Homework 6

CMPSC 360

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Question 1: Suppose $a, b \in \mathbb{Z}$. If $4|(a^2+b^2)$, then a and b are not both odd.

Proof: Suppose $a, b \in \mathbb{Z}$.
For sake of contradiction, assume that a and b are both odd.
By definition of odd, $a = 2x+1$ and $b = 2y+1$ such that $x, y \in \mathbb{Z}$
By definition of divides, $a^2+b^2 = 4z$ such that $z \in \mathbb{Z}$

$$\begin{aligned} a^2+b^2 &= (2x+1)^2+(2y+1)^2 = 4z \\ &= 4x^2+4x+1+4y^2+4y+1 = 4z \\ &= 4x^2+4x+4y^2+4y+2 = 4z \\ &= 4t+2 \text{ such that } t \in \mathbb{Z} \text{ where } t = x^2+x+y^2+y \end{aligned}$$

So, $4t+2 \neq 4z$, which means $a^2+b^2 \neq 4z$
We have arrived at a contradiction, where $a^2+b^2 = 4z$ and $a^2+b^2 \neq 4z$ when a and b are odd
Therefore, by sake of proof by contradiction, if $4|(a^2+b^2)$, then a and b are not both odd. $\square$

Question 2: Show that $\forall a, b \in \mathbb{Z},\ gcd(a,b) = b \leftrightarrow b \mid a$.

Proof:
Suppose $a, b \in \mathbb{Z}$
Case 1: $gcd(a,b) = b \rightarrow b \mid a$
By definition, $gcd(a,b) = b$ means that $b \mid b$ and $b \mid a$ where $b \neq 0$
Therefore, $gcd(a,b) = b \rightarrow b \mid a$
Case 2: $b \mid a \rightarrow gcd(a,b) = b$
Suppose $b \mid a$
We also know that $b \mid b$
By definition of gcd and since $b \mid a$ and $b \mid b$, we can say that $gcd(a,b) = b$.
Therefore, $b \mid a \rightarrow gcd(a,b) = b$
Therefore, $\forall a, b \in \mathbb{Z},\ gcd(a,b) = b \leftrightarrow b \mid a$ $\square$

Question 3: Is $\mathbb{R} = \{(x, y) \mid (x - y) \text{ is divisible by } 17\}$ an equivalence relation?

Proof: Assume that $\mathbb{R} = \{(x, y) \mid (x - y) \text{ is divisible by } 17\}$
This means that $17 \mid (x - y)$
By definition of divides $x - y = 17a$ such that $a \in \mathbb{Z}$
For a statement to be an equivalence relation, it must be reflexive, symmetric, and transitive.
Case 1: reflexive
Assume (x, x)
so $x - y = 17a$ goes to $x - x = 17a$
This means that $0 = 17a$, for which this is a valid statement since $17 \mid 0$
So $\forall x \exists y \ y = x$ is true and $(x, y) \in \mathbb{R}$
Therefore, $\mathbb{R}$ is reflexive
Case 2: Symmetric
Suppose $x - y = 17a$ for some (x, y)
To prove symmetry, take the case (y, x)
So, $y - x = 17b$ such that $b \in \mathbb{Z}$
Since this can simplify down to $-(x - y) = 17b$, we know that $b = -a$
Thus, $17|(y - x)$ which means that $(y, x) \in \mathbb{R}$
Therefore, $\mathbb{R}$ is symmetric
Case 3: Transitive
Proof for transitivity: $\exists a, b, c \ (a, b), (b, c) \in \mathbb{R} \rightarrow (a, c) \in \mathbb{R}$
Suppose $a, b, c \in \mathbb{Z}$
We can say that $17 \mid (a - b)$ and $17 \mid (b - c)$
By definition of divides, $a - b = 17p$ and $b - c = 17q$ such that $p, q \in \mathbb{Z}$
By algebra, $a = 17p + b$ and $c = b - 17q$
So for (a, c), we get $17 \mid (a - c)$
Substituting a and c, $17 \mid [(17p + b) - (b - 17q)]$

$$
\begin{aligned}
&= 17 \mid (17p + b - b + 17q) \\
&= 17 \mid (17p + 17q) \\
&= 17 \mid [17(p + q)] \\
&= 17 \mid 17t \text{ such that } t \in \mathbb{Z} \text{ where } t = p + q
\end{aligned}
$$

So, by definition of divides, we know that this is divisible by 17.
Therefore, $\mathbb{R}$ is transitive
Since $\mathbb{R}$ is reflexive, symmetric, and transitive, it is an equivalence relation. $\square$

Question 4: Suppose Neverland country (which is a fictional one), contains N cities. We define relation $\mathbb{R}$ as follows: If there is a route between two cities (c_i, c_j) for $1 \leq i, j \leq \text{N}$, then we have $(c_i, c_j) \in \mathbb{R}$. We also assume that roads are in both directions in Neverland country. Is $\mathbb{R}$ an equivalence relation? If so, what are the equivalence classes?

$\mathbb{R}$ is an equivalence relation.
$[C_a] = \{C_b \mid \text{a route exists from } C_b \text{ to } C_a\}$

Question 5: For n-dimensional vectors $x, y \in \mathbb{R}^n$, we would say $x \preceq y$ if for every $0 \leq i \leq n$, we would have $x_i \leq y_i$ where x_i is the i-th element of x. Is $\preceq$ a partial order? Prove or disprove.

Proof:
Suppose that n-dimensional vectors $x, y \in \mathbb{R}^n$.
When $0 \leq i \leq n$, we can say that $x_i \leq y_i$ such that x_i and y_i are the i-th element of x and y.
For the relation to be a partial order, it must be reflexive, transitive, and antisymmetric.

Case 1: Reflexive
For a reflexive case, suppose $y = x$, so $y_i = x_i$
That means that $x_i \leq x_i$ by substitution.
This statement is true for all $x \in \mathbb{R}^n$.
This relation is reflexive.

Case 2: Transitive
Assume case (a, b) and (b, c) where $a, b, c \in \mathbb{R}^n$
Substituting these pairs into the relation, we get $a_i \leq b_i$ and $b_i \leq c_i$.
This means that $a_i \leq b_i \leq c_i$ by combination.
This can be simplified to be that $a_i \leq c_i$.
So the ordered pair (a, c) is valid under this relation.
Therefore, this relation is transitive.

Case 3: Antisymmetric
Assume case (y, x).
Substituting into the relation we get $y_i \leq x_i$.
Since we know that $x_i \leq y_i$, the ordered pair (y, x) is not valid.
Therefore, this relation is antisymmetric.

The relationship is reflexive, transitive, and antisymmetric.
Therefore, for n-dimensional vectors $x, y \in \mathbb{R}^n$, we would say $x \preceq y$ if for every $0 \leq i \leq n$, we would have $x_i \leq y_i$ where x_i is the i-th element of x is a partial order. □

Question 6:

Question 7: Let $f(x) = 2x$ where the domain is the set of real numbers. What is

(a) $f(\mathbb{N}) \rightarrow \{x \in \mathbb{N} \mid 2x\}$
(b) $f(\mathbb{Q}) \rightarrow \{x \in \mathbb{Q} \mid 2x\}$
(c) $f(\mathbb{R}) \rightarrow \{x \in \mathbb{R} \mid 2x\}$

Question 8:

(a) no properties
(b) (4, 4)
(c) (1, 2) and (4, 1)

Question 9: Let $A = \{a_1, a_2, \ldots\}$ such that there are 4 elements in A. That is, $|A| = 4$. Similarly, let $B = \{b_1, b_2, \ldots\}$ such that $|B| = 2$. How many possible relations can be defined from A to B?

Question 10: Consider the set $A = 1, 2, 9, 11, 18$ having relation
$R = \{(1,1), (2,2), (9,9), (11,11), (18,18), (1,2), (2,1), (11,1), (1,11), (18,9), (9,18)\}$, find equivalence class of the following:

a) $[|1|] \to \{1, 2, 11\}$
b) $[|2|] \to \{2, 1\}$
c) $[|9|] \to \{9, 18\}$
d) $[|11|] \to \{11, 1\}$
e) $[|18|] \to \{18, 9\}$

Question 11: Let: $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{5x}{3} - 2$. Prove that f is a one to one function.

Proof:
Assume that x_1 and x_2 belong to $\mathbb{R}$
Suppose $f(x_1) = f(x_2)$
Definition of the function $\frac{5x_1}{3} - 2 = \frac{5x_2}{3} - 2$

$$\begin{aligned} \frac{5x_1}{3} &= \frac{5x_2}{3} && \text{Add 2 to both sides} \\ 5x_1 &= 5x_2 && \text{Multiply 3 by both sides} \\ x_1 &= x_2 && \text{Divide both sides by 5} \end{aligned}$$

Therefore, since $x_1 = x_2$, this is an injective function. $\square$

Question 12: Let: $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{x}{2} + 3$ a surjection (onto)? If it is, constructs the proof, otherwise, give a counterexample.

Proof:
Assume that $a \in \mathbb{R}$
We must show that $\exists x \in \mathbb{R}$ such that $f(x) = a$
So, $a = \frac{x}{2} + 3$

$$a - 3 = \frac{x}{2}$$
$$2a - 6 = x$$

So, substituting for x, $f(x) = \frac{2a-6}{2} + 3$

$$= a - 3 + 3$$
$$= a$$

Since $f(x) = a$, we know that it is onto. $\square$