

# Proof by Induction

mccp-dobson-2111

## Introduction

A proof by induction of  $P(n)$ , a mathematical statement involving a value  $n$ , involves these main steps:

- Prove directly that  $P$  is correct for the initial value of  $n$  (for most examples you will see this is zero or one). This is called the **base case**.
- Assume for some value  $k$  that  $P(k)$  is correct. This is called the **induction hypothesis**. We will now prove directly that  $P(k) \Rightarrow P(k+1)$ . That means prove directly that  $P(k+1)$  is correct by using the fact that  $P(k)$  is correct. This is called the **induction step**.

The combination of these steps shows that  $P(n)$  is true for all values of  $n$ .

## Example

Prove by induction that  $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$  for every positive integer  $n$ .

## Solution

Let  $P(n)$  be the statement that  $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$ . As we have an equation, at each stage of this proof we will show that the left hand side equals the right hand side.

**Base Case:** When  $n = 1$  the left hand side of the equation is 1 and the right hand side is  $\frac{1(1+1)}{2} = 1$ . So  $P(1)$  is correct.

**Induction hypothesis:** Assume that  $P(k)$  is correct for some positive integer  $k$ . That means that the left hand side of the equation equals the right hand side, so  $1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2}$ .

**Induction step:** We will now show that  $P(k+1)$  is correct by taking the left hand side of the equation and re-arranging it until it equals the right hand side. At each stage keep in mind what we are aiming for. So in this case the right hand side of the equation should be  $\frac{(k+1)((k+1)+1)}{2}$  since we have replaced the  $n$  with  $(k+1)$ .

So starting with the left hand side we have

$$\begin{aligned} 1 + 2 + 3 + \dots + k + (k+1) &= (1 + 2 + 3 + \dots + k) + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \quad \text{by the induction hypothesis} \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \quad \text{by making each part a fraction over 2} \end{aligned}$$



$$\begin{aligned}
&= \frac{k(k+1) + 2(k+1)}{2} && \text{by making it a single fraction over 2} \\
&= \frac{(k+1)(k+2)}{2} && \text{since the numerator has a common factor of } (k+1) \\
&= \frac{(k+1)((k+1)+1)}{2} && \text{and this is the right hand side.}
\end{aligned}$$

So  $P(k+1)$  is correct. Hence by mathematical induction  $P(n)$  is correct for all positive integers  $n$ .

### Example

Prove by induction that  $n^3 + 2n$  is divisible by 3 for every non-negative integer  $n$ .

### Solution

Let  $P(n)$  be the mathematical statement

$$n^3 + 2n \text{ is divisible by 3.}$$

*Base Case:* When  $n = 0$  we have  $0^3 + 0 = 0 = 3 \times 0$ . So  $P(0)$  is correct.

*Induction hypothesis:* Assume that  $P(k)$  is correct for some positive integer  $k$ . That means  $k^3 + 2k$  is divisible by 3 and hence that  $k^3 + 2k = 3m$  for some integer  $m$ .

*Induction step:* We will now show that  $P(k+1)$  is correct. So we will take the original formula and replace the  $n$  with  $k+1$  to get  $(k+1)^3 + 2(k+1)$  and we will show that this is divisible by 3. At some stage in the proof we will need to use the fact that  $k^3 + 2k = 3m$ , so when we re-arrange the formula we will try to get  $k^3 + 2k$  as part of it.

$$\begin{aligned}
(k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 2k + 2 && \text{by expanding the brackets} \\
&= k^3 + 2k + 3(k^2 + k + 1) && \text{by re-arranging the formula} \\
&= 3m + 3(k^2 + k + 1) && \text{by the induction hypothesis} \\
&= 3(m + k^2 + k + 1) && \text{since both parts of the formula} \\
&&& \text{have a common factor of 3.}
\end{aligned}$$

As  $m + k^2 + k + 1$  is an integer we have that  $(k+1)^3 + 2(k+1)$  is divisible by 3, so  $P(k+1)$  is correct. Hence by mathematical induction  $P(n)$  is correct for all non-negative integers  $n$ .

### Exercises

Prove by induction that

- $1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$  for all positive integers.
- $9^n - 1$  is divisible by 8 for all non-negative integers.
- $(1 \times 2) + (2 \times 3) + (3 \times 4) + \dots + (n \times (n+1)) = \frac{n(n+1)(n+2)}{3}$  for all positive integers.
- $2^{n+1} > n^2$  for all positive integers.

