Homework 6

CMPSC 360

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Question 1: Suppose $a, b \in \mathbb{Z}$. If $4|(a^2+b^2)$, then a and b are not both odd.

Proof: Suppose $a, b \in \mathbb{Z}$.
For sake of contradiction, assume that a and b are both odd.
By definition of odd, $a = 2x+1$ and $b = 2y+1$ such that $x, y \in \mathbb{Z}$
By definition of divides, $a^2+b^2 = 4z$ such that $z \in \mathbb{Z}$

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\begin{aligned}
a^2+b^2 &= (2x+1)^2+(2y+1)^2 = 4z \\
&= 4x^2+4x+1+4y^2+4y+1 = 4z \\
&= 4x^2+4x+4y^2+4y+2 = 4z \\
&= 4t+2 \text{ such that } t \in \mathbb{Z} \text{ where } t = x^2+x+y^2+y
\end{aligned}
$$

So, $4t+2 \neq 4z$, which means $a^2+b^2 \neq 4z$
We have arrived at a contradiction, where $a^2+b^2 = 4z$ and $a^2+b^2 \neq 4z$ when a and b are odd
Therefore, by sake of proof by contradiction, if $4|(a^2+b^2)$, then a and b are not both odd. $\square$

Question 2: Show that $\forall a, b \in \mathbb{Z},\ gcd(a,b) = b \leftrightarrow b \mid a$.

Proof:
Suppose $a, b \in \mathbb{Z}$
Case 1: $gcd(a,b) = b \rightarrow b \mid a$
By definition, $gcd(a,b) = b$ means that $b \mid b$ and $b \mid a$ where $b \neq 0$
Therefore, $gcd(a,b) = b \rightarrow b \mid a$
Case 2: $b \mid a \rightarrow gcd(a,b) = b$
Suppose $b \mid a$
We also know that $b \mid b$
By definition of gcd and since $b \mid a$ and $b \mid b$, we can say that $gcd(a,b) = b$.
Therefore, $b \mid a \rightarrow gcd(a,b) = b$
Therefore, $\forall a, b, \in \mathbb{Z},\ gcd(a,b) = b \leftrightarrow b \mid a$ $\square$

Question 3: Is $\mathbb{R} = \{(x,y) \mid (x-y) \text{ is divisible by } 17\}$ an equivalence relation?

Question 4: Suppose Neverland country (which is a fictional one), contains N cities. We define relation $\mathbb{R}$ as follows: If there is a route between two cities (c_i, c_j) for $1 \leq i, j \leq$ N, then we have $(c_i, c_j) \in \mathbb{R}$. We also assume that roads are in both directions in Neverland country. Is $\mathbb{R}$ an equivalence relation? If so, what are the equivalence classes?

Question 5: For n-dimensional vectors $x, y \in \mathbb{R}^n$, we would say $x \preceq y$ if for every $0 \leq i \leq n$, we would have $x_i \leq y_i$ where x_i is the i-th element of x. Is $\preceq$ a partial order? Prove or disprove.

Question 6:

Question 7: Let $f(x) = 2x$ where the domain is the set of real numbers. What is

(a) $f(\mathbb{N})$
(b) $f(\mathbb{Q})$
(c) $f(\mathbb{R})$

Question 8:

Question 9: Let $A = \{a_1, a_2, \ldots\}$ such that there are 4 elements in A. That is, $|A| = 4$. Similarly, let $B = \{b_1, b_2, \ldots\}$ such that $|B| = 2$. How many possible relations can be defined from A to B?

Question 10: Consider the set $A = 1, 2, 9, 11, 18$ having relation
$R = \{(1,1),(2,2),(9,9),(11,11),(18,18),(1,2),(2,1),(11,1),(1,11),(18,9),(9,18)\}$, find equivalence class of the following:

a) $[|1|]$
b) $[|2|]$
c) $[|9|]$
d) $[|11|]$
e) $[|18|]$

Question 11: Let: $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{5x}{3} - 2$. Prove that f is a one to one function.

Proof:
Assume that x_1 and x_2 belong to $\mathbb{R}$
Suppose $f(x_1) = f(x_2)$
Definition of the function $\frac{5x_1}{3} - 2 = \frac{5x_2}{3} - 2$

$$\begin{aligned} \frac{5x_1}{3} &= \frac{5x_2}{3} && \text{Add 2 to both sides} \\ 5x_1 &= 5x_2 && \text{Multiply 3 by both sides} \\ x_1 &= x_2 && \text{Divide both sides by 5} \end{aligned}$$

Therefore, since $x_1 = x_2$, this is an injective function. $\square$

Question 12: Let: $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{x}{2} + 3$ a surjection (onto)? If it is, constructs the proof, otherwise, give a counterexample.

Proof:
Assume that $a \in \mathbb{R}$
We must show that $\exists x \in \mathbb{R}$ such that $f(x) = a$
So, $a = \frac{x}{2} + 3$
$a - 3 = \frac{x}{2}$
$2a - 6 = x$
So, substituting for x, $f(x) = \frac{2a-6}{2} + 3$
$= a - 3 + 3$
$= a$
Since $f(x) = a$, we know that it is onto. $\square$