

# **Homework 8**

CMPSC 465

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**Problem 1:**

I worked with Sahil Kuwadia and Ethan Yeung  
I did not consult without anyone my group member  
I did not consult any non-class materials

**Problem 2:** Let  $G = (V, E)$  be a directed graph, with source  $s \in V$ , sink  $t \in V$  and nonnegative edge capacities  $\{c_e\}$ . Give a polynomial time algorithm to decide whether  $G$  has a unique minimum  $s$ - $t$  cut.

We can start by finding the maximum flow through  $G$  by running the Ford-Fulkerson algorithm. We know that the maximum flow through  $G$  is the minimum cut of  $G$ , set this value to  $C$ . To find if this cut is unique, we iterate through all edges taking edge  $e$ , and set its capacity to  $c_e + 1$ . We can then find the maximum flow through this new graph and set this new capacity to  $C_e$ . Once we have computed the capacity, reset  $c_e$  to its original value. If  $\forall C_e = C$ , then the cut is not unique. If  $\exists! C_e \neq C$ , then the cut is unique. Since Ford-Fulkerson runs in polynomial time, and we are running it at most  $|E| + 1$  times, this algorithm runs in polynomial time.

**Problem 3:** Let  $T$  be an MST of graph  $G$ . Given a connected subgraph  $H$  of  $G$ , show that  $T \cap H$  is contained in some MST of  $H$ .

The cut property states that  $A$  as a subset of edges from some MST of  $G$ . Let  $(S, V - S)$  be a cut that respects  $A$  and let  $e$  be the lightest edge across the cut. Then we know that  $A \cup \{e\}$  is part of some MST. Using this property, take edge  $e$  which is the lightest edge in  $H$ , thus we know that  $e \in T \cap H$ . We can then take  $e$  as the cut edge for  $H$ , so the cut set is  $(H \cap S, H - S)$  (similar to the property above). Because  $e \in T \cap H$ , the same cut exists in some MST of  $H$ , proving that  $T \cap H$  is contained in some MST of  $H$ .

**Problem 4:** Let  $T$  be a minimum spanning tree of a graph  $G = (V, E)$  and  $V'$  be a subset of  $V$ . Let  $T'$  be a subgraph of  $T$  induced by  $V'$  (i.e., an edge  $(u, v) \in T$  is present in  $T'$  iff both  $(u, v) \in V'$ ) and  $G'$  be a subgraph of  $G$  induced by  $V'$ . Show that if  $T'$  is connected, then  $T'$  is a minimum spanning tree of  $G'$ .

We know that  $T$  is an MST of  $G$ . We also know that  $T'$  is a subgraph of  $T$ . FSOC, assume that  $T'$  is not an MST for  $G'$ . Let  $S$  be a valid MST of  $G'$ . We know that  $S$  is a subgraph of  $T'$ , so  $S$  is a subgraph of  $T$ . We also know that  $S$  is a subgraph of  $G'$ . Thus,  $S$  is a subgraph of  $G$ . We know that  $S$  is a valid MST of  $G'$ . We know that  $T$  is a valid MST of  $G$ , so  $T$  is a valid MST of  $G$ . Thus,  $T = S$ , which is a contradiction. Thus,  $T'$  is an MST of  $G'$ .