Homework 5

CMPSC 360

Kinner Parikh
February 18, 2022

Question 1: We try to prove that if n is an integer, then $n^3 - 2n^2 + 5n - 1$ is divisible by 3

1. n is divisible by 3
2. 1
3. divisible by 3
4. true
5. none of the cases are true

Proof: Assume $n \in \mathbb{Z}$
Suppose n is divisible by 3 $(3 \mid n)$
By definition of divides, either $n = 3k$ where $k \in \mathbb{Z}$

$$\begin{aligned} n^3 - 2n^2 + 5n - 1 &= (3k)^3 - 2(3k)^2 + 5(3k) - 1 \\ &= 27k^3 - 18k^2 + 15k - 1 \\ &= 3(9k^3 - 6k^2 + 5k) - 1 \\ &= 3t - 1 \text{ such that } k \in \mathbb{Z} \text{ where } t = 9k^3 - 6k^2 + 5k \end{aligned}$$

By definition of divides, we know that $n^3 - 2n^2 + 5n - 1$ is not divisible by 3 when $3|n$
Therefore, the statement, when $n \in \mathbb{Z}$, then $n^3 - 2n^2 + 5n - 1$ is divisible by 3, is false. $\square$

Question 2: Prove that $\max\{x, y\} + \min\{x, y\} = x + y$. Collaborated with Sahil Kuwadia.

a. **Assumption:** x and y are real numbers $\rightarrow x, y \in \mathbb{R}$
   Conclusion: The sum of the larger value in x or y and the smaller value in x and y is equal to the sum of x and y

b. Proof: Suppose $x, y \in \mathbb{R}$
   We know that x is either greater than y or x is less than or equal to y
   Case 1: $x > y$
   $\max\{x, y\} = x$, $\min\{x, y\} = y$
   so, the sum of $\max\{x, y\}$ and $\min\{x, y\}$ is $x + y$
   therefore, $\max\{x, y\} + \min\{x, y\} = x + y$
   Case 2: $x \leq y$
   $\max\{x, y\} = y$, $\min\{x, y\} = x$
   so, the sum of $\max\{x, y\}$ and $\min\{x, y\}$ is $y + x$
   therefore, $\max\{x, y\} + \min\{x, y\} = x + y$
   The statement holds for both cases
   Therefore, $\max\{x, y\} + \min\{x, y\} = x + y$ for all $x, y \in \mathbb{R}$. $\square$

c. Proof: Suppose $x, y \in \mathbb{R}$
   $\max\{x, y\} = \frac{x+y+|x-y|}{2}$
   $\min\{x, y\} = \frac{x+y-|x-y|}{2}$
   So, $\max\{x, y\} + \min\{x, y\} = \frac{x+y+|x-y|}{2} + \frac{x+y-|x-y|}{2}$
   $= x + y + \frac{|x-y|}{2} - \frac{|x-y|}{2}$
   $= x + y$
   Therefore, $\max\{x, y\} + \min\{x, y\} = x + y$ $\square$

Question 3: Prove that there are no integer solutions to the equation: $x^{10} + y^{10} = 2022$

Assumption: $x, y \in \mathbb{Z}$
Conclusion: $x^{10} + y^{10} \neq 2022$

Proof: Suppose $x, y \in \mathbb{Z}$
So, $1^{10} = 1$, $2^{10} = 1024$, and $3^{10} = 59049$
Since $2^{10} < 2022 < 3^{10}$, x and y must be less than 3
Therefore, by definition of an integer, x and y must be either 1 or 2

Case 1: x and y are different values. Without loss of generality, $x = 1$ and $y = 2$
$$\begin{aligned} x^{10} + y^{10} &= 1^{10} + 2^{10} \\ &= 1 + 1024 \\ &= 1025 \end{aligned}$$
Therefore, when $x = 1$ and $y = 2$, the sum of $x^{10} + y^{10} \neq 2022$

Case 2: x and y are equal to 1
$$\begin{aligned} x^{10} + y^{10} &= 1^{10} + 1^{10} \\ &= 1 + 1 \\ &= 2 \end{aligned}$$
Therefore, when $x = 1$ and $y = 2$, the sum of $x^{10} + y^{10} \neq 2022$

Case 3: x and y are equal to 2
$$\begin{aligned} x^{10} + y^{10} &= 2^{10} + 1^{10} \\ &= 1024 + 1024 \\ &= 2048 \end{aligned}$$
Therefore, when $x = 1$ and $y = 2$, the sum of $x^{10} + y^{10} \neq 2022$

$x^{10} + y^{10} \neq 2022$ is true in all cases
Therefore, there are no integer solutions to the equation: $x^{10} + y^{10} = 2022$ $\square$

Question 4: Using proof by contrapositive, prove the following statement: Suppose $m, n \in \mathbb{Z}$. If both $m \cdot n$ and $m + n$ are even, then both m and n are even

Assumption: $m, n \in \mathbb{Z}$
Conclusion: m and n are even
Proof: Assume $m, n \in \mathbb{Z}$
For sake of proof by contrapositive, if m or n is odd, then $m \cdot n$ is odd or $m + n$ is odd
m and n can have the same parity or opposite parity

Case 1: m and n have the same parity, therefore are both odd
By definition of odd, $m = 2x + 1$ such that $x \in \mathbb{Z}$
By definition of odd, $n = 2y + 1$ such that $y \in \mathbb{Z}$

$$
\begin{aligned}
m \cdot n &= (2x + 1)(2y + 1) \\
&= 4xy + 2x + 2y + 1 \\
&= 2(2xy + x + y) + 1 \\
&= 2z + 1 \text{ for some } z \in \mathbb{Z} \text{ such that } z = 2xy + x + y
\end{aligned}
$$

So, by definition of odd, $m \cdot n$ is odd
Therefore, when m and n are both odd, $m \cdot n$ is odd.

Case 2: m and n have opposite parity. Without loss of generality, m is odd and n is even
By definition of even, $m = 2x$ such that $x \in \mathbb{Z}$
By definition of odd, $n = 2y + 1$ such that $y \in \mathbb{Z}$

$$
\begin{aligned}
m + n &= 2x + 2y + 1 \\
&= 2(x + y) + 1 \\
&= 2z + 1 \text{ for some } z \in \mathbb{Z} \text{ such that } z = x + y
\end{aligned}
$$

So, by definition of odd, $m + n$ is odd
Therefore, when m and n have opposite parity, $m + n$ is odd.

It is true that $m \cdot n$ is odd or $m + n$ is odd in both cases
Therefore, by proof by contrapositive, if both $m \cdot n$ and $m + n$ are even, then both m and n are even $\square$

Question 5: If $x^3 + 9x^7 + x \geq x^2 + x^6 + x^4$, then $x \geq 0$

Proof: Assume $x \in \mathbb{R}$
For sake of proof by contrapositive, if $x < 0$, then $x^3 + 9x^7 + x < x^2 + x^6 + x^4$
By definition of a negative number, $x = -n$ such that $n \in \mathbb{R}$ and $n \geq 0$

$$
\begin{aligned}
x^3 + 9x^7 + x &= (-n)^3 + 9(-n)^7 + (-n) \\
&= -n^3 - 9n^7 - n \\
&= -(n^3 + 9n^7 + n) \\
&= -k_1 \text{ such that } k_1 \in \mathbb{R} \text{ where } k_1 = n^3 + 9n^6 + n \\
x^2 + x^6 + x^4 &= (-n)^2 + (-n)^6 + (-n)^4 \\
&= n^2 + n^6 + n^4 \\
&= k_2 \text{ such that } k_2 \in \mathbb{R} \text{ where } k_2 = n^2 + n^6 + n^4
\end{aligned}
$$

since, $k_1 < k_2$, we know that $x^3 + 9x^7 + x < x^2 + x^6 + x^4$
Therefore, by proof by contraposition, it is true if $x^3 + 9x^7 + x \geq x^2 + x^6 + x^4$, then $x \geq 0$ $\square$

Question 6: There are no integers a and b such that $20a + 4b = 1$

1) There exists integers a and b such that $20a + 4b = 1$
2) There are no integers a and b such that $20a + 4b = 1$

Proof: Assume $\exists a, b \in \mathbb{Z}\ 20a + 4b = 1$
So, $20a + 4b = 1 \equiv 2(10a + 2b) = 1$
$\equiv 2(10a + 2b) = 1$
$\equiv 2k = 1$ such that $k \in \mathbb{Z}$ where $k = 10a + 2b$
By definition of even, 1 is even
We arrive at a contradiction where 1 is even while we know 1 is an odd integer
Therefore, there are no integers a and b such that $20a + 4b = 1$ $\square$

Question 7: For real number a and b, if a is rational and ab is irrational, then b is irrational

1) For $a, b \in \mathbb{R}$ and a is irrational and ab is rational, then b is rational
2) b is irrational

Proof: Suppose $a, b \in \mathbb{R}$
For sake of proof by contradiction, assume that a is rational and ab is irrational and b is rational
Then, $\exists w, x \in \mathbb{Z}$ such that $a = w/x$ where $x \neq 0$
Let the fraction be fully reduced. That means there are no common factors between w and x
Then, $\exists y, z \in \mathbb{Z}$ such that $b = y/z$ where $z \neq 0$
Let the fraction be fully reduced. That means there are no common factors between y and z
So, $ab = \frac{w}{x} \cdot \frac{y}{z}$
$= \frac{wy}{xz}$
$= \frac{m}{n}$ such that $m, n \in \mathbb{Z}$ where $m = wy$ and $n = xz$
By definition of a rational number, ab is rational
We arrive at a contradiction where ab is rational and irrational.
Therefore, by proof by contradiction, if a is rational and ab is irrational, then b is irrational. $\square$