

# Homework 8

## CMPSC 360

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**Question 1:** For all  $n \in \mathbb{N}$ :  $3 \mid 2^{2n} - 1$

Proof:

We proceed by induction on the variable  $n$ .

Let  $P(n)$  hold the property of the statement for  $n$ .

**Base Case** ( $n = 1$ ):

We need to prove  $3 \mid 2^{2(1)} - 1$

$$2^2 - 1 = 4 - 1 = 3$$

Since  $3 \mid 3$ , the base case is proved.

**Inductive Hypothesis** ( $n = k$ ):

For any arbitrary natural number  $n = k$ , assume that  $P(k)$  is true.

That means  $3 \mid 2^{2k} - 1$

Using the definition of divides, we get  $2^{2k} - 1 = 3q$  where  $q \in \mathbb{Z}$

**Inductive Step** ( $n = k + 1$ ):

We have to show that  $P(k + 1)$  is true, which means  $3 \mid 2^{2k+2} - 1$

Expanding the expression, we get:

$$\begin{aligned} 2^{2k+2} - 1 &= 4 \cdot 2^{2k} - 1 \\ &= 4 \cdot (2^{2k} - 1) + 3 \\ &= 4 \cdot 3q + 3 && \text{[inductive hypothesis]} \\ &= 3(4q + 1) && \text{[factoring out 3]} \\ &= 3t && \text{such that } t \in \mathbb{N} \text{ where } t = 4q + 1 \end{aligned}$$

Therefore, by definition of divides,  $\forall n \in \mathbb{N}, 3 \mid 2^{2n} - 1$ .

**Question 2:** Show that  $n! > 3^n$  for  $n \geq 7$

Proof:

We proceed by induction on the variable  $n$ .

**Base Case** ( $n = 7$ ):

We need to prove  $7! > 3^7$

The left hand side of the equation is 5040 and the right hand side is 2187. Since  $5040 > 2187$ , the base case is proved.

**Inductive Hypothesis** ( $n = k$ ):

For any arbitrary natural number  $n = k$  where  $k \geq 7$ , we assume that  $k! > 3^k$

**Inductive Step** ( $n = k + 1$ ):

We have to show that  $(k + 1)! > 3^{k+1}$

To show this, let's explore both sides of the equation

Expanding both sides we get:  $(k + 1) \cdot k! > 3 \cdot 3^k$

From the inductive hypothesis, we know that  $k! > 3^k$ .

We also know that  $k + 1 > 3$  because of the restriction on  $k$  that states  $k \geq 7$ .

So, we can conclude that  $(k + 1) \cdot k! > 3 \cdot 3^k$ , which means  $(k + 1)! > 3^{k+1}$  is true.

Therefore,  $\forall n \in \mathbb{N}, k! > 3^k$ .  $\square$

**Question 3:** For any positive integer  $n$ ,  $5 \mid 6^n - 1$

Proof:

We proceed by induction on the variable  $n$ .

Let  $P(n)$  hold the property of the statement for  $n$ .

**Base Case** ( $n = 1$ ):

$P(1)$  asserts that  $5 \mid 6^1 - 1$ .

By the definition of divides,  $(6^1 - 1) = 5a$  for some  $a \in \mathbb{Z}$

We get,  $5 = 5 \cdot 1 = 5$ .

The base case is proved.

**Inductive Hypothesis** ( $n = k$ ):

For any arbitrary integer  $n = k$  where  $k \geq 1$ , assume that  $P(k)$  is true.

That means  $5 \mid 6^k - 1$

Using the definition of divides, we get  $6^k - 1 = 5q$  where  $q \in \mathbb{Z}$

**Inductive Step** ( $n = k + 1$ ):

We have to show that  $P(k + 1)$  is true, which means  $5 \mid 6^{k+1} - 1$ .

Expanding the expression, we get:

$$\begin{aligned} 6^{k+1} - 1 &= 6 \cdot 6^k - 1 \\ &= 6 \cdot (6^k - 1) + 5 \\ &= 6 \cdot 5q + 5 && \text{[inductive step]} \\ &= 5 \cdot (6q + 1) && \text{[factoring 5 out]} \\ &= 5t && \text{for some } t \in \mathbb{Z} \text{ where } t = 6q + 1 \end{aligned}$$

We have  $6^{k+1} - 1 = 5t$ . By definition of divides we get  $5 \mid 6^{k+1} - 1$ .

Therefore, it is true that  $\forall n \in \mathbb{Z}, 5 \mid 6^n - 1$ .  $\square$

**Question 4:** For any  $n \in \mathbb{N}$  and any  $a \in \mathbb{R}$ , prove that  $1 + a + a^2 + a^3 + \dots + a^n = \frac{a^{n+1}-1}{a-1}$

Proof:

We proceed by induction on the variable  $n$ .

Let  $P(n)$  hold the property of the statement for  $n$ .

**Base Case** ( $n = 1$ ):

$P(1)$  asserts that  $1 + a = \frac{a^{1+1}-1}{a-1}$

Taking the right hand side:

$$\begin{aligned} \frac{a^{1+1}-1}{a-1} &= \frac{a^2-1}{a-1} \\ &= \frac{(a+1)(a-1)}{a-1} \quad [\text{factoring}] \\ &= a+1 \quad [\text{divide}] \end{aligned}$$

The base case is proved.

**Inductive Hypothesis** ( $n = k$ ):

For an arbitrary natural number  $n = k$ , we assume that  $\sum_{i=0}^k a^i = \frac{a^{k+1}-1}{a-1}$

**Inductive Step** ( $n = k+1$ ):

We have to show that  $\sum_{i=0}^{k+1} a^i = \frac{a^{k+2}-1}{a-1}$

To show this, let's explore the left hand side of the equations:

$$\begin{aligned} \sum_{i=0}^{k+1} a^i &= \sum_{i=0}^k a^i + a^{k+1} \\ &= \frac{a^{k+1}-1}{a-1} + a^{k+1} \quad [\text{inductive hypothesis}] \\ &= \frac{a^{k+1}-1 + a^{k+1} \cdot (a-1)}{a-1} \\ &= \frac{a^{k+1}-1 + a^{k+2} - a^{k+1}}{a-1} \\ &= \frac{a^{k+2}-1}{a-1} \quad [\text{subtraction}] \end{aligned}$$

Therefore, it is true that  $\forall n \in \mathbb{N}, \forall a \in \mathbb{R} \ 1 + a + a^2 + a^3 + \dots + a^n = \frac{a^{n+1}-1}{a-1}$ .  $\square$

**Question 5:** Prove that  $1^3 + 2^3 + 3^3 + \dots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

Proof:

We proceed by induction on the variable  $n$ .

Let  $P(n)$  hold the property of the statement for  $n$ .

**Base Case** ( $n = 1$ ):

$P(1)$  asserts that  $1^3 = \left(\frac{1(1+1)}{2}\right)^2$

Taking the right hand side:  $\left(\frac{1(1+1)}{2}\right)^2 = \left(\frac{1 \cdot 2}{2}\right)^2 = \left(\frac{2}{2}\right)^2 = 1^2 = 1$

The base case is proved.

**Inductive Hypothesis** ( $n = k$ ):

For an arbitrary natural number  $n = k$ , we assume that  $\sum_{i=1}^k i^3 = \left(\frac{k(k+1)}{2}\right)^2$

**Inductive Step** ( $n = k + 1$ ):

We have to show that  $\sum_{i=1}^{k+1} i^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2$

To show this, let's explore the left hand side of the equation:

$$\begin{aligned}
 \sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^k i^3 + (k+1)^3 && \text{[by making next-to-last term explicit]} \\
 &= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 && \text{[by inductive hypothesis]} \\
 &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
 &= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
 &= \frac{k^2(k+1)^2 + (k+1)^2 \cdot 4(k+1)}{4} \\
 &= \frac{(k+1)^2 \cdot (k^2 + 4(k+1))}{4} && \text{[factoring]} \\
 &= \frac{(k+1)^2 \cdot (k^2 + 4k + 4)}{4} \\
 &= \frac{(k+1)^2 \cdot (k+2)^2}{4} \\
 &= \left(\frac{(k+1)(k+2)}{2}\right)^2
 \end{aligned}$$

Therefore, it is true that  $1^3 + 2^3 + 3^3 + \dots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$ .  $\square$

**Question 6:**

a)  $F_1 + F_2 + F_3 + \dots + F_n = F_{n+2} - 1$

Proof:

We proceed by induction on  $n$ .

Let  $P(n) = \sum_{i=0}^n F_i$ , where  $F_n = F_{n-1} + F_{n-2}$  for all  $n > 1$  and  $F_0 = 0, F_1 = 1$ .

Defined recursively,  $P(n) = P(n-1) + F_n$ .

We must prove that  $P(n) = F_{n+2} - 1$

**Base Case** ( $n = 0, 1, 2$ ):

For  $n = 0$ , we know that  $P(0) = 0$ , and  $F_2 - 1 = (1 + 0) - 1 = 0$ . Therefore,  $P(0)$  is proved

For  $n = 1$ , we know that  $P(1) = 1$ , and  $F_3 - 1 = (1 + 1) - 1 = 1$ . Therefore,  $P(1)$  is proved

For  $n = 2$ , we calculate that  $P(2) = 2$ , and  $F_4 - 1 = (2 + 1) - 1 = 2$ . Therefore,  $P(2)$  is proved

**Inductive Hypothesis** ( $n = k$ ):

Let  $k$  be any arbitrary natural number and  $k > 1$ . We assume that  $P(k) = F_{k+2} - 1$  for all natural numbers  $i$  from 1 through  $k$ .

**Inductive Step** ( $n = k + 1$ ):

We need to show that  $\sum_{i=0}^{k+1} F_i = F_{k+3} - 1$

Taking the left side:

$$\begin{aligned} \sum_{i=0}^{k+1} F_i &= \sum_{i=0}^k F_i + F_{k+1} \\ &= F_{k+2} - 1 + F_{k+1} \quad [\text{inductive hypothesis}] \\ &= F_{k+3} - 1 \quad [\text{from the definition of } F_n] \end{aligned}$$

Therefore, it is true that  $F_1 + F_2 + F_3 + \dots + F_n = F_{n+2} - 1$ .  $\square$

b)  $F_2 + F_4 + F_6 + \dots + F_{2n} = F_{2n+1} - 1$

Proof:

We proceed by induction on  $n$ .

Let  $P(n) = \sum_{i=0}^n F_{2i}$ , where  $F_n = F_{n-1} + F_{n-2}$  for all  $n > 1$  and  $F_0 = 0, F_1 = 1$ .

Defined recursively,  $P(n) = P(n-1) + F_{2n}$ .

We must prove that  $P(n) = F_{2n+1} - 1$

**Base Case** ( $n = 1$ ):

For  $n = 1$ , we calculate that  $P(1) = 1$ , and  $F_3 - 1 = (1 + 1) - 1 = 1$ . Therefore,  $P(1)$  is proven.

**Inductive Hypothesis** ( $n = k$ ):

Let  $k$  be any arbitrary natural number and  $k > 1$ . We assume that  $P(k) = F_{2k+1} - 1$  for all natural numbers  $i$  from 1 through  $k$ .

**Inductive Step** ( $n = k + 1$ ):

We need to show that  $\sum_{i=0}^{k+1} F_{2i} = F_{2k+3} - 1$

Taking the left side:

$$\begin{aligned} \sum_{i=0}^{k+1} F_{2i} &= \sum_{i=0}^k F_{2i} + F_{2k+2} \\ &= F_{2k+1} - 1 + F_{2k+2} \quad [\text{inductive hypothesis}] \\ &= F_{2k+3} - 1 \quad [\text{from the definition of } F_n] \end{aligned}$$

Therefore, it is true that  $F_2 + F_4 + F_6 + \dots + F_{2n} = F_{2n+1} - 1$ .  $\square$

**Question 7:**  $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$  for all  $n \in \mathbb{N}$  where  $a_n = a_{n-1} + 2a_{n-2}$ ,  $a_1 = 1, a_2 = 8, n \geq 3$

Proof:

We proceed by induction on  $n$

Let  $P(n)$  hold the property of the statement for  $n$ .

**Base Case** ( $n = 1, 2, 3$ )

For  $n = 1$ , we know that  $a_1 = 1$ , and  $P(1) = 3 \cdot 2^0 + 2(-1)^1 = 1$ . Hence,  $P(1)$  is proven

For  $n = 2$ , we know that  $a_2 = 8$ , and  $P(2) = 3 \cdot 2^1 + 2(-1)^2 = 8$ . Hence,  $P(2)$  is proven

For  $n = 3$ , we calculate  $a_3 = 8 + 2(1) = 10$ , and  $P(3) = 3 \cdot 2^2 + 2(-1)^3 = 10$ .  $P(3)$  is proven

**Inductive Hypothesis** ( $n = k$ ):

Let  $k$  be an arbitrary natural number greater than 3. We assume that  $P(k) = 3 \cdot 2^{k-1} + 2(-1)^k$  for all natural numbers  $i$  from 1 to  $k$ .

**Inductive Step** ( $n = k + 2$ ):

We need to show that  $a_k + 2a_{k-1} = 3 \cdot 2^k + 2(-1)^{k+1}$

Taking the left side:

$$\begin{aligned} a_k + 2a_{k-1} &= 3 \cdot 2^{k-1} + 2(-1)^k + 2(3 \cdot 2^{k-2} + 2(-1)^{k-1}) \\ &= 3 \cdot 2^{k-1} + 2(-1)^k + 6 \cdot 2^{k-2} + 4(-1)^{k-1} \\ &= 3 \cdot 2^{k-1} + 2(-1)^k + 3 \cdot 2^{k-1} - 4(-1)^k \\ &= 6 \cdot 2^{k-1} - 2(-1)^k \\ &= 3 \cdot 2^k + 2(-1)^{k+1} \end{aligned}$$

Therefore, it is true that  $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$  for all  $n \in \mathbb{N}$  where  $a_n = a_{n-1} + 2a_{n-2}$ ,  $a_1 = 1, a_2 = 8, n \geq 3$

**Question 8:**