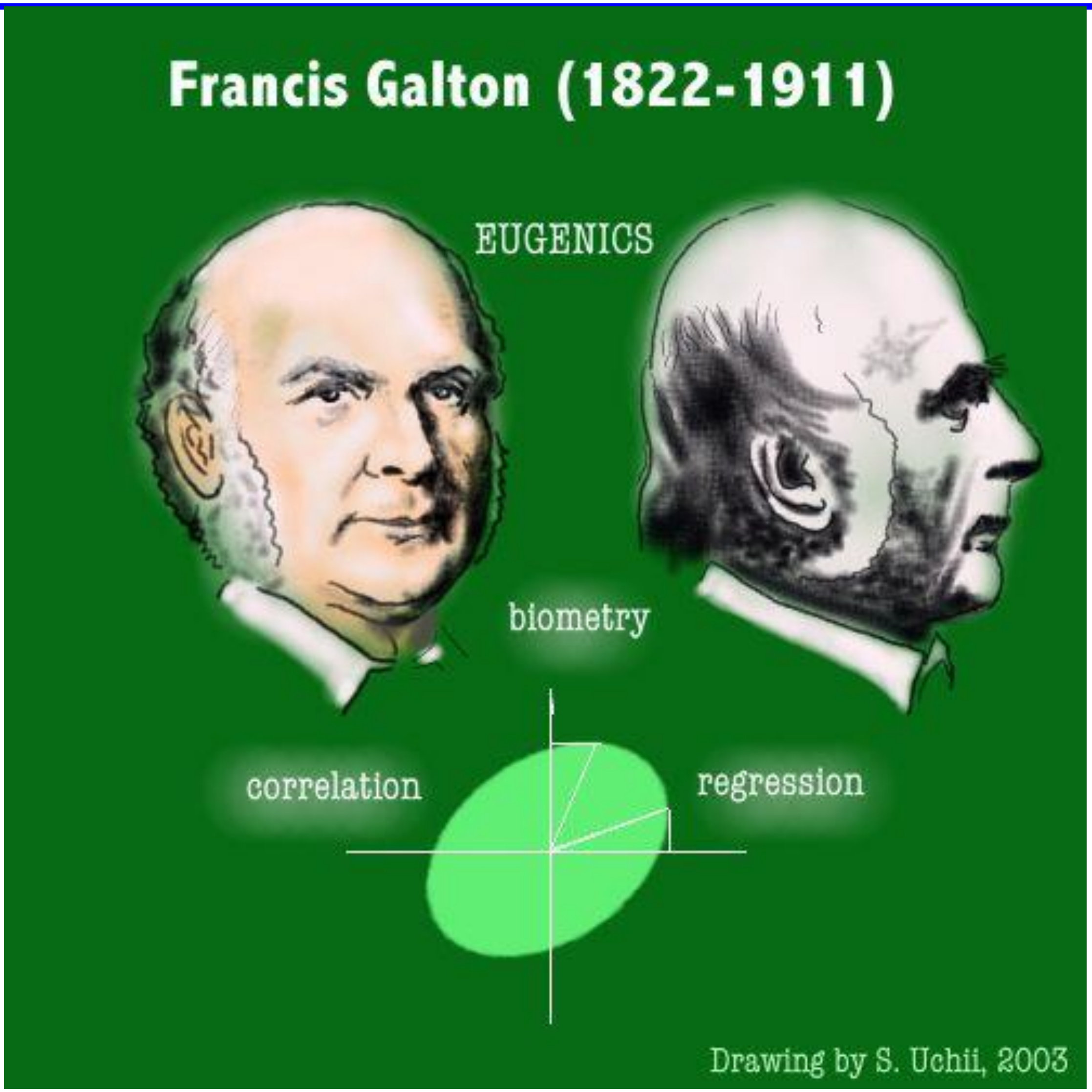
Topic 4: Simple Linear Regression-Estimation

Part A

Who Invented Regression ?



http://en.wikipedia.org/wiki/Francis_Galton

1) Linear Regression Model

- Here we are interested in estimating a linear equation that describes the relationship between 2 variables.
- A line is defined by its slope and intercept:

$$Y = \beta_0 + \beta_1 X$$

- E.g.,

$$\text{Quantity} = \beta_0 + \beta_1 \text{Price}$$

$$\text{Wage} = \beta_0 + \beta_1 \text{Education}$$

- If we can gather data on Quantity and Price, then we have information to use to estimate a linear statistical model.

1) Linear Regression Model

- The basic linear single variable model is

$$Y_i = \beta_0 + \beta_1 X_i + u_i$$

- This equation says that Y_i is equal to $\beta_0 + \beta_1 X_i$ plus some errors.

- I.e., X_i does not perfectly explain each Y_i, rather Y_i is explained by X_i with some errors.

- Interpretation of the model: keeping everything else constant, increasing one unit Xi, Yi will increase by beta1.

1) Linear Regression Model

- The basic linear single variable model is

$$Y_i = \beta_0 + \beta_1 X_i + u_i$$

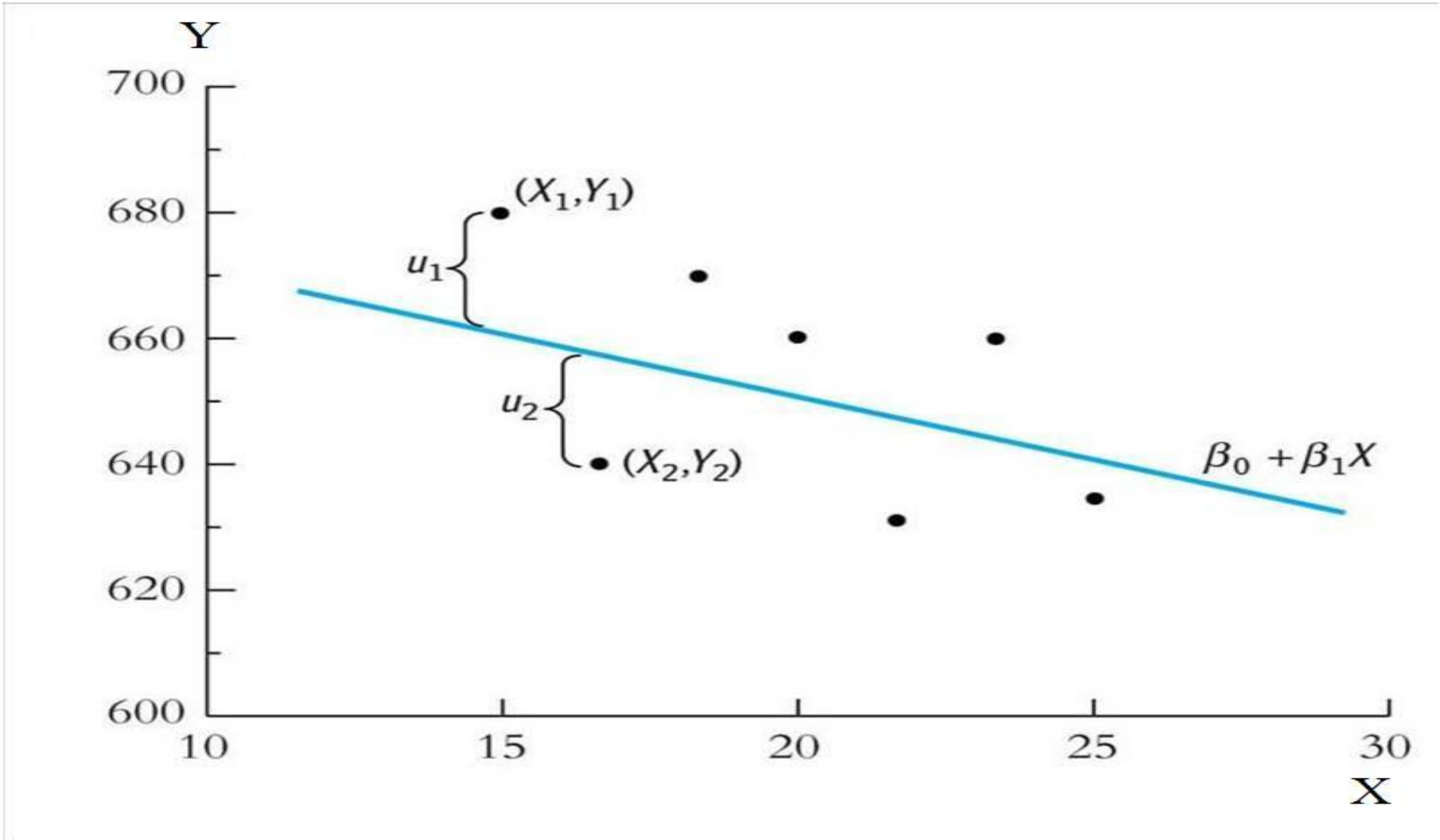
  - You have iid data (sample) :

$$\{(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)\}$$

  - Y_i is the dependent variable (random variable)
  - X_i is the independent (or explanatory) variable (random variable)
  - β_0 is the intercept or constant (unknown parameter, non-random)
  - β_1 is the slope (unknown parameter, non-random)

1) Linear Regression Model

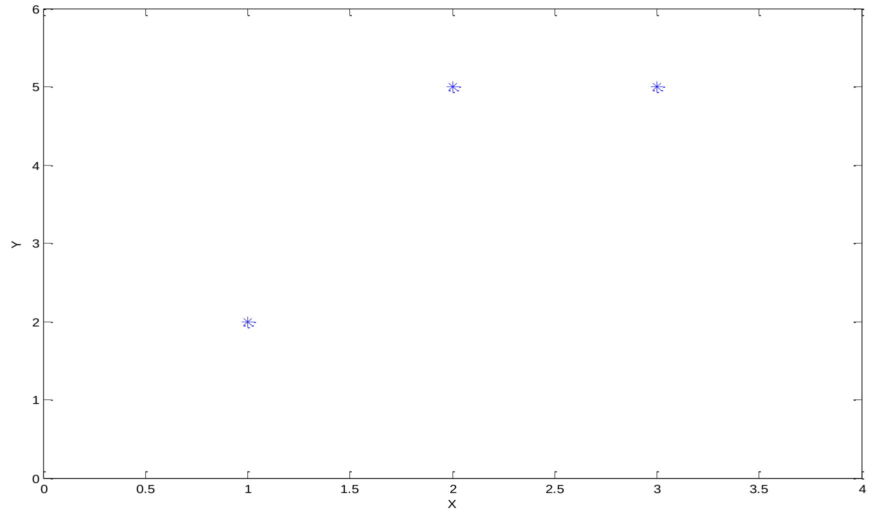
- Imagine that you know true beta0 and beta 1, then



1) Linear Regression Model

- E.g.

Xi	Yi
1	2
2	5
3	5



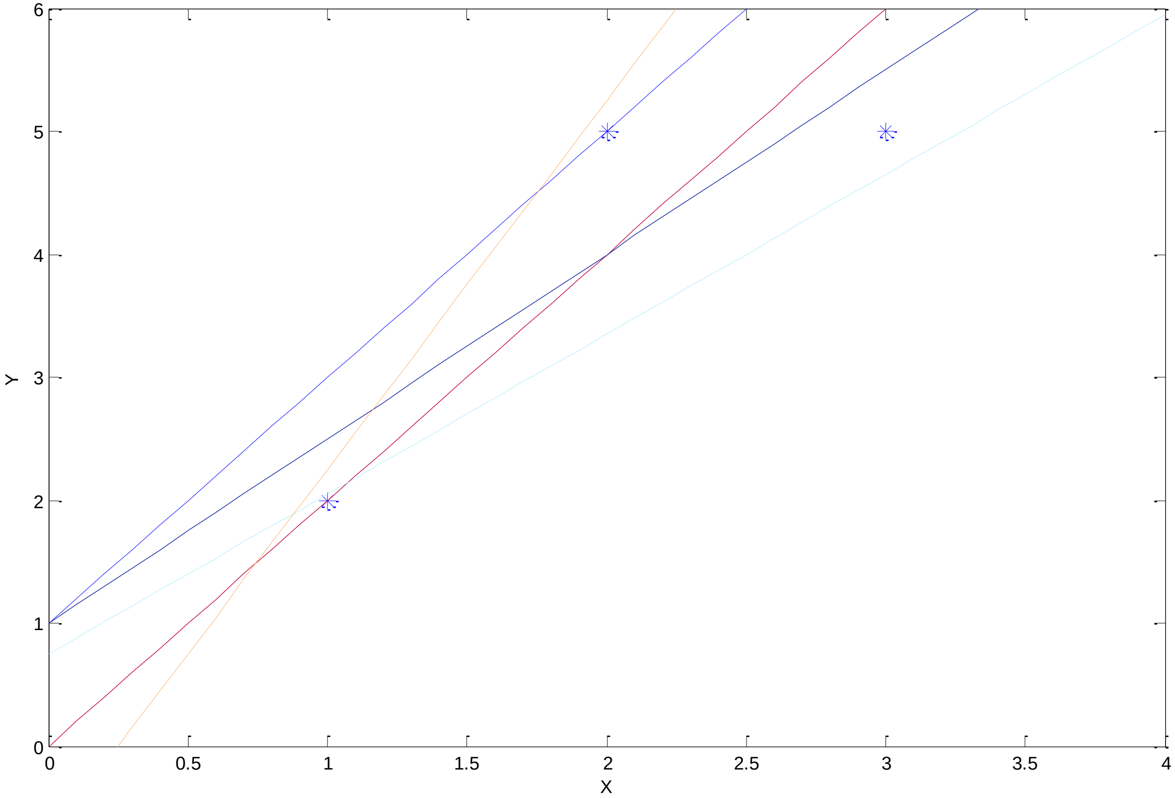
- There is positive correlation between Xi and Yi:

$$\text{Sample Correlation}\,(X, Y) = 0.866$$

- Our goal here is to come up with the equation of the line that links Yi and Xi.

1) Linear Regression Model

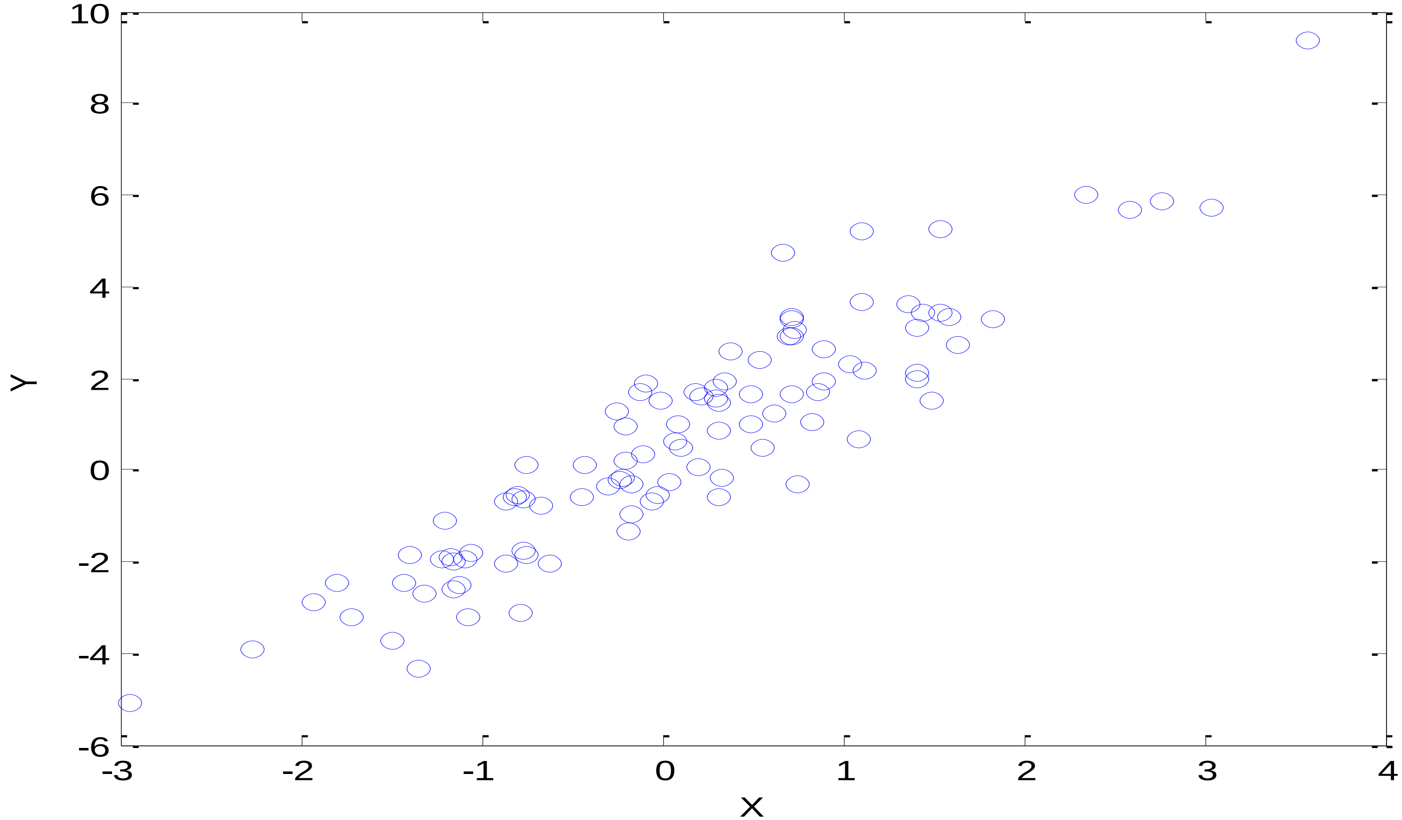
- Infinite ways to fit the line:



- which one is the best?

1) Linear Regression Model

- A computer example: I ask my computer to generate the data:
  - Step 1: X_i follows N(0,1), u_i follows N(0,1), X_i and u_i are independent
  - Step 2: Y_i is generated as $Y_i=0.5+2X_i+u_i$ (beta0=0.5, beta1=2)



- In reality, we don’t know beta0 and beta1. Need to find them.

2) Estimation: OLS

- Linear regression model:

$$Y_i = \beta_0 + \beta_1 X_i + u_i$$

- We don’t know the population parameter beta0 and beta1, so we must estimate them with the data, what are the estimators?

- We use a method called OLS (ordinary least square).

2) Estimation: OLS

- How can we estimate β_0 and β_1 from data?
- Recall that $\bar{Y}$ solves

$$\min_b \sum_{i=1}^{n} (Y_i - b)^2$$

- By analogy, we will focus on the least squares ("ordinary least squares" or "*OLS*") estimator of the unknown parameters β_0 and β_1, which solves

$$\min_{b_0, b_1} \sum_{i=1}^{n} [Y_i - (b_0 + b_1 X_i)]^2$$

2) Estimation: OLS

- The OLS estimator minimizes the average squared difference between the actual values of Y_i and the prediction ("predicted value") based on the estimated line.

$$\min_{b_0, b_1} \sum_{i=1}^{n} [Y_i - (b_0 + b_1 X_i)]^2$$

- This minimization problem can be solved using calculus.
- The result is the OLS estimators of β_0 and β_1. We denote the estimators as $\hat{\beta}_0$ and $\hat{\beta}_1$.
- We will see that OLS estimator is a "good" estimator (unbiased, consistent, efficient).

OLS

- Suppose that we are estimating a linear model:

$$Y_i = \beta_0 + X_i\beta_1 + u_i$$

- The OLS estimator $(\hat{\beta}_0, \hat{\beta}_1)$ is to minimize the objective function:

$$\sum_{i=1}^{n}[Y_i - b_0 - X_i b_1]^2$$

- Using calculus, we solve this minimization problem by solving the following two first-order conditions:

$$\frac{\partial}{\partial b_0}\sum_{i=1}^{n}[Y_i - b_0 - X_i b_1]^2 = 0$$

$$\frac{\partial}{\partial b_1}\sum_{i=1}^{n}[Y_i - b_0 - X_i b_1]^2 = 0$$

OLS

- $$\frac{\partial}{\partial b_0}\sum_{i=1}^{n}[Y_i - b_0 - X_i b_1]^2 = -2\sum_{i=1}^{n}[Y_i - b_0 - X_i b_1] = 0 \qquad (1)$$

  $$\frac{\partial}{\partial b_1}\sum_{i=1}^{n}[Y_i - b_0 - X_i b_1]^2 = -2\sum_{i=1}^{n}[Y_i - b_0 - X_i b_1]\bullet X_i = 0 \quad (2)$$

- Let's first look at equation (1):

$$-2\sum_{i=1}^{n}[Y_i - b_0 - X_i b_1] = 0 \Rightarrow \sum_{i=1}^{n} Y_i - \sum_{i=1}^{n} b_0 - b_1 \sum_{i=1}^{n} X_i = 0$$

$$\Rightarrow \left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right) - \left(\frac{1}{n}\sum_{i=1}^{n} b_0\right) - b_1\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = 0$$

$$\Rightarrow \bar{Y} - b_0 - b_1\bar{X} = 0 \Rightarrow b_0 = \bar{Y} - b_1\bar{X} \qquad (3)$$

- Now let's substitute equation (3) into equation (2). Equation (2) becomes

$$-2\sum_{i=1}^{n}[Y_i - (\bar{Y} - b_1\bar{X}) - X_i b_1] \bullet X_i = 0$$

$$\Rightarrow -2\sum_{i=1}^{n}[Y_i - \bar{Y} - b_1(X_i - \bar{X})] \bullet X_i = 0$$

$$\Rightarrow \sum_{i=1}^{n}(Y_i - \bar{Y})X_i - b_1\sum_{i=1}^{n}(X_i - \bar{X}) \bullet X_i = 0$$

$$\Rightarrow b_1 = \frac{\sum_{i=1}^{n}(Y_i - \bar{Y}) \bullet X_i}{\sum_{i=1}^{n}(X_i - \bar{X}) \bullet X_i}.$$

OLS

- We can show that

$$b_1 = \frac{\sum_{i=1}^{n}(Y_i - \bar{Y}) \cdot X_i}{\sum_{i=1}^{n}(X_i - \bar{X}) \cdot X_i} = \frac{\sum_{i=1}^{n}(Y_i - \bar{Y}) \cdot (X_i - \bar{X})}{\sum_{i=1}^{n}(X_i - \bar{X}) \cdot (X_i - \bar{X})}$$

- To show this,

(i) $$\sum_{i=1}^{n}(Y_i - \bar{Y}) \cdot (X_i - \bar{X}) = \sum_{i=1}^{n}(Y_i - \bar{Y}) \cdot X_i - \sum_{i=1}^{n}(Y_i - \bar{Y}) \cdot \bar{X}$$

$$= \sum_{i=1}^{n}(Y_i - \bar{Y}) \cdot X_i - \bar{X} \cdot \underbrace{\sum_{i=1}^{n}(Y_i - \bar{Y})}_{=0} = \sum_{i=1}^{n}(Y_i - \bar{Y}) \cdot X_i$$

(ii) $$\sum_{i=1}^{n}(X_i - \bar{X}) \cdot (X_i - \bar{X}) = \sum_{i=1}^{n}(X_i - \bar{X}) \cdot X_i - \sum_{i=1}^{n}(X_i - \bar{X}) \cdot \bar{X}$$

$$= \sum_{i=1}^{n}(X_i - \bar{X}) \cdot X_i - \bar{X} \cdot \underbrace{\sum_{i=1}^{n}(X_i - \bar{X})}_{=0} = \sum_{i=1}^{n}(X_i - \bar{X}) \cdot X_i.$$

2) Estimation: OLS

- The OLS estimator

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^{n}(X_i - \overline{X})^2} = \frac{s_{XY}}{s_X^2} \tag{4.7}$$

$$\hat{\beta}_0 = \overline{Y} - \hat{\beta}_1\overline{X}. \tag{4.8}$$

- Here, true β_0 and β_1 are unknown, we want to use data to estimate them; it turns out $\hat{\beta}_0$ and $\hat{\beta}_1$ are good estimators.
- $\hat{\beta}_0$ and $\hat{\beta}_1$ are calculated from the data, thus they are random variables. Our hope is that these two random variables are "close" to the true β_0 and β_1.

2) Estimation: OLS

- OLS predicted value and residuals

The OLS predicted values $\hat{Y}_i$ and residuals $\hat{u}_i$ are

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i,\ i = 1, \ldots, n \tag{4.9}$$

$$\hat{u}_i = Y_i - \hat{Y}_i,\ i = 1, \ldots, n. \tag{4.10}$$

- Note that is $\hat{u}_i$ different from u_i :

$$\begin{aligned} \hat{u}_i &= Y_i - \hat{Y}_i = \beta_0 + \beta_1 X_i + u_i - \left(\hat{\beta}_0 + \hat{\beta}_1 X_i\right) \\ &= u_i + \left(\beta_0 - \hat{\beta}_0\right) + \left(\beta_1 - \hat{\beta}_1\right) X_i \end{aligned}$$

- $\hat{u}_i$ is different from u_i because $\hat{\beta}_0$ and $\hat{\beta}_1$ are different from the true β_0 and β_1.

The OLS Estimator, Predicted Values, and Residuals

The OLS estimators of the slope β_1 and the intercept β_0 are

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^{n}(X_i - \overline{X})^2} = \frac{s_{XY}}{s_X^2} \tag{4.7}$$

$$\hat{\beta}_0 = \overline{Y} - \hat{\beta}_1\overline{X}. \tag{4.8}$$

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$$\hat{u}_i = Y_i - \hat{Y}_i,\ i = 1, \ldots, n. \tag{4.10}$$

The estimated intercept ($\hat{\beta}_0$), slope ($\hat{\beta}_1$), and residual ($\hat{u}_i$) are computed from a sample of n observations of X_i and Y_i, $i = 1, \ldots, n$. These are estimates of the unknown true population intercept (β_0), slope (β_1), and error term (u_i).

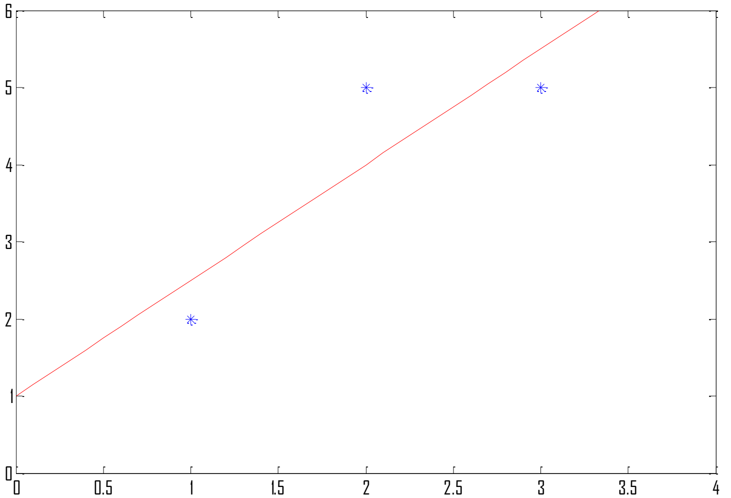
2) Estimation: OLS

- A Numerical Example:

X_i	Y_i
1	2
2	5
3	5

$\Rightarrow$

X_i	Y_i	$X_i - \bar{X}$	$Y_i - \bar{Y}$	$(X_i - \bar{X})(Y_i - \bar{Y})$	$(X_i - \bar{X})^2$
1	2	-1	-2	2	1
2	5	0	1	0	0
3	5	1	1	1	1



$$\hat{\beta}_1 = \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})^2} = \frac{2+0+1}{1+0+1} = 1.5$$

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X} = 4 - 1.5 \cdot 2 = 1$$

- So the line that "best" fits the data is:

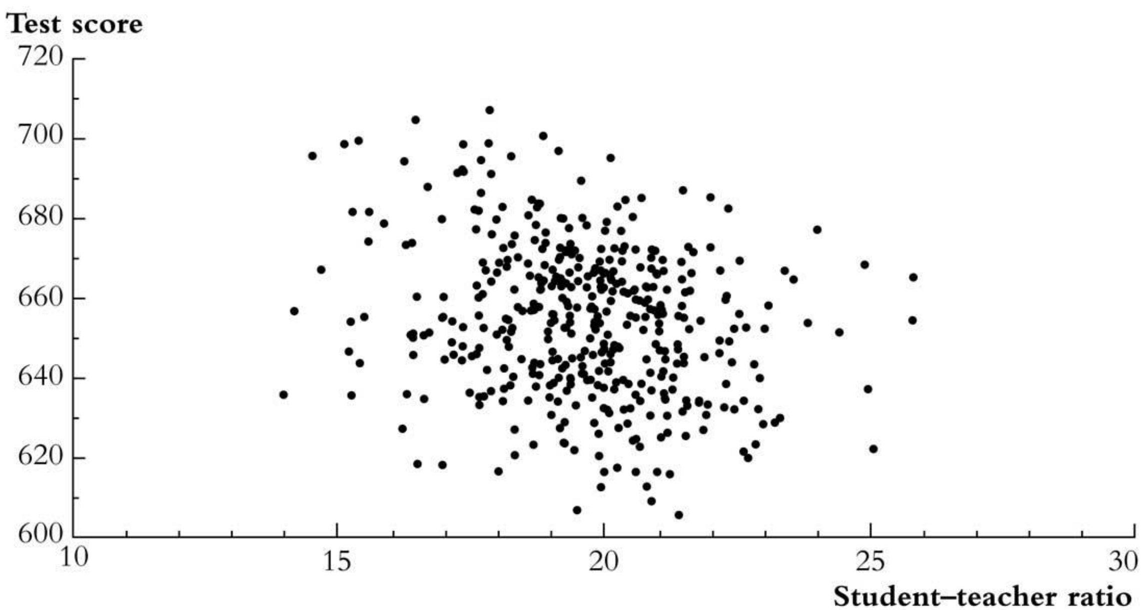
$$\hat{Y}_i = 1 + 1.5 X_i$$

2) Estimation: OLS

- Example with a lot of data:
- What is the effect on test scores of reducing STR (student-teacher ratio) by one unit?
-

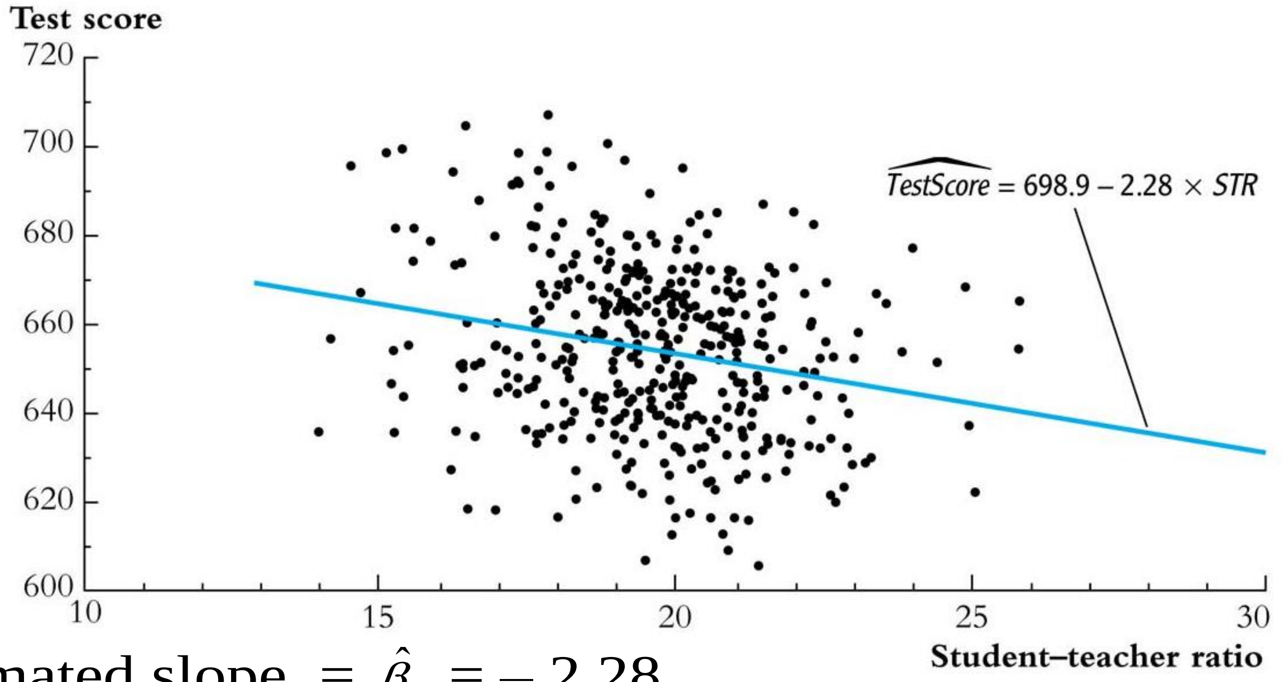
FIGURE 4.2 Scatterplot of Test Score vs. Student–Teacher Ratio (California School District Data)

Data from 420 California school districts. There is a weak negative relationship between the student–teacher ratio and test scores: The sample correlation is −0.23.



2) Estimation: OLS

- Estimation results: (Stata will do OLS, so we don't have to calculate things by hand)



Estimated slope $= \hat{\beta}_1 = -2.28$

Estimated intercept $= \hat{\beta}_0 = 698.9$

Estimated regression line: $\hat{T}estScore = 698.9 - 2.28 \times STR$

2) Estimation: Stata

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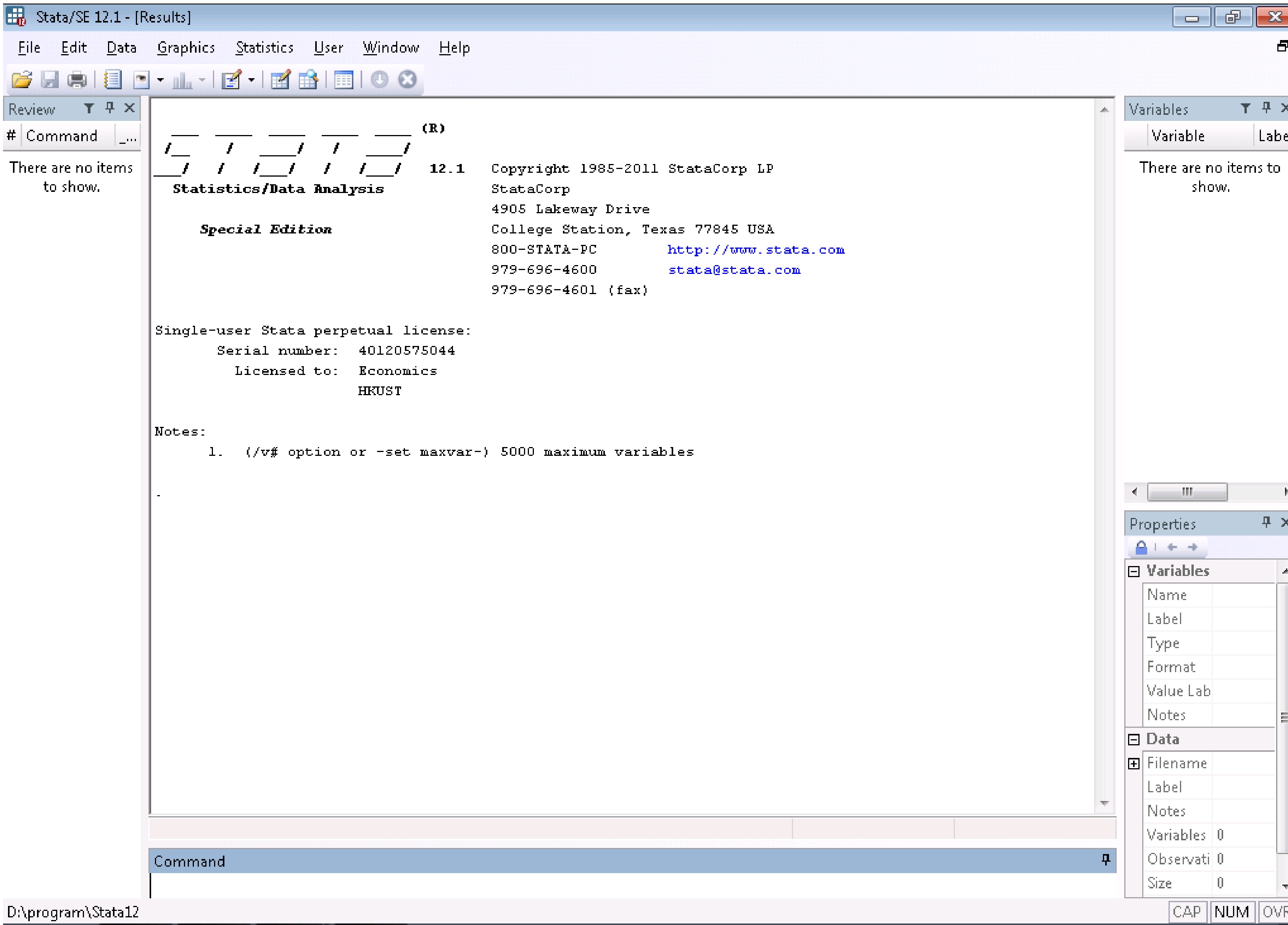
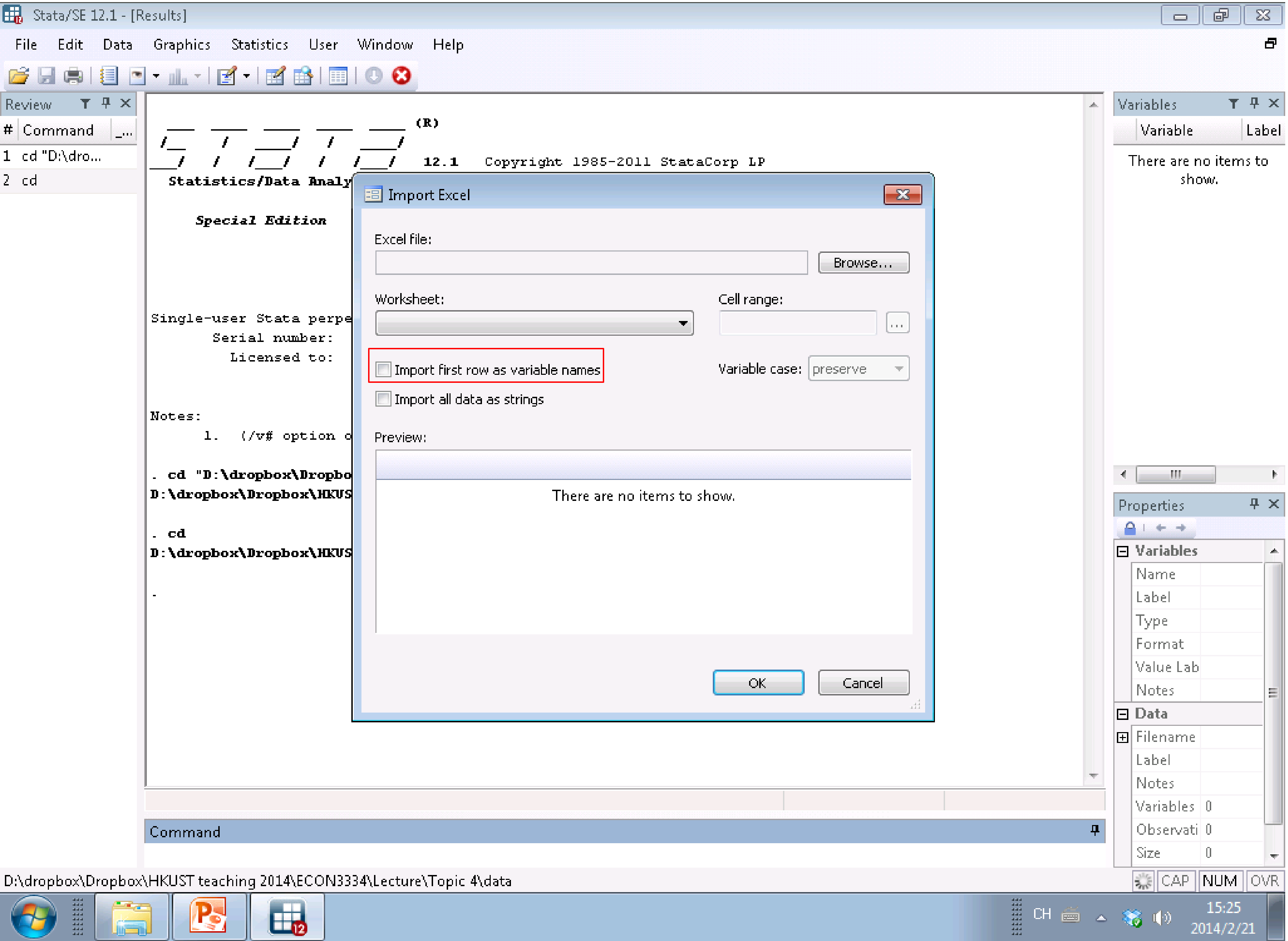
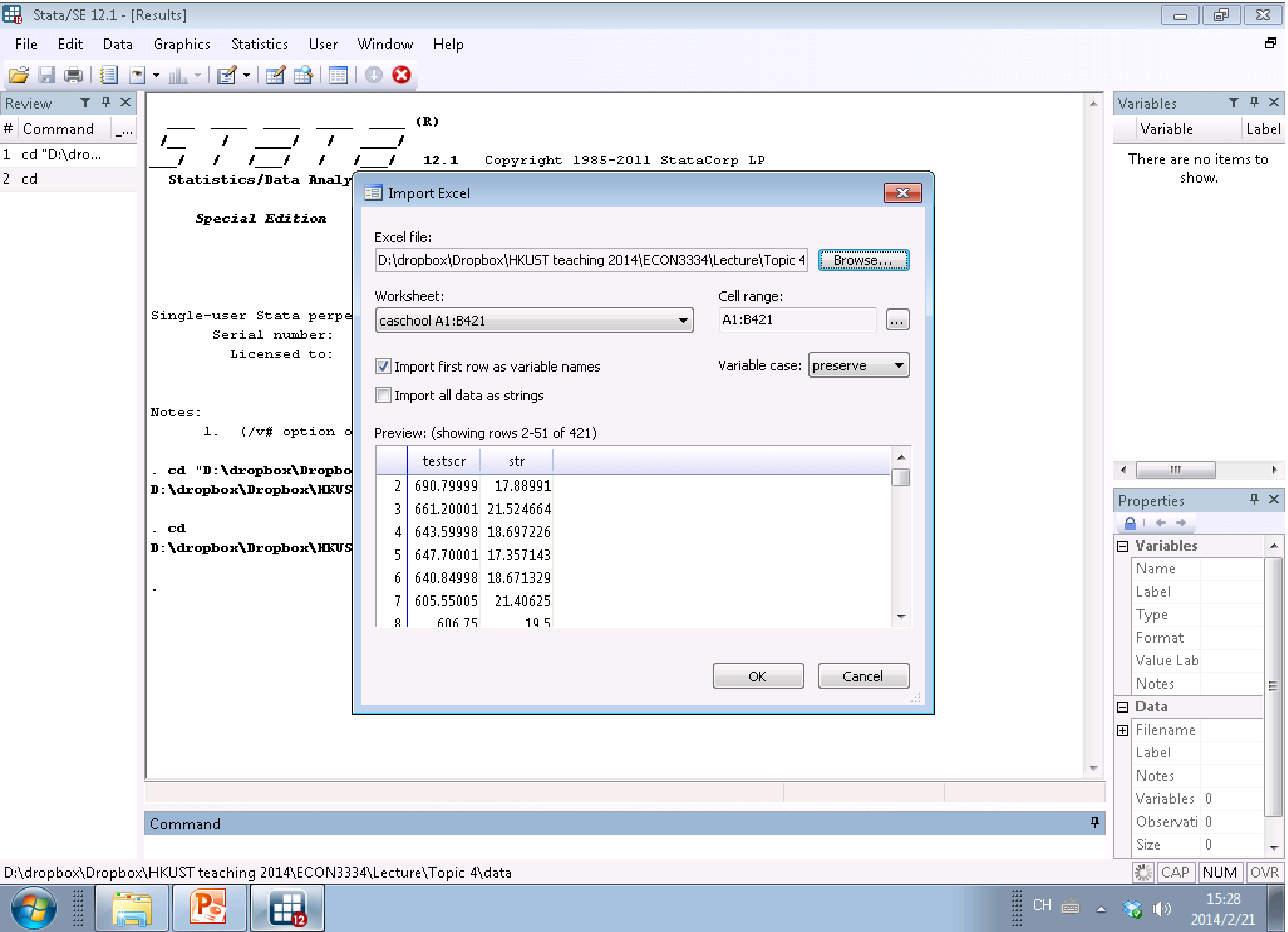


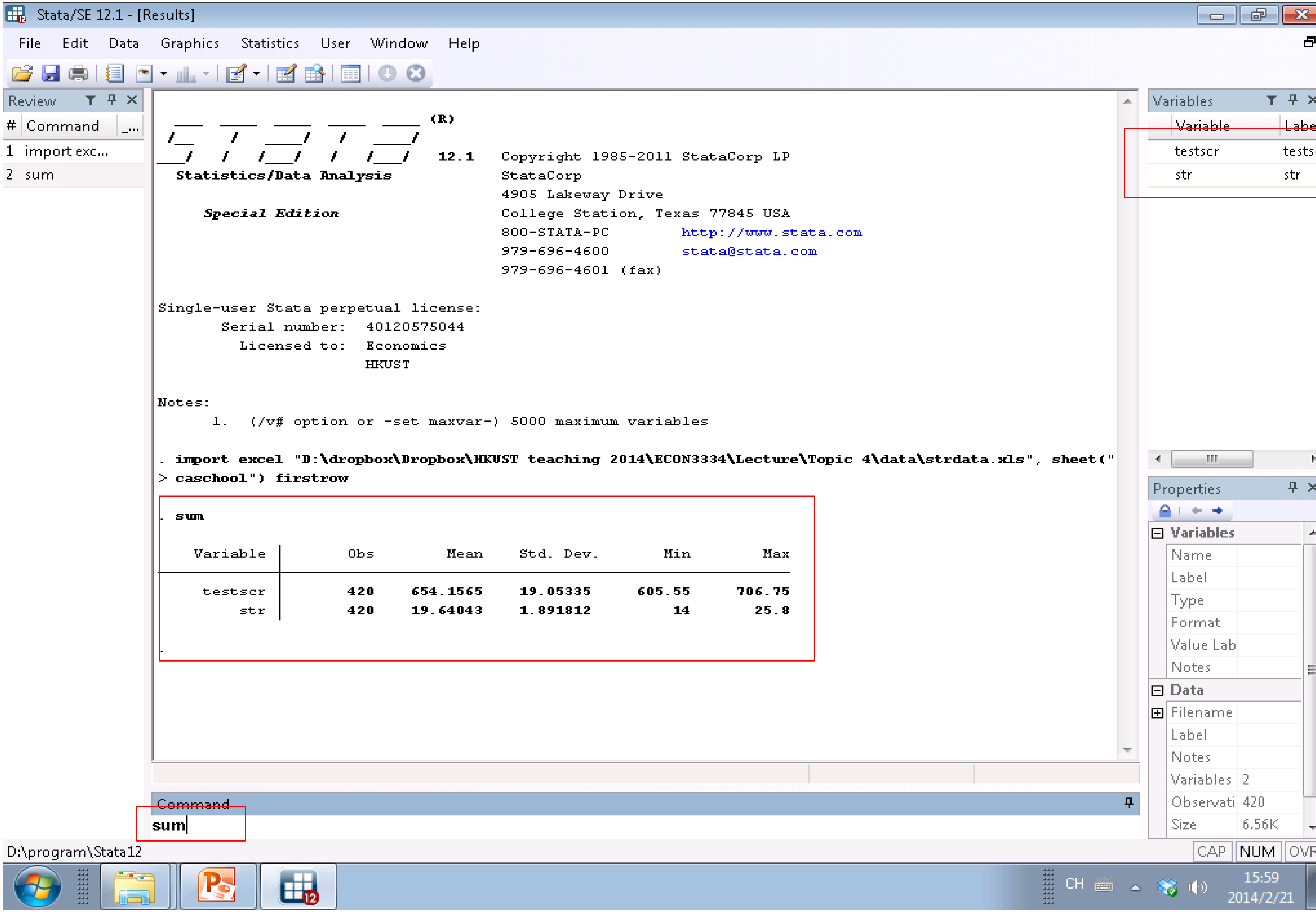
➢ Import data: File---->Import----->Excel spreadsheet



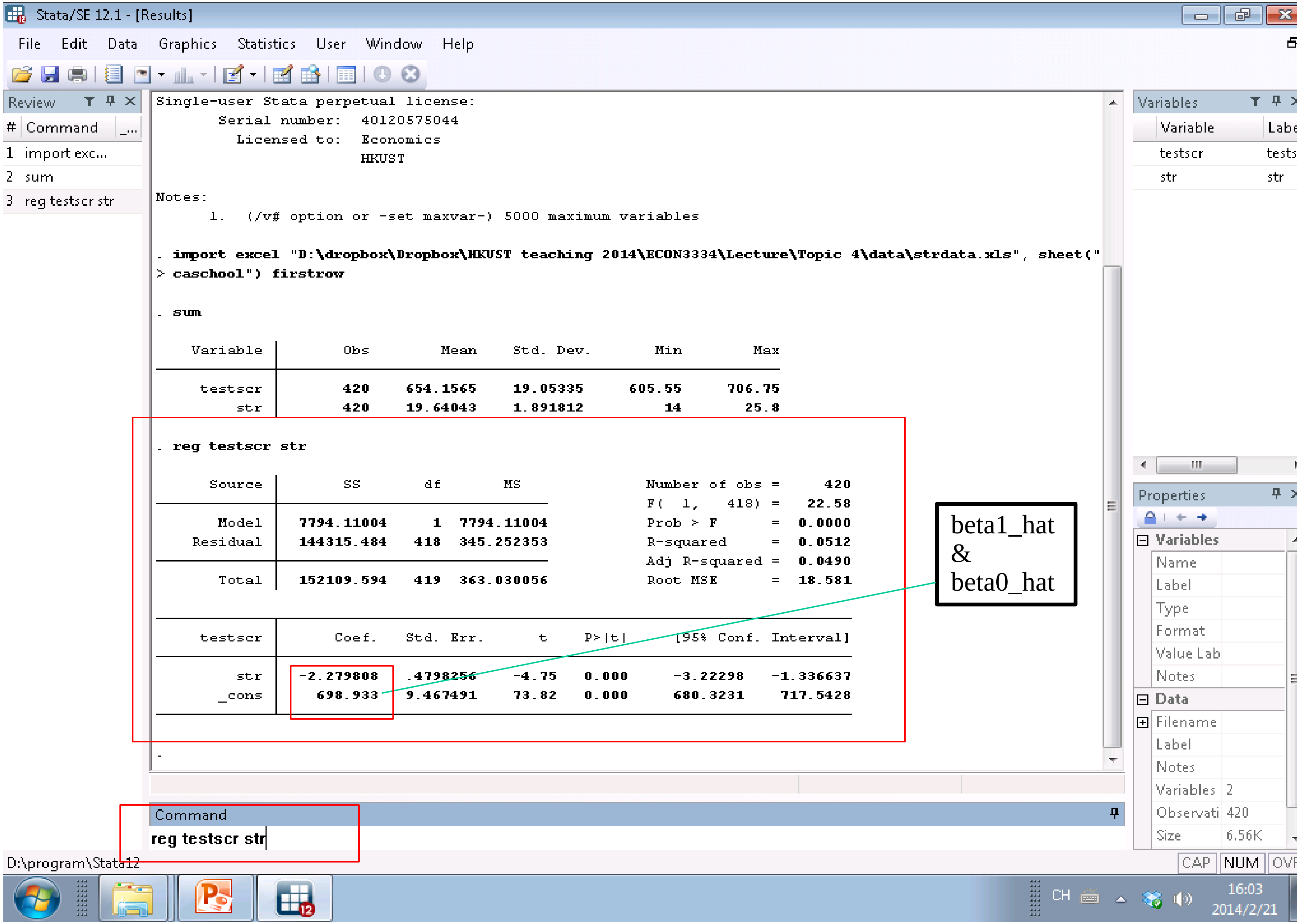
➢ Import data: File---->Import----->Excel spreadsheet



➤ Type sum (providing summary information of the data)



➢ Type reg testscr str



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Single-user Stata perpetual license:
       Serial number:  40120575044
         Licensed to:  Economics
                       HKUST

Notes:
      1.  (/v# option or -set maxvar-) 5000 maximum variables

. import excel "D:\dropbox\Dropbox\HKUST teaching 2014\ECON3334\Lecture\Topic 4\data\strdata.xls", sheet("
> caschool") firstrow

. sum

    Variable |       Obs        Mean    Std. Dev.       Min        Max
-------------+--------------------------------------------------------
     testscr |       420    654.1565    19.05335     605.55     706.75
         str |       420    19.64043    1.891812         14       25.8

. reg testscr str

      Source |       SS       df       MS              Number of obs =     420
-------------+------------------------------           F(  1,   418) =   22.58
       Model |  7794.11004     1  7794.11004           Prob > F      =  0.0000
    Residual |  144315.484   418  345.252353           R-squared     =  0.0512
-------------+------------------------------           Adj R-squared =  0.0490
       Total |  152109.594   419  363.030056           Root MSE      =  18.581

------------------------------------------------------------------------------
     testscr |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         str |  -2.279808   .4798256    -4.75   0.000     -3.22298   -1.336637
       _cons |    698.933   9.467491    73.82   0.000     680.3231    717.5428
------------------------------------------------------------------------------

.
```

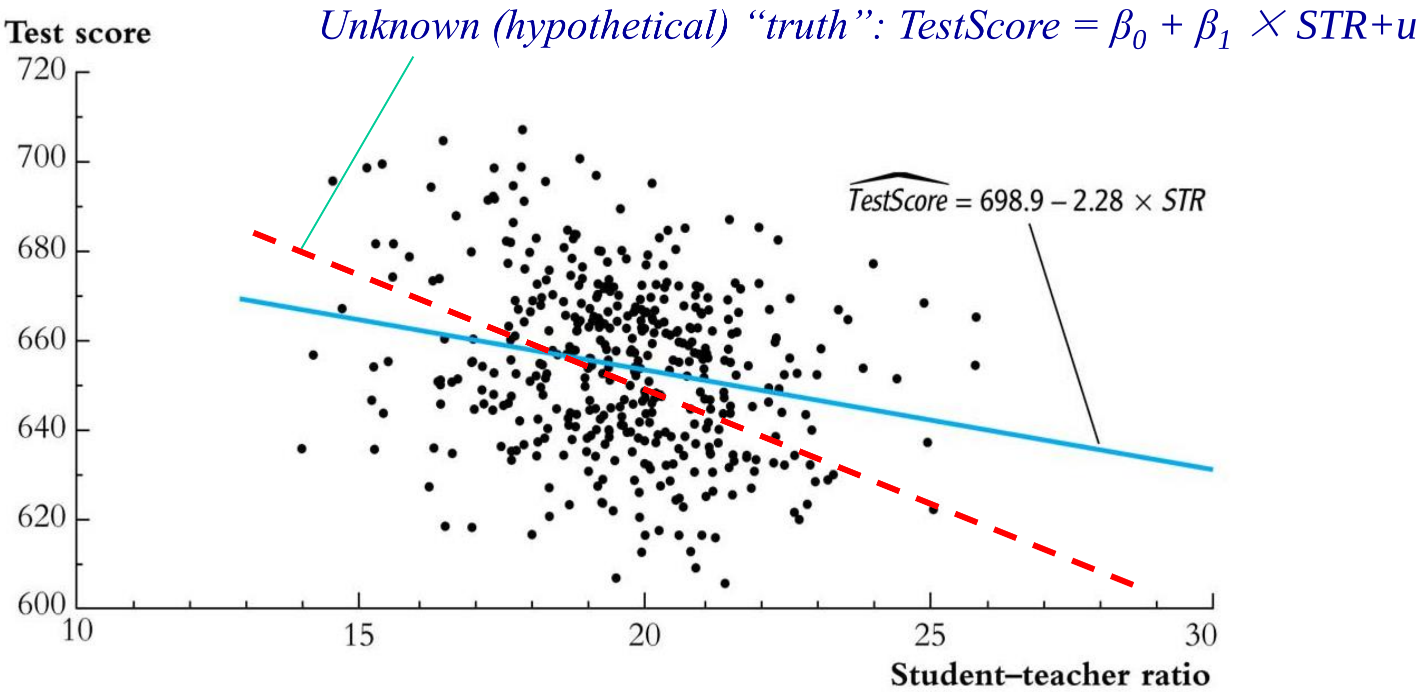

beta1_hat
&
beta0_hat

2) Estimation: OLS

$$\hat{T}estScore = 698.9 - 2.28 \times STR$$

- Districts with one more student per teacher on average have test scores that are 2.28 points lower.
- That is, $\dfrac{\Delta \text{Test score}}{\Delta STR} = -2.28$
- The intercept (taken literally) means that, according to this estimated line, districts with zero students per teacher would have a (predicted) test score of 698.9.
- This interpretation of the intercept makes no sense – it extrapolates the line outside the range of the data – here, the intercept is not economically meaningful.

2) Estimation: OLS



One of the districts in the data set is Antelope, CA, for which *STR* = 19.33 and *Test Score* = 657.8

predicted value: $\hat{Y}_{Antelope} = 698.9 - 2.28\times 19.33 = 654.8$

residual: $\hat{u}_{Antelope} = 657.8 - 654.8 = 3.0$