Comparing the speed of two methods of approximating the value of π — a computational approach

To what extent can a method of approximation of the value π be computationally more efficient than another?

Word count: 2122

Contents

1 Introduction

The value of π has been researched for many years, although under different names, and the amount of different approaches to reach the value is large. The value has been found through many processes, be it analytically, the most popular, geometrically, or through more obscure or convoluted methods, such as the possibility to approximate the value using physics akin to those from a simple game of billiards. [1]

This paper seeks to examine the extent at which two historical methods of approximation of the value π, namely the approaches suggested by the aforementioned mathematicians Madhava and Viète, differ in terms of computational speed and speed, and explain these differences. This paper does not however, suggest a method to use for computation but rather seeks to only compare the speed of a geometrically derived formula and a analytically derived one.

This research could prove useful in the field of computer science, as there is always a demand for faster and more efficient programs in an ever-changing society. Furthermore, a research based on computational speed of two different kinds approaches to the constant π has not been done to date.

2 Background

2.1 Focus on two methods

In this paper, we focus on two methods for approximating the value of π. The first by French mathematician François Viète, and another discovered by Madhava of Sangamagrama. Two methods with different approaches have been chosen for comparison and the process for original discovery of these methods will be explained. The two in question are one based on the infinite series definition of an inverse trigonometric function discovered by Madhava and one where the value is derived using geometry, by mathematician Viète. These two mathematical methods were chosen as they are both represent a first occurrence in mathematics: Madhava was the first to find the infinite series expansion of the inverse tangent function and Viète was the first mathematician to use an infinite product in calculation.

2.1.1 An analytical method: Madhava's method

Madhava is the first mathematician known to have found the series notation for the inverse tangent function, in the Kerala school of mathematics of Medieval India [2]. The research made at the Kerala school was documented by many mathematicians of their time, namely the astronomer-mathematician Jyesthadeva in his treatise *Yukti-Bhasa* written in the Malayalam language [3]. Madhava's infinite series expansion of the inverse tangent function can also be found in this treatise. R. C. Gupta, in his translation, describes this expansion as one where the arc θ is equal to the sum of a first term, "the product of the given sine and radius of the desired arc divided by the cosine of the arc"[1], followed by terms that "are obtained by a process of iteration": in which the original term is multiplied by the square of the sine and divided by the square of the cosine. Thereafter, each term is divided by an odd number in order $(1, 3, 5, 7, \dots)$. It is then said that the arc can be found by "adding and subtracting respectively the terms of odd rank and those of even rank", definition of an alternating series. [4]

This text explains what can be written in modern mathematical terms as such:

$$r\theta = \frac{r(r\sin\theta)}{1(r\cos\theta)} - \frac{r(r\sin\theta)^3}{3(r\cos\theta)^3} + \frac{r(r\sin\theta)^5}{5(r\cos\theta)^5} - \frac{r(r\sin\theta)^7}{7(r\cos\theta)^7} + \dots$$

And for a circle of radius $r = 1$, we can cancel all r terms:

$$\theta = \frac{\sin\theta}{\cos\theta} - \frac{\sin^3\theta}{3\cos^3\theta} + \frac{\sin^5\theta}{5\cos^5\theta} - \frac{\sin^7\theta}{7\cos^7\theta} + \dots$$

And because $\frac{\sin\theta}{\cos\theta} = \tan\theta$, the aforementioned expression can be simplified to:

$$\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \frac{\tan^7\theta}{7} + \dots$$

So then, if we let $\tan\theta = \alpha$, we can find the infinite series expansion of the arctangent function.

$$\arctan\alpha = \alpha - \frac{\alpha^3}{3} + \frac{\alpha^5}{5} - \frac{\alpha^7}{7} + \dots = \theta$$

Yukti-Bhasa also describes two methods for the calculation of π. Firstly, Madhava's infinite series expansion for π is described, which he obtained through the previous expansion for the arctangent function. And since $\arctan 1 = \frac{\pi}{4}$, it can be said that:

$$\frac{\pi}{4} = \arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots + \frac{(-1)^n}{2n+1}$$

Which in form of an infinite sum can be expressed as:

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

Which has come to be known as the Leibniz formula for π, after the German mathematician who discovered the same formula two decades later. [5] This series converges to π, as seen in (Figure 1) 2.1.2

However, the method with a faster convergence can be found in a further commentary describing the findings of Madhava that states a method for the calculation of the circumference c of a circle of diameter d exists. The passage in question from the commentary *Tantrasamgraha-vyakhya* of anonymous authorship states that, by following the same argument as stated before, one can find the circumference of a circle through a similar infinite sum. The first term of this sum would be "the square root of the square of

[1]Note: the old Indian meaning for the sine of θ is $r\sin\theta$, where r is the radius. The same applies for the cosine.

the diameter multiplied by twelve", followed by the first term "divided by three in each successive case", and when these "are divided in order by the odd numbers, beginning with 1", and after the even terms are subtracted from sum of the odd", one is left with the circumference of the circle (translation from C. K. Raju). [6]

This can be expressed in mathematical terms as such, where we let c be the circumference of a circle of diameter d:

$c = \sqrt{12d^2} - \frac{\sqrt{12d^2}}{3 \cdot 3} + \frac{\sqrt{12d^2}}{3^2 \cdot 5} - \frac{\sqrt{12d^2}}{3^3 \cdot 7} + \ldots$

And since $c = \pi d$, for a circle of diameter 1, $c = \pi$, all d terms cancel and the expression can be factorized as:

$c = \pi = \sqrt{12}(1 - \frac{1}{3 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \ldots)$

Or as an infinite sum:

$\pi = \sqrt{12} \sum_{n=0}^{\infty} \frac{(-3)^{-n}}{2n+1}$

2.1.2 A geometric method: Viète's method

François Viète approached the value of π from a geometric standpoint, and found an infinite product. He was able to calculate π to a place of 9 decimal points, in the year 1593 [7], using his method. His method is reminiscent of Archimedes' method, where the length of a side is calculated [8], but differs in that it consists of finding the area of a polygon of n sides in a circle of constant radius, rather than the circumference. As the value of n is increased, the area of the n-gon tends toward the area of a circle. The geometric origin of this formula can be found using simple right-angle trigonometry, by first finding the lengths OH and subsequently BD in (Figure 2) 2.1.2. With the radius of the circle with center $R = OB$,

$OH = R \cos \alpha$

and

$BD = 2BH = 2R \sin \alpha$

Since the equation for the area of a polygon is defined as $A = \frac{p \cdot a}{2}$, where p is the perimeter of the polygon and a is the apothem, in this case $BD \cdot n$ and OH respectively, let A_n equal the area of the polygon with n sides such that:

$A_n = \frac{OH \cdot BD \cdot n}{2}$

$A_n = \frac{R\cos\alpha \cdot 2R\sin\alpha \cdot n}{2} = nR^2 \sin\alpha\cos\alpha$

And if n is multiplied by 2, the angle $\angle\alpha$ is divided by 2, and the new area becomes:

$A_{2n} = 2nR^2 \sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$

So it can be written that, by definition, the ratio of the area of an n-gon to one of a $2n$-gon is

$\frac{A_n}{A_{2n}} = \frac{nR^2 \sin\alpha\cos\alpha}{2nR^2 \sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \frac{\sin 2\alpha}{2\sin\alpha}$

Which through the trigonometric identity $\sin 2\theta = 2\sin\theta\cos\theta$ can be simplified to:

$\frac{A_n}{A_{2n}} = \frac{2\sin\alpha\cos\alpha}{2\sin\alpha} = \cos\alpha$

It can be then written that, through a new variable P,

$P = \frac{A_n}{A_{2n}}\frac{A_{2n}}{A_{4n}}\frac{A_{4n}}{A_{8n}}\cdots\frac{A_{(k-2)n}}{A_{kn}}\frac{A_{kn}}{A}$

where A is the area of the circle of radius R in 2.1.2.

So $P = \frac{A_n}{A}$, since the values A_{kn} cancel, and so it is true that $P = \frac{A_n}{A} \Leftrightarrow A = \frac{A_n}{P}$. The value of $R = 1$ in this case, and since the area of a circle is defined by $A = \pi R^2$, therefore:

$\pi = \frac{A_n}{P}$

By definition, we can say that as the value k approaches infinity, the area of the kn-gon approaches that of a circle, and therefore, the value of π.

$\frac{A_n}{\cos\alpha\cos\frac{\alpha}{2}\cos\frac{\alpha}{4}\ldots} \rightarrow_{k\rightarrow\infty} \pi$

Where the value of A_n is the area of the first polygon, with $n = 4$ sides, and as such $A_n = 4\sin 45\cos 45 = 2$. We can define:

$U_0 = \cos a = \cos 45 = \frac{1}{\sqrt{2}}$

$U_1 = cos\frac{\alpha}{2}$

Which we can, through the trigonometric identity $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$, simplify as $U_1 = \sqrt{\frac{1}{2} + \frac{1}{2}U_0}$

So it can be said that $U_n = \sqrt{\frac{1}{2} + \frac{1}{2}U_{n-1}}$, which leads to a fully defined expression for the value of pi, using an infinite product:

$$\pi = \frac{2}{\prod\limits_{k=0}^{\infty} U_k}, U_0 = \frac{1}{\sqrt{2}}, U_n = \sqrt{\frac{1}{2} + \frac{1}{2}U_{n-1}}$$ [2]

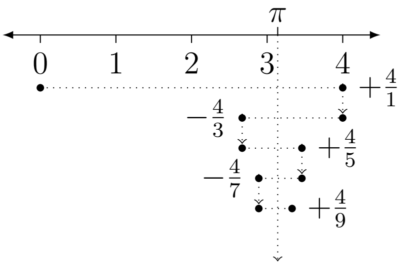


Figure 1: A number line demonstrating the convergence of Madhava's first method for the calculation of π. Adapted from (V. M. Jamkar) [9]

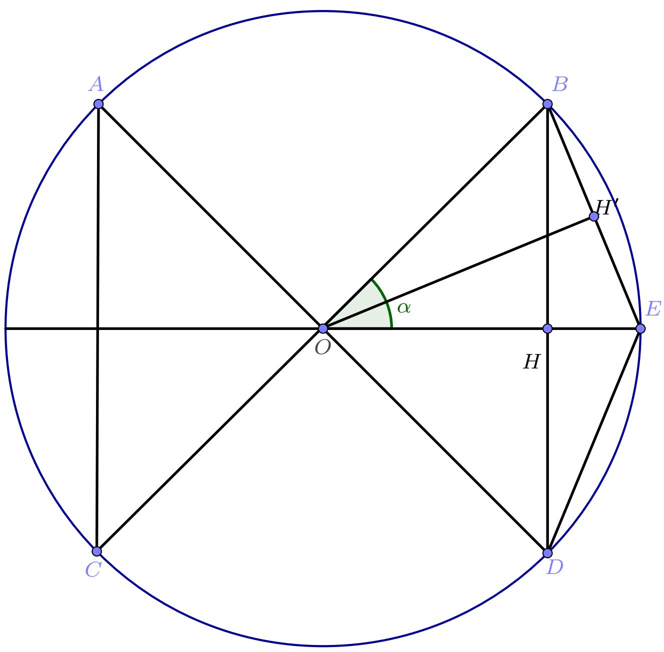


Figure 2: Circle with 1 segment from a n-gon with point H and 2 segments from an $2n$-gon, one of which on point H', inscribed in a circle of radius OB, adapted from (Boris Gourévitch) [10]

[2] where $\prod$ signifies a product. Similar expression to $\sum$

3 Computational approach

3.1 The variables

The dependent variable of this experiment is the time taken t by the program to approximate a given number n of correct decimal value of the constant π.

The value n will be altered in order to avoid possible similar convergence rates at a small amount of decimal places, and multiple trials will be run to decrease margin of error. Other variables of the experiment will be controlled. For example, the experiment will be run on a same isolated system, a virtual machine, with a minimal amount of processes running simultaneously to avoid any possible variance in results.

3.2 Implementing in Python

A Python application was programmed (see appendix) in order to run the two methods aforementioned, and manage the collection of data.

The program assigns the time before the execution of the method to a variable `t1` with the `time.time()` Python function. At each iteration of the method, the number of valid decimal places of the resultant approximation are counted and once a specified threshold is reached, a new `t2` time variable is assigned and the time taken, defined by the difference between `t2` and `t1`, is stored. This process is repeated for all specified decimal accuracies and for both methods. The times recorded were stored in a `.csv` file for further analysis.

The `mpmath` library was used for the floating point operations required for comparison between approximated values and the constant π that wouldn't have been possible using standard Python libraries. [11]

4 Analysis of the results

4.1 Presentation of the data

4.1.1 Tabular presentation

The arithmetic mean of a decimal place was caluclated for 100 trials and the data points in the table shown below were found (rounded to 3 significant figures). The values for the decimal place 0 have been dropped as it would not be logical to include them.

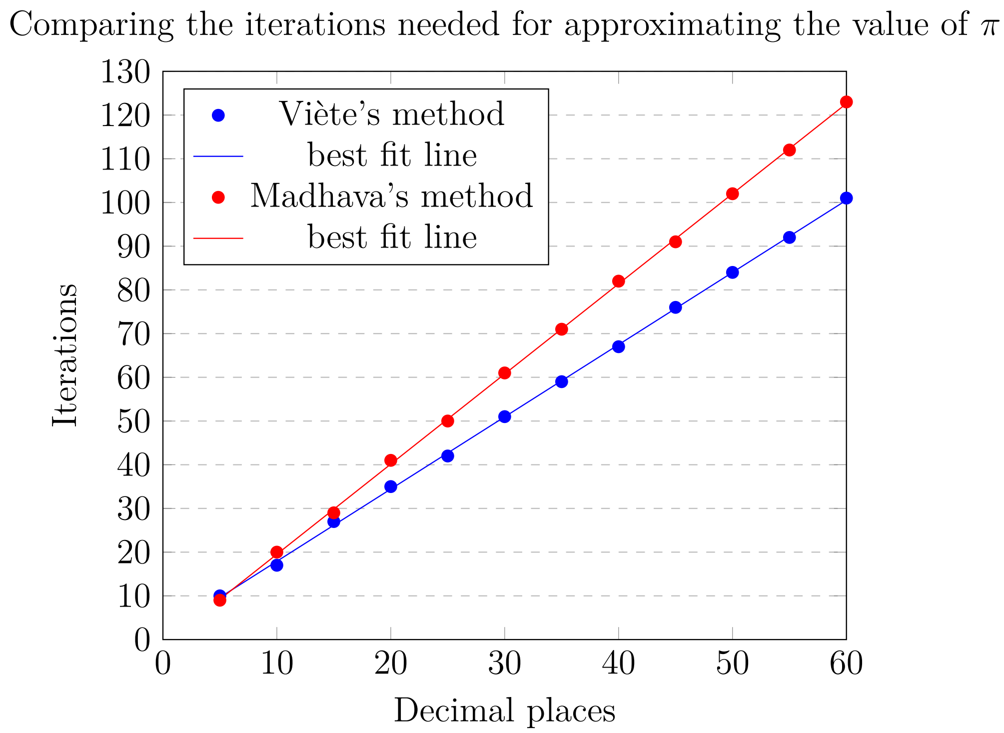
Decimal place	Average time for approximation using Viete's method, in milliseconds (ms)	Average time for approximation using Madhava's method method, in milliseconds (ms)
5	0.760	0.592
10	1.29	1.31
15	2.05	1.9
20	2.64	2.68
25	3.09	3.21
30	3.82	3.9
35	4.44	4.55
40	5.08	5.26
45	5.71	5.84
50	6.37	6.58
55	7.02	7.27
60	7.72	8.00

Decimal place	Iterations needed, Viète's method (no unit)	Iterations needed, Madhava's method (no unit)
5	10	9
10	17	20
15	27	29
20	35	41
25	42	50
30	51	61
35	59	71
40	67	82
45	76	91
50	84	102
55	92	112
60	101	123

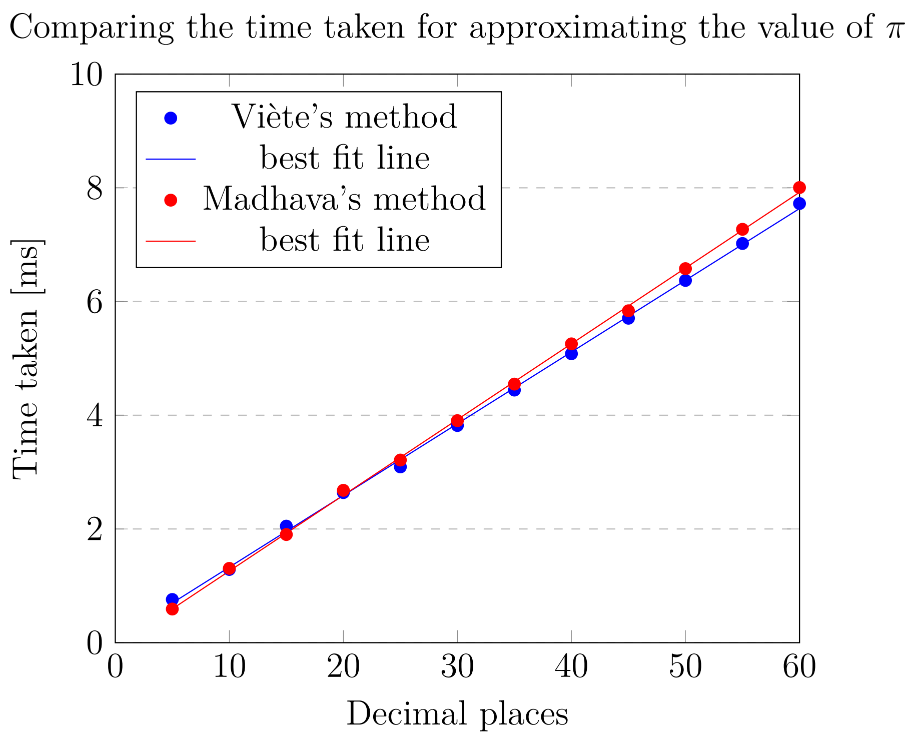
The iterations needed for both methods were also measured, using the Python program (see appendix), slightly modified to increment a variable `i` and subsequently return it, writing it in a similar manner to a `.csv` file.

4.1.2 Graphical presentation

Shown below is a graph demonstrating the correlation between the decimal places approximated and the amount of iterations needed for this specific decimal milestone. The blue markers and line of best-fit respresent the results received from Viète's method while the red represent those from Madhava's method.



To show the trend in the time data collected, see below the decimal places compared to the time taken per method. The color coding here is relevant to the one aforementioned.



4.2 Observations and analysis

The results gathered from this experiment show linear relationship between the amount of decimal places approximated and the time required for this approximation, as well as between the decimal places and the amount of iterations needed. The two lines of best fit for the two methods show similarities but also differences.

Where decimal places and iterations are compared, there is a clear difference between the two methods: Madhava's method indeed takes more iterations to converge to a specific decimal of the value π. However, in the second graph comparing time, it can be noted that Madhava's method is marginally faster until the decimal 20, when the two methods diverge. It could be assumed that this occurrence is due to differences between the types of the two methods. Madhava's method is what is called an alternating series, due to its nature of alternating between values in order to converge to a specific value, in this case π, while Viète's method is an infinite product. The difference between these two methods is underlined in (Figure 3) 3. It can be deduced that at lower decimal values Madhava's method is faster as Viète's method's approximation accuracy firstly increases in what seems similar to a curve from an exponential function $f : x \to -2x^2$. It can also be said that while Madhava's method alternates between values such that $m_n > \pi, m_{n+1} < \pi$, Viète's method approaches pi in a way that the approximation is always $v < \pi$.

More info to be added

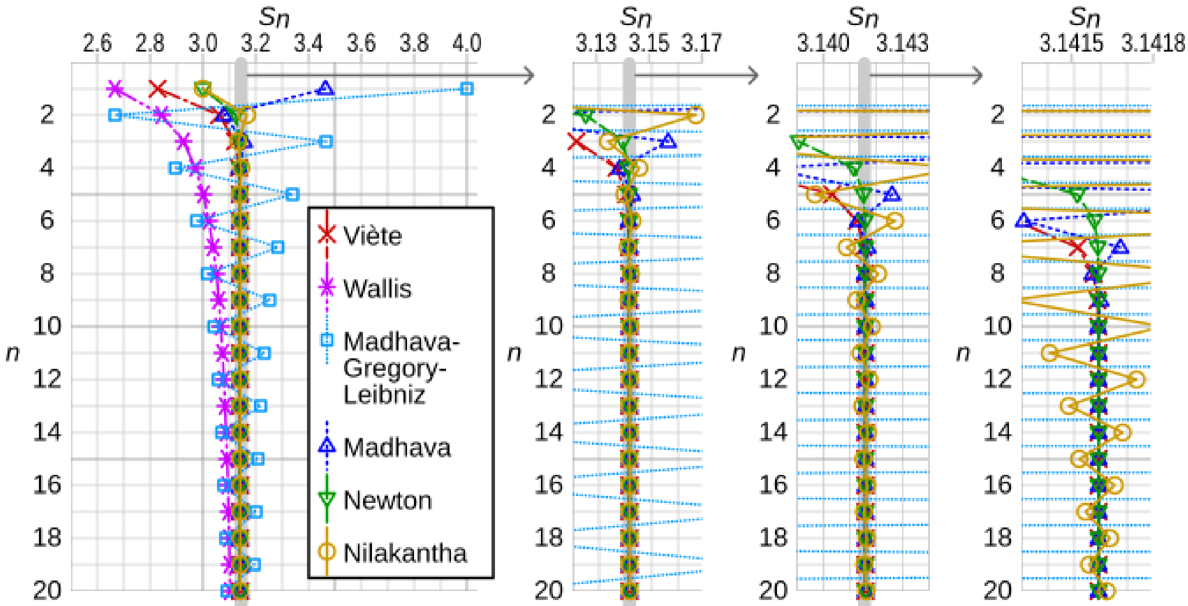


Figure 3: Comparison of historical methods of approximating the value of π. This image demonstrates their convergence rates. The line labelled Madhava in dark blue is the method used in this paper. From (Wikimedia Commons) [12]

5 Further research opportunities

5.1 Investigating newer, more efficient algorithms

As the computation of π moved from paper to machines, as all did from accounting to mathematics, newer and more efficient methods were developed, notably by Ramanujan and the Chudnovsky brothers.

More info to be added

6 Conclusion

To conclude, even though Viète's method can seem more convoluted in practice, due to its geometric origins, it can be seen through the experiment conducted in this paper that this is clearly not the case in a computational context. His method is simplified to a point where its geometric origins are no longer findable from its computational implementation. And despite the assumption that could be made based on the types of operations needed for each method, Viète's requiring arguably more computationally challenging operations. These operations, such as square-root function or the product function used on high-precision floating point numbers, can be said to be more demanding than what could be said is simple summation, in the case of Madhava. This essay could be considered in defense of what is known as geometric algebra — which can clearly produce results more capable than ones originating from an analytical method.

More info to be added

6.1 Evaluation of the experimental method

It must be noted that the experimental procedure in this paper is far from perfect. Firstly, there are many variables that would be difficult to control.

More info to be added

Works Cited

[1] G. Galperin. "PLAYING POOL WITH π (THE NUMBER π FROM A BILLIARD POINT OF VIEW)". In: *Regular and Chaotic Dynamics* 8.4 (2003), pp. 375–393. DOI: 10.1070/rd2003v008n04abeh000252. URL: http://rcd.ics.org.ru/upload/iblock/007/RCD080402.pdf.

[2] Jonathan M. Borwein, Scott T. Chapman, and Scott T. Chapman. "I Prefer Pi: A Brief History and Anthology of Articles in the American Mathematical Monthly". In: *The American Mathematical Monthly* 122.3 (2015), pp. 198–199. ISSN: 00029890, 19300972. URL: http://www.jstor.org/stable/10.4169/amer.math.monthly.122.03.195 (visited on 01/23/2020).

[3] J J O'Connor and E F Robertson. *Madhava of Sangamagramma*. Nov. 2000. URL: https://web.archive.org/web/20060514012903/http://www-gap.dcs.st-and.ac.uk/~history/Biographies/Madhava.html.

[4] R C Gupta. *The Madhava-Gregory series*. 1973, pp. 67–70.

[5] C. Henry Edwards. *The historical development of the calculus*. Springer-Verlag, 1994, p. 247.

[6] C. K. Raju. *Cultural foundations of mathematics: the nature of mathematical proof and the transmission of the calculus from India to Europe in the 16th c. CE*. Pearson Longman, Project of history of Indian Science, philosophy and culture, 2007.

[7] Rick Kreminski. "π to Thousands of Digits from Vieta's Formula". In: *Mathematics Magazine* 81.3 (2008), p. 201. ISSN: 0025570X, 19300980. URL: http://www.jstor.org/stable/27643107 (visited on 01/23/2020).

[8] RICHARD LOTSPEICH. "Archimedes' Pi—an Introduction to Iteration". In: *The Mathematics Teacher* 81.3 (1988), p. 208. ISSN: 00255769. URL: http://www.jstor.org/stable/27965770.

[9] V. M. Jamkar. “History of pi and Indian contribution”. In: 3.12 (2018), p. 264. URL: https://rrjournals.com/wp-content/uploads/2018/12/263-267_RRIJM180312058.pdf.

[10] Boris Gourévitch. *François Viete*. Apr. 2013. URL: http://www.pi314.net/fr/viete.php (visited on 01/24/2020).

[11] Fredrik Johansson et al. *mpmath: a Python library for arbitrary-precision floating-point arithmetic (version 0.18)*. Dec. 2013. URL: http://mpmath.org/.

[12] Wikimedia Commons. *Comparison pi infinite series*. File: Comparison_pi_infinite_series.svg. 2014. URL: https://upload.wikimedia.org/wikipedia/commons/f/f5/Comparison_pi_infinite_series.svg.

A Appendix

A.1 Python program

This application was run on Python version 3.9.1, on a virtual machine running the Debian operating system under QEMU/KVM on a Intel i5-2500 processor. Used the `mpmath` library for better floating-point precision [11].

```
import time
from mpmath import *
import csv



mp.dps = 100
PI_CONST = mp.pi



# Function that determines if the approximated value of
# pi is correct to a specified decimal
```

```
def decimal_is_correct(pi, decvalue):
    #pi_diff = str(abs(pi))
    zeros = 0
    pistr = str(pi)[2:]
    piconst = str(PI_CONST)[2:]
    for i in range(len(pistr)):
        if pistr[i] != piconst[i]: break
        else:
            zeros += 1
    if zeros == decvalue:
        return True
    else:
        return False


# Function that approximates pi using Madhava's method
def madhava(decimals):
    piapprox = 0
    i = 0

    t1 = time.time()

    while not decimal_is_correct(piapprox * mp.sqrt(12), decimals):
        # This is a direct mirror of the summation from the formula
        piapprox += mp.power(-3, -i) / (2*i+1)
        i += 1
    piapprox *= mp.sqrt(12)
    t2 = time.time()
```

```python
    # Return the time spent (t2-t1) getting d value of decimal places
    return t2-t1


# Function that approximates pi using Viete's method
def viete(decimals):
    piapprox = 1
    numer = 0

    t1 = time.time()
    while not decimal_is_correct((1.0 / piapprox) * 2.0, decimals):
        numer = mp.sqrt(2.0 + numer)
        piapprox *= (numer / 2.0)
    piapprox = (1.0 / piapprox) * 2.0

    t2 = time.time()

    # Return the time spent (t2-t1) getting d value of decimal places
    return t2-t1



if __name__ == "__main__":
    decimals = [i for i in range(0, 65, 5)]
    trials = 100

    # Exporting data for plotting and analysis
    f = open(r'out.csv', 'w')
    fieldnames = ['decimals', 'viete', 'madhava']
    writer = csv.DictWriter(f, fieldnames=fieldnames)
```

```python
writer.writerow({'decimals':'decimals', 'viete':'viete',
'madhava':'madhava'})

for dec in decimals:
    for t in range(trials):
        writer.writerow({'decimals':dec, 'viete':viete(dec),
        'madhava':madhava(dec)})
```