STRING THEORY

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These notes were taken for the *String Theory* course taught by R.A. Reid-Edwards at the University of Cambridge as part of the Mathematical Tripos Part III in Lent Term 2019. I live-TEXed them using Overleaf, and as such there may be typos; please send questions, comments, complaints, and corrections to itel2@cam.ac.uk.
Many thanks to Arun Debray for the LATEX template for these lecture notes: as of the time of writing, you can find him at https://web.ma.utexas.edu/users/a.debray/.

Contents

Lecture 1.

Friday, January 18, 2019

Note. This is a 24 lecture course with lectures at 11 AM M/W/F. There will be PDF notes available online somehow (TBD), and also $3+1$ problem sets plus a revision in Easter. The instructor can be reached at rar31@cam.ac.uk. Some recommended course readings[1] include "easier" texts:

- Schomerus[2]
- $(\text{Becker})^2$ and Schwarz[3]

and "harder" texts:

- Polchinski, Vol 1.[4]
- Lüst and Theisen[5]

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[1]Most of these are published by Cambridge University Press. Conspiracy– string theory was invented by CUP to sell textbooks?
[2]Available here for users with access to Cambridge University Press online: https://doi.org/10.1017/9781316672631
[3]Ditto: https://doi.org/10.1017/CBO9780511816086
[4]Here: https://doi.org/10.1017/CBO9780511816079
[5]Possibly available through Springer Link but not a CUP publication. https://link.springer.com/book/10.1007/BFb0113507

- Green, Schwarz, and Witten.[6]

Introduction Here are some of the major topics we'll be covering in this course.

- Classical theory and canonical quantization
- Path integral quantization
- Conformal field theory (CFT) and BRST quantization
- Scattering amplitudes
- Advanced topics (more on this later).

Historically, string theory emerged from ideas in QCD, the theory of the strong force. However, it really took hold as a theory of quantum gravity in the quest to reconcile quantum mechanics with general relativity. A bit of expectation management, first. Some of the motivating ideas which string theory attempts to address are as follows:

- What sets the parameters of the Standard Model?
- What sets the cosmological constant?
- Failure of perturbative GR (problems in the UV– gravity is non-renormalizable)
- The black hole information paradox (quantum information in gravitational systems)
- How do you quantize a theory in the absence of an existing causal structure? (Most of the causal structure of spacetime is encoded in the metric. But what if it's the metric itself you're trying to quantize?)

There are alternatives to string theory– for instance, one can do QFT in curved spacetime to learn about some limit of quantum gravity. There's also loop quantum gravity and causal set theory, among others, but we won't really discuss those in this course.

What is string theory? We just don't know.

In some sense, string theory is a set of rules which, given a 10-dimensional classical spacetime vacuum, allows us to do quantum perturbation theory around this vacuum. By doing perturbation theory, we seem to arrive at a unique quantum theory (details of this to be discussed more later).

In the popular science conception of string theory, we imagine replacing particles with strings, and the harmonics of these strings correspond to different particles, including the graviton. How do we reconcile this with the idea that gravity is just a function of the curvature of space time? Answer: we assume that we are close to some well-understood solution with metric $\eta_{\mu\nu}$ and take the new metric to be a perturbation,

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x).$$

Now that we have some spacetime structure, we can start to talk about interactions. We might have a propagator for strings, and also interaction vertices with some rules. We might think that an equivalent of Feynman diagrams emerges to tell us how strings can mingle and talk to each other.

In QFT, we were given some Lagrangian and from that Lagrangian, we derived interactions and Feynman rules. But in string theory, the situation is a bit backwards. It's as though we've been given some Feynman rules which do seem to reduce to the particle interactions in some limit, but we don't in some sense know the underlying theory where these rules come from.[7]

Classical theory In quantum mechanics, we have time t as a parameter and position $\hat{\mathbf{x}}$ as an operator. Of course, when we started learning quantum field theory, we were motivated to take our quantum fields $\hat{\phi}(\mathbf{x}, t)$ as operators and demote $\mathbf{x}$ to a simple label, so that $(\mathbf{x}, t)$ are both parameters. Space and time are on equal footing. This is the "second quantization" approach.

However, this isn't the only way we could do it. We could look for a formalism in which $\hat{x}^\mu = (\hat{\mathbf{x}}, \hat{t})$ are operators.

Example 1.1. Consider the *worldline formalism*. Imagine we have a massive particle propagating on a flat spacetime with metric $\eta_{\mu\nu}$. A suitable action for this theory might be

$$S[x] = -m \int_{s_1}^{s_2} ds, \tag{1.2}$$

[6] Here: rlhttps://doi.org/10.1017/CBO9781139248563

[7] "There are many reasons to study string theory. I suppose for you lot, you've got nothing better to do between the hours of 11 to 12." –R.A. Reid-Edwards

where we use natural units of $\hbar = c = 1$ and the m is some mass due to dimensional concerns. This has a sort of geodesic interpretation for some integration measure ds. We can parametrize the worldline (e.g. in terms of proper time) such that

$$S[x] = -m \int_{\tau_1}^{\tau_2} d\tau \sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}. \tag{1.3}$$

Here, dots indicate derivatives with respect to proper time. The conjugate momentum is then

$$P_\mu(\tau) = -\frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}}, \tag{1.4}$$

which obeys $P^2 + m^2 = 0$, so this is an "on-shell" formalism. We could then vary $S[x]$ with respect to trajectories $x^\mu(\tau)$ to find the equations of motion. We could imagine doing the same for an extended object and tracing out a "worldsheet" instead.

However, before we do that, let us revisit our action 1.2. In particular, we shall rewrite it as

$$S[x,e] = \frac{1}{2}\int d\tau \left(e^{-1}\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - em^2\right). \tag{1.5}$$

This new action has a sensible massless limit, unlike the previous action. For our new action, the $x^\mu(\tau)$ equation of motion is then

$$\frac{d}{d\tau}(e^{-1}\dot{x}^\mu) = 0 \tag{1.6}$$

and the $e(\tau)$ equation of motion gives

$$\dot{x}^2 + e^2m^2 = 0. \tag{1.7}$$

Now $e(\tau)$ appears algebraically, so we can substitute it back into the action to recover our previous formulation 1.2.[8]

Our theory also has some symmetry. If we shift the proper time by a function $\tau \to \tau + \zeta(\tau)$, then x and e change as

$$\delta x^\mu = \zeta \dot{x}^\mu$$
$$\delta e = \frac{d}{d\tau}(\zeta e).$$

We can use the one arbitrary degree of freedom to gauge fix $e(\tau)$ to a convenient value.

There's also a *rigid symmetry* which takes

$$x^\mu(\tau) \to \Lambda^\mu{}_\nu x^\nu(\tau) + a^\mu,$$

which we may recognize as Poincaré invariance in the background spacetime.[9]

Non-lectured aside: reparameterization invariance Here, we'll explicitly show that the action 1.5 is invariant under the transformation

$$\tau \to \tau + \xi(\tau). \tag{1.8}$$

For some reason, this is not spelled out in either David Tong's notes or the standard textbooks I've consulted so far.

[8] Explicitly, we see that

$$\begin{aligned} S[x,e] &= \frac{1}{2}\int d\tau (e^{-1}\dot{x}^2 - em^2) \\ &= \frac{1}{2}\int d\tau (e^{-1}(-e^2m^2) - em^2) \\ &= \int d\tau(-em^2) \end{aligned}$$

and by setting $e = 1/m$ we recover 1.2.

[9] We can see that the action respects this symmetry, since it only depends on $\dot{x}^\mu$ and not x^μ (so translational symmetry is preserved) and $\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \to \eta_{\mu\nu}\Lambda^\mu{}_\sigma \dot{x}^\sigma \Lambda^\nu{}_\tau \dot{x}^\tau = \eta_{\sigma\tau}\dot{x}^\sigma\dot{x}^\tau$, so $\dot{x}^2$ is also preserved under Lorentz transformations as it should be.

We make the assumption as in lecture that x and e change as

$$\delta x^\mu = \zeta \dot{x}^\mu$$
$$\delta e = \frac{d}{d\tau}(\zeta e).$$

If so, then note that

$$\delta(\dot{x}^\mu) = \frac{d}{d\tau}(\delta x^\mu) = \frac{d}{d\tau}(\lambda \dot{x}^\mu) \tag{1.9}$$

and

$$\frac{1}{e+\delta(e)} \sim \frac{1}{e} - \frac{1}{e^2}\delta(e) \implies \delta(e^{-1}) = -\frac{1}{e^2}\delta(e). \tag{1.10}$$

To perform this calculation, we'll also need the equations of motion from lecture, 1.6 and 1.7, reproduced here:

$$\frac{d}{d\tau}(e^{-1}\dot{x}^\mu) = 0$$

and

$$\dot{x}^2 + e^2 m^2 = 0.$$

Let's vary the action!

$$\begin{aligned}
\delta S[x,e] &= \frac{1}{2}\int d\tau \left[\delta(e^{-1})\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + e^{-1}\eta_{\mu\nu}\delta(\dot{x}^\mu)\dot{x}^\nu + e^{-1}\eta_{\mu\nu}\dot{x}^\mu\delta(\dot{x}^\nu) - \delta(e)m^2\right] \\
&= \frac{1}{2}\int d\tau \left[-\frac{1}{e^2}\delta(e)\dot{x}^2 + 2e^{-1}\eta_{\mu\nu}\frac{d}{d\tau}(\lambda\dot{x}^\mu)\dot{x}^\nu - \delta(e)m^2\right] \\
&= \frac{1}{2}\int d\tau \left[-\frac{1}{e^2}\delta(e)(\dot{x}^2 + m^2e^2) + 2(e^{-1}\dot{x}^\nu)\eta_{\mu\nu}\frac{d}{d\tau}(\lambda\dot{x}^\mu)\right] \\
&= \frac{1}{2}\int d\tau \frac{d}{d\tau}(\lambda e^{-1}\dot{x}^2) \\
&= 0.
\end{aligned}$$

In going from the first to the second line, we have explicitly substituted the variations for e^{-1} and for $\dot{x}^\mu$. In going from the second to the third, we simply regrouped terms into $\dot{x}^2 + m^2e^2$, which is zero by the equations of motion, and into $e^{-1}\dot{x}^\nu$, which is constant by the other equation of motion and therefore can be moved inside the total time derivative $\frac{d}{d\tau}$.

We see that after variation, what remains is simply an integral $\int d\tau$ of a total derivative, which is zero when evaluated at the endpoints of the action integral by the boundary conditions. Therefore the action is indeed invariant under reparametrization. ⊠

Lecture 2.

Monday, January 21, 2019

Last time, we introduced a *worldline action* with an einbein e (auxiliary field).

$$S[x,e] = \frac{1}{2}\int d\tau \left(e^{-1}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - em^2\right).$$

In the massless limit, this reduces to

$$S[X,e] = \frac{1}{2}d\tau e^{-1}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu, \tag{2.1}$$

where we have replaced the Minkowski metric with some generic metric. The classical equations of motion for $X^\mu(\tau)$ then give the geodesic equation,

$$\ddot{X}^\mu + \Gamma^\mu_{\nu\lambda}\dot{X}^\nu\dot{X}^\lambda = 0. \tag{2.2}$$

The $e(\tau)$ equations of motion would give some constraints. However, if we attempted to quantize this theory, we would find that the background metric $g_{\mu\nu}$ is not actually deformed in the solutions. Rather than being dynamic as in general relativity, it's sort of a thing that is given to us and sits in the background, unchanging, which is why for a particle this is not a theory of quantum gravity. As we'll see, this is *not* the case for strings.

Strings As a string moves through some flat spacetime $\mathcal{M}$ with metric $\eta_{\mu\nu}$, it sweeps out a worldsheet Σ. Assume that the string is closed, so it has a coordinate σ (along the length of the string, if you like):

$$\sigma \sim \sigma + 2n\pi, n \in \mathbb{Z}.$$

And it moves through time as parametrized by a proper time τ, so the embedding of the worldsheet is given by $X^\mu(\sigma, \tau)$. That is, σ and τ provide good coordinates for the worldsheet in $\mathcal{M}$.

Definition 2.3. We call these X^μ embedding fields. They are maps $X : \Sigma \to \mathcal{M}$ from the worldsheet to the background spacetime manifold.

We also say that the area of the worldsheet Σ is given by

$$\text{area} = \int d\tau d\sigma \sqrt{-\det(\eta_{\mu\nu}\partial_a X^\mu \partial_0 X^\nu)} \tag{2.4}$$

where $\sigma^a = (\tau, \sigma)$ so that $\partial_a = \frac{\partial}{\partial \sigma^a}$. In fact, we shall introduce an extra factor know (for historical reasons) as α' and write

$$S[X] = -\frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{-\det(\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu)}, \tag{2.5}$$

where α' is a free parameter. We often refer to the *string length*,

$$l_s \equiv 2\pi\sqrt{\alpha'} \tag{2.6}$$

or the *tension*

$$T \equiv \frac{1}{2\pi\alpha'}. \tag{2.7}$$

Definition 2.8. The object

$$G_{ab} \equiv \eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \tag{2.9}$$

is an induced metric on Σ, and the action 2.5 is called the *Nambu-Goto action*.

Having just defined this, we won't really do anything with it for the rest of the course. Bummer. However, to make up for it, let's write down a new and improved action, the *Polyakov action*.

Definition 2.10. Consider the action

$$S[X, h] = -\frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{-h}\, h^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu. \tag{2.11}$$

This should remind us of what we did with the einbein last lecture, where we introduced e into our action.

This *Polyakov action* is classically equivalent to the Nambu-Goto action, since this auxiliary h which we have introduced will turn out to be non-dynamical.

The h_{ab} equations of motion are given by a weird variation of the action,

$$-\frac{2\pi}{\sqrt{-h}} \frac{\delta S}{\delta h^{ab}} = 0. \tag{2.12}$$

These equations of motion give the vanishing of the stress tensor, $T_{ab} = 0$, where

$$T_{ab} = -\frac{1}{\alpha'}\left(\partial_a X^\mu \partial_b X_\mu - \frac{1}{2} h_{ab} \partial_c X^\mu \partial_d X_\mu h^{cd}\right). \tag{2.13}$$

Note that in two dimensions, $T_{ab}h^{ab} = 0$, i.e. T_{ab} is traceless. This is our first indication that something is different about two dimensions.

The X^μ equations of motion are

$$\frac{1}{\sqrt{-h}}(\partial_a \sqrt{-h}\, h^{ab} \partial_b X^\mu) = 0, \quad \Box X^\mu = 0. \tag{2.14}$$

Now we could imagine adding a cosmological constant (which would cause the trace of the stress tensor to change) or perhaps some sort of Einstein-Hilbert term to our metric h_{ab}. But we'll see why this might be more complicated than it initially seems.

Symmetries The Polyakov action 2.11 has the following symmetries:

- Rigid (global) symmetry, $X^\mu(\sigma,\tau) \to \Lambda^\mu{}_\nu X^\nu(\sigma,\tau) + a^\mu$ (Poincaré invariance).
- Local symmetries– the physics should be invariant under reparametrizations of the coordinates of the worldsheet, so under transformations $\sigma^a \to \sigma'^a(\sigma,\tau)$. The fields themselves transform as

$$\begin{aligned} X'^\mu(\sigma',\tau') &= X^\mu(\sigma,\tau) \\ h_{ab}(\sigma,\tau) &= \frac{\partial\sigma'^c}{\partial\sigma^a}\frac{\partial\sigma'^d}{\partial\sigma^b}h'_{cd}(\sigma',\tau'). \end{aligned}$$

  Infinitesimallly, this means that $\sigma^a \to \sigma^a - \xi^a(\sigma,\tau)$, which gives us the variations

$$\begin{aligned} \delta X^\mu &= \xi^a\partial_a X^\mu \\ \delta h_{ab} &= \xi^c\partial_c h_{ab} + \partial_a\xi^c h_{cb} + \partial_b\xi^c h_{ca} = \nabla_a\xi_b + \nabla_b\xi_a \\ \delta\sqrt{-h} &= \partial_a(\xi^a\sqrt{-h}). \end{aligned}$$

  Note this second variation, δh_{ab}, can be written in terms of some covariant derivatives for an appropriate connection, but we won't usually bother.
- Weyl transformations– we send

$$\begin{aligned} X'^\mu(\sigma,\tau) &= X^\mu(\sigma,\tau) \\ h'_{ab}(\sigma,\tau) &= e^{2\Lambda(\sigma,\tau)}h_{ab}(\sigma,\tau). \end{aligned}$$

  Thus $\delta X^\mu = 0$ and $\delta h_{ab} = 2\Lambda h_{ab}$. Under such transformations, we have three arbitrary degrees of freedom in (ξ^a,Λ) (two from the two components of ξ plus one from Λ), and we can use them to fix the three degrees of freedom in h_{ab} (there are three, since h is symmetric and 2×2).

Classical solutions Let us now use reparametrization invariance to fix

$$h_{ab} = e^{2\phi}\eta_{ab}, \quad \eta_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.15}$$

The Polyakov action then becomes

$$S[X] = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma(-\dot{X}^2 + X'^2), \tag{2.16}$$

where

$$\dot{X}^\mu \equiv \frac{\partial X^\mu}{\partial\tau}, \quad X'^\mu \equiv \frac{\partial X^\mu}{\partial\sigma} \tag{2.17}$$

and squares are taken by contracting with the metric h_{ab}. In that case, the $X^\mu(\sigma,\tau)$ equation of motion becomes the wave equation in 2D, so solutions are of the form

$$X^\mu(\sigma,\tau) = X^\mu_R(\tau-\sigma) + X^\mu_L(\tau+\sigma). \tag{2.18}$$

Moreover, since we have a wave equation it is useful to introduce modes $(\alpha^\mu_n, \bar{\alpha}^\mu_n)$ where

$$X^\mu_R(\tau-\sigma) = \frac{1}{2}x^\mu + \frac{\alpha'}{2}p^\mu(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-in(\tau-\sigma)}, \tag{2.19}$$

where x^μ, p^μ are some constants in (τ,σ) and similarly the left-going modes are

$$X^\mu_L(\tau+\sigma) = \frac{1}{2}x^\mu + \frac{\alpha'}{2}p^\mu(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\bar{\alpha}^\mu_n e^{-in(\tau+\sigma)}. \tag{2.20}$$

It's sometimes useful to define a zero-mode,

$$\alpha^\mu_0 = \bar{\alpha}^\mu_0 = \sqrt{\frac{\alpha'}{2}}p^\mu. \tag{2.21}$$

Lecture 3.

Wednesday, January 23, 2019

Two announcements. First, the official course notes will be released this weekend (I'll link them here soon). Second, today's colloquium is being given by Johanna Erdmenger, a Part III alumna working on AdS/CFT (gauge-gravity duality). The ideas in AdS/CFT were motivated by stringy concepts, and so should be relevant to our course.

Last time, we introduced the Polyakov action,

$$S[X,h] = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{-h}\,h^{ab}\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu. \tag{3.1}$$

Note that $h = \det(h_{ab})$ with h_{ab} considered as a 2×2 matrix. The equations of motion for h_{ab} gave the requirement that the stress tensor vanishes, $T_{ab} = 0$, with

$$T_{ab} = \partial_a X^\mu \partial_b X_\mu - \frac{1}{2}h_{ab}\partial^c X^\mu \partial_c X_\mu. \tag{3.2}$$

Here, a, b indices are raised and lowered with the appropriate metric h_{ab} and $\mu\nu$ indices are raised and lowered with $\eta_{\mu\nu}$.

Now, how does the Polyakov action relate to the Nambu-Goto action? Let us define the quantity

$$G_{ab} \equiv \partial_a X^\mu \partial_b X_\mu. \tag{3.3}$$

If $T_{ab} = 0$, then by 3.2,

$$G_{ab} = \frac{1}{2}h_{ab}(h^{cd}G_{cd}). \tag{3.4}$$

Taking determinants of both sides yields

$$\det(G_{ab}) = \left(\frac{1}{2}h^{cd}G_{cd}\right)^2 \det(h_{ab}) = \frac{1}{4}\left(h^{cd}G_{cd}\right)^2 h. \tag{3.5}$$

Therefore

$$2\sqrt{-\det(G_{ab})} = (h^{ab}G_{ab})\sqrt{-h} = \sqrt{-h}h^{ab}\partial_a X^\mu \partial_b X_\mu. \tag{3.6}$$

Substituting this back into the Polyakov action now gives us

$$S[X] = -\frac{1}{2\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{-\det G_{ab}},$$

the Nambu-Goto action. However, the Polyakov action is nicer to work with since it does not involve square roots of the coordinates X.

The stress tensor Recall that the conjugate momentum to X^μ is

$$P_\mu = \frac{1}{2\pi\alpha'}\dot{X}_\mu, \tag{3.7}$$

where a dot is a derivative with respect to proper time τ. We can define a Hamiltonian density $\mathcal{H}$ as

$$\mathcal{H} = P_\mu \dot{X}^\mu - \mathcal{L} = \frac{1}{4\pi\alpha'}(\dot{X}^2 + X'^2). \tag{3.8}$$

Definition 3.9. For our Hamiltonian formalism, we'll also need some *Poisson brackets* which we denote $\{\,,\,\}_{PB}$ (to contrast with another use of brackets later in the quantum theory). Given F, G defined on the phase space, we have

$$\{F,G\}_{PB} \equiv \int_0^{2\pi} d\sigma\left(\frac{\delta F}{\delta X^\mu(\sigma)}\frac{\delta G}{\delta P_\mu(\sigma)} - \frac{\delta F}{\delta P_\mu(\sigma)}\frac{\delta G}{\delta X^\mu(\sigma)}\right). \tag{3.10}$$

In particular, $\{X^\mu(\tau,\sigma), P_\nu(\tau,\sigma')\}_{PB} = \delta^\mu_\nu\delta(\sigma-\sigma')$.

Last time, we introduced a mode expansion

$$X^\mu(\sigma,\tau) = X_R^\mu(\tau-\sigma) + X_L^\mu(\tau+\sigma), \tag{3.11}$$

writing e.g. the right-going mode in terms of modes α_n^μ,

$$X_R^\mu(\tau-\sigma) = \frac{1}{2}x^\mu + \frac{\alpha'}{2}p^\mu(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in(\tau-\sigma)}, \tag{3.12}$$

and something similar holds for X_L^μ using the modes $\bar{\alpha}_n^\mu$.

Let's try to work in terms of modes rather than the embedding fields X^μ. We assert that the Poisson brackets acting on the modes $\alpha_n^\mu, \bar{\alpha}_n^\mu$ are

$$\{\alpha_m^\mu, \alpha_n^\nu\}_{PB} = -im\delta_{m,-n}\eta^{\mu\nu} \tag{3.13}$$

$$\{\alpha_m^\mu, \bar{\alpha}_n^\nu\}_{PB} = 0 \tag{3.14}$$

$$\{\bar{\alpha}_m^\mu, \bar{\alpha}_n^\nu\}_{PB} = -im\delta_{m,-n}\eta^{\mu\nu} \tag{3.15}$$

for $n \neq 0, m \neq 0$. If we define $\alpha_0^\mu = \bar{\alpha}_0^\nu = \sqrt{\frac{\alpha'}{2}}p^\mu$, we see that $\{x^\mu, p_\nu\}_{PB} = \delta^\mu_\nu$.

Let's see why this might be reasonable. We will set $\tau = 0$ so that

$$X^\mu(\sigma) = x^\mu + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^\mu e^{in\sigma} + \bar{\alpha}_n^\mu e^{-in\sigma}\right),$$

$$P^\nu(\sigma') = \frac{p^\nu}{2\pi} + \frac{1}{2\pi}\sqrt{\frac{1}{2\alpha'}}\sum_{m\neq 0}\left(\alpha_m^\nu e^{im\sigma'} + \bar{\alpha}_m^\nu e^{-im\sigma'}\right).$$

Recall that we get P^ν by deriving $X^\mu(\tau,\sigma)$ with respect to τ and dividing by a factor of 2π. (Check this expression for $P^\nu(\sigma, \tau = 0)$!)

Now we can compute the Poisson bracket: it is

$$\{X^\mu(\sigma), P_\nu(\sigma')\}_{PB} = \frac{1}{2\pi}\{x^\mu, p^\nu\} - \frac{1}{4\pi}\sum_{n,m\neq 0}\frac{1}{2m}\left(\{\alpha_m^\mu, \alpha_n^\nu\}e^{i(m\sigma+n\sigma')} + \{\bar{\alpha}_m^\mu, \bar{\alpha}_n^\nu\}e^{-i(m\sigma+n\sigma')}\right). \tag{3.16}$$

Using the Poisson bracket relations on the modes and the "periodic delta function"

$$\delta(\sigma-\sigma') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}e^{im(\sigma-\sigma')}, \tag{3.17}$$

one can show that

$$\{X^\mu(0,\tau), P^\nu(0,\sigma')\}_{PB} = \eta^{\mu\nu}\delta(\sigma-\sigma'). \tag{3.18}$$

The Wit algebra We'll quickly introduce the following concept. On our worldsheet, it will be useful to use light-cone (null) coordinates

$$\sigma^\pm = \tau \pm \sigma. \tag{3.19}$$

Thus the metric becomes

$$ds^2 = -d\tau^2 + d\sigma^2 = (d\sigma^+, d\sigma^-)\begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} d\sigma^+ \\ d\sigma^- \end{pmatrix}. \tag{3.20}$$

Derivatives become

$$\partial_\pm \equiv \frac{\partial}{\partial\sigma^\pm} = \frac{1}{2}(\partial_\tau \pm \partial_\sigma). \tag{3.21}$$

In these new coordinates, the action and equations of motion become

$$S[X] = -\frac{1}{2\pi\alpha'}\int_\Sigma d\sigma^+ d\sigma^-\, \partial_+ X^\mu \partial_- X_\mu, \quad \partial_+\partial_- X^\mu = 0. \tag{3.22}$$

The stress tensor becomes

$$T_{++} = -\frac{1}{\alpha'}\partial_+ X^\mu \partial_+ X_\mu, \quad T_{--} = -\frac{1}{\alpha'}\partial_- X^\mu \partial_- X_\mu, \tag{3.23}$$

with $T_{+-} = 0$ since this is nothing more than the trace of T_{ab}.

We can introduce modes $l_m, \bar{l}_m$ for the stress tensor, writing

$$\begin{aligned} l_n &= -\frac{1}{2\pi}\int_0^{2\pi} d\sigma T_{--} e^{-in\sigma} \\ \bar{l}_n &= -\frac{1}{2\pi}\int_0^{2\pi} d\sigma T_{++} e^{+in\sigma}. \end{aligned}$$

Again, our goal is to work with modes rather than the entire solutions.

For instance,

$$\partial_- X^\mu = \sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n^\mu e^{-in\sigma}, \text{ where } \alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu.$$

Lecture 4.

Friday, January 25, 2019

Last time we introduced the light cone coordinates on Σ, defined as $\sigma^\pm = \tau \pm \sigma$. Recall also that we want to work with modes rather than embedding fields, and for $\tau = 0$, the modes are given by

$$\begin{aligned} l_n &= -\frac{1}{2\pi}\int_0^{2\pi} d\sigma T_{--}(\sigma) e^{-in\sigma} \\ \bar{l}_n &= -\frac{1}{2\pi}\int_0^{2\pi} d\sigma T_{++}(\sigma) e^{+in\sigma}, \end{aligned}$$

with $T_{+-} = 0$.

We shall see that $l_m, \bar{l}_m$ are conserved quantities on the space $T_{ab} = 0$. Using

$$\partial_- X^\mu(\sigma) = \sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n^\mu e^{-in\sigma}, \text{ where } \alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu,$$

we would like to get expressions for the stress tensor modes l_n in terms of the string modes α_m^μ. We postulated some Poisson brackets on the modes, which will hopefully help us out in this calculation.

For instance,

$$\begin{aligned} l_n &= \frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma \partial_- X \cdot \partial_- X e^{in\sigma} \\ &= \frac{1}{4\pi}\sum_{m,p} \alpha_m \cdot \alpha_p \int_0^{2\pi} d\sigma e^{i(m+p-n)\sigma} \\ &= \frac{1}{4\pi}\sum_{m,p} \alpha_m \cdot \alpha_p (2\pi \delta_{m+p,n}) \\ \implies l_n &= \frac{1}{2}\sum \alpha_{n-m}\cdot\alpha_m, \quad \bar{l}_n = \frac{1}{2}\sum_m \bar{\alpha}_{n-m}\cdot\bar{\alpha}_m. \end{aligned}$$

Using these expressions and the PB relations for the αs, one can (and should) show that the l_n satisfy the following Poisson brackets:

$$\begin{aligned} \{l_m, l_n\}_{PB} &= (m-n) l_{m+n} \\ \{\bar{l}_m, \bar{l}_n\}_{PB} &= (m-n) \bar{l}_{m+n} \\ \{l_m, \bar{l}_n\}_{PB} &= 0. l_{m+n} \end{aligned}$$

This is often called the *Wit algebra*, and it is related to the Virasoro algebra in the quantum theory. n.b. the stress tensor modes $l_0, l_{\pm 1}, \bar{l}_0, \bar{l}_{\pm 1}$ generate the Lie algebra of $SL(2, \mathbb{C})$.

Now, the Hamiltonian may be written as

$$H = \frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma\Big((\partial_+ X)^2 + (\partial_- X)^2\Big) \tag{4.1}$$

$$= \frac{1}{2}\sum_n (\alpha_{-n}\cdot\alpha_n + \bar{\alpha}_{-n}\cdot\bar{\alpha}_n) \tag{4.2}$$

$$= l_0 + \bar{l}_0. \tag{4.3}$$

Anticipating the quantum case, we will call these l modes *Virasoro generators*.

On the constraint surface $l_n \approx 0$, one can show that $\{H, l_n\} \approx 0$, since

$$\frac{dl_n}{d\tau} = \{H, l_n\}_{PB} = -nl_n. \tag{4.4}$$

Canonical quantization We have been working entirely with the classical string so far, and our main approach will be the path integral formalism. However, it may be enlightening for us to consider how to canonically quantize the string.

In the classical theory, we have $\{X^\mu, P_\nu\}_{PB}$ the Poisson bracket, with $T_{ab} = 0$. In going to a quantum theory, we could *impose* $T_{ab} = 0$ and promote variables to operators, $\{q^\mu, \pi_\nu\}_{PB}$, and then promote the Poisson bracket to a commutator of quantum operators, $i[q^\mu, \pi_\nu]$. That is, we first constrain the phase space and then quantize. This gives us a Hilbert space $\mathcal{H}_{l.c.}$ on the light cone.

On the other hand, our approach will be a little different. We can quantize first, $\{\cdot,\cdot\}_{PB} \to i[\cdot,\cdot]$, giving us commutators $[X^\mu, P_\nu]$, and *then* impose $T_{ab} = 0$, where T_{ab} is now an operator and the constraint is $T_{ab}|\psi\rangle = 0 \forall |\psi\rangle$. This will yield another Hilbert space $\mathcal{H}_Q$, which we hope (and could prove, although it is non-trivial) is equivalent to the light cone Hilbert space.

Thus in our approach, we start by replacing *fundamental* Poisson bracket relations with canonical commutation relations,

$$\{X^\mu, P_\nu\} \to -i[X^\mu, P_\nu], \tag{4.5}$$

and can do something equivalent for the $\alpha_n^\mu, \bar{\alpha}_n^\mu$ modes.

We now introduce the *Virasoro operators*

$$L_n = \frac{1}{2}\sum_m \alpha_{n-m}\cdot\alpha_m, n \neq 0, \tag{4.6}$$

where we distinguish the L_ns from the classical l_n since the quantum Ls do not quite satisfy the Wit algebra. $\bar{L}_n$ is defined equivalently.

We also introduce a vacuum state $|0\rangle$, which we will define as the state annihilated by all α modes,

$$\alpha_n^\mu|0\rangle = 0 \text{ for } n \geq 0. \tag{4.7}$$

We think of $\alpha_n^\mu, n > 0$ as annihilation operators analogous to those of the harmonic oscillator, and $n < 0$ as creation operators.[10] What are these operators creating and annihilating? Harmonics of the string, essentially.

We now notice an ambiguity in the definition of L_0 and $\bar{L}_0$. We have

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n>0}\alpha_{-n}\cdot\alpha_n, \tag{4.8}$$

but note that the $\alpha_{-n}\cdot\alpha_n$ terms have an ordering ambiguity.

To resolve this, we define normal ordering (denoted by ::) in the usual way, moving all creation operators to the left and all annihilation operators to the right. We then define composite operators using this ordering, e.g.

$$T_{--}(\sigma^-) = -\frac{1}{\alpha'} : \partial_- X^\mu \partial_- X_\mu : . \tag{4.9}$$

[10] α_0 is a little special and has to do with the center of mass of the string, though it does annihilate the vacuum.

Physical state conditions We define the number operators $N_n, \bar{N}_n$ by

$$nN_n = \alpha_{-n} \cdot \alpha_n, \quad n\bar{N}_n = \bar{\alpha}_{-n} \cdot \bar{\alpha}_n, \tag{4.10}$$

and the total number operators as

$$N = \sum_n nN_n, \quad \bar{N} = \sum_n n\bar{N}_n. \tag{4.11}$$

The $L_0, \bar{L}_0$ may be written as

$$L_0 = \frac{\alpha'}{4}p^2 + N, \quad L_0 = \frac{\alpha'}{4}p^2 + \bar{N}. \tag{4.12}$$

Next time, we will impose the conditions

$$L_n|\phi\rangle = 0, n > 0 \text{ and } (L_0 - a)|\phi\rangle = 0 \tag{4.13}$$

for $|\phi\rangle$ to be a physical state, with $a \in \mathbb{R}$.

Lecture 5.

Monday, January 28, 2019

Last time, we began discussing the quantization of the string. We said that our approach would be to quantize the unconstrained first and then apply the quantum-ized constraint $T_{ab} = 0$ on all physical states in the Hilbert space. We do this by imposing the conditions

$$L_n|\phi\rangle = 0, \quad n > 0 \tag{5.1}$$

for $|\phi\rangle$ to be physical. Note that $\bar{L}_n|\phi\rangle = 0$ as well– for most of our theory, we'll get an exact copy of the behavior of the right-handed modes L_n in the left-handed modes $\bar{L}_n$.

We also observed that our definition of L_0 was ambiguous in the quantum theory. In the other operators, we always had products of modes α_n with different harmonics n, but for L_0 there is an ordering ambiguity. We therefor impose the physical condition that

$$(L_0 - a)|\phi\rangle = 0, \quad (\bar{L}_0 - a)|\phi\rangle = 0 \tag{5.2}$$

where $a \in \mathbb{R}$ quantifies this ordering ambiguity. We will see later (cf. BRST invariance) that the theory is consistent only if $D = 26, a = 1$. From now on we shall assume $a = 1$.

It will be useful to define

$$L_0^{\pm} = L_0 \pm \bar{L}_0, \tag{5.3}$$

so that we have

$$(L_0^+ - 2)|\psi\rangle = 0, \quad L_0^-|\psi\rangle = 0, \quad L_n|\psi\rangle = \bar{L}_n|\psi\rangle = 0, n > 0. \tag{5.4}$$

These three conditions characterize physical states. Recall that $L_0 = \frac{\alpha'}{4}p^2 + N, \bar{L}_0 = \frac{\alpha'}{4}p^2 + \bar{N}$.

The spectrum We'll start by looking at the lowest-lying modes of the theory. We haven't yet discussed the creation or destruction of strings, so the following discussion will, if you like, be centered on free propagators.

We begin by remarking that in our version of the theory, there are problems in the infrared which have to do with *tachyons*. These problems can be addressed in superstring theory, which is beyond the scope of this course.

The simplest state we can write down is the momentum eigenstate,

$$|k\rangle = e^{ik\cdot x}|0\rangle, \tag{5.5}$$

with k_μ some four-vector of our choice and x the center of mass coordinate for the string (i.e. the x such that $X^\mu(\sigma, \tau) = x^\mu + p^\mu\tau + \text{oscillations}$). The action of the center of mass momentum p_μ is then

$$p_\mu|k\rangle = k_\mu|k\rangle. \tag{5.6}$$

We could define a general state by a weighted sum of these momentum eigenstates,

$$|T\rangle = \int d^D k\, T(k)|k\rangle, \tag{5.7}$$

where $T(k)$ is a function of our choosing and we are working in D dimensions. Now the $L_0^-|\phi\rangle = 0$ condition imposes $N = \bar{N}$. This is called the "level-matching" condition. It turns out to be the only condition that relates the left-going and right-going modes– otherwise, they are totally uncoupled.

If we look at L_0^+, we get the condition

$$(L_0^+ - 2)T(k)|k\rangle = \left(\frac{\alpha'}{2}p^2 + N + \bar{N} - 2\right)T(k)|k\rangle = 0, \tag{5.8}$$

which tells us that $N = \bar{N} = 0$. Therefore

$$(L_0^+ - 2)T(k)|k\rangle = \left(\frac{\alpha'}{2}p^2 - 2\right)T(k)|k\rangle = 0, \tag{5.9}$$

which we can rewrite as a mass-shell condtion on the momentum space field $T(k)$:

$$(k^2 + M^2)T(k) = 0 \quad \text{where } M^2 = -\frac{4}{\alpha'}. \tag{5.10}$$

We notice that the field $T(k)$ is tachyonic, i.e. its mass squared is negative. (We use the mostly + sign convention for the Minkowski metric.) Note that

$$L_n|T\rangle = 0 = \bar{L}_n|T\rangle \text{ for } n > 0 \tag{5.11}$$

is satisfied trivially. A priori, tachyons need not sink our theory. It could be that we're just working relative to the wrong vacuum. This is an open question, though there are other reasons the bosonic string might not be quite the right model for our universe's physics. Having declared that superstring theory does provide some solution to this problem, we will pay it no more thought and move on.

Massless states Next, we consider states of the form

$$|\epsilon\rangle = \epsilon_{\mu\nu}(k)\alpha_{-1}^{\mu}\bar{\alpha}_{-1}^{\nu}|k\rangle, \tag{5.12}$$

where we have included both α and $\bar{\alpha}$ to satisfy level-matching, and we have thrown in an ϵ in order to kill the free indices.

The condition $(L_0^+ - 2)|\epsilon\rangle = 0$ gives $M^2 = 0$ since $N = \bar{N} = 1$. Note that $L_n|\epsilon\rangle = 0$ is satisfied trivially for $n > 1$ (and so is $\bar{L}_n|\epsilon\rangle = 0$).

What about $L_1|\epsilon\rangle = 0$? We have

$$\begin{aligned}
L_a|\epsilon\rangle &= \frac{1}{2}\sum_n \alpha_{1-n}\cdot\alpha_n\epsilon_{\mu\nu}\alpha_{-1}^{\mu}\bar{\alpha}_{-1}^{\nu}|k\rangle \\
&= \epsilon_{\mu\nu}(k)\alpha_0\cdot\alpha_1\alpha_{-1}^{\mu}\bar{\alpha}_{-1}^{\nu}|k\rangle \\
&= \sqrt{\frac{2}{\alpha'}}\epsilon_{\mu\nu}(k)k_\lambda\alpha_1^\lambda\alpha_{-1}^{\mu}\bar{\alpha}_{-1}^{\nu}|k\rangle \\
&= \sqrt{\frac{2}{\alpha'}}\epsilon_{\mu\nu}(k)k_\lambda\left([\alpha_1^\lambda,\alpha_{-1}^\lambda] + \alpha_{-1}^{\mu}\alpha_1^\lambda\right)\bar{\alpha}_{-1}^{\nu}|k\rangle.
\end{aligned}$$

We conclude that

$$\epsilon_{\mu\nu}(k)k^\mu = 0, \tag{5.13}$$

so two states related by

$$\epsilon_{\mu\nu}(k) \to \epsilon_{\mu\nu}(k) + k_\mu\xi_\nu \tag{5.14}$$

are physically equivalent since $k^2 = 0$, with ξ arbitrary. Similarly,

$$\bar{L}_1|\epsilon\rangle = 0 \implies k^\nu\epsilon_{\mu\nu}(k) = 0. \tag{5.15}$$

It is useful to decompose $\epsilon_{\mu\nu}(k)$ as follows:

$$\epsilon_{\mu\nu}(k) = \tilde{g}_{\mu\nu}(k) + \tilde{B}_{\mu\nu}(k) + \eta_{\mu\nu}\tilde{\phi}(k), \tag{5.16}$$

where $\tilde{g}_{\mu\nu}$ is traceless symmetric and $\tilde{B}_{\mu\nu}$ is antisymmetric. Now $\tilde{g}_{\mu\nu}(k)$ has the interpretation of a momentum space metric perturbation,

$$\tilde{g}_{\mu\nu}(k) \sim \tilde{g}_{\mu\nu}(k) + k_\mu\xi_\nu + \xi_\mu k_\nu, \tag{5.17}$$

which is simply (linearized) diffeomorphism invariance. What about this antisymmetric guy? We get a "B-field" which corresponds to a momentum spacetime field $\tilde{B}_{\mu\nu} = -\tilde{B}_{\nu\mu}$, where

$$\tilde{B}_{\mu\nu}(k) \sim \tilde{B}_{\mu\nu}(k) + k_\mu \lambda_\nu - k_\nu \lambda_\mu. \tag{5.18}$$

In spacetime this is a gauge invariance, where $B_{\mu\nu} \sim B_{\mu\nu} + \partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu$. Some older textbooks call this the notoph (which is nearly "photon" backwards).

Lecture 6.

Wednesday, January 30, 2019

Last time, we discovered the mildly disturbing fact that our bosonic string theory has tachyons. Having made note of this, we decided to take it on faith that superstring theory has a reasonable solution to this problem, and proceeded to define massless modes of the string by

$$|g\rangle = h_{\mu\nu} \alpha_{-1}^{(\mu} \bar{\alpha}_{-1}^{\nu)} |k\rangle \tag{6.1}$$
$$|B\rangle = B_{\mu\nu} \alpha_{-1}^{[\mu} \bar{\alpha}_{-1}^{\nu]} |k\rangle \tag{6.2}$$
$$|\phi\rangle = \phi \alpha_{-1}^{\mu} \bar{\alpha}_{-1\mu} |k\rangle. \tag{6.3}$$

These correspond sort of to a graviton ($g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$), a B-field (since $B_{\mu\nu} = -B_{\nu\mu}$), and a so-called dilaton ϕ (scalar field). One can show that these fields arise as a linear approximation to the theory described by the following spacetime action:

$$S = -\frac{1}{2K^2} \int d^D x \sqrt{-g} e^{-2\phi} \left(R - 4\partial_\mu \phi \partial^\mu \phi + \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \right), \tag{6.4}$$

where $H_{\mu\nu\lambda} = \partial_{[\mu} B_{\nu\lambda]}$ and K is a coupling constant which will be related to Newton's gravitational constant in D dimensions.

This suggests to us that the fields and modes on our worldsheet have in fact told us something about how to deform the background (until now Minkowski) metric, so we could consider the more general starting point

$$S_1[X, h] = -\frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X). \tag{6.5}$$

Moreover, it turns out that the quantum theory has Weyl symmetry if $g_{\mu\nu}(x)$ satisfies

$$R_{\mu\nu} = 0$$

to first order in α' (the only parameter in our theory, really), which are simply the Einstein equations in vacuum. That is, imposing the symmetry of the quantum theory on the worldsheet results in a condition on the background metric in all of spacetime. Higher orders will give corrections to this result – to next order in α',

$$R_{\mu\nu} + \frac{\alpha'}{2} R_{\mu\rho\lambda\sigma} R_\nu^{\rho\lambda\sigma} = 0. \tag{6.6}$$

Our theory therefore suggests that there are higher order corrections to the Einstein equations.

We could also add a term like

$$S_2 = -\frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{-h} \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu B_{\mu\nu}, \tag{6.7}$$

with ϵ^{ab} the completely antisymmetric rank two tensor. This links the action to the stress-energy tensor of our B-field.

Finally, we could add a coupling to the dilaton,

$$S_3 = \frac{1}{4\pi} \int_\Sigma d^2\sigma \sqrt{-h} \phi(X) R_\Sigma, \tag{6.8}$$

with R_Σ the worldsheet Ricci scalar.

The condition that the action

$$S = S_1 + S_2 + S_3$$

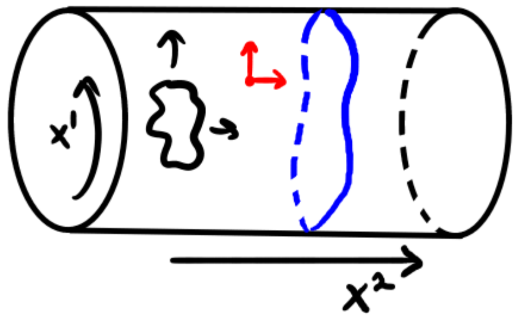

FIGURE 1. A spacetime with the topology $\mathbb{R}^2 \times S^1$. A particle (red) can move along the length of the cylinder and around its circumference. A string on the surface (black) can do the same. But this spacetime also admits the string configuration (blue) which wraps around the circumference.

gives a Weyl-invariant quantum theory results in what we might call equations of motion in spacetime for $g_{\mu\nu}, B_{\mu\nu}$, and ϕ. To leading order in α', these equations of motion may be derived from the action

$$S = -\frac{1}{2K^2}\int d^D x\sqrt{-g}e^{-2\phi}\left(R - 4\partial_\mu\phi\partial^\mu\phi + \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right) + O(\alpha').$$

If you like, the worldsheet theory couples to the metric of the background spacetime. Now, we could have just written down this action to start with. But deriving it from the worldsheet allows us to argue that any higher order terms are suppressed by the length scale of α'.

What happens if spacetime has some weird topology? Consider a theory where spacetime has the topology of $\mathbb{R}^2 \times S^1$, as in Fig. 1 Then a string can move around the spacetime just like a particle, but it can also wrap around the compact S^1 direction and probe the topology of the spacetime. Therefore something else interesting is happening which the modes we've currently defined seem totally insensitive to.

Path integral quantization Some of the details of path integral quantization are covered in *Advanced Quantum Field Theory*, and also in Polchinski (appendix in vol. 1), as well as in Ryder on QFT and Feynman and Hibbs (though this last one is broadly maligned for having errors in other sections).

Path integrals give us a conceptually different way to think about calculating amplitudes in QM and more generally in QFT. Morally speaking, a path integral is a weighted sum of paths satisfying some boundary conditions,

$$\langle x_f, t_t | x_i, t_i\rangle = \int_{x(t_i)}^{x(t_f)} \mathcal{D}x e^{iS[x]} \tag{6.9}$$

for some action $S[x] = \int_{t_i}^{t_f} dt L(x,\dot{x})$. We will be interested in the apath integral quantization of the Polyakov action.

That is, given some initial and final string states $\Psi_{i,f}$, the path integral is

$$\langle \Psi_f | \Psi_i\rangle = \int_i^f \mathcal{D}x\mathcal{D}h e^{iS[h,x]}, \tag{6.10}$$

with $S[h,x]$ the Polyakov action. But now by analogy with QFT we will have to deal with strings splitting and merging in our path integral, as shown in Fig. 2. There will be new complications when we try to compute the path integral.

Lecture 7.

Friday, February 1, 2019

Let us now continue our discussion of path integral quantization. Heuristically, we'll `import` the details of path integral quantization and see what works out. We want to understand how to make sense of

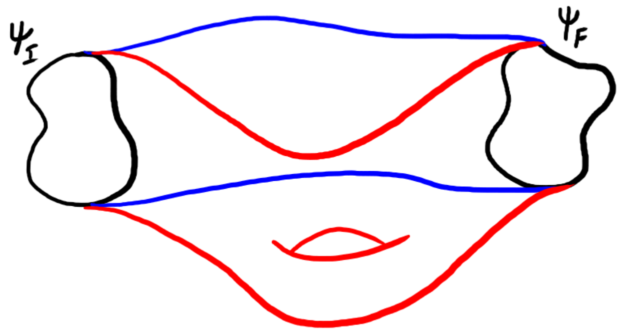

FIGURE 2. Two worldsheet configurations we might need to sum over in the path integral from Ψ_i (left) to Ψ_f (right). One worldsheet (blue) has the string propagating directly from Ψ_i to Ψ_f, while the other (red) has the string pinching off and splitting into two before merging back (the equivalent of a scattering process in QFT).

expressions like

$$\int \mathcal{D}h\mathcal{D}X\, e^{iS[h,X]} \tag{7.1}$$

where we are integrating over the space of metrics h_{ab} and embedding fields X^μs. When we do this calculation, we have to be careful not to overcount– there is a huge diffeomorphism symmetry and a Weyl symmetry in our theory relating physically equivalent states. If this path integral is to give us anything physically meaningful, we need to "quotient out" by the space of diffeomorphisms and Weyl transformations.

We would like to split the integral over all h_{ab} into integrals over physically inequivalent h_{ab} and those related by gauge transformations. Schematically,

$$\mathcal{D}h = \mathcal{D}h_{\text{phys}} \times \mathcal{J}\mathcal{D}h_{\text{Diff}\times\text{Weyl}}, \tag{7.2}$$

where $\mathcal{J}$ is a Jacobian factor whose importance we'll see in the following example.

Example 7.3. As a toy example, consider the following integral:

$$\int dxdy\, e^{-(x^2+y^2)}.$$

This isn't too hard to do– it separates into two Gaussian integrals readily. But notice that x^2+y^2 is invariant under rotations about the origin. When we pass to polar coordinates, the θ angular integral becomes trivial, so we might really be interested in this integral modulo rotations. Thus our integral can be rewritten

$$\int d\theta \int dr\, re^{-r^2}.$$

This $\int d\theta$ will always give us a factor of 2π (the "volume" of an orbit of the rotation group)– our real interest is in the dr integral.

In this example, we needed the Jacobian of the coordinate transformation: $dxdy = rdrd\theta$. The same is true of our path integral. Formally, we will take

$$\frac{1}{|\text{Diff}| \times |\text{Weyl}|} \int \mathcal{D}h\mathcal{D}X = \int \mathcal{D}h_{\text{phys}}\mathcal{D}S_{\text{phys}}\, \mathcal{J}, \tag{7.4}$$

where $\mathcal{J}$ is now a functional determinant and $|\text{Diff}|, |\text{Weyl}|$ represents the orbits of diffeomorphisms and Weyl transformations. In the same way we could write

$$\sqrt{\frac{\pi}{\det M}} = \int_V dx\, e^{-(x,Mx)}, \tag{7.5}$$

we will write $\mathcal{J}$ as a functional integral,

$$\mathcal{J} = \int \mathcal{D}b\mathcal{D}c\, e^{-S[b,c]}. \tag{7.6}$$

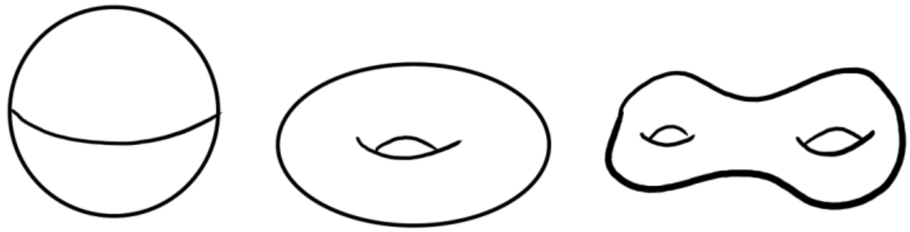

FIGURE 3. Three surfaces of genus $0, 1$, and 2, respectively. The first is the sphere S^2, the second is the torus T^2, and the final is a "handlebody" of genus two.

Global properties of the worldsheet We need to know more about what type of worldsheets appear in the path integral. This will take us on a crash course through Riemann surfaces.

We have looked at 2-dimensional Riemannian manifolds (Σ, h) modulo Weyl transformations. The set of Riemannian manifolds modulo Weyl transformations is known as *Riemann surfaces*. Quotienting out by diffeomorphisms is assumed. Note that worldsheets are Riemann surfaces.

We'll state a number of results without proof, though some of them are not too hard to prove– for more detail, see Farkas and Kra, and also Donaldson.

The first idea we'll consider is the *worldsheet genus*. For Riemann surfaces without boundary (i.e. a closed string, neglecting the initial and final string states), the relevant topological data is encoded in the *Euler characteristic*,

$$\chi = \frac{1}{4\pi} \int_\Sigma d^2\sigma \sqrt{h} R(h). \tag{7.7}$$

Here, $R(h)$ is the Ricci scalar with respect to the worldsheet metric h. The Euler characteristic captures the idea that while we can locally make the metric look however we want, in general there will be obstructions to globally bringing the metric to a required form. The *genus* g is given by

$$\chi = 2 - 2g, \tag{7.8}$$

and informally counts the "number of holes in Σ," as shown in Fig. 3. Why we care is because the genus is a topological invariant– we can't change the number of holes in a Riemann surface under smooth maps.

Moduli space of Riemann surfaces For a given genus g, the space of metrics on Σ_g modulo Weyl and diffeomorphisms is a finite-dimensional space called the *moduli space*. Schematically,

$$\mathcal{M}_g = \frac{\{\text{metrics } h_{ab}\}}{\{\text{Diff}\} \times \{\text{Weyl}\}}.$$

Both the numerator and denominator here are infinite dimensional, but our saving grace will be the following fact– the integral itself is finite-dimensional.

A useful result is the following: let s be the real dimension of the moduli space $\mathcal{M}_g$. Then

$$s = \dim \mathcal{M}_g = \begin{cases} 0, & g = 0 \\ 2, & g = 1 \\ 6g - 6, & g \geq 2. \end{cases} \tag{7.9}$$

Example 7.10. Given a metric $\hat{h}_{ab}$ on a $g = 0$ surface, we can bring any metric to the form $e^{2w}\hat{h}_{ab}$. This is not the case for a torus ($g = 1$). We can build a torus by imposing identifications on $\mathbb{C}$, i.e. under the equivalence relation

$$z \sim z + n\lambda_1 + m\lambda_2, \tag{7.11}$$

where $n, m \in \mathbb{Z}$ and λ_1, λ_2 specify the "dimensions" of the torus.

One can show that the ratio $\tau \equiv \lambda_1/\lambda_2$ is Diff and Weyl invariant. However, we can always choose λ_1, λ_2 such that $\text{Im}(\tau) \geq 0$. We also get a metric

$$ds^2 = |dz + \tau d\bar{z}|^2. \tag{7.12}$$

If we transform $\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} \to U\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}$ for some matrix U, then we can undo that change by also changing the equivalence relation numbers $(n,m) \to (n,m)U^{-1}$. For n,m to be integers under any such transformation, we require the entries of U to all be integers, i.e. $U \in SL(2,\mathbb{Z})$.

Our moduli space is

$$\mathcal{M}_1 = \frac{UHP}{SL(2,\mathbb{Z})}, \tag{7.13}$$

with UHP the upper half-plane, $\tau, \operatorname{Im}\tau \geq 0$.

Lecture 8.

Monday, February 4, 2019

We've started our lightning tour of the theory of Riemann surfaces. Soon, we'll see the emergence of our first scattering amplitudes.

Conformal Killing vectors Recall from *General Relativity* that Killing vectors are very special objects which represent symmetries of the metric. In the language of Lie derivatives, a vector K is a Killing vector if the Lie derivative of the metric with respect to K is trivial, $\mathcal{L}_K g = 0$.[11] *Conformal Killing vectors* (CKV) generalize this idea. A conformal Killing vector generates diffeomorphisms that preserve the metric up to Weyl transformations.

Our gauge transformations are

$$\delta_V h_{ab} = \nabla_a V_b + \nabla_b V_a \tag{8.1}$$

$$\delta_\omega h_{ab} = 2\omega h_{ab}. \tag{8.2}$$

We are interested in V^a such that

$$\delta_{CK} h_{ab} = \nabla_a V_b + \nabla_b V_a + 2\omega h_{ab} = 0. \tag{8.3}$$

Note the covariant derivatives are taken with respect to the metric h_{ab}. Taking the trace, we have equivalently

$$2(\nabla_a V^a) + 4\omega = 0 \implies \omega = -\frac{1}{2}(\nabla_a V^a), \tag{8.4}$$

so V^a is a conformal Killing vector if

$$\delta h_{ab} = \nabla_a V_b + \nabla_b V_a - h_{ab}(\nabla_c V^c) = 0. \tag{8.5}$$

We define

$$(Pv)_{ab} \equiv \nabla_a V_b + \nabla_b V_a - h_{ab}(\nabla_c V^c) \tag{8.6}$$

so that V^a is a conformal Killing vector if $V^a \in \operatorname{Ker} P$.

Why have we introduced these? For closed Riemann surfaces of genus g, the (real) dimension of the conformal Killing group (CKG), i.e. the subgroup of diffeomorphisms generated by the conformal Killing vectors, is known: it is

$$\kappa = |\text{CKG}| = \begin{cases} 6, & g = 0 \\ 2, & g = 1 \\ 0, & g \geq 2. \end{cases} \tag{8.7}$$

On the sphere (think of this as $\mathbb{C}$ with the point at ∞), the CKVs generate the transformations

$$z \to \frac{az+b}{cz+d} \tag{8.8}$$

and similarly for $\bar{z}$, where $a,b,c,d \in \mathbb{C}$ and $ad - bc = 1$. This is in fact the Möbius group from complex analysis. We have four parameters and one algebraic constraint on complex values (hence two real constraints). Therefore we shall fix the conformal Killing symmetry by requiring that the V^a vanish at three distinct points on Σ (i.e. imposing six real constraints, since each point on Σ comes with two coordinates).

We'll need one more mathematical preliminary before moving forward. This is the *modular group.* First, observe that the diffeomorphism group on the Riemann surface Σ_g is in general not connected. Let us

[11] In terms of covariant derivatives, $\nabla_a K_b + \nabla_b K_a = 0$.

therefore define something useful– call the connected set of diffeomorphisms that includes the identity Diff_0. The modular group $\mathcal{M}_g$ is then

$$\mathcal{M}_g = \frac{\text{Diff}}{\text{Diff}_0}. \tag{8.9}$$

For example, for the torus we have $\mathcal{M}_1 = \text{SL}(2:\mathbb{Z})$.

Then the moduli space M_g can be written schematically as

$$M_g = \frac{\{\text{metrics}\}}{\{\text{Diff}\} \times \{\text{Weyl}\}} = \frac{\{\text{metrics}\}}{\{\text{Diff}_0\} \times \{\text{Weyl}\}} / \mathcal{M}_g. \tag{8.10}$$

We often call the space

$$\mathcal{T}_g = \frac{\{\text{metrics}\}}{\{\text{Diff}_0\} \times \{\text{Weyl}\}} \tag{8.11}$$

the Teichmüller space. In this notation, $M_g = \mathcal{T}_g / \mathcal{M}_g$.

The Faddeev-Popov determinant When we do path integrals, it's usually desirable to check our answer by other means, since path integrals have a way of hiding divergences which we as self-respecting physicists ought to care about. Happily, this will be possible for the following quantity we are about to define.

The idea is to choose a "gauge slice" through the space of metrics on Σ_g. That is, we choose a gauge such that the metric on the worldsheet h_{ab} takes some nice form, $h_{ab} = \hat{h}_{ab}$ (often diagonal), such that $\text{Diff}_0 \times \text{Weyl}$ orbits then take us everywhere else in our space of metrics. We formally define the *Fadeev-Popov determinant* as

$$1 = \Delta_{FG}(\hat{h}) \int_{\text{Diff}_0 \times \text{Weyl}} \mathcal{D}(\delta h) \delta[h - \hat{h}] \prod_i \delta(v(\hat{\sigma}_i)), \tag{8.12}$$

where $\delta[h-\hat{h}]$ can be thought of as a "delta functional" and σ_i indicates points on our worldsheet Σ_g where the CKVs vanish (in order to fix the CKG). We can think of this determinant in analogy to how $\delta(f(x)) \sim \frac{\delta x}{|f'(x_i)|}$ where $f(x_i) = 0$.

In more detail, we may write

$$1 = \Delta_{FG}(\hat{h}) \int_{\mathcal{T}} d^s t \int \mathcal{D}\omega \mathcal{D}v \delta[h_{ab} - \hat{h}_{ab}] \prod_i \delta(v(\hat{\sigma}_i)), \tag{8.13}$$

where the $d^s t$ integral is taken in Teichmüller space and our path integral is now written explicitly over the space of variations of h.

We will now write the delta functions and delta functions as integrals and functional integrals. Let us introduce numbers ζ^i_a and fields $\beta^{ab}(\sigma,\tau)$ such that

$$1 = \Delta_{FG}(\hat{h}) \int_{\mathcal{T}} d^s t \int \mathcal{D}\omega \mathcal{D}v \Big(d^\kappa \zeta^i_a \mathcal{D}\beta \exp(i(\beta | h - \hat{h}) + i\zeta^i_a v^a(\hat{\sigma}_i) \Big), \tag{8.14}$$

where the inner product $(\beta | h - \hat{h})$ is defined to be

$$(\beta | h - \hat{h}) = \int_\Sigma d^2\sigma \sqrt{|h|} \beta^{ab} (h_{ab} - \hat{h}_{ab}) \tag{8.15}$$

We can write $h_{ab} - \hat{h}_{ab} = \delta_{ab}$ as

$$\begin{aligned}
\delta h_{ab} &= \underbrace{\nabla_a v_b + \nabla_b v_a}_{\text{Diffeos}} + \underbrace{2\omega h_{ab}}_{\text{Weyl}} + \underbrace{t^I \partial_I h_{ab}}_{\text{moduli}} \\
&= (Pv)_{ab} + 2(\omega + \nabla_c v^c) h_{ab} + t^I \partial_I h_{ab} \\
&= (Pv)_{ab} + 2\bar{\omega} h_{ab} + t^I \mu_{Iab}
\end{aligned}$$

where $(Pv)_{ab}$ is as defined before, $\mu_{Iab} = \partial_I h_{ab} - \text{trace}$, and $\bar{\omega}$ contains the residual trace terms.

Lecture 9.

Wednesday, February 6, 2019

The official course notes and the first example sheet are online now. Note that David Tong's notes may also supplement the notes for this course. In addition, note that problems 4 and 5 are eligible for marking, while problem 6 has a typo and therefore the instructor asks that we ignore problem 6 entirely.

Last time, we introduced the Faddeev-Popov determinant. We found that

$$\Delta_{FP}^{-1}(\hat{h}) = \int_{\mathcal{T}} d^s t \int \mathcal{D}\omega\mathcal{D}v \left(d^\kappa \zeta_a^i \mathcal{D}\beta \exp(i(\beta|Pv + 2\bar{\omega}h + t^I \mu_I)) + i\sum_{i=1}^{k} \zeta_a^i v^a(\hat{\sigma}_i) \right) \tag{9.1}$$

Grassmann quantities If you're keeping up with my *AQFT* and *Supersymmetry* notes, this will be your third time seeing Grassmann quantities/variables. These are a set of quantities θ such that any two of them anticommute,

$$\theta_1\theta_2 = -\theta_2\theta_1.$$

Equivalently their anticommutator vanishes,

$$\{\theta_1, \theta_2\} = 0.$$

Objects (such as wavefunctions) that obey Fermi statistics can naturally be described by Grassmann numbers. One bit of motivation for this is the fact that for any θ, we have $\theta^2 = -\theta^2 = 0$, which is reminiscent of the Pauli exclusion principle. This anticommuting property also holds for integration measures,

$$d\theta_1 d\theta_2 = -d\theta_2 d\theta_1.$$

One can show (e.g. by considering $(\int d\theta)^2$) that

$$\int d\theta = 0,$$

and we can consistently define

$$\int d\theta\, \theta = 1.$$

The Dirac delta function for Grassman quantities is then $\delta(\theta) = \theta$, which leads to the somewhat unusual conclusion that integration and differentiation of Grassmann variables are essentially the same process.

Note that Taylor expansions are very easy for Grassman variables, since we cannot have anything of higher degree than 1 because $\theta^2 = 0$. Thus we can write some function $f(x,\theta)$ as

$$f(x,\theta) = f_0(x) + \theta f_1(x),$$

and so an integral can be written

$$\int d\theta\, f(x,\theta) = f_1(x) = \frac{\partial f(x,\theta)}{\partial\theta}. \tag{9.2}$$

Example 9.3. Let $\theta^a = \begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}$ and $\bar{\theta}^a = (\bar{\theta}_1, \bar{\theta}_2)$. Consider the integral

$$\int d^2\theta d^2\bar{\theta} \exp\left(-\bar{\theta}^a M_{ab}\theta^b\right), \tag{9.4}$$

where M_{ab} is some normal 2×2 matrix. This exponential has a few terms but not too many. The first term is just 1, while the last term has 4 thetas and two Ms. Recalling that integration is like differentiation, we write the integral as

$$\frac{\partial^4}{\partial\theta_1\partial\theta_2\partial\bar{\theta}_1\partial\bar{\theta}_2}\{(\bar{\theta}_1 M_{11}\theta_1)(\bar{\theta}_2 M_{22}\theta_2) + (\bar{\theta}_1 M_{12}\theta_2)(\bar{\theta}_2 M_{21}\theta_1\}, \tag{9.5}$$

noting that the only nonzero term must have all four of θ_1, θ_2, and their barred versions.

Equivalently this integral is

$$\int d^2\theta d^2\bar{\theta} \exp\left(-\bar{\theta}^a M_{ab}\theta^b\right) = (M_{11}M_{22} - M_{12}M_{21}) = \det(M_{ab}). \tag{9.6}$$

This result generalizes– the equivalent of a Gaussian for Grassmann variables is

$$\int d^n\theta d^n\bar{\theta}\exp(-\bar{\theta}^a M_{ab}\theta^b) = \det(M_{ab}). \tag{9.7}$$

Note that this is a bit different from the result for $z, \bar{z}$ real, where

$$\int d^2z d^2\bar{z}\exp(-\bar{z}Mz) = \frac{1}{\det(M_{ab})}. \tag{9.8}$$

This effect of inverting the determinant when we replace commuting (bosonic) variables with Grassmann (fermionic) variables carries over to the functional case, which we will just state but not prove.

With our "new" Grassmann variables in hand, we will now rewrite the Faddeev-Popov determinant in terms of Grassmann quantities to perform these crazy path integrals. That is, promote

$$v^a \to c^a, \beta^{ab} \to b^{ab}, t^I \to \xi^I, \zeta^i_a \to \eta^i_a, \tag{9.9}$$

where c^a and $b^{ab} = b^{ba}$ are Grassmann fields on Σ.

Note also that we can apparently get rid of the $\mathcal{D}\omega$ integral by writing

$$\begin{aligned}\Delta_{FP}^{-1} &\sim \int \mathcal{D}\omega \exp[i(\beta|2\bar{\omega}h)] \\ &\sim \int \mathcal{D}\bar{\omega}\exp\left[i\int_\Sigma d^2\sigma\sqrt{h}\beta^{ab}2\bar{\omega}h_{ab}\right].\end{aligned}$$

But we can do this $\bar{\omega}$ integral– it looks like a delta function, and fixes $\beta^{ab}h_{ab} = 0$. Thus β^{ab} is traceless.

Thus we rewrite the Fadeev-Popov determinant in terms of our shiny new Grassmann variables as

$$\Delta_{FP}(\hat{h}) = \int d^s\xi \int \mathcal{D}c\mathcal{D}b d^k\eta \exp(i(b|Pc + \xi^I\mu_I) + i\sum_{i=1}^k \eta^i_a c^a(\hat{\sigma}_i)), \tag{9.10}$$

having done the ω integral as above. Note that this is really just the Faddeev-Popov determinant and not its inverse, since we have promoted everything to Grassmann variables. We can also do the η^i_a and ξ^I integrals to get

$$\begin{aligned}\Delta_{FP}(\hat{h}) &= \int \mathcal{D}c\mathcal{D}b\, e^{i(b|Pc)}\prod_{I=1}^S \delta[(b|\mu_I)]\prod_{i=1}^K \delta(c^a(\hat{\sigma}_i)) \\ &= \int \mathcal{D}c\mathcal{D}b\, e^{i(b|Pc)}\prod(b|\mu_I)\prod_{i=1}^k c^a(\hat{\sigma}_i).\end{aligned}$$

After all this computation, we therefore have

$$\Delta_{FP}(\hat{h}) = \int \mathcal{D}c\mathcal{D}b\, e^{iS[b,c]}\prod(b|\mu_I)\prod_{i=1,a=1,2}^k c^a(\hat{\sigma}_i) \tag{9.11}$$

where we have something that looks like an action,

$$S[b,c] = \int_\Sigma d^2\sigma\,\sqrt{h}b^{ab}(Pc)_{ab} = 2\int_\Sigma d^2\sigma\,\sqrt{h}b^{ab}(\nabla_a c_b). \tag{9.12}$$

Note that c^a, b_{ab} are Grassmann fields and therefore obey Fermi statistics. However, it turns out they also have integer "spin" (for some notion of spin we have not defined precisely yet). Fortunately, this is allowed because these quantities are not observables. We should think of them a bit like constraints on the observable variables of our theory, and we call them *Faddeev-Popov ghosts*.

Lecture 10.

Friday, February 8, 2019

Today, we'll wrap up our discsussion of global physics on the worldsheet. Let us return to the schematic path integral expression

$$Z = \frac{1}{|\text{Diff}| \times |\text{Weyl}|}\int \mathcal{D}X\mathcal{D}h\, e^{iS[h,X]}. \tag{10.1}$$

We will insert a factor of 1 using our expression for the Faddeev-Popov determinant:

$$1 = \Delta_{FG}(\hat{h}) \int_{\mathcal{T}_g} d^s t \int \mathcal{D}\bar{\omega}\mathcal{D}v \delta[h - \hat{h}] \prod_{i,a} \delta(v^a(\hat{\sigma}_i)). \tag{10.2}$$

The delta functional will do the $\mathcal{D}h$ integral for us, at the cost of introducing some other integrals into the picture. We rewrite

$$Z = \frac{1}{|\text{Diff}| \times |\text{Weyl}|} \int \mathcal{D}X e^{iS[X,\hat{h}]} \int_{\mathcal{T}_g} d^s t \int \mathcal{D}\bar{\omega}\mathcal{D}v \prod_{i,a} \delta(v^a(\hat{\sigma}))\Delta_{FP}(\hat{h}). \tag{10.3}$$

But now notice that

$$|\text{Weyl}| \times \frac{|\text{Diff}_0|}{|\text{CKG}|} = \int \mathcal{D}\bar{\omega} \int \mathcal{D}v \prod \pi_{i,a} \delta(v^a(\hat{\sigma}_i)).$$

That is, the delta functions are equivalent to quotienting out by the symmetries of the conformal Killing vectors, and these other integrals are taken over diffeomorphisms connected to the identity and related by Weyl transformations. This is still extremely schematic but we can "cancel" the Weyl groups and recognize $|\text{Diff}_0|/|\text{Diff}| = 1/|\mathcal{M}_g|$ so that

$$\frac{1}{|\text{Diff}| \times |\text{Weyl}|} \times |\text{Weyl}| \times \frac{|\text{Diff}_0|}{|\text{CKG}|} = \frac{1}{|\mathcal{M}_g| \times |\text{CKG}|}. \tag{10.4}$$

With this notation,

$$Z = \frac{1}{|\mathcal{M}_g||\text{CKG}|} \int_{\mathcal{T}_g} d^s t \int \mathcal{D}X e^{iS[x,\hat{h}]} \Delta_{FP}(\hat{h}). \tag{10.5}$$

We take this to mean an integral over the Teichmüller space quotiented by the modular group, i.e. over the moduli space M_g. Thus

$$\frac{1}{|\mathcal{M}_g|} \int_{\mathcal{T}_g} d^s t \equiv \int_{\mathcal{T}_g/\mathcal{M}_g} d^s t = \int_{M_g} d^s t,$$

and our full path integral is now an integral over the moduli space and the Grassmann fields b, c (substituting in our expression for Δ_{FG} explicitly):

$$Z = \frac{1}{|\text{CKG}|} \int_{M_g} d^s t \int \mathcal{D}X\mathcal{D}b\mathcal{D}c \, e^{iS[\hat{h},X,b,c]} \prod_{I=i}^{s} (b|\mu_I) \prod_{i,a} c^a(\hat{\sigma}i). \tag{10.6}$$

As before, our inner product is given by $(b|\mu_I) = \int_\Sigma d^\sigma \sqrt{|h|} b^{ab} \mu_{Iab}$ with $\mu_{Iab} = \partial_I h_{ab} -$ trace. We shall choose to define b, c such that the action takes the form

$$S[\hat{h}, X, b, c] = -\frac{1}{4\pi\alpha'} \int_\Sigma d^s\sigma \sqrt{|\hat{h}|} \hat{h}^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} + \frac{1}{2\pi} \int_\Sigma d^2\sigma \sqrt{|\hat{h}|} b^{ab} \nabla_a c_b. \tag{10.7}$$

It may be useful to consider the ghosts (bs and cs) as an integral part of the theory, rather than a hack we've added to make sense of these infinite-dimensional spaces of metrics. As we've said, these ghosts will represent important constraints, particularly when we try to figure out the dimensionality of the bigger spacetime in which our worldsheet lives.

Introduction to conformal field theory Conformal field theories (CFTs) are among the best-understood quantum field theories we have. Outside of string theory, they also have applications in condensed matter physics and other areas, and we'll see that our action as given above defines a CFT in two dimensions, which turns out to be a very special case.

We are interested in theories that are invariant under Weyl transformations. We can ask the following question: what is the natural generalization of the Poincaré group that preserves a metric up to Weyl transformations? In a general dimension $d > 1$, we are interested in transformations such that

$$\eta_{\rho\sigma} \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} = \Lambda(x) \eta_{\mu\nu}, \tag{10.8}$$

where infinitesimally, $x^\mu \to x'^\mu = x^\mu + V^\mu(x) + \ldots$. Morally, we are combining Lorentz boosts and rotations with local scale transformations.

We find that if $\Lambda(x) = e^{\omega(x)}$, then $\omega(x)$ and $v^\mu(x)$ are related by

$$\omega(x) = \frac{2}{d}\partial_\mu v^\mu(x), \tag{10.9}$$

so $v^\mu(x)$ satisfies

$$\partial_\mu v_\nu + \partial_\nu v_\mu = \frac{2}{d}\eta_{\mu\nu}\partial_\lambda v^\lambda(x). \tag{10.10}$$

We say that $V^\mu(x)$ satisfying this condition generates conformal transformations.[12]

Two dimensional CFTs Let us take

$$h_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

a metric up to a conformal factor (Wick rotation) where we have sent $t \to i\tau$ if you like. That is, we've switched from Lorentzian signature to Euclidean signature. Not a problem. We have some coordinates on the manifold given by

$$x^\mu \to \sigma^a = (\tau, \sigma). \tag{10.11}$$

The condition 10.10 now becomes

$$2\partial_\tau v_\tau = \partial_\tau v_\tau + \partial_\sigma v_\sigma \implies \partial_\tau v_\tau = \partial_\sigma v_\sigma, \tag{10.12}$$

in the case where $\mu = \nu$, and

$$\partial_\sigma v_\tau + \partial_\tau v_\sigma = 0 \tag{10.13}$$

for $\mu \neq \nu$. We write these as

$$\frac{\partial v_\tau}{\partial \tau} = \frac{\partial v_\sigma}{\partial \sigma}, \quad \frac{\partial v_\tau}{\partial \sigma} = -\frac{\partial v_\sigma}{\partial \tau}. \tag{10.14}$$

But these are just the Cauchy-Riemann equations for a complex function $v = v^\tau + iv^\sigma$, i.e. the requirement that v is holomorphic.

We conclude that in $d = 2$, the condition on $v = v^\tau + iv^\sigma$ given by 10.10 is that v is holomorphic,

$$\frac{\partial}{\partial \bar{z}} v = 0 = \bar{\partial} v \tag{10.15}$$

where $z = \tau + i\sigma, \bar{z} = \tau - i\sigma$. This tells us that it's natural to work not in worldsheet coordinates τ, σ but in the variables $z, \bar{z}$. However, we can do better– we also want variables which vary in some natural way under conformal transformations. Since all holomorphic transformations preserve our metric up to Weyl transformations, a better choice is

$$z = e^{\tau + i\sigma}, \quad \bar{z} = e^{\tau - i\sigma}. \tag{10.16}$$

In these variables, the worldsheet is mapped to the complex plane, with the infinite future mapped to the point at infinity. We can think of the worldsheet Σ as the Riemann sphere with two points removed.

In these new coordinates $(z, \bar{z})$, we find that the Polyakov action (remember that?) takes the form

$$S = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma \partial_a X^\mu \partial^a X^\nu \eta_{\mu\nu} = \frac{i}{2\pi\alpha'}\int_\Sigma d^2 z \partial X^\mu \bar{\partial} X^\mu \eta_{\mu\nu}, \tag{10.17}$$

where we have denoted $\partial \equiv \frac{\partial}{\partial z}, \bar{\partial} = \frac{\partial}{\partial \bar{z}}$. The stress tensor T_{ab} now has two non-trivial components:

$$T_{zz} \equiv T = -\frac{1}{\alpha'}\partial X^\mu \partial X^\nu \eta_{\mu\nu}, \tag{10.18}$$

$$T_{\bar{z}\bar{z}} \equiv \bar{T} = -\frac{1}{\alpha'}\bar{\partial} X^\mu \bar{\partial} X^\nu \eta_{\mu\nu}, \tag{10.19}$$

and $T_{z\bar{z}} = 0$ identically.

Finally, a quick note. In QFT we had a notion of time-ordering. For our theory, we see almost trivially that time ordering will be replaced by a "radial" ordering, i.e. curves at larger "time" τ correspond to larger radii in the complex plane.

[12] Note that v^μ looks a lot like the conformal Killing vectors we defined earlier.

Lecture 11.

Monday, February 11, 2019

Last time, we introduced conformal field theory. We found that for our two-dimensional worldsheet, we can construct a map

$$z = e^{\tau + i\sigma}, \quad \bar{z} = e^{\tau - i\sigma}.$$

We saw that the coordinates $z \to z' = f(z)$ were more generally holomorphic, i.e. we get some functions which satisfy the Cauchy-Riemann equations.

Conformal fields Here are some definitions common in the literature for conformal field theory.

Definition 11.1. A *chiral field* is a field Φ that depends on z only, i.e. $\Phi = \Phi(z)$. Similarly, an *anti-chiral field* is a field that depends only on $\bar{z}$.

Definition 11.2. The *conformal dimension* refers to how a field transforms under scalings $z \to z' = \lambda z, \bar{z} \to \bar{z}' = \bar{\lambda}\bar{z}$ ($\lambda \in \mathbb{C}$).

$$\Phi(z, \bar{z}) \to \Phi'(z', \bar{z}') = \lambda^h \bar{\lambda}^{\bar{h}} \Phi(\lambda z, \bar{\lambda}\bar{z}). \tag{11.3}$$

We shall call h and $\bar{h}$ the dimension of $\Phi(z, \bar{z})$.[13] Sometimes $h + \bar{h}$ is referred to as the dimension and $h - \bar{h}$ as the "conformal spin."

Definition 11.4. Under the conformal transformation $z \to z' = f(z)$, a *primary field* with dimension $(h, \bar{h})$ transforms as

$$\Phi(z, \bar{z}) \to \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \Phi(f(z), \bar{f}(\bar{z})). \tag{11.5}$$

That is, a primary field transforms like a tensor (with the appropriate exponents of $h, \bar{h}$).

Example 11.6. Consider an infinitesimal transformation

$$z \to z' = z + v(z) + \ldots = f(z). \tag{11.7}$$

Thus

$$\left(\frac{\partial f}{\partial z}\right)^h = (1 + \partial v)^h \tag{11.8}$$

$$\phi(f(z)) = \phi(z) + v(z)\partial\phi(z) + \ldots. \tag{11.9}$$

So for a field with $(h, \bar{h}) = (h, 0)$ we get

$$\delta\Phi(z) = (h\partial v(z) + v(z)\partial)\Phi(z) \tag{11.10}$$

where we have taken only the term to leading order in h.

Symmetries and the stress tensor For our classical theory, let us start with the action

$$S[X] = -\frac{1}{4\pi\alpha'} \int_\Sigma d^2 z \partial_a X^\mu \partial^a X^\nu \eta_{\mu\nu}. \tag{11.11}$$

Let us note that in going from τ, σ coordinates to $z, \bar{z}$, we pick up an i as the Jacobian factor, meaning that $e^{iS} \to_{(z,\bar{z})} e^{-S}$.

Consider the (conformal) transformation

$$\delta_v X^\mu = v^a \partial_a X^\mu. \tag{11.12}$$

The variation of the action is now

$$\delta_v S[X] = -\frac{1}{2\pi\alpha'} \int_\Sigma d^2 z \Big((\partial_a v^b) \partial_b X^\mu \partial^a X_\mu + v^b \partial_a (\partial_b X^\mu) \partial^a X_\mu \Big) \tag{11.13}$$

where all indices are raised and lowered with the Minkowski metric. After an integration by parts, this transformation becomes

$$\delta_v S[X] = \frac{1}{2\pi} \int_\Sigma d^2 z (\partial^a v^b) T_{ab}, \tag{11.14}$$

[13] This is a lot like what we did in *Statistical Field Theory*. In looking at the RG flows of different fields and couplings, we saw that they scaled in different ways with some scaling dimension.

with T_{ab} our old buddy the stress tensor. This tells us that $\delta S[X] = 0$ requires that

$$\partial^a T_{ab} = 0, \tag{11.15}$$

which is just Noether's theorem. That is, if the action is invariant under conformal transformations, then the stress tensor is conserved.

We could define a conserved charge

$$Q = Q_+ + Q_- \tag{11.16}$$

where

$$Q_\pm = \frac{1}{2\pi}\int_0^{2\pi} d\sigma T_{\pm\pm}(\sigma) \tag{11.17}$$

at $\tau = 0$. Classically, the symmetry transformations are generated by the charge Q:

$$\delta X^\mu = \{Q, X^\mu\}_{PB}. \tag{11.18}$$

What's the analogue of this in the quantum theory? Let's find out.

Conformal transformations and Ward identities For the following discussion, we stay in $d = 2$ but rather than focusing on our embedding fields X, we will work with more general fields $\phi(z, \bar{z})$. We shall be interested in the quantum analogue of Noether's theorem.

Consider a transformation

$$\phi \to \phi' = \phi + \delta\phi, \quad S[\phi'] = S[\phi] + \delta S[\phi]. \tag{11.19}$$

In the classical picture, we would say that if $\delta S = 0$, we've got a symmetry and that gives us some conserved quantity. But a classical action doesn't always uniquely specify a quantum action, and conversely there are some quantum actions we don't know the classical versions of. However, what we can say is that a symmetry of a quantum theory should preserve important features of that theory, and in particular it must preserve *correlation functions*.

Let us consider the correlation function

$$\langle \phi_1(z_1)\ldots\phi_n(z_n)\rangle \equiv \langle \phi_1\ldots\phi_n\rangle .$$

Here, the $\bar{z}_i$ dependence is implicit. Under a transformation, our correlation functions become

$$\begin{aligned}
\langle \phi_1\ldots\phi_n\rangle \to \langle \phi'_1\ldots\phi'_n\rangle &= \int \mathcal{D}\phi' e^{-S[\phi']}\phi'_1\ldots\phi'_n \\
&= \int \mathcal{D}\phi e^{-S[\phi]}(1 - \delta S[\phi] + \ldots)(\phi_1 + \delta\phi_1 + \ldots)\ldots(\phi_n + \delta\phi_n + \ldots) \\
&= \langle \phi_1\ldots\phi_n\rangle - \int \mathcal{D}\phi e^{-S[\phi]}\delta S[\phi]\phi_1\ldots\phi_n + \sum_{k=1}^n \int \mathcal{D}\phi e^{-S[\phi]}\phi_1\ldots\delta\phi_k\ldots\phi_n.
\end{aligned}$$

where we have assumed that $\mathcal{D}\phi' = \mathcal{D}\phi$, i.e. the transformations are such that the integration measure is unchanged. If we require that the new correlations are the same as the old, i.e. $\langle \phi_1\ldots\phi_n\rangle = \langle \phi'_1\ldots\phi'_n\rangle$, then

$$\langle \delta S[\phi]\phi_1\ldots\phi_n\rangle = \sum_{k=1}^n \langle \phi_1\ldots\delta\phi_k\ldots\phi_n\rangle . \tag{11.20}$$

We would like to draw an analogue to the classical current, so we write $\delta S[\phi]$ as

$$\delta S[\phi] = \frac{1}{2\pi i}\int_\Sigma d^2z(\partial_a v(z))j^a(z), \tag{11.21}$$

where v is the parameter of the transformation and j is the classical Noether current. Remember, our aim here is to see how classical symmetries can be promoted to quantum conservation laws. Thus

$$\frac{1}{2\pi i}\int_\Sigma d^2z\partial_a v(z)\langle j^a(z)\phi_1\ldots\phi_n\rangle = \sum_{k=1}^n \langle \phi_1\ldots\delta\phi_k\ldots\phi_n\rangle . \tag{11.22}$$

We also choose Σ and $v(z)$ to isolate a particular $\delta\phi_k$. We define $\omega = z_k$ (thus $\phi_k(z_k) = \phi(\omega)$ and two curves C_1, C_2 such that $\partial\Sigma = C_1 \cup C_2$.We choose $v(z)$ to be constant within C_1, zero outside of C_2, and arbitrary on Σ. We also require C_1, C_2 to encircle $\omega = z_k$ only so that $v = 0$ at all other points $z_{j\neq k}$, which implies that all the other $\delta\phi_j, j \neq k$ vanish. In this way, we have

$$\frac{1}{2\pi i}\int_\Sigma d^2z\partial_a v(z)\,\langle j^a(z)\phi_1\dots\phi_n\rangle = \langle\phi_1\dots\phi(\omega)\dots\phi_n\rangle\,. \tag{11.23}$$

Lecture 12.

Wednesday, February 13, 2019

Last time, we started looking at correlation functions in trying to understand how classical symmetries are promoted to quantum symmetries. We showed quite generally that a symmetry of the quantum theory means that the correlation functions are left invariant,

$$\langle\delta S[\phi]\phi_1\dots\phi_n\rangle = \sum_{k=1}^n \langle\phi_1\dots\delta\phi_k\dots\phi_n\rangle\,,$$

and we saw that under conformal transformations,

$$\delta S[\phi] = \frac{1}{2\pi}\int_\Sigma d^2\sigma(\partial^a v^b)T_{ab} \tag{12.1}$$

with T_{ab} the stress tensor.

Substituting this into our expression relating correlation functions, we have

$$\frac{1}{2\pi}\int_\Sigma d^2\sigma(\partial^a v^b)\,\langle T_{ab}\phi_1\dots\phi_n\rangle = \sum_{k=1}^n \langle\phi_1\dots\delta\phi_k\dots\phi_n\rangle\,. \tag{12.2}$$

We choose our Σ to select a single $\delta_V\phi_k$ on the RHS, i.e. define two curves C_1, C_2 with $\omega = z_k$ inside C_1, $v^a = 0$ outside and on C_2, and $v^a = (v^z(z,\bar z), v^{\bar z}(z,\bar z))$ inside and on C_1. Thus with this choice of Σ,

$$\frac{1}{2\pi}\int_\Sigma d^2\sigma(\partial^a v^b)\,\langle T_{ab}\phi_1\dots\phi_n\rangle = \langle\phi_1\dots\delta_v\phi(\omega,\bar\omega)\dots\phi_n\rangle\,. \tag{12.3}$$

We denote $v^z(z,\bar z) = v(z)$ and $v^{\bar z}(z,\bar z) = \bar v(\bar z)$, though this notation is a little misleading since $\bar v$ is not necessarily the conjugate of v. It is just the part of v^a that depends only on $\bar z$.

Integrating by parts and applying Stokes's theorem we get

$$\begin{aligned}\frac{1}{2\pi}\int_\Sigma d^2\sigma(\partial^a v^b)\,\langle T_{ab}\phi_1\dots\phi_n\rangle &= \frac{1}{2\pi}\int_\Sigma d^2\sigma\partial^a(v^b\,\langle T_{ab}\phi_1\dots\phi_n\rangle) - \frac{1}{2\pi}\int_\Sigma d^2\sigma v^b\partial^a\,\langle T_{ab}\phi_1\dots\phi_n\rangle\\ &= \frac{1}{2\pi i}\oint_{\partial\Sigma=C_1} dz\,v(z)\,\langle T(z)\phi_1\dots\phi_n\rangle - \frac{1}{2\pi i}\oint_{\partial\Sigma=C_2} d\bar z\,\bar v(\bar z)\,\langle\bar T(\bar z)\phi_1\dots\phi_n\rangle\\ &\quad - \frac{1}{2\pi}\int_\Sigma d^2\sigma\,v^b\partial^a\,\langle T_{ab}\phi_1\dots\phi_n\rangle\,,\end{aligned}$$

where we've denoted $T_{zz}(z,\bar z) \equiv T(z,\bar z = T(z)$ and $T_{\bar z\bar z}(z,\bar z) \equiv \bar T(\bar z)$. We see that the boundary term can be rewritten as two contour integrals over the boundary of our region Σ, and moreover the integral over C_2 vanishes since $v^a = 0$ outside and on C_2.

We see that

$$\partial^a\,\langle T_{ab}\phi_1\dots\phi_n\rangle = 0, \tag{12.4}$$

leaving

$$\langle\phi_1\dots\delta_v\phi(\omega,\bar\omega)\dots\phi_n\rangle = \oint_{C_1}\frac{dz}{2\pi i}v(z)\,\langle T(z)\phi_1\dots\phi(\omega,\bar\omega)\dots\phi_n\rangle - \oint_{C_2}\frac{d\bar z}{2\pi i}\bar v(\bar z)\,\langle\bar T(\bar z)\phi_1\dots\phi(\omega,\bar\omega)\dots\phi_n\rangle\,. \tag{12.5}$$

Abstractly, we have the variation

$$\delta_v\phi(\omega,\bar\omega) = \oint_{C(\omega)}\frac{dz}{2\pi i}v(z)T(z)\phi(\omega,\bar\omega) - \oint_{C(\omega)}\frac{d\bar z}{2\pi i}\bar v(\bar z)\bar T(\bar z)\phi(\omega,\bar\omega), \tag{12.6}$$

which we always think of as being inserted into a correlation function.

There are a few subtle points here. We need to take care to define the ordering of operators in this expression, since T, ϕ are operators. In addition, we can see that $T(z)$ ($\bar T(\bar z)$) generates holomorphic (resp.

anti-holomorphic) conformal transformations. Moreover, these are contour integrals, so our calculation reveals that it's the pole structure of $\lim_{z\to\omega} T(z)\phi(\omega,\bar{\omega})$ which governs the conformal transformations.

If we are interested in multiple variations $\langle \delta\phi_1\delta\phi_2\delta\phi_3\phi_4,\ldots\phi_n\rangle$, then we could choose some complicated region encircling just the corresponding points z_1, z_2, z_3.

Radial ordering Recall that we can map our worldsheet coordinates into

$$z = e^{\tau+i\sigma}, \tag{12.7}$$

where e^τ is the radial part of z, such that "time ordering" on the cylinder corresponds to radial ordering on $\mathbb{C}$. Thus $\tau_1 > \tau_2 \iff |z_1| > |z_2|$.

Last term in *Quantum Field Theory*, we computed expectation values of time-ordered objects, e.g. time-ordered correlation functions. Here, we will be interested in radially-ordered correlation functions. We define radial ordering as

$$\mathcal{R}(A(z), B(\omega)) \equiv \begin{cases} A(z)B(\omega) & |z| > |\omega| \\ B(\omega)A(z) & |\omega| > |z|. \end{cases} \tag{12.8}$$

But how should we radially order when we are integrating over some weird contour in the complex plane? For example,

$$\oint_{C(\omega)} R(a(z)b(\omega)$$

with the contour as shown in the image.

The answer is as follows. We can compute the answer in two regions where the ordering is clear, around a circle of some radius $R > |z-\omega|$ where $|z| > |\omega|$ and another circle oriented in the opposite direction with radius $R' < |z-\omega|$ where $|z| < |\omega|$. Thus we have the radial ordering

$$\oint_{C(\omega)} dz\, R(a(z)b(\omega)) = \oint_{C_1} dz\, R(a(z)b(\omega)) - \oint_{C_2} dz\, R(a(z)b(\omega)) = \oint_{C_1} dz\, a(z)b(\omega) - \oint_{C_2} dz\, b(\omega)a(z). \tag{12.9}$$

So our expression for $\delta_v\phi(\omega,\bar{\omega})$ is (once we include radial ordering)

$$\delta_v\phi(\omega) = \oint_{|\omega|<|z|} \frac{dz}{2\pi i} v(z)T(z)\phi(\omega) - \oint_{|\omega|>|z|} \frac{dz}{2\pi i}\phi(\omega)v(z)T(z) \tag{12.10}$$

(for a chiral field) where we only look at the ω dependence.

If we define

$$Q = \oint_{C(\omega)} \frac{dz}{2\pi i} v(z)T(z), \tag{12.11}$$

then we could define a bracket $[\cdot,\cdot]$ as

$$\delta_v\phi(\omega) = [Q, \phi(\omega)]. \tag{12.12}$$

Lecture 13.

Friday, February 15, 2019

Mode expansions Recall we had the expansion in σ, τ coordinates

$$X^\mu(\sigma^+,\sigma^-) = x^\mu + p^\mu\alpha'\tau + i\sqrt{\frac{\alpha}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^\mu e^{-in\sigma^-} + \bar{\alpha}_n^\mu e^{-in\sigma^+}\right), \tag{13.1}$$

and taking a derivative with respect to σ^- gives us

$$\partial_- X^\mu(\sigma^-) = \sqrt{\frac{\alpha'}{2}}\sum_n \alpha_n^\mu e^{-in\sigma^-}, \tag{13.2}$$

where α_0^μ is defined as before in terms of p^μ. We could look at the same object for a worldsheet with Euclidean signature, i.e. $\omega = \tau + i\sigma$, so that

$$\partial_\omega X^\mu(\omega) = -i\sqrt{\frac{\alpha'}{2}}\sum_n \alpha_n^\mu e^{-n\omega}. \tag{13.3}$$

But what we really want to consider is the theory on $\mathbb{C} \cup \{\infty\}$ with coordinates

$$z = e^{\omega} = e^{\tau+\sigma}. \tag{13.4}$$

Consider a chiral primary $\phi_{cyl}(\omega)$ of weight $(h, \bar{h}) = (h, 0)$ deifned on the cylinder. We expand

$$\phi_{cyl}(\omega) = \sum_n \phi_n e^{-n\omega}. \tag{13.5}$$

On the plane, we use the primary transformation law to get

$$\phi(z) = \left(\frac{\partial z}{\partial \omega}\right)^h \phi_{cyl}(\omega) = z^{-h}\phi_{cyl}(\omega) = z^{-h} \sum_n \phi_n z^{-n}. \tag{13.6}$$

Thus a natural mode expansion for $\phi(z)$ is

$$\phi(z) = \sum_n \phi_n z^{-n-h}. \tag{13.7}$$

More generally, a (primary) field of weight $(h, \bar{h})$ takes the form

$$\phi(z, \bar{z}) = \sum_{m,n} \phi_{mn} z^{-m-h} \bar{z}^{-n-\bar{h}}. \tag{13.8}$$

For instance, $T(z)$ and $\bar{T}(\bar{z})$ (which are holomorphic and antiholomorphic) have $(h, \bar{h})$ of $(2, 0)$ and $(0, 2)$ respectively, so

$$T(z) = \sum_n L_n z^{-n-2}, \quad \bar{T}(\bar{z}) = \sum_n \bar{L}_n \bar{z}^{-n-2}. \tag{13.9}$$

Note also that

$$\partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n^\mu z^{-n-1}, \tag{13.10}$$

where ∂ indicates a derivative with respect to z. Note also that

$$X^\mu(z, \bar{z}) = x^\mu - i\frac{\alpha'}{2} p^\mu \ln|z|^2 + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}(\alpha_n^\mu z^{-n} + \tilde{\alpha}_n^\mu \bar{z}^{-n}). \tag{13.11}$$

States and operators For a given physical operator $\Phi(z)$, there is a physical state $|\Phi\rangle$ given by

$$\lim_{z\to 0} \Phi(z)|0\rangle = |\Phi\rangle. \tag{13.12}$$

We shall take this as a definition for now. In the complex plane, we could imagine "inserting an operator" at the origin to produce some string state. With $\partial X^\mu(z)$ as before, consider

$$i\sqrt{\frac{2}{\alpha'}} \partial X^\mu(z)|0\rangle = \left(\ldots + \alpha_{-2}^\mu z + \alpha_{-1}^\mu + \frac{\alpha_0{}^\mu}{z} + \frac{\alpha_1}{z^2} + \ldots\right)|0\rangle. \tag{13.13}$$

For this limit to make sense, we see that some of these α_n^μs must annihilate the vacuum as we postulated earlier,

$$\alpha_n^\mu|0\rangle = 0, n \geq 0. \tag{13.14}$$

Then

$$\lim_{z=\to 0} i\sqrt{\frac{2}{\alpha'}} \partial X^\mu(z)|0\rangle = \alpha_{-1}^\mu|0\rangle. \tag{13.15}$$

For a more interesting example, we could look at

$$\lim_{z\to 0, \bar{z}\to 0} -\left(\frac{2}{\alpha'}\right) h_{\mu\nu} \partial X^\mu(z) \bar{\partial} X^\nu(\bar{z}) e^{ik\cdot X(z,\bar{z})}|0\rangle \tag{13.16}$$

whre k_μ is a momentum vector in spacetime and $h_{\mu\nu} = h_{\nu\mu}$ is a spacetime tensor. In this limit we have

$$h_{\mu\nu}\alpha_{-1}^\mu \tilde{\alpha}_{-1}^\nu |k\rangle \tag{13.17}$$

our graviton state. Note that for a field of weight $(h, \bar{h})$ we require that

$$\phi_n|0\rangle = 0 \text{ for } n > -h. \tag{13.18}$$

Normal ordering and radial ordering We shall focus on the chiral field, which we shall call

$$j^\mu(z) \equiv \partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}}\sum_n \alpha_n^\mu z^{-n-1}. \tag{13.19}$$

Let us now split $j^\mu(z)$ into creation and annihilation parts. We won't be too careful about the zero mode, since it will drop out in the end. Thus we define

$$j_+^\mu = -i\sqrt{\frac{\alpha'}{2}}\sum_{n>0} \alpha_n^\mu z^{-n-1}, \tag{13.20}$$

$$j_-^\mu = -i\sqrt{\frac{\alpha'}{2}}\sum_{n\geq 0} \alpha_{-n}^\mu z^{n-1} \tag{13.21}$$

so that $j^\mu = j_+^\mu + j_-^\mu$. Remember that normal ordering is denoted by pairs of colons, $:(\dots):$, as in QFT. For our chiral field, normal ordering is defined in an analogous way,

$$:j^\mu(z)j^\nu(\omega): = j_+^\mu(z)j_+^\nu(\omega) + j_-^\mu(z)j_+^\nu(\omega) + j_-^\nu(\omega)j_+^\mu(z) + j_-^\mu(z)j_-^\nu(\omega) \tag{13.22}$$

$$= j^\mu(z)j^\nu(\omega) + [j_-^\nu(\omega), j_+^\mu(z)]. \tag{13.23}$$

We can evaluate the commutator (as on the first examples sheet) to find

$$[j_-^\nu(\omega, j_+^\mu(z)] = -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-\omega)^2}. \tag{13.24}$$

However, in order to evaluate this commutator, we needed to sum a series, and that series only converged for $|z| > |\omega|$. Thus we see that normal ordering comes with a radial ordering requirement for the commutator to make sense. We find that

$$R(j^\mu(z)j^\nu(\omega) =: j^\mu(z)j^\nu(\omega) : -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-\omega)^2}. \tag{13.25}$$

As in QFT, it is useful to introduce the "contraction" notation

$$\overbrace{j^\mu(z)j^\nu(\omega)} = -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-\omega)^2}. \tag{13.26}$$

If you like, this is a Green's function on Σ. On the examples sheet, we computed

$$\partial_z X^\mu(z)\partial_\omega X^\nu(\omega) = -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-\omega)^2}. \tag{13.27}$$

Up to arbitrary functions of $z, \bar{z}$ we integrate to find

$$\overbrace{X^\mu(z)X^\nu(\omega)} = -\frac{\alpha'}{2}\ln(z-\omega)\eta^{\mu\nu} \tag{13.28}$$

Splitting X into its holomorphic and antiholomorphic parts,

$$X^\mu(z,\bar{z}) = X^\mu(z) + \bar{X}^\mu(\bar{z}),$$

we can also show that

$$\overbrace{\bar{X}^\mu(\bar{z})\bar{X}^\nu(\bar{\omega})} = -\frac{\alpha'}{2}\ln(\bar{z}-\bar{\omega})\eta^{\mu\nu}, \tag{13.29}$$

$$\overbrace{X^\mu(z)\bar{X}^\nu(\bar{\omega})} = 0. \tag{13.30}$$

In total, we find that

$$\begin{aligned}\overbrace{X^\mu(z,\bar{z})X^\nu(\omega,\bar{\omega})} &= (X^\nu(z) + \overbrace{\bar{X}^\mu(\bar{z}))(X^\nu(\omega) + \bar{X}^\nu(\bar{\omega})})\\ &= \left(-\frac{\alpha'}{2}\ln(z-\omega) - \frac{\alpha'}{2}\ln(\bar{z}-\bar{\omega})\right)\eta^{\mu\nu}\end{aligned}$$

where the $X^\mu(z), X^\nu(\omega)$ are also contracted over. We therefore learn that

$$\overbrace{X^\mu(z,\bar{z})X^\nu(\omega,\bar{\omega})} = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-\omega|^2, \tag{13.31}$$

which tells us the Green's function immediately:

$$\langle R(X^\mu(z,\bar{z})X^\nu(\omega,\bar{\omega})\rangle = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-\omega|^2. \tag{13.32}$$

We can use 13.31 to build contractions of more complicated operators constructed from X^μ via Wick's theorem. Notice that this Green's function diverges as $z \to \omega$, however, and its divergence also depends on this string parameter α'. This tells us that some interesting physics is captured in the particle limit as the string tension becomes infinite, $T \to \infty$, and $\alpha' \to 0$ since $T = -\frac{1}{2\pi\alpha'}$

Lecture 14.

Monday, February 18, 2019

We saw last time that a lot of the interest in our theory lies in its pole structure, i.e. the divergences that crop up when we bring two operators close together. From last term's *Quantum Field Theory*, we're familiar with Wick's theorem, which links time-ordered expressions to normal-ordered expressions with contractions. Here, we have radial ordering, so that

$$\begin{aligned} R(\phi_1(z_1)\dots\phi_n(z_n)) = {} & :\phi_1(z_1)\dots\phi_n(z_n) + \sum_{(i,j)} :\phi(z_1)\dots\overbrace{\phi_i(z_i)\dots\phi_j(z_j)}\dots\phi_n(z_n): \\ & + \sum_{(i,j),(k,l)} :\phi(z_1)\dots\overbrace{\phi_i(z_i)\dots\phi_j(z_j)}\overbrace{\phi_k(z_k)\dots\phi_l(z_l)}\dots\phi_n(z_n): + \dots \end{aligned}$$

where these sums are taken over all internal (pairwise) contractions. The contractions replace operator pairs with Green's functions, which means that there may be a lot of interest in the pole structure of this object.

We can use Wick's theorem and our knowledge of contractions to define composite operators, e.g. we found that

$$\overbrace{\partial X^\mu(z)\partial X^\nu(\omega)} = -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-\omega)^2}. \tag{14.1}$$

This gives us a natural definition for our stress tensor:

$$T(z) = \lim_{\omega\to z} -\frac{1}{\alpha'}\left(\partial X^\mu(z)\partial X_\mu(\omega) + \frac{\alpha'}{2}\frac{\eta^\mu{}_\mu}{(z-\omega)^2}\right). \tag{14.2}$$

Operator Product Expansions (OPEs) OPEs encode what happens when we bring two operators close together. Given a set of operators $\{O_i\}$, we write

$$O_i(\omega)O_j(z) = \sum_k f_{ij}^k(z-\omega)O_k(z) \tag{14.3}$$

as $\omega \to z$. Here, there's some sense of completeness in the set of operators $\{O_i\}$.

OPEs and conformal transformations For instance, let us consider the OPE $T(z)X^\mu(\omega)$ and conformal transformations. We are interested in

$$T(z)X^\mu(\omega) \text{ as } \omega \to z. \tag{14.4}$$

We have

$$T(z)X^\mu(\omega) = \frac{1}{\alpha'} :\partial X^\nu(z)\partial X_\nu(z): X^\mu(\omega), \tag{14.5}$$

and by integrating 14.1 we get

$$\partial X^\mu(z)X^\nu(\omega) = -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{z-\omega} + \dots \tag{14.6}$$

where the ... indicate terms that are finite as $z \to \omega$.

It follows that

$$\begin{aligned}T(z)X^\mu(\omega) &= -\frac{2}{\alpha'} :\overbrace{\partial X^\nu(z)\partial X_\nu(z) : X^\mu(\omega)} + \dots\\ &= -\frac{2}{\alpha'}\partial X_\nu(z)\left(-\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{z-\omega}\right) + \dots\\ &= \frac{\partial X^\mu(z)}{z-\omega} + \dots\end{aligned}$$

We then expand $\partial X^\mu(z)$ around $z = \omega$ to find

$$\partial X^\mu(z) = \partial X^\mu(\omega) + O(z-\omega),$$

so

$$T(z)X^\mu(\omega) = \frac{\partial X^\mu(\omega)}{z-\omega} + \dots \tag{14.7}$$

where the ... terms remain finite.

Recall that the conformal transformation of $X^\mu(\omega)$ may be given by

$$\delta_v X^\mu(\omega) = \oint_{z=\omega}\frac{dz}{2\pi i}R(v(z)T(z)X^\mu(\omega)), \tag{14.8}$$

where $v(z)$ is holomorphic and parametrizes our transformation.

We now substitute our OPE into $\delta_v X^\mu(\omega)$ to find

$$\delta_v X^\mu(\omega) = \oint_{z=\omega}\frac{dz}{2\pi i}v(z)\left(\frac{\partial X^\mu(\omega)}{z-\omega} + \dots\right) = v(\omega)\partial X^\mu(\omega). \tag{14.9}$$

where the contour is taken in a little loop around $z = \omega$.

Transformations of primary fields Consider a chiral primary $\phi(z)$ (where $\bar{h} = 0$). We know that

$$\delta_v\phi(z) = \oint_{C(z)}\frac{d\omega}{2\pi i}R(v(\omega)T(\omega)\phi(z)), \tag{14.10}$$

where we've swapped the z and ω in the integral to emphasize our primary field depends only on z. We want to retain the idea that a primary field transforms as a conformal tensor of weight $(h, \bar{h})$. Therefore we'll require that for $\phi(z)$ to be a chiral primary field, the OPE with $T(\omega)$ is such that

$$\delta_v\phi(z) = v(z)\partial\phi(z) + h\partial v(z)\phi(z). \tag{14.11}$$

Using the residue theorem in the following form,

$$\frac{1}{(n-1)!}\partial_z^{n-1}f(z) = \oint\frac{d\omega}{2\pi i}\frac{f(\omega)}{(\omega-z)^n}, \tag{14.12}$$

we find that the R part of the OPE can be rewritten as follows:

$$R(T(\omega)\phi(z)) = \frac{h}{(z-\omega)^2}\phi(\omega) + \frac{1}{z-\omega}\partial\phi(\omega) + \dots \tag{14.13}$$

in order to match the form of 14.11.

We could take this OPE with the stress tensor to define what we mean by a chiral primary of weight h. Thus by writing the radial ordering for some general ϕ we can read off the weight immediately.

A non-trivial OPE Consider now the OPE

$$T(z) : e^{ik\cdot X(\omega)} : \tag{14.14}$$

where $k \cdot X(\omega) = k_\mu X^\mu(\omega)$, with k_μ some constant spacetime vector. We think of this normal-ordered term in terms of its series expansion, i.e.

$$\sum_{n\geq 0}\frac{i^n}{n!}k_{\mu_1}\dots k_{\mu_n} : X^{\mu_1}(\omega)\dots X^{\mu_n}(\omega) : \tag{14.15}$$

We might wonder what the weight of $: e^{ik\cdot X}:$ is, but there's some non-trivial behavior going on in the normal ordering. Let's tack on $T(z)$ now:

$$-\frac{1}{\alpha'} : \partial X^\nu(z)\partial X_\nu(z) : \sum_{n\geq 0} \frac{i^n}{n!} k_{\mu_1} \ldots k_{\mu_n} : X^{\mu_1}(\omega) \ldots X^{\mu_n}(\omega) : \tag{14.16}$$

Single contractions contribute to this expression:

$$\sum \frac{i^n}{n!} n(k \cdot X(\omega)^{n-1} k_\nu \frac{1}{z-\omega} \partial X^\nu(\omega), \tag{14.17}$$

where we've contracted one of the ∂Xs with one of the X^{μ_i}s in the sum. Shifting the index we have

$$\sum_{m\geq 0} \frac{i^m}{m!} (k \cdot X(\omega))^m \frac{k_\nu \partial X^\nu(\omega)}{z-\omega} = \frac{1}{z-\omega} \partial_\omega (e^{ik\cdot X(\omega)}). \tag{14.18}$$

This already looks like the $\partial\phi(\omega)$ term in our expansion– we'll see how the double contractions give the other term on Wednesday.

Lecture 15.

Wednesday, February 20, 2019

Last time, we stopped mid-calculation. We were looking at the OPE for

$$T(Z) e^{ik\cdot X(\omega)}, \tag{15.1}$$

where $T(z)$ is the holomorphic part of the stress tensor, given by

$$T(z) = -\frac{1}{\alpha'} : \partial X^\mu(z) \partial X_\mu(z) : \tag{15.2}$$

and the exponential is treated as a formal power series of the operator X. We found that single contractions gave us a term

$$\frac{1}{z-\omega} \partial_\omega (e^{ik\cdot X(\omega)}. \tag{15.3}$$

What about double contractions?

Double contractions contribute

$$-\frac{1}{\alpha'} : \partial X^\mu(z) \partial X_\mu(z) : \sum_{(i,j)} \sum_{n\geq 0} \frac{(i)^n}{n!} k_{\mu_1} \ldots k_{\mu_i} \ldots k_{\mu_j} \ldots k_{\mu_n} : X^{\mu_1} \ldots X^{\mu_i} \ldots X^{\mu_j} \ldots X^{\mu_n}(\omega) :, \tag{15.4}$$

where we must now perform contractions over the $\partial X^\mu(z)$s with the X^{μ_i}s on the right. There are no triple contractions since there are only two derivatives of Xs outside the sum and the normal ordering has already taken care of contractions in the X^{μ_i}s.

We can make this more precise. There are $n(n-1)$ options for which X^{μ_i}s to contract with, so we get an overall contribution

$$-\frac{1}{\alpha'} \sum_{n\geq 2} k_{\mu_2} \ldots k_{\mu_n} \frac{(i)^n}{n!} n(n-1) \left(-\frac{\alpha'}{2}\right)^2 \frac{k^2}{(z-\omega)^2}, \tag{15.5}$$

where two of the ks have been contracted since the contraction of $\partial X_\mu(z)\ X^{\mu_i}(\omega)$ comes with an $\eta_\mu^{\mu_i}$. Cleaning up a bit more, we have

$$= -\frac{\alpha'}{4} \frac{k^2}{(z-\omega)^2} \sum_{n\geq 2} : (k \cdot X(\omega))^{n-2} : i^2 i^{n-2} : X^{\mu_2} \ldots X^{\mu_n} : \frac{n!}{n!(n-2)!} \tag{15.6}$$

$$= \frac{\alpha'}{4} \frac{k^2}{(z-\omega)^2} : e^{ik\cdot X(\omega)} : \tag{15.7}$$

In total, we have

$$T(z) : e^{ik\cdot X(\omega)} := \left(\frac{\alpha'}{4} \frac{k^2}{(z-\omega)^2} + \frac{\partial_\omega}{z-\omega}\right) : e^{ik\cdot X(\omega)} + \ldots \tag{15.8}$$

and we see that $:e^{ik\cdot X(\omega)}:$ has conformal weight

$$h = \frac{\alpha' k^2}{4}. \tag{15.9}$$

More generally $:e^{ik\cdot X(\omega,\bar{\omega})}:$ has weight

$$(h,\bar{h}) = \left(\frac{\alpha' k^2}{4}, \frac{\alpha' k^2}{4}\right). \tag{15.10}$$

Note that factors of the string tension α' go with factors of $\hbar$, which we've previously set to 1, so the relative factor of α' between the two terms in the expansion of $T(z):e^{ik\cdot X(\omega)}$ tells us that there's a quantum correction going on here so that $:e^{ik\cdot X(\omega)}:$ doesn't just transform trivially as a scalar under conformal transformations.

It is now useful to separate the notion of a primary field from the definitions of h and $\bar{h}$.

Definition 15.11. A primary field is a field $\phi(\omega)$ with an OPE with $T(z)$ of the form

$$T(z)\phi(\omega) = \frac{h}{(z-\omega)^2}\phi(\omega) + \frac{1}{z-\omega}\partial\phi(\omega). \tag{15.12}$$

However, the $(z-\omega)^{-2}\phi(\omega)$ coefficient will still be called the *weight*, regardless of the presence of higher-order poles.

OPE of $T(z)T(\omega)$ and the Virasoro Algebra Recall that

$$T(z) = -\frac{1}{\alpha'}:\partial X^\mu(z)\partial X)\mu(z):, \tag{15.13}$$

and we have the contraction

$$\overbrace{\partial X^\mu(z)\partial X^\nu(\omega)} = -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-\omega)^2}. \tag{15.14}$$

We'll just go for it, then.

$$T(z)T(\omega) = \left(-\frac{1}{\alpha'}\right)^2 :\partial X^\mu(z)\partial X_\mu(z)::\partial X^\nu(\omega)\partial X_\nu(\omega): \tag{15.15}$$

where we need to take single contractions (e.g. $\overbrace{\partial X_\mu(z)\partial X^\nu(\omega)}$) and also double contractions

$$\overbrace{\partial X_\mu(z)\partial X^\nu(\omega)}\,\overbrace{\partial X^\mu(z)\partial X_\nu(\omega)}.$$

There will be four single contractions and two double contractions. Writing it all out, we find that

$$T(z)T(\omega) = -\frac{2}{\alpha'}\frac{\eta_{\mu\nu}}{(z-\omega)^2}:\partial X^\mu(z)\partial X^\nu(\omega): + \frac{1}{2}\frac{\delta^\mu{}_\nu}{(z-\omega)^2}\frac{\delta_\mu{}^\nu}{(z-\omega)^2} + \dots$$

We now expand $\partial X^\mu(z)$ about $z=\omega$:

$$\partial X^\mu(z) = \partial X^\mu(\omega) + (z-\omega)\partial^2 X^\mu(\omega) + \dots \tag{15.16}$$

We also recall that $\delta^\mu{}_\nu\delta^\nu{}_\mu = D$ the dimension of spacetime. Thus

$$T(z)T(\omega) = \frac{D/2}{(z-\omega)^4} - \frac{2}{\alpha'}\frac{1}{(z-\omega)^2}:\partial X^\nu(\omega)\partial X_\nu(\omega): - \frac{2}{\alpha'}\frac{1}{(z-\omega)}:\partial^2 X^\mu(\omega)\partial X_\mu(\omega): + \dots \tag{15.17}$$

using the expansion of $\partial X^\mu(z)$, 15.16. We arrive at

$$T(z)T(\omega) = \frac{D/2}{(z-\omega)^4} - \frac{2}{\alpha'}\frac{1}{(z-\omega)^2}T(\omega) - \frac{2}{\alpha'}\frac{1}{(z-\omega)}\partial T(\omega) + \dots \tag{15.18}$$

Clearly, this has weight $h=2$, so $T(z)$ is of weight $(2,0)$. However, it is only a primary if $D=0$. And of course there's no way to embed a nontrivial worldsheet in $D=0$, so it seems like something very bad has happened.

The Virasoro algebra We've just show that $T(z)$ has $h = 2$, so we expand it in modes as

$$T(z) = \sum_n L_n z^{-n-2}, \quad \bar{T}(z) = \sum_n \bar{L}_n \bar{z}^{-n-2}. \tag{15.19}$$

We can invert these expressions to find

$$L_m = \oint_{z=0} \frac{dz}{2\pi i} z^{m+1} T(z). \tag{15.20}$$

Let's now consider the commutator of two L_ns– to wit(t),

$$[L_m, L_n] = \oint_{\omega=0} \frac{d\omega}{2\pi i} \omega^{n+1} \oint_{z=0} \frac{dz}{2\pi i} z^{m+1} [T(z), T(\omega)]. \tag{15.21}$$

What do we mean by this commutator of operators? Let's just look at the dz integral first. In our discussion of radial ordering, we split up the contour integral as

$$\oint_{z=0} \frac{dz}{2\pi i} z^{m+1} [T(z), T(\omega)] := \oint_{|z|>|\omega|} z^{m+1} T(z) T(\omega) - \oint_{|z|<|\omega|} \frac{dz}{2\pi i} z^{m+1} T(\omega) T(z) \tag{15.22}$$

$$= \oint_{z=\omega} \frac{dz}{2\pi i} R(T(z)T(\omega)) z^{m+1}. \tag{15.23}$$

Using our $T(z)T(\omega)$ OPE, we have

$$[L_m, L_n] = \oint_{\omega=0} \frac{d\omega}{2\pi i} \omega^{n+1} \oint_{z=\omega} \frac{dz}{2\pi i} z^{m+1} \left(\frac{D/2}{(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial T(\omega)}{z-\omega} \right) \tag{15.24}$$

$$= \oint_{\omega=0} \frac{d\omega}{2\pi i} \omega^{n+1} \left(\frac{D/2}{3!} \frac{\partial^3}{\partial z^3} z^{m+1} + 2T(\omega) \frac{\partial}{\partial z} z^{m+1} + z^{m+1} \partial T(\omega) \right)_{z=\omega} \tag{15.25}$$

where we have dropped the ... from our OPE, as the contour integral will be evaluated by the residue theorem, and the residue theorem only cares about the pole structure of the thing we are integrating. Taking the derivatives, we get

$$\begin{aligned}
[L_m, L_n] &= \oint_{\omega=0} \frac{d\omega}{2\pi i} \left(\frac{D}{12}(m^3 - m)\omega^{n+m-1} + 2(m+1)\omega^{m+n+1} T(\omega) - \omega^{m+n+2} \partial T(\omega) \right) \\
&= \oint_{\omega=0} \frac{d\omega}{2\pi i} \left(\frac{D}{12}(m^3 - m)\omega^{n+m-1} + (m-n)\omega^{m+n+1} T(\omega) \right) \\
&= \frac{D}{12}(m^3 - m)\delta_{m+n,0} + (m-n) L_{m+n}.
\end{aligned}$$

where in the second line we have integrated the final term by parts and simplified. This final result is something we were given on the first examples sheet. It looks almost like the Witt algebra, except there is an anomaly, the $D/12$ term. We call it the *Virasoro algebra*, and it is a consequence of the conformal symmetry of our quantum theory. It's sometimes called a central extension of the Witt algebra.

In fact, there's a complication that we've missed. Our theory isn't just defined by the Xs– there are also the ghosts, and as constraints, we expect that those ghosts will act like "negative degrees of freedom." They will contribute to this commutator and show that our theory can be consistent in $D \neq 0$