Class Quiz 5

CMPSC 360

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Question 5: *For all real numbers x and y, if $x + y \geq 2$, then either $x \geq 1$ or $y \geq 1$*

Proof:
Assume that $x, y \in \mathbb{R}$
For sake of proof by contradiction, prove that if $x + y \geq 2$, then $x < 1$ and $y < 1$
Assume for the sake of argument that the maximum value of x and y is 1.
So, $x + y = 2$.
However, applying the bounding rules, we know that $x + y < 2$.
We have arrived at a contradiction.
Therefore, for all real numbers x and y, if $x + y \geq 2$, then either $x \geq 1$ or $y \geq 1$.

Question 6: *A relation* $\mathbb{R}$ *is defined on integers as follows:* $\forall a, b \in \mathbb{Z}, a \ \mathbb{R} \ b \leftrightarrow 3 \mid (a^2 - b^2)$. *Determine if R is an equivalence relation.*

Proof:
Assume that $\forall a, b \in \mathbb{Z} \ 3 \mid (a^2 - b^2)$
By definition of divides, $a^2 - b^2 = 3t$ for some $t \in \mathbb{Z}$
An equivalence relation must be reflexive, symmetric, and transitive
Case 1: Reflexive
Assume that $b = a$
So, $a^2 - b^2 = a^2 - a^2$ plugging in a for b
$= 0$ Subtraction
This means that $0 = 3t$. This statement is true.
Therefore, this relation is reflexive.
Case 2: Symmetric
Suppose $a^2 - b^2 = 3t$ for some (a, b)
To prove symmetry, take the case (b, a)
So, $3 \mid b^2 - a^2$
By definition of divides $b^2 - a^2 = 3k$ for some $k \in \mathbb{Z}$
Rearranging, $-(a^2 - b^2) = 3k$ for some $k \in \mathbb{Z}$
So, $k = -t$. Therefore, the relation is valid for (a, b) and (b, a)
Therefore, the relation is symmetric.
Case 3: Transitive
Proof for transitivity: $\exists a, b, c \ (a, b), (b, c) \in \mathbb{R} \rightarrow (a, c) \in \mathbb{R}$
Suppose $x, y, z \in \mathbb{Z}$
We can say that $3 \mid (x^2 - y^2)$ and $3 \mid (y^2 - z^2)$
By definition of divides, $x^2 - y^2 = 3p$ and $y^2 - z^2 = 3q$ such that $p, q \in \mathbb{Z}$
By algebra, $x^2 = 3p + y^2$ and $z = y^2 - 3q$
So for the case (x, z) we get $3 \mid (x^2 - z^2)$

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\begin{aligned}
&= 3 \mid (3p + y^2 - y^2 - 3q) \\
&= 3 \mid (3p - 3q) \\
&= 3 \mid 3(p - q) \\
&= 3 \mid 3t \text{ such that } t \in \mathbb{Z} \text{ where } t = p - q
\end{aligned}
$$

So, by definition of divides, we know that this is divisible by 3.
Therefore, the relation is transitive.

Since the relation is reflexive, symmetric, and transitive, we know that R is an equivalence relation.