3. Profit Maximization

Profit maximization and supply of a price-taking firm

Price is fixed at p for this firm, the firm chooses the output level q by

$$\max_q \pi(q) = p \times q - C(q).$$

FOC yields

$$\pi'(q) = p - MC(q) = 0 \Leftrightarrow p = MC(q)$$

The output level q^* is determined by the equality

$$MC(q^*) = p$$

Note that, q^* maximizes the profit requires second-order condition (SOC):

$$\pi''(q) = -\frac{dMC}{dq} = -C''(q) < 0$$

$$\Leftrightarrow C''(q) > 0 \text{ (convex cost function) or },MC(q) \text{ is increasing.}$$

The production technology is DRS.

From the conditon $MC(q^*) = p$, we can derive the supply function as

$$S(p) = \begin{cases} MC^{-1}(p) & \text{if } p \text{ is greater than the shut-down price} \\ 0 & \text{otherwise.} \end{cases}$$

In the long run,

$$S(p) = \begin{cases} MC^{-1}(p) & \text{if } p \text{ is greater than the zero-profit price} \\ 0 & \text{otherwise.} \end{cases}$$

If in the long run, price is below the zero-profit point, then firm will exit the industry. See later in long-run equilibrium.

Example: $SC(w, v, q, k_1) = vk_1 + wq^{\frac{1}{\beta}} k_1^{-\frac{\alpha}{\beta}}$
$w = 12, v = 3, k_1 = 80, \alpha = \beta = 0.5$

$$\begin{aligned} C(q) &= 3 \times 80 + 12 \times q^2 \times 80^{-1} \\ &= 240 + \frac{3}{20} q^2 \end{aligned}$$

This cost function is convex, so there will be an interior solution to profit maximization.

$$MC(q) = \frac{3}{10}q,$$

which increases in q.

Profit maximization

$$\max_q \{p \times q - 240 - \frac{3}{20}q^2\}$$

$$p = MC(q) = \frac{3}{10}q$$

So we get the supply function

$$q^* = MC^{-1}(p) = S(p) = \frac{10}{3}p.$$

For this cost function, $VC(q) = \frac{3}{20}q^2$, the average variable cost is

$$AVC(q) = \frac{VC(q)}{q} = \frac{3}{20}q.$$

It is an increasing function, its minimum is at $q = 0$. So the shut-down point is at $q = 0$ and $p = 0$.

So the supply function starts at $p = 0$.

Given a market price p_1, what is the profit?

Firm will produce $q_1 = S(p_1) = \frac{10}{3}p_1$. At the optimal quantity, the profit is

$$\begin{aligned}
\pi_1^* &= p_1 q_1 - C(q_1) \\
&= \frac{10}{3}p_1^2 - 240 - \frac{3}{20}(\frac{10}{3}p_1)^2 \\
&= \frac{10}{3}p_1^2 - 240 - \frac{3}{20}\frac{10 \times 10}{3 \times 3}p_1^2 \\
&= \frac{10}{3}p_1^2 - 240 - \frac{10}{2 \times 3}p_1^2 \\
&= \frac{5}{3}p_1^2 - 240
\end{aligned}$$

What is the minimum p_1 to guarantee zero profit in the long run?

$$\pi_1^* = 0 = \frac{5}{3}p_1^2 - 240$$

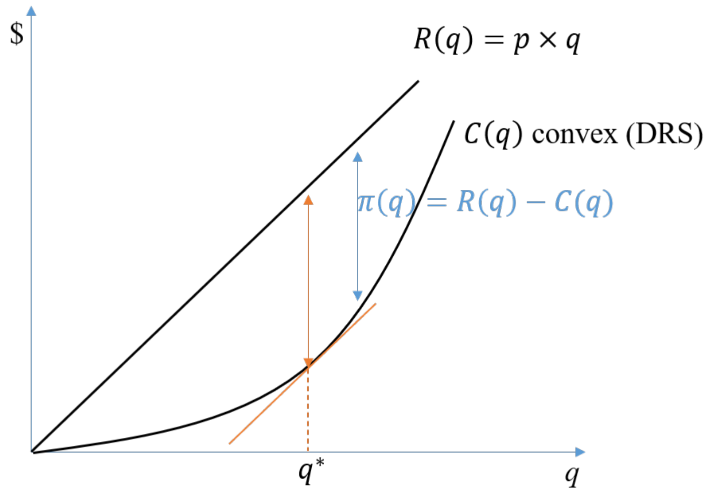
$$p_1^2 = \frac{720}{5} = 144$$

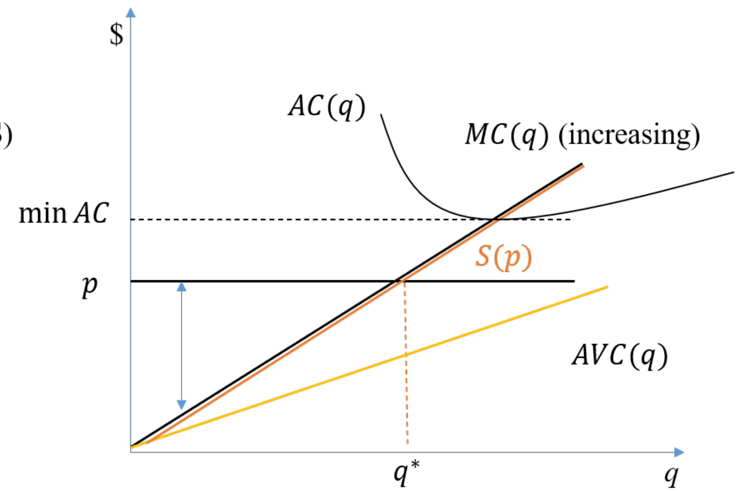
$$p_1 = \sqrt{144} = 12$$

If $p_1 > 12$, firm will earn a positive economic profit in the long run.

The same zere profit condition can be acheived by finding the minimum of AC

$$AC(q) = \frac{240}{q} + \frac{3}{20}q.$$



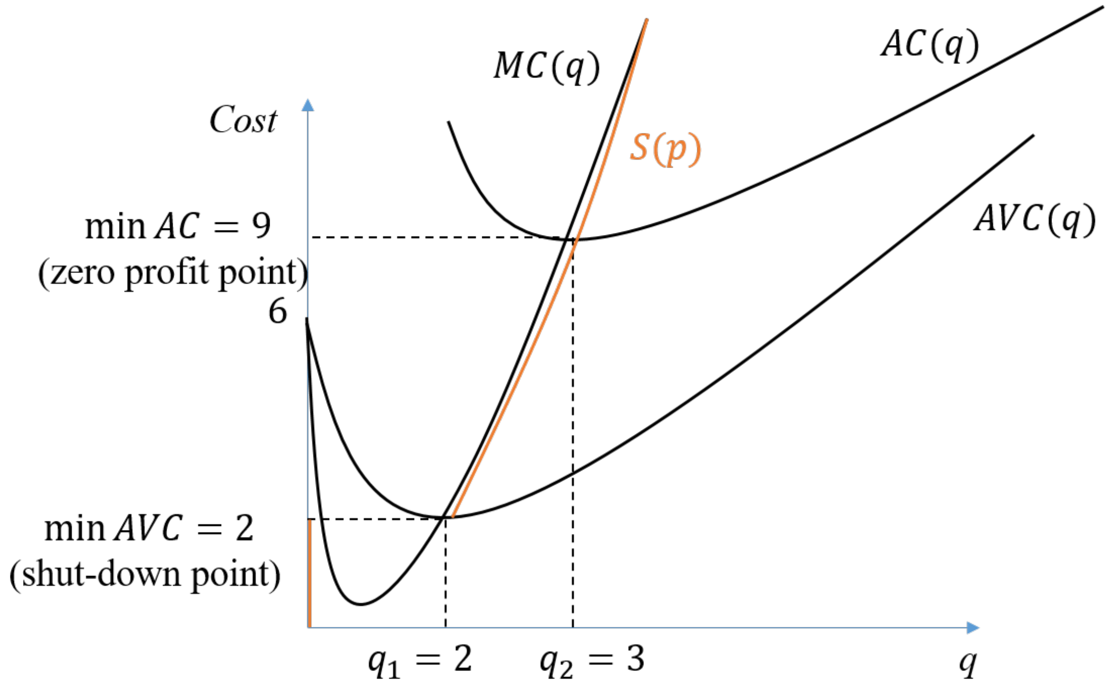


Example: $C(q) = q^3 - 4q^2 + 6q + 18$. In note 2 we have shown that
The shut-down point is at $p_1 = 2$, $q_1 = 2$.
The zero-profit point is at $p_2 = 9$, $q_2 = 3$.
Firm will supply where q satisfies

$$p = MC(q) = 3q^2 - 8q + 6$$

$$\begin{aligned}
3(q^2 - \frac{8}{3}q + \frac{16}{9}) + \frac{2}{3} &= p \\
3(q - \frac{4}{3})^2 &= p - \frac{2}{3} \\
(q - \frac{4}{3})^2 &= \frac{1}{3}p - \frac{2}{9} \\
q - \frac{4}{3} &= \sqrt{\frac{1}{3}p - \frac{2}{9}} \\
q^* &= \sqrt{\frac{1}{3}p - \frac{2}{9}} + \frac{4}{3}
\end{aligned}$$

$$S(p) = \begin{cases} \sqrt{\frac{1}{3}p - \frac{2}{9}} + \frac{4}{3} & \text{if } p \geq 2 \\ 0 & \text{if } p < 2 \end{cases}$$



Profit maximization of a price-taking firm

We can express the problem as

$$\max_{k,l} p \times f(k,l) - vk - wl$$

This is a unconstraint maximization problem, the **necessary** condition of the solution are two FOCs

$$\begin{cases} p \times f_k - v = 0 \\ p \times f_l - w = 0 \end{cases} \Leftrightarrow \begin{cases} pMP_k = v \\ pMP_l = w \end{cases}$$

Divide these two FOCs

$$\frac{MP_k}{MP_l} = \frac{v}{w}.$$

The solution is input demand

$$k^*(p,v,w), \;\; l^*(p,v,w)$$

Plug in the objective function, we get the value function / profit function

$$\pi(p,v,w) = p \times f(k^*(p,v,w), l^*(p,v,w)) - vk^*(p,v,w) - wl^*(p,v,w)$$

The maximization requires second-order condition (SOC)

For one variable function, SOC means the second-order derivative is negative (concave). $\pi(q)$, SOC is $\frac{d^2\pi}{dq^2} < 0$.

*For two variable function, SOC means that the Hessian matrix is negative definite.

$$H = \begin{bmatrix} f_{kk} & f_{kl} \\ f_{lk} & f_{ll} \end{bmatrix}$$

Negative definite requires that $f_{kk} < 0$, $|H| = f_{kk}f_{ll} - f_{lk}f_{kl} > 0$.

There is one important restriction of this profit max framework: it only works for the case of DRS.

If the production technology exhibits CRS, then if we double the inputs, we can double the profit:

$$p \times f(2k, 2l) - v2k - w2l = 2 \times [p \times f(k,l) - vk - wl]$$

The solution becomes infinite.

Producer surplus

With the supply function, we can express producer surplus by integration

$$PS(p_1) = \int_{p_0}^{p_1} S(p)dp$$

p_0 is the shut-down point.

Example

$$S(p) = \frac{10}{3}p$$

At $p = 12$, the producer surplus is

$$PS(12) = \int_0^{12} \frac{10}{3} p dp = \frac{10}{3} \int_0^{12} p dp = \frac{10}{3} \left[\frac{p^2}{2}\right]_0^{12} = \frac{10}{3}\frac{144}{2} = 5 \times 48 = 240$$

Using the original profit function, we can get the same result

$$\max_q \{p \times q - 240 - \frac{3}{20}q^2\}$$

Firm will optimally choose q,

$$\begin{aligned} MC(q) &= p \\ \frac{3}{10}q &= p = 12 \\ q &= 40 \end{aligned}$$

$$\begin{aligned} \pi &= \overbrace{12 \times 40 - \frac{3}{20} \times 40^2}^{PS} - \overbrace{240}^{F} \\ &= 480 - 3 \times 2 \times 40 - 240 \\ &= 240 - 240 \\ &= PS - F = 0 \end{aligned}$$

If price rises, $p = 15$, then we will see

$$\pi = PS - F > 0$$

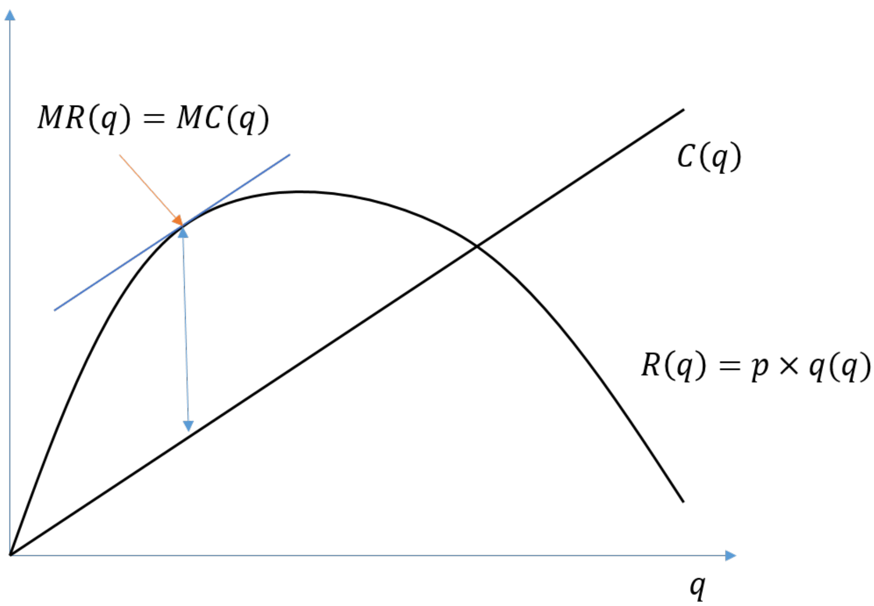
Firm facing downward-sloping demand

Firm choosing q to maximize profit

$$\max_q \pi(q) = q \times p(q) - C(q) = R(q) - C(q).$$

$C(q)$ is obtained from cost minimization.
FOC is

$$\frac{d\pi}{dq} = MR(q) - MC(q) = 0 \Leftrightarrow MR(q) = MC(q)$$



Price elasticity of demand,

$$\varepsilon = e_{q,p} = \frac{\frac{dq}{q}}{\frac{dp}{p}} = \frac{p}{q}\frac{dq}{dp}$$

$$\begin{cases} e_{q,p} < -1,\ |e_{q,p}| > 1 & \text{elastic demand} \\ e_{q,p} = -1,\ |e_{q,p}| = 1 & \text{unit elastic} \\ e_{q,p} \in (-1,0),\ |e_{q,p}| < 1 & \text{inelastic demand} \end{cases}$$

The FOC can be written in a way with demand elasticity

$$
\begin{aligned}
MR(q) &= p + q \times \frac{dp}{dq} \\
&= p + p\frac{q}{p} \times \frac{dp}{dq} = p \times \left\{ 1 + \frac{q}{p} \times \frac{dp}{dq} \right\} \\
&= p \times \left\{ 1 + \frac{1}{\frac{p}{q} \times \frac{dq}{dp}} \right\} = p \times \left\{ 1 + \frac{1}{e_{q,p}} \right\} \\
&= p \times \left\{ 1 - \frac{1}{|e_{q,p}|} \right\}
\end{aligned}
$$

FOC requires

$$MR(q) = p \times \left\{ 1 - \frac{1}{|e_{q,p}|} \right\} = MC(q) > 0$$

It requires that

$$1 - \frac{1}{|e_{q,p}|} > 0 \Leftrightarrow 1 > \frac{1}{|e_{q,p}|} \Leftrightarrow |e_{q,p}| > 1 \text{ elastic}$$

Firm will always supply at the elastic portion of the demand curve.

If the firm is at an inelastic portion of the demand curve, by raising price, the firm can increase revenue, save cost, and increase profit.

$$p \uparrow\uparrow \Rightarrow q \downarrow \begin{cases} \Rightarrow R \uparrow \\ \Rightarrow C \downarrow \end{cases} \quad \Rightarrow \pi \uparrow$$

Example of linear demand

$$q(p) = a - bp$$

Inverse demand

$$
\begin{aligned}
p(q) &= \frac{a}{b} - \frac{1}{b}q \\
R(q) = q \times p(q) &= \frac{a}{b}q - \frac{1}{b}q^2 \\
MR(q) &= \frac{a}{b} - 2 \times \frac{1}{b}q
\end{aligned}
$$

For linear demand, marginal revenue is twice as steep as the demand. Revenue maximized at $q = \frac{a}{2}$, the midpoint.

MR and MC will intersect at the left-hand side of the midpoint of the demand curve. This is the elastic portion of the demand curve.

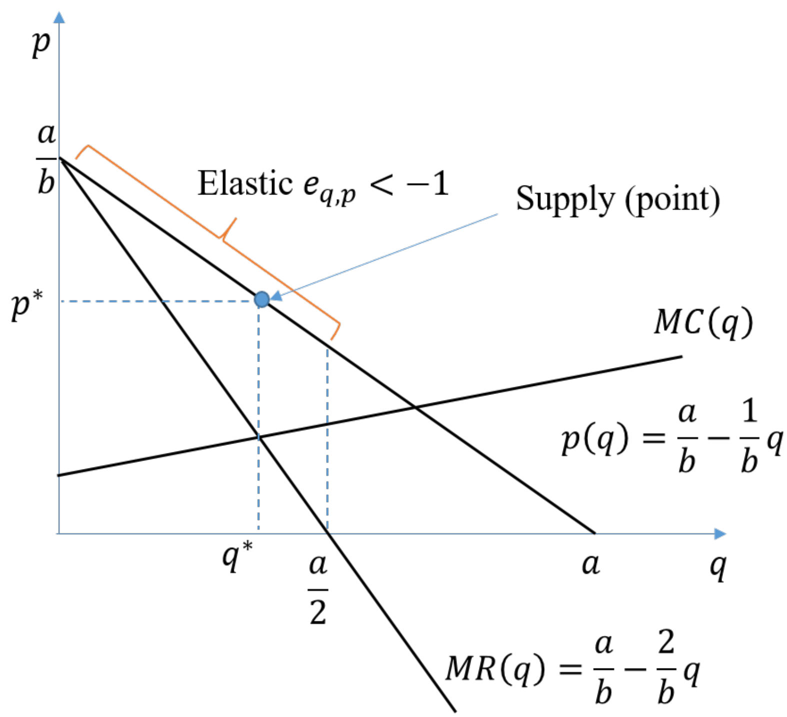
$$
\begin{aligned}
e_{q,p} &= \frac{p}{q} \times \frac{dq}{dp} = \frac{\frac{a}{b} - \frac{1}{b}q}{q} \times (-b) \\
&= [\frac{a}{bq} - \frac{1}{b}] \times (-b) \\
&= -\frac{a}{q} + 1
\end{aligned}
$$

To have elastic demand, it requires

$$e_{q,p} = -\frac{a}{q} + 1 < -1$$

$$-\frac{a}{q} < -2 \Leftrightarrow \frac{a}{q} > 2 \Leftrightarrow q < \frac{a}{2}$$

Therefore, for linear demand function, the left-hand side of the midpoint is elastic.



Markup pricing rule

$$MR(q) = p \times \left\{1 - \frac{1}{|e_{q,p}|}\right\} = MC$$

$$p - p\frac{1}{|e_{q,p}|} = MC$$

$$p - MC = p\frac{1}{|e_{q,p}|}$$

Rewrite this formula so that the left-hand side is a markup (profit margin)

$$\frac{p - MC}{p} = \frac{1}{|e_{q,p}|} = \begin{cases} \frac{1}{2} & \text{when } |e_{q,p}| = 2 \\ \frac{1}{3} & \text{when } |e_{q,p}| = 3 \\ 0 & \text{when } |e_{q,p}| = \infty \end{cases}$$

When consumers become elastic ($|e_{q,p}|$ is large), the markup is smaller.
In the extreme case, $|e_{q,p}| = \infty$, this is the perfect competitive market with zero markup.

Example, constant elasticity demand

$$q(p) = ap^b, \quad b < 0$$

$$e_{q,p} = \frac{p}{q} \times \frac{dq}{dp} = \frac{p}{ap^b} \times abp^{b-1} = \frac{p^b}{ap^b} \times ab = b$$

Apply the formula above, we get

$$\frac{p - MC}{p} = \frac{1}{|b|}$$