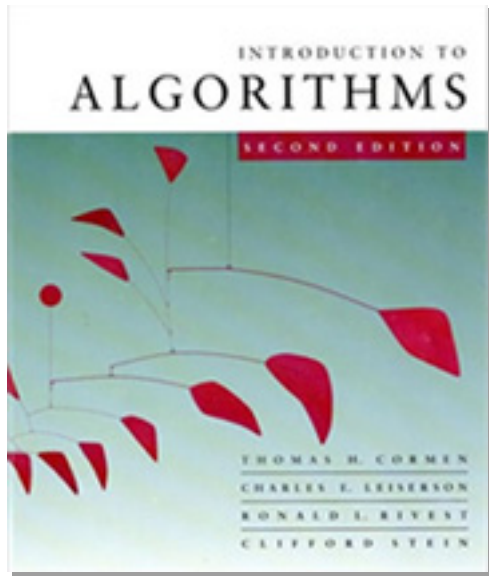# *Introduction to Algorithms*
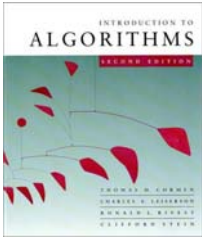
6.046J/18.401J

LECTURE 4

Quicksort
- Divide and conquer
- Partitioning
- Worst-case analysis
- Intuition
- Randomized quicksort
- Analysis

## Prof. Charles E. Leiserson

Quicksort

- Proposed by C.A.R. Hoare in 1962.

- Divide-and-conquer algorithm.

- Sorts "in place" (like insertion sort, but not like merge sort).

- Very practical (with tuning).

Divide and conquer
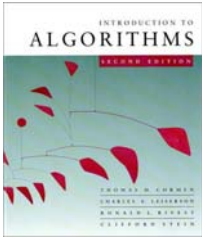
Quicksort an n-element array:

1. ***Divide:*** Partition the array into two subarrays around a ***pivot*** x such that elements in lower subarray $\leq x \leq$ elements in upper subarray.

$\leq x$	x	$\geq x$
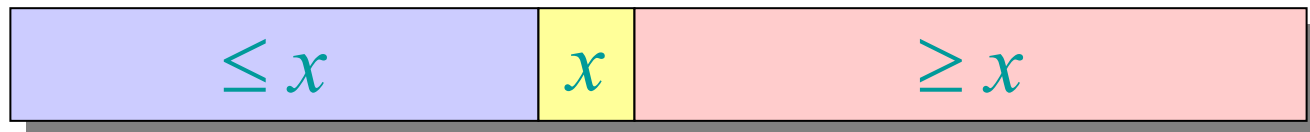
2. ***Conquer:*** Recursively sort the two subarrays.

3. ***Combine:*** Trivial.

Key: *Linear-time partitioning subroutine.*

Partitioning subroutine
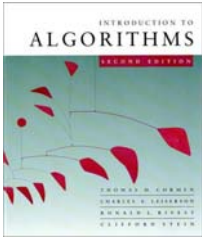
$\textsc{Partition}(A, p, q)$ ▷ $A[p \, . \, . \, q]$
     $x \leftarrow A[p]$ ▷ pivot $= A[p]$
     $i \leftarrow p$
     for $j \leftarrow p + 1$ **to** q
        do if $A[j] \leq x$
            then $i \leftarrow i + 1$
                 exchange $A[i] \leftrightarrow A[j]$
     exchange $A[p] \leftrightarrow A[i]$
     return i

> Running time $= O(n)$ for n elements.
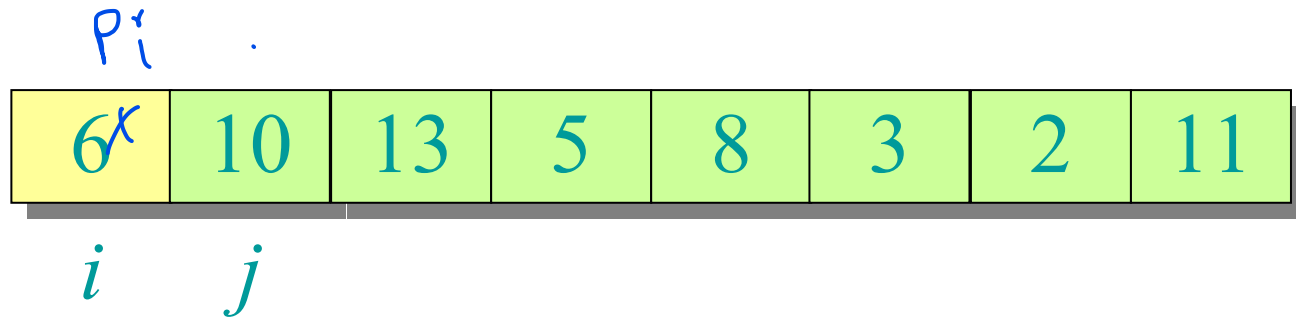
Invariant:
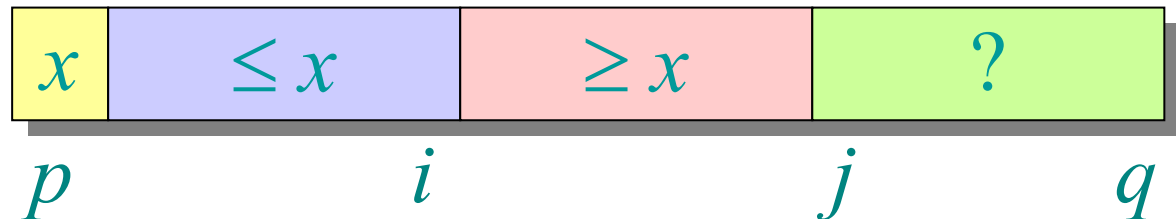
x	$\leq x$	$\geq x$	?
p	i	j	q

Example of partitioning

Pi

6 x	10	13	5	8	3	2	11

i   *j*

Example of partitioning

6	10	13	5	8	3	2	11

i $\bullet \longrightarrow$ j

Example of partitioning

6	10	13	5	8	3	2	11

i $\longrightarrow$ j

Example of partitioning

6	10	13	5	8	3	2	11

6	5	13	10	8	3	2	11

i $\quad\quad\quad\quad\quad$ j

Example of partitioning

6	10	13	5	8	3	2	11

6	5	13	10	8	3	2	11

i $\bullet\longrightarrow j$

Example of partitioning

| 6 | 10 | 13 | 5 | 8 | 3 | 2 | 11 |

| 6 | 5 | 13 | 10 | 8 | 3 | 2 | 11 |

i j →

Example of partitioning

6	10	13	5	8	3	2	11

6	5	13	10	8	3	2	11

6	5	3	10	8	13	2	11

$\longrightarrow i$ j

Example of partitioning

6	10	13	5	8	3	2	11

6	5	13	10	8	3	2	11

6	5	3	10	8	13	2	11

i $\longrightarrow$ j

Example of partitioning

6	10	13	5	8	3	2	11

6	5	13	10	8	3	2	11

6	5	3	10	8	13	2	11

6	5	3	2	8	13	10	11

i j

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Example of partitioning

| 6 | 10 | 13 | 5 | 8 | 3 | 2 | 11 |

| 6 | 5 | 13 | 10 | 8 | 3 | 2 | 11 |

| 6 | 5 | 3 | 10 | 8 | 13 | 2 | 11 |

| 6 | 5 | 3 | 2 | 8 | 13 | 10 | 11 |

i $\longrightarrow$ j

Example of partitioning

6	10	13	5	8	3	2	11

6	5	13	10	8	3	2	11

6	5	3	10	8	13	2	11

6	5	3	2	8	13	10	11

i j

Example of partitioning

| 6 | 10 | 13 | 5 | 8 | 3 | 2 | 11 |

| 6 | 5 | 13 | 10 | 8 | 3 | 2 | 11 |

| 6 | 5 | 3 | 10 | 8 | 13 | 2 | 11 |

| 6 | 5 | 3 | 2 | 8 | 13 | 10 | 11 |

| 2 | 5 | 3 | 6 | 8 | 13 | 10 | 11 |

i

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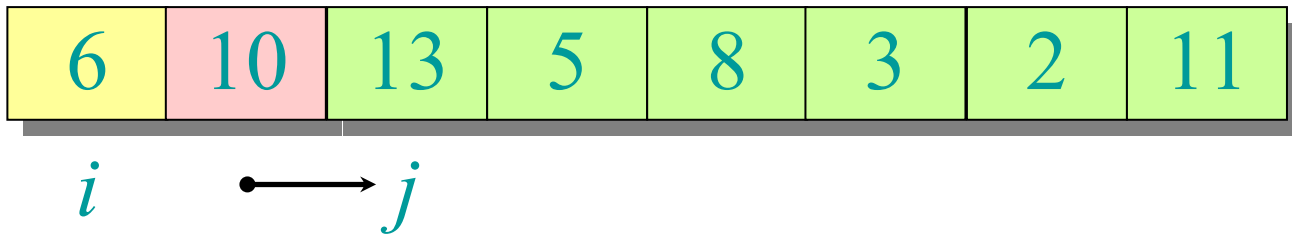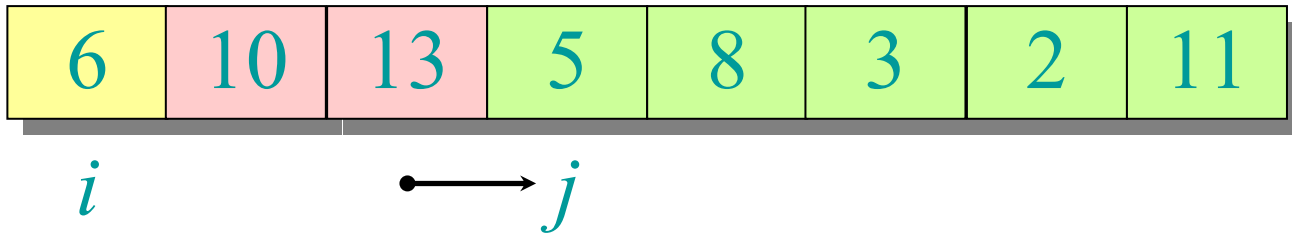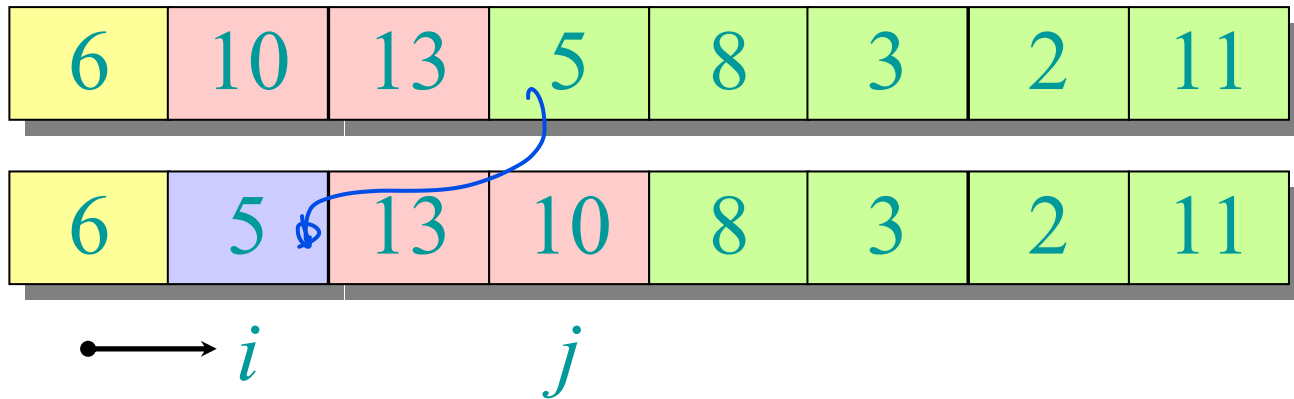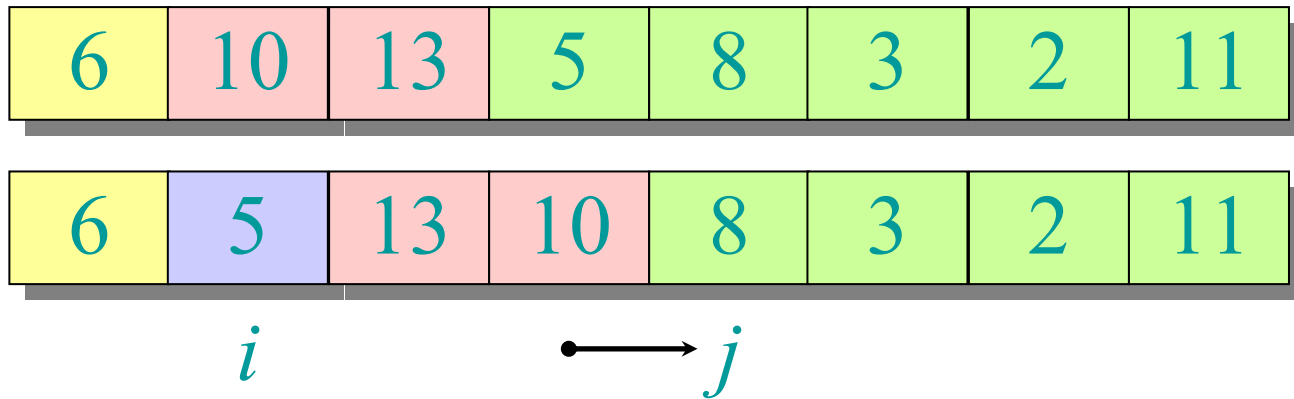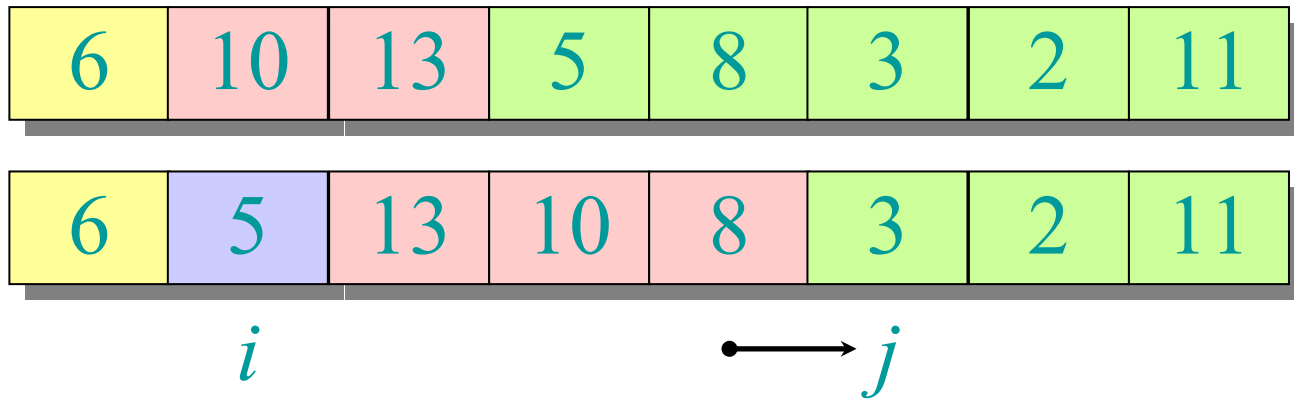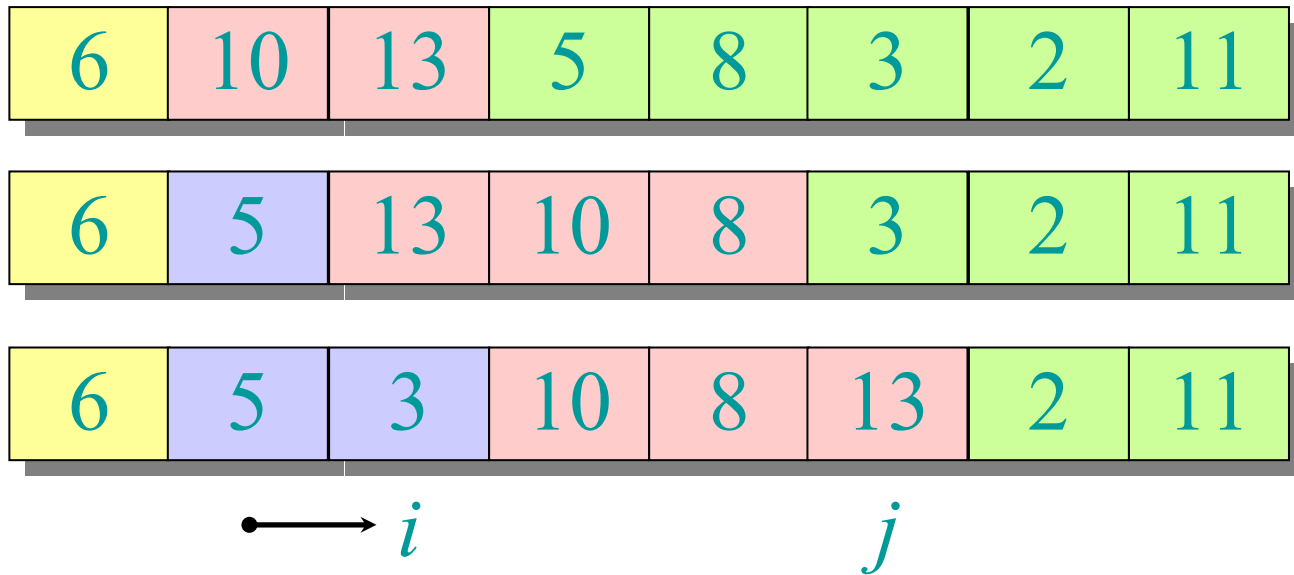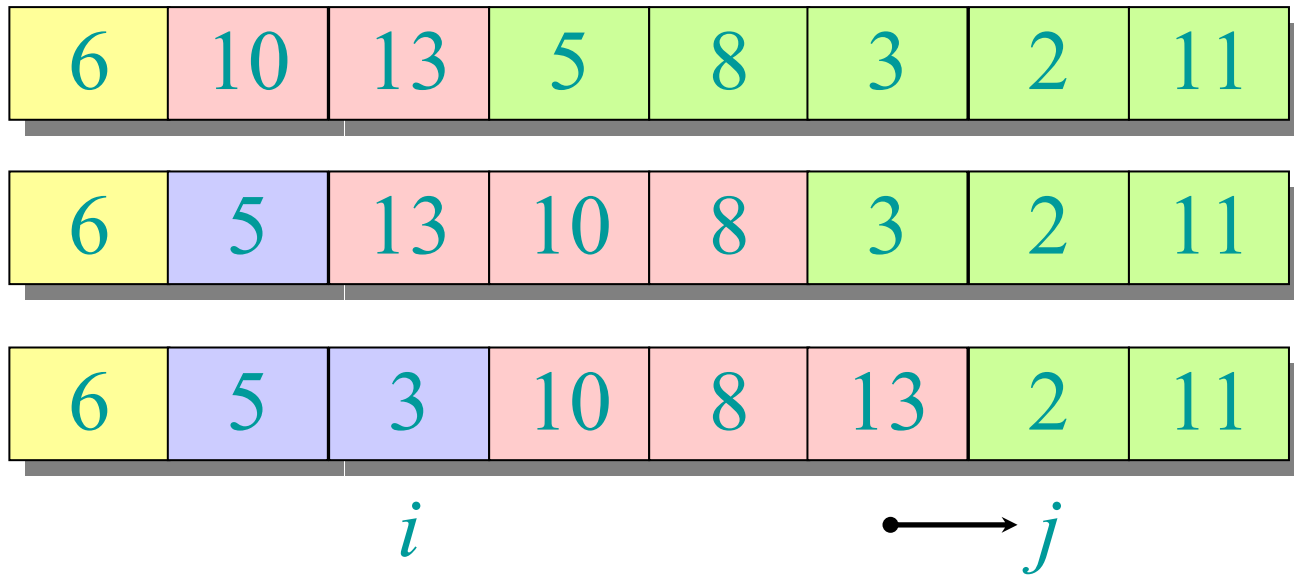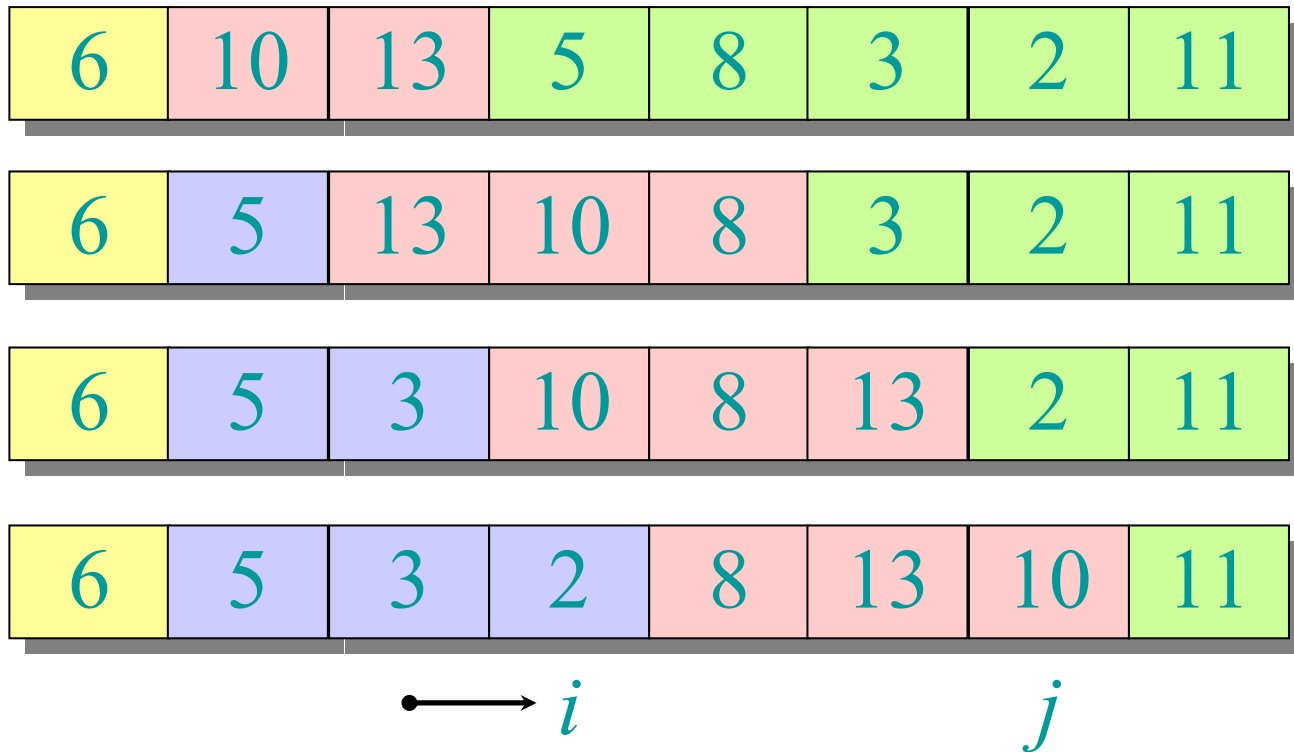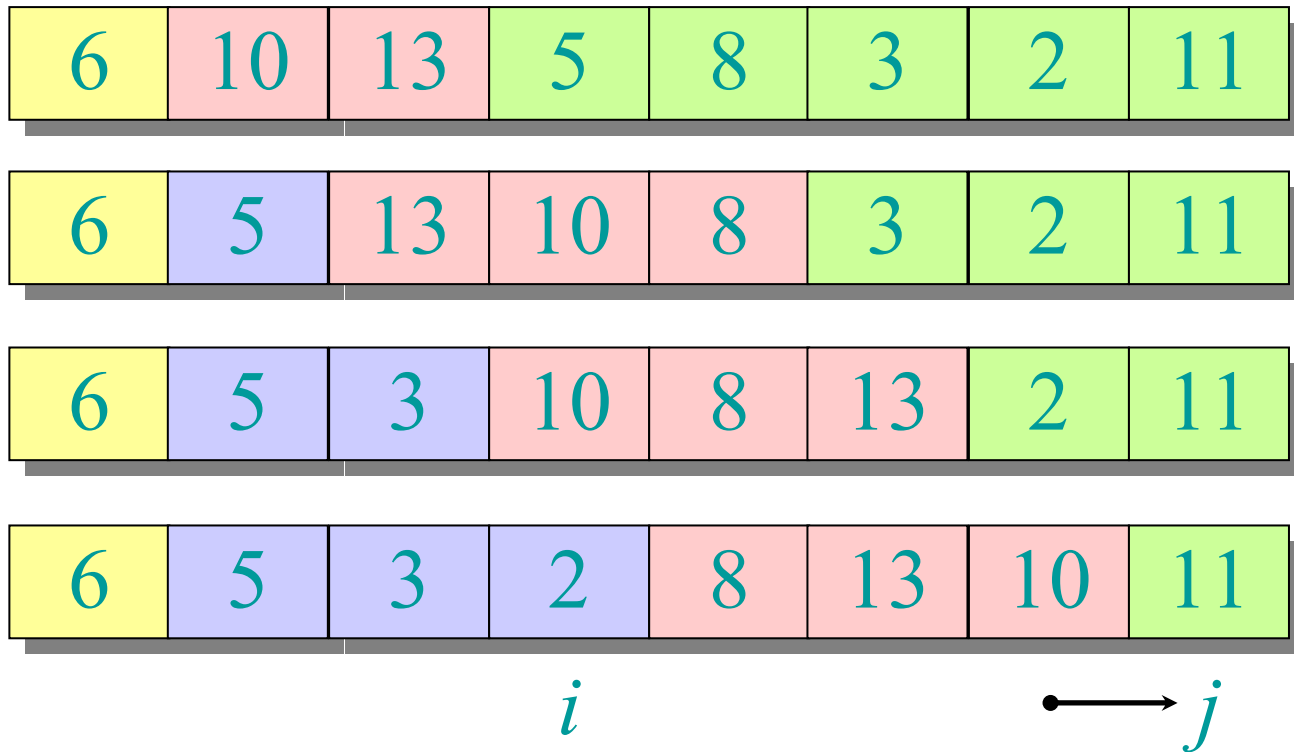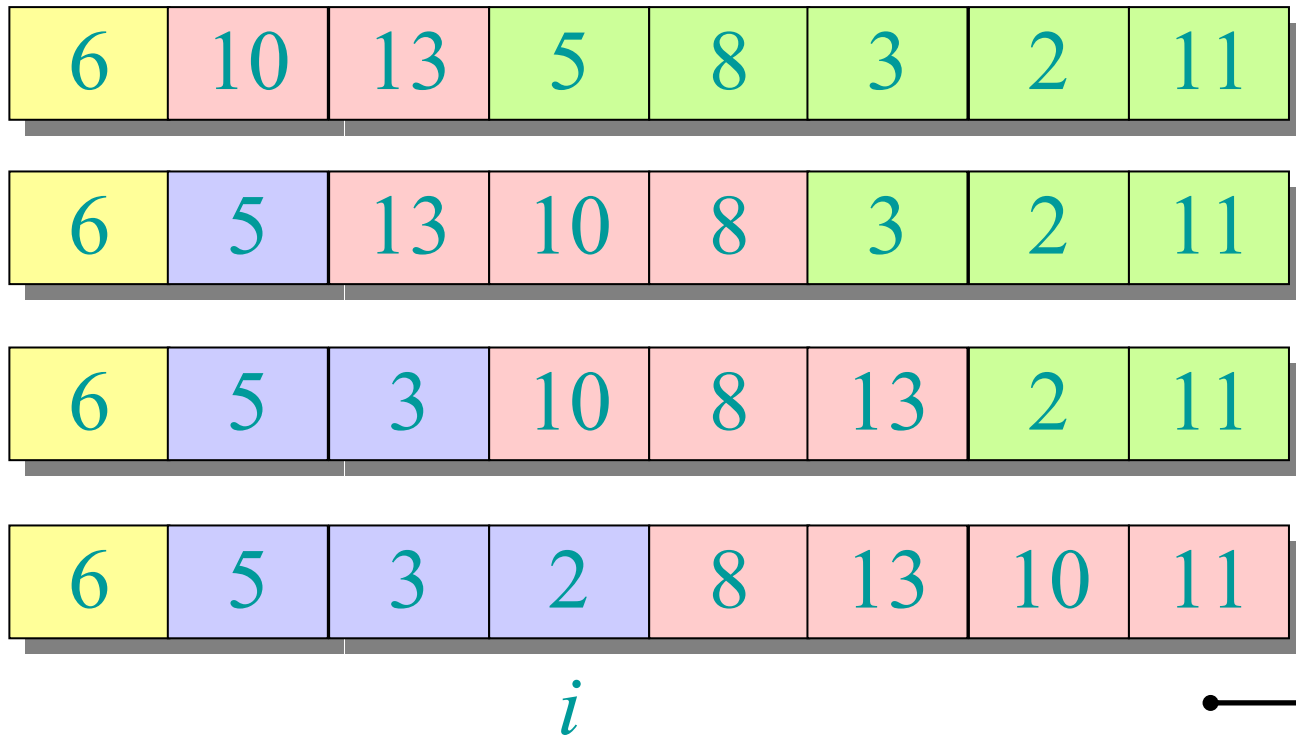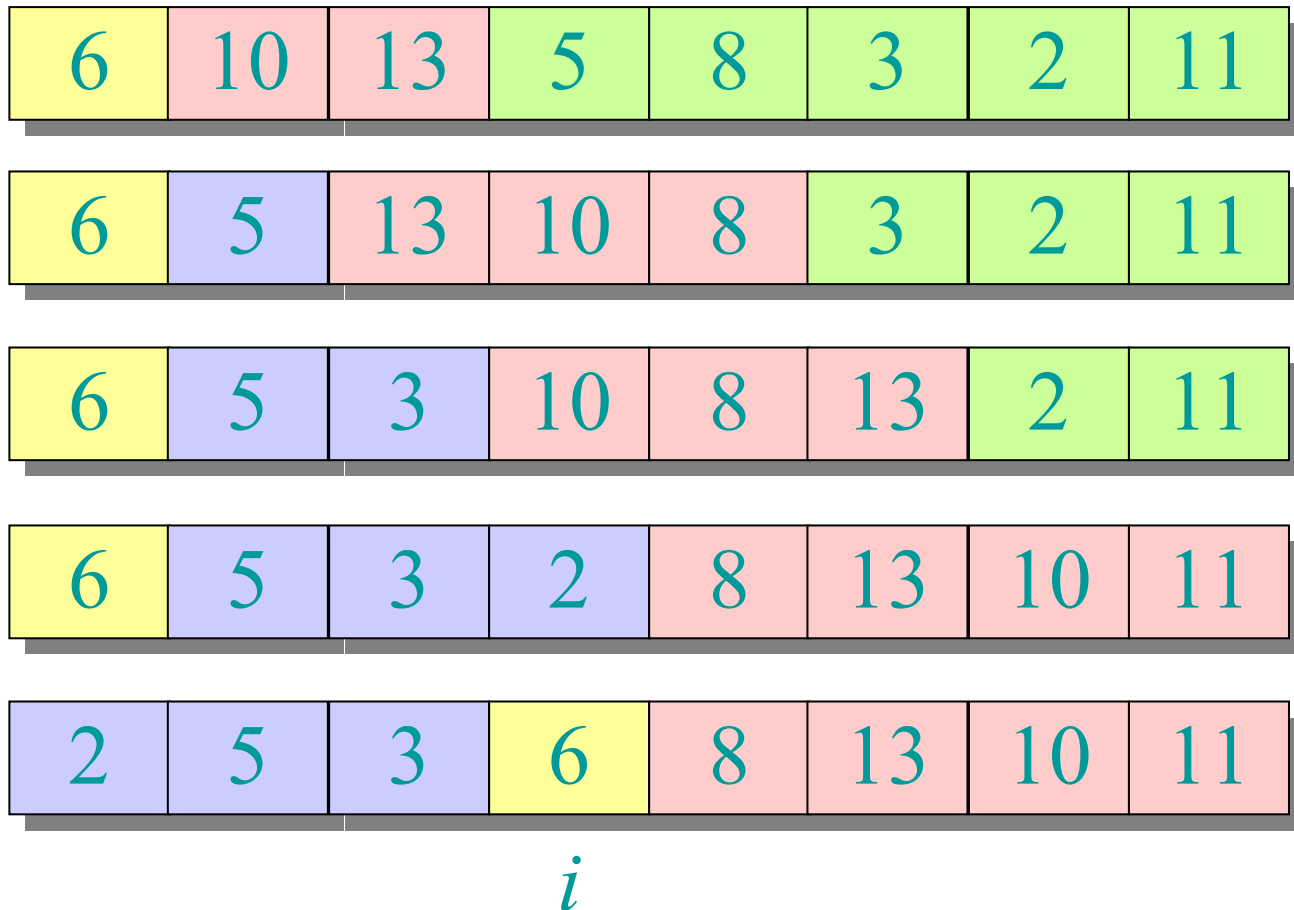
Pseudocode for quicksort

QUICKSORT(A, p, r)
    if $p < r$
        then $q \leftarrow$ PARTITION(A, p, r)
            QUICKSORT($A, p, q{-}1$)
            QUICKSORT($A, q{+}1, r$)

Initial call: QUICKSORT($A, 1, n$)
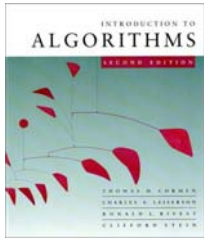
Analysis of quicksort

- Assume all input elements are distinct.

- In practice, there are better partitioning algorithms for when duplicate input elements may exist.

- Let $T(n)$ = worst-case running time on an array of n elements.

Worst-case of quicksort

- Input sorted or reverse sorted.
- Partition around min or max element.
- One side of partition always has no elements.

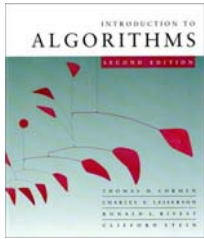$$T(n) = T(0) + T(n-1) + \Theta(n)$$
$$= \Theta(1) + T(n-1) + \Theta(n)$$
$$= T(n-1) + \Theta(n)$$
$$= \Theta(n^2) \quad \textit{(arithmetic series)}$$
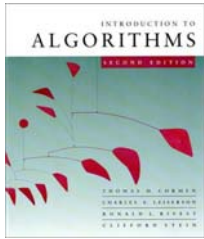
Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$
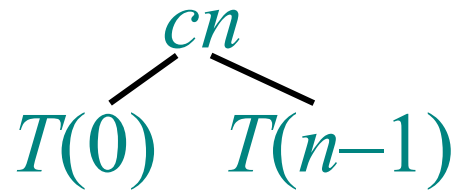
Worst-case recursion tree
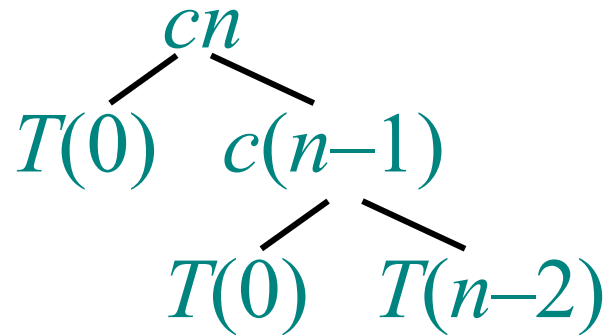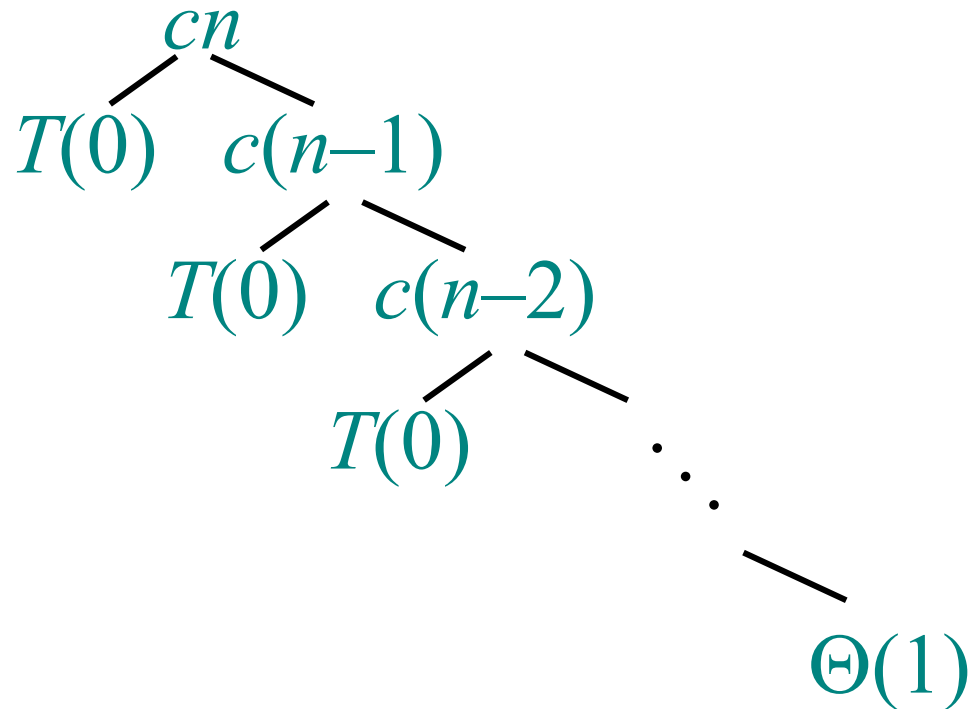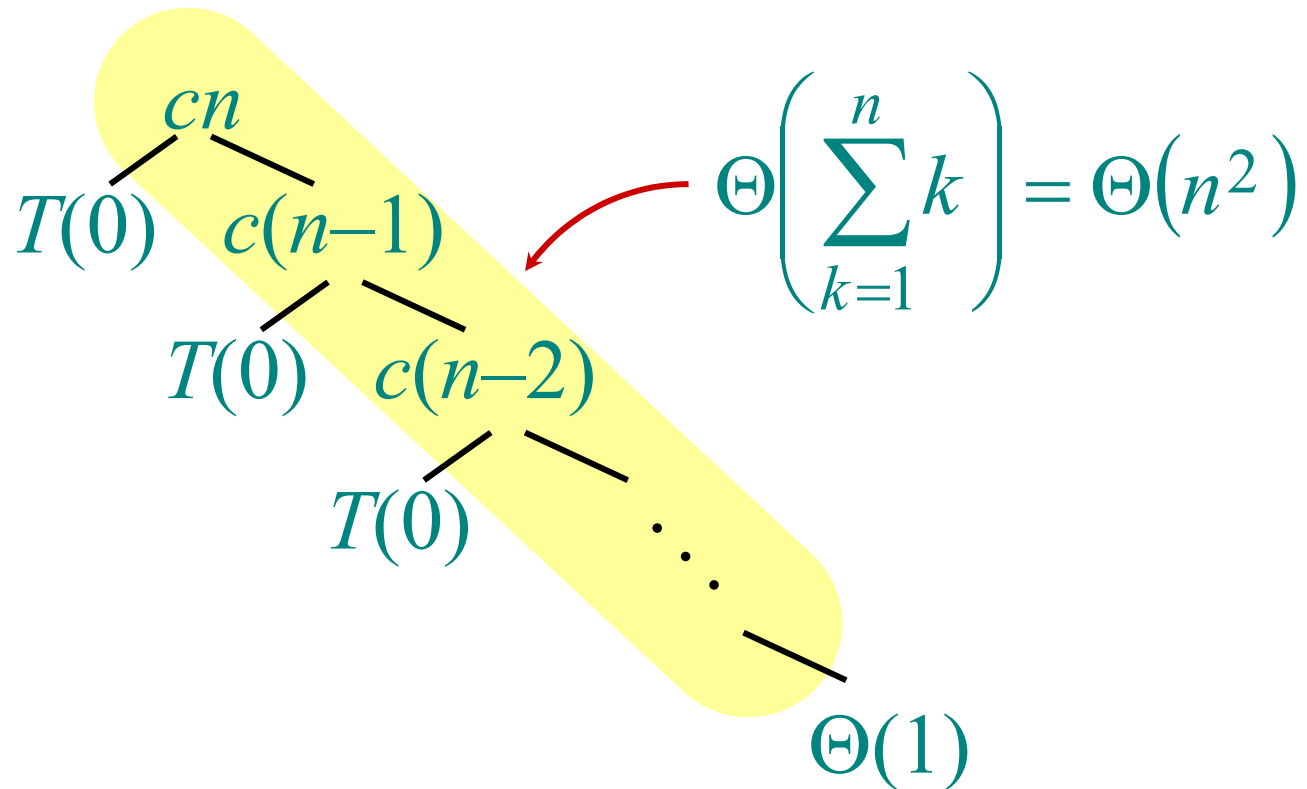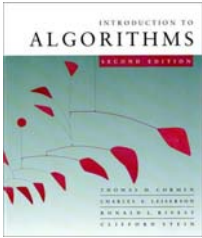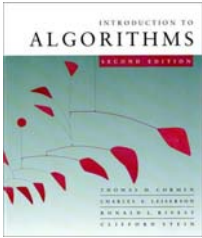
$$T(n) = T(0) + T(n-1) + cn$$

$T(n)$

Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$

cn

$T(0)$ $T(n-1)$

Worst-case recursion tree

$$T(n) = T(0) + T(n–1) + cn$$

cn

$T(0)$ $c(n–1)$

$T(0)$ $T(n–2)$

Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$

September 21, 2005

Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$

cn

$T(0)$ $c(n-1)$

$T(0)$ $c(n-2)$

$T(0)$ $\ldots$

$\Theta(1)$

$$\Theta\left(\sum_{k=1}^{n} k\right) = \Theta(n^2)$$

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Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$

$h = n$

cn

$\Theta(1) \quad c(n-1)$

$\Theta(1) \quad c(n-2)$

$\Theta(1)$

$\cdots$

$\Theta(1)$

$$\Theta\left(\sum_{k=1}^{n} k\right) = \Theta(n^2)$$

$$T(n) = \Theta(n) + \Theta(n^2)$$
$$= \Theta(n^2)$$

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Best-case analysis
(For intuition only!)

If we're lucky, PARTITION splits the array evenly:

$$T(n) = 2T(n/2) + \Theta(n)$$
$$ = \Theta(n \lg n) \qquad \text{(same as merge sort)}$$

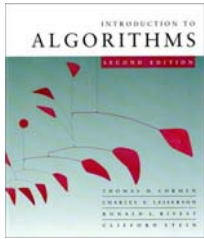What if the split is always $\frac{1}{10} : \frac{9}{10}$?

$$T(n) = T\left(\tfrac{1}{10}n\right) + T\left(\tfrac{9}{10}n\right) + \Theta(n)$$

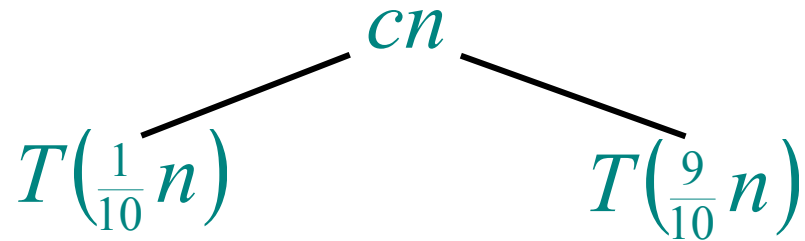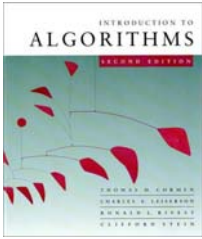What is the solution to this recurrence?
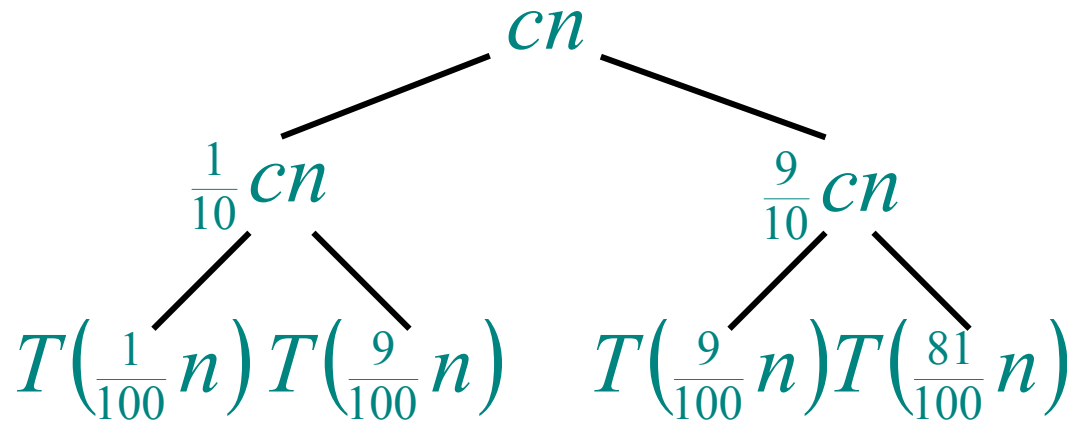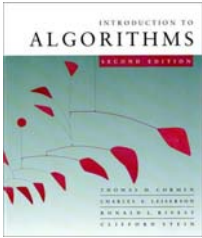
Analysis of "almost-best" case

$$T(n)$$

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Analysis of "almost-best" case

$$cn$$

$$T\left(\tfrac{1}{10}n\right) \qquad\qquad T\left(\tfrac{9}{10}n\right)$$

Analysis of "almost-best" case

$$cn$$

$$\frac{1}{10}cn \qquad \frac{9}{10}cn$$

$$T\left(\frac{1}{100}n\right) \ T\left(\frac{9}{100}n\right) \qquad T\left(\frac{9}{100}n\right) T\left(\frac{81}{100}n\right)$$

Analysis of "almost-best" case

$$cn \quad\text{------------------}\quad cn$$

$$\frac{1}{10}cn \qquad\qquad \frac{9}{10}cn \quad\text{------------}\quad cn$$

$$\log_{10/9}n$$

$$\frac{1}{100}cn \quad \frac{9}{100}cn \qquad \frac{9}{100}cn \quad \frac{81}{100}cn \quad\text{------}\quad cn$$

$$\Theta(1)$$

$$O(n) \text{ leaves}$$

$$\Theta(1)$$

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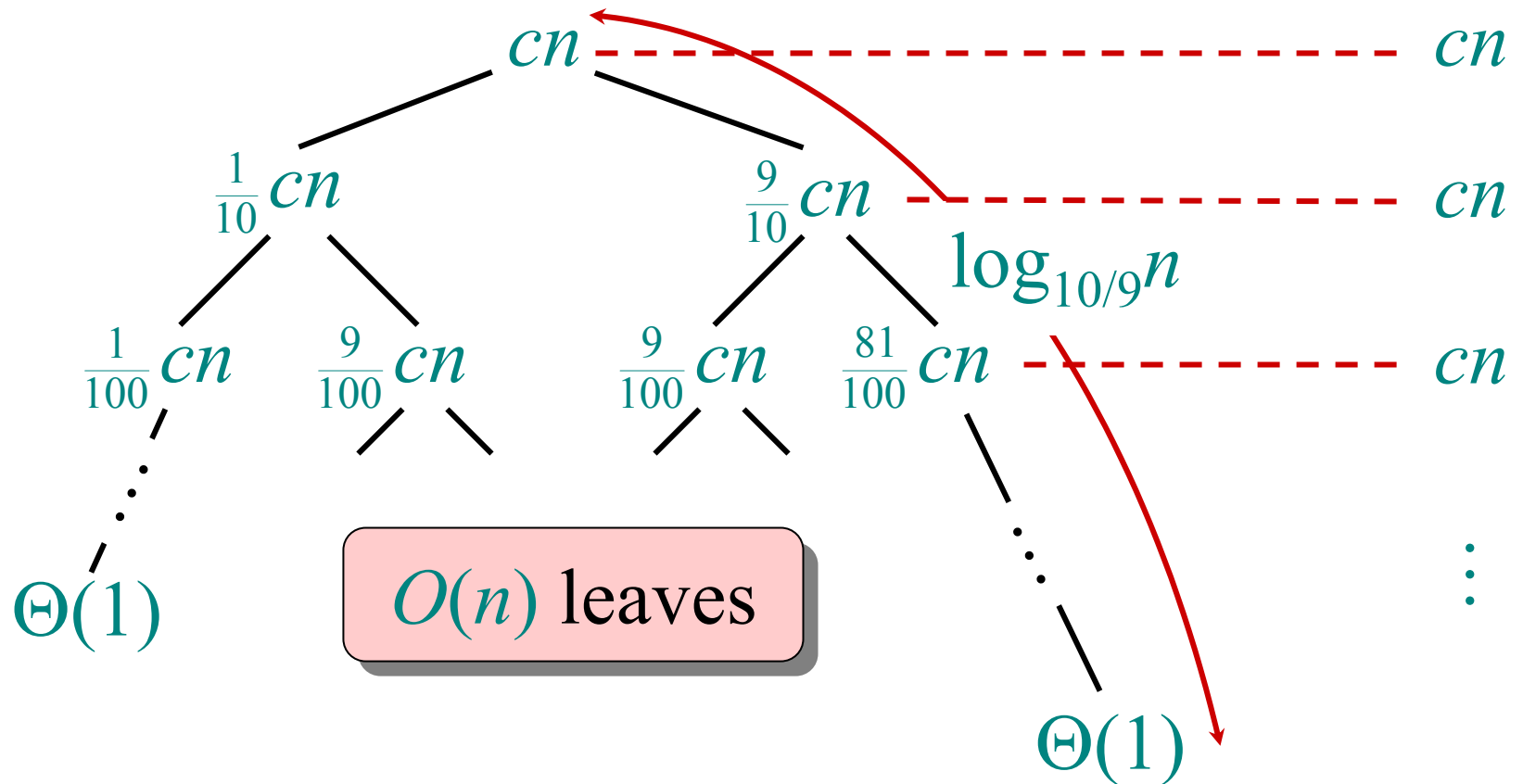
Analysis of "almost-best" case

$$cn \log_{10} n \leq T(n) \leq cn \log_{10/9} n + O(n)$$

$$\Theta(n \lg n)$$
Lucky!

$O(n)$ leaves
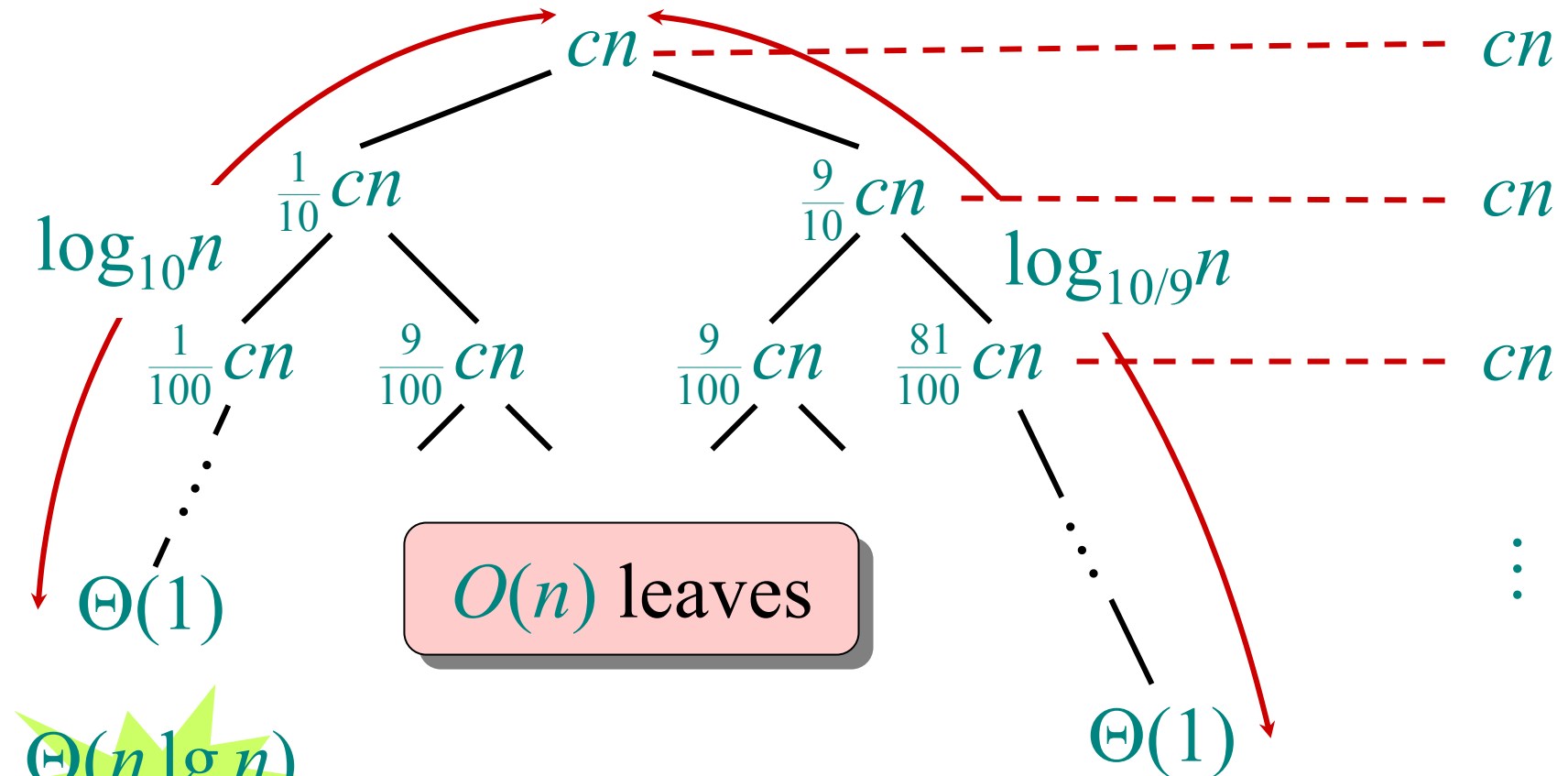
More intuition

Suppose we alternate lucky, unlucky, lucky, unlucky, lucky, ….

$L(n) = 2U(n/2) + \Theta(n)$ *lucky*

$U(n) = L(n-1) + \Theta(n)$ *unlucky*

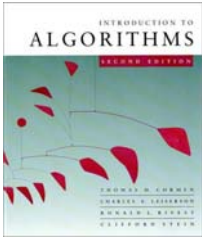Solving:

$L(n) = 2(L(n/2 - 1) + \Theta(n/2)) + \Theta(n)$

$= 2L(n/2 - 1) + \Theta(n)$

$= \Theta(n \lg n)$ *Lucky!*

How can we make sure we are usually lucky?

Randomized quicksort

IDEA: Partition around a *random* element.

- Running time is independent of the input order.

- No assumptions need to be made about the input distribution.

- No specific input elicits the worst-case behavior.

- The worst case is determined only by the output of a random-number generator.
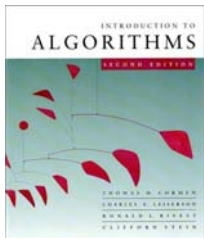
Randomized quicksort analysis

Let $T(n) =$ the random variable for the running time of randomized quicksort on an input of size n, assuming random numbers are independent.

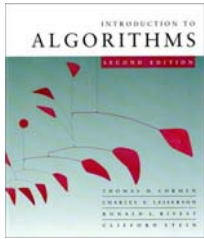For $k = 0, 1, \ldots, n-1$, define the *indicator random variable*

$$X_k = \begin{cases} 1 & \text{if PARTITION generates a } k : n-k-1 \text{ split,} \\ 0 & \text{otherwise.} \end{cases}$$

$E[X_k] = \Pr\{X_k = 1\} = 1/n$, since all splits are equally likely, assuming elements are distinct.
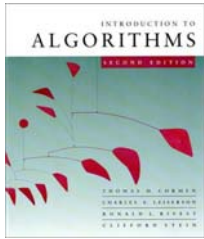
Analysis (continued)

$$T(n) = \begin{cases} T(0) + T(n{-}1) + \Theta(n) & \text{if } 0 : n{-}1 \text{ split,} \\ T(1) + T(n{-}2) + \Theta(n) & \text{if } 1 : n{-}2 \text{ split,} \\ \quad\quad\vdots \\ T(n{-}1) + T(0) + \Theta(n) & \text{if } n{-}1 : 0 \text{ split,} \end{cases}$$

$$= \sum_{k=0}^{n-1} X_k \left(T(k) + T(n - k - 1) + \Theta(n) \right)$$

Calculating expectation

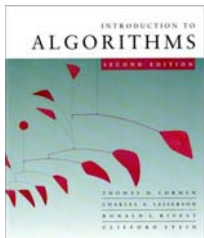$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n) \right) \right]$$

Take expectations of both sides.

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Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \big(T(k) + T(n-k-1) + \Theta(n) \big) \right]$$

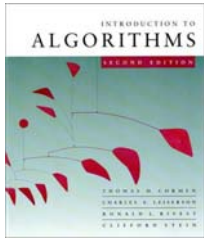$$= \sum_{k=0}^{n-1} E\big[X_k \big(T(k) + T(n-k-1) + \Theta(n) \big) \big]$$

Linearity of expectation.

Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E\left[X_k\left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E\left[X_k\right] \cdot E\left[T(k) + T(n-k-1) + \Theta(n)\right]$$

Independence of X_k from other random choices.

Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

$$= \sum_{k=0}^{n-1} E\big[X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\big]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E\big[T(k) + T(n-k-1) + \Theta(n)\big]$$

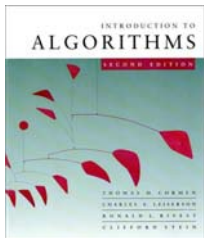$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$

Linearity of expectation; $E[X_k] = 1/n$.

Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k (T(k) + T(n-k-1) + \Theta(n))\right]$$

$$= \sum_{k=0}^{n-1} E[X_k (T(k) + T(n-k-1) + \Theta(n))]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)]$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$

$$= \frac{2}{n}\sum_{k=1}^{n-1} E[T(k)] + \Theta(n) \qquad$$ Summations have identical terms.

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Hairy recurrence

$$E[T(n)] = \frac{2}{n} \sum_{k=2}^{n-1} E[T(k)] + \Theta(n)$$

(The $k = 0, 1$ terms can be absorbed in the $\Theta(n)$.)

Prove: $E[T(n)] \leq a\,n\lg n$ for constant $a > 0$.

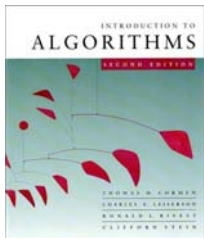• Choose a large enough so that $a\,n\lg n$ dominates $E[T(n)]$ for sufficiently small $n \geq 2$.

Use fact: $\displaystyle\sum_{k=2}^{n-1} k\lg k \leq \frac{1}{2}n^2\lg n - \frac{1}{8}n^2$ (exercise).

Substitution method

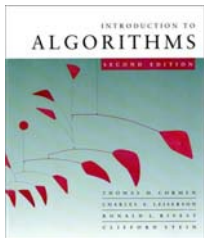$$E[T(n)] \le \frac{2}{n} \sum_{k=2}^{n-1} ak \lg k + \Theta(n)$$

Substitute inductive hypothesis.

Substitution method

$$E[T(n)] \leq \frac{2}{n} \sum_{k=2}^{n-1} ak \lg k + \Theta(n)$$

$$\leq \frac{2a}{n} \left(\frac{1}{2} n^2 \lg n - \frac{1}{8} n^2 \right) + \Theta(n)$$

Use fact.

Substitution method

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak\lg k + \Theta(n)$$

$$\le \frac{2a}{n}\left(\frac{1}{2}n^2\lg n - \frac{1}{8}n^2\right) + \Theta(n)$$

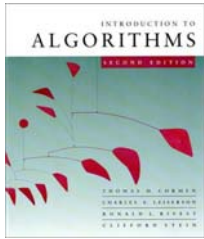$$= an\lg n - \left(\frac{an}{4} - \Theta(n)\right)$$

Express as *desired* − *residual*.

Substitution method

$$E[T(n)] \leq \frac{2}{n} \sum_{k=2}^{n-1} ak \lg k + \Theta(n)$$

$$= \frac{2a}{n} \left(\frac{1}{2} n^2 \lg n - \frac{1}{8} n^2 \right) + \Theta(n)$$
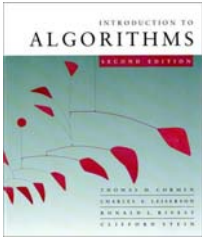
$$= an \lg n - \left(\frac{an}{4} - \Theta(n) \right)$$

$$\leq an \lg n \, ,$$

if *a* is chosen large enough so that *an*/4 dominates the $\Theta(n)$.

Quicksort in practice

- Quicksort is a great general-purpose sorting algorithm.

- Quicksort is typically over twice as fast as merge sort.

- Quicksort can benefit substantially from *code tuning*.

- Quicksort behaves well even with caching and virtual memory.