ADVANCED QUANTUM FIELD THEORY

IAN LIM
LAST UPDATED JANUARY 22, 2019

These notes were taken for the *Advanced Quantum Field Theory* course taught by Matthew Wingate at the University of Cambridge as part of the Mathematical Tripos Part III in Michaelmas Term 2018. I live-TEXed them using Overleaf, and as such there may be typos; please send questions, comments, complaints, and corrections to itel2@cam.ac.uk. Many thanks to Arun Debray for the LATEX template for these lecture notes: as of the time of writing, you can find him at https://web.ma.utexas.edu/users/a.debray/.

Contents

Lecture 1.

Saturday, January 19, 2019

Note. There will not be official typed course notes, but there will be scanned handwritten notes (which I will link here as they become available). Previous lecturers' notes are currently online (Skinner, Osborn).

Today we introduce path integrals in a QFT context. There are some benefits to working with path integrals– some computations are simplified or more straightforward, and Lorentz invariance is manifest (unlike in the canonical formalism).

Path integrals in quantum mechanics Rather than trying to tackle the full machinery of QFT, we'll start with $0+1$ dimensional non-relativistic quantum mechanics (cf. Osborn § 1.2. We'll set $\hbar = 1$ for now, though we may restore it later in order to make arguments when $\hbar \ll 1$ in a classical limit. In these units,

$$[E][t] = [\hbar] = [p][x]$$

using uncertainty relations.

Let us consider a Hamiltonian in 1 spatial dimension,

$$\hat{H} = H(\hat{x}, \hat{p}) \quad \text{with } [\hat{x}, \hat{p}] = i.$$

We'll further assume for simplicity that the Hamiltonian has a kinetic term and a potential based only on position,

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}).$$

Now the Schrödinger equation takes the form

$$i\frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle \tag{1.1}$$

which has formal solution

$$|\psi(t)\rangle = e^{-i\hat{H}t} |\psi(0)\rangle . \tag{1.2}$$

Let us consider some position eigenstates $|x, t\rangle$ such that

$$\hat{x}(t) |x, t\rangle = x |x, t\rangle , \quad x \in \mathbb{R},$$

where these states obey some normalization

$$\langle x', t | x, t\rangle = \delta(x' - x).$$

In the Schrödinger picture, states depend on time, while operators are constant. In terms of fixed (time-independent) eigenstates $\{|x\rangle\}$ of the position operator $\hat{x}$, we may write the wavefunction as

$$\psi(x,t) = \langle x|\ \psi(t)\rangle\,, \tag{1.3}$$

so that applying the Hamiltonian to the wavefunction $\psi(x,t)$ yields

$$\hat{H}\psi(x,t) = (-\frac{1}{2m}\frac{\partial^2}{\partial x^2} + V(x))\psi(x,t). \tag{1.4}$$

This is the traditional presentation of quantum mechanics and the wavefunction. In the path integral formalism, we'll consider a more particle-like treatment, where we express time evolution as a sum over all trajectories (meeting some boundary conditions) appropriately weighted (by an action).

Recall that our formal solution 1.2 tells us what $|\psi(t)\rangle$ is– we can therefore rewrite the wavefunction as

$$\psi(x,t)\ \langle x|\, e^{-i\hat{H}t}\, |\psi(0)\rangle\,. \tag{1.5}$$

By inserting a complete set of (position eigen)states, $1 = \int dx_0\, |x_0\rangle\, \langle x_0|$, we get

$$\begin{aligned}\psi(x,t) &= \int dx_0\, \langle x|\, e^{-i\hat{H}t}\, |x_0\rangle\, \langle x_0|\ \psi(0)\rangle \\ &= \int dx_0\, K(x,x_0;t)\psi(x_0,0),\end{aligned}$$

where we have defined $K(x,x_0;t) \equiv \langle x|\, e^{-i\hat{H}t}\, |x_0\rangle$. Let us further consider time evolution in discrete steps, with $0 \equiv t_0 < t_1 < t_2 < \ldots < t_n < t_{n+1} \equiv T$ so that

$$e^{-i\hat{H}T} = e^{-i\hat{H}(t_{n+1}-t_n)} \ldots e^{-i\hat{H}(t_1-t_0)}.$$

As before, we insert complete sets of states, finding that our generic time evolution from any x_0 to an x of our choosing:

$$K(x,x_0;T) = \int \left[\prod_{r=1}^{n} dx_r\, \langle x_{r+1}|\, e^{-i\hat{H}(t_{r+1}-t_r)}\, |x_r\rangle\right] \langle x_1|\, e^{-i\hat{H}t_1}\, |x_0\rangle\,. \tag{1.6}$$

That is, we integrate over all intermediate positions x_r for each t_r. Naturally, dx_{n+1} must be x.

Let's look at the free theory first to understand what we've done, $V(x) = 0$. Now this weird K_0 object we've defined takes the form

$$K_0(x,x';t) = \langle x|\, e^{-i\frac{\hat{p}^2}{2m}t}\, |x'\rangle\,. \tag{1.7}$$

We'll instead insert a complete set of momentum eigenstates $|p\rangle$ with the normalization

$$\int \frac{dp}{2\pi}\, |p\rangle\, \langle p| = 1,$$

recalling that $\langle x|\ p\rangle = e^{ipx}$ are simply plane waves. Then

$$K_0(x,x';t) = \int \frac{dp}{2\pi} e^{-ip^2t/2m} e^{ip(x-x')}.$$

We can compute this– completing the square with a change of variables to $p' = p - \frac{m(x-x')}{t}$, K_0 becomes a gaussian integral,

$$\begin{aligned}K_0(x,x';t) &= e^{im(x-x')^2/2t} \int_{-\infty}^{\infty} \exp\left[-\frac{i(p')^2t}{2m}\right] \\ &= e^{im(x-x')^2/2t}\sqrt{\frac{m}{2\pi it}}.\end{aligned}$$

Note that as $t \to 0$,[1]

$$\lim_{t\to 0} K_0(x,x';t) = \delta(x-x'),$$

which agrees with the fact that $\langle x'|\ x\rangle = \delta(x-x')$.

[1] This was more obvious from the original expression for K_0 where $K_0(x,x';t=0) = \int \frac{dp}{2\pi} e^{ip(x-x')}$.

For $V(\hat{x}) \neq 0$, we still need small time steps but since operators generically do not compute, exponentials don't add in the usual way:

$$e^{\hat{A}}e^{\hat{B}} = \exp(\hat{A} + \hat{B} + \frac{1}{2}[\hat{A}, \hat{B}] + \ldots) \neq e^{\hat{A}+\hat{B}} \quad \text{when } [\hat{A}, \hat{B}] \neq 0.$$

This is the Baker-Campbell-Hausdorff (BCH) formula. However, for small ϵ we can write

$$e^{\epsilon\hat{A}}e^{\epsilon\hat{B}} = \exp(\epsilon\hat{A} + \epsilon\hat{B} + O(\epsilon^2)),$$

or equivalently

$$e^{\epsilon(\hat{A}+\hat{B})} = e^{\epsilon\hat{A}}e^{\epsilon\hat{B}}(1 + O(\epsilon^2)),$$

so we conclude that

$$e^{\hat{A}+\hat{B}} = \lim_{n\to\infty} \left(e^{\hat{A}/n}e^{\hat{B}/n}\right)^n.$$

Suppose now that we divide our time into n time steps so that $t_r - 1 - t_r = \delta t$, with $T = n\delta t$. Then one of the intermediate time evolution steps looks like

$$\begin{aligned}\langle x_{r+1}| e^{-i\hat{H}\delta t} |x_r\rangle &= e^{-iV(x_r)\delta t} \langle x_{r+1}| e^{-i\hat{p}^2\delta t/2m} |x_r\rangle \\ &= \sqrt{\frac{m}{2\pi i\delta t}} \exp\left[\frac{i}{2}m\left(\frac{x_{r+1} - x_r}{\delta t}\right)^2 \delta t - iV(x_r)\delta t\right].\end{aligned}$$

Taking $T = n\delta t$, we find that the entire K becomes

$$K(x, x_0; T) = \int \left(\prod_{r=1}^{n} dx_r\right) \left(\frac{m}{2\pi i\delta t}\right)^{\frac{n+1}{2}} \exp\left(i\sum_{r=0}^{n}\left[\frac{m}{2}\left(\frac{x_{r+1} - x_r}{\delta t}\right)^2 - V(x_r)\right]\delta t\right). \tag{1.8}$$

Now we take the limit as $n \to \infty, \delta t \to 0$ with T fixed. Then the argument of the exponential becomes

$$\int_0^T \frac{m}{2}\dot{x}^2 - V(x)]dt = \int_0^t L dt, \tag{1.9}$$

where $L(x, \dot{x})$ is the classical Lagrangian and this integral is nothing more than the action. We conclude that

$$K(x, x_0; T) = \langle x| e^{-i\hat{H}t} |x_0\rangle = \int \mathcal{D}x\, e^{iS[x]}, \tag{1.10}$$

where $S[x] = \int_0^T L(x, \dot{x})dt$ is the classical action and the $\mathcal{D}$ conceals all our sins (the continuum limit) in a cute integration measure. Note that the action has units of energy$\times$ time, so if we restore $\hbar$, we see that this integral becomes

$$K(x, x_0; T) = \int \mathcal{D}x\, e^{iS/\hbar}, \tag{1.11}$$

and in the $\hbar \to 0$ limit (the classical limit), the integral is dominated by paths x which minimize the classical action, and we recognize this as Hamilton's principle from classical mechanics.