Homework 4: Nonlinear equations and finding extrema

Due Feb 27

1. Write a program to find the roots of one-dimensional equations using a) the Bisection method (AKA Binary search) and b) the Newton-Raphson method (sometimes called just 'Newton's method').

   Test your implementations on the following equations:

   (a) A simple test case: $x^2 = a$ (choose an a, say `2.0`, and solve for x).

   (b) Kepler's equation, used for determining Solar System orbits:

   $$M = E - e\sin(E).$$

   Here M is the "mean anomaly", an angle that increases linearly in time; E is the "eccentric anomaly", the position of the body from the center (not the focus) of the ellipse, and e is the eccentricity of the ellipse. (For more information on this equation and its meaning, see e.g. Wikipedia.) Choose $M = 1.5$ and $e = 0.5$ and solve for E in the interval $0 \leq E < 2\pi$. Try it again for $M = 1.5$ and $e = 0.9$, but be careful where you start the Newton-Raphson.

   For each attempt, demonstrate how quickly the method converges by plotting the error (i.e., the difference between the current guess and the true answer) as a function of iteration number. I suggest using a logarithmic axis for the error.

2. It is frequently required that the Kepler problem be solved to the maximum precision available as quickly as possible. Attempt to do this with your Kepler solver. Which method, Bisection or Newton-Raphson, is better? Use your solver to plot $\sin(E)$ vs. $\cos(E)$ for 20 equally spaced values of M between 0 and 2π for an $e = 0.9$ orbit.

3. The same techniques can also be used to find the extrema of a function. I have some data $d(t_i)$, indexed by t_i, and some model $m(t|p)$ where p is the parameter of the model. Explain *in just words and equations*

how you would find the parameter $p_\star$ of the model that best fits the data. The best fit model is the model the minimizes the quantity

$$\chi^2 = \sum_{i=0}^{N} \frac{[d(t_i) - m(t_i|p)]^2}{\sigma_i^2}, \tag{1}$$

where σ_i is the error on the measurements. See the Extra Credit problem for a specific example of what $m(T|p)$ might look like.

Note that Newton's method can be generalized to multidimensional problems and is used often to find the best fit parameters. (This method is also related method called gradient descent methods which get around having to evaluate second derivatives.)

4. **Extra credit: (Please read problem, even if you don't do it.)** The rotation curves (that is, rotation velocity vs. distance from the center) for galaxies are observed to rise linearly close to the center, and to be constant far from the center. A possible function which can be fit to such a rotation curve is;

$$v_{model}(r) = v_{inf}(1 - e^{-r/r_0}),$$

where v_{inf} is the asymptotic velocity and r_0 is a characteristic radius.

Using a method of your choice described in class or in the book, and assuming that v_{inf} is 100 km/s, find the r_0 that gives the best fit of the above formula to the following "data" (i.e. that minimizes χ^2):

r_{obs} (kiloparsecs)	v_{obs}(km/s)
1.0	12.09
2.0	47.53
3.0	51.80
4.0	63.28
5.0	90.33
6.0	84.32
7.0	92.23
8.0	94.84
9.0	99.37
10.0	94.42

Assume the errors, σ_i are the same across the sample such that you can set $\sigma_i = 1$. Note that some of these methods can be persnickety!

For even more extra credit, simultaneously fit both r_0 and v_{inf} using a program of your own construction or using a routine that exists (such as in numpy or scipy).

Note that the above data were generated from a model with $r_0 = 3$, $v_{inf} = 100$ and an error with standard deviation of 10 km/s.

By the way, the flatness of galactic rotation curves ($v_{\mathrm{model}} \to v_{inf}$ at large radii) is how people realized there is dark matter in the Universe! (although now we have many other lines of evidence)