2).$$

Then we observe that under $\psi \to e^{i\beta}\psi, \psi^* \to e^{-i\beta}\psi^*$, the Lagrangian is invariant. The differential changes are $\Delta\psi = i\psi$ (think of expanding $\psi \to e^{i\beta}\psi$ to leading order) and similarly $\Delta\psi^* = -i\psi^*$ (here we find that $X^\mu = 0$).

We add the currents from ψ, ψ^* to find

$$j^\mu = i\{\psi\partial_\mu\psi^* - \psi^*\partial_\mu\psi\}.$$

[3]Copied here for quick reference: Show directly that if $\phi(x)$ satisfies the Klein-Gordon equation, then $\phi(\Lambda^{-1}x)$ also satisfies this equation for any Lorentz transformation Λ.

This is enough to do questions 2 and 3 on the example sheet.

Example 2.14. Under infinitesimal translation $x^\mu \to x^\mu - \alpha\epsilon^\mu$, we have $\phi(x) \to \phi(x) + \alpha\epsilon^\mu\partial_\mu\phi(x)$ by Taylor expansion (similar for $\partial_\mu\phi$). If the Lagrangian doesn't depend explicitly on x, then $\mathcal{L}(x) \to \mathcal{L}(x) + \alpha\epsilon^\mu\partial_\mu\mathcal{L}(x)$.

Rewriting to match the form $\mathcal{L} + \alpha\partial_\mu X^\mu$, we see that our new Lagrangian takes the form $L(x) + \alpha\epsilon^\nu\partial_\mu(\delta^\mu_\nu L)$. We get one conserved current for each component of ϵ^ν, so that

$$(j^\mu)_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}$$

with $\partial_\mu(j^\mu)_\nu = 0$. We write this as $j^\mu_\nu \equiv T^\mu_\nu$, the energy-momentum tensor.

Definition 2.15. The *energy-momentum tensor* (sometimes *stress-energy tensor*)is the conserved current corresponding to translations in time and space. It takes the form

$$T^{\mu\nu} \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L},$$

where we have raised an index with the Minkowski metric as is conventional. The conserved charges from integrating $\int d^3xT^{0\nu}$ end up being the total energy $E = \int d^3xT^{00}$ and the three components of momentum $P^i = \int d^3xT^{0i}$.

Lecture 3.

Tuesday, October 9, 2018

Last time, we used Noether's theorem to find the stress-energy tensor

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}.$$

To better understand this object, we might ask: what is $T^{\mu\nu}$ for free scalar field theory? Recall the Lagrangian for this theory is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2.$$

Then by explicit computation, the stress-energy tensor is

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}.$$

The energy is given by

$$E = \int d^3x\left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right]$$

(from integrating the T^{00} component) and the conserved momentum components are (from T^{0i})

$$p^i = \int d^3x\dot{\phi}(\partial^i\phi).$$

Note that the original Lagrangian terms don't show up here, since $\eta^{\mu\nu}$ is diagonal.

We'll note that $T^{\mu\nu}$ for this theory is symmetric in μ, ν, but a priori it doesn't have to be. If $T^{\mu\nu}$ is not symmetric initially, we can massage it into a symmetric form by adding $\partial_\rho\Gamma^{\rho\mu\nu}$ where $\Gamma^{\mu\rho\nu} = -\Gamma^{\rho\mu\nu}$ (antisymmetric in the first two indices). Then $\partial_\mu\left(\partial_\rho\Gamma^{\rho\mu\nu}\right) = 0$, which means that adding this term will not affect the conservation of $T^{\mu\nu}$. This is sufficient to attempt questions 1-6 of the first examples sheet.

Canonical quantization Here, we'll follow Dirac's lead and attempt to quantize our field theories. Recall that the Hamiltonian formalism also accommodates field theories (as well as our garden-variety QM).

Definition 3.1. We define the *conjugate momentum*

$$\pi(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

and the *Hamiltonian density* corresponding to a Lagrangian $\mathcal{L}$ is then

$$\mathcal{H} = \pi(x)\dot{\phi}(x) - \mathcal{L}(x).$$

As in classical mechanics, we eliminate $\dot{\phi}$ in favor of π everywhere in $\mathcal{H}$.

Example 3.2. For $\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi)$ (and writing in terms of $\pi(x) = \dot{\phi}(x)$) we get

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi).$$

The Hamiltonian is just the integral of the Hamiltonian density: $H = \int d^3x\mathcal{H}$. Hamlton's equations then yield the equations of motion:

$$\dot{\phi} = \frac{\partial H}{\partial \pi}, \dot{\pi} = -\frac{\partial H}{\partial \phi}.$$

Working these out explicitly for the free theory will give us back the Klein-Gordon equation. Note that H agrees with the total field energy E that we computed above.

There's a slight snag in working in the Hamiltonian formalism– because t is special in our equations, the theory is not manifestly Lorentz invariant (compare to the ∂_μs and variations with respect to $\delta\partial_\mu\phi$ in the Lagrangian formalism). Our original theory was LI, so our new theory is still LI– it just doesn't look LI.

Now let's recall that in quantum mechanics, canonical quantization takes the coordinates q_a and momenta p_a and promotes them to operators. We also replace the Poisson bracket $\{,\}$ with commutators $[,]$. In QM, we had

$$[q_a, p^b] = i\delta_a^b,$$

working in units where $\hbar = 1$. We'll do the same for our fields ϕ_a and the conjugate momenta π_b.

Definition 3.3. A *quantum field* is an operator-valued function of space obeying the commutation relations

$$[\phi_a(\mathbf{x}), \phi_b(\mathbf{y})] = 0 \tag{3.4}$$

$$[\pi_a(\mathbf{x}), \pi_b(\mathbf{y})] = 0 \tag{3.5}$$

$$[\phi_a(\mathbf{x}), \pi^b(\mathbf{y})] = i\delta^3(\mathbf{x}-\mathbf{y})\delta_a^b. \tag{3.6}$$

Note that $\phi_a(x), \pi^b(x)$ don't depend on t, since we are in the Schrödinger picture. All the t dependence sits in the states which evolve by the usual time-dependent Schrödinger equation

$$i\frac{d}{dt}|\psi\rangle = H|\psi\rangle.$$

We have an infinite number of degrees of freedom, at least one for each x in space. For some theories (free theories), the coordinates evolve independently. Free field theories have L quadratic in ϕ_a (plus derivatives thereof), which implies linear equations of motion.

We saw that the simplest free theory leads to the classical Klein-Gordon equation for a real scalar field $\phi(\mathbf{x}, t)$, i.e. $\partial_\mu\partial^\mu\phi + m^2\phi = 0$. To see why this is free, take the Fourier tranform

$$\phi(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}}\phi(\mathbf{p}, t).$$

Then we get the equation of motion

$$\left[\frac{\partial^2}{\partial t^2} + (|\mathbf{p}|^2 + m^2)\right]\phi(\mathbf{p}, t) = 0.$$

We see that the solution is a harmonic oscillator with frequency $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$, so the general solution is a superposition of simple harmonic oscillators each vibrating at different frequencies $\omega_{\mathbf{p}}$. To quantize our field $\phi(\mathbf{x}, t)$, we have to quantize these harmonic oscillators.

Review of 1D harmonic oscillators Recall that the Hamiltonian for the simple harmonic oscillator is

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2,$$

subject to the quantization condition

$$[q, p] = i.$$

It's certainly possible to solve this system by the series method, but the algebraic method is much more elegant by far. Our approach is as follows– we'd like to factor the Hamiltonian, but we know that it doesn't quite work because p and q do not commute. Therefore, we define the following operators:

- The creation or raising operator, $a^\dagger \equiv -\frac{i}{\sqrt{2\omega}}p + \sqrt{\frac{\omega}{2}}q$
- The annihilation or lowering operator, $a \equiv +\frac{i}{\sqrt{2\omega}}p + \sqrt{\frac{\omega}{2}}q$.

Note that we can equivalently solve for p and q in terms of a and $a^\dagger$: $q = \frac{1}{\sqrt{2\omega}}(a + a^\dagger)$ and $p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger)$. Substituting p and q into the quantization condition yields the commutator of $a, a^\dagger$,

$$[a, a^\dagger] = 1.$$

A little more algebra allows us to rewrite the Hamiltonian as

$$H = \frac{1}{2}\omega(aa^\dagger + a^\dagger a) = \omega(a^\dagger a + \frac{1}{2}).$$

Computing the commutators $[H, a]$ and $[H, a^\dagger]$ reveals that

$$[H, a^\dagger] = \omega a^\dagger, [H, a] = -\omega a,$$

which tells us that $a, a^\dagger$ take us between energy eigenstates.[4] More specifically, they take us up and down a ladder of equally spaced energy eigenstates so that if we have one eigenstate with energy E, then we can reach a whole set of eigenstates with energy $\ldots E + 2\omega, E + \omega, E, E - \omega, E - 2\omega, \ldots$.

If we further postulate that the energy is bounded from below, this implies the existence of a ground state $|0\rangle$ such that the lowering operator acting on $|0\rangle$ kills the state: $a|0\rangle = 0$. In our original Hamiltonian, this ground state has energy given by

$$H|0\rangle = \omega(a^\dagger a + \frac{1}{2})|0\rangle = \frac{\omega}{2}|0\rangle,$$

so the ground state energy (or *zero point energy*) of the system is $\omega/2$. For our quantum theory it's really differences in energy which matter more than their absolute values,[5] so we can just as easily write an equivalent Hamiltonian $H = \omega a^\dagger a$ and set the ground state energy to 0.

We only need one state to construct our full ladder of energy eigenstates, and we can do so by passing our equation back to q-space (real coordinates) and further writing $p = i\frac{\partial}{\partial q}$. If we plug these back into the Hamiltonian, knowing that $H|0\rangle = 0$ now, we can solve for the ground state, finding that it is a Gaussian in q with some appropriate variance and normalization. Then we simply need to apply $a^\dagger$ to get all the other states, labeling them as $|n\rangle \equiv (a^\dagger)^n|0\rangle$ with $H|n\rangle = n\omega|n\rangle$. (Here we've disregarded normalization, but it's easy enough to add some scaling factor so that $\langle n|m\rangle = \delta_{nm}$.)

That's about all there is to the quantum harmonic oscillator! We have recovered the quantized energy levels and defined operators to move between them. Next time, we'll repeat the same procedure with quantum fields.

[4]Explicitly, consider an eigenstate $|E\rangle$ with energy E. Then $Ha^\dagger|E\rangle = (a^\dagger H + \omega a^\dagger)|E\rangle = (E + \omega)a^\dagger|E\rangle$, so $a^\dagger|E\rangle$ is an eigenstate with energy $E + \omega$. The computation for a is similar.

[5]Remark: gravity is different! Gravity couples directly to energy, not to differences in energy. But in a simple theory like the 1D harmonic oscillator, all we care about is the spacing of the energy levels.

Lecture 4.

Thursday, October 11, 2018

Recall that for a free scalar field

$$\phi(\mathbf{x},t)=\int\frac{d^3\mathbf{p}}{(2\pi)^3}e^{i\mathbf{p}\cdot\mathbf{x}}\phi(\mathbf{p},t),$$

where

$$\left[\frac{\partial^2}{\partial t^2}+(\mathbf{p}^2+m^2)\right]\phi(\mathbf{p},t)=0.$$

We also defined $\omega_{\mathbf{p}}^2\equiv\mathbf{p}^2+m^2$. Our theory has plane wave solutions. Let's apply the simple harmonic oscillator quantization process to free fields now, defining

$$\phi(\mathbf{x})=\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}+a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}).$$

We also have the related conjugate momentum to the field,

$$\pi(\mathbf{x})=\int\frac{d^3\mathbf{p}}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\mathbf{p}}}{2}}(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}-a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}).$$

In the *second quantization* process, we've written our infinite number of harmonic oscillators in momentum space. We want to impose the equivalent of

$$[a_{\mathbf{p}},a_{\mathbf{q}}]=[a_{\mathbf{p}}^\dagger,a_{\mathbf{q}}^\dagger]=0$$

and

$$[a_{\mathbf{p}},a_{\mathbf{q}}^\dagger]=(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}).$$

Thus in the field theory context we have instead

$$[\phi(\mathbf{x}),\phi(\mathbf{y})]=[\pi(\mathbf{x}),\pi(\mathbf{y})]=0$$

and

$$[\phi(\mathbf{x}),\pi(\mathbf{y})]=i\delta^3(\mathbf{x}-\mathbf{y}).$$

It's a good exercise to check this, but we can for instance check this one way: assume the $a,a^\dagger$ commutation relations:

$$[\phi(\mathbf{x}),\pi(\mathbf{y})]=\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{q}}{(2\pi)^3}\frac{(-1)}{2}\sqrt{\frac{\omega_{\mathbf{q}}}{\omega_{\mathbf{p}}}}\{-[a_{\mathbf{p}},a_{\mathbf{q}}^\dagger]e^{i\mathbf{p}\cdot\mathbf{x}-i\mathbf{q}\cdot\mathbf{y}}+[a_{\mathbf{p}}^\dagger,a_{\mathbf{q}}]e^{-i\mathbf{p}\cdot\mathbf{x}+i\mathbf{q}\cdot\mathbf{y}}\}.$$

Using these commutation relations, we can rewrite and do the integral over $\mathbf{q}$ to get a delta function setting $\mathbf{p}=\mathbf{q}$,

$$\int\frac{d^3\mathbf{p}}{(2\pi)^3}\left(\frac{-i}{2}\right)\{-e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}-e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}=i\delta^3(\mathbf{x}-\mathbf{y})\}$$

since $\delta^3(\mathbf{x})=\int\frac{d^3\mathbf{p}}{(2\pi)^3}e^{i\mathbf{p}\cdot\mathbf{x}}$.

Now we compute H in terms of $a_{\mathbf{p}}a_{\mathbf{p}}^\dagger$ to find (after some work with δ functions which you should check) that

$$\begin{aligned}H&=\frac{1}{2}\int d^3x\left(\pi^2+(\nabla\phi)^2+m^2\phi^2\right)\\&=\frac{1}{2}\int d^3x\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{q}}{(2\pi)^3}\{\frac{-\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}{2}(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}-a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}})(a_q e^{iq\cdot x}-a_q^\dagger e^{-iq\cdot x})+\frac{1}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}(ipa_pe^{ip\cdot x}-ipa_p^\dagger e^{-ip\cdot x})\}\end{aligned}$$

There's a lot of algebraic manipulation (details in David Tong's notes) but the net result is that

$$H=\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}(a_{\mathbf{p}}a_{\mathbf{p}}^\dagger+a_{\mathbf{p}}^\dagger a_{\mathbf{p}}).$$

This is simply the Hamiltonian for an infinite number of uncoupled simple harmonic oscillators with frequency ω_p, just as expected.

Now we can define a vacuum state $|0\rangle$ as the state which is annihilated by all $a_{\mathbf{p}}$:

$$a_{\mathbf{p}} |0\rangle = 0 \forall \mathbf{p}.$$

Then computing the vacuum state energy $H|0\rangle$ yields

$$\begin{aligned} H|0\rangle &= \int \frac{d^3p}{(2\pi)^3} \omega_p (a_p^\dagger a_p + \frac{1}{2}[a_p, a_p^\dagger]) |0\rangle \\ &= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \omega_p [a_p, a^\dagger p] |0\rangle \\ &= \int d^3p \omega_p \delta^3(\mathbf{0}) |0\rangle, \end{aligned}$$

which is infinite. Oh no!

What's happened is that $\int d^3p\omega_p$ is the sum of ground state energies for each harmonic oscillator, but $\omega_p = \sqrt{|\mathbf{p}|^2 + m^2} \to \infty$ as $|\mathbf{p}| \to \infty$, so we call this a high-frequency or "ultraviolet divergence." That is, at very high frequencies/short distances, our theory breaks down and our theory should cut off at high momentum. Of course, there's another way to handle this divergence in our theory– just redefine the Hamiltonian to set the ground state energy to zero. "We're not interested in gravity, only energy differences, so we can just subtract ∞."

Thus, we redefine the Hamiltonian for our free scalar field theory to be

$$H = \int \frac{d^3p}{(2\pi)^3} \omega_p a_p^\dagger a_p,$$

such that $H|0\rangle = 0$. Nice. Subtractin' infinities. Because we're physicists.

More formally, the difference between the old and new Hamiltonians can be seen as due to an ordering ambiguity in moving from the classical theory to the quantum one. We could have written the classical Hamiltonian as

$$H = \frac{1}{2}(\omega q - ip)(\omega q + ip)$$

which is classically the same as the original simple harmonic oscillator but becomes

$$\omega a^\dagger a$$

when we quantize.

Definition 4.1. We define a *normal ordered* string of operators $\phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2)\ldots\phi_n(\mathbf{x}_n)$ as follows: written with the notation

$$: \phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2)\ldots\phi_n(\mathbf{x}_n) :$$

we simply move all annihilation operators to the righthand side of the expression (so all the creation operators are on the left). Note that we totally ignore commutation relations in normal ordering! Just move the operators around (well, there are sign flip subtleties when we come to working with fermions).

Example 4.2. For our Hamiltonian,

$$\begin{aligned} :H: &= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \omega_p : (a_p a_p^\dagger + a_p^\dagger a_p) : \\ &= \int \frac{d^3p}{(2\pi)^3} \omega_p a_p^\dagger a_p. \end{aligned}$$

We'd like to recover particles from this theory. Recall that $\forall p, a_p |0\rangle = 0$, so $H|0\rangle = 0$ (where now H means the normal-ordered version of the Hamiltonian). It's easy to verify (exercise) that

$$[H, a_p^\dagger] = \omega_p a_p^\dagger$$

and similarly

$$[H, a_p] = -\omega_p a_p.$$

Let $|p'\rangle = a_{p'}^\dagger |0\rangle$. Then

$$H|p'\rangle = \int \frac{d^3p}{(2\pi)^3} \omega_p a_p^\dagger [a_p, a_{p'}^\dagger] |0\rangle = \omega_{p'} |p'\rangle.$$

Therefore the energy is given by $\omega_{p'} = \sqrt{p'^2 + m^2}$, the relativistic dispersion relation for a particle of mass m and momentum p'. We may thus interpret $|p\rangle$ as a momentum eigenstate of a single particle of mass m and momentum p. Recognizing ω_p as the energy, we'll write E_p instead of ω_p.

We can also write the (single-particle) momentum operator P such that

$$\mathbf{P} |\mathbf{p}\rangle = \mathbf{p} |\mathbf{p}\rangle .$$

P is simply the quantized version of the momentum operator from the stress-energy tensor:

$$\mathbf{P} = -\int \pi(x) \nabla \phi(\mathbf{x}) d^3x = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \mathbf{p} a^\dagger_{\mathbf{p}} a_{\mathbf{p}}.$$

Lecture 5.

Saturday, October 13, 2018

We previously found that we could write the field momentum operator (not the conjugate momentum!) as

$$\mathbf{P} = -\int \pi(x) \nabla \phi(\mathbf{x}) d^3x = \int \frac{d^3p}{(2\pi)^3} \mathbf{p} a^\dagger_{\mathbf{p}} a_{\mathbf{p}}.$$

We could also act on our momentum eigenstates with the angular momentum operator J^i, and what we find is that

$$J^i |\mathbf{p}\rangle = 0,$$

so the scalar field theory represents a spin 0 (scalar) boson.

In general we could imagine cooking up the multi-particle state

$$|\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n\rangle = a^\dagger_{\mathbf{p}_1} a^\dagger_{\mathbf{p}_2} \ldots a^\dagger_{\mathbf{p}_n} |0\rangle .$$

But it follows that

$$|\mathbf{p}, \mathbf{q}\rangle = |\mathbf{q}, \mathbf{p}\rangle ,$$

since the creation operators for different momenta commute. So our states are symmetric under interchange, which means these particles are bosons. The full Hilbert space is spanned by

$$|0\rangle , a^\dagger_{\mathbf{p}} |0\rangle , a^\dagger_{\mathbf{p}_1} a^\dagger_{\mathbf{p}_2} |0\rangle , \ldots$$

and this is called Fock space.

If we use the number operator

$$N \equiv \int \frac{d^3p}{(2\pi)^3} a^\dagger_{\mathbf{p}} a_{\mathbf{p}}$$

which counts the number of particles in a state, we find (exercise)

$$N |\mathbf{p}_1, \ldots, \mathbf{p}_n\rangle = n |\mathbf{p}_1, \ldots, \mathbf{p}_n\rangle .$$

But it's easy to check that (and you should check this using the commutation relations)

$$[N, H] = 0,$$

which means that the number of particles is conserved in the free theory (this is not true once we add interactions).

Let's also note that our momentum eigenstates are *not* localized in space. We can describe a spatially localized state by a Fourier transform,

$$|\mathbf{x}\rangle = \int \frac{d^3p}{(2\pi)^3} e^{-i\mathbf{p}\cdot\mathbf{x}} |\mathbf{p}\rangle .$$

More generally we describe a wavepacket partially localized in position and momentum space, e.g. by

$$|\psi\rangle = \int \frac{d^3p}{(2\pi)^3} e^{-i\mathbf{p}\cdot\mathbf{x}} \psi(\mathbf{p}) |\mathbf{p}\rangle \langle\mathbf{p}| .$$

Note that neither $|\mathbf{x}\rangle$ nor $|\psi\rangle$ are eigenstates of the Hamiltonian like in QM.

We consider now relativistic normalization. We define the vacuum such that $\langle 0|\, 0\rangle = 1$, which certainly must be Lorentz invariant (1 is just a number). So in general

$$\langle \mathbf{p}|\, \mathbf{q}\rangle = \langle 0|\, [a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger]\, |0\rangle = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}).$$

Is this Lorentz invariant? Under the Lorentz transformation, we have

$$p^\mu \to \Lambda^\mu_\nu p^\nu \equiv p'^\mu.$$

We want the two states to be related by a unitary transformation so that the inner product $\langle \mathbf{p}|\, \mathbf{q}\rangle$ is Lorentz invariant (i.e. $\langle \mathbf{p}|\, \mathbf{q}\rangle \to \langle \mathbf{p}'|\, \mathbf{q}'\rangle = \langle \mathbf{p}|\, U(\Lambda)^\dagger U(\Lambda)\, |\mathbf{q}\rangle) = \langle \mathbf{p}|\, \mathbf{q}\rangle$ by unitarity).

To figure this out, we'll need to look at a Lorentz invariant object, e.g. the identity operator on 1-particle states.

$$1 = \int \frac{d^3p}{(2\pi)^3}\, |\mathbf{p}\rangle \langle \mathbf{p}|\, .$$

Either half of this (the d^3p part and the $|\mathbf{p}\rangle \langle \mathbf{p}|$ part) is not LI, but somehow the whole thing is (since it's equal to 1).

How do we prove this? We start by claiming that

$$\int \frac{d^3p}{2E_{\mathbf{p}}}$$

is Lorentz invariant. This follows because $\int d^4p$ is LI, since $\Lambda \in SO(1,3)$ (i.e. $\det\Lambda = 1$) so the factor of $\det\Lambda$ we would normally pick up from doing the coordinate transformation is just 1– $\int d^4p = \int d^4p'$. It's also true that $p_0^2 = \mathbf{p}^2 + m^2$ is Lorentz invariant (in particular, it expresses the length of a four-vector $p_\mu p^\mu = m^2$). The solutions for p_0 have two branches, positive and negative:

$$p_0 = \pm\sqrt{\mathbf{p}^2 + m^2}.$$

But our choice of branch is also Lorentz invariant (we can't go from the positive to negative solutions via Lorentz transformation). Therefore combining the last few facts, we get

$$\int d^4p\delta(p_0^2 - \mathbf{p}^2 - m^2)|_{p^0>0} = \int \frac{d^3p}{2p_0|_{p_0=E_p}},$$

where we have used the fact that

$$\delta(g(x)) = \sum_{x_i \text{ roots of g}} \frac{\delta(x-x_i)}{|g'(x_i)|}.$$

(To see why this is true, consider Taylor expanding $\delta(g(x))$ around its roots to leading order.)

We make the next claim: $2E_p\delta^3(\mathbf{p}-\mathbf{q})$ is the Lorentz invariant version of a δ-function. The proof is as follows:

$$\int \frac{d^3p}{2E_p} 2E_p\delta^3(\mathbf{p}-\mathbf{q}) = 1.$$

But we showed that $\int d^3p/2E_p$ was Lorentz invariant and 1 is certainly Lorentz invariant, so it follows that $2E_p\delta^3(\mathbf{p}-\mathbf{q})$ is also Lorentz invariant.

We therfore learn that the correctly normalized states are

$$|p\rangle \equiv \sqrt{2E_p}\, |\mathbf{p}\rangle = \sqrt{2E_p} a_{\mathbf{p}}^\dagger\, |0\rangle\, ,$$

(where p is now the four-vector p, not the three-vector $\mathbf{p}$) with the inner product

$$\langle p|\, q\rangle = (2\pi)^3 2\sqrt{E_p E_q}\delta^3(\mathbf{p}-\mathbf{q}).$$

We can then rewrite the 1-particle identity operator as an integral over the normalized states,

$$1 = \int \frac{d^3p}{2E_p(2\pi)^3}\, |p\rangle \langle p|\, .$$

(To see this is the identity, try acting on the normalized $|q\rangle$.)

Free $\mathbb{C}$ scalar field We could also look at the free complex scalar field ψ, with Lagrangian

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - \mu^2 \psi^* \psi.$$

We can compute the Euler-Lagrange equations varying ψ, ψ^* separately to find

$$\partial_\mu \partial^\mu \psi + \mu^2 \psi = 0, \partial_\mu \partial^\mu \psi^* + \mu^2 \psi^* = 0$$

(the second equation is simply the complex conjugate of the first). Now we ought to write our field as a sum of two different creation and annihilation operators:

$$\psi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (b_\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{x}} + c^\dagger_\mathbf{p} e^{-i\mathbf{p}\cdot x})$$

and similarly

$$\psi^\dagger = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (b^\dagger_\mathbf{p} e^{-i\mathbf{p}\cdot\mathbf{x}} + c_\mathbf{p} e^{+i\mathbf{p}\cdot x})$$

so that

$$\pi(x) = \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{E_p}{2}} (b^\dagger_\mathbf{p} e^{-i\mathbf{p}\cdot\mathbf{x}} - c_\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{x}}).$$

The conjugate momentum to $\psi^\dagger$ is equivalently $\pi^\dagger$. The commutation relations are then (exercise)

$$[\psi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^3(\mathbf{x}-\mathbf{y})$$
$$\implies [b_\mathbf{p}, b^\dagger_\mathbf{q}] = (2\pi)^3 \delta^3(\mathbf{p}-\mathbf{q}) = [c_\mathbf{p}, c^\dagger_\mathbf{q}].$$

The interpretation of these equations is that different types of particle are created by the $b^\dagger_\mathbf{p}$ and $c^\dagger_\mathbf{p}$ operators. They are both spin 0 and of mass μ, so we should interpret them as a particle-antiparticle pair. This doesn't work for electrons, which have spin 1/2, but it would describe something like a charged pion.

Indeed, if we compute the conserved charges in this theory by applying Noether's theorem, we get a conserved charge of the form $Q = i \int d^3x \dot{\psi}^* \psi - \psi^* \dot{\psi}$ or equivalently in terms of the conjugate momentum (since $\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \dot{\psi}^*$)

$$Q = i \int d^3x [\pi\psi - \psi^\dagger \pi^\dagger].$$

After normal ordering (exercise) one can write

$$Q = \int \frac{d^3p}{(2\pi)^3} (c^\dagger_\mathbf{p} c_\mathbf{p} - b^\dagger_\mathbf{p} b_\mathbf{p}) = N_c - N_b,$$

which shows that our conserved quantity has the interpretation of particle number (counting antiparticles as -1).

Since there are two real scalar fields in this theory, the Hamiltonian for this theory takes the form

$$H = \int \frac{d^3p}{(2\pi)^3} E_p (b^\dagger_\mathbf{p} b_\mathbf{p} + c^\dagger_\mathbf{p} c_\mathbf{p}).$$

As an exercise one can check that $[Q, H] = 0$ using the commutation relations, and therefore Q is conserved. This is also true in the interacting theory. N_c, N_b are individually conserved in the free theory, but in the interacting theory they aren't– instead, they can be created and destroyed in particle-antiparticle pairs so that $N_c - N_b$ is constant.

Lecture 6.

Tuesday, October 16, 2018

We've been working in the Schrödinger picture where the states evolve in time, but now it will be useful to pass to the Heisenberg picture, where the states are fixed and the *operators* evolve in time.

In the Schrödinger picture, it's not obvious how our theory is Lorentz invariant. We seem to have picked out time as a special dimension when we write things down (even though we started with a Lorentz invariant theory, so our final theory should still be Lorentz invariant). The operators $\phi(\mathbf{x})$ don't depend on t, but the states evolve as

$$i\frac{d\,|p\rangle}{dt} = H\,|p\rangle = E_p\,|p\rangle \implies |p(t)\rangle = e^{-iE_pt}\,|p(0)\rangle\,.$$

In the Heisenberg picture, things are a bit better– time dependence is moved into the operators. Denoting Heisenberg picture operators as O_H and Schrödinger picture operators as O_S, we have[6]

$$O_H(t) \equiv e^{iHt}O_Se^{-iHt}.$$

Taking the time derivative of each side, one finds that[7]

$$\frac{dO_H}{dt} = i[H, O_H].$$

This is the general time evolution of operators in the Heisenberg picture. It's clear that $O_H(t=0) = O_S$, so our operators agree at $t = 0$ (but in general nowhere else). The field commutators then become *equal time commutation relations*:

$$[\phi(\mathbf{x},t), \phi(\mathbf{y},t)] = [\pi(\mathbf{x},t), \pi(\mathbf{y},t)] = 0$$

and

$$[\phi(\mathbf{x},t), \pi(\mathbf{y},t)] = i\delta^3(\mathbf{x}-\mathbf{y}).$$

Exercise 6.1. One should check (exercise) that $\frac{d\phi}{dt} = i[H,\phi]$ now means that the Heisenberg picture operator ϕ_H satisfies the Klein-Gordon equation, $\partial_\mu\partial^\mu\phi + m^2\phi = 0$.

We now write the Fourier transform of $\phi(x)$ (where x is now a four-vector) by deriving

$$d^{iHt}a_{\mathbf{p}}e^{-iHt} = e^{-iE_pt}a_{\mathbf{p}}$$

and

$$d^{iHt}a^\dagger_{\mathbf{p}}e^{-iHt} = e^{+iE_pt}a^\dagger_{\mathbf{p}}.$$

You should also check this (exercise) using the commutation relation $[H, a_{\mathbf{p}}] = -E_pa_{\mathbf{p}}$.

Therefore we can now write

$$\phi(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\{a_{\mathbf{p}}e^{-ip\cdot x} + a^\dagger_{\mathbf{p}}e^{+ip\cdot x}\}$$

where x and p are now four-vectors and $p_0 = E_p$.

Causality We might be concerned about the causal structure of this theory, since ϕ and π satisfy equal-time commutation relations. In general a Lorentz transform might mix up events which in one frame take place at "equal times." So what about arbitrary space-time separations? It turns out that causality requires that the commutators of spacelike separated operators is zero, i.e. two events which are spacelike separated cannot impact one another.

$$[O_1(x), O_2(y)] = 0\forall(x-y)^2 < 0.$$

Does this condition hold? Let's define

$$\Delta(x-y) \equiv [\phi(x), \phi(y)]$$

and expand in the Fourier basis.

$$\begin{aligned}\Delta(x-y) &= \int \frac{d^3p}{(2\pi)^6}\frac{d^3p'}{\sqrt{4E_pE_{p'}}}\left([a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}]e^{-i(p\cdot x - p'\cdot y)} + [a^\dagger_{\mathbf{p}}, a_{\mathbf{p}'}]e^{i(p\cdot x - p'\cdot y)}\right) \\ &= \int \frac{d^3p}{2E_p(2\pi)^3}\left(e^{-ip\cdot(x-y)} - e^{ip'\cdot(x-y)}\right)\end{aligned}$$

Remarkably, this is just a c-number– it's not an operator at all but a (classical) number. It is Lorentz invariant since the integration measure $d^3p/(2E_p)$ is and the integrand is (it depends on $p\cdot(x-y)$, so

[6]Here, the exponential of an operator is simply defined in terms of the power expansion of e, e.g. $e^{iHt} = \sum_{n=0}^\infty \frac{(iHt)^n}{n!}$.

[7]Explicitly, $\frac{dO_H(t)}{dt} = iHe^{iHt}O_Se^{-iHt} + e^{iHt}O_S(-iH)e^{-iHt} = ie^{iHt}[H, O_S]e^{-iHt} = i[H, O_H]$ since $e^{iHt}He^{-iHt} = H$. We also see from this that it doesn't matter to the Hamiltonian itself what picture we're in, since $H_S = H_H$.

totally contracted). Moreover, each term is separately Lorentz invariant. In addition, if $x-y$ is spacelike then $x-y$ can be Lorentz transformed to $y-x$ in the first term, giving 0. It does not vanish for timelike separations, e.g.

$$[\phi(\mathbf{x},0),\phi(\mathbf{x},t)] = \int \frac{d^3p}{(2\pi)^3 2E_p}(e^{-imt}-e^{+imt}) \neq 0.$$

And at equal times

$$[\phi(\mathbf{x},t),\phi(\mathbf{y},t)] = \int \frac{d^3p}{(2\pi)^3 2E_p}(e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}-e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}) = 0$$

(since we can send the integration variable $\mathbf{p}\to-\mathbf{p}$). One can also see in this way that the commutator for spacelike separated operators vanishes, since a general spacelike separation can always be transformed into a frame where the two events take place at equal times.

Definition 6.2. We can then introduce the idea of a *propagator*– if we initially prepare a particle at point y, what is the amplitude to find it at x? We can write this as

$$\begin{aligned}
\langle 0|\,\phi(x)\phi(y)\,|0\rangle &= \int \frac{d^3p d^3p'}{(2\pi)^6\sqrt{4E_pE_{p'}}}\langle 0|\,a_{\mathbf{p}}a^\dagger_{\mathbf{p}'}\,|0\rangle\, e^{-ip\cdot x+ip'\cdot y}\\
&= \int \frac{d^3p d^3p'}{(2\pi)^6\sqrt{4E_pE_{p'}}}\langle 0|\,[a_{\mathbf{p}}a^\dagger_{\mathbf{p}'}]\,|0\rangle\, e^{-ip\cdot x+ip'\cdot y}\\
&= \int \frac{d^3p}{(2\pi)^3 2E_p}e^{-ip\cdot(x-y)} \equiv D(x-y),
\end{aligned}$$

where we have used the fact that $a_{\mathbf{p}}$ kills the ground state (so we can freely subtract off $a^\dagger_{\mathbf{p}'}a_{\mathbf{p}}$ to get a commutator) and used the resulting delta function to integrate over d^3p'.

In fact, one can show[8] that for spacelike separations $(x-y)^2<0$, the propagator decays as $D(x-y)\sim e^{-m|\mathbf{x}-\mathbf{y}|}$. The quantum field seems to "leak" out of the light cone. But we also computed that

$$\Delta(x-y) = [\phi(x),\phi(y)] = D(x-y) - D(y-x) = 0$$

if $(x-y)^2<0$. We can interpret this to mean that there's no Lorentz invariant way to order the two events at x and y. A particle can travel as easily from $y\to x$ as $x\to y$, so in a quantum measurement these two amplitudes cancel. With a complex scalar field, the story is more interesting. We find instead that the amplitude for a particle to go from $x\to y$ is cancelled by the amplitude for an anti-particle to go from $y\to x$.[9] This is also the case for the real scalar field, except the particle is its own antiparticle.

Definition 6.3. We now introduce the *Feynman propagator* Δ_F, which is like a regular propagator but with time ordering baked in. That is,

$$\Delta_F = \begin{cases}\langle 0|\,\phi(x)\phi(y)\,|0\rangle & \text{for } x^0>y^0\\ \langle 0|\,\phi(y)\phi(x)\,|0\rangle & \text{for } y^0>x^0.\end{cases}$$

We claim the Feynman propagator can also be written as

$$\Delta_F = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2}e^{-ip\cdot(x-y)}.$$

Note that this is Lorentz invariant– the volume element is certainly Lorentz invariant, and everything else is scalars. But there's an issue– this integral has a pole whenver $p^2=m^2$, or equivalently for each value of $\mathbf{p}$, $p^2-m^2=(p^0)^2-\mathbf{p}^2-m^2=0$ when $p^0=\pm E_{\mathbf{p}}=\pm\sqrt{\mathbf{p}^2+m^2}$. We would like to integrate over p^0 to recover the earlier form of the propagator, so we can either deform the contour or push the poles of the real p^0 axis with an *$i\epsilon$ prescription*.

[8]The easiest way to do this is to set $y=0$ and take x and y at equal times, $x^0=y^0=0$. This gets rid of p^0, and from here you can switch to spherical coordinates, rewriting $\mathbf{p}\cdot(x)$ as $|p||x|\cos\theta$.

[9]See also Wheeler's "one-electron universe"– `https://en.wikipedia.org/wiki/One-electron_universe`.

We'll finish the proof next time, but by analytically continuing p^0 to the complex plane, making this $i\epsilon$ prescription, and closing the contour appropriately we can do the p^0 integral and find that what we get is exactly the Feynman propagator as defined earlier in terms of time ordering.

Proof of Exercise 6.1 Let's find the equation of motion for ϕ. Recall that $[\phi(\mathbf{x}), \phi(\mathbf{y})] = 0$. We can also show that $\nabla\phi(y)$ and $\phi(x)$ commute:

$$\nabla\phi(\mathbf{y})\phi(\mathbf{x}) = \nabla_y(\phi(\mathbf{y})\phi(\mathbf{x})) = \nabla_y(\phi(\mathbf{x})\phi(\mathbf{y})) = \phi(\mathbf{x})\nabla\phi(\mathbf{y})$$

so the only term in the Hamiltonian we need to worry about is the π^2 term.

$$\begin{aligned}
\dot{\phi} &= i[H, \phi] \\
&= \frac{i}{2}\int d^3y \left[\pi^2(y) + (\nabla\phi(y))^2 + m^2\phi(y)^2, \phi(x)\right] \\
&= \frac{i}{2}\int d^3y(\pi^2(y)\phi(x) - \phi(x)\pi^2(y)) \\
&= \frac{i}{2}\int d^3y(\pi(y)(-[\phi(x), \pi(y)] + \phi(x)\pi(y)) - \phi(x)\pi^2(y)) \\
&= \frac{i}{2}\int d^3y(-i\delta^3(x-y)\pi(y) + \pi(y)\phi(x)\pi(y) - \phi(x)\pi^2(y) \\
&= \frac{i}{2}\int d^3y(-2i\delta^3(x-y)\pi(y)) \\
&= \pi(x).
\end{aligned}$$

We can also compute the time evolution for π. Here, we do have to worry about the $\nabla\phi$ terms as well as the ϕ terms.

$$\begin{aligned}
\dot{\pi} &= i[H, \pi] \\
&= \frac{i}{2}\int d^3y \left[\pi^2(y) + (\nabla\phi(y))^2 + m^2\phi(y)^2, \pi(x)\right] \\
&= \frac{i}{2}\int d^3y\nabla\phi(y)\nabla_y(\phi(y)\pi(x)) - \nabla_y(\pi(x)\phi(y))\nabla\phi(y) + 2im^2\delta^3(x-y)\phi(y) \\
&= \frac{i}{2}\int d^3y\nabla\phi(y)\nabla_y([\phi(y), \pi(x)]) - \nabla_y(-[\phi(y), \pi(x)])\nabla\phi(y) + 2im^2\delta^3(x-y)\phi(y) \\
&= \frac{i}{2}\int d^3y\nabla\phi(y)\nabla_y(i\delta^3(y-x)) + \nabla_y(i\delta^3(y-x))\nabla\phi(y) + 2im^2\delta^3(x-y)\phi(y) \\
&= \frac{i}{2}\int d^3y \left(-2i\delta^3(x-y)\nabla^2\phi(y) + 2im^2\delta^3(x-y)\phi(y)\right) \\
&= \nabla^2\phi - m^2\phi.
\end{aligned}$$

(where we have integrated by parts to move the ∇ from the delta function to ϕ). Thus ϕ obeys the equation

$$\ddot{\phi} = \dot{\pi} = \nabla^2\phi - m^2\phi$$

or equivalently

$$\ddot{\phi} - \nabla^2\phi + m^2 = \partial_\mu\partial^\mu\phi + m^2 = 0.$$

Therefore ϕ satisfies the Klein-Gordon equation. (This is also in David Tong's notes.) ⊠

We'll also make note of a potentially useful identity which can be proved by induction: if $[a, b] = \alpha$, then $[a^n, b] = n\alpha a^{n-1}$.

Proof of Heisenberg picture $a_p, a_p^\dagger$ Here, we'll show that

$$e^{iHt}a_pe^{-iHt} = e^{-iE_pt}a_p$$

using the commutation relation $[H, a_p] = -E_pa_p$. First, I'll claim that

$$H^na_p = a(-E_p + H)^n.$$

Let's prove it by induction: for the base case, $n = 1$ and

$$Ha_p = [H, a_p] + a_pH = -E_pa_p + a_pH = a_p(-E_p + H).$$

Now the inductive step: suppose the hypothesis holds for n. Then

$$H^{n+1}a_p = H(H^n a_p) = Ha_p(-E_p + H)^n = a_p(-E_p + H)^{n+1}.$$

Therefore we can use this in the expansion of e^{iHt}.

$$\begin{aligned} e^{iHt}a_p e^{-iHt} &= \sum_{n=0}^{\infty} \frac{(iHt)^n}{n!} a_p e^{-iHt} \\ &= a_p \sum_{n=0}^{\infty} \frac{(it(-E_p + H))^n}{n!} e^{-iHt} \\ &= a_p e^{-iE_p t} e^{iHt} e^{-iHt} \\ &= a_p e^{-iE_p t}. \end{aligned}$$

Rather than repeating this whole calculation, we can simply take the hermitian conjugate of each side (since H is hermitian) to get

$$e^{iHt} a_p^\dagger e^{-iHt} = e^{+iE_p t} a_p^\dagger.$$

Note that the sign flip in the exponent of $e^{\pm iHt}$ and the reversing of order from taking the hermitian conjugate cancel out. So the operators $a, a^\dagger$ do evolve in a nice way that allows us to write ϕ in terms of a four-vector product in the exponent, $p \cdot x$, and in turn this helps us to see that our theory has a sensible causal structure under Lorentz transformations. ⊠