

Homework 5

CMPSC 360

Kinner Parikh
February 18, 2022

Question 1: We try to prove that if n is an integer, then $n^3 - 2n^2 + 5n - 1$ is divisible by 3

1. n is not divisible by 3
2. 2
3. divisible by 3
4. neither true or false \rightarrow even if the cases pass, there is a case where $n \% 3 == 0$
5. all cases are true

Proof: Assume $n \in \mathbb{Z}$

Suppose n is not divisible by 3 ($3 \nmid n$)

By definition of divides, either $n = 3k + 1$ or $n = 3k + 2$ where $k \in \mathbb{Z}$

Case 1: $n = 3k + 1$

$$\begin{aligned} n^3 - 2n^2 + 5n - 1 &= (3k + 1)^3 - 2(3k + 1)^2 + 5(3k + 1) - 1 \\ &= 27k^3 + 27k^2 + 9k + 1 - 18k^2 - 12k - 2 + 15k + 5 - 1 \\ &= 27k^3 + 9k^2 + 12k + 3 \\ &= 3(9k^3 + 3k^2 + 4k + 1) \\ &= 3t \text{ such that } t \in \mathbb{Z} \text{ where } t = 9k^3 + 3k^2 + 4k + 1 \end{aligned}$$

By definition of divides, $n^3 - 2n^2 + 5n - 1$ is divisible by 3

Therefore, when $n = 3k + 1$, $n^3 - 2n^2 + 5n - 1$ is divisible by 3

Case 2: $n = 3k + 2$

$$\begin{aligned} n^3 - 2n^2 + 5n - 1 &= (3k + 2)^3 - 2(3k + 2)^2 + 5(3k + 2) - 1 \\ &= 27k^3 + 54k^2 + 36k + 8 - 18k^2 - 24k - 8 + 15k + 10 - 1 \\ &= 27k^3 + 36k^2 + 27k + 9 \\ &= 3(9k^3 + 12k^2 + 9k + 3) \\ &= 3t \text{ such that } t \in \mathbb{Z} \text{ where } t = 9k^3 + 12k^2 + 9k + 3 \end{aligned}$$

By definition of divides, $n^3 - 2n^2 + 5n - 1$ is divisible by 3

Therefore, when $n = 3k + 2$, $n^3 - 2n^2 + 5n - 1$ is divisible by 3

Question 2: Prove that $\max\{x, y\} + \min\{x, y\} = x + y$. Collaborated with Sahil Kuwadia.

a. **Assumption:** x and y are real numbers $\rightarrow x, y \in \mathbb{R}$

Conclusion: The sum of the larger value in x or y and the smaller value in x and y is equal to the sum of x and y

b. Proof: Suppose $x, y \in \mathbb{R}$

We know that x is either greater than y or x is less than or equal to y

Case 1: $x > y$

$$\max\{x, y\} = x, \min\{x, y\} = y$$

so, the sum of $\max\{x, y\}$ and $\min\{x, y\}$ is $x + y$

$$\text{therefore, } \max\{x, y\} + \min\{x, y\} = x + y$$

Case 2: $x \leq y$

$$\max\{x, y\} = y, \min\{x, y\} = x$$

so, the sum of $\max\{x, y\}$ and $\min\{x, y\}$ is $y + x$

$$\text{therefore, } \max\{x, y\} + \min\{x, y\} = x + y$$

The statement holds for both cases

Therefore, $\max\{x, y\} + \min\{x, y\} = x + y$ for all $x, y \in \mathbb{R}$. \square

c. Proof: Suppose $x, y \in \mathbb{R}$

$$\max\{x, y\} = \frac{x+y+|x-y|}{2}$$

$$\min\{x, y\} = \frac{x+y-|x-y|}{2}$$

$$\begin{aligned} \text{So, } \max\{x, y\} + \min\{x, y\} &= \frac{x+y+|x-y|}{2} + \frac{x+y-|x-y|}{2} \\ &= x + y + \frac{|x-y|}{2} - \frac{|x-y|}{2} \\ &= x + y \end{aligned}$$

Therefore, $\max\{x, y\} + \min\{x, y\} = x + y$ \square

Question 3: Prove that there are no integer solutions to the equation: $x^{10} + y^{10} = 2022$

Assumption: $x, y \in \mathbb{Z}$

Conclusion: $x^{10} + y^{10} \neq 2022$

Proof: Suppose $x, y \in \mathbb{Z}$

So, $1^{10} = 1$, $2^{10} = 1024$, and $3^{10} = 59049$

Since $2^{10} < 2022 < 3^{10}$, x and y must be less than 3

Therefore, by definition of an integer, x and y must be either 1 or 2

Case 1: x and y are different values. Without loss of generality, $x = 1$ and $y = 2$

$$\begin{aligned}x^{10} + y^{10} &= 1^{10} + 2^{10} \\&= 1 + 1024 \\&= 1025\end{aligned}$$

Therefore, when $x = 1$ and $y = 2$, the sum of $x^{10} + y^{10} \neq 2022$

Case 2: x and y are equal to 1

$$\begin{aligned}x^{10} + y^{10} &= 1^{10} + 1^{10} \\&= 1 + 1 \\&= 2\end{aligned}$$

Therefore, when $x = 1$ and $y = 2$, the sum of $x^{10} + y^{10} \neq 2022$

Case 3: x and y are equal to 2

$$\begin{aligned}x^{10} + y^{10} &= 2^{10} + 1^{10} \\&= 1024 + 1024 \\&= 2048\end{aligned}$$

Therefore, when $x = 1$ and $y = 2$, the sum of $x^{10} + y^{10} \neq 2022$

$x^{10} + y^{10} \neq 2022$ is true in all cases

Therefore, there are no integer solutions to the equation: $x^{10} + y^{10} = 2022$ \square

Question 4: Using proof by contrapositive, prove the following statement: Suppose $m, n \in \mathbb{Z}$. If both $m \cdot n$ and $m + n$ are even, then both m and n are even

Assumption: $m, n \in \mathbb{Z}$

Conclusion: m and n are even

Proof: Assume $m, n \in \mathbb{Z}$

For sake of proof by contrapositive, if m or n is odd, then $m \cdot n$ is odd or $m + n$ is odd

m and n can have the same parity or opposite parity

Case 1: m and n have the same parity, therefore are both odd

By definition of odd, $m = 2x + 1$ such that $x \in \mathbb{Z}$

By definition of odd, $n = 2y + 1$ such that $y \in \mathbb{Z}$

$$\begin{aligned} m \cdot n &= (2x + 1)(2y + 1) \\ &= 4xy + 2x + 2y + 1 \\ &= 2(2xy + x + y) + 1 \\ &= 2z + 1 \text{ for some } z \in \mathbb{Z} \text{ such that } z = 2xy + x + y \end{aligned}$$

So, by definition of odd, $m \cdot n$ is odd

Therefore, when m and n are both odd, $m \cdot n$ is odd.

Case 2: m and n have opposite parity. Without loss of generality, m is odd and n is even

By definition of even, $m = 2x$ such that $x \in \mathbb{Z}$

By definition of odd, $n = 2y + 1$ such that $y \in \mathbb{Z}$

$$\begin{aligned} m + n &= 2x + 2y + 1 \\ &= 2(x + y) + 1 \\ &= 2z + 1 \text{ for some } z \in \mathbb{Z} \text{ such that } z = x + y \end{aligned}$$

So, by definition of odd, $m + n$ is odd

Therefore, when m and n have opposite parity, $m + n$ is odd.

It is true that $m \cdot n$ is odd or $m + n$ is odd in both cases

Therefore, by proof by contrapositive, if both $m \cdot n$ and $m + n$ are even, then both m and n are even \square

Question 5: If $x^3 + 9x^7 + x \geq x^2 + x^6 + x^4$, then $x \geq 0$

Proof: Assume $x \in \mathbb{R}$

For sake of proof by contrapositive, if $x < 0$, then $x^3 + 9x^7 + x < x^2 + x^6 + x^4$

By definition of a negative number, $x = -n$ such that $n \in \mathbb{R}$ and $n \geq 0$

$$\begin{aligned} x^3 + 9x^7 + x &= (-n)^3 + 9(-n)^7 + (-n) \\ &= -n^3 - 9n^7 - n \\ &= -(n^3 + 9n^7 + n) \\ &= -k_1 \text{ such that } k_1 \in \mathbb{R} \text{ where } k_1 = n^3 + 9n^7 + n \\ x^2 + x^6 + x^4 &= (-n)^2 + (-n)^6 + (-n)^4 \\ &= n^2 + n^6 + n^4 \\ &= k_2 \text{ such that } k_2 \in \mathbb{R} \text{ where } k_2 = n^2 + n^6 + n^4 \end{aligned}$$

since, $k_1 < k_2$, we know that $x^3 + 9x^7 + x < x^2 + x^6 + x^4$

Therefore, by proof by contraposition, it is true if $x^3 + 9x^7 + x \geq x^2 + x^6 + x^4$, then $x \geq 0$ \square

Question 6: There are no integers a and b such that $20a + 4b = 1$

- 1) There exists integers a and b such that $20a + 4b = 1$
- 2) There are no integers a and b such that $20a + 4b = 1$
- 3)

Proof: Assume $\exists a, b \in \mathbb{Z} \ 20a + 4b = 1$

$$\text{So, } 20a + 4b = 1 \equiv 2(10a + 2b) = 1$$

$$\equiv 2(10a + 2b) = 1$$

$$\equiv 2k = 1 \text{ such that } k \in \mathbb{Z} \text{ where } k = 10a + 2b$$

By definition of even, 1 is even

A contradiction has been found

Therefore, there are no integers a and b such that $20a + 4b = 1$ \square

Question 7: For real number a and b , if a is rational and ab is irrational, then b is irrational

- 1) For $a, b \in \mathbb{R}$ and a is irrational and ab is rational, then b is rational
- 2) b is irrational
- 3)

Proof: Suppose $a, b \in \mathbb{R}$

For sake of proof by contradiction, assume that a is rational and ab is irrational and b is rational

Then, $\exists w, x \in \mathbb{Z}$ such that $a = w/x$ where $x \neq 0$

Let the fraction be fully reduced. That means there are no common factors between w and x

Then, $\exists y, z \in \mathbb{Z}$ such that $b = y/z$ where $z \neq 0$

Let the fraction be fully reduced. That means there are no common factors between y and z

$$\text{So, } ab = \frac{w}{x} \cdot \frac{y}{z}$$

$$= \frac{wy}{xz}$$

$$= \frac{m}{n} \text{ such that } m, n \in \mathbb{Z} \text{ where } m = wy \text{ and } n = xz$$

By definition of a rational number, ab is rational

We arrive at a contradiction where ab is rational and irrational.

Therefore, by proof by contradiction, if a is rational and ab is irrational, then b is irrational. \square