Homework 8

CMPSC 360

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Question 1: For all $n \in \mathbb{N}$: $3 \mid 2^{2n-1}$

Proof:
We proceed by induction on the variable n.
Let $P(n)$ hold the property of the statement for n.
Base Case (n = 1):
We need to prove $3|2^{2(1)-1}$

Question 2: Show that $n! > 3^n$ for $n \geq 7$

Proof:
We proceed by induction on the variable n.
Base Case ($n = 7$):
We need to prove $7! > 3^7$
The left hand side of the equation is 5040 and the right hand side is 2187. Since $5040 > 2187$, the base case is proved.
Inductive Hypothesis ($n = k$):
For any arbitrary natural number $n = k$ where $k \geq 7$, we assume that $k! > 3^k$
Inductive Step ($n = k + 1$):
We have to show that $(k + 1)! > 3^{k+1}$
To show this, let's explore both sides of the equation
Expanding both sides we get: $(k + 1) \cdot k! > 3 \cdot 3^k$
From the inductive hypothesis, we know that $k! > 3^k$.
We also know that $k + 1 > 3$ because of the restriction on k that states $k \geq 7$.
So, we can conclude that $(k + 1) \cdot k! > 3 \cdot 3^k$, which means $(k + 1)! > 3^{k+1}$ is true.
Therefore, $\forall n \in \mathbb{N}, k! > 3^k$. $\square$

Question 3: For any positive integer n, $5 \mid 6^n - 1$

Proof:
We proceed by induction on the variable n.
Let $P(n)$ hold the property of the statement for n.
Base Case ($n = 1$):
$P(1)$ asserts that $5 \mid 6^1 - 1$.
By the definition of divides, $(6^1 - 1) = 5a$ for some $a \in \mathbb{Z}$
We get, $5 = 5 \cdot 1 = 5$.
The base case is proved.
Inductive Hypothesis ($n = k$):
For any arbitrary integer $n = k$ where $k \geq 1$, assume that $P(k)$ is true.
That means $5 \mid 6^k - 1$
Using the definition of divides, we get $6^k - 1 = 5q$ where $q \in \mathbb{Z}$
Inductive Step ($n = k + 1$):
We have to show that $P(k+1)$ is true, which means $5 \mid 6^{k+1} - 1$.
Expanding the expression, we get:

$$\begin{aligned} 6^{k+1} - 1 &= 6 \cdot 6^k - 1 \\ &= 6 \cdot (6^k - 1) + 5 \\ &= 6 \cdot 5q + 5 && \text{[inductive step]} \\ &= 5 \cdot (6q + 1) && \text{[factoring 5 out]} \\ &= 5t && \text{for some } t \in \mathbb{Z} \text{ where } t = 6q + 1 \end{aligned}$$

We have $6^{k+1} - 1 = 5t$. By definition of divides we get $5 \mid 6^{k+1} - 1$.
Therefore, it is true that $\forall n \in \mathbb{Z}, 5 \mid 6^n - 1$. $\square$

Question 4: For any $n \in \mathbb{N}$ and any $a \in \mathbb{R}$, prove that $1 + a + a^2 + a^3 + ... + a^n = \frac{a^{n+1}-1}{a-1}$

Proof:
We proceed by induction on the variable n.
Let $P(n)$ hold the property of the statement for n.
Base Case ($n = 1$):
$P(1)$ asserts that $1 + a = \frac{a^{1+1}-1}{a-1}$
Taking the right hand side:

$$\begin{aligned} \frac{a^{1+1}-1}{a-1} &= \frac{a^2-1}{a-1} \\ &= \frac{(a+1)(a-1)}{a-1} \quad \text{[factoring]} \\ &= a+1 \qquad\qquad\quad \text{[divide]} \end{aligned}$$

The base case is proved.
Inductive Hypothesis ($n = k$):
For an arbitrary natural number $n = k$, we assume that $\sum_{i=0}^{k} a^i = \frac{a^{k+1}-1}{a-1}$
Inductive Step ($n = k + 1$):
We have to show that $\sum_{i=0}^{k+1} a^i = \frac{a^{k+2}-1}{a-1}$
To show this, let's explore the left hand side of the equations:

$$\begin{aligned} \sum_{i=0}^{k+1} a^i &= \sum_{i=0}^{k} a^i + a^{k+1} \\ &= \frac{a^{k+1}-1}{a-1} + a^{k+1} \qquad \text{[injective hypothesis]} \\ &= \frac{a^{k+1}-1+a^{k+1}\cdot(a-1)}{a-1} \\ &= \frac{a^{k+1}-1+a^{k+2}-a^{k+1}}{a-1} \\ &= \frac{a^{k+2}-1}{a-1} \qquad \text{[subtraction]} \end{aligned}$$

Therefore, it is true that $\forall n \in \mathbb{N}, \forall a \in \mathbb{R}\ 1 + a + a^2 + a^3 + ... + a^n = \frac{a^{n+1}-1}{a-1}$. $\square$

Question 5: Prove that $1^3 + 2^3 + 3^3 + ... + n^3 = (\frac{n(n+1)}{2})^2$

Proof:
We proceed by induction on the variable n.
Let $P(n)$ hold the property of the statement for n.
Base Case ($n = 1$):
$P(1)$ asserts that $1^3 = (\frac{1(1+1)}{2})^2$
Taking the right hand side: $(\frac{1(1+1)}{2})^2 = (\frac{1 \cdot 2}{2})^2 = (\frac{2}{2})^2 = 1^2 = 1$
The base case is proved.
Inductive Hypothesis ($n = k$):
For an arbitrary natural number $n = k$, we assume that $\sum_{i=1}^{k} i^3 = (\frac{k(k+1)}{2})^2$
Inductive Step ($n = k + 1$):
We have to show that $\sum_{i=1}^{k+1} i^3 = (\frac{(k+1)(k+2)}{2})^2$
To show this, let's explore the left hand side of the equation:

$$\begin{aligned}
\sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 && \text{[by making next-to-last term explicit]} \\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 && \text{[by inductive hypothesis]} \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{k^2(k+1)^2 + (k+1)^2 \cdot 4(k+1)}{4} \\
&= \frac{(k+1)^2 \cdot (k^2 + 4(k+1))}{4} && \text{[factoring]} \\
&= \frac{(k+1)^2 \cdot (k^2 + 4k + 4)}{4} \\
&= \frac{(k+1)^2 \cdot (k+2)^2}{4} \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^2
\end{aligned}$$

Therefore, it is true that $1^3 + 2^3 + 3^3 + ... + n^3 = (\frac{n(n+1)}{2})^2$. $\square$

Question 6:

a) $F_1 + F_2 + F_3 + ... + F_n = F_{n+2} - 1$
Proof:
We proceed by induction on n.
Let $P(n) = \sum_{i=0}^{n} F_i$, where $F_n = F_{n-1} + F_{n-2}$ for all $n > 1$ and $F_0 = 0, F_1 = 1$.
Defined recursively, $P(n) = P(n-1) + F_n$.
We must prove that $P(n) = F_{n+2} - 1$
Base Case ($n = 0, 1, 2$):
For $n = 0$, we know that $P(0) = 0$, and $F_2 - 1 = (1 + 0) - 1 = 0$. Therefore, $P(0)$ is proved
For $n = 1$, we know that $P(1) = 1$, and $F_3 - 1 = (1 + 1) - 1 = 1$. Therefore, $P(1)$ is proved
For $n = 2$, we calculate that $P(2) = 2$, and $F_4 - 1 = (2 + 1) - 1 = 2$. Therefore, $P(1)$ is proved
Inductive Hypothesis ($n = k$):
Let k be any arbitrary natural number and $k > 1$. We assume that $P(k) = F_{k+2} - 1$ for all natural numbers i from 1 through k.
Inductive Step ($n = k + 1$):
We need to show that $\sum_{i=0}^{k+1} F_i = F_{k+3} - 1$
Taking the left side:

$$\begin{aligned}
\sum_{i=0}^{k+1} F_i &= \sum_{i=0}^{k} F_i + F_{k+1} \\
&= F_{k+2} - 1 + F_{k+1} && \text{[inductive hypothesis]} \\
&= F_{k+3} - 1 && \text{[from the definition of } F_n\text{]}
\end{aligned}$$

Therefore, it is true that $F_1 + F_2 + F_3 + ... + F_n = F_{n+2} - 1$. $\square$

b) $F_2 + F_4 + F_6 + ... + F_{2n} = F_{2n+1} - 1$
Proof:
We proceed by induction on n.
Let $P(n) = \sum_{i=0}^{n} F_{2i}$, where $F_n = F_{n-1} + F_{n-2}$ for all $n > 1$ and $F_0 = 0, F_1 = 1$.
Defined recursively, $P(n) = P(n-1) + F_{2n}$.
We must prove that $P(n) = F_{2n+1} - 1$
Base Case ($n = 1$):
For $n = 1$, we calculate that $P(1) = 1$, and $F_3 - 1 = (1 + 1) - 1 = 1$. Therefore, $P(1)$ is proven.
Inductive Hypothesis ($n = k$):
Let k be any arbitrary natural number and $k > 1$. We assume that $P(k) = F_{2k+1} - 1$ for all natural numbers i from 1 through k.
Inductive Step ($n = k + 1$):
We need to show that $\sum_{i=0}^{k+1} F_{2i} = F_{2k+3} - 1$
Taking the left side:

$$\begin{aligned}
\sum_{i=0}^{k+1} F_{2i} &= \sum_{i=0}^{k} F_{2i} + F_{2k+2} \\
&= F_{2k+1} - 1 + F_{2k+2} && \text{[inductive hypothesis]} \\
&= F_{2k+3} - 1 && \text{[from the definition of } F_n\text{]}
\end{aligned}$$

Therefore, it is true that $F_2 + F_4 + F_6 + ... + F_{2n} = F_{2n+1} - 1$. $\square$