

Homework 8

CMPSC 360

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Question 1: For all $n \in \mathbb{N}$: $3 \mid 2^{2n-1}$

Proof:

We proceed by induction on the variable n .

Let $P(n)$ hold the property of the statement for n .

Base Case ($n = 1$):

We need to prove $3 \mid 2^{2(1)-1}$

Question 2: Show that $n! > 3^n$ for $n \geq 7$

Proof:

We proceed by induction on the variable n .

Base Case ($n = 7$):

We need to prove $7! > 3^7$

The left hand side of the equation is 5040 and the right hand side is 2187. Since $5040 > 2187$, the base case is proved.

Inductive Hypothesis ($n = k$):

For any arbitrary natural number $n = k$ where $k \geq 7$, we assume that $k! > 3^k$

Inductive Step ($n = k + 1$):

We have to show that $(k + 1)! > 3^{k+1}$

To show this, let's explore both sides of the equation

Expanding both sides we get: $(k + 1) \cdot k! > 3 \cdot 3^k$

From the inductive hypothesis, we know that $k! > 3^k$.

We also know that $k + 1 > 3$ because of the restriction on k that states $k \geq 7$.

So, we can conclude that $(k + 1) \cdot k! > 3 \cdot 3^k$, which means $(k + 1)! > 3^{k+1}$ is true.

Therefore, $\forall n \in \mathbb{N}, k! > 3^k$. \square

Question 3: For any positive integer n , $5 \mid 6^n - 1$

Proof:

We proceed by induction on the variable n .

Let $P(n)$ hold the property of the statement for n .

Base Case ($n = 1$):

$P(1)$ asserts that $5 \mid 6^1 - 1$.

By the definition of divides, $(6^1 - 1) = 5a$ for some $a \in \mathbb{Z}$

We get, $5 = 5 \cdot 1 = 5$.

The base case is proved.

Inductive Hypothesis ($n = k$):

For any arbitrary integer $n = k$ where $k \geq 1$, assume that $P(k)$ is true.

That means $5 \mid 6^k - 1$

Using the definition of divides, we get $6^k - 1 = 5q$ where $q \in \mathbb{Z}$

Inductive Step ($n = k + 1$):

We have to show that $P(k + 1)$ is true, which means $5 \mid 6^{k+1} - 1$.

Expanding the expression, we get:

$$\begin{aligned} 6^{k+1} - 1 &= 6 \cdot 6^k - 1 \\ &= 6 \cdot (6^k - 1) + 5 \\ &= 6 \cdot 5q + 5 && \text{[inductive step]} \\ &= 5 \cdot (6q + 1) && \text{[factoring 5 out]} \\ &= 5t && \text{for some } t \in \mathbb{Z} \text{ where } t = 6q + 1 \end{aligned}$$

We have $6^{k+1} - 1 = 5t$. By definition of divides we get $5 \mid 6^{k+1} - 1$.

Therefore, it is true that $\forall n \in \mathbb{Z}, 5 \mid 6^n - 1$. \square

Question 4: For any $n \in \mathbb{N}$ and any $a \in \mathbb{R}$, prove that $1 + a + a^2 + a^3 + \dots + a^n = \frac{a^{n+1}-1}{a-1}$

Proof:

We proceed by induction on the variable n .

Let $P(n)$ hold the property of the statement for n .

Base Case ($n = 1$):

$P(1)$ asserts that $1 + a = \frac{a^{1+1}-1}{a-1}$

Taking the right hand side:

$$\begin{aligned} \frac{a^{1+1}-1}{a-1} &= \frac{a^2-1}{a-1} \\ &= \frac{(a+1)(a-1)}{a-1} \quad [\text{factoring}] \\ &= a+1 \quad [\text{divide}] \end{aligned}$$

The base case is proved.

Inductive Hypothesis ($n = k$):

For an arbitrary natural number $n = k$, we assume that $\sum_{i=0}^k a^i = \frac{a^{k+1}-1}{a-1}$

Inductive Step ($n = k+1$):

We have to show that $\sum_{i=0}^{k+1} a^i = \frac{a^{k+2}-1}{a-1}$

To show this, let's explore the left hand side of the equations:

$$\begin{aligned} \sum_{i=0}^{k+1} a^i &= \sum_{i=0}^k a^i + a^{k+1} \\ &= \frac{a^{k+1}-1}{a-1} + a^{k+1} \quad [\text{inductive hypothesis}] \\ &= \frac{a^{k+1}-1 + a^{k+1} \cdot (a-1)}{a-1} \\ &= \frac{a^{k+1}-1 + a^{k+2} - a^{k+1}}{a-1} \\ &= \frac{a^{k+2}-1}{a-1} \quad [\text{subtraction}] \end{aligned}$$

Therefore, it is true that $\forall n \in \mathbb{N}, \forall a \in \mathbb{R} \ 1 + a + a^2 + a^3 + \dots + a^n = \frac{a^{n+1}-1}{a-1}$. \square

Question 5: Prove that $1^3 + 2^3 + 3^3 + \dots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

Proof:

We proceed by induction on the variable n .

Let $P(n)$ hold the property of the statement for n .

Base Case ($n = 1$):

$P(1)$ asserts that $1^3 = \left(\frac{1(1+1)}{2}\right)^2$

Taking the right hand side: $\left(\frac{1(1+1)}{2}\right)^2 = \left(\frac{1 \cdot 2}{2}\right)^2 = \left(\frac{2}{2}\right)^2 = 1^2 = 1$

The base case is proved.

Inductive Hypothesis ($n = k$):

For an arbitrary natural number $n = k$, we assume that $\sum_{i=1}^k i^3 = \left(\frac{k(k+1)}{2}\right)^2$

Inductive Step ($n = k + 1$):

We have to show that $\sum_{i=1}^{k+1} i^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$

To show this, let's explore the left hand side of the equation:

$$\begin{aligned}
 \sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^k i^3 + (k+1)^3 && \text{[by making next-to-last term explicit]} \\
 &= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 && \text{[by inductive hypothesis]} \\
 &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
 &= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
 &= \frac{k^2(k+1)^2 + (k+1)^2 \cdot 4(k+1)}{4} \\
 &= \frac{(k+1)^2(k^2 + 4(k+1))}{4} && \text{[factoring]} \\
 &= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\
 &= \frac{(k+1)^2(k+2)^2}{4} \\
 &= \left(\frac{(k+1)(k+2)}{2}\right)^2
 \end{aligned}$$

Therefore, it is true that $1^3 + 2^3 + 3^3 + \dots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$. \square