Comparing two methods of approximating the value of π in a computational context

To what extent can a method of approximation of the value π be computationally more efficient than another?

Word count:

Contents

1 Introduction

The value of π has been researched for many years, although under different names, and the amount of different approaches to reach the value is large. The value has been found through many processes, be it geometrically, algebraically or through other means.

This paper seeks to examine the extent at which two historical methods of approximation of the value π, namely the approaches suggested by the aforementioned mathematicians Madhava and Viète, differ in terms of computational efficiency and speed, and explain these differences. This paper does not however, suggest a method to use for computation but rather seeks to compare the efficiency of a geometrically derived formula and a algebraically derived one.

2 Background Information

2.1 Focus on two methods

For the sake of this paper, one by French mathematician François Viète, and another supposedly discovered by Madhava of Sangamagrama, and rediscovered by Swiss mathematician Leibniz. Two methods with different approaches have been chosen for comparison, the process for the original discovery of these methods will be explained. The two methods in question are one based on the infinite series definition of an inverse trigonometric function and one where the value is derived using geometry, by mathematician Viète. These two mathematical methods were chosen as they are both represent a first occurence in mathematics: Madhava was the first to find the series notation of the arctangent function and Viète was one of the first mathematicians to use infinite series in his calculation. Furthermore, a research based on computational speed of two different kinds approaches to the constant π has not been done to date.

2.1.1 Madhava-Gregory-Leibniz method

Madhava, having indentified the integral and series for the arctan function, in the Kerala school during medieval India [1]

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2.1.2 Viète's method

François Viète, having approached the value of π from a geometric standpoint, found the following formula:

$$\pi = 2\frac{2}{\sqrt{2}}\frac{2}{\sqrt{2+\sqrt{2}}}\frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}}\cdots$$

He was able to calculate π to a place of 9 decimal points, in the year 1593 [3], using his method. His method is reminiscent of Archimedes' method, where the length of a side is calculated [4], but differs in that it consists of finding the area of a polygon of n sides in a circle of constant radius, rather than the circumference. As the value of n is increased, the area of the n-gon tends toward the area of a circle. The geometric origin of this formula can be found using simple right-angle trigonometry, by first finding the lengths OH and subsequently BD in 2.1.2.

With the radius of the circle with center $R = OB$,

$OH = R\cos\alpha$

and

$BD = 2BH = 2R\sin\alpha$

Since the equation for the area of a polygon is defined as $A = \frac{p \cdot a}{2}$, where p is the perimeter of the polygon and a is the apothem, in this case $BD \cdot n$ and OH respectively, let A_n equal the area of the polygon with n sides such that:

$A_n = \frac{OH \cdot BD \cdot n}{2}$

$A_n = \frac{R\cos\alpha \cdot 2R\sin\alpha \cdot n}{2} = nR^2 \sin\alpha\cos\alpha$

And if n is multiplied by 2, the angle $\angle\alpha$ is divided by 2, and the new area becomes:

$A_{2n} = 2nR^2 \sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$

So it can be written that, by definition, the ratio of the area of an n-gon to one of a $2n$-gon is

$\frac{A_n}{A_{2n}} = \frac{nR^2\sin\alpha\cos\alpha}{2nR^2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \frac{\sin 2\alpha}{2\sin\alpha}$

Which through the trigonometric identity $\sin 2\theta = 2\sin\theta\cos\theta$ can be simplified to:

$\frac{A_n}{A_{2n}} = \frac{2\sin\alpha\cos\alpha}{2\sin\alpha} = \cos\alpha$

It can be then written that, through a new variable P,

$P = \frac{A_n}{A_{2n}}\frac{A_{2n}}{A_{4n}}\frac{A_{4n}}{A_{8n}} \dots \frac{A_{(k-2)n}}{A_{kn}}\frac{A_{kn}}{A}$

where A is the area of the circle of radius R in 2.1.2.

So $P = \frac{A_n}{A}$, since the values A_{kn} cancel, and followingly, it is true that $P = \frac{A_n}{A} \Leftrightarrow A = \frac{A_n}{P}$.

The value of $R = 1$ in this case, and since the area of a circle is defined by $A = \pi R^2$, therefore:

$\pi = \frac{A_n}{P}$

By definition, we can say that as the value k approaches infinity, the area of the kn-gon aproaches that of a circle, and therefore, the value of π.

$\frac{A_n}{\cos\alpha \cos\frac{\alpha}{2} \cos\frac{\alpha}{4} \ldots} \rightarrow_{k\rightarrow\infty} \pi$

Where the value of A_n is the area of the first polygon, with $n = 4$ sides, and as such $A_n = 4 \sin 45 \cos 45 = 2$. We can define:

$U_0 = \cos a = \cos 45 = \frac{1}{\sqrt{2}}$

$U_1 = cos\frac{\alpha}{2}$

Which we can, through the trigonometric identity $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$, simplify as $U_1 = \sqrt{\frac{1}{2} + \frac{1}{2}U_0}$

So it can be said that $U_n = \sqrt{\frac{1}{2} + \frac{1}{2}U_{n-1}}$, which leads to a fully defined expression for the value of pi:

$\pi = \frac{2}{\prod_{k=0}^{\infty} U_k}, U_0 = \frac{1}{\sqrt{2}}, U_n = \sqrt{\frac{1}{2} + \frac{1}{2}U_{n-1}}$ [1]

These expressions, when under a single expression result in the aforementioned formula with nested roots.

[1] where $\prod$ signifies a product. Similar expression to $\sum$

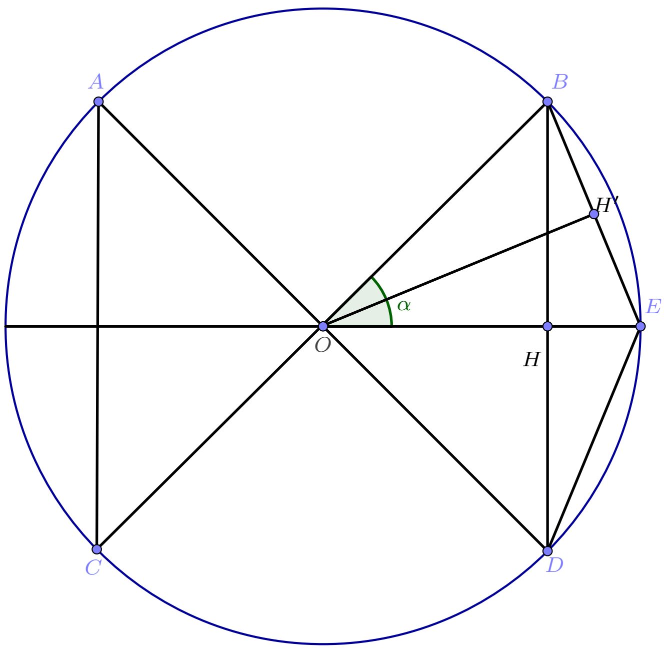

Figure 1: Circle with 1 segment from a n-gon with point H and 2 segments from an $2n$-gon, one of which on point H', inscribed in a circle of radius OB, adapted from Boris Gourévitch [2]

3 Computational approach

3.1 The variables

The dependent variable of this experiment is the time taken t by the program to approximate a given number n of correct decimal value of the constant π.

The value n will be altered in order to avoid possible similar convergence rates at a small amount of decimal places, and multiple trials will be run to decrease margin of error. Other variables of the experiment will be controlled. For example, the experiment will be run on a same isolated system, a virtual machine, with a minimal amount of processes running to avoid any possible variance in results.

3.2 Implementing in Python

The main source for data in this experiment is primary. A Python application was programmed (see appendix) in order to run the two methods aforementioned, and manage the collection of data.

The program made assigns the time before the execution of the method to a variable `t1` with the `time.time()` function. At each iteration of the method, the number of valid decimal places of the resultant approximation are counted and once a specified threshold is reached, a new `t2` time variable is assigned and the time taken, defined by the difference between `t2` and `t1` is stored. This process is repeated for all specified decimal accuracies and for both methods.

4 Analysis of the results

4.1 Presentation of the data

4.2 Analysis of the data

Works Cited

[1] Jonathan M. Borwein, Scott T. Chapman, and Scott T. Chapman. "I Prefer Pi: A Brief History and Anthology of Articles in the American Mathematical Monthly". In: *The American Mathematical Monthly* 122.3 (2015), pp. 198–199. ISSN: 00029890, 19300972. URL: http://www.jstor.org/stable/10.4169/amer.math.monthly.122.03.195 (visited on 01/23/2020).

[2] Boris Gourévitch. *François Viete*. Apr. 2013. URL: http://www.pi314.net/fr/viete.php (visited on 01/24/2020).

[3] Rick Kreminski. "π to Thousands of Digits from Vieta's Formula". In: *Mathematics Magazine* 81.3 (2008), p. 201. ISSN: 0025570X, 19300980. URL: http://www.jstor.org/stable/27643107 (visited on 01/23/2020).

[4] RICHARD LOTSPEICH. "Archimedes' Pi—an Introduction to Iteration". In: *The Mathematics Teacher* 81.3 (1988), p. 208. ISSN: 00255769. URL: http://www.jstor.org/stable/27965770.

A Appendix

A.1 Python program

```
def test():
    sqdqdsqsd
```