Homework 7

CMPSC 360

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Question 1:

Base Case: $a_1 = 1$
Recurrence relation: $a_n = a_{n-1} \cdot n$ for $n \geq 2$

Question 2:

a) Base Case: $a_1 = 15$
Recurrence relation: $a_n = a_{n-1} - 7$ for $n \geq 2$

b) $(-1)^{n-1} \cdot \frac{n-1}{n}$ for all integers $n \geq 1$

Question 3:

a) Domain: $(-16, -4) \cup [4, \infty)$
b) Co-domain: $\mathbb{R}$
c) Image: $(-8, -2) \cup [-16, \infty]$
d) Prove it is injective
Proof: Let $x_1, x_2 \in (-16, -4) \cup [4, \infty)$

$$f(x) = \begin{cases} -x^2, & \text{if } x \geq 0 \\ \frac{x}{2}, & \text{if } x < 0 \end{cases}$$

Case 1: $x_1, x_2 \in [4, \infty)$
From the definition of the function, $-x_1^2 = -x_2^2$
$x_1^2 = x_2^2$ [divide by -1]
$x_1 = x_2$ [square root]
So, when $x_1, x_2 \in [4, \infty)$, f is injective

Case 2: $x_1, x_2 \in (-16, -4)$
From the definition of the function, $\frac{x_1}{2} = \frac{x_2}{2}$
$x_1 = x_2$ [multiply by 2]
So, when $x_1, x_2 \in (-16, -4)$, f is injective

Case 3: Without loss of generality, $x_1 \in (-16, -4)$ and $x_2 \in [4, \infty)$
From the definition of the function, $f(x_1) = \frac{x_1}{2}, f(x_2) = -x_2^2$
Taking the inverse of $f(x_1)$, we get $f^{-1}(x_1) = 2x_1$
Taking the inverse of $f(x_2)$, we get $f^{-1}(x_2) = \sqrt{x_2}$
Since both functions are invertible, this function is injective

The function is injective in all cases
Therefore f is injective. $\square$

e) Proof: From the definitions of the codomain and the image above, we notice that they are not the same.
Therefore, f is not surjective.

Question 4: Let $f : C \to B$ and $g : A \to C$. Suppose that $f \circ g$ is bijective.

a) Proof: Assume that $f \circ g$ is bijective.
By definition of bijective, we know that $f \circ g$ is both injective and surjective.
Case 1: g is injective
For the sake of contradiction, assume that g is not injective
By definition, we know that $\exists x, y$ where $x \neq y$ such that $g(x) = g(y)$
We know that $f \circ g$ is injective, so we know that there cannot be a case where $g(x) = g(y)$
We have arrived at a contradiction, where $\exists x, y\ g(x) = g(y)$ and $\forall x, y\ g(x) \neq g(y)$
Therefore, g must be injective.
Case 2: f is surjective
Take some arbitrary $b \in B$ and some arbitrary $a \in A$ such that $f(g(a)) = b$
Set $g(a) = x$, so $f(x) = b$
Thus, f is surjective
Therefore, g is injective and f is surjective when $f \circ g$ is bijective. $\square$
b) Consider $A = \{\alpha\}, B = \{\beta\}, C = \{\gamma_1, \gamma_2\}$
Take the case of $f(g(\alpha))$. $g(\alpha) = \gamma_1$ so $f(g(\alpha)) = f(\gamma_1) = \beta$.
Since the codomain of $f \circ g$ is equal to its image, the function is surjective.
Furthermore, since the domain is α and as we know from above that the codomain is equal to the image, the function is injective.
So, since $f \circ g$ is both surjective and injective, it is bijective.
However, $g(\alpha)$ can result in either γ_1 or γ_2. So when plugged into f, we see that $f(\gamma_1) = f(\gamma_2)$.
Therefore, f is not injective.

Question 5:

a) $d_{1,\texttt{blue}} = p_{1,\texttt{blue}} + p_{0,\min(\texttt{yellow, green})}$
$d_{1,\texttt{yellow}} = p_{1,\texttt{yellow}} + p_{0,\min(\texttt{green, blue})}$
$d_{1,\texttt{green}} = p_{1,\texttt{green}} + p_{0,\min(\texttt{yellow, blue})}$
b) Base Case: $d_{0,c} = p_{0,c}$
Recurrence relation: $d_{i,c} = d_{i-1,c} + p_{i,c}$ for all $i \geq 1$

Question 6:

1)

$$\begin{aligned}
f(x) &= \frac{7-x}{6} \\
y &= \frac{7-x}{6} \text{ (by definition of f)} \\
6y &= 7-x \\
6y-7 &= -x \\
f^{-1}(x) &= 7-6y
\end{aligned}$$

2)

$$\begin{aligned}
g(x) &= \sqrt[3]{x+5}+6 \\
y &= \sqrt[3]{x+5}+6 \text{ (by definition of f)} \\
y-6 &= \sqrt[3]{x+5} \\
(y-6)^3 &= x+5 \\
(y-6)^3 &= x+5 \\
f^{-1}(x) &= (y-6)^3-5
\end{aligned}$$

3)

$$\begin{aligned}
h(x) &= \frac{x+6}{x+2} \\
y &= \frac{x+6}{x+2} \text{ (by definition of f)} \\
y(x+2) &= x+6 \\
yx+2y-6 &= x \\
2y-6 &= x-yx \\
2y-6 &= x(1-y) \\
f^{-1}(x) &= \frac{2y-6}{1-y}
\end{aligned}$$

4)

$$\begin{aligned}
f(x) &= 4-6x^7 \\
y &= 4-6x^7 \text{ (by definition of f)} \\
y-4 &= -6x^7 \\
\frac{4-y}{6} &= x^7 \\
f^{-1}(x) &= \sqrt[7]{\frac{4-y}{6}}
\end{aligned}$$

Question 7:

$f \circ g$ = {(a, 3), (b, 8), (c, 2), (d, 3)}
f^{-1} = {(2, 3), (3, 2), (8, 1)}
$f \circ f^{-1}$ = {(8, 8), (3,3), (2,2)}

Question 8: $f(x) = x + 4$, $g(x) = 5 - x^2$

a) $(f \circ g)(x) = f(5 - x^2) = 5 - x^2 + 4 = 9 - x^2$
b) $(g \circ f)(x) = g(x + 4) = 5 - (x + 4)^2 = 5 - (x^2 + 8x + 16) = -x^2 - 8x - 11$
c) $(f \circ g)(-1) = 9 - (-1)^2 = 9 - 1 = 8$

Question 9:

If $F : X \to Y$ is bijective, then function F has an inverse
Proof: Let $F : X \to Y$
For the sake of proof by contrapositive, assume that F does not have an inverse.
That means for an arbitrary $\exists p, q$ where $p \neq q$ that satisfies the condition $F(p) = F(q)$
So, this means that the function is not injective.
Thus, the function is not bijective.
Therefore, from proof by contrapositive, we know that if $F : X \to Y$ is bijective, then function F has an inverse. $\square$

Question 10: If you randomly choose five numbers from the integers 1 through 8, then two of them must add up to 9.

Proof:
For this set of integers, there are an invertible set of pairs of numbers that add up to 9.
The set is: {(1, 8), (2, 7), (3, 6), (4, 5)}
There are $k = 4$ cases in which the two values can add up to 9.
When choosing $n = 5$ random integers, we see that $k < n$.
By applying the Pigeonhole Theorem, we can conclude that at least two of the randomly chosen integers must sum to 9. $\square$

Question 11:

Let $s_n = \sum_{x=1}^{n} w_x$ where w_x are the total number of wins on the x^{th} day.
Since there is at least one win per day, we know that $s_1 < s_2 < ... < s_{77}$, where 77 days have passed in 11 weeks
Since we know that the maximum total wins is a week is 12, we know that $s_{77} \leq 12 * 11 = 132$
We also know that he wins at least once per day, so $1 \leq s_1$
So we can say that $1 \leq s_1 < s_2 < ... < 132$
Adding 21 to this, we get $22 \leq s_1 + 21 < s_2 + 21 < ... < 153$
Since the sequence $\{s_1, s_2, ..., s_{77}\}$ is distinct and has no repeats, we know that after adding 21 the sequence will be distinct.
Therefore, we can conclude that there must be some $s_i - s_j = 21$ for some i, j. $\square$