

Comparing two methods of approximating the value of π in a computational context

To what extent can a method of approximation of the value π be computationally more efficient than another?

Word count:

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1 Introduction

The value of π has been researched for many years, although under different names, and the amount of different approaches to reach the value is large. The value has been found through many processes, be it geometrically, algebraically or through other means.

This paper seeks to examine the extent at which two historical methods of approximation of the value π , namely the approaches suggested by the aforementioned mathematicians Madhava and Viète, differ in terms of computational efficiency and speed, and explain these differences. This paper does not however, suggest a method to use for computation but rather seeks to compare the efficiency of a geometrically derived formula and a algebraically derived one.

2 Theoretical approach

2.1 Focus on two methods

For the sake of this paper, two different methods with similar convergence rates but different approaches have been chosen for comparison, the process for the original discovery of these methods will be explained. The two methods in question are one based on the infinite series definition of an inverse trigonometric function and one where the value is derived using geometry, by mathematician Viète. These two mathematical methods were chosen as they both represent a first occurrence in mathematics: Madhava was the first to find the series notation of the arctan function and Viète was one of the first mathematicians to use an infinite series in calculation.

2.1.1 Madhava-Gregory-Leibniz method

Madhava, having identified the integral and series for the arctan function, in the Kerala school during medieval India [\[1\]](#)

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2.1.2 Viète's method

The French mathematician, François Viète, approached the value of π from a geometric standpoint, finding the following formula:

$$\pi = 2 \frac{2}{\sqrt{2}} \frac{2}{\sqrt{2 + \sqrt{2}}} \frac{2}{\sqrt{2 + \sqrt{2 + \sqrt{2}}}} \dots$$

He was able to calculate π to a place of 9 decimal points, in the year 1593 [\[3\]](#), using his method. His method is reminiscent of Archimedes' method, but differs in that it consists of finding the area of a polygon of n sides in a circle of constant radius, rather than the circumference. As the value of n is increased, the area of the n -gon tends toward the area

of a circle. The geometric origin of this formula can be found using simple right-angle trigonometry, by first finding the lengths OH and subsequently BD in 2.1.2.

With the radius of the circle with center $R = OB$,

$$OH = R \cos \alpha$$

and

$$BD = 2BH = 2R \sin \alpha$$

Since the equation for the area of a polygon is defined as $A = \frac{p \cdot a}{2}$, where p is the perimeter of the polygon and a is the apothem, in this case $BD \cdot n$ and OH respectively, let A_n equal the area of the polygon with n sides such that:

$$A_n = \frac{OH \cdot BD \cdot n}{2}$$

$$A_n = \frac{R \cos \alpha \cdot 2R \sin \alpha \cdot n}{2} = nR^2 \sin \alpha \cos \alpha$$

And if n is multiplied by 2, the angle $\angle \alpha$ is divided by 2, and the new area becomes:

$$A_{2n} = 2nR^2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}$$

So it can be written that, by definition, the ratio of the area of an n -gon

$$\frac{A_n}{A_{2n}} = \frac{nR^2 \sin \alpha \cos \alpha}{2nR^2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}} = \frac{\sin 2\alpha}{2 \sin \alpha}$$

Which through the trigonometric identity $\sin 2\alpha = 2 \sin \alpha \cos \alpha$ can be simplified to:

$$\frac{A_n}{A_{2n}} = \frac{2 \sin \alpha \cos \alpha}{2 \sin \alpha} = \cos \alpha$$

It can be then written that, through a new variable P ,

$$P = \frac{A_n}{A_{2n}} \frac{A_{2n}}{A_{4n}} \frac{A_{4n}}{A_{8n}} \dots \frac{A_{(k-2)n}}{A_{kn}} \frac{A_{kn}}{A}$$

where A is the area of the circle of radius R in 2.1.2.

So $P = \frac{A_n}{A}$, since the values A_{kn} cancel, and followingly, it is true that $P = \frac{A_n}{A} \Leftrightarrow A = \frac{A_n}{P}$.

The value of R being constant in this case, and since the area of a circle is defined by

$A = \pi R^2$, therefore:

$$\pi = \frac{A_n}{P}$$

By definition, we can say that as the value k approaches infinity, the area of the kn -gon approaches that of a circle, and therefore, the value of π .

$$\frac{A_n}{\cos \alpha \cos \frac{\alpha}{2} \cos \frac{\alpha}{4} \dots} \rightarrow_{k \rightarrow \infty} \pi$$

There is however, a neater expression for this same formula. To simplify to it, firstly we let $U_0 = \cos \alpha$. Then, we define $U_1 = \cos \frac{\alpha}{2}$, following the formula stated before.

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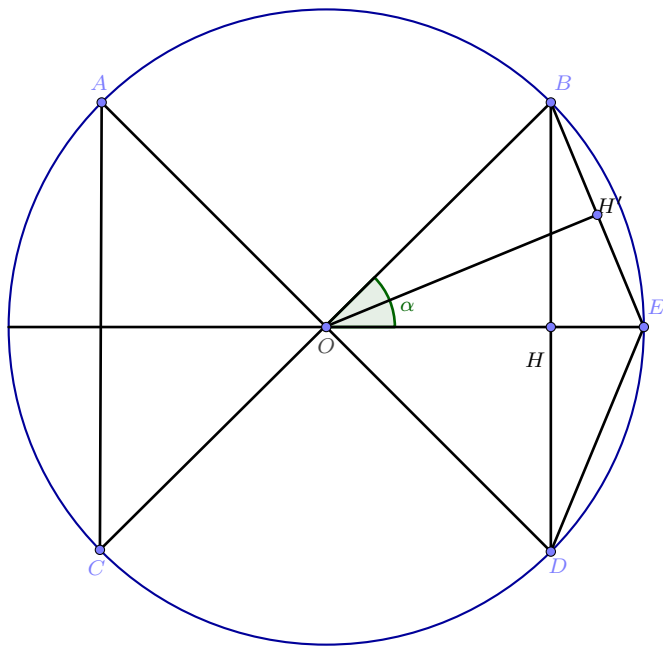


Figure 1: Circle with 1 segment from a n -gon with point H and 2 segments from an $2n$ -gon, one of which on point H' , inscribed in a circle of radius OB , adapted from [2]

3 Computational approach

3.1 Implementing

3.2 The variables

4 Analysis of the results

Works Cited

- [1] Jonathan M. Borwein, Scott T. Chapman, and Scott T. Chapman. “I Prefer Pi: A Brief History and Anthology of Articles in the American Mathematical Monthly”. In: *The American Mathematical Monthly* 122.3 (2015), pp. 198–199. ISSN: 00029890, 19300972. URL: <http://www.jstor.org/stable/10.4169/amer.math.monthly.122.03.195>.
- [2] Boris Gourévitch. *François Viète*. Apr. 2013. URL: <http://www.pi314.net/fr/viete.php>.
- [3] Rick Kreminski. “ π to Thousands of Digits from Vieta’s Formula”. In: *Mathematics Magazine* 81.3 (2008), p. 201. ISSN: 0025570X, 19300980. URL: <http://www.jstor.org/stable/27643107>.