Network Science

Class 4: Scale-free property

Ganesh Bagler

— Adapted from —
Albert-László Barabási
(With Roberta Sinatra)

Introduction

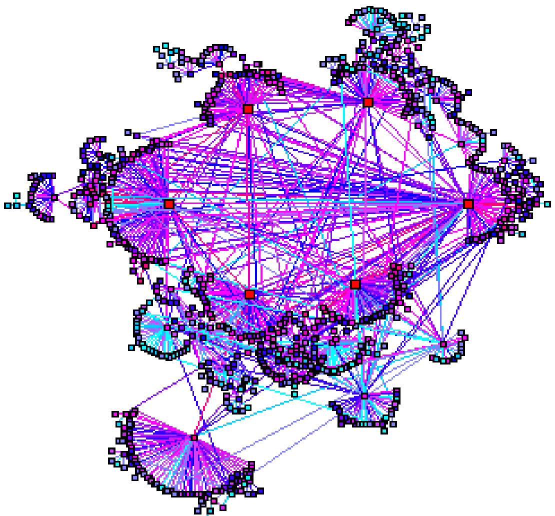
Power laws and scale-free networks

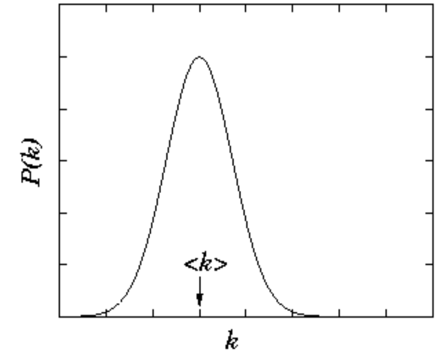
WORLD WIDE WEB

Nodes: **WWW documents**
Links: **URL links**

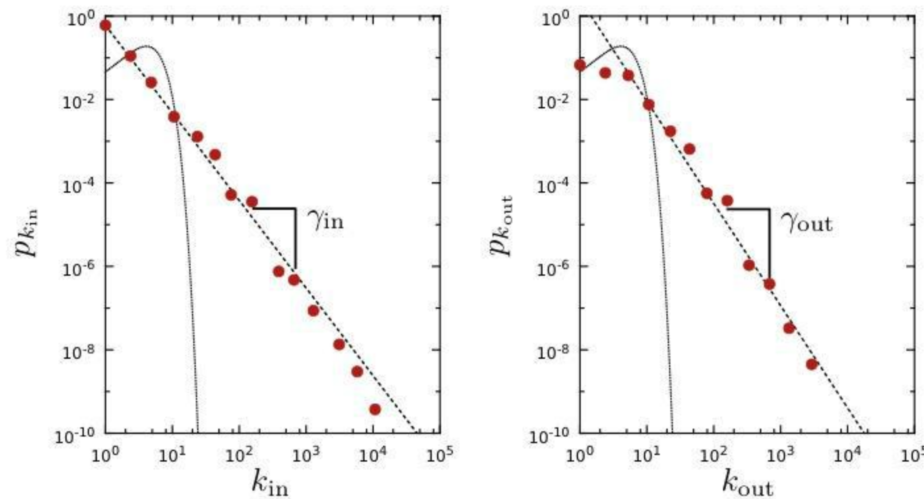
Over 3 billion documents

ROBOT: collects all URL's found in a document and follows them recursively





Expected



R. Albert, H. Jeong, A-L Barabasi, *Nature*, 401 130 (1999).

Discrete vs. Continuum formalism

Discrete Formalism

As node degrees are always positive integers, the discrete formalism captures the probability that a node has exactly k links:

$$p_k = Ck^{-\gamma}.$$

$$\sum_{k=1}^{\infty} p_k = 1.$$

$$C\sum_{k=1}^{\infty} k^{-\gamma} = 1 \qquad C = \frac{1}{\sum_{k=1}^{\infty} k^{-\gamma}} = \frac{1}{\zeta(\gamma)},$$

$$p_k = \frac{k^{-\gamma}}{\zeta(\gamma)}$$

Continuum Formalism

In analytical calculations it is often convenient to assume that the degrees can take up any positive real value:

$$p(k) = Ck^{-\gamma}.$$

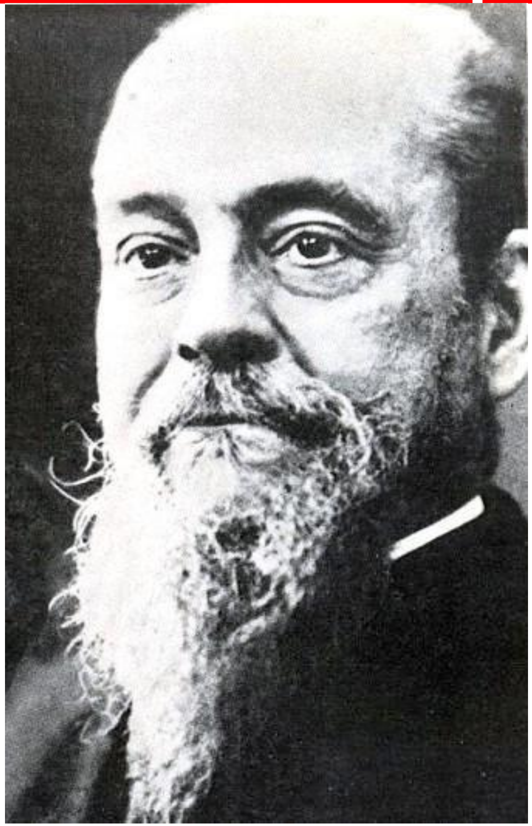
$$\int_{k_{\min}}^{\infty} p(k)dk = 1$$

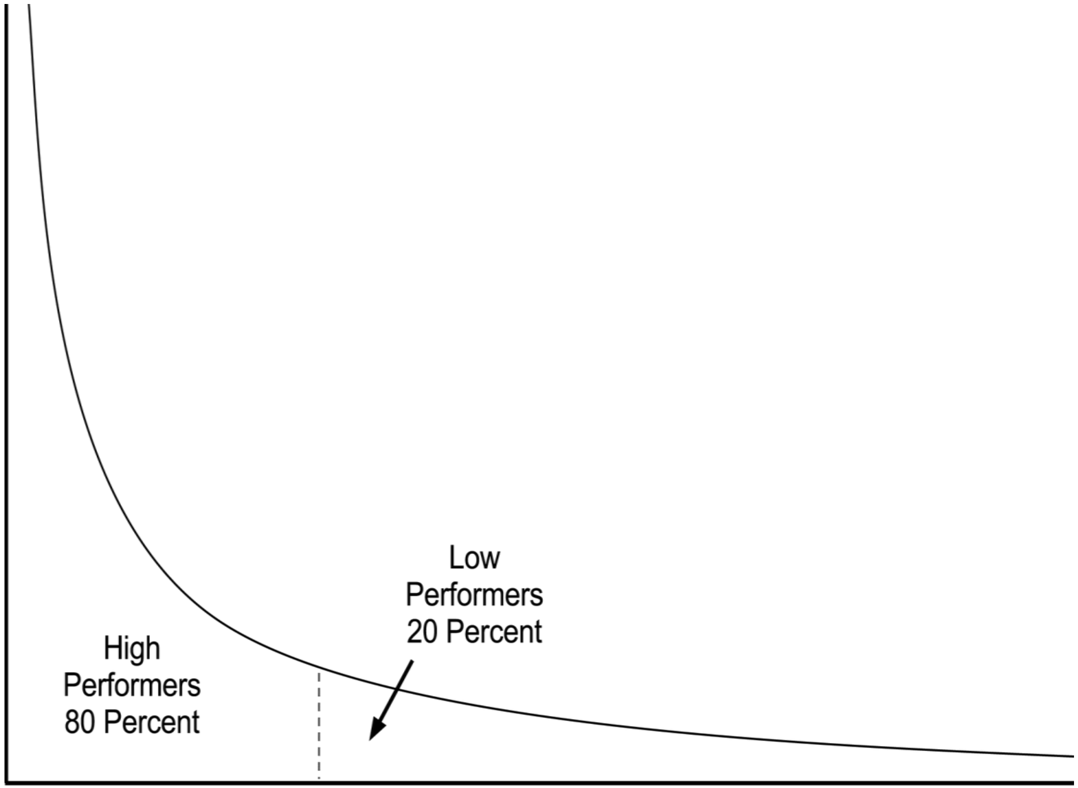
$$C = \frac{1}{\int_{k_{\min}}^{\infty} k^{-\gamma}dk} = (\gamma-1)k_{\min}^{\gamma-1}$$

$$p(k) = (\gamma-1)k_{\min}^{\gamma-1}k^{-\gamma}.$$

INTERPRETATION: p_k $\int_{k_1}^{k_2} p(k)dk$

80/20 RULE





Vilfredo Federico Damaso Pareto (1848 – 1923), Italian economist, political scientist and philosopher, who had important contributions to our understanding of income distribution and to the analysis of individuals choices. A number of fundamental principles are named after him, like Pareto efficiency, Pareto distribution (another name for a power-law distribution), the Pareto principle (or 80/20 law).

80/20 RULES?

Inequality gap widens as 42 people hold same wealth as 3.7bn poorest

Oxfam calls for action on gap as wealthiest people gather at World Economic Forum in Davos

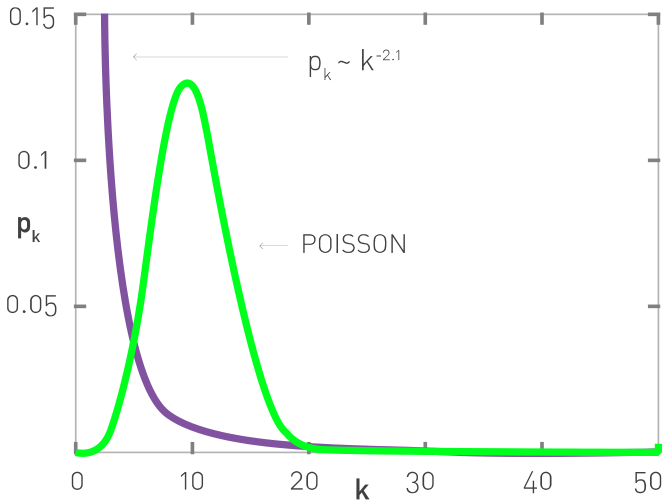
The development charity Oxfam has called for action to tackle the growing gap between rich and poor as it launched a new report showing that 42 people hold as much wealth as the 3.7 billion who make up the poorest half of the world's population.

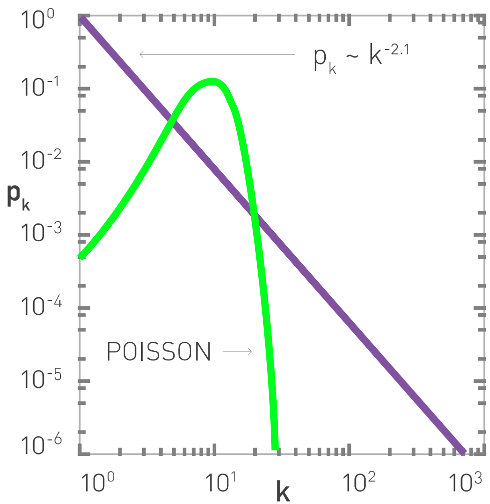
Income inequality gets worse; India's top 1% bag 73% of the country's wealth, says Oxfam

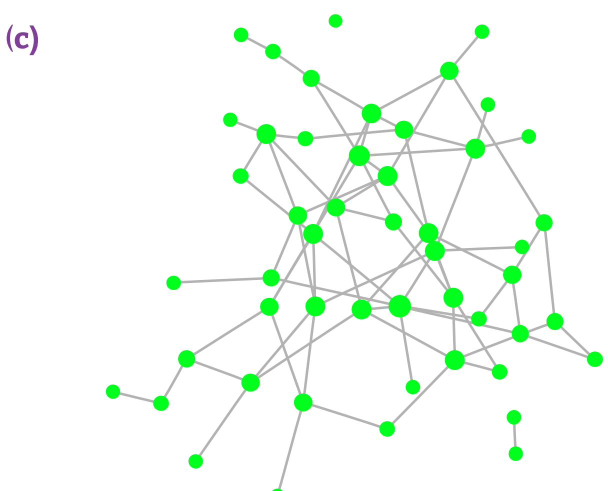
BusinessToday.in Last Updated: January 22, 2018 | 12:53 IST

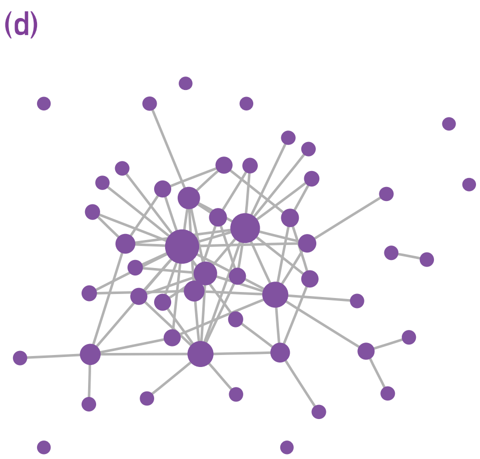
Hubs

The difference between a power law and an exponential distribution









The difference between a power law and an exponential distribution

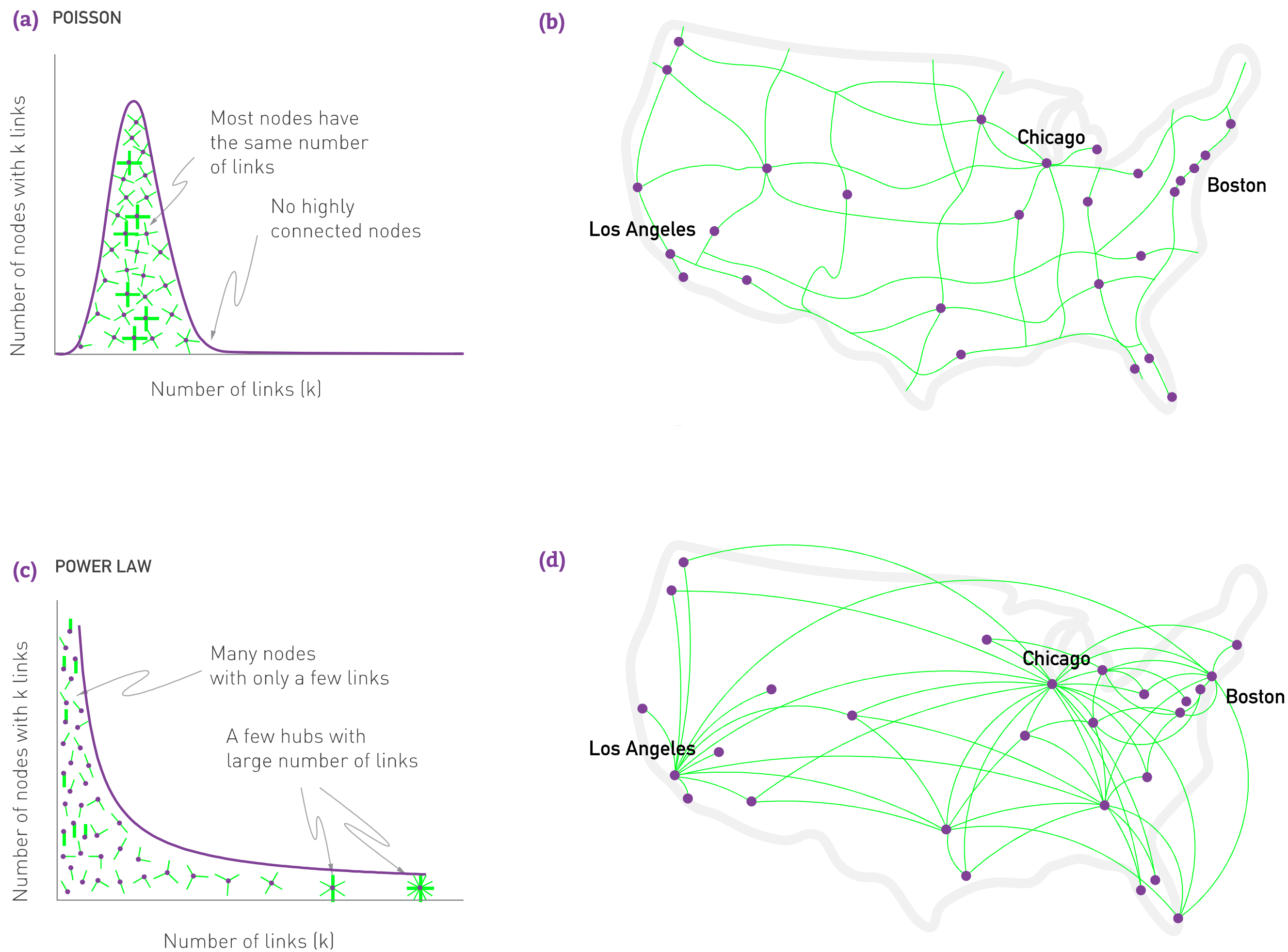
Let us use the WWW to illustrate the properties of the high-*k regime.*
The probability to have a node with k~100 is

- *About* $p_{100} \simeq 10^{-30}$ *in a Poisson distribution*

- *About* $p_{100} \simeq 10^{-4}$ *if* p_k *follows a power law.*

- *Consequently, if the WWW were to be a random network, according to the Poisson prediction we would expect* 10^{-18} *k>100 degree nodes, or none.*

- *For a power law degree distribution, we expect about* $N_{k>100} = 10^9$ *k>100 degree nodes*


(a) POISSON
Number of nodes with k links
Number of links (k)
Most nodes have the same number of links
No highly connected nodes
(b)
Chicago
Boston
Los Angeles
(c) POWER LAW
Number of nodes with k links
Number of links (k)
Many nodes with only a few links
A few hubs with large number of links
(d)
Chicago
Boston
Los Angeles

The size of the biggest hub

All real networks are finite → let us explore its consequences.
→ We have an expected maximum degree, k_{max}

Estimating k_{max}

$$\int_{k_{\max}}^{\infty} P(k)\,dk \approx \frac{1}{N}$$

The probability to have a node larger than k_{max} should not exceed the prob. to have one node, i.e. *1/N* fraction of all nodes

$$\int_{k_{\max}}^{\infty} P(k)\,dk = (\gamma - 1)k_{\min}^{\gamma-1}\int_{k_{\max}}^{\infty} k^{-\gamma}\,dk = \frac{(\gamma-1)}{(-\gamma+1)}k_{\min}^{\gamma-1}\left[k^{-\gamma+1}\right]_{k_{\max}}^{\infty} = \frac{k_{\min}^{\gamma-1}}{k_{\max}^{\gamma-1}} \approx \frac{1}{N}$$

$$k_{\max} = k_{\min} N^{\frac{1}{\gamma-1}}$$

The size of the biggest hub

$$k_{\max} = k_{\min} N^{\frac{1}{g-1}}$$

To illustrate the difference in the maximum degree of an exponential and a scale-free network let us return to the WWW sample of Figure 4.1, consisting of $N \simeq 3 \times 10^5$ nodes. As $k_{min} = 1$, if the degree distribution were to follow an exponential, (4.17) predicts that the maximum degree should be $k_{max} \simeq 13$. In a scale-free network of similar size and $\gamma = 2.1$, (4.18) predicts $k_{max} \simeq 85{,}000$, a remarkable difference. Note that the largest in-degree of the WWW map of Figure 4.1 is 10,721, which is comparable to k_{max} predicted by a scale-free network. This reinforces our conclusion that *in a random network hubs are effectivelly forbidden, while in scale-free networks they are naturally present.*

Finite scale-free networks

Expected maximum degree, k_{max}

$$k_{max} = k_{min} N^{\frac{1}{g-1}}$$

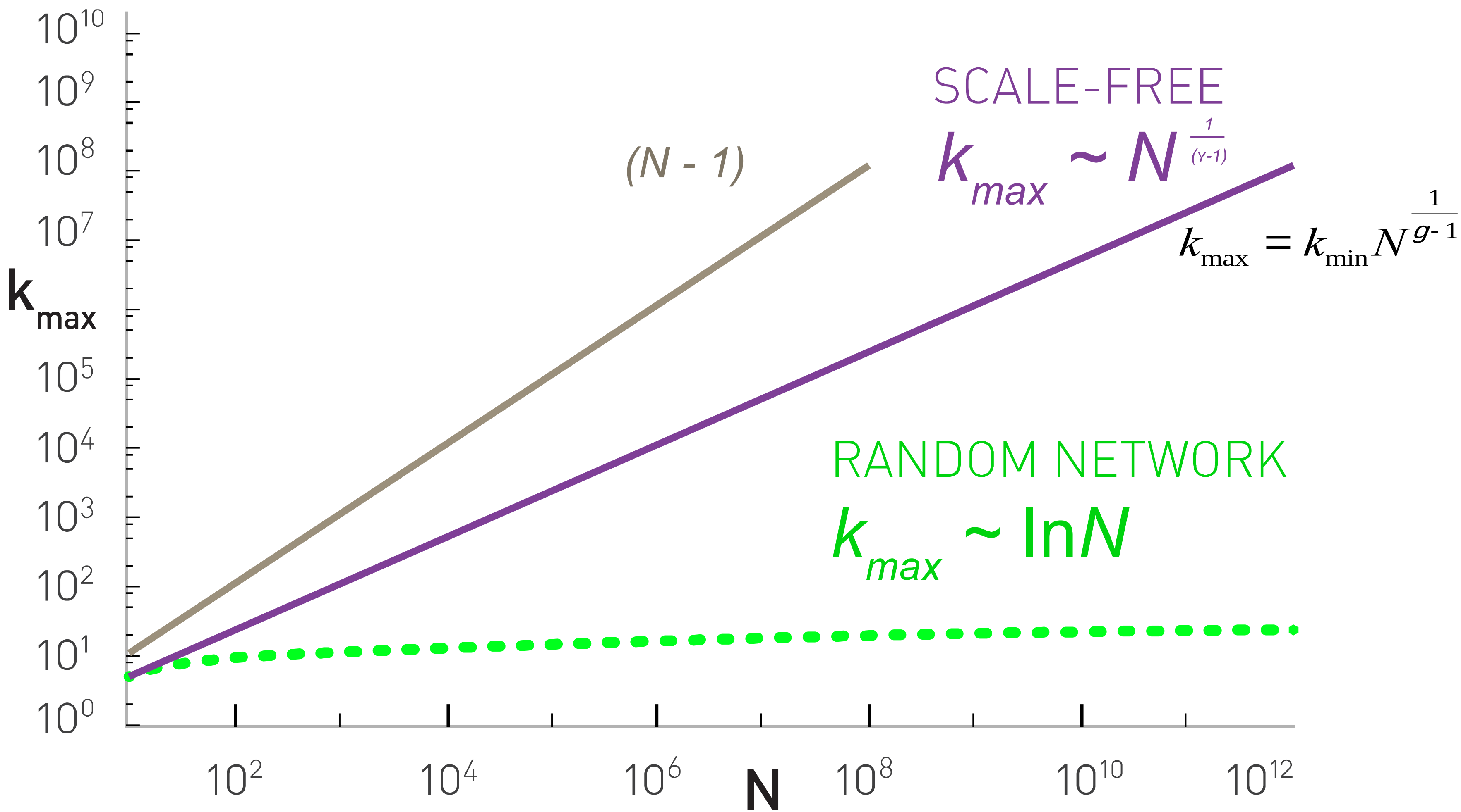
- k_{max}, increases with the size of the network
  → the larger a system is, the larger its biggest hub

- For γ>2 k_{max} increases slower than N
  → the largest hub will contain a decreasing fraction of links as N increases.

- For γ=2 k_{max}~N.
  → The size of the biggest hub is O(N)

- For γ<2 k_{max} increases faster than N: condensation phenomena
  → the largest hub will grab an increasing fraction of links. Anomaly!

The size of the largest hub



The meaning of scale-free

Scale-free networks: Definition

Definition:

Networks with a power law tail in their degree distribution are called ‘scale-free networks’

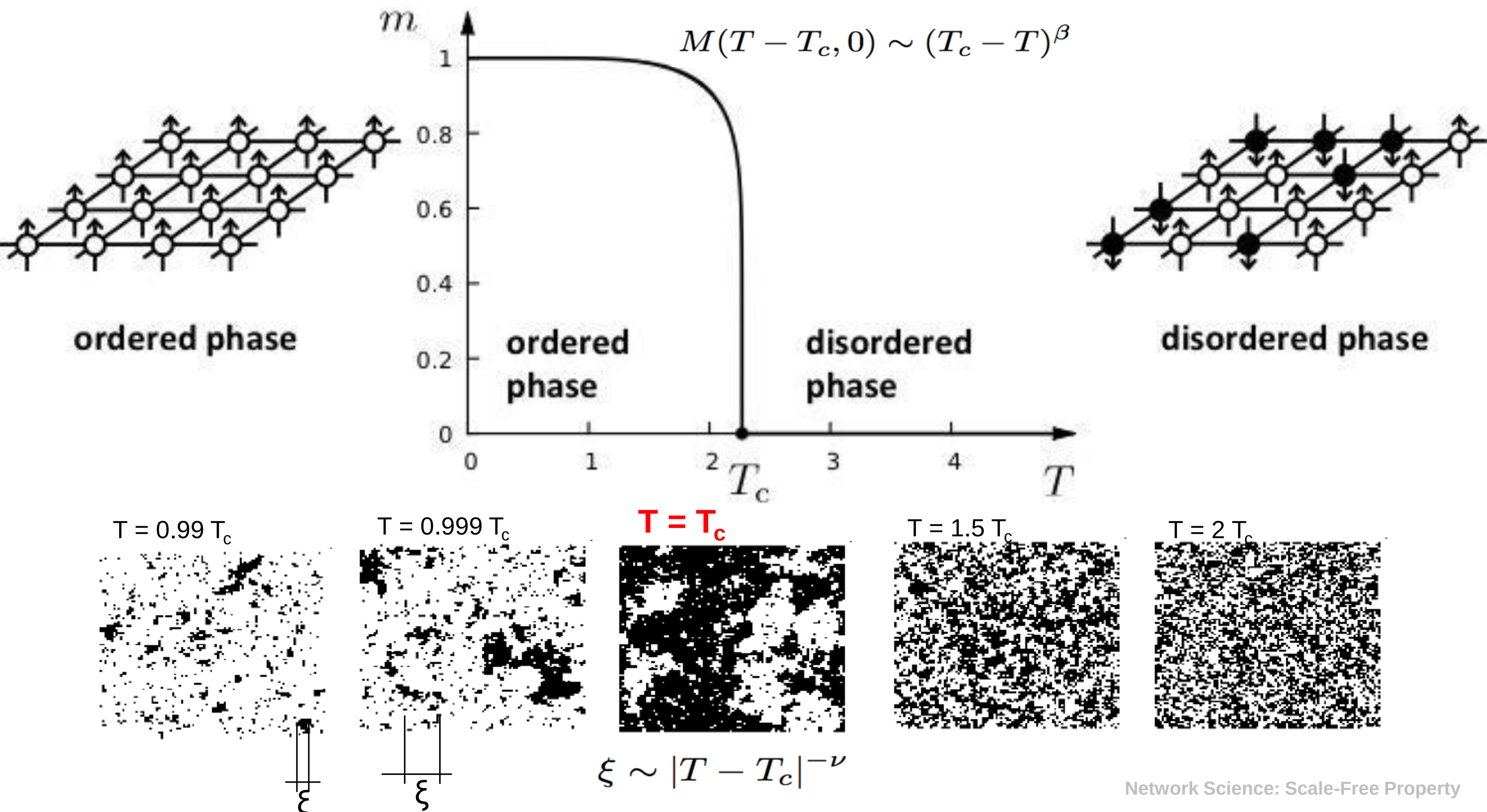
Where does the name come from?

Critical Phenomena and scale-invariance

(a detour)

Slides after **Dante R. Chialvo**

Phase transitions in complex systems I: Magnetism



Scale-free behavior in space

$$\xi \sim |T - T_c|^{-\nu}$$

At T = Tc:

correlation length diverges

Fluctuations emerge at all scales:

scale-free behavior

CRITICAL PHENOMENA

- Correlation length diverges at the critical point: the whole system is correlated!

- **Scale invariance**: there is no characteristic scale for the fluctuation **(scale-free behavior).**

- **Universality**: exponents are independent of the system's details.

Divergences in scale-free distributions

$$P(k) = Ck^{-\gamma} \qquad k = [k_{\min}, \infty) \qquad \int_{k_{\min}}^{\infty} P(k)dk = 1 \qquad C = \frac{1}{\int_{k_{\min}}^{\infty} k^{-\gamma}\, dk} = (\gamma - 1)k_{\min}^{\gamma-1}$$

$$P(k) = (\gamma - 1)k_{\min}^{\gamma-1}k^{-\gamma}$$

$$< k^m >= \int_{k_{\min}}^{\infty} k^m P(k)dk \qquad < k^m >= (\gamma - 1)k_{\min}^{\gamma-1}\int_{k_{\min}}^{\infty} k^{m-\gamma}\, dk = \frac{(\gamma - 1)}{(m - \gamma + 1)} k_{\min}^{\gamma-1}\left[k^{m-\gamma+1}\right]_{k_{\min}}^{\infty}$$

If m-γ+1<0: $$< k^m >= -\frac{(\gamma - 1)}{(m - \gamma + 1)} k_{\min}^{m}$$

If m-γ+1>0, the integral diverges.

For a fixed γ this means that all moments with m>γ-1 diverge.

DIVERGENCE OF THE HIGHER MOMENTS

$$< k^m >= (g - 1)k_{\min}^{g-1} \int_{k_{\min}}^{\infty} k^{m-/}\, dk = \frac{(g-1)}{(m-g+1)} k_{\min}^{g-1} \left[k^{m-g+1} \right]_{k_{\min}}^{\infty}$$

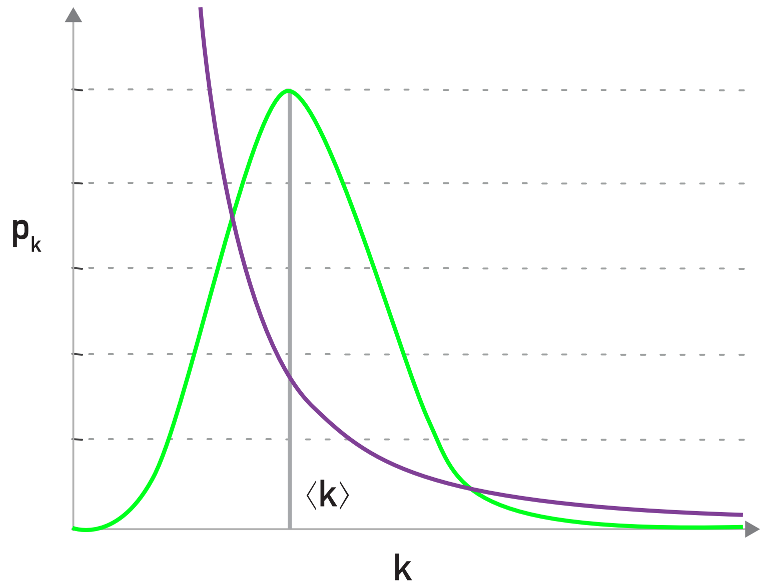
For a fixed λ this means all moments m>γ-1 diverge.

Network	Size	$\langle k \rangle$	κ	γ_{out}	γ_{in}
WWW	325 729	4.51	900	2.45	2.1
WWW	4×10^7	7		2.38	2.1
WWW	2×10^8	7.5	4000	2.72	2.1
WWW, site	260 000				1.94
Internet, domain*	3015–4389	3.42–3.76	30–40	2.1–2.2	2.1–2.2
Internet, router*	3888	2.57	30	2.48	2.48
Internet, router*	150 000	2.66	60	2.4	2.4
Movie actors*	212 250	28.78	900	2.3	2.3
Co-authors, SPIRES*	56 627	173	1100	1.2	1.2
Co-authors, neuro.*	209 293	11.54	400	2.1	2.1
Co-authors, math.*	70 975	3.9	120	2.5	2.5
Sexual contacts*	2810			3.4	3.4
Metabolic, *E. coli*	778	7.4	110	2.2	2.2
Protein, *S. cerev.**	1870	2.39		2.4	2.4
Ythan estuary*	134	8.7	35	1.05	1.05
Silwood Park*	154	4.75	27	1.13	1.13
Citation	783 339	8.57			3
Phone call	53×10^6	3.16		2.1	2.1
Words, co-occurrence*	460 902	70.13		2.7	2.7
Words, synonyms*	22 311	13.48		2.8	2.8

Many degree exponents are smaller than 3

→ $<k^2>$ diverges in the N→∞ limit!!!

The meaning of scale-free



Random Network

Randomly chosen node: $k = \langle k \rangle \pm \langle k \rangle^{1/2}$

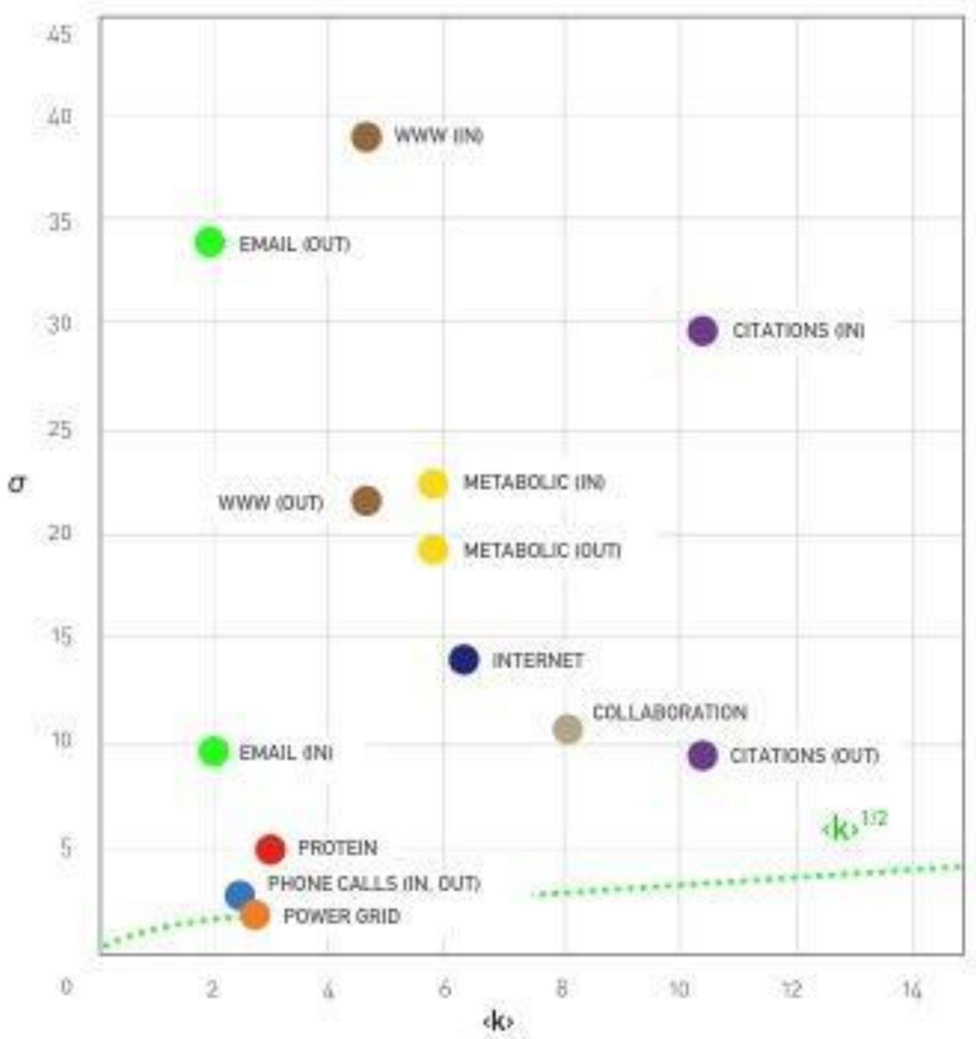
Scale: $\langle k \rangle$

Scale-Free Network

Randomly chosen node: $k = \langle k \rangle \pm \infty$

Scale: none

The meaning of scale-free

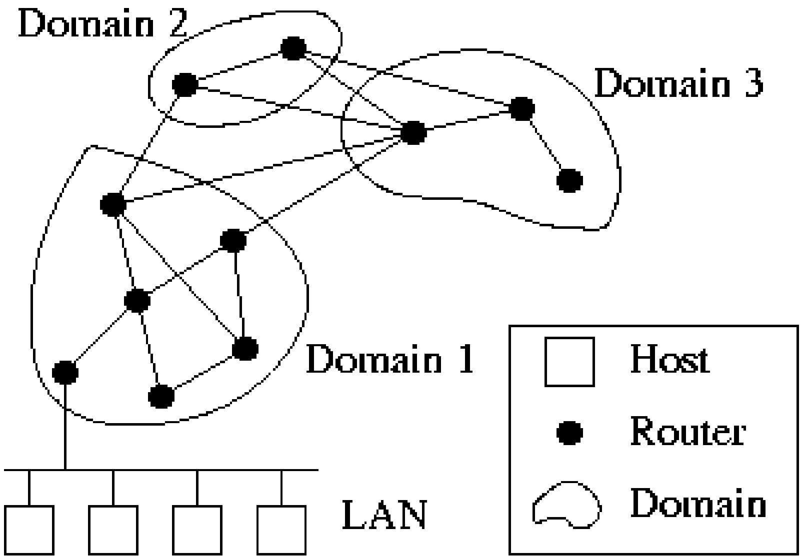


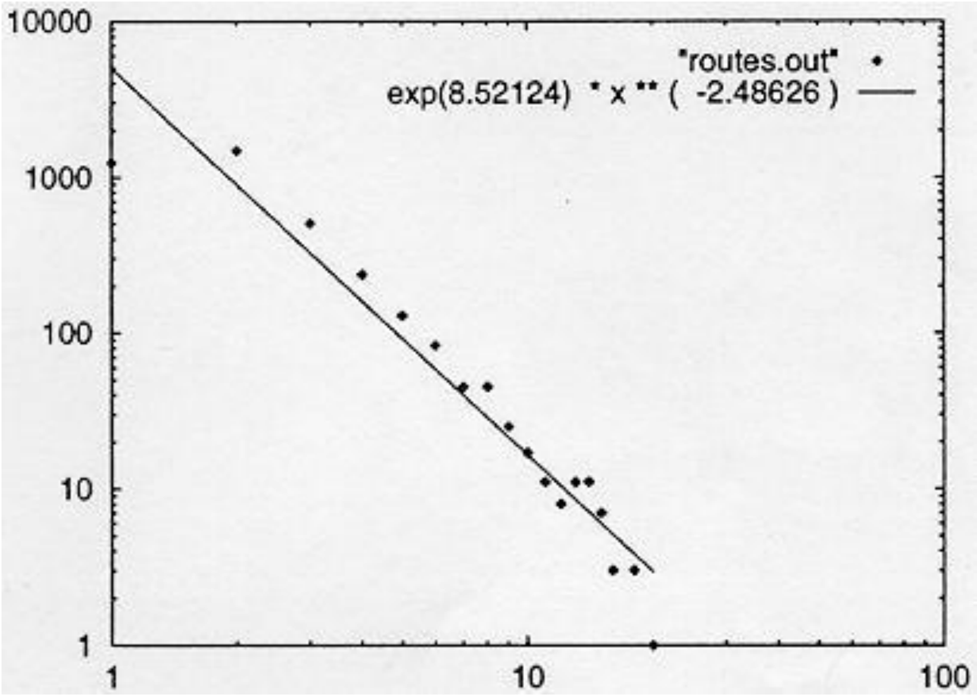
$$k = \langle k \rangle \pm \sigma_k$$

universality

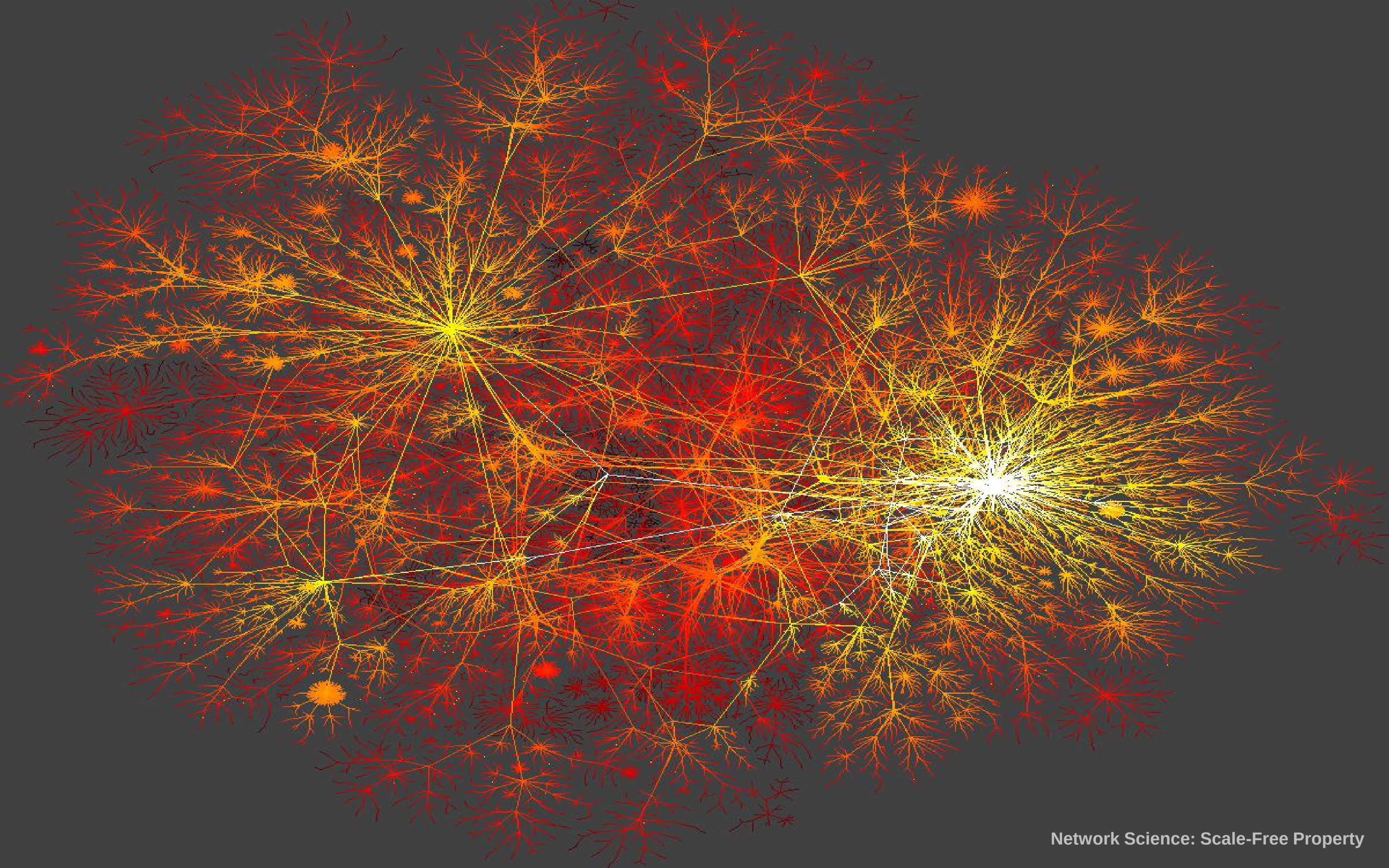
INTERNET BACKBONE

Nodes: computers, routers
Links: physical lines





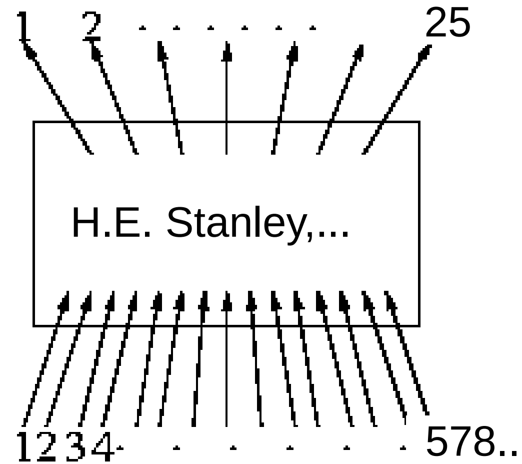
(Faloutsos, Faloutsos and Faloutsos, 1999)



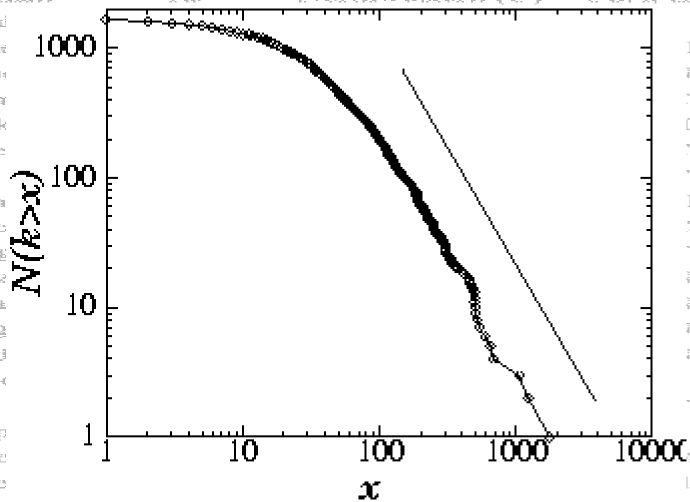
SCIENCE CITATION INDEX

Out of over 500,000 Examined
(see http://www.ssl.nrel.gov)

Nodes: papers
Links: citations



1736 PRL papers (1988)



$P(k) \sim k^{-\gamma}$

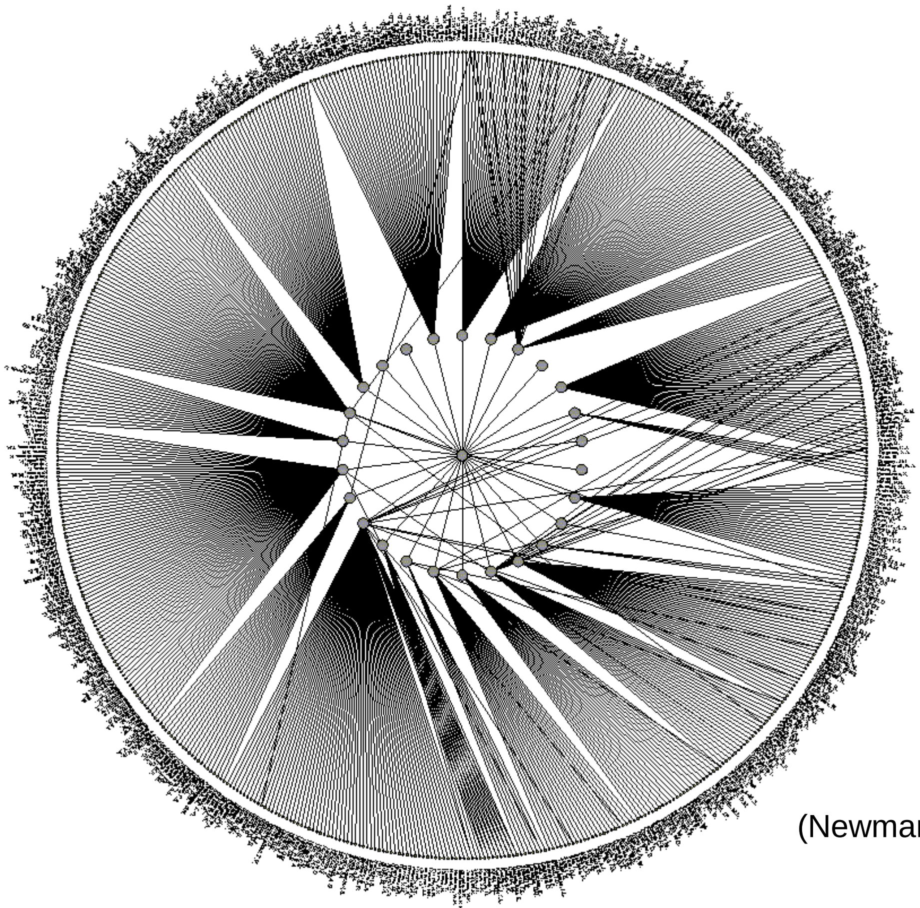
$(\gamma = 3)$

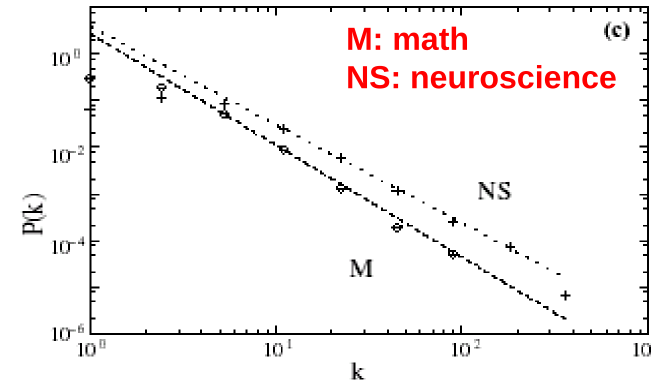
(S. Redner, 1998)

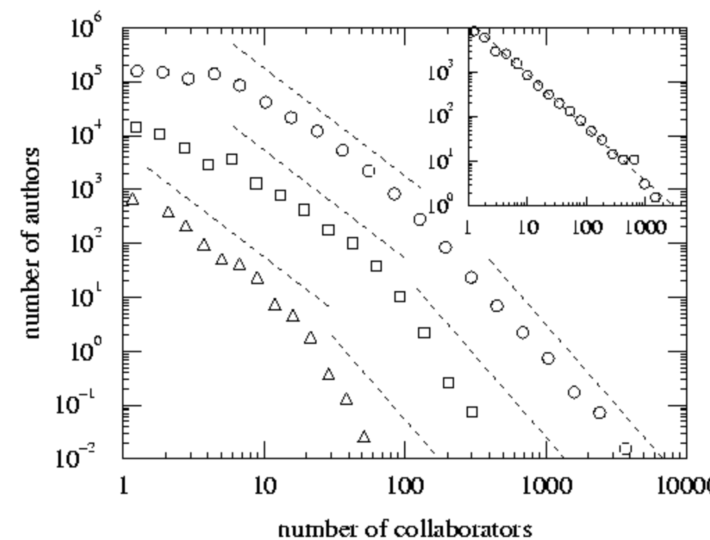
* citation total may be skewed because of multiple authors with the same name

SCIENCE COAUTHORSHIP

Nodes: scientist (authors)
Links: joint publication





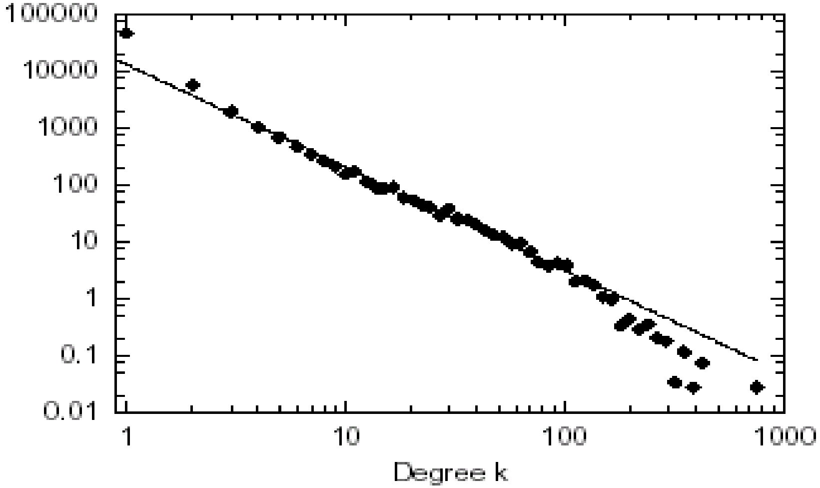


(Newman, 2000, Barabasi et al 2001)

ONLINE COMMUNITIES

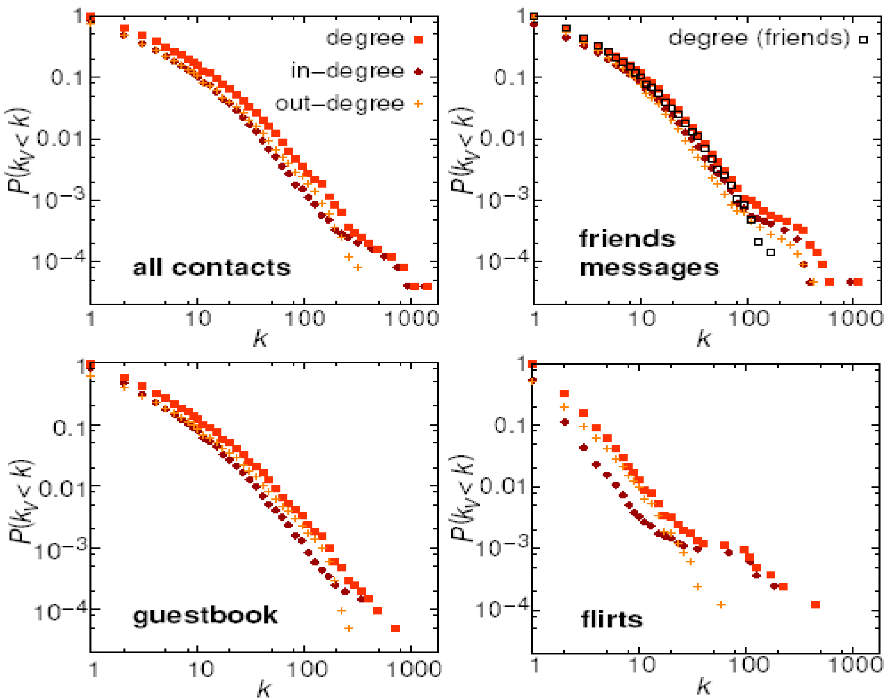
Nodes: online user
Links: email contact

Kiel University log files
112 days, N=59,912 nodes



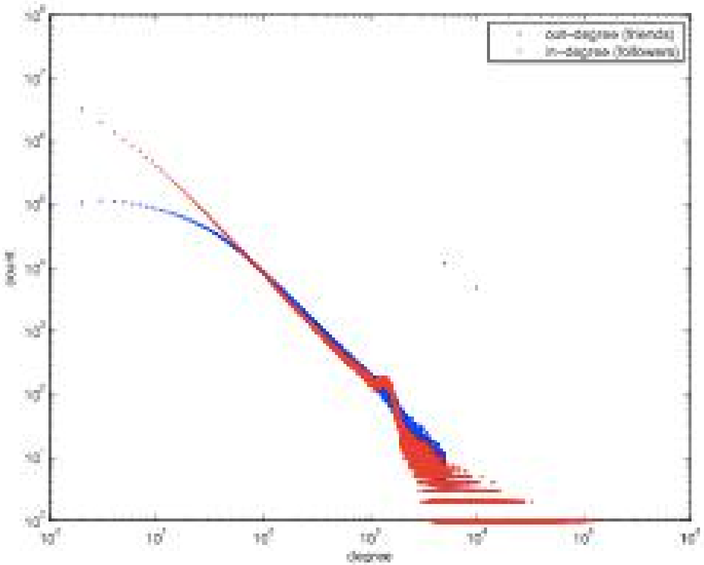
Ebel, Mielsch, Bornholdtz, PRE 2002.

Pussokram.com online community;
512 days, 25,000 users.

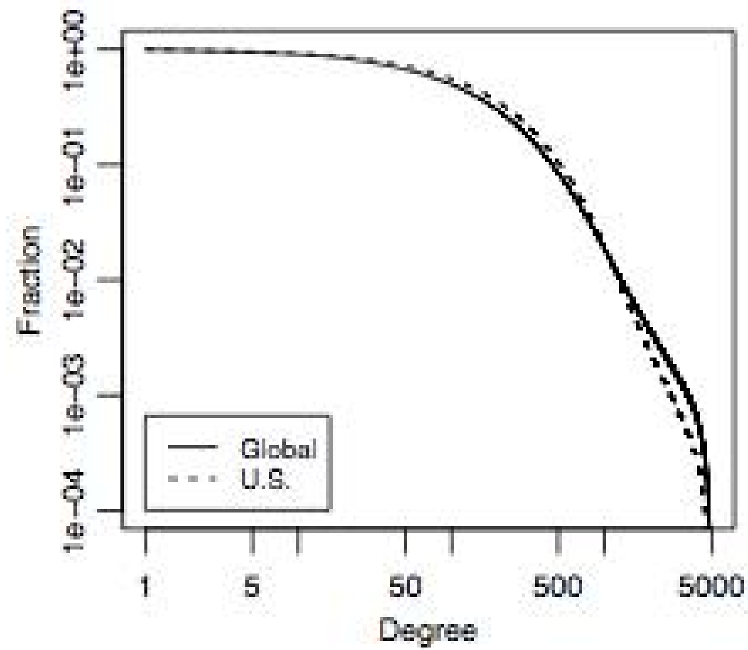


Holme, Edling, Liljeros, 2002.

Twitter:

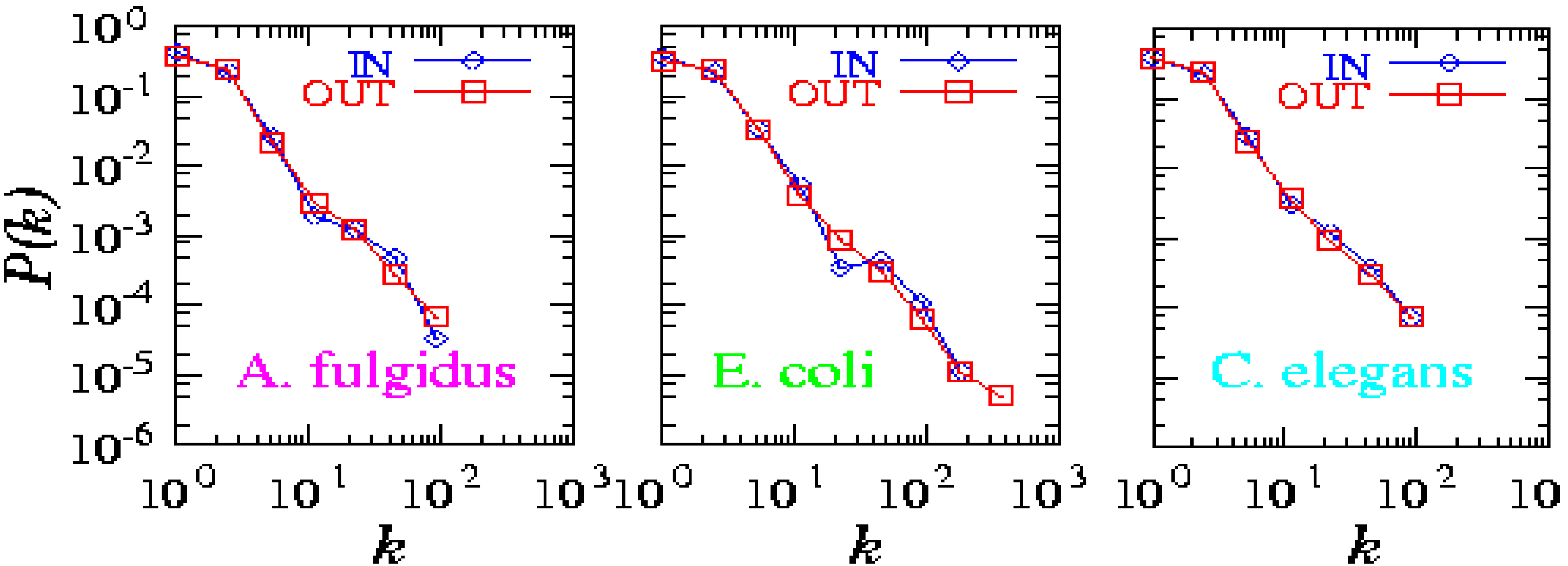


Facebook



Brian Karrer, Lars Backstrom, Cameron Marlowm 2011

METABOLIC NETWORK



Archaea

Bacteria

Eukaryotes

Organisms from all three domains of life are scale-free!

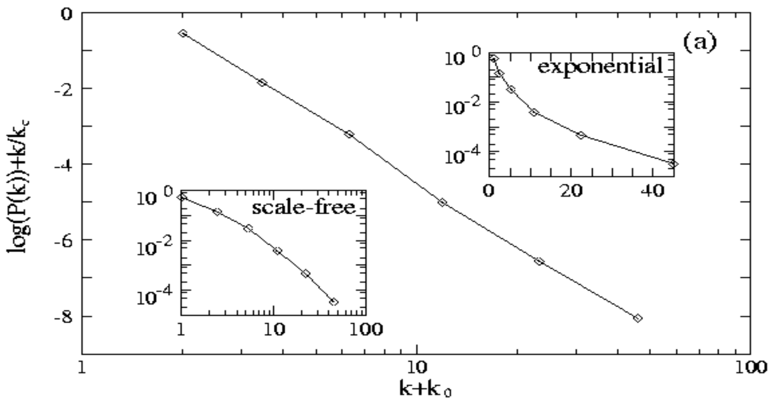
$$P_{in}(k) \approx k^{-2.2}$$

$$P_{out}(k) \approx k^{-2.2}$$

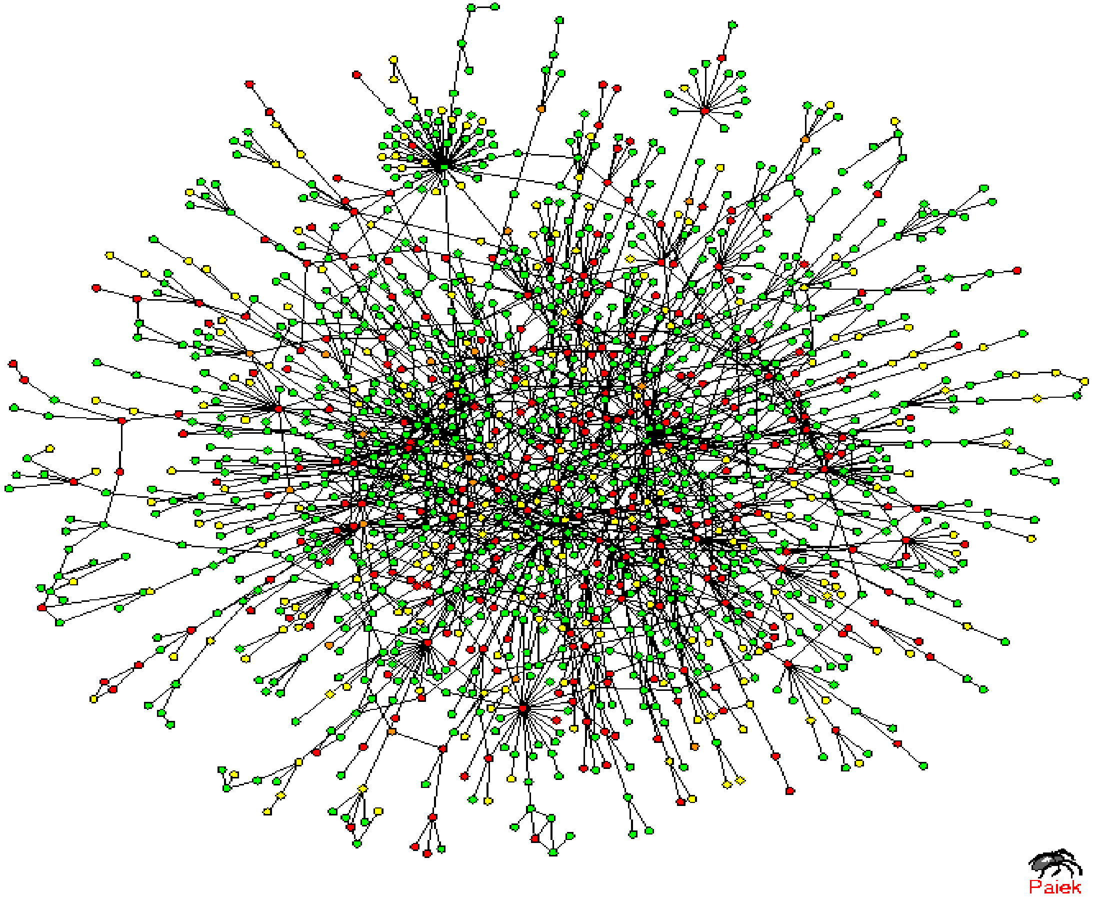
H. Jeong, B. Tombor, R. Albert, Z.N. Oltvai, and A.L. Barabasi, *Nature*, 407 651 (2000)

TOPOLOGY OF THE PROTEIN NETWORK

Nodes: proteins
Links: physical interactions-binding

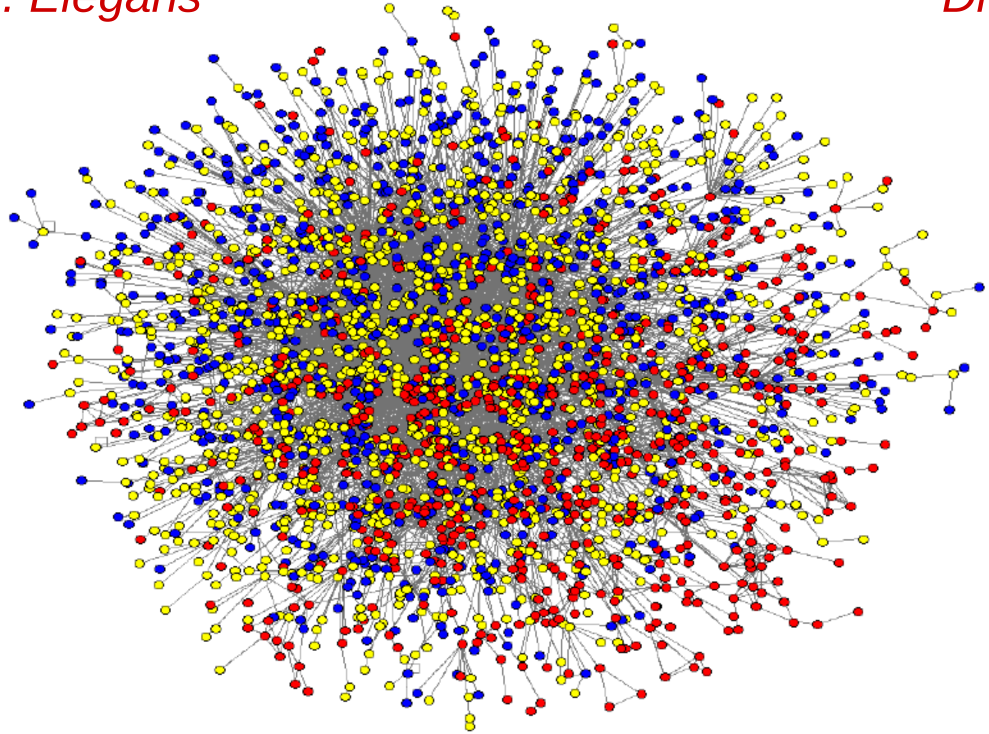


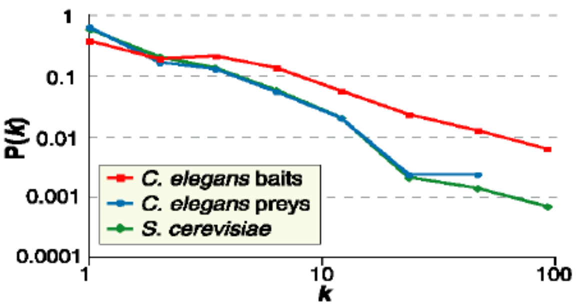
$$P(k) \sim (k+k_0)^{-\gamma} \exp\left(-\frac{k+k_0}{k_\tau}\right)$$



H. Jeong, S.P. Mason, A.-L. Barabasi, Z.N. Oltvai, Nature 411, 41-42 (2001)

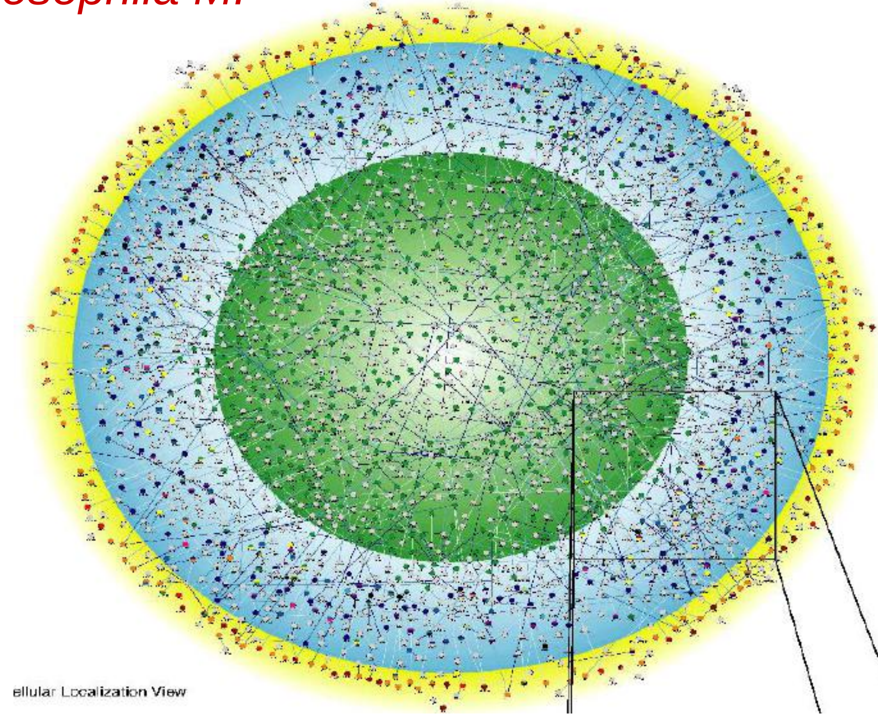
C. Elegans

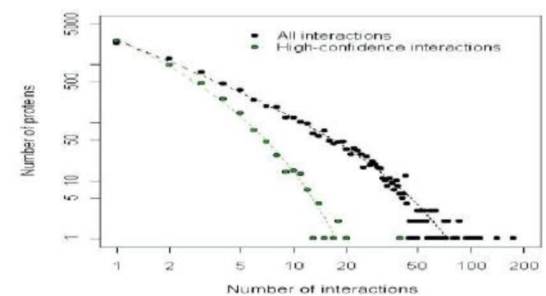




Li *et al.* Science 2004

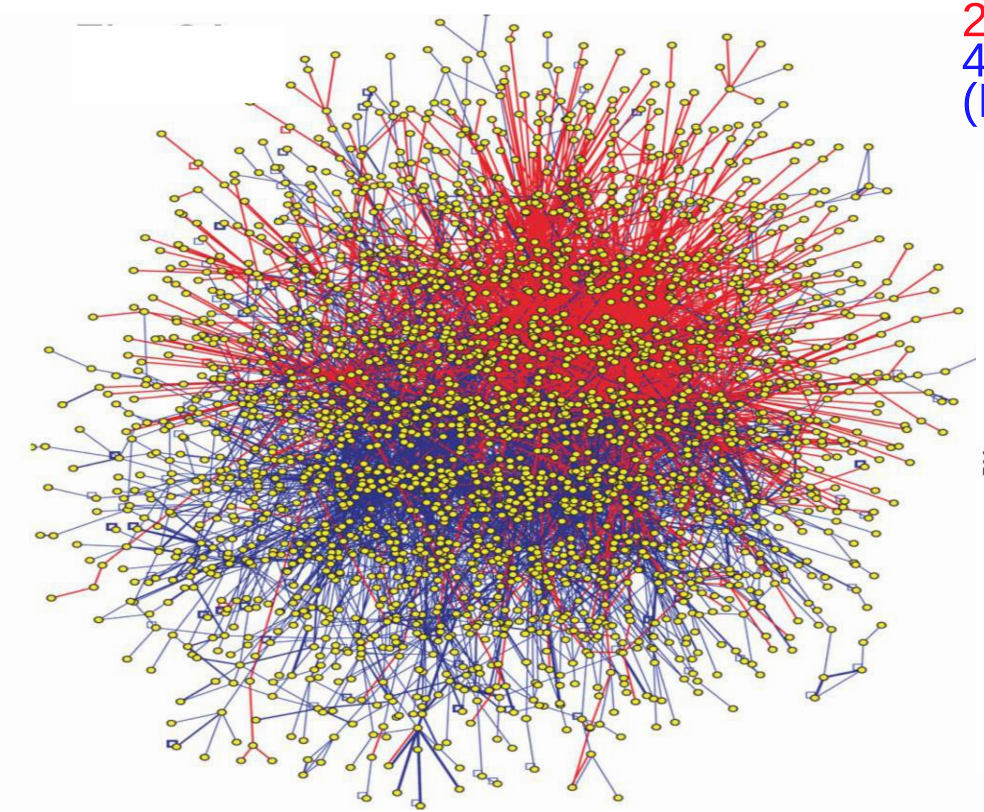
Drosophila M.



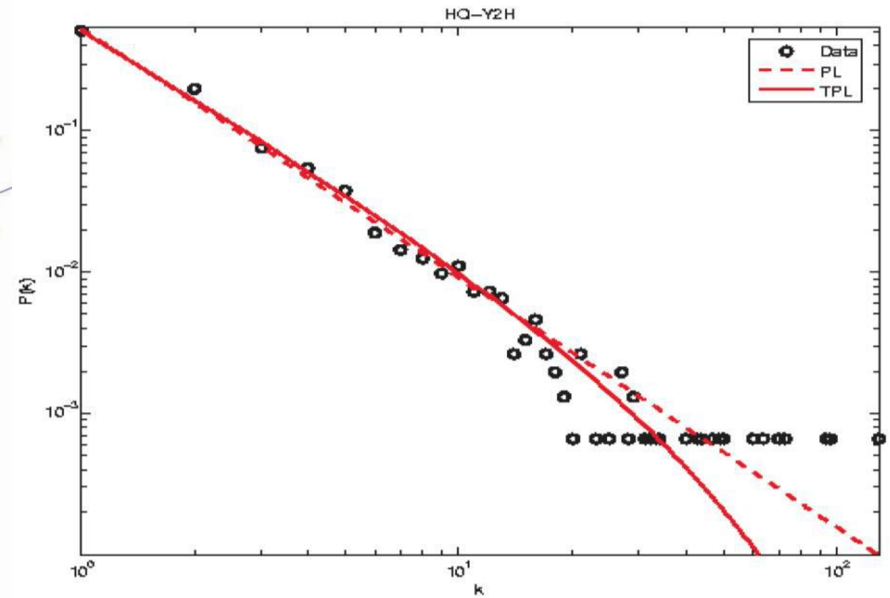


Giot *et al.* Science 2003

HUMAN PROTEIN INTERACTION NETWORK



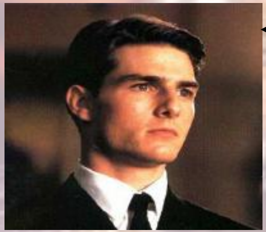
2,800 Y2H interactions
4,100 binary LC interactions
(HPRD, MINT, BIND, DIP, MIPS)



Rual *et al.* Nature 2005; Stelze *et al.* Cell 2005

ACTOR NETWORK

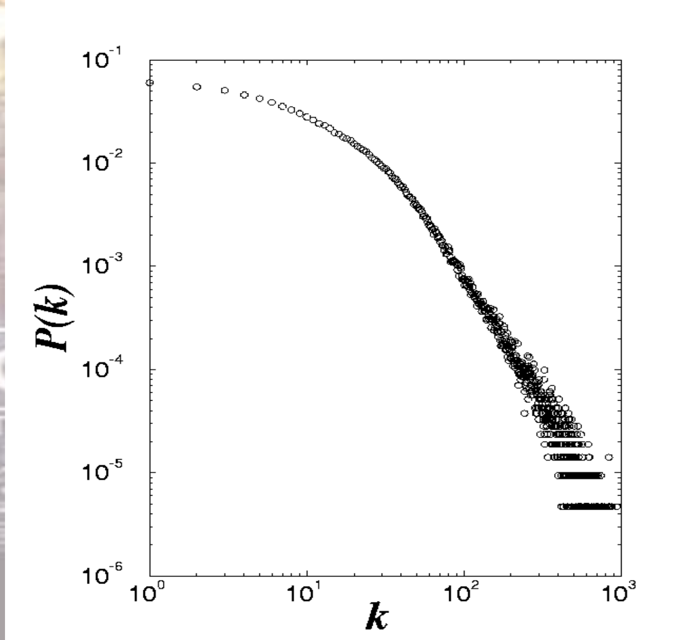
Nodes: actors
Links: cast jointly



Days of Thunder *(1990)*
Far and Away *(1992)*
Eyes Wide Shut *(1999)*

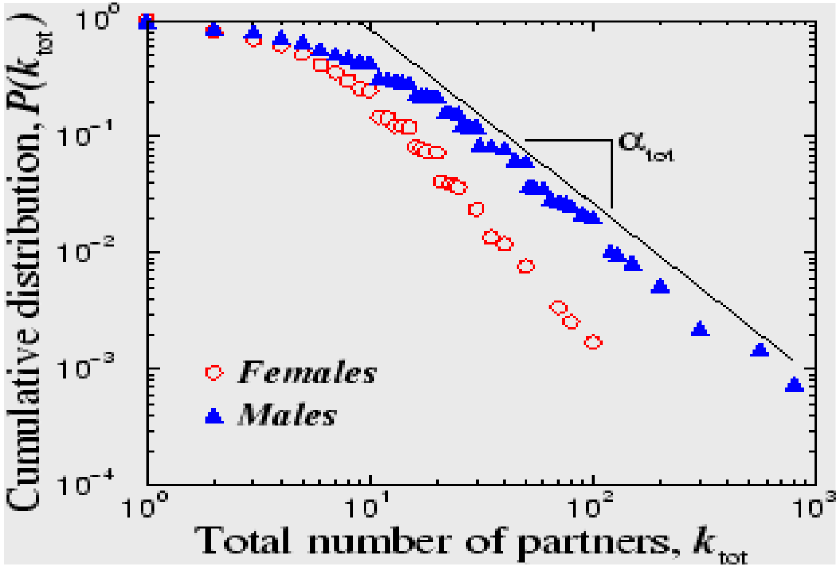
N = 212,250 actors
$\langle k \rangle = 28.78$

$P(k) \sim k^{-\gamma}$
$\gamma = 2.3$



SWEDISH SEX-WEB

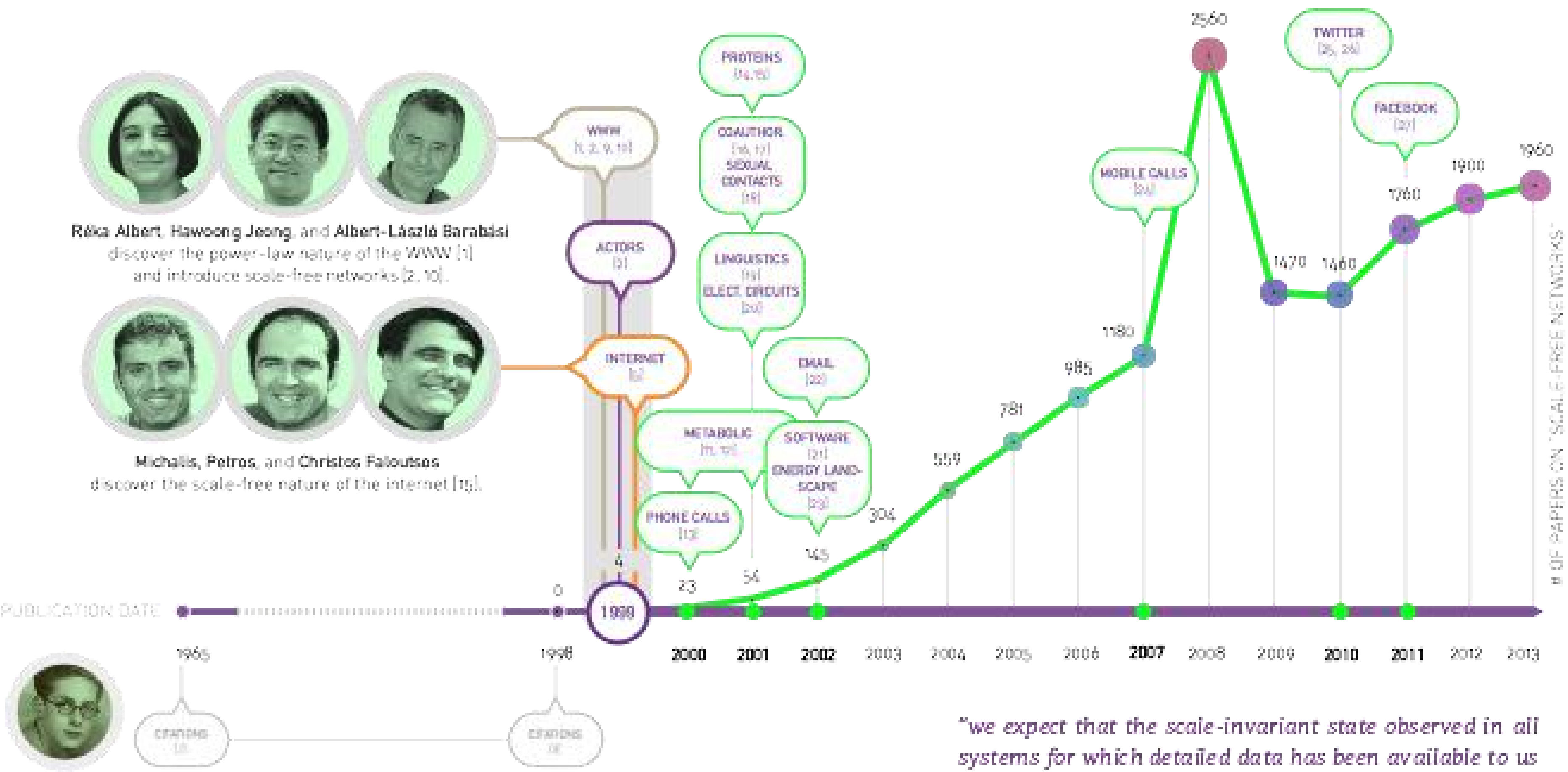
Nodes: people (Females; Males)
Links: sexual relationships



4781 Swedes; 18-74;
59% response rate.

Liljeros et al. Nature 2001

TIMELINE: SCALE-FREE NETWORKS

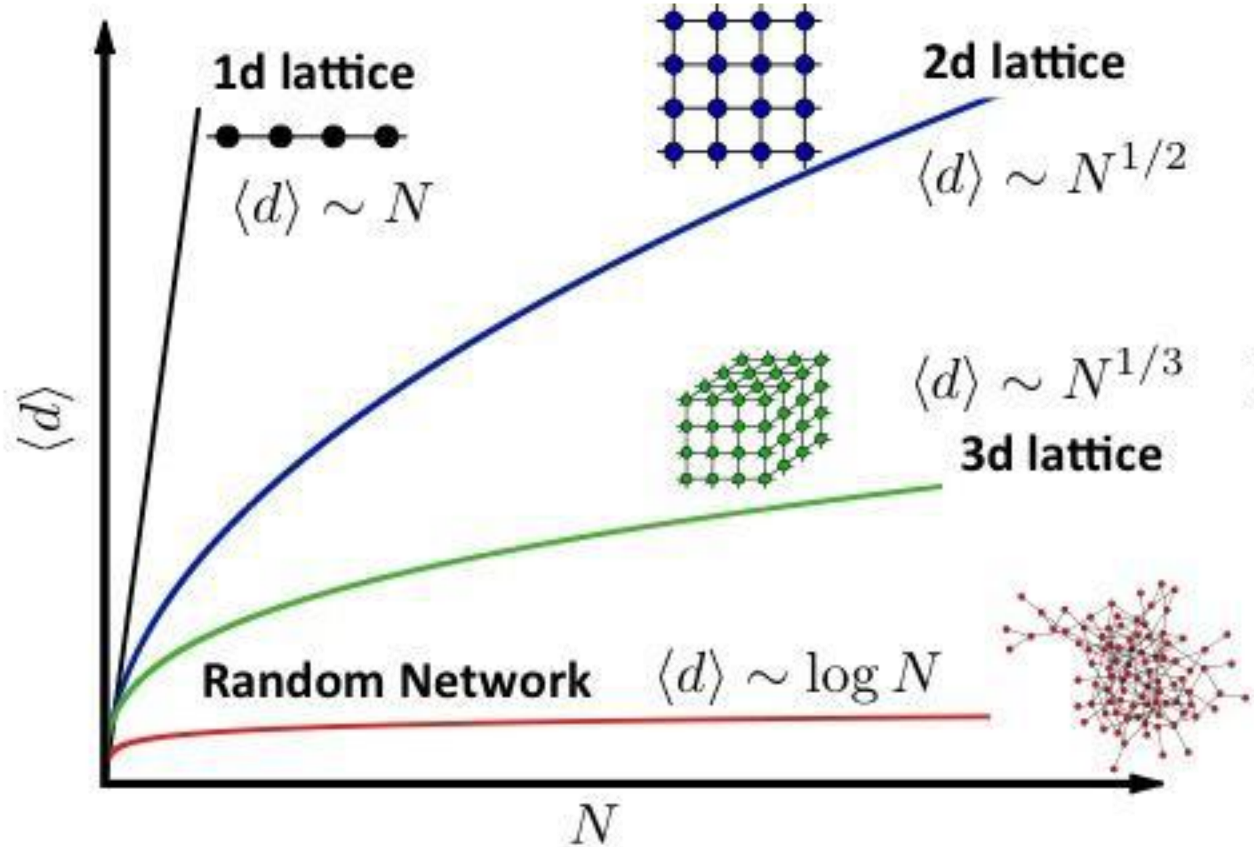


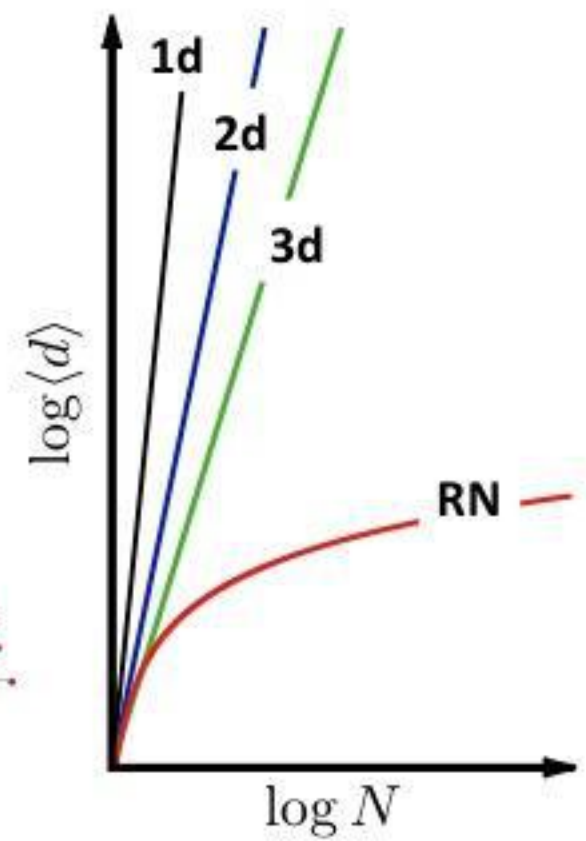
Derek de Solla Price (1922 - 1983)
discovers that citations follow a power-law distribution [7], a finding later attributed to the scale-free nature of the citation network [2].

"we expect that the scale-invariant state observed in all systems for which detailed data has been available to us is a generic property of many complex networks, with applicability reaching far beyond the quoted examples."

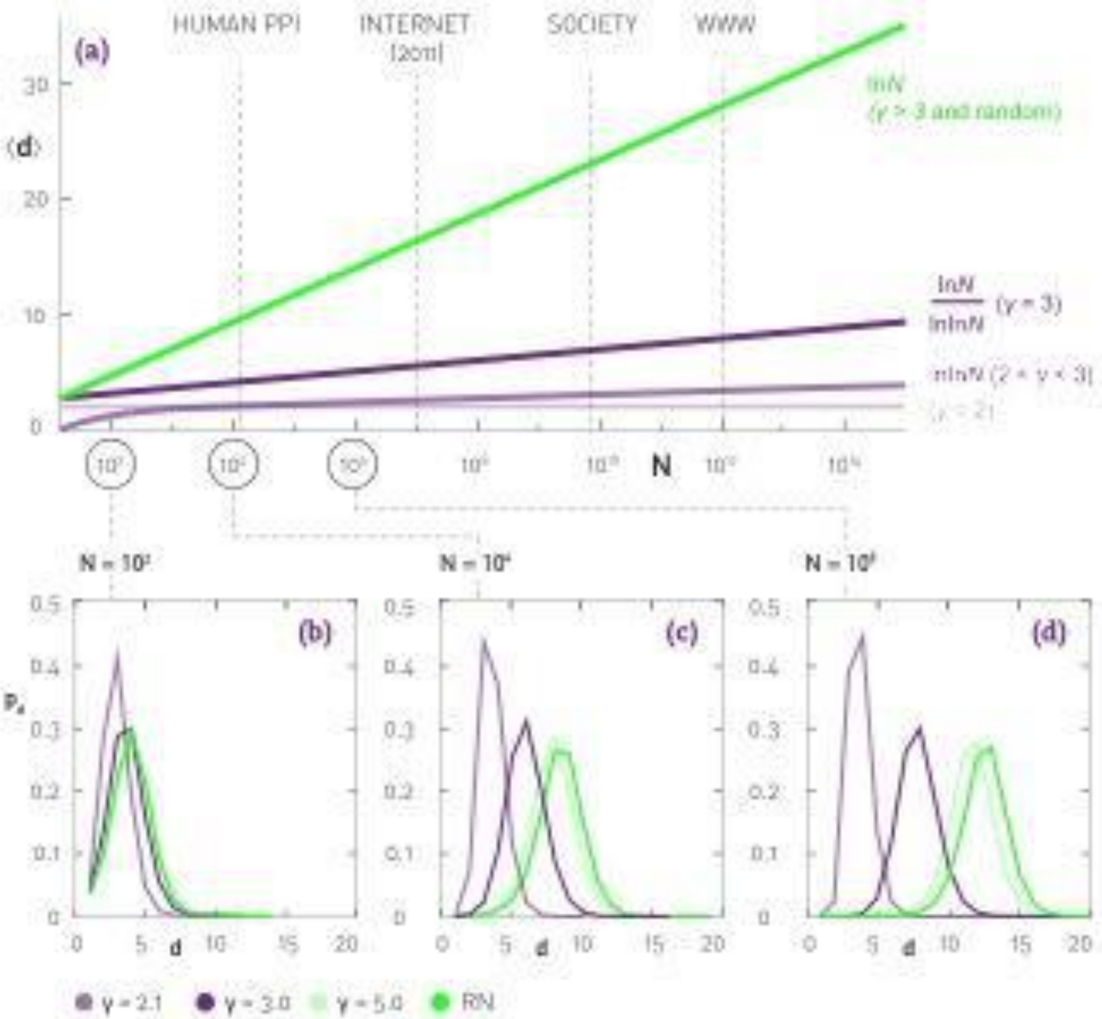
Barabási and Albert, 1999

Why are small worlds surprising? Suprising compared to what?





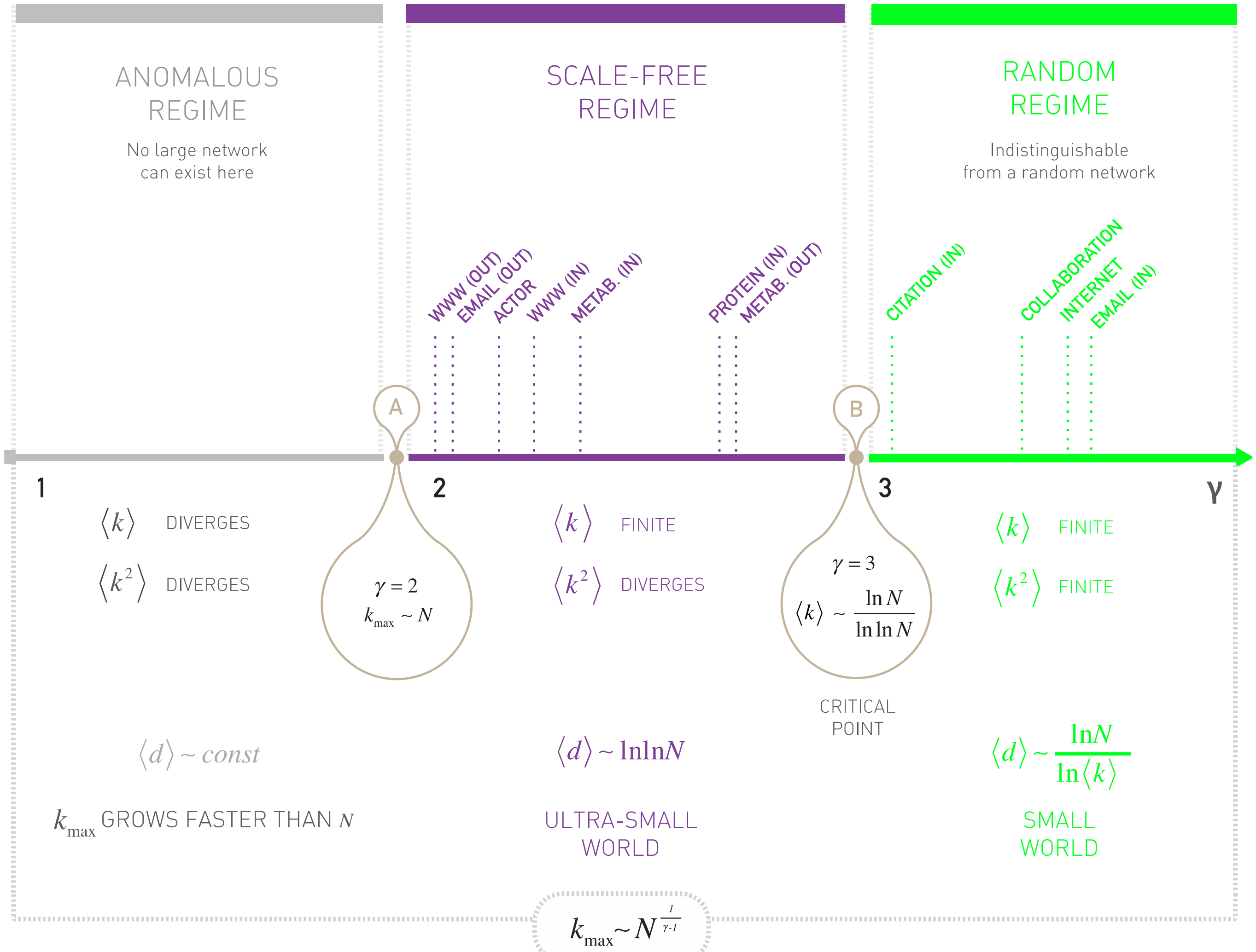
SMALL WORLD BEHAVIOR IN SCALE-FREE NETWORKS



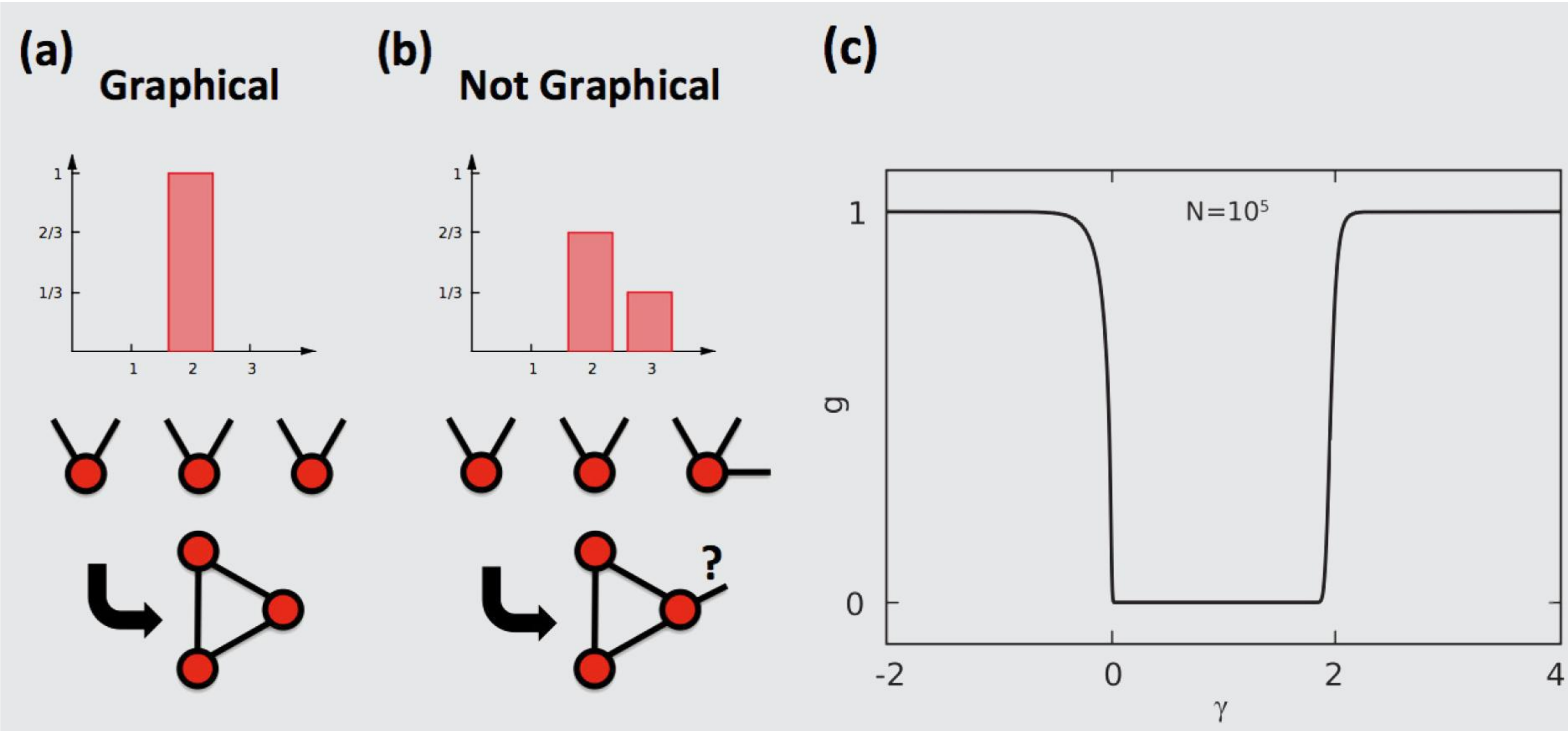
$$
\langle d \rangle \sim \begin{cases} \text{const.} & \gamma = 2, \\ \dfrac{\ln \ln N}{\ln(\gamma - 1)} & 2 < \gamma < 3, \\ \dfrac{\ln N}{\ln \ln N} & \gamma = 3, \\ \ln N & \gamma > 3. \end{cases}
$$

The role of the degree exponent

SUMMARY OF THE BEHAVIOR OF SCALE-FREE NETWORKS



Graphicality: No large networks for γ<2



In scale-free networks: $k_{\max} = k_{\min} N^{\frac{1}{g-1}}$ For γ<2: 1/(γ-2)>1

Why don’t we see networks with exponents in the range of γ=4,5,6, etc?

In order to document a scale-free networks, we need 2-3 orders of magnitude scaling. That is, $K_{max} \sim 10^3$

However, that constrains on the system size we require to document it.

For example, to measure an exponent *γ=5*,we need to maximum degree a system size of the order of—

$$K_{\max} = K_{\min} N^{\frac{1}{\gamma - 1}}$$

$$N = \left(\frac{K_{\max}}{K_{\min}} \right)^{\gamma - 1} \approx 10^8$$

Mobile Call Network

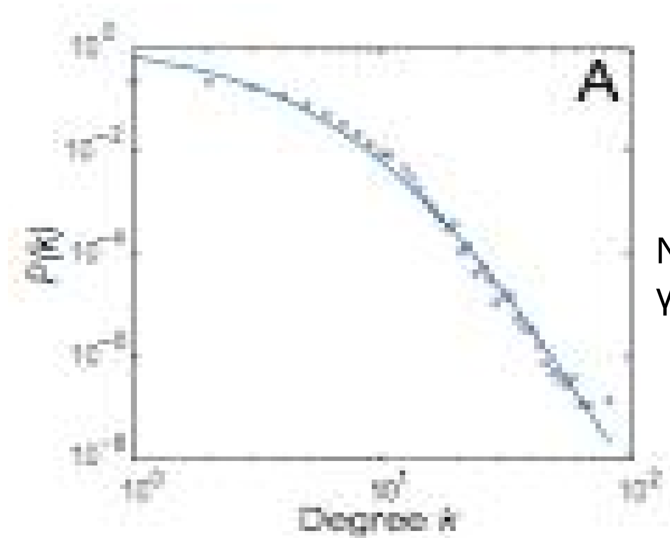


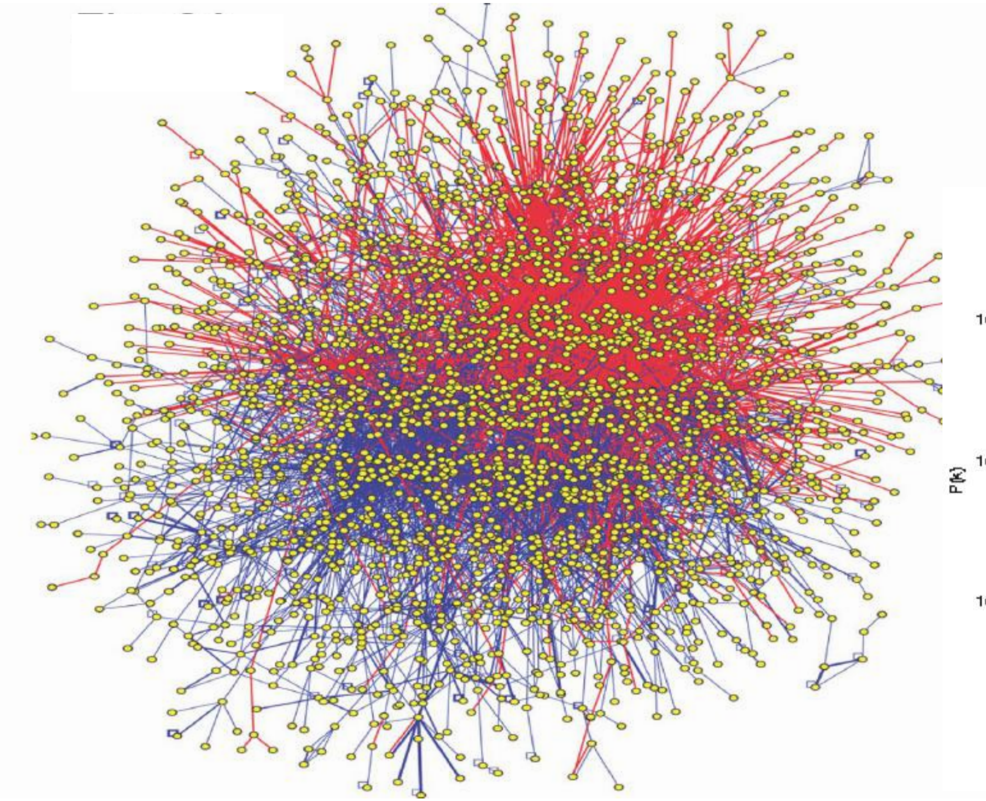
Fig. 1. Characterizing the large-scale structure and the tie strengths of the mobile call graph. (A and B) Vertex degree (A) and tie strength distribution (B). Each distribution was fitted with $P(x) = a(x + x_0)^{-\gamma_x} \exp(-x/x_c)$, shown as a blue curve, where x corresponds to either k or w. The parameter values for the fits are $k_0 = 10.9$, $\gamma_k = 8.4$, $k_c = \infty$ (A, degree), and $w_0 = 280$, $\gamma_w = 1.9$, $w_c = 3.45 \times$

$N=4.6 \times 10^6$
$\gamma=8.4$

Onella et al. PNAS 2007

PLOTTING POWER LAWS

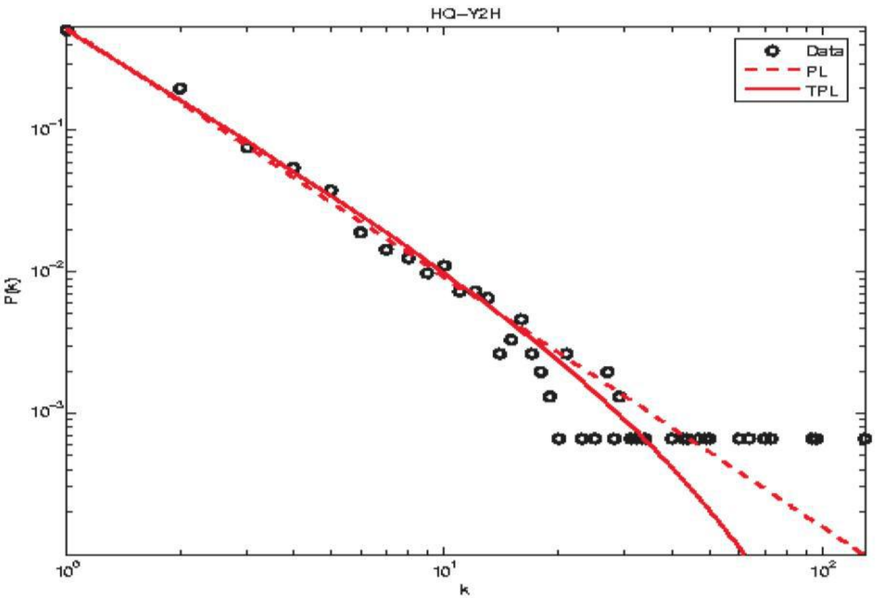
HUMAN INTERACTION NETWORK



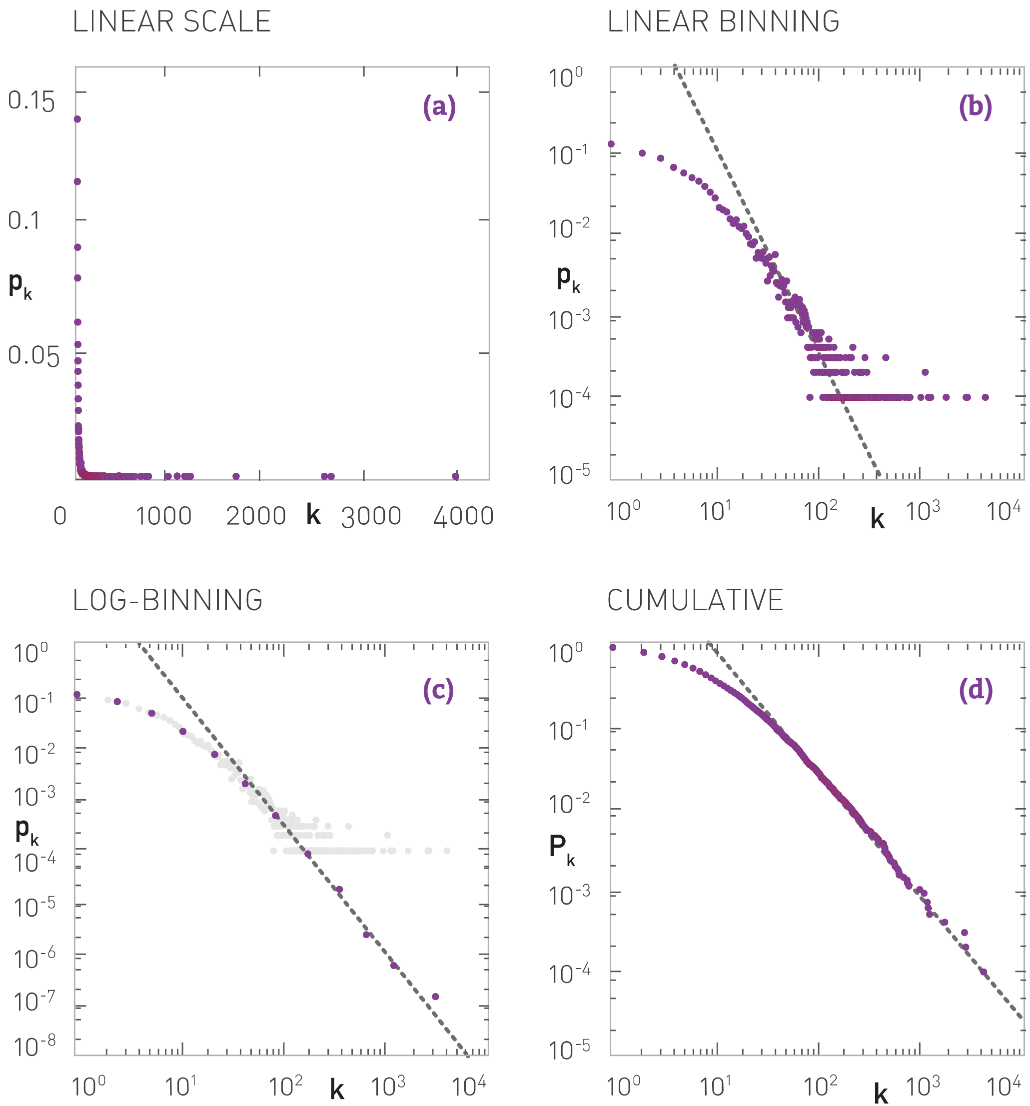
2,800 Y2H interactions

4,100 binary LC interactions

(HPRD, MINT, BIND, DIP, MIPS)



Rual et al. Nature 2005; Stelze et al. Cell 2005



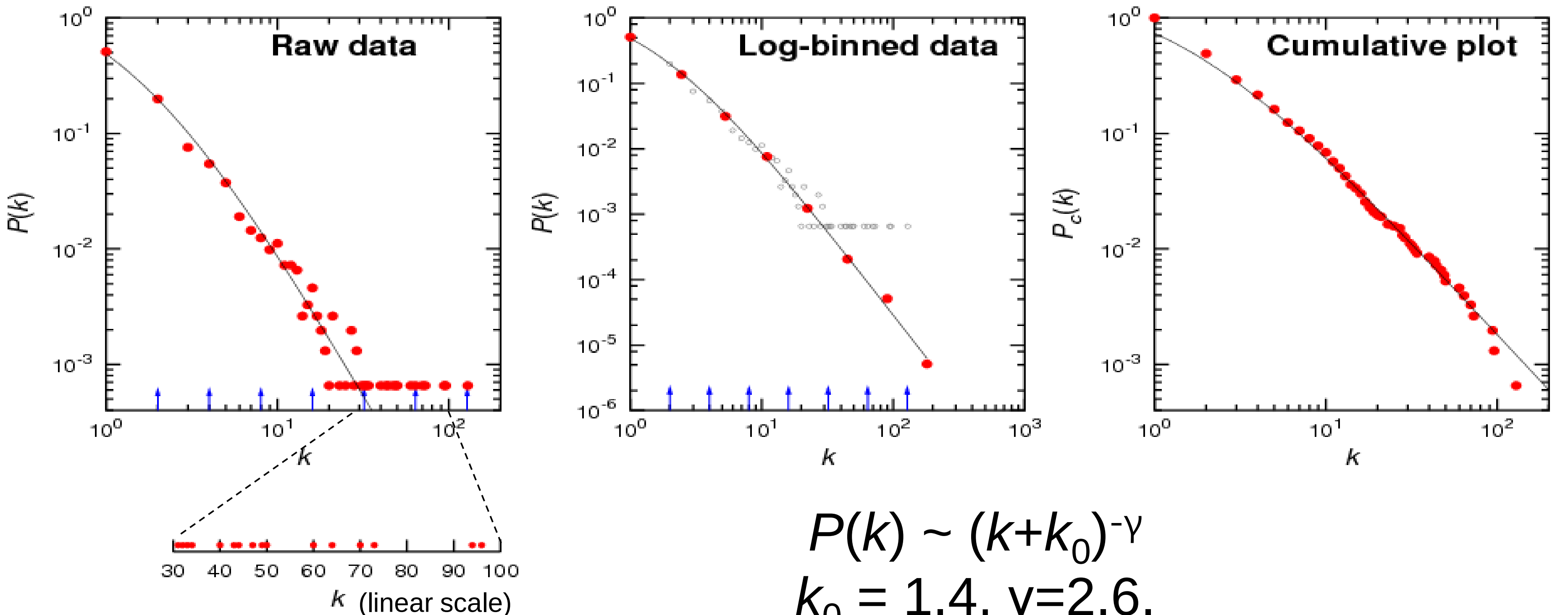
Use a Log-Log Plot

Avoid Linear Binning

Use Logarithmic Binning

Use Cumulative Distribution

HUMAN INTERACTION DATA BY RUAL ET AL.



$$P(k) \sim (k+k_0)^{-\gamma}$$
$$k_0 = 1.4,\ \gamma=2.6.$$

WHY SCALE FREE ARCHITECTURE

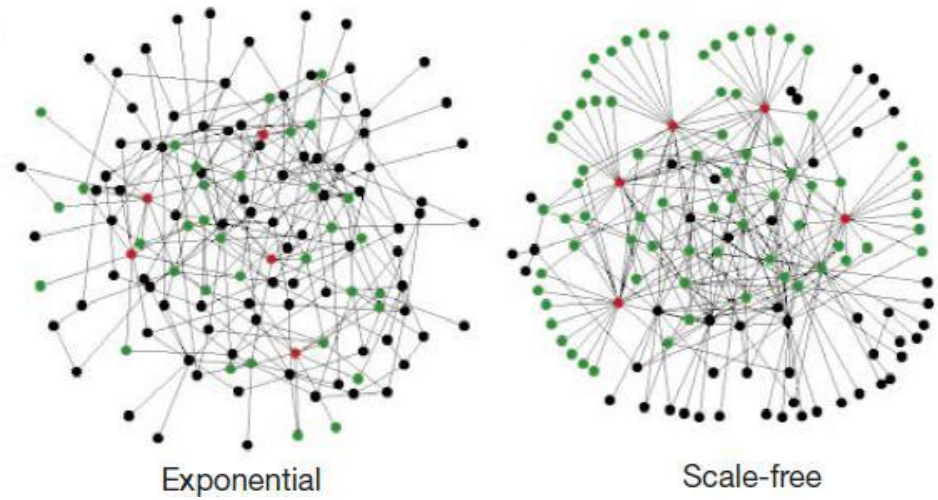
(Also See Section 8.3.3.)

Error and attack tolerance of complex networks

Réka Albert, Hawoong Jeong & Albert-László Barabási

Department of Physics, 225 Nieuwland Science Hall, University of Notre Dame, Notre Dame, Indiana 46556, USA

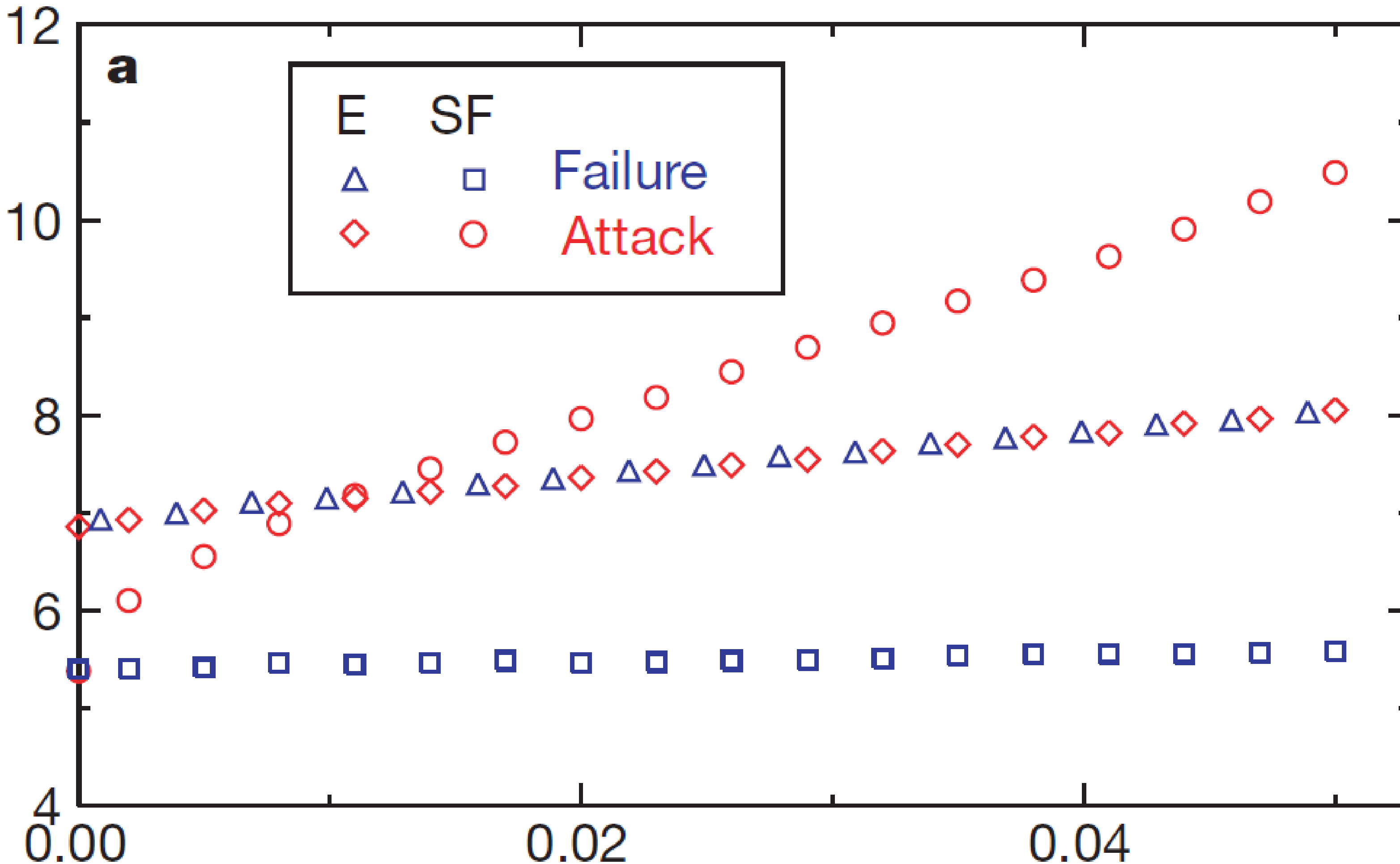
Many complex systems display a surprising degree of tolerance against errors. For example, relatively simple organisms grow, persist and reproduce despite drastic pharmaceutical or environmental interventions, an error tolerance attributed to the robustness of the underlying metabolic network[1]. Complex communication networks[2] display a surprising degree of robustness: although key components regularly malfunction, local failures rarely lead to the loss of the global information-carrying ability of the network. The stability of these and other complex systems is often attributed to the redundant wiring of the functional web defined by the systems' components. Here we demonstrate that error tolerance is not shared by all redundant systems: it is displayed only by a class of inhomogeneously wired networks,



Random Network Scale free network

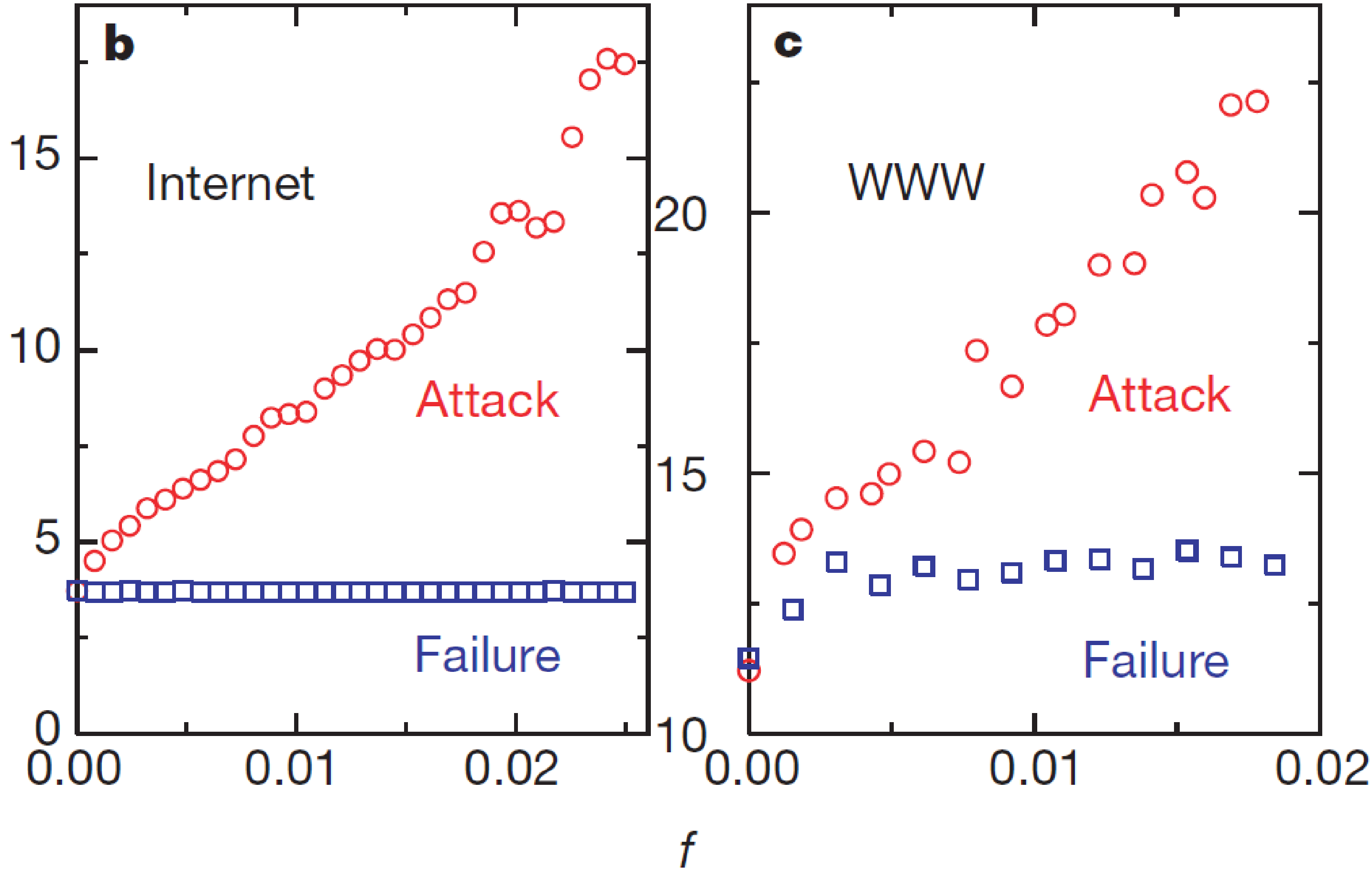
Response of Random and Scale free networks to:
- Random errors &
- Targeted attacks.

Albert et al, “Error and attack tolerance of complex networks”, 406, Nature, 378-382 (2000).



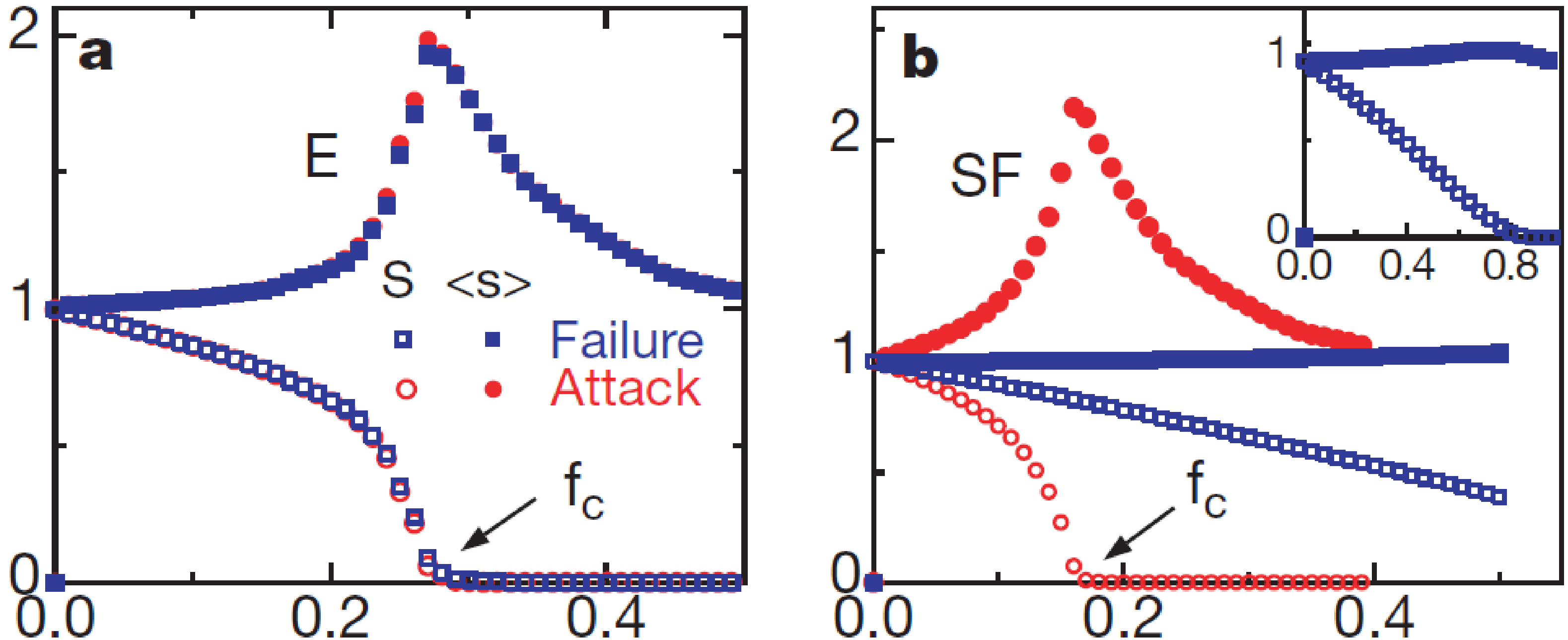
Changes in diameter (Characteristic Path Length) of the network as a function of fraction of nodes removed. (a) Comparison between exponential (E) and scale-free (SF) network models. N = 10,000 and E = 20,000

Albert et al, "Error and attack tolerance of complex networks", 406, Nature, 378-382 (2000).



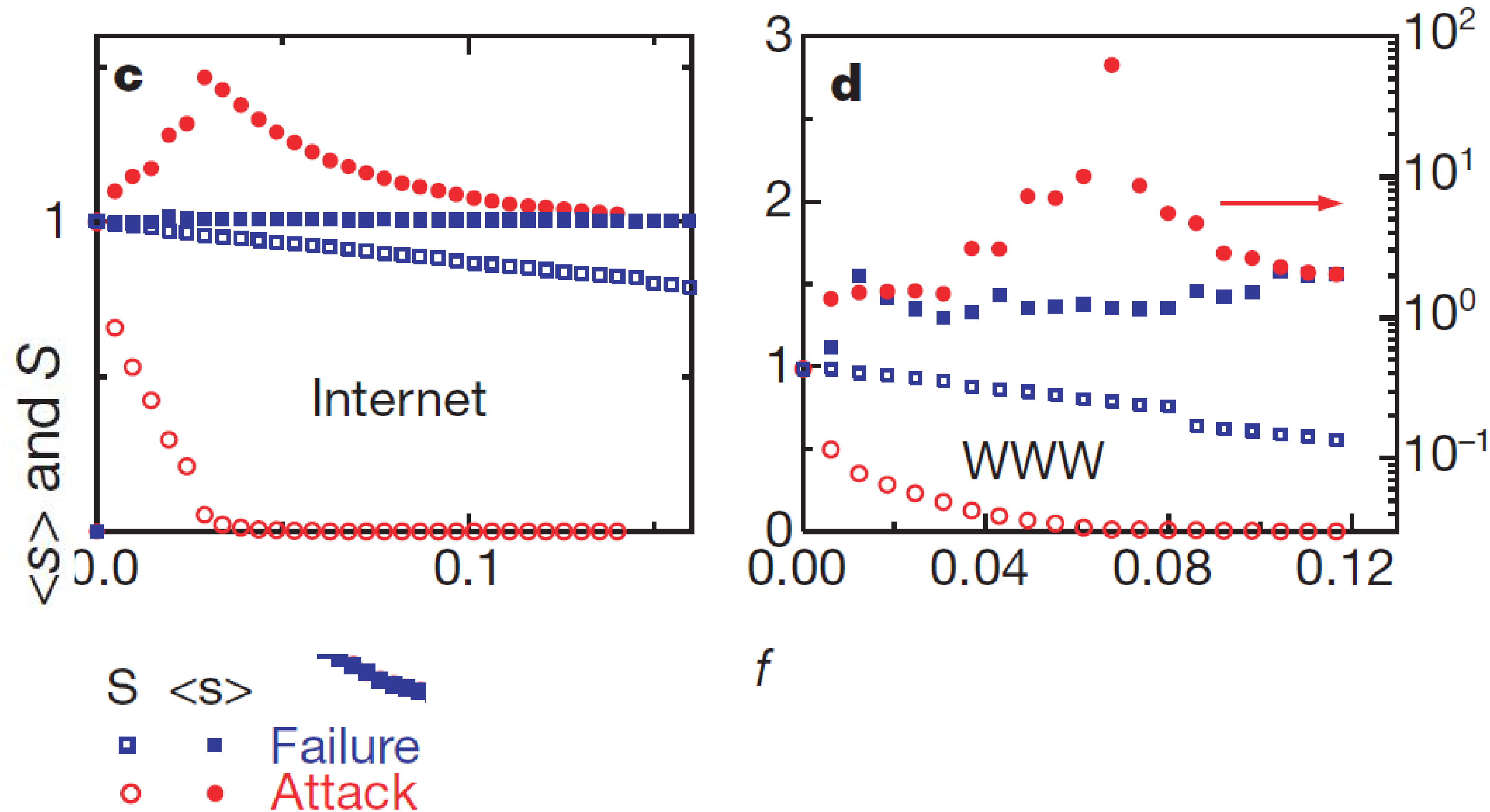
Changes in diameter (Characteristic Path Length) of the network as a function of fraction of nodes removed. (b) Internet, containing 6,209 nodes and 12,200 links. (c) World-Wide Web, containing 325,729 nodes and 1,498,353 links.

Albert et al, "Error and attack tolerance of complex networks", 406, Nature, 378-382 (2000).



Network fragmentation under random failures and attacks. The relative size of the largest cluster S (open symbols) and the average size of the isolated clusters <s> (filled symbols) as a function of the fraction of removed nodes f for the same systems.

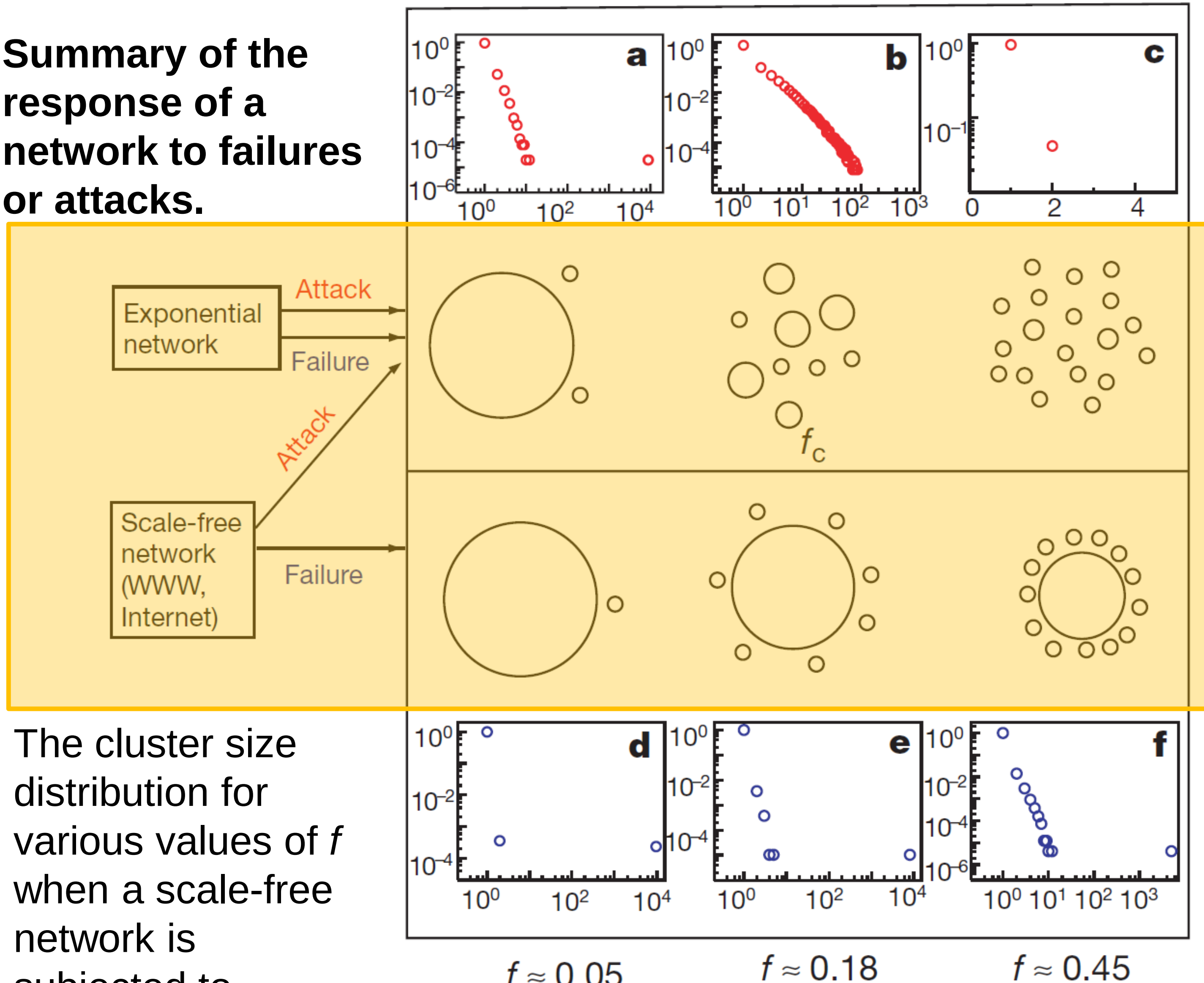
Albert et al, “Error and attack tolerance of complex networks”, 406, Nature, 378-382 (2000).



Network fragmentation under random failures and attacks. The relative size of the largest cluster S (open symbols) and the average size of the isolated clusters <s> (filled symbols) as a function of the fraction of removed nodes f for the same systems.

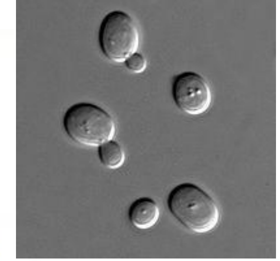
Albert et al, "Error and attack tolerance of complex networks", 406, Nature, 378-382 (2000).

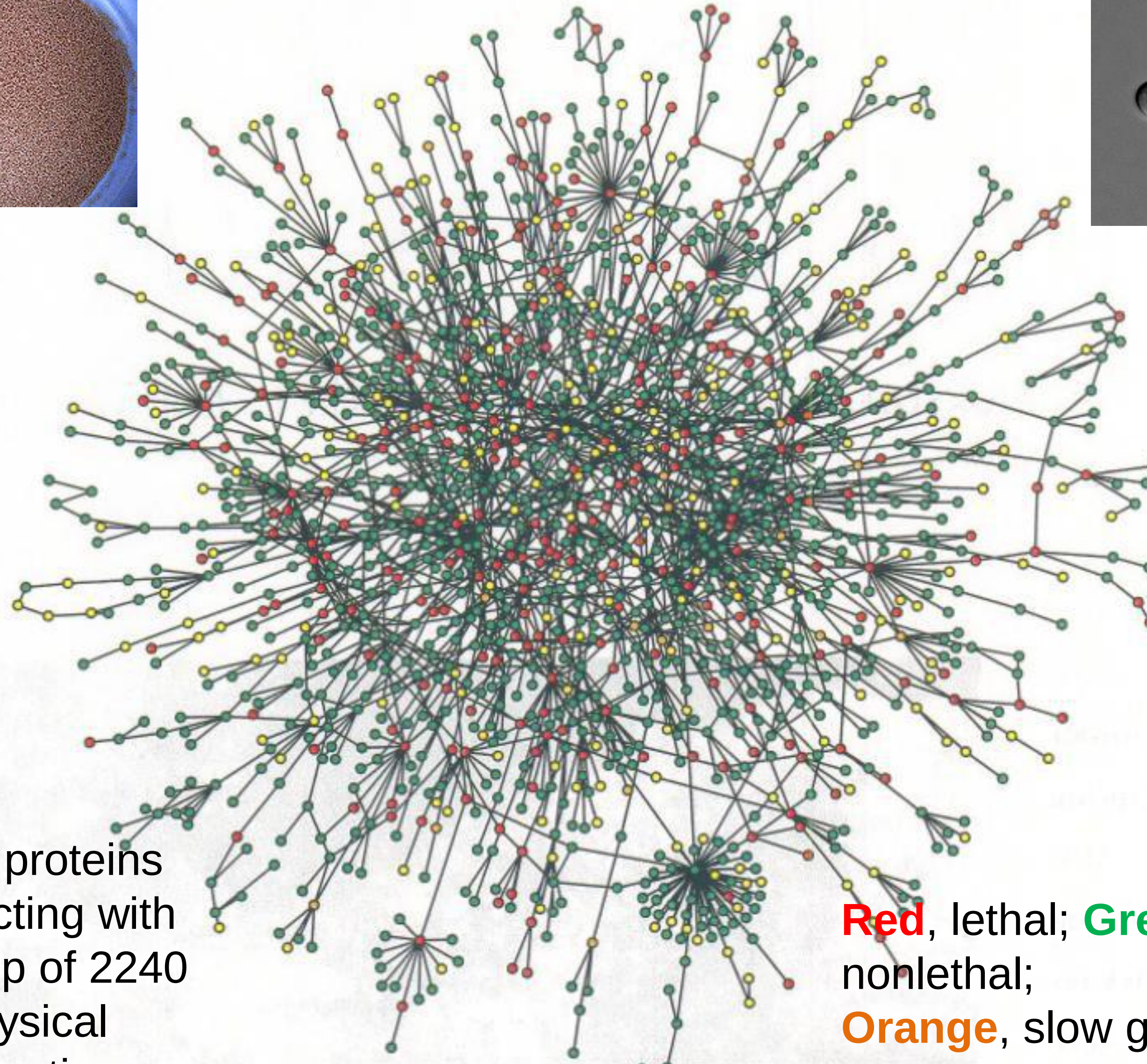
Summary of the response of a network to failures or attacks.



The cluster size distribution for various values of *f* when a scale-free network is subjected to

Albert et al, “Error and attack tolerance of complex networks”, 406, Nature, 378-382 (2



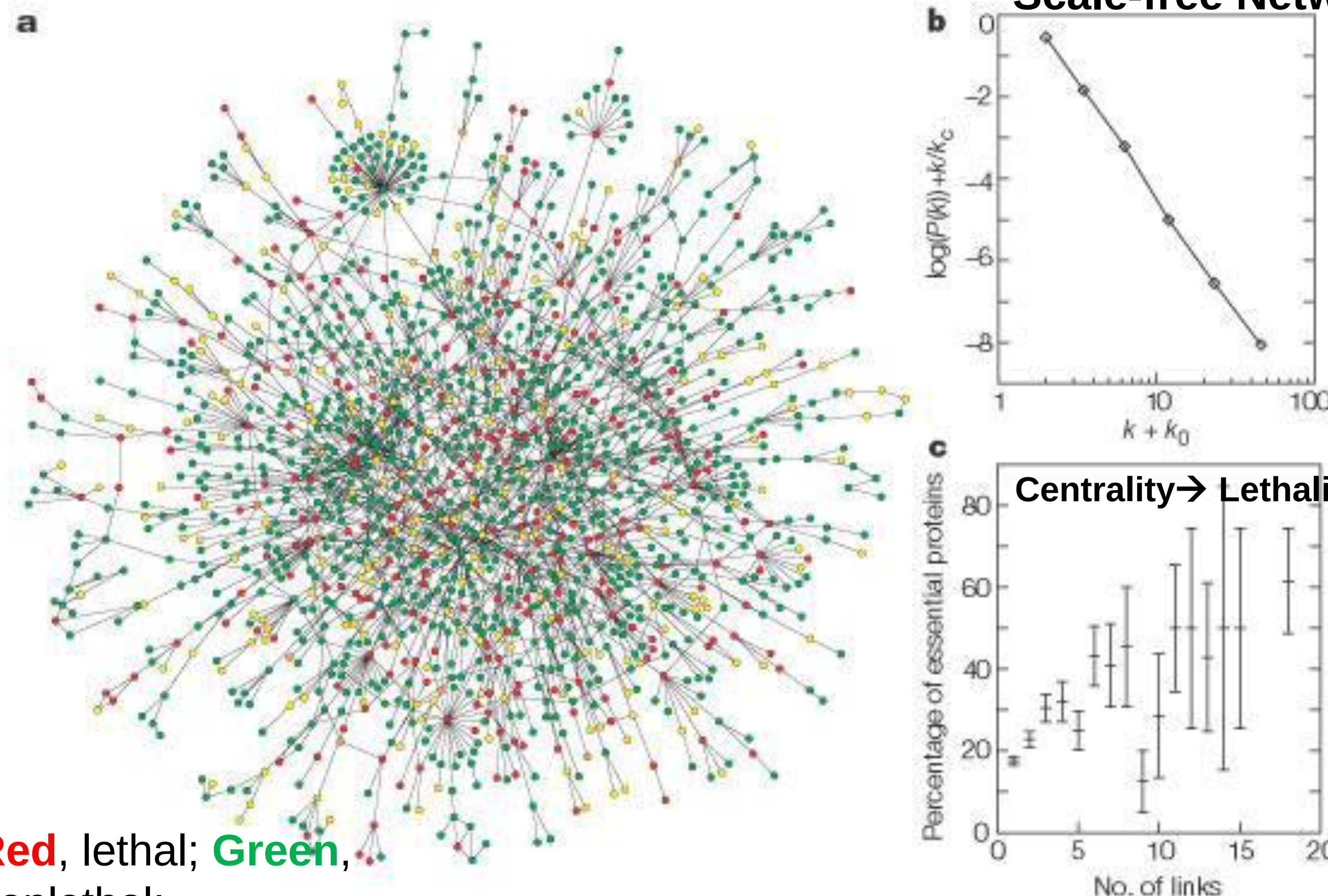


1870 proteins interacting with the help of 2240 physical interactions

Red, lethal; **Green**, nonlethal; **Orange**, slow growth; **Yellow**, unknown.

'Lethality and centrality in protein networks', H Jeong, Sp Meson and A-L Barabasi *Nature* 411, 41-42 (2001).

Scale-free Network



Centrality→ Lethality

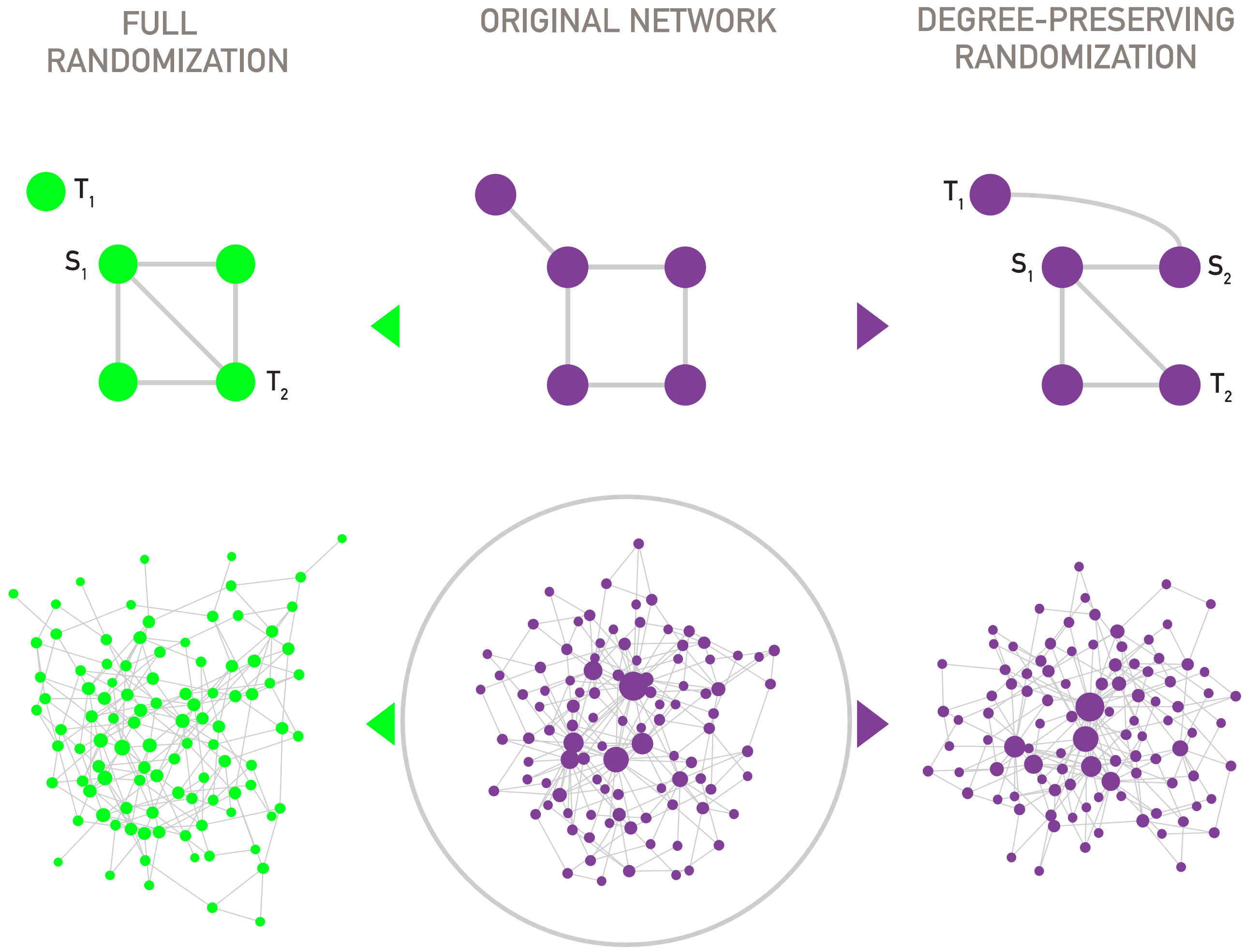
Red, lethal; **Green**, nonlethal;
Orange, slow growth;
Yellow, unknown

‘Lethality and centrality in protein networks’, H Jeong, Sp Meson and A-LBarabasi, *Nature* 411, 41-42 (2001).

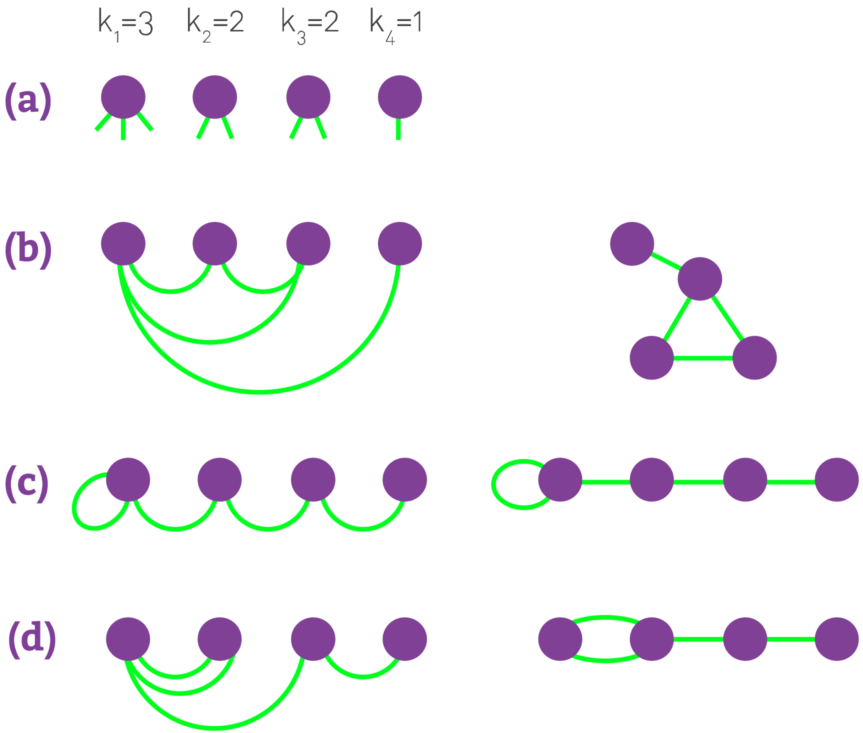
- Scale-free networks are robust to random errors, but are vulnerable to targeted attacks.
- The response of exponential (random) networks for random errors and targeted attacks is indistinguishable.
- The hub proteins in the yeast protein interaction network tend to be most critical.

Generating networks with a pre-defined p_k

Degree Preserving randomization



Configuration model



$$p_{ij} = \frac{k_i k_j}{2L - 1}$$

(1) ***Degree sequence****: Assign a degree to each node, represented as stubs or half-links. The degree sequence is either generated analytically from a preselected distribution (Box 4.5), or it is extracted from the adjacency matrix of a real network. We must start from an even number of stubs, otherwise we will be left with unpaired stubs.*

(2) ***Network assembly****: Randomly select a stub pair and connect them. Then randomly choose another pair from the remaining stubs and connect them. This procedure is repeated until all stubs are paired up. Depending on the order in which the stubs were chosen, we obtain different networks. Some networks include cycles (2a), others self-edges (2b) or multi-edges (2c). Yet, the expected number of self- and multi-edges goes to zero in the limit.*

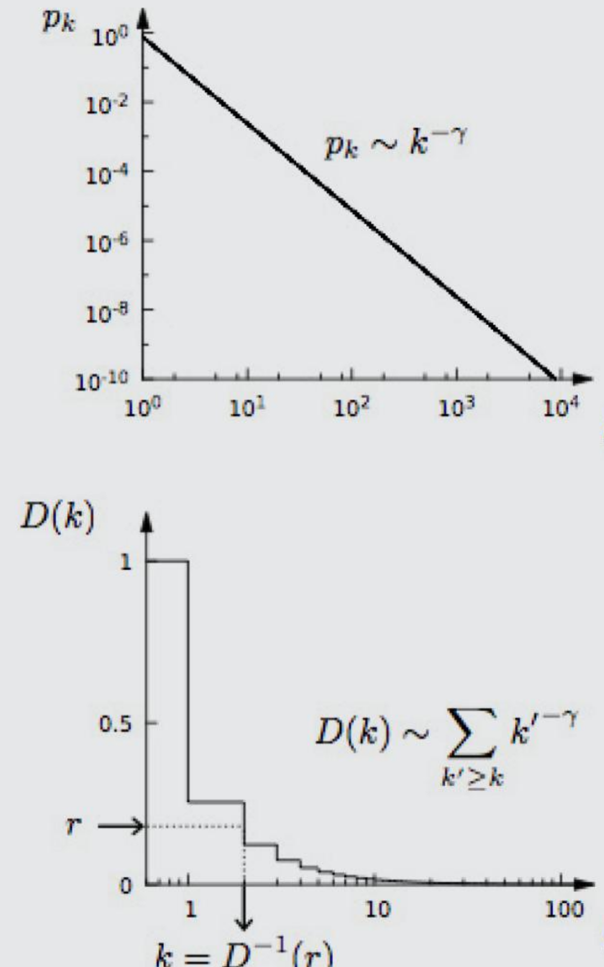
BOX 4.5

GENERATING A DEGREE SEQUENCE WITH POWER-LAW DISTRIBUTION

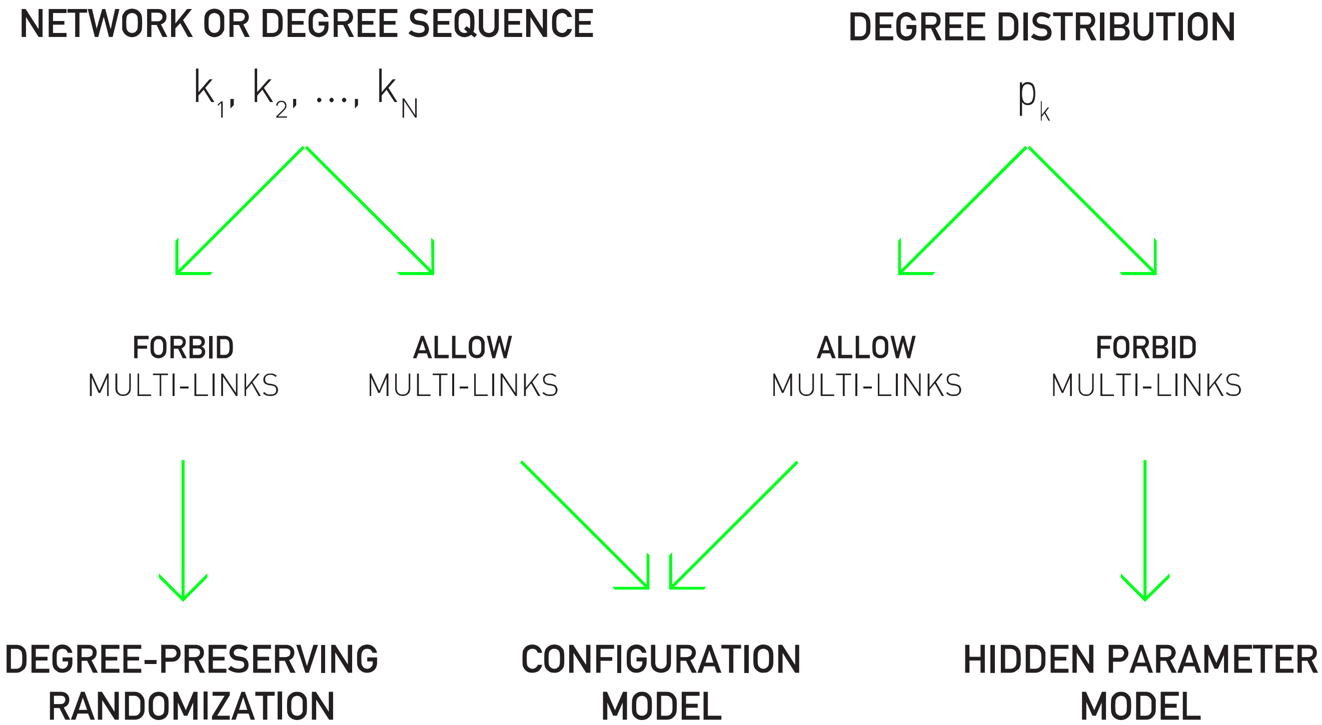
The degree sequence of an undirected network is a non-increasing sequence of the node degrees. For example, the degree sequence of each of the networks shown in Fig. 4.15a is {3, 2, 2, 1}. As Fig. 4.15a illustrates, the degree sequence in general does not uniquely identify a graph. There can be multiple graphs with the same degree sequence. We often need to generate a degree sequence from a pre-defined degree distribution. Our purpose here is to provide the tools to achieve this. We start from an analytically pre-defined degree distribution, like $p_k \sim k^{-\gamma}$, shown in panel (a). Our goal is to generate a degree sequence $\{k_1, k_2, ..., k_N\}$ of N degrees that follow the distribution p_k. We start by calculating the complementary cumulative distribution function

$$D(k) = \sum_{k' \geq k} p_{k'}, \tag{4.25}$$

shown in (b). $D(k)$ is between 0 and 1, and the step size at any k equals p_k. Therefore, to generate a sequence of N random numbers following a pre-defined p_k distribution, we generate N random numbers r_i, i = 1, ... , N, chosen from the (0, 1) interval. For each r_i we use the plot in (b) to assign a degree k_i. The obtained $k_i = D^{-1}(r_i)$ set will follow the desired p_k distribution. Note that the degree sequence assigned to a p_k is not unique - we can generate multiple sets of $\{k_1, ..., k_N\}$ sequences compatible with the same p_k.



Decision tree



BOX 4.4

SCALE-FREE NETWORK WITH $\gamma < 2$ DO NOT EXIST

To see why networks with $\gamma < 2$ are problematic, we need to attempt to build one. A degree sequence that can be turned into simple graph (i.e. a graph lacking multilinks or self-loops) is called graphical [32]. Yet, not all degree sequences are graphical: if for example the number of stubs is odd, then we will always have an unmatched stub, as shown in Fig. 4.13b.

The graphicality of a degree sequence can be tested with an algorithm proposed by Erdős and Gallai [32, 33, 34, 35]. If we apply the algorithm to scale-free networks we find that the number of graphical degree sequences drops to zero for $\gamma < 2$. Hence degree distributions with $\gamma < 2$ cannot be turned into a network. Indeed, for networks in this regime the largest hub grows faster than N. If we do not allow self-loops and multi-links, then the degree of the largest hub cannot exceed $N - 1$.

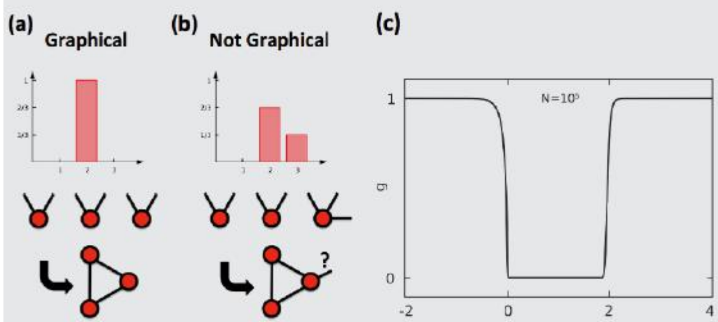


Figure 4.13
Networks with $\gamma < 2$ are not graphical

(a-b) Two degree distributions and the corresponding degree sequences. The difference is limited to the degree of a single node. While we can build a network consistent with the degree distribution (a), it is impossible to build one from (b), as one stub always remains unmatched. Hence (a) is graphical, while (b) is not.

(c) Fraction of networks with a given γ that are graphical. A large number of degree sequences with degree exponent γ and $N = 10^5$ were generated, testing the graphicality of each network.

While virtually all networks with $\gamma > 2$ are graphical, it is impossible to find graphical networks with $0 < \gamma < 2$.

SMALL WORLD BEHAVIOR IN SCALE-FREE NETWORKS

$$k_{\max} = k_{\min} N^{\frac{1}{g-1}}$$

$$< l > \sim \begin{cases} const. & \gamma = 2 \\ \dfrac{\ln \ln N}{\ln(\gamma - 1)} & 2 < \gamma < 3 \\ \dfrac{\ln N}{\ln \ln N} & \gamma = 3 \\ \ln N & \gamma > 3 \end{cases}$$

Ultra Small World

Small World

Size of the biggest hub is of order O(N). Most nodes can be connected within two layers of it, thus the average path length will be independent of the system size.

The **average path length increases slower than logarithmically**. In a random network all nodes have comparable degree, thus most paths will have comparable length. In a scale-free network the vast majority of the path go through the few high degree hubs, reducing the distances between nodes.

Some key models produce γ=3, so the result is of particular importance for them. This was first derived by Bollobas and collaborators for the network diameter in the context of a dynamical model, but it holds for the average path length as well.

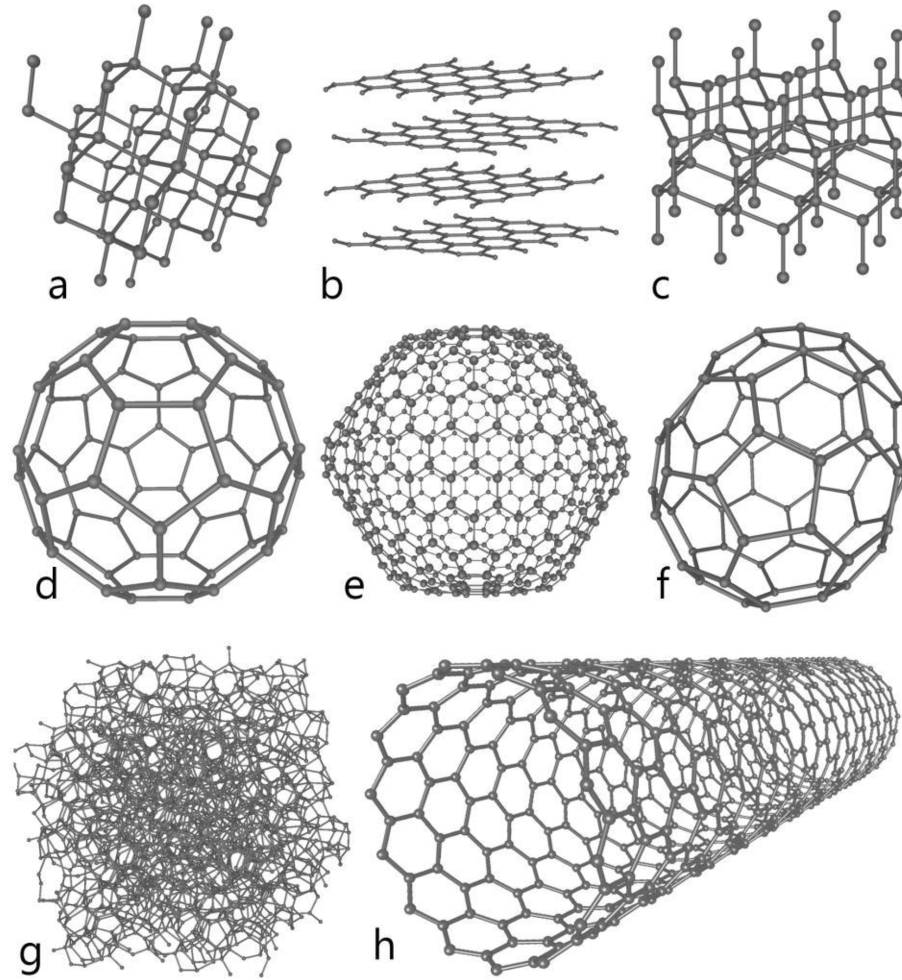
The second moment of the distribution is finite, thus in many ways the network behaves as a random network. Hence the average path length follows the result that we derived for the random network model earlier.

Cohen, Havlin Phys. Rev. Lett. 90, 58701(2003); Cohen, Havlin and ben-Avraham, in Handbook of Graphs and Networks, Eds. Bornholdt and Shuster (Willy-VCH, NY, 2002) Chap. 4; Confirmed also by: Dorogovtsev et al (2002), Chung and Lu (2002); (Bollobas, Riordan, 2002; Bollobas, 1985; Newman, 2001

Not all networks are scale-free

- *Networks appearing in material science, like the network describing the bonds between the atoms in crystalline or amorphous materials, where each node has exactly the same degree.*

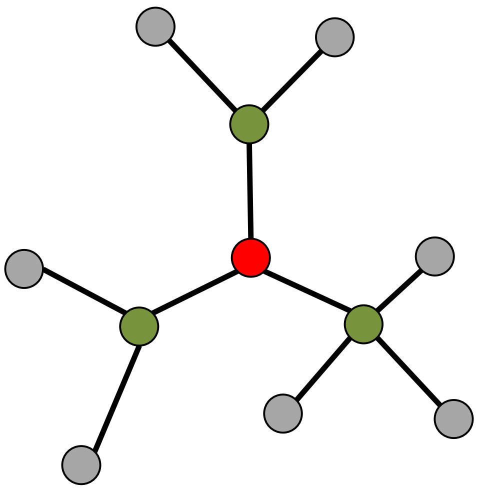
- *The power grid, consisting of generators and switches connected by transmission lines*



Ultra-small property

DISTANCES IN RANDOM GRAPHS

Random graphs tend to have a tree-like topology with almost constant node degrees.



- nr. of first neighbors: $N_1 \cong \langle k \rangle$
- nr. of second neighbors: $N_2 \cong \langle k \rangle^2$
- nr. of neighbours at distance d: $N_d \cong \langle k \rangle^d$
- estimate maximum distance:

$$1 + \sum_{l=1}^{l_{max}} \langle k \rangle^i = N \Rightarrow l_{max} = \frac{\log N}{\log \langle k \rangle}$$

summary

DEGREE DISTRIBUTION

Discrete form:

$$p_k = \frac{k^{-\gamma}}{\zeta(\gamma)}.$$

Continuous form:

$$p(k) = (\gamma - 1)k_{min}^{\gamma-1}\, k^{-\gamma}$$

SIZE OF THE LARGEST HUB

$$k_{max} \sim k_{min} N^{\frac{1}{\gamma-1}}.$$

MOMENTS OF p_k *for* $N \to \infty$

$2 < \gamma < 3$: $\langle k \rangle$ finite, $\langle k^2 \rangle$ diverges.

$\gamma > 3$: $\langle k \rangle$ and $\langle k^2 \rangle$ finite.

DISTANCES

$$\langle d \rangle \sim \begin{cases} \text{const.} & \gamma = 2, \\ \dfrac{\ln \ln N}{\ln(\gamma - 1)} & 2 < \gamma < 3, \\ \dfrac{\ln N}{\ln \ln N} & \gamma = 3, \\ \ln N & \gamma > 3. \end{cases}$$

Bounded Networks

We call a network *bounded* if its degree distribution decrease exponentially or faster for high k. As a consequence $<k^2>$ is smaller than $<k>$, implying that we lack significant degree variations. Examples of p_k in this class include the Poisson, Gaussian, or the simple exponential distribution (Table 4.2). The Erdős-Rényi and the Watts-Strogatz networks are the best known network models belonging to this class. Bounded networks lack outliers, consequently most nodes have comparable degrees. Real networks in this class include highway networks and the power grid.

Unbounded Networks

We call a network *unbounded* if its degree distribution has a fat tail in the high-k region. As a consequence $<k^2>$ is much larger than $<k>$, resulting in considerable degree variations. Scale-free networks with a power-law degree distribution (4.1) offer the best known example of networks belonging to this class. Outliers, or exceptionally high-degree nodes, are not only allowed but are expected in these networks. Networks in this class include the WWW, the Internet, the protein interaction networks, and most social and online networks.