

# **Homework 1**

CMPSC 465

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I did not work in a group

I did not consult without anyone my group member

I did not consult any non-class materials

**Question 1:** In each of the following situations, indicate whether  $f = O(g)$ , or  $f = \Omega(g)$ , or both (in which case  $f = \Theta(g)$ ). Give a one sentence justification for each of your answers.

- a)  $f(n) = n^{1.5}$ ,  $g(n) = n^{1.3} \rightarrow \underline{f = \Omega(g)}$  and not  $O(g)$  because the exponent of  $f$  is greater than the exponent in  $g$ .
- b)  $f(n) = 2^{n-1}$ ,  $g(n) = 2^n \rightarrow \underline{f = \Theta(g)}$  because  $f(n) = \frac{1}{2} \cdot 2^n$ , and since we disregard the coefficient, we know that asymptotically  $f$  is  $2^n$ , which is equivalent to  $g$ .
- c)  $f(n) = n^{1.3 \log(n)}$ ,  $g(n) = n^{1.5} \rightarrow \underline{f = \Omega(g)}$  because simply comparing the exponents shows that  $f$ 's exponent will grow faster than  $g$ 's because  $f$ 's exponent grows based on  $n$  while  $g$ 's is constant.
- d)  $f(n) = 3^n$ ,  $g(n) = n \cdot 2^n \rightarrow \underline{f = \Omega(g)}$  because based on the rule of exponential functions for asymptotic growth, the larger the base of the exponent, the faster it will grow. Since  $3 > 2$ ,  $f$  will grow faster than  $g$ .
- e)  $f(n) = (\log n)^{100}$ ,  $g(n) = n^{0.1} \rightarrow \underline{f = O(g)}$  because  $\log(n)$  grows slower than  $n$ , and since the function is dependent on the base of the exponent,  $g$  will grow faster than  $f$ .
- f)  $f(n) = n$ ,  $g(n) = (\log n)^{\log \log n} \rightarrow \underline{f = O(g)}$  because  $g$  is an exponential function and  $f$  is linear, thus  $g$  will grow faster than  $f$ .
- g)  $f(n) = 2^n$ ,  $g(n) = n! \rightarrow \underline{f = O(g)}$  because factorial grows much faster than any exponential function.
- h)  $f(n) = \log(e^n)$ ,  $g(n) = n \log n \rightarrow \underline{f = O(g)}$  because we can simplify  $f$  down to  $n \log(e)$ . Since  $\log(e)$  is a numerical value, we can drop it and see that  $f$  is a linear function, thus grows slower than  $n \log n$ .
- i)  $f(n) = n + \log n$ ,  $g(n) = n + (\log n)^2 \rightarrow \underline{f = \Theta(g)}$  because  $(\log n)^2$  grows slower than  $n$ , thus both  $f$  and  $g$  exhibit linear growth.
- j)  $f(n) = 5n + \sqrt{n}$ ,  $g(n) = \log n + n \rightarrow \underline{f = \Theta(g)}$  because both  $\sqrt{n}$  and  $\log n$  grow slower than linear growth, thus, both  $f$  and  $g$  exhibit linear growth.

**Question 2:**

a) Prove that  $R(i) \geq 3^{i/2} \forall i \geq 2$  where  $R(i) = R(i-1) + R(i-2) + R(i-3)$

Proof:

We proceed by induction on the variable  $i$ .

**Base Case** ( $i = 2, 3, 4$ ):

$R(2) = 3$  and  $3^{2/2} = 3$ , thus the case holds

$R(3) = R(2) + R(1) + R(0) = 3 + 2 + 1 = 6$  and  $3^{3/2} \approx 5.196$ , thus the case holds

$R(4) = R(3) + R(2) + R(1) = 6 + 3 + 2 = 11$  and  $3^{4/2} = 9$ , thus the case holds

**Inductive Hypothesis** ( $i = n$ ):

For any arbitrary positive integer  $i = n$  where  $n \geq 2$ , assume that  $R(n) \geq 3^{n/2}$

This means that  $R(n-1) + R(n-2) + R(n-3) \geq 3^{n/2}$

**Inductive Step** ( $i = n+1$ ):

We have to show that  $R(n+1) \geq 3^{(n+1)/2}$

Expanding this, we get  $R(n) + R(n-1) + R(n-2) \geq 3^{(n+1)/2}$

From the inductive hypothesis, we know that  $R(n), R(n-1), R(n-2)$  holds.

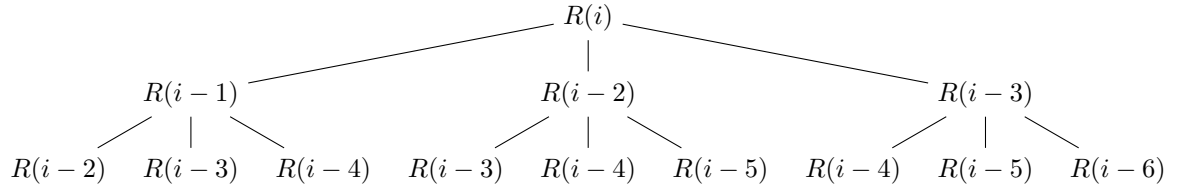
$$\begin{aligned} \text{Plugging this in, we can say that } R(n+1) &\geq 3^{0.5n} + 3^{0.5n-0.5} + 3^{0.5n-1} \\ &= 3^{0.5n}(1 + 3^{-0.5} + 3^{-1}) \\ &= 3^{0.5n}\left(\frac{3+4\sqrt{3}}{3\sqrt{3}}\right) \\ &= 3^{0.5n}(1.911) \end{aligned}$$

So,  $R(n+1) \geq 3^{0.5n}(1.911) \geq 3^{0.5n}(\sqrt{3})$

Since  $1.911 > \sqrt{3}$ , the statement  $R(n+1) \geq 3^{(n+1)/2}$  holds.

Therefore,  $R(i) \geq 3^{i/2} \forall i \geq 2$ .  $\square$

b) Recursion Tree



**Question 3:** *Prove that  $O_1(g(n)) = O_2(g(n)) \forall g$*

$$O_1(g(n)) = \{f(n) : \exists c_1, n_0 > 0 \text{ s.t. } f(n) \leq c_1 g(n), \forall n \geq n_0\}$$

$$O_2(g(n)) = \{f(n) : \exists c_2 > 0 \text{ s.t. } f(n) \leq c_2 g(n), \forall n > 0\}$$