ECON3133 Microeconomic Theory II

Tutorial #3: More on the Cost Function

$$\sigma = \frac{f_l f_k}{f f_{kl}}$$

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Do not need to know where this comes from!

Today's tutorial:

- announcements!

- More on the production function
  - Deriving the equation for an isoquant and finding its slope and curvature, then drawing it
- The cost function:
  - Deriving contingent input demand functions from a production function
  - Deriving the cost function from a production function
  - Verifying Shephard's Lemma
  - Verifying the properties of the cost function
  - Finding the short run cost function and illustrating the envelope property
- Finding the production function from a given cost function: textbook exercise 10.6

1. production function → cost function

2. cost function → production

Re-cap: the cost minimization problem

- The cost minimization problem is:
  - $\min_{k,l} vk + wl \text{ s.t. } \bar{q} = f(k,l)$
- The solution to the problem are the contingent/conditional demand functions for k and l
  - $k^c = k^c(v, w, \bar{q})$
  - $l^c = l^c(v, w, \bar{q})$
- Then the cost function is given by $C^* = C^*(v, w, \bar{q})$
- Note that the cost function and contingent/conditional input demand functions depend only on v, w and $\bar{q}$
- Shephard's Lemma:
  - $\frac{\partial C^*}{\partial v} = k^c(v, w, \bar{q})$
  - $\frac{\partial C^*}{\partial w} = l^c(v, w, \bar{q})$

$v, w, \bar{q}$

$c = vk + wl$

$c = v k^c(v, w, \bar{q}) + w l^c(v, w, \bar{q})$

$C^* = c(v, w, \bar{q})$

· cost function.

Re-cap: the shape of the isoquants of a given production function

- Example: Consider the cost minimisation problem with production function $q = \left(k^{1/2} + l^{1/2}\right)^2$

CES $\rho = \frac{1}{2}$.

$\sigma = 2$

1) Find an equation for the isoquants, and its slope. Draw the isoquants
2) Find the contingent demand functions for k and l
3) Verify Shephard's Lemma with regard to both v and w

Re-cap: the shape of the isoquants of a given production function

1) Find an equation for the isoquants, and its slope. Draw the isoquants

- To find an isoquant, consider the function $\bar{q} = (k^{1/2} + l^{1/2})^2$
- We re-arrange the function to express k in terms of l:

$$\bar{q}^{1/2} = k^{1/2} + l^{1/2}$$

$$k^{1/2} = \bar{q}^{1/2} - l^{1/2}$$

$$k = \left(\bar{q}^{1/2} - l^{1/2}\right)^2$$

q fixed.

$$q = \left(k^{1/2} + l^{1/2}\right)^2$$

fix q at $\bar{q}$

$$\bar{q} = \left(k^{1/2} + l^{1/2}\right)^2$$

Re-cap: the shape of the isoquants of a given production function

1) Find an equation for the isoquants, and its slope. Draw the isoquants

1. $l = 0 \Rightarrow K = \bar{q}$
2. $K = 0 \Rightarrow l = \bar{q}$

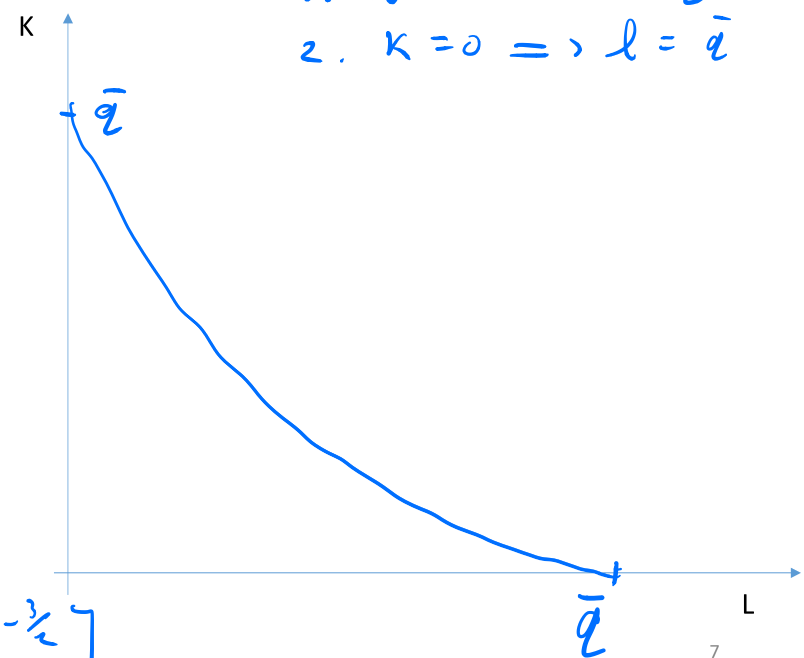
- We then find its slope:

$$k = \left(q^{\frac{1}{2}} - l^{\frac{1}{2}}\right)^2$$

$$\frac{dk}{dl} = 2\left(q^{\frac{1}{2}} - l^{\frac{1}{2}}\right)(-)\frac{1}{2}\, l^{-\frac{1}{2}}$$

$$= -\left(q^{\frac{1}{2}} - l^{\frac{1}{2}}\right) l^{-\frac{1}{2}}$$

$$= -k^{\frac{1}{2}} l^{-\frac{1}{2}}$$



- We may also find its curvature:

$$\frac{d^2k}{dl^2} = -\frac{d}{dl}\left[k(l)^{\frac{1}{2}}\, l^{-\frac{1}{2}}\right]$$

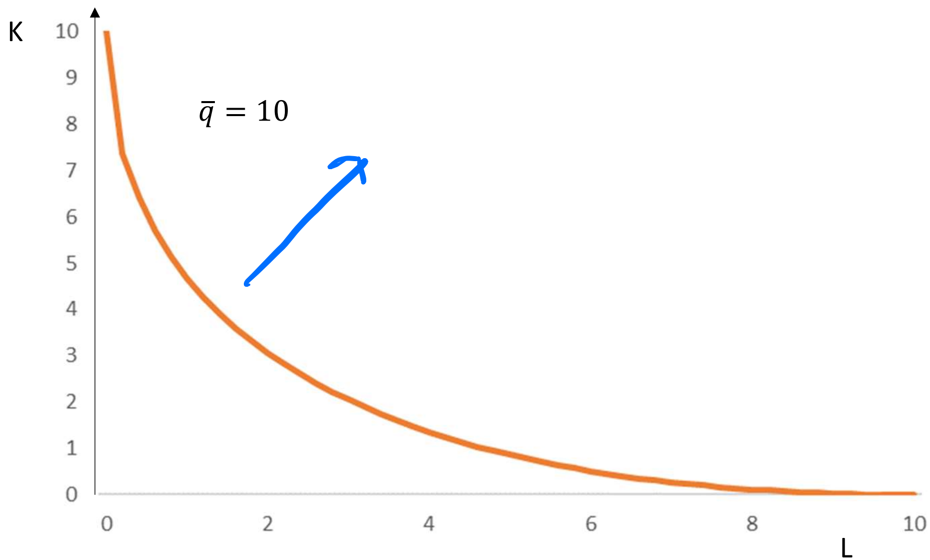
$$= -\left[\frac{1}{2} k^{-\frac{1}{2}} \frac{dk}{dl}\, l^{-\frac{1}{2}} + k^{\frac{1}{2}}(-)\, l^{-\frac{3}{2}}\right]$$

$$= -\left[\frac{1}{2}(kl)^{-\frac{1}{2}} \underbrace{\frac{dk}{dl}}_{<0} \underbrace{-\,k^{\frac{1}{2}} l^{-\frac{3}{2}}}_{<0}\right] > 0$$

Re-cap: the shape of the isoquants of a given production function

1) Find an equation for the isoquants, and its slope. Draw the isoquants

- An isoquant for $q = 10$ is shown below:



The cost function: finding the contingent demand functions and the cost function

2) Find the contingent demand functions for k and l

$\min \underbrace{vk + wl}_{c}$ s.t. $\bar{q} = f(k, l)$

Step1: Form the Lagrangian

$$\mathcal{L} = vk + wl + \lambda\left[\bar{q} - \left(k^{1/2} + l^{1/2}\right)^2\right]$$

Step 2: Find the first order conditions

$$\frac{\partial \mathcal{L}}{\partial k} = v - \lambda\left(k^{1/2} + l^{1/2}\right)k^{-1/2} \qquad \underline{= 0}$$

$$\frac{\partial \mathcal{L}}{\partial l} = w - \lambda\left(k^{1/2} + l^{1/2}\right)l^{-1/2} \qquad \underline{= 0}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \bar{q} - \left(k^{1/2} + l^{1/2}\right)^2 \qquad = 0$$

The cost function: finding the contingent demand functions and the cost function

- Step 3: Divide 1) by 2) and solve for l in terms of k

$$\frac{v}{w} = \frac{\lambda\left(k^{1/2}+l^{1/2}\right)k^{-1/2}}{\lambda\left(k^{1/2}+l^{1/2}\right)l^{-1/2}}$$

$$= \frac{k^{-1/2}}{l^{-1/2}}$$

$$= \left(\frac{l}{k}\right)^{1/2}$$

$$\Rightarrow \frac{v}{w}k^{1/2} = l^{1/2} \Rightarrow \boxed{l = \frac{v^2}{w^2}k}$$

$$l = \frac{v^2}{w^2} k$$

The cost function: finding the contingent demand functions and the cost function

- Step 4: Substitute into equation 3) to find l in terms of $v, w, \bar{q}$

$$\bar{q} = \left(k^{1/2} + l^{1/2}\right)^2$$

$$= \left[k^{1/2} + \left(\frac{v^2}{w^2} k\right)^{1/2}\right]^2$$

$$= \left(k^{1/2} + \frac{v}{w} k^{1/2}\right)^2$$

$$= k\left(1 + \frac{v}{w}\right)^2$$

$$= k\left(\frac{w+v}{w}\right)^2$$

Then:

$$k^c = \bar{q}\frac{w^2}{(v+w)^2}$$

- This is the contingent demand function for k

- By symmetry, we also have the contingent demand function for l

$$l^c = \bar{q}\frac{v^2}{(v+w)^2}$$

Verify Shephard's lemma for the given cost function

3) Verify Shephard's Lemma with respect to v ie $\frac{\partial C^*}{\partial v} = k^c(v, w, \bar{q})$

Step 1: Find the cost function $C(v, w, \bar{q})$

$$C^* = vk^c + wl^c$$

$$= v\bar{q}\frac{w^2}{(v+w)^2} + w\bar{q}\frac{v^2}{(v+w)^2}$$

$$= \frac{\bar{q}}{(v+w)^2}\left[vw^2 + wv^2\right]$$

$$= \frac{\bar{q}vw[v+w]}{(v+w)^2} = \frac{\bar{q}vw}{v+w}$$

$$C^* = \frac{\bar{q}vw}{v+w}$$ the cost function

Verify Shephard's lemma for the given cost function

Step 2: Differentiate this cost function with respect to v

$$\frac{\partial C^*(v,w,\bar{q})}{\partial v} = \frac{\partial}{\partial v}\left[\frac{\bar{q}vw}{v+w}\right]$$

$$= \frac{\partial}{\partial v}\left[\frac{\bar{q}\,vw}{v+w}\right]$$

$$= \frac{\partial}{\partial v}\left[\bar{q}\,vw(v+w)^{-1}\right]$$

$$= \bar{q}\,w(v+w) - \bar{q}\,vw(v+w)^{-2}$$

$$= \frac{\bar{q}\,w(v+w) - q\,vw}{(v+w)^2}$$

$$= \frac{\bar{q}\,[wv - wv + w^2]}{(v+w)^2}$$

$$= \bar{q}\,\frac{w^2}{(v+w)^2} = K^c$$

Verify Shephard's lemma for the given cost function

- We may argue that by symmetry $\frac{\partial C^*(v,w,\bar{q})}{\partial w} = l^c$

$$l^c = \bar{q} \frac{v^2}{(v+w)^2}$$

Verify that a given cost function satisfies the properties of a cost function

4) Verify that the cost function $C^*(v, w, \bar{q}) = \frac{\bar{q}vw}{v+w}$ satisfies the properties of a cost function:

- Homogeneity of degree 1 in input prices
- Non-decreasing in q, v, w
- Concavity in input prices
- Average cost and marginal cost are homogeneous of degree 1

Verify that a given cost function satisfies the properties of a cost function

4.1) Homogeneity of degree 1 in input prices

- Definition: A cost function is homogenous of degree 1 in input prices if the following holds for $t \geq 1$:
  - $tC(v, w, \bar{q}) = C(tv, tw, \bar{q})$
- We have $C^*(v, w, \bar{q}) = \frac{\bar{q}vw}{v+w}$

let $C^{*\prime} = \dfrac{\bar{q}(tv)(tw)}{tv + tw}$

$$= \frac{\bar{q}\, t^2 vw}{t(v+w)}$$

$$= t\, \frac{\bar{q}\, vw}{v+w} = t\, C^*$$

Verify that a given cost function satisfies the properties of a cost function

4.2) Non-decreasing in v, w and $\bar{q}$

- We have $C^*(v, w, \bar{q}) = \frac{\bar{q}vw}{v+w}$
- Then:
- $\frac{\partial C^*(v,w,\bar{q})}{\partial v} =$ $k^c = \bar{q}\left(\frac{w}{v+w}\right)^2 > 0$
- $\frac{\partial C^*(v,w,\bar{q})}{\partial w} =$ $l^c = \bar{q}\left(\frac{v}{v+w}\right)^2 > 0$
- $\frac{\partial C^*(v,w,\bar{q})}{\partial \bar{q}} =$ $\frac{vw}{v+w} > 0$

Verify that a given cost function satisfies the properties of a cost function

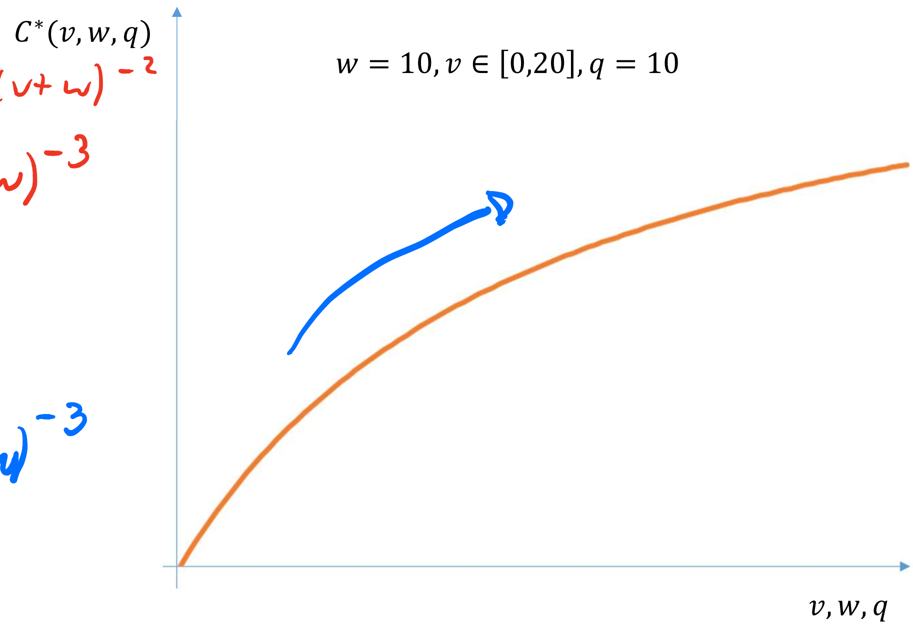
4.3) Concave in v, w and $\bar{q}$

- We have:
- $\frac{\partial C^*(v,w,\bar{q})}{\partial v} = k^c = \bar{q}\frac{w^2}{(v+w)^2} = \bar{q}\, w^2 (v+w)^{-2}$
- $\frac{\partial^2 C^*(v,w,\bar{q})}{\partial v^2} = -2\bar{q}\, w^2 (v+w)^{-3}$

  < 0

- $\frac{\partial C^*(v,w,\bar{q})}{\partial w} = l^c = \bar{q}\frac{v^2}{(v+w)^2}$
- $\frac{\partial^2 C^*(v,w,\bar{q})}{\partial w^2} = -2\bar{q}\, v^2 (v+w)^{-3}$

  < 0



Verify that a given cost function satisfies the properties of a cost function

4.4) Average and marginal costs are homogeneous of degree 1 in v, w and $\bar{q}$

- We have:
- Total cost $= C^*(v,w,q) = \frac{qvw}{v+w}$
- Average Cost $= C^*(v,w,q) = \frac{vw}{v+w}$

let $AC' = \frac{tvtw}{tv+tw}$

$= t^2 \frac{vw}{t(v+w)} = \frac{t\,vw}{v+w} = AC$

- Marginal cost $= \frac{\partial C^*(v,w,q)}{\partial q} = \frac{vw}{v+w}$

The Envelope Relation and Cost functions

- What is the relationship between total costs in the short run and the long run?
  - Given production function $q = f(k, l)$ ✓
  - The long run: all factor inputs may be changed k, l
  - The short run: at least one input factor is held constant $k = k_1$
- We then have cost minimisation problems:

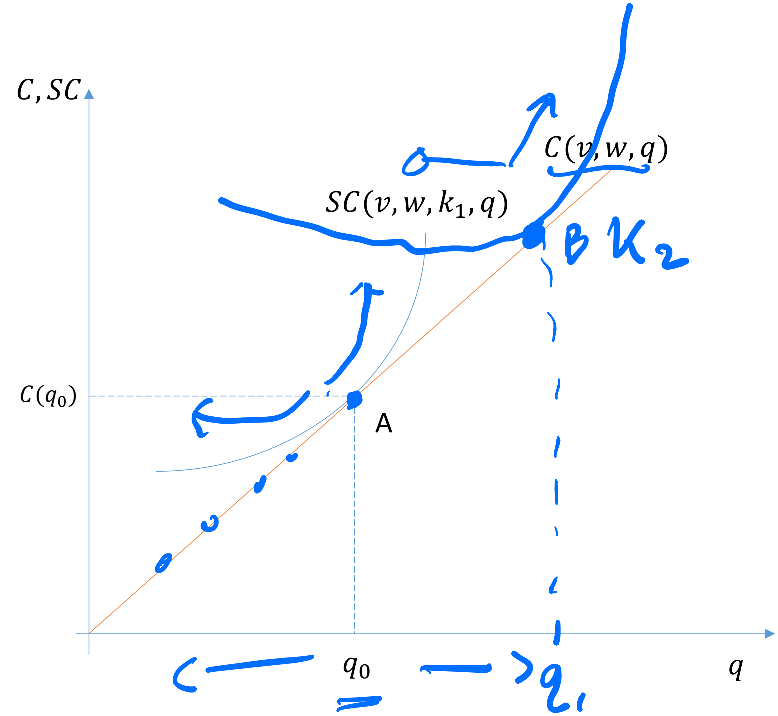
- Long run:
  - $\min_{k,l} vk + wl \text{ s.t. } \bar{q} = f(k, l)$
  - Both k and l may be changed
- The long run problem has solution:
  - $k^c = k^c(v, w, \bar{q})$
  - $l^c = l^c(v, w, \bar{q})$
- And long run cost function $C(v, w, q)$

- Short run:
  - $\min_{l} vk_1 + wl \text{ s.t. } \bar{q} = f(k_1, l)$
  - Only l may be changed; k fixed at k_1
- The short run problem has solution:
  - $k = k_1$
  - $l_s^c = l_s^c(v, w, k_1, \bar{q})$
- And short run cost function $SC(v, w, k_1, q)$

· SC depends on k_1.

The Envelope Relation and Cost functions

- In the long run, k and l are chosen to minimise costs on $C(v, w, q)$ for each level of output, q
- For example, at point A, long run costs are minimised and q_0 is produced
- In the short run, k is fixed at k_1
- Starting at A, if we wanted to produce a different amount ($q \neq q_0$), then we would minimise costs with $k = k_1$
- It is most likely that for $q \neq q_0$ short run minimum costs (with k fixed) would be greater than long run costs (with k flexible)
- Therefore each SC curve tangent to the C curve at one point only eg point A for $k = k_1$



The Envelope Relation and Cost functions

Example: For the production function $q = \left(k^{1/2} + l^{1/2}\right)^2$ calculate long run and short run total costs (with k fixed at $k_1 = 16$) for $v = 4, w = 12$

Step 1:

- For the long run case, calculate contingent input demand functions k^c, l^c and total costs $C(v, w, q)$:
- $k^c = q\frac{w^2}{(v+w)^2} = \frac{144}{256}q = \frac{9}{16}q$
- $l^c = q\frac{v^2}{(v+w)^2} = \frac{1}{16}q$
- $C = q\frac{vw}{v+w} = \frac{48q}{16} = 3q.$

q	k^c	l^c	C
5	5	2.8	0.3
10	10	5.6	0.6
15	15	8.4	0.9
20	20	11.3	1.3
25	25	14.1	1.6
30	30	16.9	1.9
35	35	19.7	2.2
40	40	22.5	2.5
45	45	25.3	2.8
50	50	28.1	3.1

The Envelope Relation and Cost functions

Step 2:

- For fixed k, $k_1 = 16$, derive and calculate the short run total cost function

(a) Express demand for labour in terms of the production function

We have:

- $q = \left(k^{1/2} + l^{1/2}\right)^2$
- In the short run, this becomes:

$$q = \left(k_1^{\frac{1}{2}} + l^{\frac{1}{2}}\right)^2.$$

The Envelope Relation and Cost functions

Step 2: Substitute the expression for l into the short run cost relation:

We have:

- $l = \left(q^{1/2} - k_1^{1/2}\right)^2$
- Then:
- $SC(v, w, q, k_1) =$ $vk_1 + w\left(q^{\frac{1}{2}} - k_1^{\frac{1}{2}}\right)^2$

$= 64 + 12\left(q^{\frac{1}{2}} - 4\right)^2.$

The Envelope Relation and Cost functions

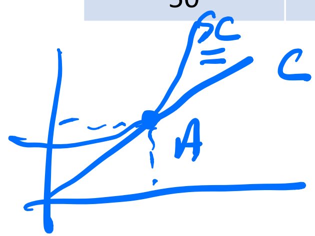
- We have $SC(v, w, q, k_1\ \ ) = vk_1 + w\left(q^{1/2} - k_1^{1/2}\right)^2$
- $SC(12{,}4, q, 16) \qquad = 4 \times 16 + 12 \times \left(q^{1/2} - 4\right)^2$

$SC \quad = 64 + 12\left(q^{1/2} - 4\right)^2$

- For $k_1 = 16$, both long run and short run costs are minimized at $q = 28.4$
- Minimum costs at this point are $C = SC = 85.3$

$$C' = vk + wl$$
$$= q\,\frac{vw}{v+w}$$

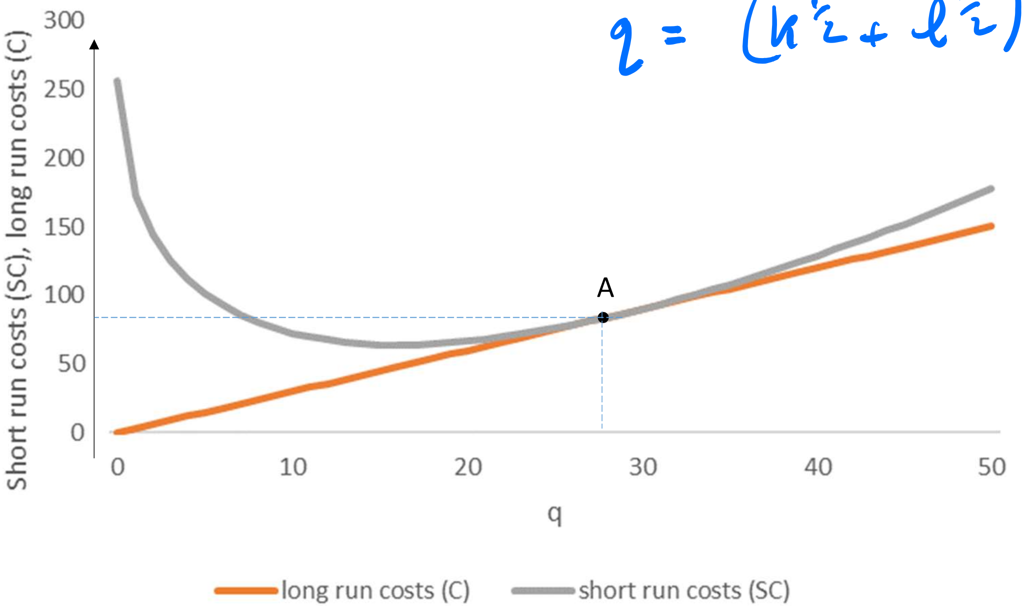
q	$SC(4, 12, q, 16)$	$C = 3q$
5	15	101.3
10	30	72.4
15	45	64.2
20	60	66.7
25	75	76.0
28.4	85.3	85.3
30	90	90.2
35	105	108.1
40	120	128.8
45	135	152.0
50	150	177.2



The Envelope Relation and Cost functions

- For k_1, short run costs are tangent to long run costs at $q = 28.4$ and $C = 85.3$

$$q = \left(k^{\frac{1}{2}} + l^{\frac{1}{2}}\right)^2.$$



Finding the production function from a cost function: textbook exercise 10.6

- Suppose that the total cost function for a firm is given by
  - $C = qw^{2/3}v^{1/3}$

1) Use Shephard's lemma to find the contingent input demand functions for k and l
2) Use the results from 1) to find the underlying production function for q

1) By Shephard's lemma:

$$\frac{\partial C}{\partial w} = l^c = \frac{2}{3} q w^{-1/3} v^{1/3}$$

$$\frac{\partial C}{\partial v} = k^c = \frac{1}{3} q w^{2/3} v^{-2/3}$$

Finding the production function from a cost function: textbook exercise 10.6

2) From part 1) we have:

$$l^c = \frac{2}{3} q \left(\frac{v}{w}\right)^{1/3} \qquad 1)$$

$$k^c = \frac{1}{3} q \left(\frac{w}{v}\right)^{2/3} \qquad 2)$$

- Then:

$$\frac{3k^c}{q} = \left(\frac{w}{v}\right)^{2/3}$$

$$= \left[\left(\frac{v}{w}\right)^{-1/3}\right]^2$$

$$= \left(\frac{2q}{3l^c}\right)^2$$

Then: $\frac{3k^c}{q} = \left(\frac{2q}{3l^c}\right)^2$

$$3k^c = \frac{4}{q}\frac{q^3}{l^{c^2}}$$

$$\frac{27}{4} k^c l^{c^2} = q^3$$

$$q = 3\left(\frac{k^c l^{c^2}}{4}\right)^{1/3}$$

- which is the required production function.

Finding the production function from a cost function: textbook exercise 10.6