

# ADVANCED QUANTUM FIELD THEORY

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These notes were taken for the *Advanced Quantum Field Theory* course taught by Matthew Wingate at the University of Cambridge as part of the Mathematical Tripos Part III in Michaelmas Term 2018. I live-TeXed them using Overleaf, and as such there may be typos; please send questions, comments, complaints, and corrections to [it12@cam.ac.uk](mailto:it12@cam.ac.uk).

Many thanks to Arun Debray for the L<sup>A</sup>T<sub>E</sub>X template for these lecture notes: as of the time of writing, you can find him at <https://web.ma.utexas.edu/users/a.debray/>.

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Lecture 1.

#### Saturday, January 19, 2019

*Note.* There will not be official typed course notes, but there will be scanned handwritten notes (which I will link here as they become available). Previous lecturers' notes are currently online (Skinner, Osborn).

Today we introduce path integrals in a QFT context. There are some benefits to working with path integrals—some computations are simplified or more straightforward, and Lorentz invariance is manifest (unlike in the canonical formalism).

**Path integrals in quantum mechanics** Rather than trying to tackle the full machinery of QFT, we'll start with  $0+1$  dimensional non-relativistic quantum mechanics (cf. Osborn § 1.2. We'll set  $\hbar = 1$  for now, though we may restore it later in order to make arguments when  $\hbar \ll 1$  in a classical limit. In these units,

$$[E][t] = [\hbar] = [p][x]$$

using uncertainty relations.

Let us consider a Hamiltonian in 1 spatial dimension,

$$\hat{H} = H(\hat{x}, \hat{p}) \quad \text{with } [\hat{x}, \hat{p}] = i.$$

We'll further assume for simplicity that the Hamiltonian has a kinetic term and a potential based only on position,

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}).$$

Now the Schrödinger equation takes the form

$$i \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle \tag{1.1}$$

which has formal solution

$$|\psi(t)\rangle = e^{-i\hat{H}t} |\psi(0)\rangle. \tag{1.2}$$

Let us consider some position eigenstates  $|x, t\rangle$  such that

$$\hat{x}(t) |x, t\rangle = x |x, t\rangle, \quad x \in \mathbb{R},$$

where these states obey some normalization

$$\langle x', t | x, t \rangle = \delta(x' - x).$$

In the Schrödinger picture, states depend on time, while operators are constant. In terms of fixed (time-independent) eigenstates  $\{|x\rangle\}$  of the position operator  $\hat{x}$ , we may write the wavefunction as

$$\psi(x, t) = \langle x | \psi(t) \rangle, \quad (1.3)$$

so that applying the Hamiltonian to the wavefunction  $\psi(x, t)$  yields

$$\hat{H}\psi(x, t) = \left(-\frac{1}{2m} \frac{\partial^2}{\partial x^2} + V(x)\right)\psi(x, t). \quad (1.4)$$

This is the traditional presentation of quantum mechanics and the wavefunction. In the path integral formalism, we'll consider a more particle-like treatment, where we express time evolution as a sum over all trajectories (meeting some boundary conditions) appropriately weighted (by an action).

Recall that our formal solution 1.2 tells us what  $|\psi(t)\rangle$  is— we can therefore rewrite the wavefunction as

$$\psi(x, t) = \langle x | e^{-i\hat{H}t} | \psi(0) \rangle. \quad (1.5)$$

By inserting a complete set of (position eigen)states,  $1 = \int dx_0 |x_0\rangle \langle x_0|$ , we get

$$\begin{aligned} \psi(x, t) &= \int dx_0 \langle x | e^{-i\hat{H}t} | x_0 \rangle \langle x_0 | \psi(0) \rangle \\ &= \int dx_0 K(x, x_0; t) \psi(x_0, 0), \end{aligned}$$

where we have defined  $K(x, x_0; t) \equiv \langle x | e^{-i\hat{H}t} | x_0 \rangle$ . Let us further consider time evolution in discrete steps, with  $0 \equiv t_0 < t_1 < t_2 < \dots < t_n < t_{n+1} \equiv T$  so that

$$e^{-i\hat{H}T} = e^{-i\hat{H}(t_{n+1}-t_n)} \dots e^{-i\hat{H}(t_1-t_0)}.$$

As before, we insert complete sets of states, finding that our generic time evolution from any  $x_0$  to an  $x$  of our choosing:

$$K(x, x_0; T) = \int \left[ \prod_{r=1}^n dx_r \langle x_{r+1} | e^{-i\hat{H}(t_{r+1}-t_r)} | x_r \rangle \right] \langle x_1 | e^{-i\hat{H}t_1} | x_0 \rangle. \quad (1.6)$$

That is, we integrate over all intermediate positions  $x_r$  for each  $t_r$ . Naturally,  $dx_{n+1}$  must be  $x$ .

Let's look at the free theory first to understand what we've done,  $V(x) = 0$ . Now this weird  $K_0$  object we've defined takes the form

$$K_0(x, x'; t) = \langle x | e^{-i\frac{\hat{p}^2}{2m}t} | x' \rangle. \quad (1.7)$$

We'll instead insert a complete set of momentum eigenstates  $|p\rangle$  with the normalization

$$\int \frac{dp}{2\pi} |p\rangle \langle p| = 1,$$

recalling that  $\langle x | p \rangle = e^{ipx}$  are simply plane waves. Then

$$K_0(x, x'; t) = \int \frac{dp}{2\pi} e^{-ip^2 t/2m} e^{ip(x-x')}.$$

We can compute this— completing the square with a change of variables to  $p' = p - \frac{m(x-x')}{t}$ ,  $K_0$  becomes a gaussian integral,

$$\begin{aligned} K_0(x, x'; t) &= e^{im(x-x')^2/2t} \int_{-\infty}^{\infty} \exp \left[ -\frac{i(p')^2 t}{2m} \right] \\ &= e^{im(x-x')^2/2t} \sqrt{\frac{m}{2\pi i t}}. \end{aligned}$$

Note that as  $t \rightarrow 0$ ,<sup>1</sup>

$$\lim_{t \rightarrow 0} K_0(x, x'; t) = \delta(x - x'),$$

which agrees with the fact that  $\langle x' | x \rangle = \delta(x - x')$ .

<sup>1</sup>This was more obvious from the original expression for  $K_0$  where  $K_0(x, x'; t=0) = \int \frac{dp}{2\pi} e^{ip(x-x')}.$

For  $V(\hat{x}) \neq 0$ , we still need small time steps but since operators generically do not commute, exponentials don't add in the usual way:

$$e^{\hat{A}}e^{\hat{B}} = \exp(\hat{A} + \hat{B} + \frac{1}{2}[\hat{A}, \hat{B}] + \dots) \neq e^{\hat{A}+\hat{B}} \quad \text{when } [\hat{A}, \hat{B}] \neq 0.$$

This is the Baker-Campbell-Hausdorff (BCH) formula. However, for small  $\epsilon$  we can write

$$e^{\epsilon\hat{A}}e^{\epsilon\hat{B}} = \exp(\epsilon\hat{A} + \epsilon\hat{B} + O(\epsilon^2)),$$

or equivalently

$$e^{\epsilon(\hat{A}+\hat{B})} = e^{\epsilon\hat{A}}e^{\epsilon\hat{B}}(1 + O(\epsilon^2)),$$

so we conclude that

$$e^{\hat{A}+\hat{B}} = \lim_{n \rightarrow \infty} \left( e^{\hat{A}/n} e^{\hat{B}/n} \right)^n.$$

Suppose now that we divide our time into  $n$  time steps so that  $t_r - 1 - t_r = \delta t$ , with  $T = n\delta t$ . Then one of the intermediate time evolution steps looks like

$$\begin{aligned} \langle x_{r+1} | e^{-i\hat{H}\delta t} | x_r \rangle &= e^{-iV(x_r)\delta t} \langle x_{r+1} | e^{-i\hat{p}^2\delta t/2m} | x_r \rangle \\ &= \sqrt{\frac{m}{2\pi i\delta t}} \exp \left[ \frac{i}{2} m \left( \frac{x_{r+1} - x_r}{\delta t} \right)^2 \delta t - iV(x_r)\delta t \right]. \end{aligned}$$

Taking  $T = n\delta t$ , we find that the entire  $K$  becomes

$$K(x, x_0; T) = \int \left( \prod_{r=1}^n dx_r \right) \left( \frac{m}{2\pi i\delta t} \right)^{\frac{n+1}{2}} \exp \left( i \sum_{r=0}^n \left[ \frac{m}{2} \left( \frac{x_{r+1} - x_r}{\delta t} \right)^2 - V(x_r) \right] \delta t \right). \quad (1.8)$$

Now we take the limit as  $n \rightarrow \infty, \delta t \rightarrow 0$  with  $T$  fixed. Then the argument of the exponential becomes

$$\int_0^T \frac{m}{2} \dot{x}^2 - V(x) dt = \int_0^T L dt, \quad (1.9)$$

where  $L(x, \dot{x})$  is the classical Lagrangian and this integral is nothing more than the action. We conclude that

$$K(x, x_0; T) = \langle x | e^{-i\hat{H}T} | x_0 \rangle = \int \mathcal{D}x e^{iS[x]}, \quad (1.10)$$

where  $S[x] = \int_0^T L(x, \dot{x}) dt$  is the classical action and the  $\mathcal{D}$  conceals all our sins (the continuum limit) in a cute integration measure. Note that the action has units of energy  $\times$  time, so if we restore  $\hbar$ , we see that this integral becomes

$$K(x, x_0; T) = \int \mathcal{D}x e^{iS/\hbar}, \quad (1.11)$$

and in the  $\hbar \rightarrow 0$  limit (the classical limit), the integral is dominated by paths  $x$  which minimize the classical action, and we recognize this as Hamilton's principle from classical mechanics.