# Quiz 7

## CMPSC 360

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**Question 2**: Consider the statement $5 \mid (n^5 - n)$ for all $n \geq 0$

Proof:
We proceed by induction on the variable $n$.
Let $P(n)$ hold the property statement for $n$.
**Base Case** ($n = 0$):
We need to prove $5 \mid (0^5 - 0)$.
By definition of divides, we get $0^5 - 0 = 5a$ for some $a \in \mathbb{Z}$.
We get $0 - 0 = 0 = 5 \cdot 0 = 0$
The base case is proved.
**Inductive Hypothesis** ($n = k$):
For any arbitrary integer $n = k$ where $k \geq 0$, assume that $P(k)$ is true
This means $5 \mid (k^5 - k)$
Using the definition of divides, we get $k^5 - k = 5q$ where $q \in \mathbb{Z}$.
**Inductive Step** ($n = k + 1$):
We have to show that $P(k + 1)$ is true, which means $5 \mid [(k + 1)^5 - (k + 1)]$
Expanding the expression, we get:

$$\begin{aligned}
(k+1)^5 - (k+1) &= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1 && \text{[algebra]} \\
&= k^5 + 5k^4 + 10k^3 + 10k^2 + 4k \\
&= (k^5 - k) + 5k^4 + 10k^3 + 10k^2 + 5k \\
&= 5q + 5k^4 + 10k^3 + 10k^2 + 5k && \text{[inductive step]} \\
&= 5(q + k^4 + 2k^3 + 2k^2 + k) \\
&= 5t \text{ for some } t \in \mathbb{Z} \text{ and } t = q + k^4 + 2k^3 + 2k^2 + k
\end{aligned}$$

We have $(k + 1)^5 - (k + 1) = 5t$. By definition of divides, we get $5 \mid (k + 1)^5 - (k + 1)$
Therefore, it is true that $\forall n \in \mathbb{Z}, n \geq 0, 5 \mid (n^5 - n)$. $\square$

**Question 3**: $S_n = S_{n-1} + S_{n-2} + S_{n-3}$, where $n > 2$. $S_0 = 0, S_1 = 1, S_2 = 1$. Prove $\forall n \geq 0, S_n < 2^n$

Proof:
We proceed by strong induction on $n$.
Let $S_n$ be defined as $S_n = S_{n-1} + S_{n-2} + S_{n-3}$ for $n > 2$ where $S_0 = 0, S_1 = 1, S_2 = 1$
Let $P(n)$ be the proposition that $S_n < 2^n$.
**Base Case** ($n = 0, n = 1, n = 2, n = 3$):
When $n = 0$, we know that $0 < 2^0 = 0 < 1$. $P(n)$ holds for $n = 0$.
When $n = 1$, we know that $1 < 2^1 = 1 < 2$. $P(n)$ holds for $n = 1$.
When $n = 2$, we know that $1 < 2^2 = 1 < 4$. $P(n)$ holds for $n = 2$.
When $n = 3$, we know that $(1 + 1 + 0) < 2^3 = 2 < 8$. $P(n)$ holds for $n = 3$.
The base case is proved.
**Inductive Hypothesis** ($n = k$):
Suppose $k$ is an arbitrary integer greater than 2. Assume that $P(i)$ is true for all $1 \leq i \leq k$ for some integer $k$.
**Inductive Step** ($n = k + 1$):
We have to prove $P(k + 1)$ is true. This means we have to show that $S_{k+1} < 2^{k+1}$.
Let's explore both sides of the equation:

$$\begin{aligned} S_{k+1} &< 2^{k+1} \\ S_{(k+1)-1} + S_{(k+1)-2} + S_{(k+1)-3} &< 2^{k+1} \\ S_k + S_{k-1} + S_{k-2} &< 2^{k+1} \\ S_k + S_{k-1} + S_{k-2} &< 2 \cdot 2^k \\ S_k + S_{k-1} + S_{k-2} + S_{k-3} - S_{k-3} &< 2 \cdot 2^k \\ S_k + S_k - S_{k-3} &< 2 \cdot 2^k \text{ [inductive step]} \\ 2S_k - S_{k-3} &< 2 \cdot 2^k \end{aligned}$$

Since we know that $P(k)$ is true, we can say for sure that $2S_k < 2 \cdot 2^k$.
This means that it is true that $2S_k - S_{k-3} < 2 \cdot 2^k$.
Therefore, $\forall n \geq 0, S_n < 2^n$. $\square$