Homework 8

CMPSC 360

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Question 1: For all $n \in \mathbb{N}$: $3 \mid 2^{2n} - 1$

Proof:
We proceed by induction on the variable n.
Let $P(n)$ hold the property of the statement for n.
Base Case (n = 1):
We need to prove $3|2^{2(1)} - 1$
$2^2 - 1 = 4 - 1 = 3$
Since $3 \mid 3$, the base case is proved.
Inductive Hypothesis ($n = k$):
For any arbitrary natural number $n = k$, assume that $P(k)$ is true.
That means $3 \mid 2^{2k} - 1$
Using the definition of divides, we get $2^{2k} - 1 = 3q$ where $q \in \mathbb{Z}$
Inductive Step ($n = k + 1$):
We have to show that $P(k + 1)$ is true, which means $3 \mid 2^{2k+2} - 1$
Expanding the expression, we get:

$$\begin{aligned}
2^{2k+2} - 1 &= 4 \cdot 2^{2k} - 1 \\
&= 4 \cdot (2^{2k} - 1) + 3 \\
&= 4 \cdot 3q + 3 && \text{[inductive hypothesis]} \\
&= 3(4q + 1) && \text{[factoring out 3]} \\
&= 3t && \text{such that } t \in \mathbb{N} \text{ where } t = 4q + 1
\end{aligned}$$

Therefore, by definition of divides, $\forall n \in \mathbb{N}, 3 \mid 2^{2n} - 1$. $\square$

Question 2: Show that $n! > 3^n$ for $n \geq 7$

Proof:
We proceed by induction on the variable n.
Base Case ($n = 7$):
We need to prove $7! > 3^7$
The left hand side of the equation is 5040 and the right hand side is 2187. Since $5040 > 2187$, the base case is proved.
Inductive Hypothesis ($n = k$):
For any arbitrary natural number $n = k$ where $k \geq 7$, we assume that $k! > 3^k$
Inductive Step ($n = k + 1$):
We have to show that $(k + 1)! > 3^{k+1}$
To show this, let's explore both sides of the equation
Expanding both sides we get: $(k + 1) \cdot k! > 3 \cdot 3^k$
From the inductive hypothesis, we know that $k! > 3^k$.
We also know that $k + 1 > 3$ because of the restriction on k that states $k \geq 7$.
So, we can conclude that $(k + 1) \cdot k! > 3 \cdot 3^k$, which means $(k + 1)! > 3^{k+1}$ is true.
Therefore, $\forall n \in \mathbb{N}, k! > 3^k$. $\square$

Question 3: For any positive integer n, $5 \mid 6^n - 1$

Proof:
We proceed by induction on the variable n.
Let $P(n)$ hold the property of the statement for n.
Base Case $(n = 1)$:
$P(1)$ asserts that $5 \mid 6^1 - 1$.
By the definition of divides, $(6^1 - 1) = 5a$ for some $a \in \mathbb{Z}$
We get, $5 = 5 \cdot 1 = 5$.
The base case is proved.
Inductive Hypothesis $(n = k)$:
For any arbitrary integer $n = k$ where $k \geq 1$, assume that $P(k)$ is true.
That means $5 \mid 6^k - 1$
Using the definition of divides, we get $6^k - 1 = 5q$ where $q \in \mathbb{Z}$
Inductive Step $(n = k + 1)$:
We have to show that $P(k + 1)$ is true, which means $5 \mid 6^{k+1} - 1$.
Expanding the expression, we get:

$$\begin{aligned} 6^{k+1} - 1 &= 6 \cdot 6^k - 1 \\ &= 6 \cdot (6^k - 1) + 5 \\ &= 6 \cdot 5q + 5 && \text{[inductive step]} \\ &= 5 \cdot (6q + 1) && \text{[factoring 5 out]} \\ &= 5t && \text{for some } t \in \mathbb{Z} \text{ where } t = 6q + 1 \end{aligned}$$

We have $6^{k+1} - 1 = 5t$. By definition of divides we get $5 \mid 6^{k+1} - 1$.
Therefore, it is true that $\forall n \in \mathbb{Z}, 5 \mid 6^n - 1$. $\square$

Question 4: For any $n \in \mathbb{N}$ and any $a \in \mathbb{R}$, prove that $1 + a + a^2 + a^3 + ... + a^n = \frac{a^{n+1}-1}{a-1}$

Proof:
We proceed by induction on the variable n.
Let $P(n)$ hold the property of the statement for n.
Base Case ($n = 1$):
$P(1)$ asserts that $1 + a = \frac{a^{1+1}-1}{a-1}$
Taking the right hand side:

$$\begin{aligned} \frac{a^{1+1}-1}{a-1} &= \frac{a^2-1}{a-1} \\ &= \frac{(a+1)(a-1)}{a-1} && \text{[factoring]} \\ &= a+1 && \text{[divide]} \end{aligned}$$

The base case is proved.
Inductive Hypothesis ($n = k$):
For an arbitrary natural number $n = k$, we assume that $\sum_{i=0}^{k} a^i = \frac{a^{k+1}-1}{a-1}$
Inductive Step ($n = k + 1$):
We have to show that $\sum_{i=0}^{k+1} a^i = \frac{a^{k+2}-1}{a-1}$
To show this, let's explore the left hand side of the equations:

$$\begin{aligned} \sum_{i=0}^{k+1} a^i &= \sum_{i=0}^{k} a^i + a^{k+1} \\ &= \frac{a^{k+1}-1}{a-1} + a^{k+1} && \text{[inductive hypothesis]} \\ &= \frac{a^{k+1}-1+a^{k+1}\cdot(a-1)}{a-1} \\ &= \frac{a^{k+1}-1+a^{k+2}-a^{k+1}}{a-1} \\ &= \frac{a^{k+2}-1}{a-1} && \text{[subtraction]} \end{aligned}$$

Therefore, it is true that $\forall n \in \mathbb{N}, \forall a \in \mathbb{R}\ 1 + a + a^2 + a^3 + ... + a^n = \frac{a^{n+1}-1}{a-1}$. $\square$

Question 5: Prove that $1^3 + 2^3 + 3^3 + ... + n^3 = (\frac{n(n+1)}{2})^2$

Proof:
We proceed by induction on the variable n.
Let $P(n)$ hold the property of the statement for n.
Base Case ($n = 1$):
$P(1)$ asserts that $1^3 = (\frac{1(1+1)}{2})^2$
Taking the right hand side: $(\frac{1(1+1)}{2})^2 = (\frac{1 \cdot 2}{2})^2 = (\frac{2}{2})^2 = 1^2 = 1$
The base case is proved.
Inductive Hypothesis ($n = k$):
For an arbitrary natural number $n = k$, we assume that $\sum_{i=1}^{k} i^3 = (\frac{k(k+1)}{2})^2$
Inductive Step ($n = k + 1$):
We have to show that $\sum_{i=1}^{k+1} i^3 = (\frac{(k+1)(k+2)}{2})^2$
To show this, let's explore the left hand side of the equation:

$$\begin{aligned}
\sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 && \text{[by making next-to-last term explicit]} \\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 && \text{[by inductive hypothesis]} \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{k^2(k+1)^2 + (k+1)^2 \cdot 4(k+1)}{4} \\
&= \frac{(k+1)^2 \cdot (k^2 + 4(k+1))}{4} && \text{[factoring]} \\
&= \frac{(k+1)^2 \cdot (k^2 + 4k + 4)}{4} \\
&= \frac{(k+1)^2 \cdot (k+2)^2}{4} \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^2
\end{aligned}$$

Therefore, it is true that $1^3 + 2^3 + 3^3 + ... + n^3 = (\frac{n(n+1)}{2})^2$. $\square$

Question 6a: $F_1 + F_2 + F_3 + ... + F_n = F_{n+2} - 1$

Proof:
We proceed by induction on n.
Let $P(n) = \sum_{i=0}^{n} F_i$, where $F_n = F_{n-1} + F_{n-2}$ for all $n > 1$ and $F_0 = 0, F_1 = 1$.
Defined recursively, $P(n) = P(n-1) + F_n$.
We must prove that $P(n) = F_{n+2} - 1$
Base Case ($n = 0, 1, 2$):
For $n = 0$, we know that $P(0) = 0$, and $F_2 - 1 = (1 + 0) - 1 = 0$. Therefore, $P(0)$ is proved
For $n = 1$, we know that $P(1) = 1$, and $F_3 - 1 = (1 + 1) - 1 = 1$. Therefore, $P(1)$ is proved
For $n = 2$, we calculate that $P(2) = 2$, and $F_4 - 1 = (2 + 1) - 1 = 2$. Therefore, $P(2)$ is proved
Inductive Hypothesis ($n = k$):
Let k be any arbitrary natural number and $k > 1$. We assume that $P(k) = F_{k+2} - 1$ for all natural numbers i from 1 through k.
Inductive Step ($n = k + 1$):
We need to show that $\sum_{i=0}^{k+1} F_i = F_{k+3} - 1$
Taking the left side:

$$\begin{aligned}
\sum_{i=0}^{k+1} F_i &= \sum_{i=0}^{k} F_i + F_{k+1} \\
&= F_{k+2} - 1 + F_{k+1} \quad \text{[inductive hypothesis]} \\
&= (F_{k+2} + F_{k+1}) - 1 \\
&= F_{k+3} - 1 \qquad\qquad \text{[from the definition of } F_n\text{]}
\end{aligned}$$

Therefore, it is true that $F_1 + F_2 + F_3 + ... + F_n = F_{n+2} - 1$. $\square$

Question 6b: $F_2 + F_4 + F_6 + ... + F_{2n} = F_{2n+1} - 1$

Proof:
We proceed by induction on n.
Let $P(n) = \sum_{i=0}^{n} F_{2i}$, where $F_n = F_{n-1} + F_{n-2}$ for all $n > 1$ and $F_0 = 0, F_1 = 1$.
Defined recursively, $P(n) = P(n-1) + F_{2n}$.
We must prove that $P(n) = F_{2n+1} - 1$
Base Case ($n = 1$):
For $n = 1$, we calculate that $P(1) = 1$, and $F_3 - 1 = (1+1) - 1 = 1$. Therefore, $P(1)$ is proven.
Inductive Hypothesis ($n = k$):
Let k be any arbitrary natural number and $k > 1$. We assume that $P(k) = F_{2k+1} - 1$ for all natural numbers i from 1 through k.
Inductive Step ($n = k + 1$):
We need to show that $\sum_{i=0}^{k+1} F_{2i} = F_{2k+3} - 1$
Taking the left side:

$$\begin{aligned}
\sum_{i=0}^{k+1} F_{2i} &= \sum_{i=0}^{k} F_{2i} + F_{2k+2} \\
&= F_{2k+1} - 1 + F_{2k+2} \quad \text{[inductive hypothesis]} \\
&= (F_{2k+2} + F_{2k+1}) - 1 \\
&= F_{2k+3} - 1 \qquad\qquad\quad \text{[from the definition of } F_n\text{]}
\end{aligned}$$

Therefore, it is true that $F_2 + F_4 + F_6 + ... + F_{2n} = F_{2n+1} - 1$. $\square$

Question 7: $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for all $n \in \mathbb{N}$ where $a_n = a_{n-1} + 2a_{n-2}, a_1 = 1, a_2 = 8, n \geq 3$

Proof:
We proceed by induction on n
Let $P(n)$ hold the property of the statement for n.
Base Case ($n = 1, 2, 3$)
For $n = 1$, we know that $a_1 = 1$, and $P(1) = 3 \cdot 2^0 + 2(-1)^1 = 1$. Hence, $P(1)$ is proven
For $n = 2$, we know that $a_2 = 8$, and $P(2) = 3 \cdot 2^1 + 2(-1)^2 = 8$. Hence, $P(2)$ is proven
For $n = 3$, we calculate $a_3 = 8 + 2(1) = 10$, and $P(3) = 3 \cdot 2^2 + 2(-1)^3 = 10$. $P(3)$ is proven
Inductive Hypothesis ($n = k$):
Let k be an arbitrary natural number greater than 3. We assume that $P(k) = 3 \cdot 2^{k-1} + 2(-1)^k$ for all natural numbers i in $3 < i < k$.
Inductive Step ($n = k + 1$):
From the recursive definition of the function, we get that $a_{k+1} = a_k + 2a_{k-1}$
We need to show that $a_k + 2a_{k-1} = 3 \cdot 2^k + 2(-1)^{k+1}$
Taking the left side:

$$\begin{aligned}
a_k + 2a_{k-1} &= 3 \cdot 2^{k-1} + 2(-1)^k + 2(3 \cdot 2^{k-2} + 2(-1)^{k-1}) && \text{[inductive hypothesis]} \\
&= 3 \cdot 2^{k-1} + 2(-1)^k + 6 \cdot 2^{k-2} + 4(-1)^{k-1} \\
&= 3 \cdot 2^{k-1} + 2(-1)^k + 3 \cdot 2^{k-1} - 4(-1)^k \\
&= 6 \cdot 2^{k-1} - 2(-1)^k && \text{[addition and subtraction]} \\
&= 3 \cdot 2^k + 2(-1)^{k+1}
\end{aligned}$$

Therefore, it is true that $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ for all $n \in \mathbb{N}$ where $a_n = a_{n-1} + 2a_{n-2}$, $a_1 = 1, a_2 = 8, n \geq 3$. $\square$

Question 8: Use strong induction to show that every positive integer n can be written as a sum of distinct powers of two, that is, as a sum of a subset of the integers $2^0 = 1, 2^1 = 2, 2^2 = 4, ...$

Proof:
We proceed by strong induction on n.
Let $P(n)$ be the proposition that n can be written as a sum of distinct powers of two.
We will prove $P(n)$ for all $n \in \mathbb{Z}^+$.
Base Case ($n = 1$):
$P(1)$ holds the property of the statement because $2^0 = 1$.
The base case is proved.
Inductive Hypothesis ($n = k$):
Suppose k is an arbitrary integer greater than 0. Assume that $P(i)$ is true for all $1 \leq i \leq k$ for some integer k.
Inductive Step ($n = k + 1$):
We have to show that $P(k + 1)$ is even or odd.
Case 1: $k + 1$ is even.
Since it is even, we can say that $\frac{k+1}{2}$ is an integer.
We know that $1 \leq \frac{k+1}{2} \leq k$
Based on this, from the inductive hypothesis, we can say that $P(\frac{k+1}{2})$ is true.
So this means that $k + 1$ can be written as $2 \cdot \frac{k+1}{2}$.
Since we know that $P(\frac{k+1}{2})$ can be written as a sum of distinct powers of two, we can say that all the powers of two in $2 \cdot \frac{k+1}{2}$ are distinct too.
Case 2: $k + 1$ is odd.
When $k + 1$ is odd, we know that k is even.
From the inductive hypothesis, we know that $P(k)$ holds.
Since k is even, the sum cannot include 1, or 2^0.
Thus, $k + 1$ can be written as $P(k + 1) = 2^0 + P(k)$.
So we can say that $P(k + 1)$ can be written as a sum of distinct powers of two.
Therefore, $P(k + 1)$ is true for all positive integers. $\square$

Question 9: Assume we know that for each natural number $n > 1$, there is a prime number p such that $n < p < 2n$. We call such a prime number a pseudo-prime number. Prove that every natural number $n > 2$ can be written as the summation of distinct pseudo-prime numbers.

Proof:
We proceed by strong induction on n.
Let $P(n)$ be the proposition that n can be written as a sum of distinct pseudo-prime numbers.
Let $Q(n) = t$ such that $n < t < 2n$ where $t \in \mathbb{N}, t \neq 1, t \neq n$, and $t \nmid n$
We will prove $P(n)$ for all $n \in \mathbb{N}, n \geq 3$.
Base Case $(n = 3)$:
We know that $Q(2) = 3$.
This means that $P(3)$ holds because $3 = Q(2)$, which means 3 can be written as a sum of distinct pseudo-prime numbers. The base case is proved.
Inductive Hypothesis $(n = k)$:
For any integer $k \geq 3$ assume that all natural numbers from 3 through k can be written as a sum of distinct prime numbers.
This means, we assume that any integer $k \geq 3$, $P(i)$ is true for all natural numbers $3 \leq i \leq k$.
Inductive Step $(n = k + 1)$:
We need to show that $P(k + 1)$ is true. That means $k + 1$ can be written as a sum of distinct pseudo-prime numbers.
There are two cases: $k + 1$ is a prime itself, or $k + 1$ is a composite number.
Case 1: $k + 1$ is a prime number.
When $k + 1$ is a prime number, then $P(k + 1)$ will be true.
Case 2: $k + 1$ is a composite number.
We can say that $Q(\lfloor \frac{k+1}{2} \rfloor)$ is less than $k + 1$
So, $k + 1 - Q(\lfloor \frac{k+1}{2} \rfloor) < k$
From the inductive hypothesis we know that $P(k + 1 - Q(\lfloor \frac{k+1}{2} \rfloor))$ holds true
This means that $P(k + 1) = P(k + 1 - Q(\lfloor \frac{k+1}{2} \rfloor)) + Q(\lfloor \frac{k+1}{2} \rfloor)$
Since $Q(\lfloor \frac{k+1}{2} \rfloor)$ cannot exist in $P(k + 1 - Q(\lfloor \frac{k+1}{2} \rfloor))$, we know that all primes are distinct.
So, when $k + 1$ is a composite number, $P(k + 1)$ will be true.
Therefore, we know that $k + 1$ can be written as a sum of distinct pseudo-prime numbers. $\square$