The probability that a random network has exactly L links is the product of three terms;

1. The probability that L of the attempts to connect the $N(N-1)/2$ pairs of nodes have resulted in a link, which is p^L.

2. The probability that the remaining $N(N-1)/2 - L$ attempts have not resulted in a link, which is
$$(1-p)^{N(N-1)/2 - L}.$$

3. A combinatorial factor
$$^{N(N-1)/2}C_L$$
counting the number of different ways we can place L links among $N(N-1)/2$ node pairs.

We can therefore write the probability that a particular realization of a random network has exactly L links as

$$\boxed{p_L = {}^{N(N-1)}C_L \cdot p^L \cdot (1-p)^{N(N-1)/2 - L}} \quad \text{— } (3.1)$$

DERIVING THE POISSON DISTRIBUTION

In a random network, the probability that node i has exactly k links is given by:

$$p_k = {}^{N-1}C_k \; p^k \; (1-p)^{N-1-k} \qquad \text{— (3.7)/(3.22)}$$

The number of ways we can select k links from $N-1$ potential links that a node can have.

The probability that k of its links are present

The probability that the remaining $(N-1-k)$ links are missing.

FIRST TERM: We can rewrite the first term as,

$$ {}^{N-1}C_k = \frac{(N-1)(N-1-1)(N-1-2)\dots(N-1-k+1)}{k!}$$

$$\approx \frac{(N-1)^k}{k!} \qquad \text{— (3.23)}$$

LAST TERM: Since, $\ln\left[(1-p)^{(N-1)-k}\right] = (N-1-k)\ln\left(1-\frac{\langle k\rangle}{N-1}\right)$

Using series expansion,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \dots \quad \forall \; |x| \leq 1,$$

we obtain,

$$\ln\left[(1-p)^{N-1-k}\right] \approx (N-1-k)\frac{\langle k\rangle}{N-1}$$

$$= -\langle k\rangle\left[1-\frac{k}{N-1}\right] \approx -\langle k\rangle$$

$$\therefore (1-p)^{N-1-k} = e^{-\langle k\rangle} \qquad \text{— (3.24)}$$

Therefore, the expected number of links in a random graph is

$$\langle L \rangle = \sum_{L=0}^{\frac{N(N-1)}{2}} L \cdot P_L = p \cdot \frac{N(N-1)}{2} \qquad \text{— (3.2)}$$

L_{max} is given by $\langle L \rangle$ for maximum value of p,

$$L_{max} = \frac{N(N-1)}{2}$$

Average degree of a random graph is given by,

$$\langle k \rangle = \frac{2\langle L \rangle}{N} = \frac{2}{N} \cdot \frac{N(N-1)}{2} \cdot p \quad \text{(using 3.2)}$$

$$\langle k \rangle = p \cdot (N-1) \qquad \text{— (3.3)}$$

Substituting (3.23) & (3.24) in (3.22), we get,

$$p_k = {}^{N-1}C_k \cdot p^k \cdot (1-p)^{N-1-k}$$

$$p_k = \frac{(N-1)}{k!} \cdot p^k \cdot e^{-\langle k \rangle}$$

$$p_k = \frac{(N-1)^k}{k!} \left(\frac{\langle k \rangle}{N-1}\right)^k e^{-\langle k \rangle}$$

$$\boxed{p_k = e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}} \quad \text{—(3.25)}$$