Homework 8
CMPSC 360

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Question 1: For all $n \in \mathbb{N}$: $3 \mid 2^{2n-1}$

Proof:
We proceed by induction on the variable n.
Let $P(n)$ hold the property of the statement for n.
Base Case (n = 1):
We need to prove $3 \mid 2^{2(1)-1}$

Question 2: Show that $n! > 3^n$ for $n \geq 7$

Proof:
We proceed by induction on the variable n.
Base Case $(n = 7)$:
We need to prove $7! > 3^7$
The left hand side of the equation is 5040 and the right hand side is 2187. Since $5040 > 2187$, the base case is proved.
Inductive Hypothesis $(n = k)$:
For any arbitrary natural number $n = k$ where $k \geq 7$, we assume that $k! > 3^k$
Inductive Step $(n = k + 1)$:
We have to show that $(k+1)! > 3^{k+1}$
To show this, let's explore both sides of the equation
Expanding both sides we get: $(k+1) \cdot k! > 3 \cdot 3^k$
From the inductive hypothesis, we know that $k! > 3^k$.
We also know that $k + 1 > 3$ because of the restriction on k that states $k \geq 7$.
So, we can conclude that $(k+1) \cdot k! > 3 \cdot 3^k$, which means $(k+1)! > 3^{k+1}$ is true.
Therefore, $\forall n \in \mathbb{N}, k! > 3^k$. $\square$

Question 3: For any positive integer n, $5 \mid 6^n - 1$

Proof:
We proceed by induction on the variable n.
Let $P(n)$ hold the property of the statement for n.
Base Case $(n = 1)$:
$P(1)$ asserts that $5 \mid 6^1 - 1$.
By the definition of divides, $(6^1 - 1) = 5a$ for some $a \in \mathbb{Z}$
We get, $5 = 5 \cdot 1 = 5$.
The base case is proved.
Inductive Hypothesis $(n = k)$:
For any arbitrary integer $n = k$ where $k \geq 1$, assume that $P(k)$ is true.
That means $5 \mid 6^k - 1$
Using the definition of divides, we get $6^k - 1 = 5q$ where $q \in \mathbb{Z}$
Inductive Step $(n = k + 1)$:
We have to show that $P(k+1)$ is true, which means $5 \mid 6^{k+1} - 1$.
Expanding the expression, we get:

$$
\begin{aligned}
6^{k+1} - 1 &= 6 \cdot 6^k - 1 \\
&= 6 \cdot (6^k - 1) + 5 \\
&= 6 \cdot 5q + 5 && \text{[inductive step]} \\
&= 5 \cdot (6q + 1) && \text{[factoring 5 out]} \\
&= 5t && \text{for some } t \in \mathbb{Z} \text{ where } t = 6q + 1
\end{aligned}
$$

We have $6^{k+1} - 1 = 5t$. By definition of divides we get $5 \mid 6^{k+1} - 1$.
Therefore, it is true that $\forall n \in \mathbb{Z}, 5 \mid 6^n - 1$. $\square$

Question 4: For any $n \in \mathbb{N}$ and any $a \in \mathbb{R}$, prove that $1 + a + a^2 + a^3 + ... + a^n = \frac{a^{n+1}-1}{a-1}$

Proof:

We proceed by induction on the variable n.

Let $P(n)$ hold the property of the statement for n.

Base Case $(n = 1)$:

$P(1)$ asserts that $1 + a = \frac{a^{1+1}-1}{a-1}$

Taking the right hand side:

$$\frac{a^{1+1}-1}{a-1} = \frac{a^2-1}{a-1}$$
$$= \frac{(a+1)(a-1)}{a-1} \quad \text{[factoring]}$$
$$= a + 1 \quad\quad \text{[divide]}$$

The base case is proved.

Inductive Hypothesis $(n = k)$:

For an arbitrary natural number $n = k$, we assume that $\sum_{i=0}^{k} a^i = \frac{a^{k+1}-1}{a-1}$

Inductive Step $(n = k + 1)$:

We have to show that $\sum_{i=0}^{k+1} a^i = \frac{a^{k+2}-1}{a-1}$

To show this, let's explore the left hand side of the equations:

$$\sum_{i=0}^{k+1} a^i = \sum_{i=0}^{k} a^i + a^{k+1}$$
$$= \frac{a^{k+1}-1}{a-1} + a^{k+1} \quad\quad \text{[injective hypothesis]}$$
$$= \frac{a^{k+1}-1+a^{k+1}\cdot(a-1)}{a-1}$$
$$= \frac{a^{k+1}-1+a^{k+2}-a^{k+1}}{a-1}$$
$$= \frac{a^{k+2}-1}{a-1} \quad\quad\quad \text{[subtraction]}$$

Therefore, it is true that $\forall n \in \mathbb{N}, \forall a \in \mathbb{R} \ 1 + a + a^2 + a^3 + ... + a^n = \frac{a^{n+1}-1}{a-1}$. $\square$

Question 5: Prove that $1^3 + 2^3 + 3^3 + ... + n^3 = (\frac{n(n+1)}{2})^2$

Proof:

We proceed by induction on the variable n.

Let $P(n)$ hold the property of the statement for n.

Base Case $(n = 1)$:

$P(1)$ asserts that $1^3 = (\frac{1(1+1)}{2})^2$

Taking the right hand side: $(\frac{1(1+1)}{2})^2 = (\frac{1 \cdot 2}{2})^2 = (\frac{2}{2})^2 = 1^2 = 1$

The base case is proved.

Inductive Hypothesis $(n = k)$:

For an arbitrary natural number $n = k$, we assume that $\sum_{i=1}^{k} i^3 = (\frac{k(k+1)}{2})^2$

Inductive Step $(n = k + 1)$:

We have to show that $\sum_{i=1}^{k+1} i^3 = (\frac{(k+1)(k+2)}{2})^2$

To show this, let's explore the left hand side of the equation:

$$\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3 \qquad \text{[by making next-to-last term explicit]}$$

$$= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \qquad \text{[by inductive hypothesis]}$$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

$$= \frac{k^2(k+1)^2 + (k+1)^2 \cdot 4(k+1)}{4}$$

$$= \frac{(k+1)^2 \cdot (k^2 + 4(k+1))}{4} \qquad \text{[factoring]}$$

$$= \frac{(k+1)^2 \cdot (k^2 + 4k + 4)}{4}$$

$$= \frac{(k+1)^2 \cdot (k+2)^2}{4}$$

$$= \left(\frac{(k+1)(k+2)}{2}\right)^2$$

Therefore, it is true that $1^3 + 2^3 + 3^3 + ... + n^3 = (\frac{n(n+1)}{2})^2$. $\square$

Question 6:

a) $F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1$

Proof:

We proceed by induction on n.

Let $P(n) = \sum_{i=0}^{n} F_i$, where $F_n = F_{n-1} + F_{n-2}$ for all $n > 1$ and $F_0 = 0, F_1 = 1$.

Defined recursively, $P(n) = P(n-1) + F_n$.

We must prove that $P(n) = F_{n+2} - 1$

Base Case $(n = 0, 1, 2)$:

For $n = 0$, we know that $P(0) = 0$, and $F_2 - 1 = (1 + 0) - 1 = 0$. Therefore, $P(0)$ is proved

For $n = 1$, we know that $P(1) = 1$, and $F_3 - 1 = (1 + 1) - 1 = 1$. Therefore, $P(1)$ is proved

For $n = 2$, we calculate that $P(2) = 2$, and $F_4 - 1 = (2 + 1) - 1 = 2$. Therefore, $P(1)$ is proved

Inductive Hypothesis $(n = k)$:

Let k be any arbitrary natural number and $k > 1$. We assume that $P(k) = F_{k+2} - 1$ for all natural numbers i from 1 through k.

Inductive Step $(n = k + 1)$:

We need to show that $\sum_{i=0}^{k+1} F_i = F_{k+3} - 1$

Taking the left side:

$$\sum_{i=0}^{k+1} F_i = \sum_{i=0}^{k} F_i + F_{k+1}$$

$$= F_{k+2} - 1 + F_{k+1} \quad \text{[inductive hypothesis]}$$

$$= F_{k+3} - 1 \qquad \text{[from the definition of } F_n]$$

Therefore, it is true that $F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1$. $\square$

b) $F_2 + F_4 + F_6 + ... + F_{2n} = F_{2n+1} - 1$

Proof:

We proceed by induction on n.

Let $P(n) = \sum_{i=0}^{n} F_{2i}$, where $F_n = F_{n-1} + F_{n-2}$ for all $n > 1$ and $F_0 = 0, F_1 = 1$.

Defined recursively, $P(n) = P(n-1) + F_{2n}$.

We must prove that $P(n) = F_{2n+1} - 1$

Base Case $(n = 1)$:

For $n = 1$, we calculate that $P(1) = 1$, and $F_3 - 1 = (1+1) - 1 = 1$. Therefore, $P(1)$ is proven.

Inductive Hypothesis $(n = k)$:

Let k be any arbitrary natural number and $k > 1$. We assume that $P(k) = F_{2k+1} - 1$ for all natural numbers i from 1 through k.

Inductive Step $(n = k + 1)$:

We need to show that $\sum_{i=0}^{k+1} F_{2i} = F_{2k+3} - 1$

Taking the left side:

$$\sum_{i=0}^{k+1} F_{2i} = \sum_{i=0}^{k} F_{2i} + F_{2k+2}$$
$$= F_{2k+1} - 1 + F_{2k+2} \quad \text{[inductive hypothesis]}$$
$$= F_{2k+3} - 1 \quad\quad\quad \text{[from the definition of } F_n]$$

Therefore, it is true that $F_2 + F_4 + F_6 + ... + F_{2n} = F_{2n+1} - 1$. $\square$

Question 7: $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for all $n \in \mathbb{N}$ where $a_n = a_{n-1} + 2a_{n-2}, a_1 = 1, a_2 = 8, n \geq 3$

Proof:

We proceed by induction on n

Let $P(n)$ hold the property of the statement for n.

Base Case $(n = 1, 2, 3)$

For $n = 1$, we know that $a_1 = 1$, and $P(1) = 3 \cdot 2^0 + 2(-1)^1 = 1$. Hence, $P(1)$ is proven

For $n = 2$, we know that $a_2 = 8$, and $P(2) = 3 \cdot 2^1 + 2(-1)^2 = 8$. Hence, $P(2)$ is proven

For $n = 3$, we calculate $a_3 = 8 + 2(1) = 10$, and $P(3) = 3 \cdot 2^2 + 2(-1)^3 = 10$. $P(3)$ is proven

Inductive Hypothesis $(n = k)$:

Let k be an arbitrary natural number greater than 3. We assume that $P(k) = 3 \cdot 2^{n-1} + 2(-1)^n$ for all natural numbers i from 1 to k.

Inductive Step $(n = k + 2)$:

We need to show that $a_k + 2a_{k-1} = 3 \cdot 2^k + 2(-1)^{k+1}$

Taking the left side:

$$
\begin{aligned}
a_k + 2a_{k-1} &= 3 \cdot 2^{k-1} + 2(-1)^k + 2(3 \cdot 2^{k-2} + 2(-1)^{k-1}) \\
&= 3 \cdot 2^{k-1} + 2(-1)^k + 6 \cdot 2^{k-2} + 4(-1)^{k-1} \\
&= 3 \cdot 2^{k-1} + 2(-1)^k + 3 \cdot 2^{k-1} - 4(-1)^k \\
&= 6 \cdot 2^{k-1} - 2(-1)^k \\
&= 3 \cdot 2^k + 2(-1)^{k+1}
\end{aligned}
$$

Therefore, it is true that $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for all $n \in \mathbb{N}$ where $a_n = a_{n-1} + 2a_{n-2}$, $a_1 = 1, a_2 = 8, n \geq 3$

Question 8: