Neuroprothetics Exercise 2
Mathematical Basics 1

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1 Slope fields

The first part of this exercise consists of plotting the slope fields of two given differential equations. The task sheet kindly gave an example of implementing such plots in Python. In addition to the slope fields, lines of constant slope (isoclines) were inserted. Figure 1 shows the slope field of the differential equation

$$\frac{dV}{dt} = -10 - V - t\,. \tag{1}$$

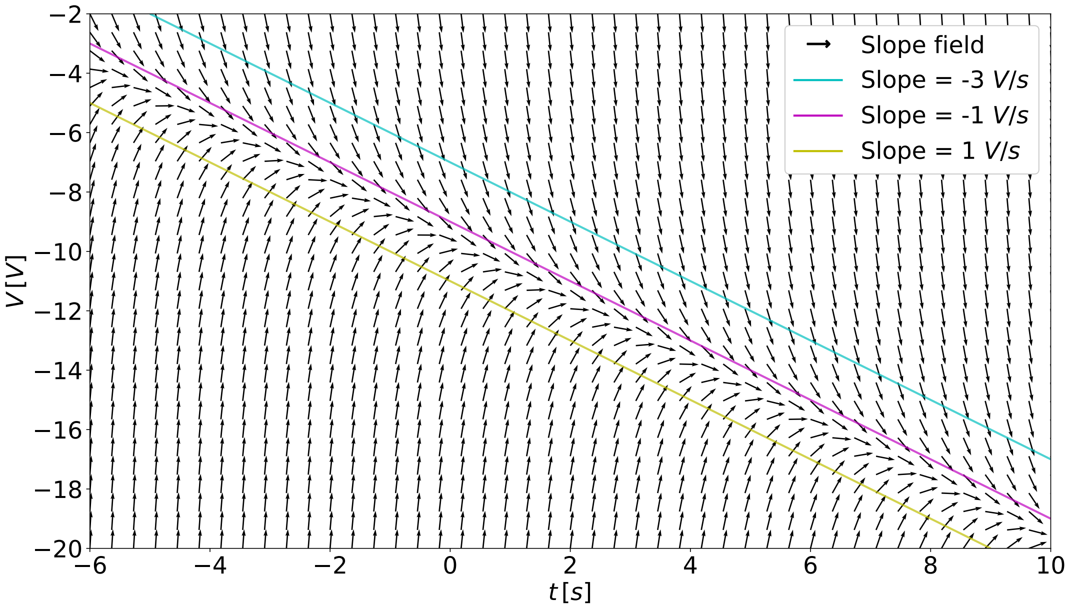

Figure 1: Slope field for equation 1 with isoclines for selected slope values.

Likewise, Figure 2 shows the slope field of the differential equation

$$\frac{dV}{dt} = \cos(t) - \frac{1}{2}V + 20\,. \tag{2}$$

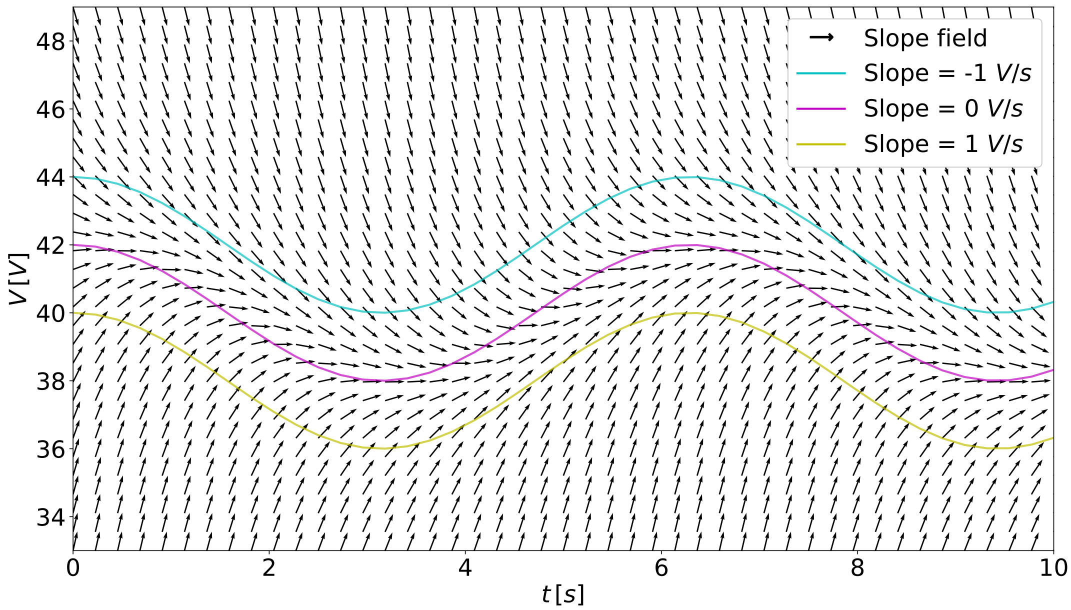

Figure 2: Slope field for equation 2 with isoclines for selected slope values.

2 Simple cell model

A simple electrical model of a biological cell should be implemented in the second task. Figure 3 shows the electrical equivalent circuit.

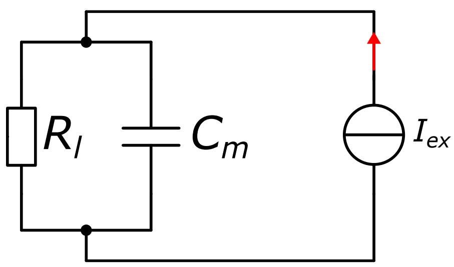

Figure 3: Simple cell model. $R_l \mathrel{\hat=}$ membrane leak resistance, $C_m \mathrel{\hat=}$ membrane capacity, $I_{ex} \mathrel{\hat=}$ stimulating current.

2.1 Derivation of differential equation

To analyze the model, the differential equation describing the model must first be derived.

The current of a capacitor is described by:

$$I_C(t) = C\frac{dV(t)}{dt} \, . \tag{3}$$

Applying Kirchhoff's laws, we receive the following expression for the current in the circuit:

$$I_{ex}(t) = I_C(t) + I_R(t) = C_m\frac{dV(t)}{dt} + I_R(t) \, . \tag{4}$$

According to Ohm's law, the current in the resistor can also be expressed as

$$I_R(t) = \frac{V_R(t)}{R_l} \, , \tag{5}$$

where $V_R(t) = V_C(t) = V(t)$, since capacitor and resistor are in parallel. After inserting Equation 5 in 4 and rearranging the equation, we receive:

$$\frac{dV(t)}{dt} = \frac{I_{ex}(t)}{C_m} - \frac{V(t)}{R_l C_m} \, . \tag{6}$$

With $I_{ex}(t) = I_{max} * \sin(t)$, we get the final differential equation that describes the simple cell model circuit:

$$\frac{dV(t)}{dt} = \frac{1}{C_m}\left(I_{max} * \sin(t) - \frac{V(t)}{R_l}\right) . \tag{7}$$

When also considering a constant current D that is superimposed to the sinusoidal current $I_{ex}(t)$, the equation becomes

$$\frac{dV(t)}{dt} = \frac{1}{C_m}\left(I_{max} * \sin(t) + D - \frac{V(t)}{R_l}\right) . \tag{8}$$

2.2 Slope fields of simple cell model

To visualize the model shown in Figure 3, described by equation 7, the slope fields for the following two sets of values are shown.

- $R_l = 1\,\Omega$; $C_m = 2\,\mathrm{F}$; $I_{max} = 0\,\mathrm{A}$ (Figure 4)
- $R_l = 1\,\Omega$; $C_m = 2\,\mathrm{F}$; $I_{max} = 10\,\mathrm{A}$ (Figure 5)

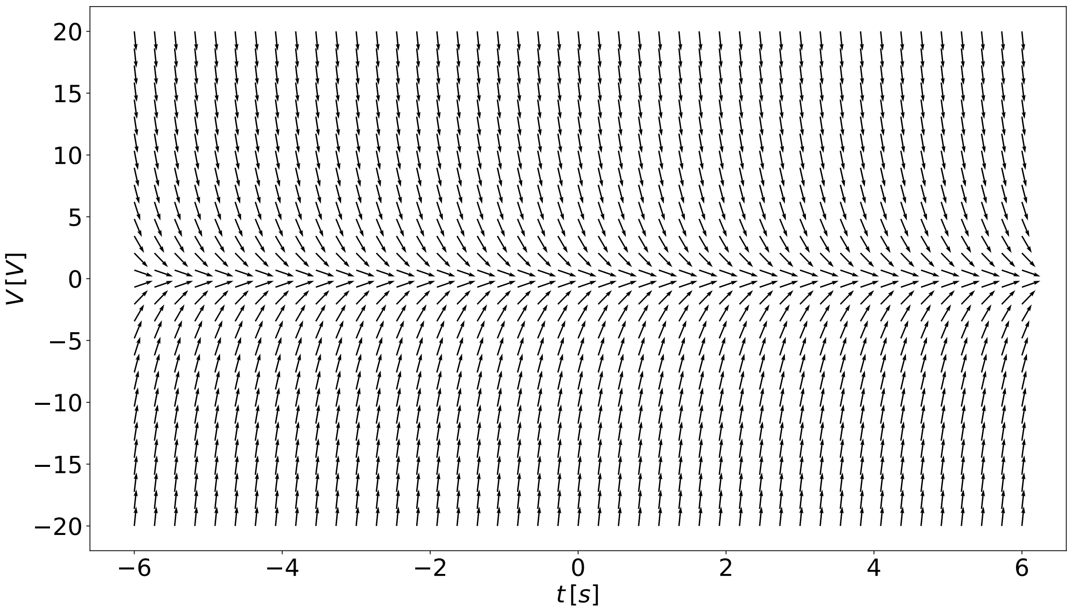

Figure 4: Slope field of simple cell model (Figure 3) without constant current offset for values $R_l = 1\,\Omega$, $C_m = 2\,\mathrm{F}$ and $I_{max} = 0\,\mathrm{A}$.

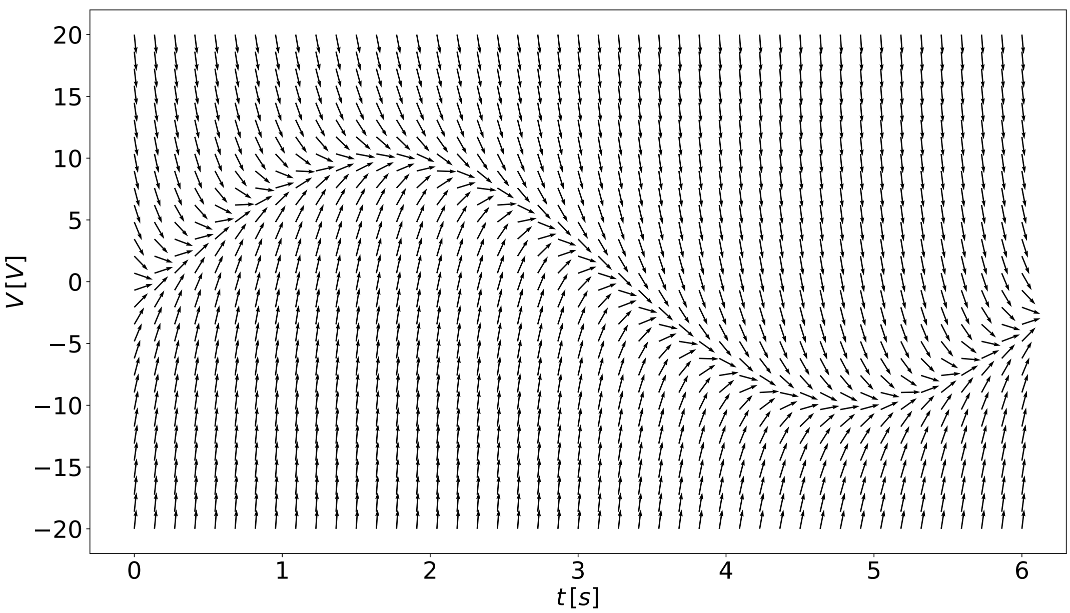

Figure 5: Slope field of simple cell model (Figure 3) without constant current offset for values $R_l = 1\,\Omega$, $C_m = 2\,\mathrm{F}$ and $I_{max} = 10\,\mathrm{A}$.

Correspondingly, the model considering the constant term D is visualized with slope fields with the following sets of values:

- $R_l = 1\,\Omega$; $C_m = 2\,\mathrm{F}$; $I_{max} = 0\,\mathrm{A}$; $D = 5\,\mathrm{A}$ (Figure 6)
- $R_l = 1\,\Omega$; $C_m = 2\,\mathrm{F}$ $I_{max} = 10\,\mathrm{A}$; $D = 5\,\mathrm{A}$ (Figure 7)

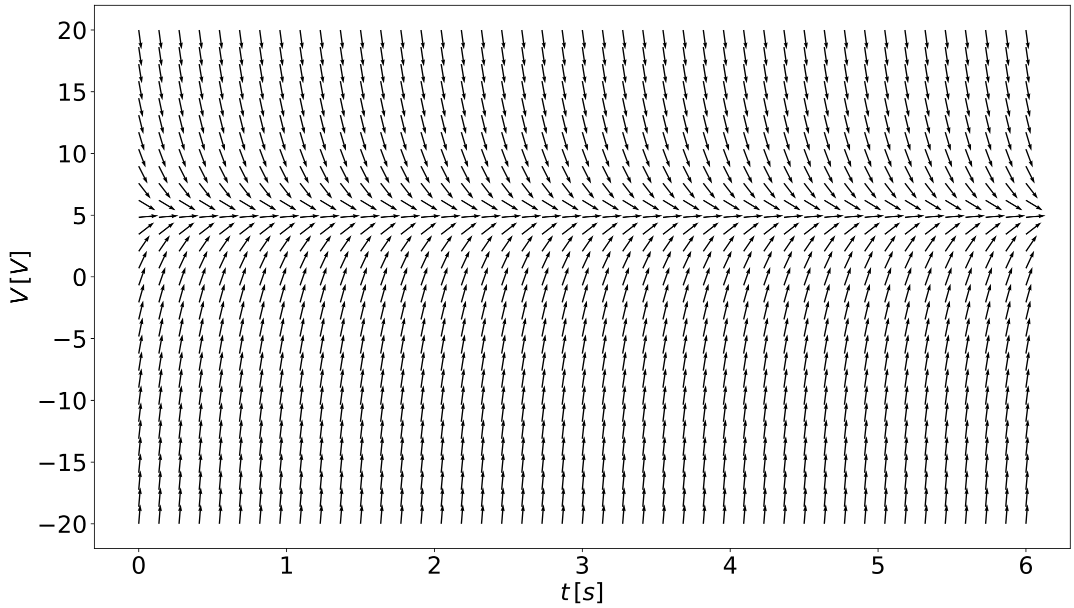

Figure 6: Slope field of simple cell model (Figure 3) with constant current offset for values $R_l = 1\,\Omega$, $C_m = 2\,\mathrm{F}$, $I_{max} = 0\,\mathrm{A}$ and $D = 5\,\mathrm{A}$.

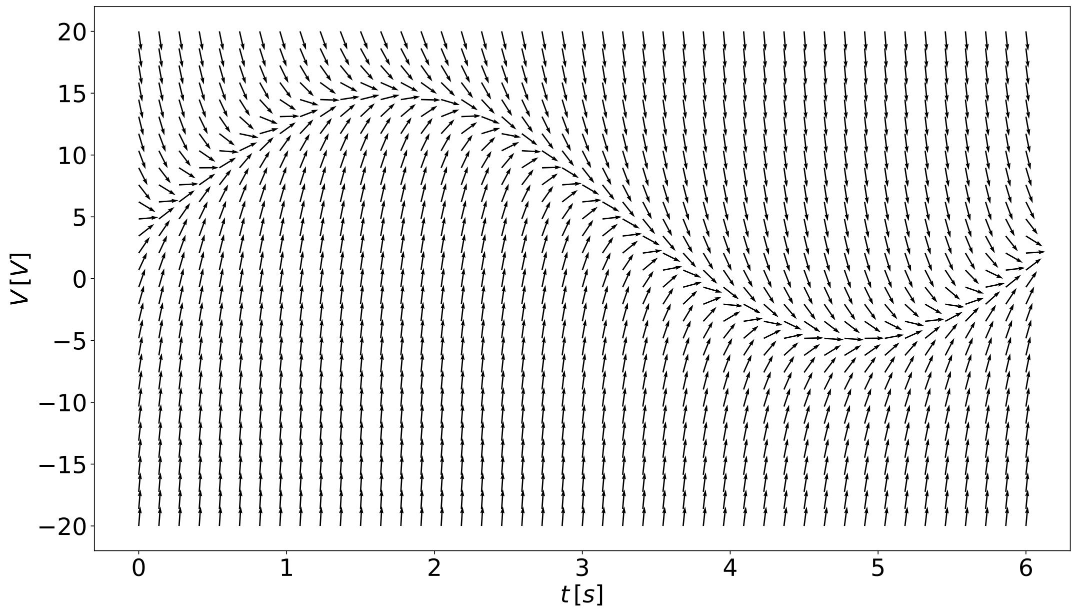

Figure 7: Slope field of simple cell model (Figure 3) with constant current offset for values $R_l = 1\,\Omega$, $C_m = 2\,\mathrm{F}$, $I_{max} = 10\,\mathrm{A}$ and $D = 5\,\mathrm{A}$.

3 Interpretation

Slope fields can help to obtain an intuitive understanding of the behaviour of a system without actually solving the differential equation. In the previous examples, one only has to follow the arrows for any given time and voltage to see how the system will develop. Slope fields are great for visualizing and understanding even more complex differential equations without solving them.