BLACK HOLES

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These notes were taken for the *Black Holes* course taught by Jorge Santos at the University of Cambridge as part of the Mathematical Tripos Part III in Michaelmas Term 2018. I live-TEXed them using Overleaf, and as such there may be typos; please send questions, comments, complaints, and corrections to itel2@cam.ac.uk.
Many thanks to Arun Debray for the LATEX template for these lecture notes: as of the time of writing, you can find him at https://web.ma.utexas.edu/users/a.debray/.

Contents

Lecture 1.

Friday, January 18, 2019

> *"The integral curves of the timelike Killing vector don't intersect, or else you could go back in time and kill your own grandmother... which would make you a WEIRDO." –Jorge Santos*

Note. Some very important admininistrative details for this course! Lectures will be Monday, Wednesday, *Thursday*, and Friday, with M/W/F lectures from 12:00-13:00 and Thursday lectures from 13:00-14:00. There will be no classes from 4th February to 15th February, due to Prof. Santos anticipating a baby.

Some useful readings include

- Harvey Reall's notes on black holes and general relativity
- Wald's "General Relativity"
- Witten's review, "Light Rays, Singularities and All That"

To begin with, some conventions. Naturally, we set $c = G = 1$. We use the $- + ++$ sign convention for the Minkowski metric. We shall use the abstract tensor notation where tensor expressions with Greek indices μ, ν, σ are only valid in a particular coordinate basis, while Latin indices a, b, c are valid in any basis, e.g. the Riemann scalar is defined to be $R = g^{ab}R_{ab}$, whereas the Christoffel connection takes the form $\Gamma^\mu_{\nu\rho} = \frac{g^{\mu\epsilon}}{2}(g_{\epsilon\nu,\rho} + g_{\epsilon\rho,\nu} - g_{\nu\rho,\epsilon})$. We also define $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$.

Stars Black holes are one possible endpoint of a star's life cycle. Let's start by assuming spherical stars. Now, stars radiate energy and burn out. However, even very cold stars can avoid total gravitational collapse because of *degeneracy pressure*. If you make a star out of fermions (e.g. electrons) then the Pauli exclusion principle says they can't be in the same state (or indeed get too close), and it might be that the degeneracy pressure is enough to balance the gravitational forces. When this happens, we call the star a *white dwarf*. It turns out this can only happen for stars up to $1.4M_\odot$ (solar masses). If a star is instead made of neutrons (naturally we call these *neutron stars*) then the pressure of the neutrons can prevent gravitational collapse in a mass range from $1.4M_\odot < 3M_\odot$. *Beyond* $3M_\odot$, *stars are doomed to collapse into black holes.* We'll spend some time understanding this limit.

Spherical symmetry A normal sphere is invariant under rotations in 3-space, $SO(3)$. The line element on the 2-sphere of unit radius is

$$d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2.$$

It is also invariant under reflections sending $\theta \to \pi - \theta$ (the full group $O(3)$), and perhaps some other symmetries.

Definition 1.1. A spacetime (M, g) is *spherically symmetric* if it possesses the same group of isometries as the round two-sphere $d\Omega_2^2$. That is, it has an $SO(3)$ symmetry where the orbits are S^2s (two-spheres). Important remark– there are spacetimes such as Taub-NUT spacetime which enjoy $SO(3)$ symmetry but are *not* spherically symmetric.

In a spherically symmetric spacetime, we shall define a "radius" $r : M \to \mathbb{R}^+$ defined by

$$r(p) = \sqrt{\frac{A(p)}{4\pi}},$$

where $A(p)$ is the area of the S^2 orbit from a point p. This only makes good sense to define under spherical symmetry, but the idea is that we invert the old relationship $A = 4\pi r^2$ to define a radius given an area.

Definition 1.2. A spacetime (M, g) is *stationary* if it admits a Killing vector field K^a which is everywhere timelike. That is,

$$K^a K^b g_{ab} < 0.$$

Using the assumptions of time independence and spherical symmetry, we'll show some constraints on the resulting spacetime. Let us pick a hypersurface Σ which is nowhere tangent to the Killing vector K. We assign coordinates t, x^i where x^i is defined on the hypersurface, and t then describes a distance along the integral curves of K^a through each point on Σ. That is, we follow the curves such that $\frac{dx^a}{dt} = K^a$.

But in this coordinate system, K^a now takes the wonderfully simple form

$$K^a = \left(\frac{\partial}{\partial t}\right)^a$$

Since K^a is a Killing vector, the Lie derivative of the metric with respect to K vanishes, $\mathcal{L}_K g = 0$. (See Harvey Reall's notes for the definition of a Lie derivative– it's just a derivative, "covariant-ized.") In this case, $K^c \partial_c g_{ab} + K^c{}_{,a} g_{cb} + K^c{}_{,b} g_{ac} = 0$.

With these assumptions, our metric takes the form

$$ds^2 = g_{tt}(x^k)dt^2 + 2g_{ti}(x^k)dt dx^i + g_{ij}(x^k)dx^i dx^j,$$

where $g_{tt}(x^k) < 0$ (stationarity).

We shall also consider *static spacetimes*. Take Σ to be a hypersurface defined by $f(x) = 0$ for some function $f : M \to \mathbb{R}, df \neq 0$. Then df is orthogonal to Σ. Let's prove this.

Proof. Take Z^a to be tangent to Σ. Thus

$$(df)(Z) = Z(f) = Z^\mu \partial_\mu f = 0$$

on Σ. A useful example might be to compute this for the two-sphere.

Now take a general 1-form normal to Σ. This 1-form can be written as

$$m = g df + f m'. \tag{1.3}$$

That is, on the surface Σ the one-form m is precisely normal to Σ, but if we go off Σ a little bit then we can smoothly extend m off by a bit. We require that g is a smooth function and that m' is smooth but otherwise arbitrary.

Then the differential of m is

$$dm = dg \wedge df + df \wedge m' + f \wedge dm' \implies dm|_\Sigma = (dg - m') \wedge df.$$

We find that $m|_\Sigma = g\,df \implies (m \wedge dm)|_\Sigma = 0$– this follows since $df \wedge df$ is zero. So if Σ is a hypersurface with m orthogonal then $(m \wedge dm)_\Sigma = 0$. ⊠

The converse is also true (a theorem due to Frobenius)– if m is a non-zero 1-form such that $m \wedge dm = 0$ everywhere, then there exists f, g such that $m = g\,df$.

Definition 1.4. A spacetime is *static* if it admits a hypersurface-orthogonal timelike Killing vector field. In particular, static $\implies$ stationary.

In practice, for a static spacetime, we know that K^a is hypersurface orthogonal, so when defining coordinates we shall choose Σ to be orthogonal to K^a. Equivalently, this means that we can choose a hypersurface Σ to be the surface $t = 0$, which implies that $K_\mu \propto (1, 0, 0, 0)$. Of course, this means that $K_i = 0$. But recall that

$$K^a = \left(\frac{\partial}{\partial t}\right)^a \implies g_{ti} = K_i = 0.$$

So our generic metric simplifies considerably– the cross-terms g_{ti} go away and spherical symmetry will further constrain the spatial g_{ij} terms.

Lecture 2.

Monday, January 21, 2019

"Not only are we going to make our cow spherical, we're going to shoot it down so it doesn't move."
–Jorge Santos

So far, we discussed two key concepts. We discussed the condition of a spacetime being static, i.e. stationary and enjoying invariance under $t \leftrightarrow -t$, which forced the line element to take the form

$$ds^2 = g_{tt}(x^k)dt^2 + g_{ij}(x^k)dx^i dx^j. \tag{2.1}$$

We also required spherical symmetry, i.e. the $SO(3)$ orbits of points p in the manifold are S^2s.

Let us see what we get when we combine static and spherically symmetric solutions. We know from staticity that there is a timelike Killing vector $K = \left(\frac{\partial}{\partial t}\right)^a$. Suppose we take a hypersurface Σ_t which is normal to our timelike Killing vector K. Then take any point $p \in \Sigma_t$. By spherical symmetry, its $SO(3)$ orbit is a sphere S^2. Assign angular coordinates θ, ϕ on the S^2 orbit. Using spacelike geodesics normal to the sphere S^2, we can then extend θ, ϕ to the entire hypersurface, a process which is shown in Fig. 1.

Thus our metric on Σ_t takes the form

$$ds^2_{\Sigma_t} = e^{2\psi(r)}dr^2 + r^2 d\Omega_2^2, \tag{2.2}$$

where the coefficient of dr^2 must only depend on r by spherical symmetry, and r is given by our old area relation, $r : \mathcal{M} \to \mathbb{R}^+$ with $r(p) = \sqrt{\frac{A(p)}{4\pi}}$. Now using the property our spacetime is static, we can write down the full spacetime metric,

$$ds^2 = -e^{2\Phi(r)}dt^2 + ds^2_{\Sigma_t}. \tag{2.3}$$

So far we have two degrees of freedom, $(\psi(r), \Phi(r))$. For fluids, recall that the stress-energy tensor takes the form

$$T_{ab} = (\rho + p)U_a U_b + p g_{ab} \tag{2.4}$$

where U_a is a four-velocity, ρ is an energy density and p is a pressure. By spherical symmetry, ρ and p can only be functions of the radial coordinate r, so $\rho = \rho(r)$ and $p = p(r)$. We take

$$U^a U_a = -1 \implies U_t^2 g^{tt} = -1 \implies U^a = e^{-\Phi}\left(\frac{\partial}{\partial t}\right)^a \tag{2.5}$$

so that $p, \rho > 0$. This is an energy condition.

We want to describe spherical stars (with finite spatial extent), so outside the star both the pressure and energy density must vanish,

$$p = \rho = 0 \text{ for } r > R \tag{2.6}$$

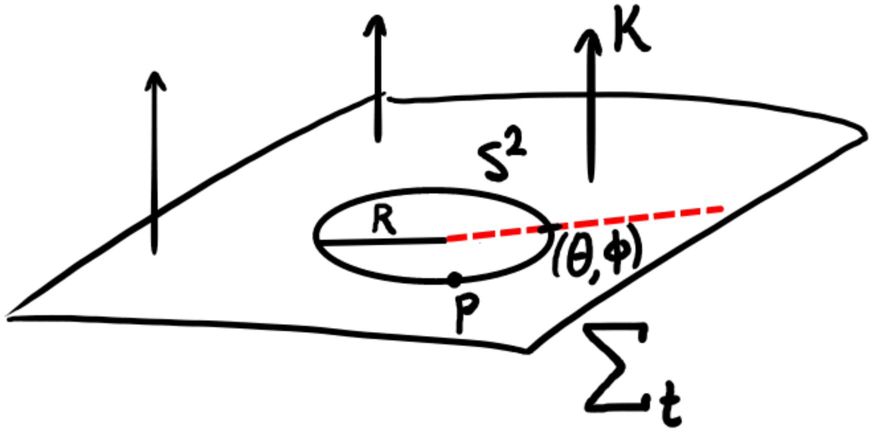

FIGURE 1. An illustration of our coordinates for static, spherically symmetric solutions. We can always choose a hypersurface Σ_t which is orthogonal to the timelike Killing vector K. On Σ_t, choose a point p and trace out its S^2 orbit (drawn here as a circle, S^1) under the action of the $SO(3)$ symmetry. On the S^2 orbit, we can define angular coordinates (θ, ϕ), and we can then extend these to the rest of Σ_t by defining θ, ϕ to be constant on spacelike geodesics normal to the S^2 orbit (red dashed line). The radial coordinate r is given by the area formula $r = \sqrt{A(p)/4\pi}$. This defines coordinates on Σ_t, which we can extend to the entire manifold by following the integral curves of K.

with R the radius of the star. Now, we know that the defining property of stress-energy is that it is conserved– $\nabla^a T_{ab} = 0$. But the Einstein equation says that

$$R_{ab} - \frac{R}{2} g_{ab} = T_{ab}, \tag{2.7}$$

and by the contracted Bianchi identity we know that the divergence of the LHS always vanishes, so it suffices to look at the Einstein equation since it automatically implies the conservation equation for fluids. This is not generally true for other energy content since

Let's look at a specific example, the tt componetn of the Einstein equations.

$$G_{tt} = \frac{e^{2(\Phi-\psi)}}{r^2}[e^{2\psi} + 2r\psi' - 1], \tag{2.8}$$

where the prime indicates a $\frac{\partial}{\partial R}$.

Let us also define a function $m(r)$, given by

$$e^{2\psi} \equiv \left[1 - \frac{2m(r))}{r}\right]^{-1}. \tag{2.9}$$

From the various components we learn that

$$tt : m' = 4\pi r^2 \rho(r) \tag{2.10}$$

$$nn : \Phi' = \frac{m + 4\pi r^3 p}{r(r-2m)} \tag{2.11}$$

$$\theta\theta : p' = -(p+\rho)\frac{m + 4\pi n^3 p}{r[r - 2m(r)]}. \tag{2.12}$$

We call these the Tollman-Oppenheimer-Volkoff equations (TOV for short). We have three equations but four unknowns: m, Φ, p and ρ. We need one more bit of information– namely, an equation of state relating the pressure and energy density. Normally, p depends on ρ and also T the temperature. But for our purposes, we will assume cold stars so that p is only a function of the energy density ρ.

What can we figure out before imposing any sort of conditions on $p(\rho)$? Well, outside the star, $r > R$, we have $p = \rho = 0$. But we see immediately that $m' = 0 \implies m = M$ constant. This in turn implies that

$$\psi(r) = -\frac{1}{2}\log\left(1 - \frac{2M}{r}\right) = -\Phi(r). \tag{2.13}$$

We can now write down the line element,

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega_2^2. \tag{2.14}$$

This is the *Schwarzschild line element*. We associate the parameter M with the mass of the system.

There's a bit of physics to extract from this– for *stars*, we need $R > 2M$ to keep the signs correct in the metric. For the sun, we have $2M_\odot = 3\,\text{km}$ and $R \simeq 7 \times 10^5\,\text{km}$, so this is a (very loose) bound which is easily satisfied.

Inside the star, life is not so easy. The mass now depends on the radius, and it has a solution

$$m(r) = 4\pi \int_0^r \rho(\tilde{r})\tilde{r}^2 d\tilde{r} + m_*, \tag{2.15}$$

with m_* some integration constant. Fortunately, we note that by physical concerns, $m(r) \to 0$ as $r \to 0$ in order to preserve regularity (the metric should look flat), which tells us that this integration constant is zero, $m_* = 0$.

At the surface of the star ($r = R$), the metric is continuous. This tells us that

$$M = 4\pi \int_0^R \rho(r) r^2 dr, \tag{2.16}$$

so the mass M is related to an integration of the energy density. It is not however the total energy, which is given by

$$E = \int_V \rho r^2 \sin\theta e^{\psi} > M.$$

The total energy differs by a factor which corresponds to the gravitational binding energy.

Restoring units to our $R < 2M$ bound on the star radius, we write

$$\frac{GM}{c^2 R} < \frac{1}{2}. \tag{2.17}$$

This isn't hard to satisfy but it's a start, considering we haven't assumed anything about the equations of state.

Let's add some assumptions. For reasonable matter,

$$\frac{dp}{d\rho} > 0, \quad \frac{dp}{dr} \leq 0 \implies \frac{d\rho}{dr} \leq 0. \tag{2.18}$$

This first condition says that more stuff (density) means more pressure, and the second says that pressure decreases as we go towards the surface of the star. The $\theta\theta$ component then tells us that

$$\frac{m(r)}{r} < \frac{2}{9}\left[1 - 6\pi r^2 p + (1 + 6\pi r^2 p)^{1/2}\right], \tag{2.19}$$

which we will prove on Example Sheet 1. Knowing that the pressure vanishes at the surface of the star, $r = R$, we arrive at the Buchdahl bound,

$$R > \frac{9}{4}M. \tag{2.20}$$

This already improves on our naïve bound.

Now using the TOV equations, we could just consider the m' and p' equations. Recall that p is a function of ρ, so we can consider these as two first-order equations for p and m. Normally, each of these conditions would require a boundary condition. But recall we have one integration constant (our m_* from earlier) fixed to be zero, so really we just need to specify one boundary condition, $\rho(r = 0)$.

By the form of p', we see that the pressure decreases as we go towards the surface, so we just integrate outwards until p vanishes and we hit the surface of the star at some value R. This tells us that $M(\rho(0))$ and $R(\rho(0))$, so all the physical parameters of the star are fixed by just one number– the energy density at the center of the star, $\rho(0)$.

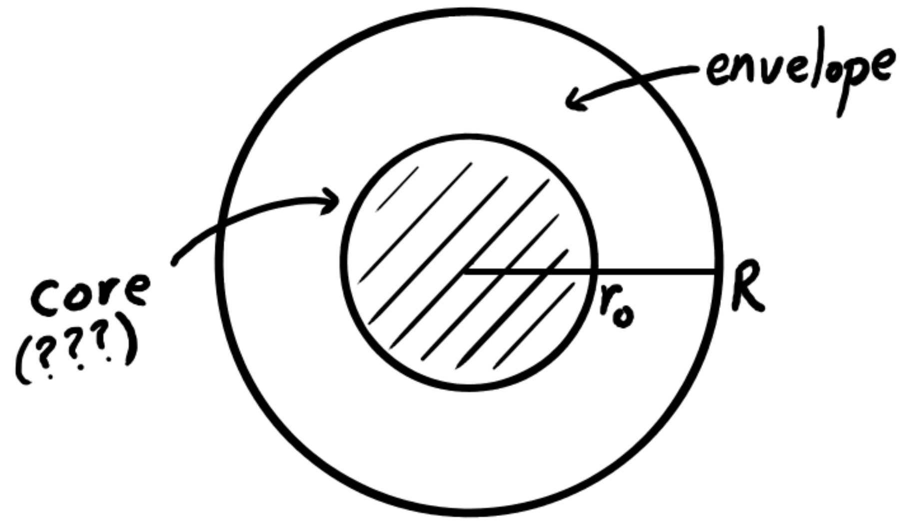

FIGURE 2. A schematic drawing of the interior + envelope model for a star. The interior region extends from $0 < r \leq r_0$ and the exterior region (envelope) goes from $r_0 < r < R$, to the surface of the star.

We could now introduce an equation of state, in principle. But let's try to be a bit more clever and deduce something independent of the equation of state of whatever this star is made of. This star could be super dense in its core, and maybe we don't know anything about physics in the interior, up to some radius r_0. But outside the core there's some envelope region $r_0 < r < R$ where we do know what's happening– see Fig. 2 for an illustration.

What could happen in the interior? If ρ takes on some value $\rho(r_0) = \rho_0$ on the surface of the core, then by integrating we can put the bound

$$m_0 \geq \frac{4\pi}{3} r_0^3 \rho_0 \tag{2.21}$$

on m_0 the mass contained in the core. That is, in the best case $\rho(r)$ is constant in the core region– the star certainly cannot be less dense in its core. But we have another inequality on $m(r)$, the Buchdahl limit 2.19, which we can see is a decreasing function of p. So we evaluate this condition at $r = r_0, m(r_0) = m_0$, noting that the most general bound we can put on m_0 in terms of r_0 occurs when $p = 0$. We find that

$$\frac{m_0}{r_0} < \frac{4}{9}. \tag{2.22}$$

These two inequalities in the space of core masses m_0 and core radii r_0 plus a value for the core density ρ_0 tell us that there is a limit on the total core mass– taking $\rho_0 = 5 \times 10^{14}\ \mathrm{g/cm}^3$, the density of nuclear matter, we find that $m_0 < 5M_\odot$. Strictly, these are only limits on the core mass, but it turns out that the envelope region is generally insignificant, so

$$M \approx m_0 < 5M_\odot. \tag{2.23}$$

Lecture 3.

Wednesday, January 23, 2019

"If anyone starts calculating geodesics from here [the geodesic equation], I will give you a zero! You should pay for your own stupidity." –Jorge Santos

Today is an exciting class! For the first time, we will look at the Schwarzschild line element. We will not actually define what a black hole is for another ten lectures, but we'll plow ahead and learn some thing about them anyway.

In Schwarzschild coordinates $t, r, \theta, \phi)$, the line element of a Schwarzschild black hole takes the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2. \tag{3.1}$$

For stars, we said this solution was only valid for $r > 2M$, but as you may already know, the $r = 2M$ singularity is just a coordinate singularity and can be defined away by a good choice of coordinates (as we'll do today). We take the parameter M to be a physical mass right now, and thus $M > 0$ (we will treat the case $M < 0$ later). The radius $r = 2M$ is known as the *Schwarzschild radius.*

Last time, we assumed our solutions were both spherically symmetric and stationary. What if we drop the assumption that the solution is stationary? The answer is the following theorem by Birkhoff.

Theorem 3.2 (Birkhoff's theorem)**.** *Any spherically symmetric solution of the vacuum Einstein equations is isometric to Schwarzschild.*

Proof. What could the spherically symmetric line element look like? The angular bit must look like $r^2 d\Omega_2^2$. The rest of it is unconstrained– we can have dt^2, dr^2, and $drdt$ cross terms. Thus

$$ds^2 = -f(t,r)[dt + \chi dr]^2 + \frac{dr^2}{g} + r^2 d\Omega_2^2. \tag{3.3}$$

Here, χ, f, and g can all depend on t, r. Let us begin by rescaling time t to set the factor $\chi = 0$. If you like, $dt + \chi dr = dt'$. We can still send $t \to p(t)$ a function of t. Now our line element reduces to

$$ds^2 = -f(t,r)dt^2 + \frac{dr^2}{g(t,r)} + r^2 d\Omega_2^2. \tag{3.4}$$

From the tr component of the vacuum Einstein equations, we find that

$$0 = R_{tr} - g_{tr}\frac{R}{2} \implies \frac{\partial}{\partial t} g = 0 \implies g(r), \tag{3.5}$$

g does not depend on time. Looking at the tt component we get

$$1 - g - rg' = 0 \implies g(r) = 1 - \frac{2M}{r}, \tag{3.6}$$

where M is an integration parameter. From the rr component we get instead

$$1 - \frac{1}{g} + r\frac{f'}{f} = 0 \implies f(t,r) = C(t)\left[1 - \frac{2M}{r}\right]. \tag{3.7}$$

This is almost good– all we need to do is reparametrize t and we can set this function $C(t) = 1$. We find that with f, g defined in this way, our general spherically symmetric metric has been put into the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2. \qquad \boxtimes$$

Gravitational redshift Suppose we have two observers Alice and Bob. They sit at some constant coordinates (r_A, θ, ϕ) and (r_B, θ, ϕ) respectively. Now Alice sends some signals (light pulses) to Bob, separated by a coordinate time Δt. Because $\frac{\partial}{\partial t}$ is a Killing vector, our scenario has time translation symmetry– each pulse will be the same as the last, just translated in time, so Alice and Bob agree on the difference in coordinate time Δt.

Let us notice that Alice measures the photons as being separated by a proper time

$$\Delta\tau_A = \sqrt{1 - \frac{2M}{r_A}}\Delta t. \tag{3.8}$$

An equivalent expression is true for Bob at r_B. Eliminating Δt, we find that

$$\frac{\Delta\tau_B}{\Delta\tau_A} = \sqrt{\frac{1 - \frac{2M}{r_B}}{1 - \frac{2M}{r_A}}} > 1 \tag{3.9}$$

for $r_B > r_A$. This interval is a proxy for a perceived wavelength, $\lambda_B > \lambda_A$. If Bob is far away, $R_B \gg 2M$, then the measured redshift z is given by

$$1 + z \equiv \frac{\lambda_B}{\lambda_A} = \frac{1}{\sqrt{1 - \frac{2M}{r_A}}}. \tag{3.10}$$

For an observer outside a physical star, $R = 9M/4$, $Z = 2$ is the maximum redshift Bob can measure as Alice gets close to the surface of the star. But $R = 2M$ represents a surface of infinite redshift, so it seems to represent something very strange. This is the *event horizon*. We'll explore the consequences of this calculation more as we go on.

Geodesics of Schwarzschild We'll introduce some nice coordinates to let us cross the event horizon. Let $x^\mu(\lambda)$ be an affinely parametrized geodesic with tangent vector

$$U^\mu \equiv \frac{dx^\mu}{d\tau}.$$

Since we have two Killing vectors $K = \frac{\partial}{\partial t}$ and $m = \frac{\partial}{\partial \phi}$, we get two conserved charges,

$$E = -K \cdot U = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} \tag{3.11}$$

$$h = m \cdot U = r^2 \sin^2\theta \frac{d\phi}{d\tau}. \tag{3.12}$$

Note that $U^a \nabla_a U_b = 0$ defines an affinely parametrized geodesic, and Killing vectors obey $\nabla_a K_b \nabla_b K_a = 0$. These two facts are enough to prove the existence of a conserved charge:

$$U^c \nabla_C (U^a K_a) = (U^c \nabla_c U^a) K_a + U^c U^a \nabla_C K_a. \tag{3.13}$$

But this first term vanishes by the definition of an affinely parametrized geodesic, and the second can be symmetrized since U^c and U^a commute, so the second term vanishes by Killing's equation. Thus $\nabla_U (U^a K_a) = 0$.

For timelike particles, if τ is the proper time, then E has the interpretation of energy per unit mass, and h is the angular momentum per unit mass. For null geodesics, the quantity

$$b = \left|\frac{h}{E}\right|$$

is the physical impact factor.

Note that there is a third conserved charge– it is the value $U^a U_a = \pm 1, 0$ depending on if the geodesic is spacelike, timelike, or null.

We can write an action

$$S = \int d\tau \dot{x}^a \dot{x}^b g_{ab}. \tag{3.14}$$

which naturally gives us a Lagrangian that we can apply our conserved charges to and calculate Euler-Lagrange equations for. In Schwarzschild, the action takes the form

$$S = \int d\tau \Big[g_{tt}\dot{t}^2 + g_{rr}\dot{r}^2 + \dot{\theta}^2 r^2 + + \dot{\theta}^2 r^2 + r^2 \sin^2\theta \dot{\phi}^2 \Big]. \tag{3.15}$$

Note also that

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \implies r^2 \frac{d}{d\tau}(r^2\dot{\theta}) - \frac{\cos\theta}{\sin^3\theta} h^2 = 0. \tag{3.16}$$

WLOG we can always set $\theta(0) = \pi/2$ and $\dot{\theta}(0) = 0$, which means that $\ddot{\theta} = 0 \implies \theta(\tau) = \pi/2$ for all τ.

Substituting in our conserved quantities, we have only one equation which we need to consider: $\dot{r}$. For a geodesic such that

$$g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -\tilde{v}, \tag{3.17}$$

we have

$$\dot{r}^2/2 + V(r) = E^2/2, \ \text{with} V(r) = \frac{1}{2}\left(\tilde{v} + \frac{h^2}{r^2}\right)\left(1 - \frac{2M}{r}\right). \tag{3.18}$$

What are the radial free-fall paths of Schwarzschild? For radial trajectories, $h = 0$, so that

$$E = 1 \implies \dot{t} = \frac{1}{1 - \frac{2M}{r}}, \dot{r} = \pm 1. \tag{3.19}$$

For the upper sign, $\dot{r} = +1$, we have $\dot{t}/\dot{r} > 0, r > 2M$, which represents outgoing trajectories. For the lower sign, $\dot{t}/\dot{r} < 0, r > 2M$ (ingoing). In any case, from $\dot{r} = -1$ we see that we can reach $r = 2M$ without a problem.[1]

In Schwarzschild, we can then write

$$\frac{dt}{dr} = \pm \left(1 - \frac{2M}{r}\right)^{-1}. \tag{3.20}$$

We define Regge-Wheeler coordinates (AKA tortoise coordinates) as follows:

$$dr_* = \frac{dr}{1 - \frac{2M}{r}} \implies r_* = r + 2M \log\left|\frac{r}{2M} - 1\right|. \tag{3.21}$$

We will show that in classical gravity, we can indeed cross the horizon in finite proper time.

Lecture 4.

Thursday, January 24, 2019

"This [tidal force divergence] is related to something that happens in AdS/CFT. ... don't tell anyone I mentioned that." –Jorge Santos

Quick announcement– office hours from this course will be held at 14:00 on Tuesdays. If you plan to attend the office hours, however, do send an email in advance.

Last time, we started looking at the geodesics of Schwarzschild. We found two particularly simple ones: the ingoing and outgoing radial null geodesics, with equation

$$\frac{dt}{dr} = \pm \left(1 - \frac{2M}{r}\right)^{-1}, r > 2M. \tag{4.1}$$

We also defined the tortoise (Regge-Wheeler) coordinate r_* such that

$$dr_* = \frac{dr}{1 - \frac{2M}{r}}. \tag{4.2}$$

Thus our null geodesic equation becomes

$$\frac{dt}{dr_*} = \pm 1. \tag{4.3}$$

In tortoise coordinates, null geodesics therefore obey

$$t = \pm r_* + \text{constant}. \tag{4.4}$$

This seems kind of trivial. But now we introduce a new coordinate, the *ingoing Eddington-Finkelstein coordinate*, defining

$$v \equiv t + r_*. \tag{4.5}$$

This is clearly constant on ingoing geodesics, just by looking at the previous equation. That is, $dv|_{\text{geodesic}} = dt + dr_* = 0$. Good. (We could have done the same for outgoing geodesics taking $u \equiv t - r_*$.) Now we eliminate the original coordinate time t from the line element using $dt = dv - \frac{dr}{1 - \frac{2M}{r}}$. We get the new line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2 dv dr + r^2 d\Omega_2^2. \tag{4.6}$$

[1] At least from a pure gravity perspective. Once we introduce quantum mechanical effects, we have to worry about firewalls and such, and all bets are off.

Let's write this in matrix notation. Nothing fancy.

$$g_{\mu\nu} = \begin{pmatrix} -1+\frac{2M}{r} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}. \tag{4.7}$$

We haven't done anything too extreme– just made a change of coordinates. But we see something very nice– none of the metric components are singular at $r = 2M$. In fact, the determinant of the metric is still perfectly nice at $r = 2M$– by an explicit computation, $\det g = -r^2 \sin^2\theta$. This only vanishes at $\theta = 0$ (the regular coordinate badness of spherical coordinates) and at $r = 0$, which may (a priori, we don't know yet) be a real problem.

So we have found some coordinates which appear to extend r not just from $r > 2M$ but to all $r > 0$. Our metric is real and analytic so it is a nice analytic continuation of the old (bad) Schwarzschild coordinates. This is related to the problem of extendibility– are there other metrics which cover more of the spacetime manifold which are compatible with the solution that we've found?

However, something really bad does happen as $r \to 0$. The Kretchmann scalar $R^{abcd}R_{abcd} = \frac{48M^2}{r^6} \to \infty$ as $r \to 0$, and scalars by definition are invariant under coordinate transformations. So we cannot get rid of this by a simple redefinition of coordinates.

Moreover, let us observe that $\frac{\partial}{\partial t} = \frac{\partial}{\partial v}$. Thus our Killing vector field becomes

$$K = \frac{\partial}{\partial v}, \quad K^2 = -\left(1 - \frac{2M}{r}\right). \tag{4.8}$$

So our metric appears to be no longer static or stationary in these coordinates.

We now introduce the Finkelstein diagram. Recall that for $r = 2M$, on outgoing geodesics we have $t - r_* =$constant $\implies v = 2r + 4M \log\left|\frac{r}{2M} - 1\right|$+constant. Let us draw a plot in the $t_* \equiv v - r$,r plane. What we see is that the ingoing geodesics follow 45° paths, while the outgoing geodesics follow some curved paths in these coordinates.

What we observe is that in these coordinates, the light cones "tip over" at $r = 2M$– even the "outgoing" null geodesics are forced to proceed towards $r = 0$ for $r < 2M$. Now, this does not yet prove that this is a black hole. We've only examined radial geodesic trajectories, and it's not clear that adding even a little bit of angular momentum can't save us from a spaghettified death by black hole-like object.

Suppose that we[2] sit on the surface of a collapsing star. What we can show is that the time to singularity is $\Delta t = \mathbb{T}M$, where $M = M_\odot \implies \Delta t = 10 \times 10^{-5}$ s. It's a weird fact that we can indeed cross the horizon and hit the singularity in finite time, and yet because the event horizon is a surface of infinite redshift, a far-away observer will never see us actually cross the event horizon.

Black hole region How do we define a black hole? Before we can do that, we'll need some preliminaries.

Definition 4.9. A vector is *causal* if it is null or timelike and nonzero. A curve is causal if its tangent vector is everywhere causal.

Definition 4.10. A spacetime is *time-orientable* if it admits a time orientation, i.e. a causal vector field T^a. Another causal vector field x^a is then *future-directed* if it lies in the same light cone as T^a and is past-directed otherwise (i.e. $x^a T_a \leq 0$).

Note that the old coordinate time t in Schwarzschild was really a bad choice inside the event horizon. This is because $\frac{\partial}{\partial t}$ becomes spacelike inside the horizon, related to the change of sign of g_{tt} for $r < 2M$. So let us take $\pm\frac{\partial}{\partial r}$, and note that in our EF coordinates, $g_{rr} = 0$. Therefore $\frac{\partial}{\partial r}\frac{\partial}{\partial r} = 0$, meaning that we've found a vector which is null everywhere.

In fact, recall our timelike Killing vector outside, K. This gave us a good sense of time outside. We see now that

$$K \cdot \left(-\frac{\partial}{\partial r}\right) = -g_{vr} = -1,$$

[2] "Imagine that you, not me, are situated at the surface of this star. I don't want to go there." – Jorge Santos

where $K \equiv \frac{\partial}{\partial v}$. Therefore we've found a timelike coordinate which is good everywhere and defines a good time orientation.

Proposition 4.11. *Let $x^\mu(\lambda)$ be any future-directed causal curve, i.e. one whose tangent vector is everywhere future-directed and causal. Assume $r(\lambda_0) \leq 2M$. Then $r(\lambda) \leq 2M$ for any $\lambda \geq \lambda_0$.*

Let us define the tangent vector $V^\mu = \frac{dx^\mu}{d\lambda}$. Since $\left(-\frac{\partial}{\partial r}\right)$ and V^a are both future-directed, we have

$$0 \geq \left(-\frac{\partial}{\partial r}\right) V = -g_{r\mu} V^\mu = -V^v = -\frac{dv}{d\lambda} \implies \frac{dv}{d\lambda} \geq 0. \tag{4.12}$$

Now

$$V^2 \equiv V^a V_a = -\left(1 - \frac{2M}{r}\right)\left(\frac{dv}{d\lambda}\right)^2 + 2(dvd\lambda)\left(\frac{dr}{d\lambda}\right) + r^2 \left(\frac{d\Omega}{d\lambda}\right)^2, \tag{4.13}$$

where $\left(\frac{d\Omega}{d\lambda}\right)^2 \equiv \left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta \left(\frac{d\phi}{d\lambda}\right)^2$. Then

$$-2\left(\frac{dv}{d\lambda}\right)\left(\frac{dr}{d\lambda}\right) = -V^2 + \left(\frac{2M}{r} - 1\right)\left(\frac{dv}{d\lambda}\right)^2 + r^2\left(\frac{d\Omega}{d\lambda}\right)^2. \tag{4.14}$$

But if V is causal and $r \leq 2M$, then the right side of 4.14 is non-negative, so it follows that

$$\frac{dv}{d\lambda}\frac{dr}{d\lambda} \leq 0. \tag{4.15}$$

Let us assume that $\frac{dr}{d\lambda} > 0$ (our curve at any point is directed towards larger r). Then $\frac{dv}{d\lambda} = 0$, which means that by 4.14, $V^2 = 0$ and $\frac{d\Omega}{d\lambda} = 0$. But then the only nonvanishing component of V is $V^r = \frac{dr}{d\lambda} > 0 \implies V$ is a positive multiple of $\frac{\partial}{\partial r}$, and hence is past-directed. We have reached a contradiction.

Therefore $\frac{dr}{d\lambda} \leq 0$ if $r \leq 2M$. If $r < 2M$, the equality must be strict. If $\frac{dr}{d\lambda} = 0$ then by 4.14, $\frac{d\Omega}{d\lambda} = \frac{dv}{d\lambda} = 0 \implies V^\mu = 0$. Hence if $r(\lambda_0) < 2M$, then $r(\lambda)$ is monotonically decreasing for all $\lambda \geq \lambda_0$.

Lecture 5.

Friday, January 25, 2019

"[The white hole region is] like a norovirus. Nothing can stay in. Everything has to come out." –Jorge Santos

Last time, we made the proposition that the Schwarzschild spacetime really does feature a black hole region, i.e. a region where any future-directed causal curves with $r(\lambda) \leq 2M$ are forced to have $r(\lambda) \leq 2M \forall \lambda \geq \lambda_0$. We proved last time that if you are inside the horizon, $r(\lambda_0) < 2M$, the $r(\lambda)$ is monotonically decreasing for $\lambda \geq \lambda_0$.

Formally, we must consider the case $r(\lambda_0) = 2M$. Note that if $\frac{dr}{d\lambda} < 0|_{\lambda=\lambda_0}$, then we're done. In the next ϵ time later, we'll be inside the horizon and our proof holds. If $\frac{dr}{d\lambda} = 0$, then we sit at $r = 2M$ forever and we're also done.

So there's only one case we have to consider, $dr/d\lambda > 0$ at $\lambda = \lambda_0$.

We've seen that

$$-2\left(\frac{dv}{d\lambda}\right)\left(\frac{dr}{d\lambda}\right) = -V^2 + \left(\frac{2M}{r} - 1\right)\left(\frac{dv}{d\lambda}\right)^2 + r^2\left(\frac{d\Omega}{d\lambda}\right)^2.$$

(4.14 from last time). At $\lambda = \lambda_0$, this vanishes, so $\frac{d\Omega}{d\lambda} = V^2 = 0$. This means that $\frac{dv}{d\lambda} \neq 0$, or else $V^\mu = 0$ (which is a contradiction). Then

$$\frac{dv}{d\lambda}|_{\lambda=\lambda_0} > 0,$$

since we proved last time that $\frac{dv}{d\lambda} \geq 0$.

Hence at least near $|lambda = \lambda_0$, we can use v instead of λ as a parameter along the curve with $r = 2M$ at $v = v_0 = v(\lambda_0)$. Dividing 4.14 by $\left(\frac{dv}{d\lambda}\right)^2$ gives

$$-2\frac{dr}{dv} \geq \frac{2M}{r} - 1 \implies 2\frac{dr}{dv} \leq 1 - \frac{2M}{r}. \tag{5.1}$$

Hence for v_2, v_1 greater than v_0 with $v_2 > v_1$, we have

$$2\int_{r(v_1)}^{r(v_2)} \frac{dr}{1-\frac{2M}{r}} \leq v_2 - v_1. \tag{5.2}$$

This completes the $r = 2M$ case. ☒

Technically, this doesn't prove that we have a black hole because it is a local statement. To establish a global event horizon will take more work.

Detecting black holes Note the following facts.

(a) There is no upper bound on the mass of a black hole.
(b) Black holes are very small. For instance, a black hole with the mass of the Earth would have a radius of 0.9 cm.

We observe black holes by noticing their gravitational pull on stars, for instance, and considering their incredible compactness. Fun fact– there are massive and supermassive (billions of solar mass) black holes in the universe which we have detected astrophysically, and we have no idea where they came from. It seems like there isn't enough time in the age of the universe for them to have formed.

Orbits around a black hole Consider a timelike geodesic around a black hole. The turning points of the potential are given by points $\dot{r} = 0$ in the potential

$$V(r) = \frac{E^2}{2} = \frac{1}{2}\left(\tilde{v} + \frac{h^2}{r^2}\right)\left(1 - \frac{2M}{r}\right), \tag{5.3}$$

and where

$$V'(r) = 0 \implies r_\pm = \frac{h^2 \pm \sqrt{h^4 - 12h^2\tilde{v}M^2}}{2M\tilde{v}}, \tilde{v} = 1. \tag{5.4}$$

If $h^2 < 12m$, then we are in free-fall– there are no turning points. However, if $h^2 > 12M^2$, then there is a local minimum of the potential at r_+, and a local maximum at r_-. One can show that

$$3M < r_- < 6M < r_+, \tag{5.5}$$

where $r = 6M$ is known as the ISCO (innermost stable circular orbit). There is no Newtonian analogue to this– $r = 6M$ lies well within the star.

The energy of the orbit is then

$$E_\pm = \frac{r_\pm - 2M}{r_\pm^{1/2}(r_\pm - 3M)^{1/2}}, \tag{5.6}$$

and for $r_+ \gg M$, we find that

$$E_+ \approx 1 - \frac{M}{2r_+} \to m - \frac{Mm}{(2r_+,)} \tag{5.7}$$

tells us that the energy at this orbit is the relativistic mass-energy $E = m$ minus a correction for the gravitational energy.

Let us approximate the accretion disk of the black hole as non-interacting so that particles basically travel along geodesics. Orbits will radiate off energy, decreasing towards $r \to 0$. When a particle hits $r = 6M$, it will suddenly turn towards the singularity, releasing a burst of energy (cf. brehmsstrahlung). It is these bursts of energy that we can detect in astrophysical systems.

White holes We looked at ingoing null geodesics in EF coordinates. What about outgoing (radial) null geodesics? We have the outgoing EF coordinate

$$u \equiv t - r_*, \tag{5.8}$$

which is constant along radial outgoing null geodesics. Here, $\frac{dt}{dr_*} = 1$. Then the metric takes the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right)du^2 - 2du dr + r^2 d\Omega_2^2. \tag{5.9}$$

Importantly, this is *not* the same region that we found before! We have a new $r < 2M$ region, but outgoing null geodesics are forced to leave this region. However, we just spent all that time proving that geodesics inside the event horizon were trapped inside, so this cannot possibly be the same region as in the ingoing coordinates. For constant u (outgoing null radial geodesics), the metric tells us that

$$\frac{dr}{d\tau} = 1. \tag{5.10}$$

So these can propagate from the curvature singularity at $r = 0$ through the surface $r = 2M$. Weak cosmic censorship says that naked singularities are forbidden in nature, and to define a white hole we must have an initial singularity, so this seems to be unphysical. Another argument for this involves time reversal– it has been proved in some generality that black holes are stable to perturbations, so after perturbation they will settle down to another black hole state. If we take the time reversal of this statement, this suggests that white holes will be highly unstable to perturbation, so they are probably not physical.

The Kruskal extension Can we get both the black and white hole regions in a single set of coordinates? Yes! We start in the exterior region $r > 2M$, and define *Kruskal-Szekeres* coordinates (U, V, θ, ϕ), defining

$$U = -e^{-u/4M}, \quad V = e^{v/4M} \tag{5.11}$$

in terms of the EF coordinates from before. So for $r > 2M, U < 0$ and $V > 0$.

Now we can directly compute

$$UV = -e^{\frac{+r}{2M}}\left(\frac{r}{2M} - 1\right). \tag{5.12}$$

Let us observe that the RHS of this equation is a monotonic function of r, and hence determines $r(U, V)$ uniquely. We also have

$$\frac{V}{U} = -e^{t/2M}, \tag{5.13}$$

which determines $t(U, V)$.

Computing the differentials, we find that

$$dU = \frac{1}{4M}e^{-u/4M}du, \quad dV = \frac{1}{4M}e^{v/4M}dv. \tag{5.14}$$

Therefore

$$\begin{aligned}
dUdV &= \frac{1}{16M^2}\exp\left(\frac{v-u}{4M}\right)dudv \\
&= \frac{1}{16M^2}\exp\left(\frac{r_*}{2M}\right)(dt^2 - dr_*^2) \\
&= \frac{1}{16M^2}\exp\left(\frac{r_*}{2M}\right)\left[dt^2 - \frac{dr^2}{\left(1-\frac{2M}{r}\right)^2}\right].
\end{aligned}$$

This is almost what we want– now we just multiply by the appropriate factors to find that our new line element is

$$ds^2 = -32M^3\frac{\exp\left(-\frac{r(U,V)}{2M}\right)}{r(U,V)}dUdV + r^2(U,V)d\Omega_2^2. \tag{5.15}$$

Lecture 6.

Monday, January 28, 2019

"What about the bifurcation sphere?" "It will be the throat of an Einstein-Rosen bridge. Those of you who watched Thor. . . this will be familiar." –a student and Jorge Santos

Last time, we tried to extend the Schwarzschild spacetime. Our first attempt took us inside the black hole with the EF coordinates, and we then defined the Kruskal extension. Defining Kruskal coordinates U and V, we observe that

$$UV = -e^{\frac{+r}{2M}}\left(\frac{r}{2M} - 1\right), \quad \frac{V}{U} = -e^{t/2M}. \tag{6.1}$$

Note that $U < 0$ and $V > 0$ for $r > 2M$ in Schwarzschild coordinates. But here's another surprise– if we switch the signs of U *and* V, we can still sensibly define $r(U, V)$ for $U \geq 0, V \leq 0$ via 6.1.

This is an entirely new region of spacetime, which is isometric to our black hole exterior region with one notable caveat– time runs backwards. The surface $R = 2M$ is actually achieved by either taking $U = 0$ or $V = 0$. These lines intersect at $U = V = 0$, and note that at $r = 0$, we have $UV = +1$.

If we're interested in AdS/CFT, we should take all four regions seriously. However, if we're interested in astrophysical black holes, then most of the diagram is really the interior of the star as it proceeds to collapse along a geodesic.

Now recall that we have a Killing vector

$$K = \frac{\partial}{\partial t} = \frac{1}{4M}\left(V\frac{\partial}{\partial v} - U\frac{\partial}{\partial U}\right). \tag{6.2}$$

What happens at the point $U = V = 0$?

The Einstein-Rosen bridge Let us make the following coordinate transformation. Define the coordinate ρ by

$$r = \rho + M + \frac{M^2}{4\rho} \tag{6.3}$$

so that as $\rho \to +\infty, r \to +\infty$. Naturally this is quadratic(ish) in ρ so we'll get two solutions for each value of r. We choose $\rho > M/2$ in region I, the exterior region, and $0 < \rho < M/2$ in region IV. Then our metric becomes

$$ds^2 = -\left[\frac{1-\frac{M}{2\rho}}{1+\frac{M}{2\rho}}\right]^2 dt^2 + \left(1+\frac{M}{2\rho}\right)^4 \underbrace{(d\rho^2 + \rho^2 d\Omega_2^2)}_{\mathbb{R}^3} \tag{6.4}$$

So we've put the metric in a form where the spatial part looks like $\mathbb{R}^3$ up to a scaling factor. It's an exercise to check that the transformation $\rho \to M^2/4\rho$ leaves this metric unchanged.

Now what does a line of constant t look like?

$$ds^2_{\Sigma_t} = \left(1+\frac{M}{2\rho}\right)^4 \underbrace{(d\rho^2 + \rho^2 d\Omega_2^2)}_{\mathbb{R}^3}, \tag{6.5}$$

As $\rho \to +\infty$, we get one asymptotically flat region, and as $\rho \to 0$ we get another asymptotically flat region. Connecting them we get a wormhole where there is an S^2 at $\rho = M/2$ with some minimum (nonzero) radius.

Now, note that from our Kruskal diagram, this wormhole is non-traversable. In order to get to region IV, we must enter the black hole region. And once we're in the black hole region (region II), we are bound to hit the singularity. Tough luck.

Extendability and singularities

Definition 6.6. A spacetime $(\mathcal{M}, g)$ is *extendible* if it is isometric to a proper subset of another spacetime $(\mathcal{M}', g)$. The latter is called an *extension* of $(\mathcal{M}, g)$.

Thus the Kruskal spacetime is an extension of the Schwarzschild spacetime, and moreover it is the maximal analytic extension of Schwarzschild.

Let's talk a bit about singularities now. There are different sorts of singularities we might be interested in.

- Coordinate singularities: the metric (or determinant) is not smooth in some coordinate chart. Nothing physically bad has (necessarily) happened, but we just chose bad coordinates.
- Curvature singularities: a curvature scalar becomes singular. These are physically significant, since we cannot define them away by a coordinate choice.
- Conical singularities: consider the line element

$$g = dr^2 + \lambda^2 r^2 d\phi^2, \tag{6.7}$$

with $\lambda > 0$. We can always make this look like $dr^2 + r^2 d\tilde{\phi}^2$, which looks like $\mathbb{R}^2$. But suppose $\lambda \neq 1$. Then the period of our new $\tilde{\phi} = \lambda\phi$ coordinate is not 2π. If we take a "circle" of radius ϵ, we see that $\frac{\textit{circumference}}{\text{radius}} = \frac{2\pi\lambda\epsilon}{\epsilon} = 2\pi\lambda$, which does not go to 2π as $\epsilon \to 0$. What results is a conical singularity at the origin.

- There are certain metrics such that the components of the Riemann tensor are singular in every coordinate system, which means that the tidal forces become infinite as you approach the singularity. We will see an example of this on the examples sheet.

Definition 6.8. A *curve* is a smooth map from some interval of the real line to our manifold, $\gamma : (a, b) \to \mathcal{M}$, with $a, b \in \mathbb{R}$.

Definition 6.9. A point $p \in \mathcal{M}$ is a *future endpoint* of a future-directed causal curve if for any neighborhood $\mathcal{O}$ of p, there exists t_0 such that $\gamma(t) \in \mathcal{O}$ for all $t > t_0$. That is, our curve gets trapped arbitrarily close to p at late times.

Definition 6.10. We say that γ is future-inextendible if it has no future endpoints. Equivalent notions of past endpoints and past-inextendible can be defined by replacing future with past, and we say a curve is inextendible if it is future-inextendible and past-inextendible.

Example 6.11. Let $\gamma : (-\infty, 0) \to \mathcal{M}$, with $\gamma(t) = (t, 0, 0, 0)$. Clearly, this has a future endpoint at $(0, 0, 0, 0)$. If we remove that point from the manifold, however, there is no future endpoint– no other point on our manifold will satisfy the future endpoint condition since we can always take γ at some time arbitrarily close to zero. So the notion of extendibility depends crucially on our choice of manifold.

Definition 6.12. A *geodesic* is complete if an affine parameter for the geodesic exists to $\pm\infty$. A spacetime is *geodesically complete* if all inextendible causal geodesics are complete. A spacetime is *singular* if if it is inextendible and geodesically incomplete.

Example 6.13. The Kruskal spacetime is inextendible, and it has plenty of geodesics which hit the $r = 0$ singularity and are therefore incomplete. Therefore by our definition, the Kruskal spacetime is singular.[3]

Lecture 7.

Wednesday, January 30, 2019

"I could draw here AdS, but I know most of you here are allergic to that." –Jorge Santos

Today, we will introduce the initial value problem in general relativity. In classical physics, it's natural to say that we set initial conditions and (using some differential equations) evolve them to the future. But in GR, the question is more complicated because we are trying to evolve things to "the future," yet the future is part of what we are solving for! It will take some care to define precisely what we mean.

Definition 7.1. Let $(\mathcal{M}, g)$ be a time-orientable spacetime. A *partial Cauchy surface* Σ is a hypersurface for which no two points $p, p' \in \Sigma$ are connected by a causal curve in $\mathcal{M}$.

Definition 7.2. The *future domain of dependence* of Σ, denoted $D^+(\Sigma)$, is the set $p \in \mathcal{M}$ such that every past-inextendible causal curve through p intersects Σ. The past domain of dependence $D^-(\Sigma)$ is defined equivalently for future-inextendible causal curves, and the domain of dependence is then $D(\Sigma) = D^+(\Sigma) \cup D^-(\Sigma)$.

Example 7.3. Consider Minkowski space, with a partial Cauchy surface $\Sigma : t = 0, x > 0$. Then the domain of dependence $D(\Sigma)$ is the 45-degrees region with $x \geq |t|$.

Definition 7.4. As far as we (as physicists) are concerned, a *hyperbolic partial differential equation* (second order) is defined to be a differential equation of the form

$$g^{ef} \nabla_e \nabla_f T^{ab\ldots}{}_{cd\ldots} = \tilde{G}^{ab\ldots}{}_{cd}, \tag{7.5}$$

where the tensor we've put on the RHS depends only on T and first derivatives of T in some smooth way. That is, the second derivative is the highest derivative, and it appears only linearly. These are sometimes called quasilinear equations.

[3] In fact, incomplete geodesics are a sufficient but not necessary condition for a spacetime to be singular. We should also in principle be concerned about accelerating observers falling off the edge of spacetime into a singularity– cf. Hawking and Ellis, and also work by Geroch. According to Hawking and Ellis, the more appropriate notion is not geodesic completeness but a more general idea of "b-completeness," completeness of general inextendible causal curves in the spacetime.

Definition 7.6. A *Cauchy surface* for a spacetime $(\mathcal{M}, g)$ is a partial Cauchy surface Σ with $D(\Sigma) = \mathcal{M}$ the entire manifold. Spacetimes $(\mathcal{M}, g)$ which admit Cauchy surfaces are called *globally hyperbolic*.

Note that if $D(\Sigma) \neq \mathcal{M}$, then the solution of hyperbolic equations will *not* be uniquely specified on $\mathcal{M} \setminus D(\Sigma)$ by data on Σ.

Example 7.7. A very simple example of a spacetime that is not globally hyperbolic is $\mathbb{R}^{1,1} \setminus 0$. There are points where a past-directed inextendible causal curve can drop off the manifold (i.e. hit the point we have removed).

Theorem 7.8 (Wald)**.** *Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime. Then*

(i) *There exists a global-time function $t : \mathcal{M} \to \mathbb{R}$ such that $-(dt)^a$ (the normal to surfaces of constant t) is future-directed and timelike.*

(ii) *Surfaces of constant t, Σ_t, are Cauchy surfaces and all have the same topology.*

(iii) *The topology of $\mathcal{M}$ is $\mathbb{R} \times \Sigma$.*

Remark: spacetimes which have singularities can still be globally hyperbolic. Consider the Kruskal spacetime– we can set initial conditions on the Cauchy surface $U + V =$ constant.

Extrinsic curvature Let Σ denote a spacelike or timelike hypersurface with unit normal n_a (i.e. such that $n^a n_a = \pm 1$).

Lemma 7.9. *For any $p \in \Sigma$, let $h^a{}_b = \delta^a{}_b \mp n^a n_b$ so that $h^a{}_b n^b = 0$ (where upper/lower signs apply for spacelike/timelike respectively). Then*

(a) $h^a{}_c h^c b = h^a b$

(b) *Any vector x^a at p can be written as $x^a_\parallel + x^a_\perp$, where $x^a_\parallel = h^a{}_b x^b$ and $x^a_\perp = \pm m_b x^b m^a$.*

(c) *If X^a, Y^a are tangent to Σ, then $h^{ab} X_a Y_b = g^{ab} X_a Y_b$. h is sometimes called the first fundamental form.*

That is, let N^a be a normal to Σ at p. If we parallel transport N^a on Σ along X^a (that is, $X^b \nabla_b N_a = 0$), will the parallel-transported N^a still be normal? Take Y tangent to Σ, so that $Y^a N_a = 0$ at the point p. Then

$$\nabla_X (Y^a N_a) = X^b \nabla_b (Y^a N_a) = N_a X^b \nabla_b Y^a. \tag{7.10}$$

Definition 7.11. Up to now, n_a has been defined only on Σ. First, let us extend it to a neighborhood of Σ in an arbitrary way. The *extrinsic curvature* (second fundamental form) is defined at $p \in \Sigma$ by

$$K(X, Y) = -n_a (\nabla_{X_\parallel} Y_\parallel)^a. \tag{7.12}$$

Here, X and Y need not be tangent to Σ, but we are interested in their projections parallel to Σ.

Lemma 7.13. *K_{ab} is independent of how n_a is extended off Σ, and in particular*

$$K_{ab} = h_a{}^c h_b{}^d \nabla_c n_d. \tag{7.14}$$

Proof. The RHS of $K(X, Y)$ is

$$-n_d X^c_\parallel \nabla_c Y^d_\parallel = X^c_\parallel Y^d \nabla_c n_d \tag{7.15}$$

by the Leibniz rule, since $n_d Y^d_\parallel = 0$. So

$$K(X, Y) = X^c_\parallel Y^d_\parallel \nabla_c n_d = X^a Y^b h_a{}^c h_b{}^d \nabla_c n_d. \tag{7.16}$$

Therefore

$$K_{ab} = h_a{}^c h_b{}^d \nabla_c n_d. \tag{7.17}$$

To demonstrate that K_{ab} is independent of how n_a is extended, consider a different extension n'_a, and let $m_a \equiv n'_a - n_a$. Note that on $\Sigma, m_a = 0$. Then on Σ,

$$\begin{aligned} X^a Y^b (K'_{ab} - K_{ab}) &= X^c_\parallel Y^d_\parallel \nabla_c m_d \\ &= \nabla_{x_\parallel} (Y^d_\parallel m_d) = 0 \end{aligned}$$

since $Y^d_\parallel m_d = 0$ on Σ. ⊠

We can use

$$n^b \nabla_c n_b = \frac{1}{2} \nabla_c (n_b n^b) = 0 \tag{7.18}$$

and conclude that

$$K_{ab} = h_a{}^c \nabla_c n_b. \tag{7.19}$$

At home, we should show that the extrinsic curvature is given by a Lie derivative,

$$K_{ab} = \frac{1}{2} \mathcal{L}_m (h_{ab}) \tag{7.20}$$

where h here is the first fundamental form.

Lecture 8.

Thursday, January 31, 2019

"I know that you all really hate my handwriting. And our equations have hundreds of indices. So today I will receive threatening emails by the end of the day." –Jorge Santos

Last time, we introduced the extrinsic curvature,

$$K_{ab} h_a{}^c h_b{}^d \nabla_c n_d, \tag{8.1}$$

where $n^a n_a = \pm 1$, with n^a the unit normal to the hypersurface Σ. We also define the projection

$$h_{ab} = g_{ab} \mp n_a n_b. \tag{8.2}$$

Lemma 8.3. *$K_{ab} = K_{ab}$, so K is a symmetric 2-tensor.*

Proof. Let $f : \mathcal{M} \to \mathbb{R}$ be constant on Σ with $df \neq 0$ on Σ. Let X^a be tangent to Σ. Thus

$$X(f) = X^a \nabla_a f = 0. \tag{8.4}$$

This implies that $(df)^a$ is normal to Σ. Thus on Σ we can write

$$n_a = \alpha (df)_a, \tag{8.5}$$

where α is chosen such that $n_a n^a = \pm 1$. It follows that

$$\nabla_c n_d = \alpha \nabla_c \nabla_d f + \frac{(\nabla_c \alpha)}{\alpha} n_d. \tag{8.6}$$

But this tells us that

$$K_{ab} = h_a{}^c h_b{}^d \nabla_c n_d = \alpha h_a{}^c h_b{}^d \nabla_c \nabla_d f, \tag{8.7}$$

and for all torsion-free spacetimes, covariant derivatives commute on scalars, so K_{ab} is symmetric. $\boxtimes$

As it turns out, the property that

$$K_{ab} = \frac{1}{2} (\mathcal{L}_m h)_{ab} \tag{8.8}$$

means that the extrinsic curvature has the right form for an initial condition in the initial value problem of general relativity.

The Gauss-Coducci equation A tensor at a point $p \in \Sigma$ is invarinat under projection $h_a{}^b$ if

$$T^{a_1 \dots a_n}{}_{b_1 \dots b_s} = h_{c_1}{}^{a_1} \dots h_{c_n}{}^{a_n} h_{b_1}{}^{d_1} \dots h_{b_s}{}^{d_s} T^{c_1 \dots c_n}{}_{d_1} \dots {}_{d_s}. \tag{8.9}$$

Proposition 8.10. *A covariant derivative D on Σ can be identified by projection of the covariant derivative on $\mathcal{M}$. Thus*

$$D_a \Gamma^{b_1 \dots b_n}{}_{c_1 \dots c_s} = h_a{}^d h_{e_1}{}^{b_1} \dots h_{e_n}{}^{b_n} h_{c_1}{}^{f_1} \dots h_{c_s}{}^{f_s} \nabla_d T^{e_1 \dots e_n}{}_{f_1 \dots f_s}. \tag{8.11}$$

Lemma 8.12. *The covariant derivative D is precisely the Levi-Civita (metric) connection associated to the metric h_{ab} on the submanifold Σ:*

$$D_a h_{bc} = 0, \tag{8.13}$$

and D is torsion-free.

Proof. Let's expand out the following derivative:

$$\nabla_a h_{bc} = \mp n_c \nabla_a n_b \mp n_b \nabla_a n_c. \tag{8.14}$$

But recall that $h_a{}^c n_c = 0$, so $D_a h_{bc} = 0$.

To prove it is torsion-free, let $f : \Sigma \to \mathbb{R}$, and extend to a function $f : \mathcal{M} \to \mathbb{R}$. Then consider $D_a D_b f$. This is

$$\begin{aligned} D_a D_b f &= h_a{}^c h_b{}^d \nabla_c (h_d{}^e \nabla_e f) \\ &= h_a{}^c h_b{}^e \nabla_c \nabla_e f + h_a{}^c h_b{}^d \nabla_c h^e{}_d \nabla_e f. \end{aligned}$$

This first term is already manifestly symmetric in a and b since the original connection was torsion free, so let us rewrite the second term as follows:

$$\begin{aligned} h_a{}^c h_b{}^d \nabla_c h_d{}^e &= g^{ef} h_a{}^c h_b{}^d \nabla_c h_{df} \\ &= \mp g^{ef} h_a{}^c h_b{}^d n_f \nabla_c n_d \\ &= \mp n^e K_{ab}, \end{aligned}$$

where we have grouped together $h_a{}^c h_b{}^d \nabla_c n_d = K_{ab}$. We conclude that

$$D_{[a} D_{b]} f = 0. \tag{8.15}$$

⊠

We'd like to relate the extrinsic curvature to our old-fashioned Riemann curvature. How do we do this?

Proposition 8.16. *Denote the Riemann tensor associated with D_a on Σ by $R'^a{}_{bcd}$. This is given by Gauss's equation:*

$$R'^a{}_{bcd} = h^a{}_e h_b{}^f h_c{}^g h_d{}^h R^e{}_{fgh} \pm 2K_{[c}{}^a K_{d]b}. \tag{8.17}$$

Proof. Let x^a be tangent to Σ. Then the Ricci identity for D is

$$R'^a{}_{bcd} X^b = 2D_{[c} D_{d]} X^a. \tag{8.18}$$

Let us compute the RHS of this equation.

$$\begin{aligned} D_c D_d X^a &= h_c{}^e h_d{}^f h_g{}^a \nabla_e (D_f X^g) && (8.19) \\ &= h_c{}^e h^f{}_d h^a{}_g \nabla_e (h_f{}^h h_i{}^g \nabla_h X^i) && (8.20) \\ &= h_c{}^e h^f{}_d h^a{}_g \nabla_e \nabla_h X^i + h_c{}^e h^f{}_d h^a{}_g (\nabla_e h^h{}_f) \nabla_h X^i + h_c{}^e h^f{}_d h^a{}_g (\nabla_e h^g{}_i) \nabla_h X^i. && (8.21) \end{aligned}$$

This first term is already looking good. Note that we have to project the indices everywhere, so in the first line we project c, d, a indices, and in the second line we project f, g indices, and in the final line we expand out the covariant derivative.

We have seen that

$$h^c{}_d h^d{}_b \nabla_c h^e{}_d = \mp n^e K_{ab}. \tag{8.22}$$

We can use this identity in the last two terms of 8.21 to get

$$D_c D_d X^a = h^e{}_c h_d{}^h h^a{}_i \nabla_e \nabla_h X^i \mp K_{cd} h^a{}_i n^h \nabla_h X^i \mp K_c{}^a n_a h^h{}_d \nabla_h X^i. \tag{8.23}$$

This second term already drops out since K_{cd} is symmetric and we're antisymmetrizing over c and d in the final expression. The last term can be recast as

$$K_c{}^a h^h{}_d \nabla_h (n_i X^i) \pm K^a{}_c X^i h^h{}_d \nabla_h n_i = \pm K_c{}^a K_{bd} X^b. \tag{8.24}$$

Antisymmetrizing, we find that

$$\begin{aligned} R'^a{}_{bcd} X^b &= 2h_{[c}{}^e h_{d]}{}^f h^a{}_g \nabla_e \nabla_f X^g \pm 2K_{[c}{}^a K_{d]b} X^b \\ &= h_c{}^e h_d{}^f h_g{}^a h^a{}_b R^g{}_{hef} x^b \pm 2K_{[c}{}^a K_{a]b} X^b. \end{aligned}$$

Since X^b is arbitrary, this holds as a tensor identity. ⊠

Lemma 8.25. *The Ricci scalar of Σ is*

$$R' = R \mp 2R_{ab} n^a n^b \pm K^2 \mp K^{ab} K_{ab}, \tag{8.26}$$

where $K = g^{ab} K_{ab}$, the trace of K_{ab}.

Proposition 8.27 (Codacci's equation)**.** $D_a K_{bc} - D_b K_{ac} = h_a{}^d h_b{}^e h_c{}^f n^a R_{defg}$.

The proof of Codacci's equation is homework but simple compared to Gauss's equation.

Lemma 8.28. *$D_a K^a{}_b - D_b K = h_b{}^c R_{cd} n^d$. This is sometimes referred to as the Codacci equation, though we will call it the contracted Codacci equation (if we refer to it by name at all).*

The constraint equations We now have several definitions and identities. What are they good for? Assume the hypersurface Σ is spacelike, with a timelike normal n^a. The Einstein equation is just

$$R_{ab} - \frac{R}{2} g_{ab} \equiv G_{ab} = 8\pi T_{ab}.$$

We're going to contract it with $n^a n^b$. Using the same notation as before, the Einstein equation takes the form

$$R' - K^{ab} K_{ab} + K^2 = 16\pi\rho. \tag{8.29}$$

A priori, we might have thought that we had two free choices for our initial conditions on Σ– we could have chosen a metric on Σ and also an extrinsic curvature (like a first derivative of the metric moving off of Σ). But in fact the choice is not entirely free.

The equation 8.29 is known as the *Hamiltonian constraint*. Contracting with n^a and projecting with h gives us instead

$$D_b K^b{}_a - D_a K = 8\pi h_a{}^b T_{bc} n^c, \tag{8.30}$$

which is sometimes called the momentum constraint.[4]

Theorem 8.31 (Choquet-Bruhat and Geroch (1969)**.** *Let (Σ, h, K) be initial data satisfying the vacuum Hamiltonian and momentum constraints ($T_{ab} = 0$). Then there exists a unique (up to diffeomorphism) spacetime $(\mathcal{M}, g_{ab})$ called the maximal Cauchy development of (Σ, h, K) such that*

(i) *$(\mathcal{M}, g)$ satisfies the Einstein equation,*
(ii) *$(\mathcal{M}, g)$ is globally hyperbolic with Cauchy surface Σ,*
(iii) *the induced metric and the extrinsic curvature of Σ are h and K, respectively,*
(iv) *and any other spacetime satsfying (i), (ii), and (iii) is isometric to a subset of $(\mathcal{M}, g)$.*

Analogous conditions exist for matter that obeys reasonable energy conditions. However, *it is possible that the maximal Cauchy development is extendible.* In that case, the region of the manifold in the complement of the maximal Cauchy development is not unique. Physics is not predictable outside the maximal Cauchy development, and the boundary of this region is called a *Cauchy horizon.* Note that in order to avoid trivial Cauchy horizons from poor choices of Σ, we require the initial data to be inextendible.

A final note. Look at the Schwarzschild solution with negative M, such that the line element is

$$ds^2 = -\left(1 + \frac{2|M|}{r}\right) dt^2 + \frac{dr^2}{1 + \frac{2|M|}{r}} + r^2 d\Omega_2^2. \tag{8.32}$$

But if we take any $t = 0$ surface, it is impossible for us to avoid the $r = 0$ singularity. Our initial data should not be singular! Look at outgoing geodesics,

$$\frac{dt}{dr} = \frac{1}{1 + \frac{2|M|}{r}} \implies t \simeq t_0 + \frac{r^2}{4|M|}. \tag{8.33}$$

This tells us that a Cauchy horizon has appeared, and this one we cannot avoid by picking a different Cauchy surface.

[4]It's diffeomorphism invariance that gives us this constraint, analogous to the constraint on trying to solve QED in Coulomb gauge.

Lecture 9.

Friday, February 1, 2019

"Look at this word, the most naughty word in mathematics. 'Generically.'"–Jorge Santos

Last time, we formulated the initial value problem in general relativity. We said that some initial data is bad– we require that it is at a minimum inextendible and geodesically complete. However, something else bad can happen. See the figure: if our initial data is prescribed on an asymptotically null surface, then it could be our domain of dependence is cut off by a light cone. Therefore, we requre that the initial data is also *asymptotically flat*. We haven't yet defined what this means, though.

Definition 9.1. An initial data set (Σ, h, K) has an asymptotically flat end if

(i) Σ is diffeomorphic to $\mathbb{R}^3 \setminus B$, where B is a closed ball centered on the origin in $\mathbb{R}^3$,
(ii) If we pull back the $\mathbb{R}^3$ coordinates to define coordinates x^i on Σ, then $h_{ij} = \delta_{ij} + O(1/r)$ (the Euclidean metric plus $1/r$ terms) and $K_{ij} = O(1/r^2)$ as $r \to +\infty$, where $r = \sqrt{x^i x_i}$.
(iii) Derivatives of the latter expression also hold, e.g.

$$\partial_k h_{ij} = O(1/r^2).$$

Definition 9.2. An initial data set is asymptotically flat with N ends if it is the union of a compact set with N asymptotically flat ends (e.g. the Kruskal spacetime has two asymptotically flat ends).

Having defined our wish list for an initial data set, it would be extremely disturbing if we made this data set as nice as we like, and yet ended up with an extendible spacetime as our maximal Cauchy development. That such spacetimes do not (generically) occur is the content of the strong cosmic censorship conjecture.

Theorem 9.3. *Let (Σ, h, K) be a geodesically complete, asymptotically flat, initial data set for the vacuum Einstein equations. Then generically the maximal Cauchy development if the initial data is inextendible.*

This has been proved (with much work) that strong cosmic censorship is true in asymptotically flat space (i.e. Minkowski space is stable to small perturbations). It has nearly been proved (by Dafermos et al) for the Kerr black hole. And violations are known for Reissner-Nordström solution in de Sitter space.

Singularity theorems Now, in Newtonian gravity the formation of singularities is generally avoidable. If we drop a bit of matter in a Newtonian potential, it will fall straight to $r = 0$. But if we add a bit of angular momentum, the angular momentum will prevent the matter from falling straight in, and so our chunk of stuff cannot actually hit the $r = 0$ singularity. This is *not* true in general relativity.

It is the focus of an amazing set of theorems originally due to Penrose, and later generalized in conjunction with Hawking,[5] that the formation of singularities is generic in general relativity given a set of very reasonable conditions.[6]

Definition 9.4. A *null hypersurface* is a hypersurface whose normal is everywhere null. For example, take the metric

$$g^{\mu\nu} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 - \frac{2M}{r} & 0 & 0 \\ 0 & 0 & 1/r^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{9.5}$$

The 1-form $n = dr$ is normal to the surfaces of constant Ω. Thus note that

$$n^2 = g^{\mu\nu} n_\mu n_\nu = g^{rr} = \left(1 - \frac{2M}{r}\right). \tag{9.6}$$

So the surface $r = 2M$ is a null hypersurface.

Now let n_a be normal to a null hypersurface $\mathcal{N}$. Then any nonzero vector X^a tangent to the hypersufrace obeys

$$X^a n_a = 0. \tag{9.7}$$

[5] It's kind of great to learn about black holes from a guy who's on a first-name basis with Stephen Hawking.
[6] I gave a talk on this last term!

But in particular we could have $X^a = n_a$, which implies that either X^a is spacelike or X^a is parallel to n^a. In partuclar, note that n^a is tangent to the hypersurface. Hence on $\mathcal{N}$, the integral curves of n^a lie within $\mathcal{N}$.

Proposition 9.8. *The integral curves of n^a are null geodesics. These are called the generators of $\mathcal{N}$.*

Proof. Let $\mathcal{N}$ be given by an equation f =constant for some function f with $df \neq 0$. df is everywhere normal to $\mathcal{N}$, so we must have $n = h\,df$. Define $N \equiv df$. Since $\mathcal{N}$ is null, we must have

$$N_a N^a = 0 \text{ on } \mathcal{N}. \tag{9.9}$$

Hence the function $(N^a N_a)$ is constant on $\mathcal{N}$. This implies that the gradient of this function is normal to $\mathcal{N}$. Thus

$$\nabla_a (N_b N^b)|_{\mathcal{N}} = 2\alpha N_a \tag{9.10}$$

for some proportionality constant α. Now, we also have that

$$\nabla_a N_a = \nabla_a \nabla_b f = \nabla_b \nabla_a f = \nabla_b N_a,$$

where we have used the assumption that our spacetime is torsion-free. But then we conclude that

$$N^b \nabla_b N_a|_{\mathcal{N}} = \alpha N_a, \tag{9.11}$$

which tells us that N_a is a non-affinely parametrized geodesic. We conclude that the integral curves are null geodesics. ⊠

Geodesic deviation We saw this last term in *General Relativity*– for "nearby" geodesics, we can define relative "velocities" and "accelerations" leading to tidal forces. We'll make this notion more precise now.

Definition 9.12. A 1-parameter family of geodesics is a map $\gamma : I \times I' \to \mathcal{M}$, where I, I' are both open intervals in $\mathbb{R}$, such that

(i) for fixed s, $\gamma(s, \lambda)$ is a geodesic with affine parameter λ,
(ii) the map $(s, \lambda) \mapsto \gamma(s, \lambda)$ is smooth and one-to-one with smooth inverse. Thus γ defines a surface in $\mathcal{M}$.

Let U^a be the tangent vector to the geodesics and S^a be the vector tangent to curves of constant λ (i.e. takes us between neighboring geodesics). In a chart x^μ, the geodesics are specified in coordinates $x^\mu(s, \lambda)$ where $S^\mu = \frac{\partial x^\mu}{\partial s}$. Hence

$$x^\mu(s + \delta s, \lambda) = x^\mu(s, \lambda) + \delta s S^\mu + O(\delta s^2),$$

where S^μ points towards the next geodesic at $s + \delta s$. For this reason, we call S^μ the *deviation vector*.

On the surface (i.e. the image of γ), we can use s and λ coordinates. This gives a coordinate chart in which

$$S^\mu = \left(\frac{\partial}{\partial s}\right)^\mu, \quad U^\mu = \left(\frac{\partial}{\partial \lambda}\right)^\mu \tag{9.13}$$

on the surface. But in this coordinate system, these are just partial derivatives, so they automatically commute, and the commutator is covariant, so S^a and U^a commute in general, independent of basis. Thus

$$0 = [S, U] \iff U^b \nabla_b S^a = S^b \nabla_b U^a. \tag{9.14}$$

This implies that

$$U^c \nabla_c (U^b \nabla_b S^a) = R^a{}_{bcd} U^b U^c S^d. \tag{9.15}$$

Solutions of 9.15 are called *Jacobi fields*.

Geodesic congruence We'll introduce a final definition for today.

Definition 9.16. Let $U \subset \mathcal{M}$ be open. A *geodesic congruence* on U is a family of geodesics such that exactly one geodesic passes through each $p \in U$.