Exercise 10 solution

2. Lagrange function

$$L(x,\lambda,\mu) = x_2 x_3 + x_1 x_3 + \lambda\left(x_2^2 + x_3^2 - 1\right) + \mu\left(x_1 x_3 - 3\right)$$

$G(x) = \begin{pmatrix} 0 & 2x_2 & 2x_3 \\ x_3 & 0 & x_1 \end{pmatrix}$ is full rank since $x_3 \neq 0$

FOC:

$$\begin{cases} \frac{\partial L}{\partial x_1} = x_3 + \mu x_3 = 0 \\ \frac{\partial L}{\partial x_2} = x_3 + 2\lambda x_2 = 0 \\ \frac{\partial L}{\partial x_3} = x_1 + x_2 + 2\lambda x_3 + \mu x_1 = 0 \end{cases}$$

from first equation: $\mu = -1$ (since $x_3 \neq 0$), thus we have

$$\begin{cases} x_3 + 2\lambda x_2 = 0 \\ x_2 + 2\lambda x_3 = 0 \end{cases} \implies x_2^2 = x_3^2$$

plug this into $x_2^2 + x_3^2 = 1 \implies x_2^2 = x_3^2 = 1/2$

solution 1: $\begin{cases} x_1^* = 3\sqrt{2} \\ x_2^* = \frac{1}{\sqrt{2}} \\ x_3^* = \frac{1}{\sqrt{2}} \\ \lambda^* = -\frac{1}{2} \\ \mu^* = -1 \end{cases}$, solution 2: $\begin{cases} x_1^* = -3\sqrt{2} \\ x_2^* = -\frac{1}{\sqrt{2}} \\ x_3^* = -\frac{1}{\sqrt{2}} \\ \lambda^* = -\frac{1}{2} \\ \mu^* = -1 \end{cases}$,

solution 3: $\begin{cases} x_1^* = -3\sqrt{2} \\ x_2^* = \frac{1}{\sqrt{2}} \\ x_3^* = -\frac{1}{\sqrt{2}} \\ \lambda^* = \frac{1}{2} \\ \mu^* = -1 \end{cases}$, solution 4: $\begin{cases} x_1^* = 3\sqrt{2} \\ x_2^* = -\frac{1}{\sqrt{2}} \\ x_3^* = \frac{1}{\sqrt{2}} \\ \lambda^* = \frac{1}{2} \\ \mu^* = -1 \end{cases}$

bordered Hessian:

$$B = \begin{pmatrix} 0 & 0 & 0 & 2x_2^* & 2x_3^* \\ 0 & 0 & x_3^* & 0 & x_1^* \\ 0 & x_3^* & 0 & 0 & 0 \\ 2x_2^* & 0 & 0 & 2\lambda^* & 1 \\ 2x_3^* & x_1^* & 0 & 1 & 2\lambda^* \end{pmatrix}$$

$$b_5 = |B| = (-x_3^*) \begin{vmatrix} 0 & 0 & 2x_2^* & 2x_3^* \\ 0 & x_3^* & 0 & x_1^* \\ 2x_2^* & 0 & 2\lambda^* & 1 \\ 2x_3^* & 0 & 1 & 2\lambda^* \end{vmatrix} = -(x_3^*)^2 \begin{vmatrix} 0 & 2x_2^* & 2x_3^* \\ 2x_2^* & 2\lambda^* & 1 \\ 2x_3^* & 1 & 2\lambda^* \end{vmatrix}$$

$$= -(x_3^*)^2 \left[8x_2^* x_3^* - 8\lambda^* (x_2^*)^2 - 8\lambda^* (x_3^*)^2\right] = -\frac{1}{2}\left[8x_2^* x_3^* - 8\lambda^*\right]$$

therefore, $|B| = -4 < 0$ for solution 1 and 2, $|B| = 4 > 0$ for solution 3 and 4

Conclusion: $\begin{cases} \text{local maximum: Solutions 1 and 2} \\ \text{local minimum: Solutions 3 and 4} \end{cases}$

3. Lagrange function: $L(x, \lambda) = x_1 x_2 + \lambda (1 - x_1^2 - x_2^2)$

FOC: $x_2 - 2\lambda x_1 = 0$; $x_1 - 2\lambda x_2 = 0$

KTC: $\lambda (1 - x_1^2 - x_2^2) = 0$

Case 1: $\lambda = 0 \implies x_1 = x_2 = 0$ which can't be a local maximum
since $f(0,0) < f(x_1, x_2)$ as long as x_1, x_2 have same sign

Case 2: $\lambda > 0 \implies x_1^2 = x_2^2 = 1/2$

solution: $\lambda^* = 1/2$, $x_1^* = x_2^* = \frac{1}{\sqrt{2}}$ or $x_1^* = x_2^* = -\frac{1}{\sqrt{2}}$

bordered Hessian: $B = \begin{pmatrix} 0 & -2x_1^* & -2x_2^* \\ -2x_1^* & -2\lambda^* & 1 \\ -2x_2^* & 1 & -2\lambda^* \end{pmatrix}$

$|B| = 8x_1^* x_2^* + 8\lambda^* \left[(x_1^*)^2 + (x_1^*)^2\right] = 8 > 0$ for both solutions

thus both $(x_1^*, x_2^*) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $(x_1^*, x_2^*) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ are local maximizer