Topic 4:

Optimization of single variable objective functions

Outline

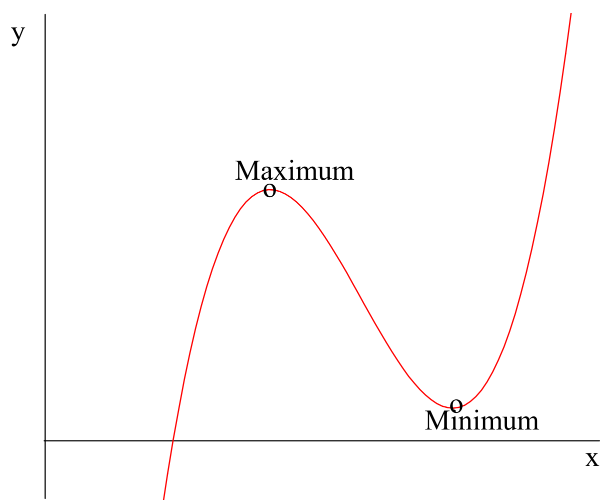
1. Necessary condition of local Maximum/minimum points
2. Sufficient condition 1: First derivative test
3. Sufficient condition 2: Second derivative test
4. Global optimization
5. Taylor Expansion (Series)

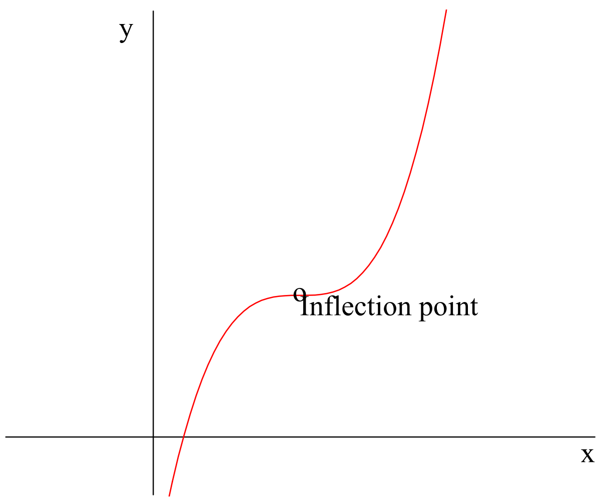
1. Necessary condition of local Maximum/minimum points

- If f has domain D, then
  - $c \in D$ is a (global) maximum point of $f \Leftrightarrow f(x) \leq f(c)$ for all $x \in D$
  - $d \in D$ is a (global) minimum point of $f \Leftrightarrow f(x) \geq f(d)$ for all $x \in D$
  - A point is called (global) extreme point if it is either a minimum or a maximum point

- If there is a neighborhood I of c such that c is the maximum (minimum) point of f on I, then c is called a local maximum (minimum)

- Suppose that a function f is differentiable in an interval I, if an interior point x_0 of I is a local minimum/maximum point , then the tangent line must be horizontal at x_0, i.e., $f'(x_0) = 0$
  - If $f'(x_0) = 0$, then x_0 is called the stationary point of f.
  - The condition $f'(x_0) = 0$ is referred to as the first order condition (FOC)





- For a differentiable function, $f'(x_0) = 0$ is a necessary condition for x_0 to be local minimum or maximum point.
- For a non-differentiable function, above is not true. e.g., $f(x) = |x|, x_0 = 0$ is a local minimum point, but $x_0 = 0$ is not a stationary point.

2. Sufficient condition 1: First derivative test

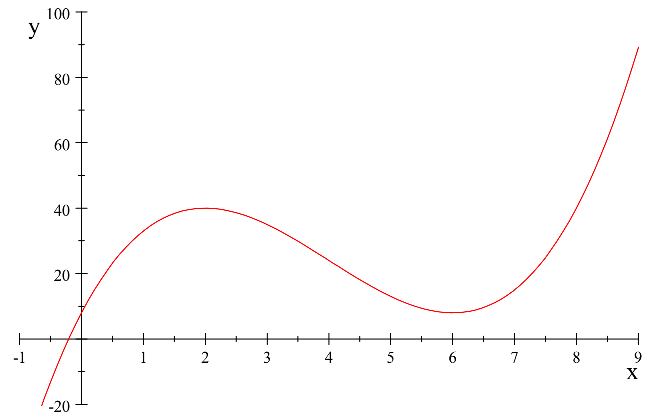
- Recall

  $$f'(x) \geq 0 \text{ on } I \Leftrightarrow f \text{ is increasing on } I$$
  $$f'(x) \leq 0 \text{ on } I \Leftrightarrow f \text{ is decreasing on } I$$

- (First derivative test) Suppose that f is differentiable and x_0 is stationary point, if there is a small neighborhood I of x_0 such that for $x \in I$
  - If $f'(x) \geq 0$ for $x \leq x_0$ and $f'(x) \leq 0$ for $x \geq x_0$, then, $x = x_0$ is a (local) maximum point
  - If $f'(x) \leq 0$ for $x \leq x_0$ and $f'(x) \geq 0$ for $x \geq x_0$, then, $x = x_0$ is a (local) minimum point

- **Example**: Find the local extreme points of the function

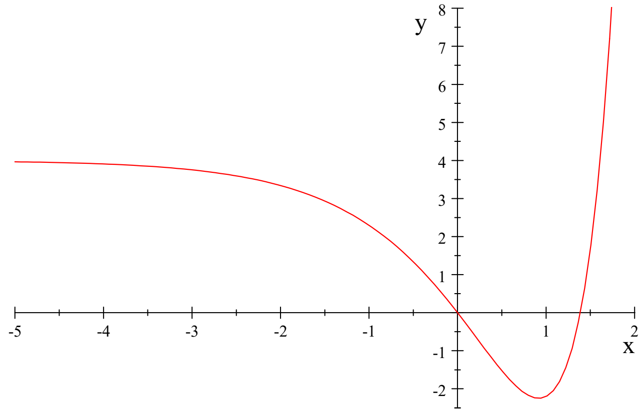
$$f(x) = x^3 - 12x^2 + 36x + 8$$



- FOC: $f'(x) = 3x^2 - 24x + 36 = 3(x-2)(x-6) = 0$
- Stationary points: $x = 2$ or $x = 6$
- In addition, $f'(x)\begin{cases} > 0 \text{ when } x < 2 \\ < 0 \text{ when } 2 < x < 6 \\ > 0 \text{ when } x > 6 \end{cases}$
- from first derivative test, ${x_1}^* = 2$ is a local maximum point and ${x_2}^* = 6$ is a local minimum point.

- Example: Find the local maximum/minimum/inflection points of the function

$$f(x) = e^{2x} - 5e^{x} + 4$$



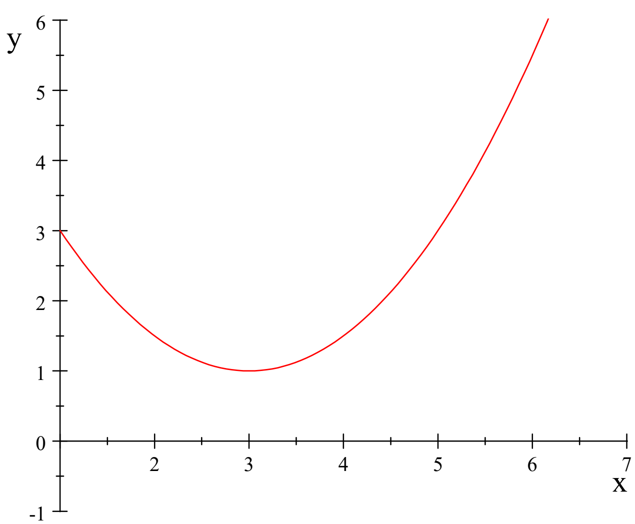
- FOC: $f'(x) = 2e^{2x} - 5e^{x} = e^{x}(2e^{x} - 5) = 0 \text{ when } x = \ln(2.5).$

- In addition $f'(x)\begin{cases} < 0 \text{ when } x < \ln(2.5) \\ > 0 \text{ when } x > \ln(2.5) \end{cases}$

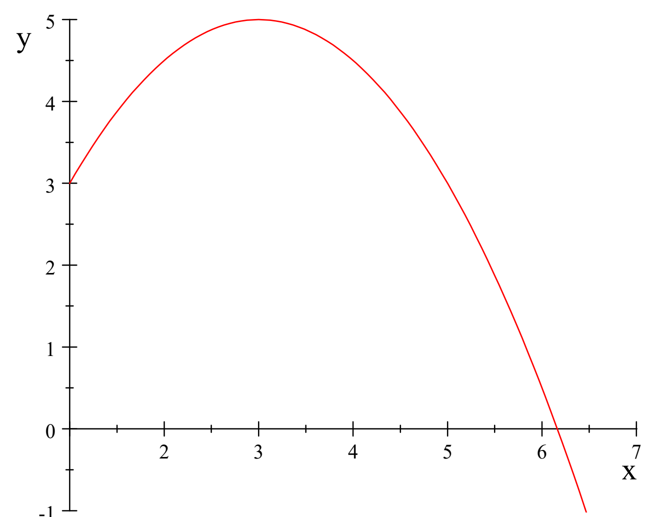
- Therefore, $x = \ln(2.5)$ is local (global?) minimum point.

3. Sufficient condition 2: Second derivative test

- Suppose that $f \in C^2$ and x_0 is a stationary point, I is a small neighborhood of x_0
  - If $f''(x) \geq 0$ for $x \in I$, $f'(x)$ is increasing , $f'(x) \leq 0$ for $x \leq x_0$ and $f'(x) \geq 0$ for $x \geq x_0$, x_0 is minimum point in I
  - If $f''(x) \leq 0$ for $x \in I$, $f'(x)$ is decreasing, $f'(x) \geq 0$ for $x \leq x_0$ and $f'(x) \leq 0$ for $x \geq x_0$, x_0 is maximum point in I



$f''(x) \geq 0$

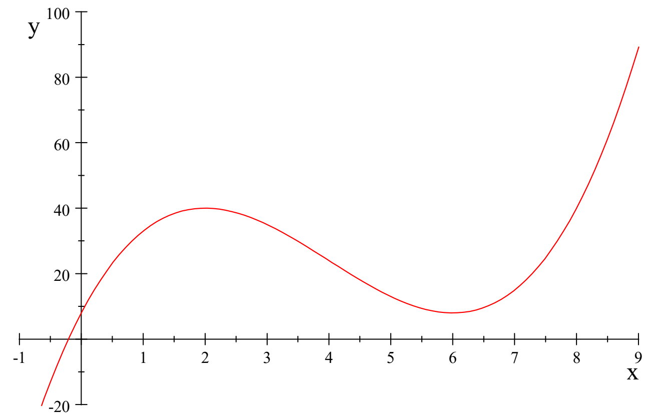


$f''(x) \leq 0$

- [Second order conditions for local extremes] Suppose that $f \in C^2$ and x_0 is a stationary point in I
  - A sufficient condition for x_0 to be local minimum is $f''(x_0) > 0$.
  - A sufficient condition for x_0 to be local maximum is $f''(x_0) < 0$.
  - A necessary condition for x_0 to be local minimum is $f''(x_0) \geq 0$.
  - A necessary condition for x_0 to be local maximum is $f''(x_0) \leq 0$.
  - If $f''(x_0) = 0$, then, x_0 can be a local minimum, a local maximum or an inflection point.

- **Example (revisit)**: Find the local extreme points of the function

$$f(x) = x^3 - 12x^2 + 36x + 8$$



- Recall: Stationary points: $x = 2$ or $x = 6$

- $f''(x) = 6x - 24, f''(2) = -12 < 0, f''(6) = 12 > 0.$

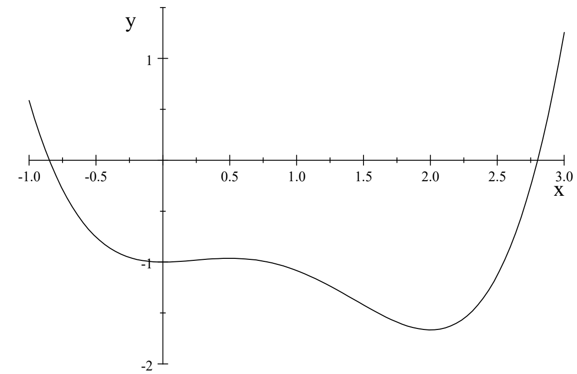
- from second derivative test, $x_1{}^* = 2$ is a local maximum point and $x_2{}^* = 6$ is a local minimum point.

4. Global optimization

- Every extreme point in an interval I must belong to one of the following:
  - Interior points in I where $f'(x) = 0$
  - End points in I
  - Interior points in I where f' does not exist
- Any point satisfying one of the above three conditions is a candidate for extreme points.

- **Example**: Find the (global) extreme values of

$$f(x) = \frac{1}{4}x^4 - \frac{5}{6}x^3 + \frac{1}{2}x^2 - 1, x \in [-1,3]$$

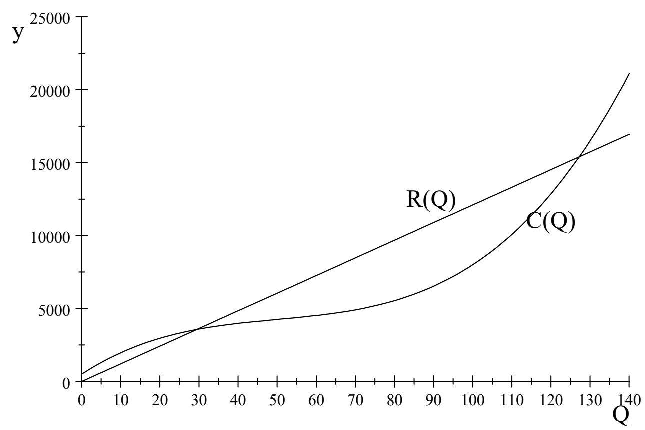


  - $f'(x) = x^3 - \frac{5}{2}x^2 + x = \frac{1}{2}x(2x-1)(x-2)$. $f''(x) = 3x^2 - 5x + 1$
  - Stationary points: $x = 0, \frac{1}{2}, 2$. $f''(0) = 1 > 0, f''(\frac{1}{2}) = -\frac{3}{4} < 0,\ f''(2) = 3 > 0$
  - Therefore, $x = 0,2$ are local minimum and $x = \frac{1}{2}$ is local maximum
  - Candidate for global minimum: $0,2,-1,3$. $f(0) = -1, f(2) = -\frac{5}{3}, f(-1) = \frac{7}{12}$, $f(3) = \frac{5}{4}, x = 2$ is global minimum point.
  - Candidate for global maximum: $\frac{1}{2}, -1,3$. Since $f\left(\frac{1}{2}\right) = -\frac{185}{192}, x = 3$ is global maximum point.
  - Note: If the interval considered in the above example is $(-1,3)$, then f does not have a global maximum.

Profit maximization

- Let $R = R(Q)$ be the total revenue function and $C = C(Q)$ is the total cost function, where Q is the output level, the profit function is
$$\pi(Q) = R(Q) - C(Q).$$
all functions are second-order differentiable.
- The profit-maximizing output level Q^* should satisfy the first-order condition:
$$\pi'(Q^*) = R'(Q^*) - C'(Q^*) = 0$$
- or
$$R'(Q^*) = C'(Q^*)$$
- In order to maximize profit, a firm must equate marginal cost and marginal revenue (MR=MC).

- Example: Suppose that the firm obtains a fixed price P=121 per unit and that the cost function is $C(Q) = 0.02Q^3 - 3Q^2 + 175Q + 500$, Find the production level that maximizes profits, and compute the maximum profit.



- $\pi(Q) = 121Q - C(Q) = -0.02Q^3 + 3Q^2 - 54Q - 500,$

- FOC

$$\frac{d\pi}{dQ} = -0.06Q^2 + 6Q - 54 = -0.06\left(Q^2 - 100Q + 900\right)$$

$$= -0.06(Q - 10)(Q - 90) = 0$$

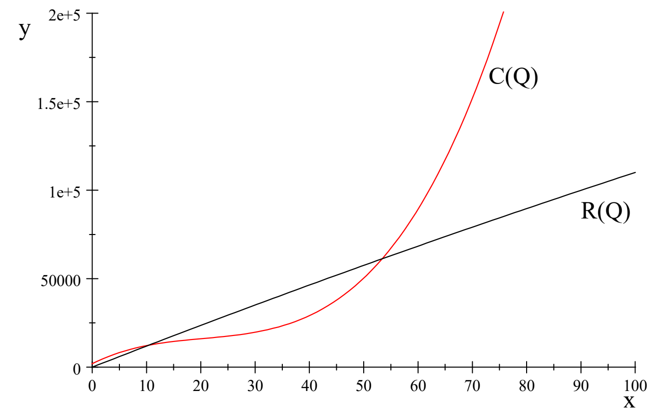
- the stationary points are $Q^* = 10$ or $Q^{**} = 90$

$$\frac{d\pi}{dQ}\begin{cases} < 0 \text{ when } Q < 10 \\ > 0 \text{ when } 10 < Q < 90 \\ < 0 \text{ when } Q > 90 \end{cases} \quad \Rightarrow Q^{**} = 90 \text{ is local maximum}$$

- The profit maximizing output is $Q^{**} = 90$ and the maximized profit is $\pi(Q^{**}) = 4360$

- **Example**: Suppose the firm has a monopoly in the sale of the commodity, assume that the price per unit $P(Q) = 1200 - 2Q$, Suppose the cost function is

$$C(Q) = Q^3 - 61.25Q^2 + 1528.5Q + 2000$$



- Find the production level that maximizes profits, and compute the maximum profit.
- $\pi(Q) = QP(Q) - C(Q) = -Q^3 + 59.25Q^2 - 328.5Q - 2000$,
- FOC: $\frac{d\pi}{dQ} = -3Q^2 + 118.5Q - 328.5 = 0$ when $Q = 3$ or $Q = 36.5$

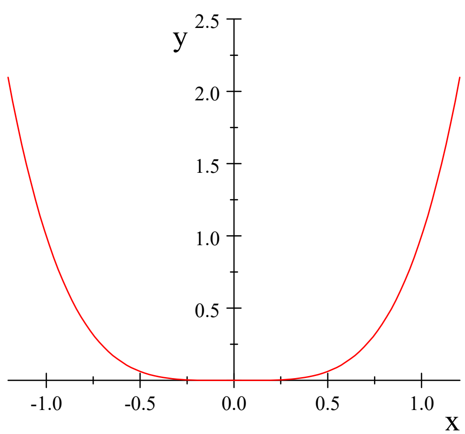
$$\frac{d^2\pi}{dQ^2} = -6Q + 118.5 \begin{cases} > 0 \text{ when } Q = 3 \\ < 0 \text{ when } Q = 36.5 \end{cases}$$

- The profit maximizing output is $Q^* = 36.5$ and the maximized profit is $\pi(Q^*) = 16388.44$

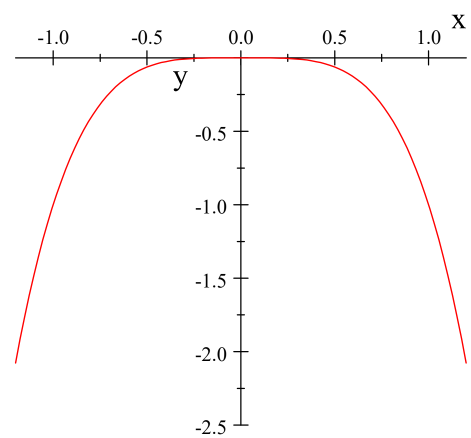
5. Taylor Expansion (Series)

- Consider the following three functions at the point $x = 0$

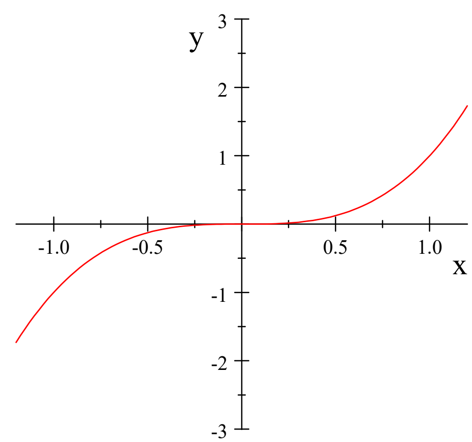
$f(x)$	$f'(x)$	$f''(x)$	$f'(0)$	$f''(0)$	**Point $x = 0$**
x^4	$4x^3$	$12x^2$	0	0	minimum
$-x^4$	$-4x^3$	$-12x^2$	0	0	maximum
x^3	$3x^2$	$6x$	0	0	inflection



$f(x) = x^4$



$f(x) = -x^4$



$f(x) = x^3$

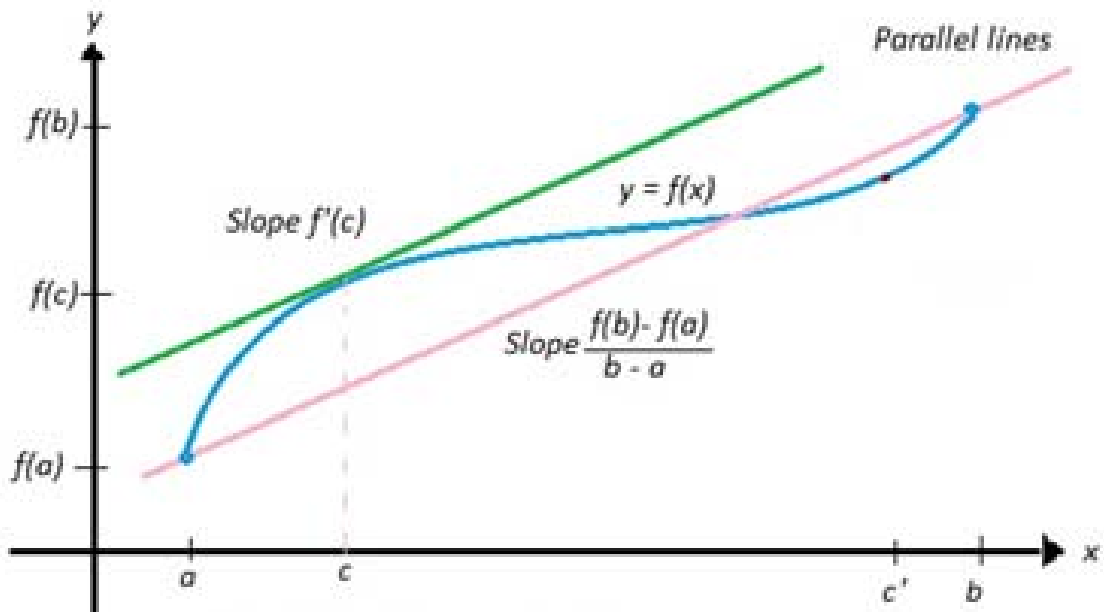
- Second derivative test does not work here
- We can justify the above observation through Taylor expansion of a function $f \in C^{\mathrm{n}}$.

$$f(x) = \frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(\mathrm{n})}(\xi)}{\mathrm{n}!}(x - x_0)^n$$

where ξ is a point between x and x_0

- In the Taylor expansion, set $n = 1$, you get the Mean Value Theorem: If $f \in C^1$, then there is at least one c between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Mean Value Theorem Graph

- Justification of second derivative test: For $f \in C^2$:

$$f(x) = \frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(\xi)}{2!}(x - x_0)^2$$

where ξ is between x and x_0.

  - If $f'(x_0) = 0$ and $f''(x_0) < 0$, then $f''(\xi) < 0$ when x is close to x_0. therefore $f(x) < f(x_0)$: x_0 is a local maximum.
  - If $f'(x_0) = 0$ and $f''(x_0) > 0$, then $f''(\xi) > 0$ when x is close to x_0. therefore $f(x) > f(x_0)$: x_0 is a local minimum.

- If $f'(x_0) = 0$, $f''(x_0) = 0$ and $f'''(x_0) > 0$, set $n = 3$ in the Taylor expansion,

$$\begin{aligned} f(x) &= \frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(\xi)}{3!}(x - x_0)^3 \\ &= f(x_0) + \frac{f'''(\xi)}{3!}(x - x_0)^3 \end{aligned}$$

- when x is close to x_0, $f'''(\xi) > 0$, but $(x - x_0)^3 \begin{cases} > 0 \text{ when } x > x_0 \\ < 0 \text{ when } x < x_0 \end{cases}$

- x_0 can not be maximum or minimum point

- A general result: Suppose $f^{(k)}(x_0) = 0$ for $k = 1,2, \dots, n-1$ and $f^{(\mathrm{n})}(x_0) \neq 0$, then,
  - if n is odd, x_0 is a inflection point;
  - if n is even and $f^{(\mathrm{n})}(x_0) > 0$, x_0 is a local minimum point;
  - if n is even and $f^{(\mathrm{n})}(x_0) < 0$, x_0 is a local maximum point.

- **Example**: $f(x) = x^{\mathrm{n}}$ for $n \geq 2$. If n is odd, then $x_0 = 0$ is not maximum/minimum point; if n is even, then $x_0 = 0$ is a minimum point.

- **Example**: Find the stationary points of
$$f(x) = (x-1)^4 - 4(x-1)^3 - 1$$
are they local maximum, local minimum, or inflection points?