# Homework 5

## CMPSC 360

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**Question 1**: We try to prove that if $n$ is an integer, then $n^3 - 2n^2 + 5n - 1$ is divisible by 3

1. n is not divisible by 3
2. 2
3. divisible by 3
4. neither true nor false $\rightarrow$ even if the cases pass, there is a case where $n\%3 == 0$
5. all cases are true

Proof: Assume $n \in \mathbb{Z}$
Suppose $n$ is not divisible by 3 ($3 \nmid n$)
By definition of divides, either $n = 3k + 1$ or $n = 3k + 2$ where $k \in \mathbb{Z}$
**Case 1:** $n = 3k + 1$

$$\begin{aligned} n^3 - 2n^2 + 5n - 1 &= (3k+1)^3 - 2(3k+1)^2 + 5(3k+1) - 1 \\ &= 27k^3 + 27k^2 + 9k + 1 - 18k^2 - 12k - 2 + 15k + 5 - 1 \\ &= 27k^3 + 9k^2 + 12k + 3 \\ &= 3(9k^3 + 3k^2 + 4k + 1) \\ &= 3t \text{ such that } t \in \mathbb{Z} \text{ where } t = 9k^3 + 3k^2 + 4k + 1 \end{aligned}$$

By definition of divides, $n^3 - 2n^2 + 5n - 1$ is divisible by 3
Therefore, when $n = 3k + 1$, $n^3 - 2n^2 + 5n - 1$ is divisible by 3
**Case 2:** $n = 3k + 2$

$$\begin{aligned} n^3 - 2n^2 + 5n - 1 &= (3k+2)^3 - 2(3k+2)^2 + 5(3k+2) - 1 \\ &= 27k^3 + 54k^2 + 36k + 8 - 18k^2 - 24k - 8 + 15k + 10 - 1 \\ &= 27k^3 + 36k^2 + 27k + 9 \\ &= 3(9k^3 + 12k^2 + 9k + 3) \\ &= 3t \text{ such that } t \in \mathbb{Z} \text{ where } t = 9k^3 + 12k^2 + 9k + 3 \end{aligned}$$

By definition of divides, $n^3 - 2n^2 + 5n - 1$ is divisible by 3
Therefore, when $n = 3k + 2$, $n^3 - 2n^2 + 5n - 1$ is divisible by 3

**Question 2**: Prove that $\max\{x, y\} + \min\{x, y\} = x + y$. Collaborated with Sahil Kuwadia.

a. **Assumption:** $x$ and $y$ are real numbers $\rightarrow x, y \in \mathbb{R}$
   **Conclusion:** The sum of the larger value in $x$ or $y$ and the smaller value in $x$ and $y$ is equal to the sum of $x$ and $y$

b. Proof: Suppose $x, y \in \mathbb{R}$
   We know that $x$ is either greater than $y$ or $x$ is less than or equal to $y$
   **Case 1:** $x > y$
   $\max\{x, y\} = x$, $\min\{x, y\} = y$
   so, the sum of $\max\{x, y\}$ and $\min\{x, y\}$ is $x + y$
   therefore, $\max\{x, y\} + \min\{x, y\} = x + y$
   **Case 2:** $x \leq y$
   $\max\{x, y\} = y$, $\min\{x, y\} = x$
   so, the sum of $\max\{x, y\}$ and $\min\{x, y\}$ is $y + x$
   therefore, $\max\{x, y\} + \min\{x, y\} = x + y$
   The statement holds for both cases
   Therefore, $\max\{x, y\} + \min\{x, y\} = x + y$ for all $x, y \in \mathbb{R}$. $\square$

c. Proof: Suppose $x, y \in \mathbb{R}$
   $\max\{x, y\} = \frac{x+y+|x-y|}{2}$
   $\min\{x, y\} = \frac{x+y-|x-y|}{2}$
   $$\begin{aligned} \text{So, } \max\{x, y\} + \min\{x, y\} &= \frac{x+y+|x-y|}{2} + \frac{x+y-|x-y|}{2} \\ &= x + y + \frac{|x-y|}{2} - \frac{|x-y|}{2} \\ &= x + y \end{aligned}$$
   Therefore, $\max\{x, y\} + \min\{x, y\} = x + y$ $\square$

**Question 3**: Prove that there are no integer solutions to the equation: $x^{10} + y^{10} = 2022$

**Assumption:** $x, y \in \mathbb{Z}$
**Conclusion:** $x^{10} + y^{10} \neq 2022$

Proof: Suppose $x, y \in \mathbb{Z}$
So, $1^{10} = 1$, $2^{10} = 1024$, and $3^{10} = 59049$
Since $2^{10} < 2022 < 3^{10}$, $x$ and $y$ must be less than 3
Therefore, by definition of an integer, $x$ and $y$ must be either 1 or 2

**Case 1:** $x$ and $y$ are different values. Without loss of generality, $x = 1$ and $y = 2$
$x^{10} + y^{10} = 1^{10} + 2^{10}$
$= 1 + 1024$
$= 1025$
Therefore, when $x = 1$ and $y = 2$, the sum of $x^{10} + y^{10} \neq 2022$
**Case 2:** $x$ and $y$ are equal to 1
$x^{10} + y^{10} = 1^{10} + 1^{10}$
$= 1 + 1$
$= 2$
Therefore, when $x = 1$ and $y = 2$, the sum of $x^{10} + y^{10} \neq 2022$
**Case 3:** $x$ and $y$ are equal to 2
$x^{10} + y^{10} = 2^{10} + 1^{10}$
$= 1024 + 1024$
$= 2048$
Therefore, when $x = 1$ and $y = 2$, the sum of $x^{10} + y^{10} \neq 2022$

$x^{10} + y^{10} \neq 2022$ is true in all cases
Therefore, there are no integer solutions to the equation: $x^{10} + y^{10} = 2022$ $\square$

**Question 4**: Using proof by contrapositive, prove the following statement: Suppose $m, n \in \mathbb{Z}$. If both $m \cdot n$ and $m + n$ are even, then both $m$ and $n$ are even

**Assumption:** $m, n \in \mathbb{Z}$
**Conclusion:** $m$ and $n$ are even
Proof: Assume $m, n \in \mathbb{Z}$
For sake of proof by contrapositive, if $m$ or $n$ is odd, then $m \cdot n$ is odd or $m + n$ is odd
$m$ and $n$ can have the same parity or opposite parity

**Case 1:** $m$ and $n$ have the same parity, therefore are both odd
By definition of odd, $m = 2x + 1$ such that $x \in \mathbb{Z}$
By definition of odd, $n = 2y + 1$ such that $y \in \mathbb{Z}$
$m \cdot n = (2x + 1)(2y + 1)$
$= 4xy + 2x + 2y + 1$
$= 2(2xy + x + y) + 1$
$= 2z + 1$ for some $z \in \mathbb{Z}$ such that $z = 2xy + x + y$
So, by definition of odd, $m \cdot n$ is odd
Therefore, when $m$ and $n$ are both odd, $m \cdot n$ is odd.

**Case 2:** $m$ and $n$ have opposite parity. Without loss of generality, $m$ is odd and $n$ is even
By definition of even, $m = 2x$ such that $x \in \mathbb{Z}$
By definition of odd, $n = 2y + 1$ such that $y \in \mathbb{Z}$
$m + n = 2x + 2y + 1$
$= 2(x + y) + 1$
$= 2z + 1$ for some $z \in \mathbb{Z}$ such that $z = x + y$
So, by definition of odd, $m + n$ is odd
Therefore, when $m$ and $n$ have opposite parity, $m + n$ is odd.

It is true that $m \cdot n$ is odd or $m + n$ is odd in both cases
Therefore, by proof by contrapositive, if both $m \cdot n$ and $m + n$ are even, then both $m$ and $n$ are even $\square$

**Question 5**: If $x^3 + 9x^7 + x \geq x^2 + x^6 + x^4$, then $x \geq 0$

Proof: Assume $x \in \mathbb{R}$
For sake of proof by contrapositive, if $x < 0$, then $x^3 + 9x^7 + x < x^2 + x^6 + x^4$
By definition of a negative number, $x = -n$ such that $n \in \mathbb{R}$ and $n \geq 0$
$x^3 + 9x^7 + x = (-n)^3 + 9(-n)^7 + (-n)$
$= -n^3 - 9n^7 - n$
$= -(n^3 + 9n^7 + n)$
$= -k_1$ such that $k_1 \in \mathbb{R}$ where $k_1 = n^3 + 9n^6 + n$
$x^2 + x^6 + x^4 = (-n)^2 + (-n)^6 + (-n)^4$
$= n^2 + n^6 + n^4$
$= k_2$ such that $k_2 \in \mathbb{R}$ where $k_2 = n^2 + n^6 + n^4$
since, $k_1 < k_2$, we know that $x^3 + 9x^7 + x < x^2 + x^6 + x^4$
Therefore, by proof by contraposition, it is true if $x^3 + 9x^7 + x \geq x^2 + x^6 + x^4$, then $x \geq 0$ $\square$

**Question 6**: There are no integers $a$ and $b$ such that $20a + 4b = 1$

1) There exists integers $a$ and $b$ such that $20a + 4b = 1$
2) There are no integers $a$ and $b$ such that $20a + 4b = 1$

Proof: Assume $\exists a, b \in \mathbb{Z}\ 20a + 4b = 1$

$$\begin{aligned} \text{So, } 20a + 4b = 1 &\equiv 2(10a + 2b) = 1 \\ &\equiv 2(10a + 2b) = 1 \\ &\equiv 2k = 1 \text{ such that } k \in \mathbb{Z} \text{ where } k = 10a + 2b \end{aligned}$$

By definition of even, 1 is even
We arrive at a contradiction where 1 is even while we know 1 is an odd integer
Therefore, there are no integers $a$ and $b$ such that $20a + 4b = 1$ $\square$

**Question 7**: For real number $a$ and $b$, if $a$ is rational and $ab$ is irrational, then $b$ is irrational

1) For $a, b \in \mathbb{R}$ and $a$ is irrational and $ab$ is rational, then $b$ is rational
2) $b$ is irrational

Proof: Suppose $a, b \in \mathbb{R}$
For sake of proof by contradiction, assume that $a$ is rational and $ab$ is irrational and $b$ is rational
Then, $\exists w, x \in \mathbb{Z}$ such that $a = w/x$ where $x \neq 0$
Let the fraction be fully reduced. That means there are no common factors between $w$ and $x$
Then, $\exists y, z \in \mathbb{Z}$ such that $b = y/z$ where $z \neq 0$
Let the fraction be fully reduced. That means there are no common factors between $y$ and $z$

$$\begin{aligned} \text{So, } ab &= \tfrac{w}{x} \cdot \tfrac{y}{z} \\ &= \tfrac{wy}{xz} \\ &= \tfrac{m}{n} \text{ such that } m, n \in \mathbb{Z} \text{ where } m = wy \text{ and } n = xz \end{aligned}$$

By definition of a rational number, $ab$ is rational
We arrive at a contradiction where $ab$ is rational and irrational.
Therefore, by proof by contradiction, if $a$ is rational and $ab$ is irrational, then $b$ is irrational. $\square$