Homework 1

CMPSC 465

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Problem 1:

I did not work in a group
I did not consult without anyone my group member
I did not consult any non-class materials

Problem 2: *In each of the following situations, indicate whether $f = O(g)$, or $f = \Omega(g)$, or both (in which case $f = \Theta(g)$). Give a one sentence justification for each of your answers.*

a) $f(n) = n^{1.5}$, $g(n) = n^{1.3} \rightarrow \underline{f = \Omega(g)}$ and not $O(g)$ because the exponent of f is greater than the exponent in g.

b) $f(n) = 2^{n-1}$, $g(n) = 2^n \rightarrow \underline{f = \Theta(g)}$ because $f(n) = \frac{1}{2} \cdot 2^n$, and since we disregard the coefficient, we know that asymptotically f is 2^n, which is equivalent to g.

c) $f(n) = n^{1.3\log(n)}$, $g(n) = n^{1.5} \rightarrow \underline{f = \Omega(g)}$ because simply comparing the exponents shows that f's exponent will grow faster than g's because f's exponent grows based on n while g's is constant.

d) $f(n) = 3^n$, $g(n) = n \cdot 2^n \rightarrow \underline{f = \Omega(g)}$ because based on the rule of exponential functions for asymptotic growth, the larger the base of the exponent, the faster it will grow. Since $3 > 2$, f will grow faster than g.

e) $f(n) = (\log n)^{100}$, $g(n) = n^{0.1} \rightarrow \underline{f = O(g)}$ because $\log(n)$ grows slower than n, and since the function is dependent on the base of the exponent, g will grow faster than f.

f) $f(n) = n$, $g(n) = (\log n)^{\log \log n} \rightarrow \underline{f = O(g)}$ because g is an exponential function and f is linear, thus g will grow faster than f.

g) $f(n) = 2^n$, $g(n) = n! \rightarrow \underline{f = O(g)}$ because factorial grows much faster than any exponential function.

h) $f(n) = \log(e^n)$, $g(n) = n\log n \rightarrow \underline{f = O(g)}$ because we can simplify f down to $n\log(e)$. Since $\log(e)$ is a numerical value, we can drop it and see that f is a linear function, thus grows slower than $n\log n$.

i) $f(n) = n + \log n$, $g(n) = n + (\log n)^2 \rightarrow \underline{f = \Theta(g)}$ because $(\log n)^2$ grows slower than n, thus both f and g exhibit linear growth.

j) $f(n) = 5n + \sqrt{n}$, $g(n) = \log n + n \rightarrow \underline{f = \Theta(g)}$ because both $\sqrt{n}$ and $\log n$ grow slower than linear growth, thus, both f and g exhibit linear growth.

Problem 3:

a) Prove that $R(i) \geq 3^{i/2}$ $\forall i \geq 2$ where $R(i) = R(i-1) + R(i-2) + R(i-3)$

Proof:
We proceed by induction on the variable i.
Base Case ($i = 2, 3, 4$):
$R(2) = 3$ and $3^{2/2} = 3$, thus the case holds
$R(3) = R(2) + R(1) + R(0) = 3 + 2 + 1 = 6$ and $3^{3/2} \approx 5.196$, thus the case holds
$R(4) = R(3) + R(2) + R(1) = 6 + 3 + 2 = 11$ and $3^{4/2} = 9$, thus the case holds

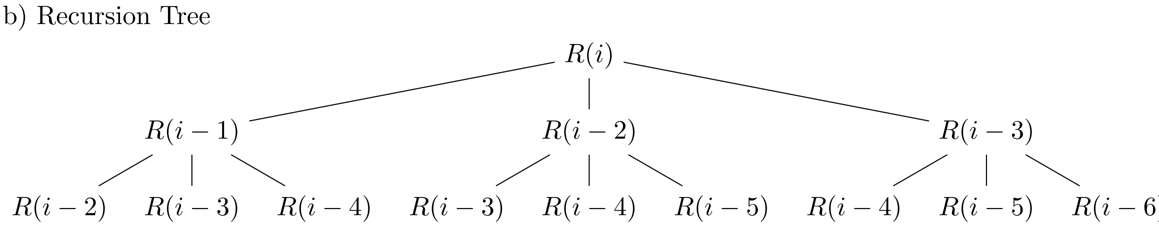
Inductive Hypothesis ($i = n$):
For any arbitrary positive integer $i = n$ where $n \geq 2$, assume that $R(n) \geq 3^{n/2}$
This means that $R(n-1) + R(n-2) + R(n-3) \geq 3^{n/2}$

Inductive Step ($i = n+1$):
We have to show that $R(n+1) \geq 3^{(n+1)/2}$
Expanding this, we get $R(n) + R(n-1) + R(n-2) \geq 3^{(n+1)/2}$
From the inductive hypothesis, we know that $R(n), R(n-1), R(n-2)$ holds.
Plugging this in, we can say that

$$\begin{aligned} R(n+1) &\geq 3^{0.5n} + 3^{0.5n-0.5} + 3^{0.5n-1} \\ &= 3^{0.5n}(1 + 3^{-0.5} + 3^{-1}) \\ &= 3^{0.5n}(\tfrac{3+4\sqrt{3}}{3\sqrt{3}}) \\ &= 3^{0.5n}(1.911) \end{aligned}$$

So, $R(n+1) \geq 3^{0.5n}(1.911) \geq 3^{0.5n}(\sqrt{3})$
Since $1.911 > \sqrt{3}$, the statement $R(n+1) \geq 3^{(n+1)/2}$ holds.
Therefore, $R(i) \geq 3^{i/2}$ $\forall i \geq 2$. $\square$

b) Recursion Tree



Problem 4: *Prove that* $O_1(g(n)) = O_2(g(n))\ \forall g$

$O_1(g(n)) = \{f(n) : \exists c_1, n_0 > 0 \text{ s.t. } f(n) \leq c_1 g(n), \forall n \geq n_0\}$
$O_2(g(n)) = \{f(n) : \exists c_2 > 0 \text{ s.t. } f(n) \leq c_2 g(n), \forall n > 0\}$

Proof:
In O_1, we have that $\exists n_0 > 0$. We also know that $\forall n \geq n_0$, so we can say that $\forall n \geq n_0 > 0$. Thus, we can conclude $n > 0$. Substituting this to the definition of O_1, we can say that:
$O_1(g(n)) = \{f(n) : \exists c_1, n_0 > 0 \text{ s.t. } f(n) \leq c_1 g(n), \forall n > 0\}$. Furthermore, since both set builders state that $\exists c_1, c_2 > 0$, we can say that $c_1 = c_2 = c_{all}$. Thus, the set builders will be:
$O_1(g(n)) = \{f(n) : \exists c_{all} > 0 \text{ s.t. } f(n) \leq c_{all} g(n), \forall n > 0\}$
$O_2(g(n)) = \{f(n) : \exists c_{all} > 0 \text{ s.t. } f(n) \leq c_{all} g(n), \forall n > 0\}$
Thus, we can conclude that $O_1 = O_2$ $\square$