Homework 8 Solution

Choose the best answer

1. Each of the following provides incentives to reduce a negative externality except
   a. merger with affected firms.
   b. subsidizing consumption of the good being produced.
   c. bargaining among firms.
   d. taxation of the externality.

2. Externalities between two firms can be "internalized" if:
   I. The two firms merge.
   II. The property right arrangement is clear and transaction costs are zero.
   III. The externalities affect each firm equally.
   IV. Marginal costs for both firms are constant.

   Which statement(s) correctly complete the sentence?
   a. Only II.
   b. All except III.
   c. I and II, but not III and IV.
   d. I and IV, but not II and III. 3. A non-exclusive good is a good which

   a. is sold in low-price markets.
   b. is impossible to keep people from enjoying the benefits the good provides.
   c. is produced by a perfectly competitive firm.
   d. is produced at the lowest possible cost.

4. A non-rival good is a good which
   a. is produced by a monopoly.
   b. is produced by a cartel.
   c. can provide benefits to additional users at a zero marginal cost.
   d. is sold in a single market.

Analytical questions

1. Consider the case describe in Varian Ch 35: There is a steel firm (S) and fishery firm (F) located along the same river. Denote x as the amount of pollution. Firm S's cost of producing s unit of steel is $c_s(s,x) = s^2x^{-\frac{1}{2}}$. Firm F's cost of fishing f unit of fish is $c_f(f,x) = f^2x^{\frac{1}{2}}$. The maximum amount of pollution firm S can emit is $\bar{x} = 16$. The price of steel is $p_s = 4$ per unit and the price of fish is $p_f = 8$ per unit.

a. Write down the profit function of firm S and find out the level of steel production and pollution.

$$\max_{s,x} \pi_s(s,x) = 4s - s^2x^{-\frac{1}{2}}$$

$$\frac{\partial \pi_s}{\partial s} = 4 - 2sx^{-\frac{1}{2}} = 0$$

$$\frac{\partial \pi_s}{\partial x} = -\frac{\partial c_s}{\partial x} = \frac{1}{2} s^2 x^{-\frac{3}{2}} > 0.$$

Because profit is rising with respect to x, so the steel firm will use up all his allowance for pollution. Pollution is free input for him. Therefore $x^* = 16$.

$$4 - 2s \times 16^{-\frac{1}{2}} = 0 \Rightarrow s^* = 8.$$

So firm S use all pollution allowance and produce 8 unit of steel.

b. Given the profit maximizing behavior of firm S, how many units of fish will firm F produce?

$$\max_f \pi_f(f, x) = 8f - f^2 x^{\frac{1}{2}}$$

$$\frac{\partial \pi_f}{\partial f} = 8 - 2f x^{\frac{1}{2}} = 0$$

Since $x = 16$, we can solve that $f^* = 1$.

c. Find the Pareto (social) optimal level of production and pollution (s^*, f^*, x^*).

We can find it by assuming two firms merge and make decision to maximize joint profit

$$\max_{s,f,x} \pi_s(s, x) + \pi_f(f, x) == 4s + 8f - s^2 x^{-\frac{1}{2}} - f^2 x^{\frac{1}{2}}$$

There are three first-order conditions

$$\frac{\partial \pi_s}{\partial s} = 4 - 2s x^{-\frac{1}{2}} = 0$$

$$\frac{\partial \pi_f}{\partial f} = 8 - 2f x^{\frac{1}{2}} = 0$$

$$\frac{\partial \pi_s}{\partial x} = \frac{1}{2} s^2 x^{-\frac{3}{2}} - \frac{1}{2} f^2 x^{-\frac{1}{2}} = 0$$

(the second term denotes how externality is now internalized). Solve them obtain

$$x^* = 4, \; s^* = 4, \; f^* = 2.$$

Compare to part (a) and (b), at Pareto efficient outcome, more fish shall be produced, and the amount of steel and pollution shall be reduced.

2. Two roommates $i = 1, 2$ are considering to buy a TV and use it together. TV is a public good and the spending is contributed by two roommates, $G = g_1 + g_2$. Each roommate need to divide its income w_i into his private spending x_i and public spending g_i, that is, $x_i + g_i = w_i$. Roommate 1 has income $w_1 = 100$ and his utility function is

$$u_1(x_1, G) = x_1^{\frac{1}{2}} G^{\frac{1}{2}}.$$

Roommate 2 has income $w_2 = 60$ and his utility function is

$$u_2(x_2, G) = x_2^{\frac{1}{3}} G^{\frac{2}{3}}.$$

a. Assume that roommate 2 decides not to pay for the TV, $g_2 = 0$. Roommate 1 is selfish and maximizes his utility. How he will choose g_1 and x_1?

Roommate 1 will solve his own utility maximization problem

$$\max_{x_1, g_1} u_1(x_1, G) = x_1^{\frac{1}{2}}(g_1 + 0)^{\frac{1}{2}}, \quad \text{s.t. } x_1 + g_1 = 100.$$

Use MRS condition

$$MRS_1 = \frac{\frac{\partial u_1}{\partial g_1}}{\frac{\partial u_1}{\partial x_1}} = \frac{x_1}{g_1} = 1$$
$$x_1 = g_1.$$

Plug in budget constraint, we get

$$x_1 + g_1 = g_1 + g_1 = 100$$

$$g_1 = 50, \ x_1 = 50$$

b. Solve for the non-cooperative (Nash equilibrium) allocation, $(x_1^{NE}, g_1^{NE}, x_2^{NE}, g_2^{NE})$.

Roommate 1's utility maximization problem is

$$\max_{x_1, g_1} u_1(x_1, G) = x_1^{\frac{1}{2}}(g_1 + g_2)^{\frac{1}{2}}, \quad \text{s.t. } x_1 + g_1 = 100.$$

Use MRS condition

$$MRS_1 = \frac{x_1}{g_1 + g_2} = 1 \Rightarrow x_1 = g_1 + g_2.$$

Plug in the budget constraint

$$g_1 + g_2 + g_1 = 100$$
$$g_1 = BR_1(g_2) = 50 - \frac{1}{2}g_2.$$

Roommate 2's utility maximization problem is

$$\max_{x_2, g_2} u_2(x_2, G) = x_2^{\frac{1}{3}}(g_1 + g_2)^{\frac{2}{3}}, \quad \text{s.t. } x_2 + g_2 = 60.$$

$$MRS_2 = \frac{\frac{\partial u_2}{\partial g_2}}{\frac{\partial u_2}{\partial x_2}} = \frac{2x_2}{G} = \frac{2x_2}{g_1 + g_2} = 1$$
$$x_2 = \frac{1}{2}g_1 + \frac{1}{2}g_2$$

Plug in budget constraint, we get

$$\frac{1}{2}g_1 + \frac{1}{2}g_2 + g_2 = 60$$

$$g_2 = BR_2(g_1) = 40 - \frac{1}{3}g_1.$$

Solve for the Nash equilibrium

$$\begin{cases} g_1 = 50 - \frac{1}{2} g_2 \\ g_2 = 40 - \frac{1}{3} g_1 \end{cases} \Rightarrow \begin{cases} g_1^{NE} = 36 \\ g_2^{NE} = 28 \end{cases}$$

$$x_1^{NE} = w_1 - g_1^{NE} = 100 - 36 = 64$$

$$x_2^{NE} = w_2 - g_2^{NE} = 60 - 28 = 32$$

So the allocation is $(x_1^{NE}, g_1^{NE}, x_2^{NE}, g_2^{NE}) = (64, 36, 32, 28)$.

c. Fix roommate 2's utility at $\bar{u}_2$. Find three equations characterize the Pareto efficient allocation $(x_1^{PE}, x_2^{PE}, G^{PE})$.

[You won't be able to solve for a specific allocation because the equations will contain $\bar{u}_2$.]

$$\max_{x_1, x_2, G} u_1(x_1, G), \quad \text{s.t.} \begin{cases} u_2(x_2, G) = \bar{u}_2 \\ x_1 + x_2 + G = w_1 + w_2 = 160 \end{cases}$$

*Students do not need to show the following derivation.

Set up Lagangian

$$\mathcal{L} = u_1(x_1, G) + \lambda[u_2(x_2, G) - \bar{u}_2] + \mu[160 - x_1 - x_2 - G].$$

Differentiate w.r.t. x_1, x_2, G, we get

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial u_1(x_1, G)}{\partial x_1} - \mu = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \lambda \frac{\partial u_2(x_2, G)}{\partial x_2} - \mu = 0$$

$$\frac{\partial \mathcal{L}}{\partial G} = \frac{\partial u_1(x_1, G)}{\partial G} + \lambda \frac{\partial u_2(x_2, G)}{\partial G} - \mu = 0.$$

From first equation

$$\mu = \frac{\partial u_1(x_1, G)}{\partial x_1}.$$

From second equation

$$\lambda = \frac{\mu}{\frac{\partial u_2(x_2,G)}{\partial x_2}} = \frac{\frac{\partial u_1(x_1,G)}{\partial x_1}}{\frac{\partial u_2(x_2,G)}{\partial x_2}}.$$

Plug in third equation, we have

$$\frac{\partial u_1(x_1, G)}{\partial G} + \frac{\frac{\partial u_1(x_1,G)}{\partial x_1}}{\frac{\partial u_2(x_2,G)}{\partial x_2}} \frac{\partial u_2(x_2, G)}{\partial G} - \frac{\partial u_1(x_1, G)}{\partial x_1} = 0.$$

Divide both side by $\frac{\partial u_1(x_1,G)}{\partial x_1}$, we obtain the condition:

$$MRS_1 + MRS_2 = \frac{\frac{\partial u_1(x_1,G)}{\partial G}}{\frac{\partial u_1(x_1,G)}{\partial x_1}} + \frac{\frac{\partial u_2(x_2,G)}{\partial G}}{\frac{\partial u_2(x_2,G)}{\partial x_2}} = 1.$$

For the specific utility function in this example, we have

$$MRS_1 = \frac{\frac{\partial u_1(x_1,G)}{\partial G}}{\frac{\partial u_1(x_1,G)}{\partial x_1}} = \frac{x_1}{G}$$

$$MRS_2 = \frac{\frac{\partial u_2(x_2,G)}{\partial G}}{\frac{\partial u_2(x_2,G)}{\partial x_2}} = \frac{2x_2}{G}.$$

Plug in the constraints, we get the three equations

$$\begin{cases} \frac{x_1}{G} + \frac{2x_2}{G} = 1 \\ x_1 + x_2 + G = 160 \\ u_2(x_2, G) = x_2^{\frac{1}{3}} G^{\frac{2}{3}} = \bar{u}_2 \end{cases}$$

3. Consider a society with an electricity company and a representative consumer (all households). The electricity company uses labor l units and coal x units to produce electricity q units. The company's production function is

$$q = f(l, x) = 12 \times l^{\frac{1}{3}} x^{\frac{1}{2}}.$$

So its profit is

$$\pi(l, x) = 12 l^{\frac{1}{3}} x^{\frac{1}{2}} - wl - vx$$

The representative consumer's payoff is

$$u(l, x) = (w - 2)l - x,$$

which is measured in dollars. So the consumer is harmed by pollution from coal burning. There is no market power: All prices are considered as fixed, and all electricity produced are being consumed. Each unit of labor costs $w = 4$, and each unit of coal costs $v = 3$. Electricity price is $p = 1$.

a. To maximize profit, how will the electricity company choose l and x. Compute the company's profit and the consumer's payoff.

This is straightforward by profit maximization problem

$$\max_{l,x} \pi(l, x) = p \times f(l, x) - wl - vx = 12 l^{\frac{1}{3}} x^{\frac{1}{2}} - 4l - 3x.$$

First-order condition yields,

$$\begin{cases} \frac{\partial \pi}{\partial l} = 4l^{-\frac{2}{3}}x^{\frac{1}{2}} - 4 = 0 \\ \frac{\partial \pi}{\partial x} = 6l^{\frac{1}{3}}x^{-\frac{1}{2}} - 3 = 0 \end{cases} \Rightarrow \begin{cases} l_a = 8 \\ x_a = 16 \end{cases}$$

The company's profit is

$$\pi_a = 12 \times l_a^{\frac{1}{3}} \times x_a^{\frac{1}{2}} - 4l_a - 3x_a = 96 - 4 \times 8 - 3 \times 16 = 16.$$

The consumer's payoff is

$$u_a = 2l_a - x_a = 16 - 16 = 0.$$

b. Consider the company and the consumer together as a society. Determine the social efficient level of l and x.

[Hint: the social welfare is the summation of the company's profit and the representative consumer's payoff.]

$$\begin{aligned} \max_{l,o} W &= \pi(l,x) + u(l,x), \quad \text{s.t. } e = 12 \times l^{\frac{1}{3}}x^{\frac{1}{2}} \\ &= 12l^{\frac{1}{3}}x^{\frac{1}{2}} - 4l - 3x + 2l - x \\ &= 12l^{\frac{1}{3}}x^{\frac{1}{2}} - 2l - 4x \end{aligned}$$

First-order condition yields,

$$\begin{cases} \frac{\partial \pi}{\partial l} + \frac{\partial u}{\partial l} = 4l^{-\frac{2}{3}}x^{\frac{1}{2}} - 2 = 0 \\ \frac{\partial \pi}{\partial x} + \frac{\partial u}{\partial x} = 6l^{\frac{1}{3}}x^{-\frac{1}{2}} - 4 = 0 \end{cases} \Rightarrow \begin{cases} l_b = 27 \\ x_b = 20.25 \end{cases}$$

c. Compute the social welfare in part (a) and part (b) and compare them.

$$W_a = \pi_a + u_a = 16$$

$$\pi_b = 12 \times l_b^{\frac{1}{3}} x_b^{\frac{1}{2}} - 4l_b - 3x_b = 162 - 4 * 27 - 3 * 20.25 = -6.75$$

$$u_b = 2l_b - x_b = 33.75$$

$$W_b = \pi_b + u_b = -6.75 + 33.75 = 27$$

So $W_b > W_a$.

d. Suppose the electricity company is selfish and only care its own profit. Using the concept of externality, how will you describe the electricity company's usage of coal? As a policy maker, how to fix the probem?

[Proposing one solution is enough.]

Using coal has a **negative externality** to the society. One of the following policy can fix the problem:

(i) The policy maker can charge $t = 1$ on using coal.

(ii)The policy maker can ask the company to pay $p_x = 1$ to the consumer when using coal.

(iii) The policy maker can "merge" these two parties. For example, let the consumer own the electricity company.

*Students do not need to answer the following explanation

The tax or price help raise the private marginal cost $v = 3$ to the social marginal cost 4. The first-order conditions of coal in part (a) after tax is

$$\frac{\partial \pi}{\partial x} = 6l^{\frac{1}{3}}x^{-\frac{1}{2}} - 3 - t = 6l^{\frac{1}{3}}x^{-\frac{1}{2}} - 4,$$

which becomes the same as that in part (b).

*Note that the company is making a loss in part (c). So policy (iii) is the most practical.

4. There are $n \geq 2$ people in a residential community indexed by $i = 1, 2, ..., n$. Each of them needs to contribute to a joint account for the maintenance of the facilities in the community. This joint account can be considered as a public good.

Each person i needs to determine to spend his income w_i between his private consumption y_i and joint account contribution x_i, i.e., $x_i + y_i = w_i$. Each person has the same utility function

$$U_i(X, y_i) = Xy^2,$$

where $X = \sum_{i=1}^{n} x_i$. Each person has an income $w_i = 120$.

a. Consider that n people voluntarily contributes to the joint account. Find the non-cooperative equilibrium $(x_1^{NE}, x_2^{NE}, ..., x_n^{NE}, y_1^{NE}, y_2^{NE}, ..., y_n^{NE})$. Note that the solution depends on n.

For each person i,

$$\max_{x_i, y_i} U_i(X, y_i) = (\sum_{i=1}^{n} x_i)y_i^2, \quad \text{s.t. } x_i + y_i = 120.$$

The MRS condition is

$$MRS_i = \frac{\frac{\partial U_i}{\partial x_i}}{\frac{\partial U_i}{\partial y_i}} = \frac{y_i^2}{2(\sum_{i=1}^{n} x_i)y_i} = \frac{y_i}{2\sum_{i=1}^{n} x_i} = 1$$

$$\Rightarrow 2\sum_{i=1}^{n} x_i = 2X = y_i.$$

Together with the budget constraint

$$x_i + y_i = 120,$$

we have

$$x_i + 2\sum_{i=1}^{n} x_i = x_i + 2X = 120, \quad i = 1, 2, ..., n.$$

Sum up the above n conditions

$$\sum_{i=1}^{n}(x_i + 2X) = \sum_{i=1}^{n} 120$$

$$\sum_{i=1}^{n} x_i + 2\sum_{i=1}^{n} X = X + 2nX = 120n$$

$$(2n+1)X = 120n$$

$$\Rightarrow X = \frac{120n}{2n+1}.$$

Because all agents are symmetric

$$x_1^{NE} = x_2^{NE} = \cdots = x_n^{NE} = \frac{120}{2n+1}.$$

$$y_1^{NE} = y_2^{NE} = \cdots = y_n^{NE} = 120 - \frac{120}{2n+1} = \frac{240n}{2n+1}.$$

b. Assume that each person contribute the equal amount. What is the social optimal (Pareto efficient) allocation $(X^{SO}, y_1^{SO}, y_2^{SO}, ..., y_n^{SO})$? The solution also depends on n.

We can directly use the formula

$$MRS_1 + MRS_2 + \cdots + MRS_n = 1$$

$$\frac{y_1}{2X} + \frac{y_2}{2X} + \cdots + \frac{y_n}{2X} = 1.$$

$$\Rightarrow 2X = y_1 + y_2 + \cdots + y_n.$$

Together with the resource constraint

$$X + y_1 + y_2 + \cdots + y_n = 120n,$$

we get

$$X + 2X = 120n$$

$$X^{SO} = 40n$$

$$y_1 + y_2 + \cdots + y_n = 80n.$$

Because each agent contribute the equal amount,

$$y_1^{SO} = y_2^{SO} = \cdots = y_n^{SO} = 80.$$

c. Compare the joint account amount under non-cooperative equilibrium (X^{NE}) and social optimal level (X^{SO}). When the number of people n increases, does the difference becomes larger or smaller? Does the free-rider problem become more severe when there are more people in the community?

$$X^{NE} = \frac{120n}{2n+1}$$
$$X^{SO} = 40n$$

$$\begin{aligned} X^{SO} - X^{NE} &= 40n - \frac{120n}{2n+1} \\ &= \frac{(80n + 40 - 120)n}{2n+1} \\ &= \frac{80(n^2 - n)}{2n+1} \end{aligned}$$

The difference is greater than zero for $n \geq 2$. Moreover, for $n \geq 2$,

$$\begin{aligned} \frac{d}{dn}(\frac{80(n^2 - n)}{2n+1}) &= \frac{80[(2n-1)(2n+1) - 2(n^2 - n)]}{(2n+1)^2} \\ &= \frac{80[4n^2 - 1 - 2n^2 + 2n)]}{(2n+1)^2} \\ &= \frac{80[2n^2 + 2n - 1)]}{(2n+1)^2} > 0. \end{aligned}$$

Therefore, the difference becomes larger when n increases. The free-rider problem become more severe.