ECON3113
Microeconomic Theory I

Tutorial #11

Risk and Uncertainty

Today's tutorial

- The Problem with expected value
- From preferences to a utility function that we can use in an environment of uncertainty
  - Compound lotteries
  - The Independence and continuity axioms
  - The Von-Neumann Morgernstern Theorem and utility function
- Two issues with the approach:
  - the Allais Paradox
  - Invariance of the Von-Neumann Morgernstern utility function up to a linear transformation only
- Risk attitude
- Risk premium and certainty equivalent income

The problem with expected value

- How much would we pay for a ticket to play the following lottery?

	Pay-off		
	$1mn	$100,000	$0
Probability	0.1	0.2	0.7
Expected value	$120,000		

- Answer (1): We would pay the expected value of the lottery ie $120,000. Any less and we would expect to win free money on average
- But what if our whole net worth was $120,000? How much would we pay in this case?

The problem with expected value

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Expected value		$120,000	

- Answer (1): We would pay the expected value of the lottery ie $120,000. Any less and we would expect to win free money on average
- But what if our whole net worth was $120,000? How much would we pay in this case?
  - We would pay (a lot) less than the fair value of the gamble
- It's not how much money we win or lose, but how much we <u>value</u> the money that we win or lose
- <u>In general</u>, how much we would pay will not be given by the expected value of the lottery

Towards a utility function in an environment of uncertainty

- We would like a utility function that we can use when we have uncertainty
- We want to be able to derive this from a preference relation
  - We need to state a preference relation and then use some axioms to derive a utility function
- The same approach as we used to derive the utility function in an environment of certainty
- Our approach: The Von Neumann Morgernstern Theorem and utility function

The Von Neumann Morgernstern Theorem and utility function

- The Von Neumann Morgernstern (vN-M) approach defines objects to be chosen as lotteries
  - A lottery is defined in terms of a state space, the prize with each state and the probability of each state occurring
  - For simplicity, define each state as the prize that is won if that state occurs
- A typical lottery takes the form:

Prize/state	x_1	x_2	…	x_n
Probability	P_1	P_2	…	P_n

- We assume that a consumer has a complete and transitive preference relation $\succsim$ over a given set of lotteries
  - Complete: The preference relation may be applied to any pair of lotteries in the set of lotteries
  - Transitive: If we have lotteries A, B and C such that $A \succsim \mathrm{B}$ and $B \succsim \mathrm{C}$ then we must have $A \succsim \mathrm{C}$
- If the preference $\succsim$ satisfies some properties, then it can be represented by a utility function and expected utility

Two types of lottery: degenerate and compound

- A degenerate lottery gives all probability to a single prize

Prize/state	x_1	x_2	x_3	…	x_{n-1}	x_n
Probability	0	0	1	…	0	0

- A compound lottery is a two-step lottery as follows:
    - First, a draw is made between two lotteries L and L' with the same state space and probability of choosing each α and $1-\alpha$, respectively ($\alpha \in [0,1]$)
    - Second, a draw is made in the chosen lottery, with prize and probability determined according to that lottery
- Given lotteries L and L' and $\alpha \in [0,1]$, a compound lottery is given by $\alpha L + (1-\alpha)L'$

Compound lotteries

- A compound lottery can be reduced to a single lottery
- With lotteries $L = (p_1, p_2, \dots, p_n)$ and $L' = (p_1', p_2', \dots, p_n')$ and $\alpha \in [0,1]$, the compound lottery $\alpha L + (1-\alpha)L'$ has prize x_i occurring with probability $\alpha p_1 + (1-\alpha)p_1'$.
- Example:
    - Consider the two lotteries below
    - Assume $\alpha = 0.25$

Prize/state	x_1	x_2	x_3	x_n
p_1	0.1	0.2	0.3	0.4
$p_1{}'$	0.4	0.3	0.2	0.1
$\alpha p_1 + (1-\alpha)p_1'$	0.325	0.275	0.225	0.175

- A consequentialist individual is one who views the compound lottery and the reduced lottery as identical objects

Two essential axioms

- To be able to go from a preference relation $\succsim$ to a utility function, we need two axioms:

Definition

Preference $\succsim$ over lotteries satisfies the **independence axiom** if for any three lotteries L, L', and L'', and any $\alpha \in [0, 1]$,

$$L \succsim L' \Rightarrow \alpha L + (1 - \alpha) L'' \succsim \alpha L' + (1 - \alpha) L''.$$

- The axiom says that preference between two lotteries should be invariant to the introduction of a third lottery and the resulting compound lotteries

Definition

Preference $\succsim$ over lotteries satisfies the **continuity axiom** if for any three lotteries such that $L'' \succsim L \succsim L'$, there is a $\alpha \in [0, 1]$ such that $L \sim \alpha L' + (1 - \alpha) L''$.

Is the Independence axiom reasonable? The Allais Paradox

- Which lottery do you prefer, lottery A or lottery B?

	\$5mn	**\$1mn**	**\$0**
Lottery A probabilities	0.0	1.0	0.0
Lottery B probabilities	0.98	0.00	0.02

Is the Independence axiom reasonable? The Allais Paradox

- Which lottery do you prefer, lottery C or lottery D?

	\$5mn	\$1mn	\$0
Lottery C probabilities	0.00	0.50	0.50
Lottery D probabilities	0.49	0.00	0.51

Is the Independence axiom reasonable? The Allais Paradox

- A common finding is that people prefer lottery A to lottery B, and lottery D to lottery C

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- Note that $A \succ B$ and $D \succ C$ violates the Independence axiom

Is the Independence axiom reasonable? The Allais Paradox

- Note that $A \succ B$ and $D \succ C$ violates the Independence axiom
  - Let L_0 be the degenerate lottery:

	\$5mn	\$1mn	\$0
L_0 probabilities	0.00	0.00	1.00

  - Then we have:
    - $C = 0.5A + 0.5L_0$
    - $D = 0.5B + 0.5L_0$
  - And the independence axiom requires that $A \succ B \Rightarrow C \succ D$
- Whether the Independence axiom is violated is (still) a major research topic in economics

The Von Neumann Morgernstern Theorem and utility function

Theorem

If a complete and transitive preference $\succsim$ over lotteries satisfies the independence axiom and the continuity axiom, then it can be represented by some utility function $u(x)$ over prizes, that is, for any pair of lotteries $L = (p_1, p_2, ..., p_n)$ and $L' = (p'_1, p'_2, ..., p'_n)$,

$$L \succsim L' \Leftrightarrow \sum_{i=1}^{n} p_i u(x_i) \geq \sum_{i=1}^{n} p'_i u(x_i).$$

- Summary:
  - A preference relation has to be complete and transitive
  - It also has to satisfy the Continuity and Independence axioms
  - Then given two lotteries L and L', then if L is preferred to L' then the preferences may be represented by a utility function and the expected utility from L is greater than or equal to the expected utility from L'
  - Therefore, we may rank lotteries in terms of <u>expected</u> utility

How invariant is Von Neumann Morgernstern utility?

- Re-call that in a world of certainty, utility was a purely ordinal concept
  - In particular, the ranking given by a utility function was invariant to a transformation by any strictly increasing function
- Von Neumann Morgernstern utility, however, is only invariant up to a positive linear transformation:
  - That is, two vN-M utility functions represent the same preferences if and only if one is a positive linear transformation of the other
  - So if $u(x)$ represents $\succcurlyeq$, then so does $A + Bu(x)$, with $\mathrm{A}, B \in \mathbb{R}$ and $B > 0$

How invariant is Von Neumann Morgernstern utility?

- Example:
- Consider two lotteries and vN-M utility function $u(x) = 50 + \frac{5}{6}x$:

Lottery A	**200**	**40**	$u(200)$	$u(40)$	E(u)
Probabilities	0.25	0.75	216.67	83.33	116.67

Lottery B	**300**	**10**	$u(300)$	$u(10)$	E(u)
Probabilities	0.25	0.75	300	58.33	118.75

- So $E\big(u(B)\big) > E\big(u(A)\big)$
- Now consider what happens when we transform $u(x)$ by the strictly increasing $v(x) = \sqrt{u(x)}$

Lottery A	**200**	**40**	$v(200)$	$v(40)$	E(v)
Probabilities	0.25	0.75	14.72	9.13	10.53

Lottery B	**300**	**10**	v(300)	$v(10)$	E(v)
Probabilities	0.25	0.75	17.32	7.64	10.06

- Now $E\big(v(B)\big) > E\big(v(A)\big)$, and so u is not invariant to the transformation to v

Attitudes to risk

- Re-call that in a world of certainty, utility was a purely ordinal concept
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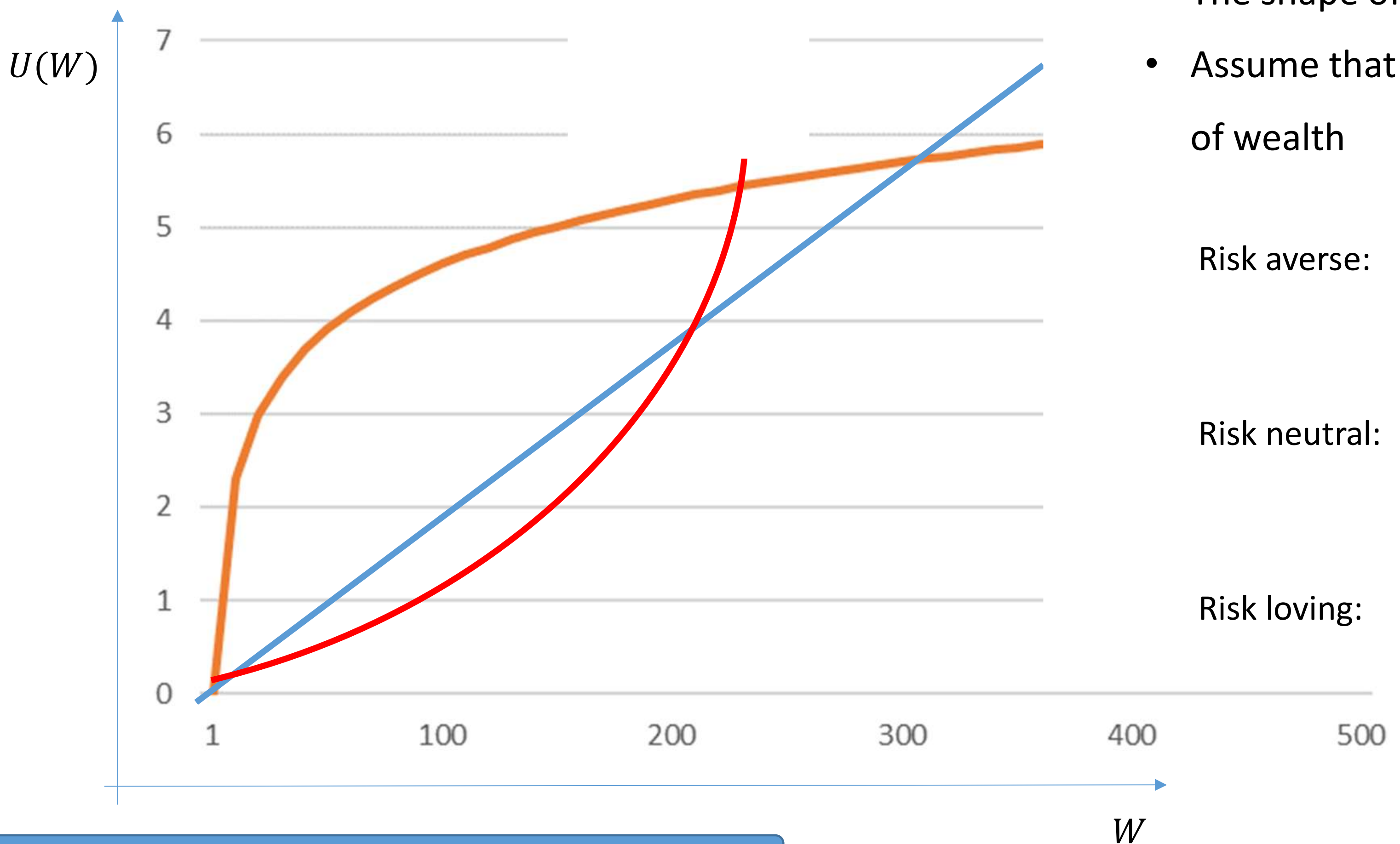
Risk attitude



- The shape of vN-M utility functions
- Assume that we are considering utility of wealth

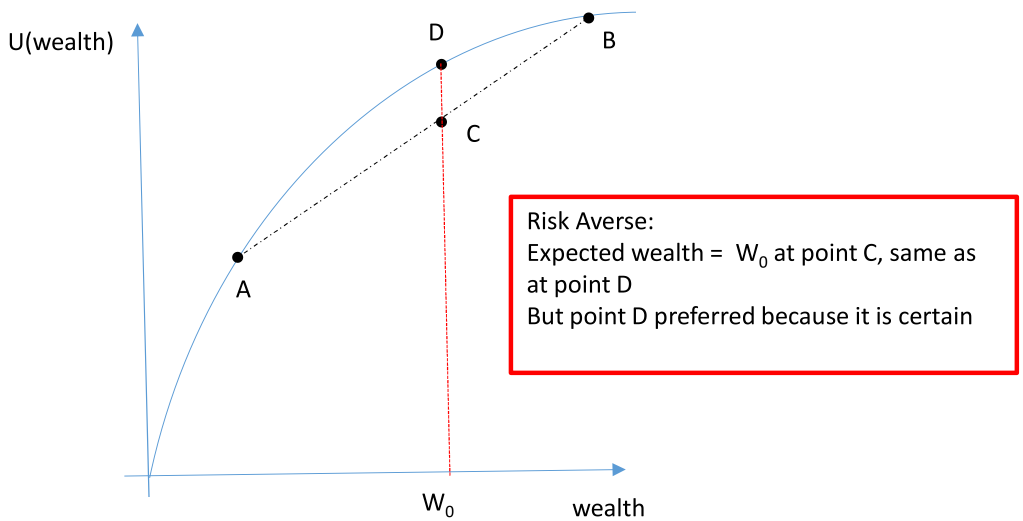
Risk averse:

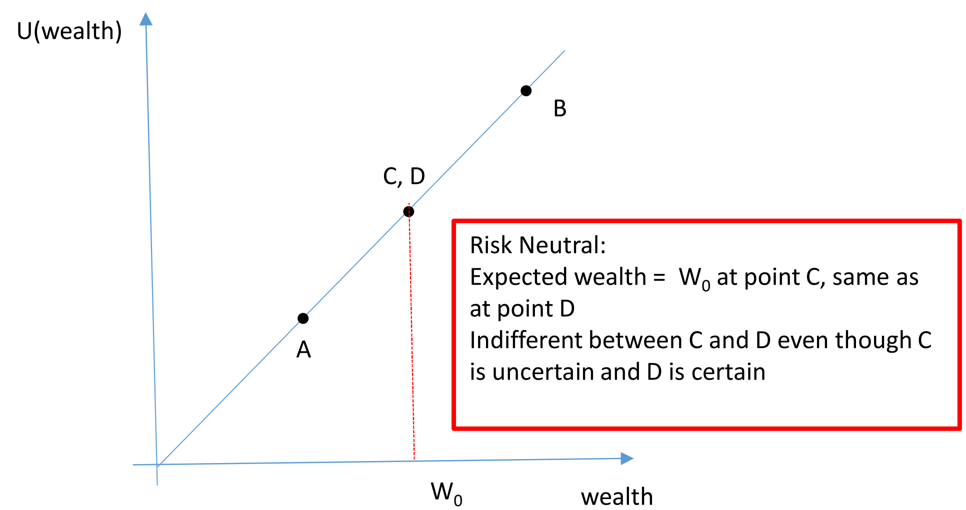
Risk neutral:

Risk loving:

Risk attitude

- Compare two pay-offs under risk aversion and risk neutrality:
  - (i) a pay-off of W with certainty
  - (i) a pay-off with an expected value of W





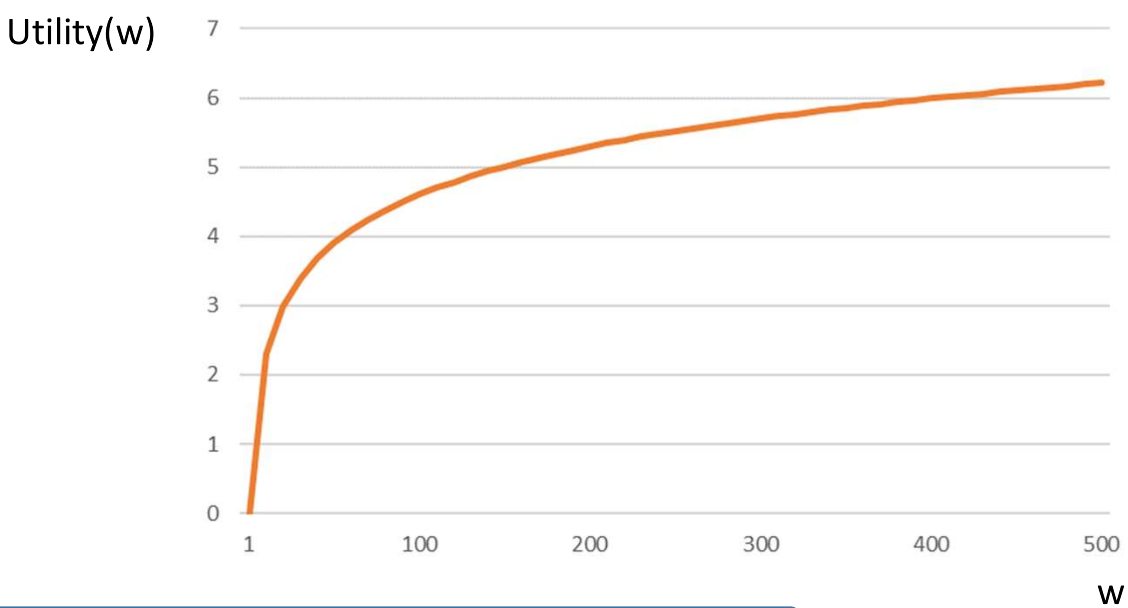
Expected utility: example

- Assume that you have graduated and have been offered two jobs
  - Joint owner of a Fintech start-up
  - Tax accountant at one of the major Accountancy/Consultancy practices
- Over the next 25 years, your average annual income depends on whether business conditions over your 25 year career are Good or Bad
  - Probability of Good business conditions =   0.5
  - Probability of Bad business conditions   =   0.5
- The income per year that you can earn under each scenario is as follows:

	Good (USD 000s)	Bad (USD 000s)
Fintech	1,000	50
Accountant	400	300

Expected utility: example

- Assume that your utility function depends on average annual income over the next 25 years:
  - $U(w) = \ln(\mathrm{w})$, $\quad \mathrm{w} \coloneqq$ average income over the next 25 years



- Risk averse, neutral or lover?

Expected utility: example

- What is your expected annual income over the next 25 years?

- Fintech
  - $EY(w, \boldsymbol{p}) = \frac{1}{2} \times 1{,}000 + \frac{1}{2} \times 50 = 525$
- Accountant
  - $EY(w, \boldsymbol{p}) = \frac{1}{2} \times 400 + \frac{1}{2} \times 300 = 350$

	Good (USD 000s)	Bad (USD 000s)
Fintech	1,000	50
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$P_{good} = p_{bad} = 0.5$

- So expected income is higher if you choose Fintech

- Is this the criteria you would use to decide on your career?

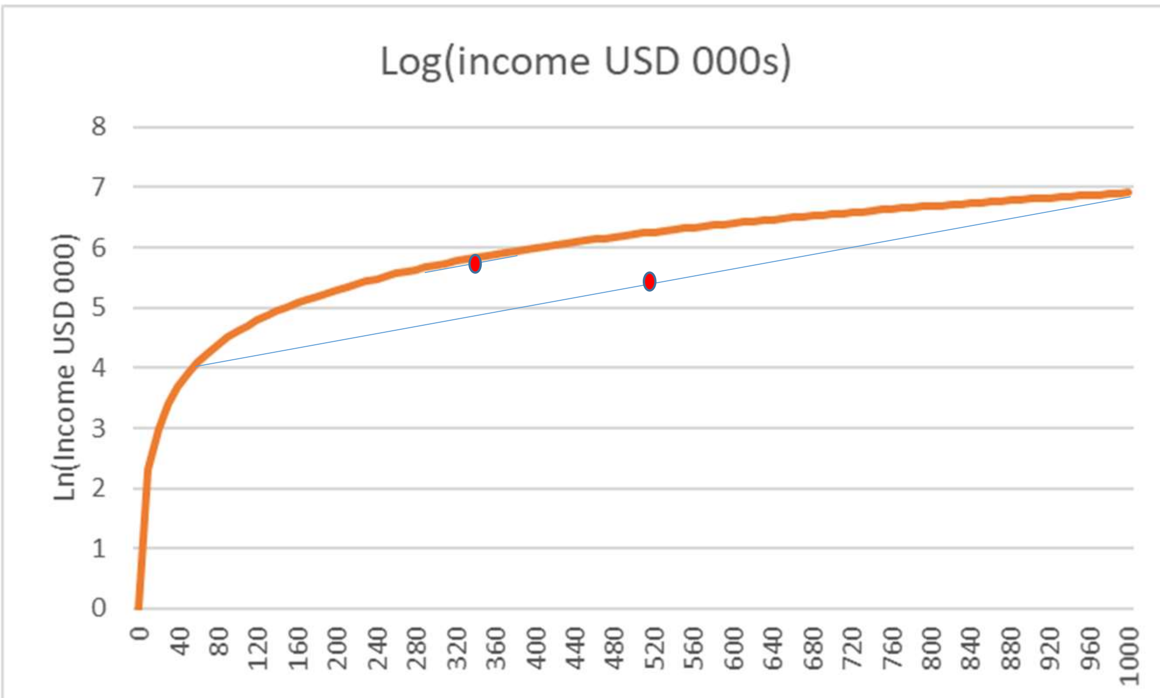
  - The role of expected utility and the costs of uncertainty

Expected utility: example

- What is your expected utility over the next 25 years?
- Fintech
  - $EU(w, p) = \frac{1}{2} \times \ln(1{,}000) + \frac{1}{2} \times \ln(50) \approx 5.41$
- Accountant
  - $EU(w, p) = \frac{1}{2} \times \ln(400) + \frac{1}{2} \times \ln(300) \approx 5.85$
- So expected utility is higher if you choose to be an accountant
  - Even though expected income from this choice is lower
  - You will be (materially) poorer but happier
- What’s going on here?

Expected utility: example

- Utility and average annual income over the next 25 years



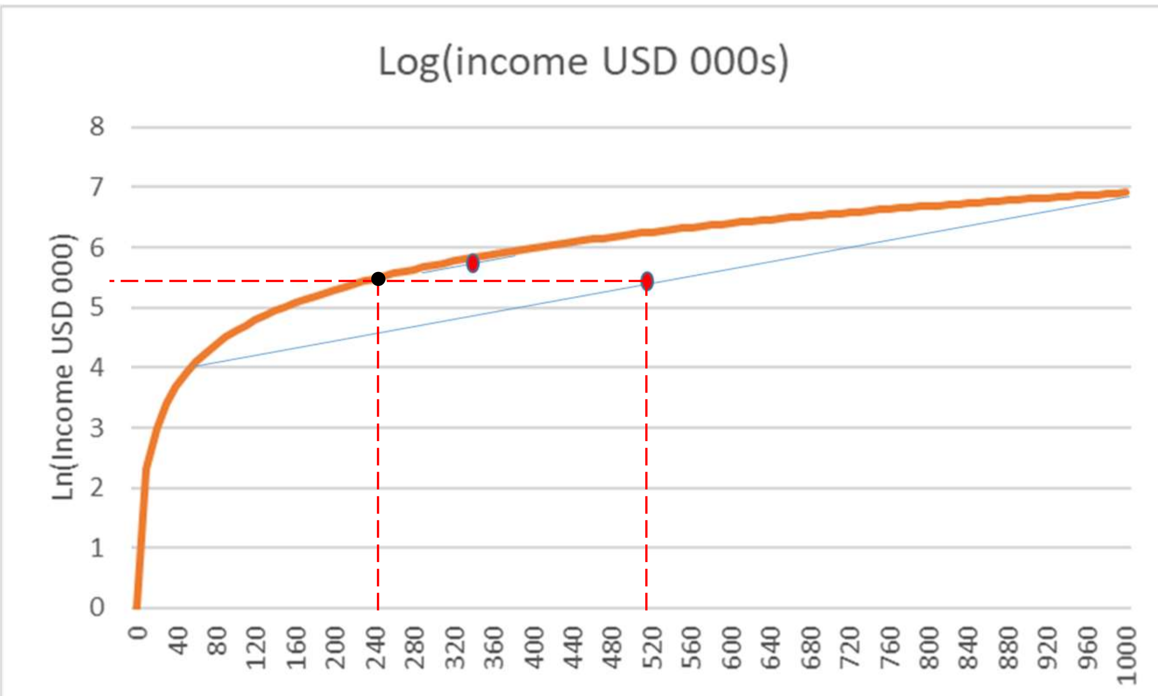
- EU(Fintech) = 5.41
- EU(Accountant) = 5.85

- Depends on:
- Utility of wealth, not wealth itself
- probability of Good business conditions
  - The volatility of outcomes

Average income (25 years)

Expected utility: example

- If you could find a career that paid a salary for 25 years with certainty, how much would it have to pay for you to be indifferent between it and the career in Fintech?

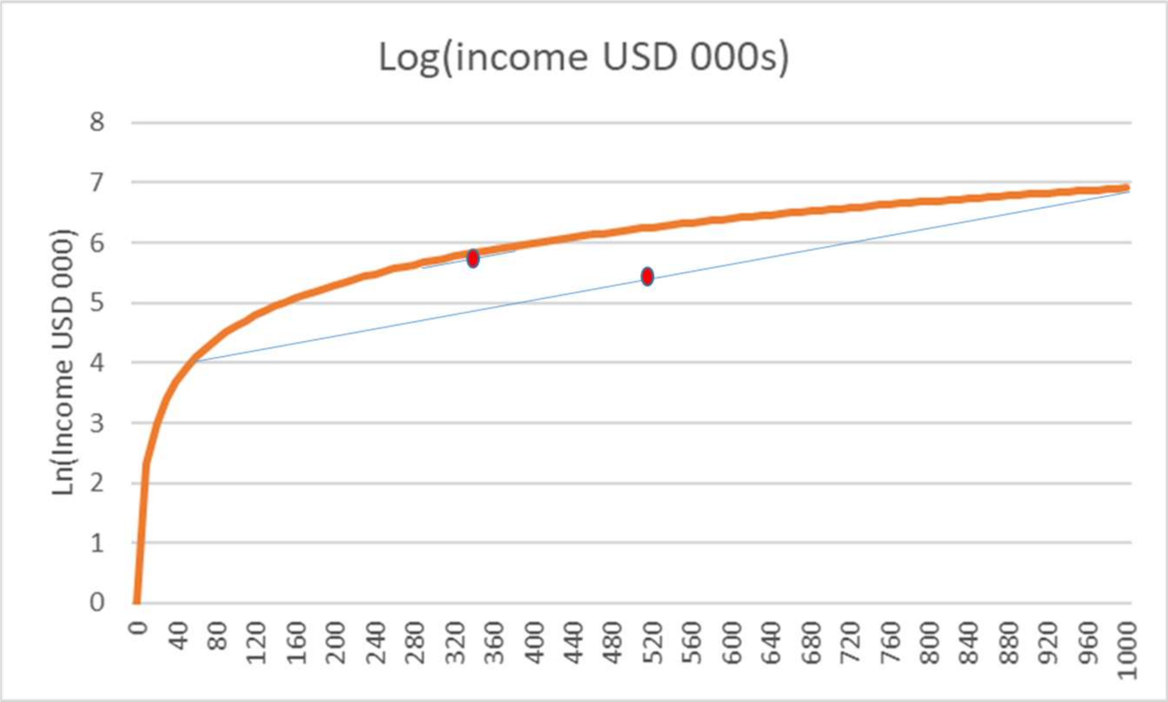


- We have:
  - $E(u(Fintech)) = 5.41$
- Then $u(W_{certain})$=5.41
- ie $\ln(W_{certain}) = 5.41 \Rightarrow$
- $W_{certain} = e^{5.41} \approx 223$
- So you would need to be paid $223,000 to be indifferent between this career and the career in Fintech
- This is the Certainty Equivalent Income
- And the risk premium is $525,000-$223,000 $= \$302,000$

Average income (25 years)

Expected utility: example

- At what probability of Good business conditions would you be indifferent between being a Fintech owner or an accountant?



- We need to move to the right on both Expected Utility lines so that EU(Fintech)>EU(Accountant)
- That is, we need to increase the probability of Good business conditions

Average income (25 years)

Expected utility: example

- To calculate the probabilities

- Expected utility

	Good (USD 000s)	Bad (USD 000s)
Fintech	1,000	50
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Expected utility: example

- To calculate the probabilities
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Expected utility: example

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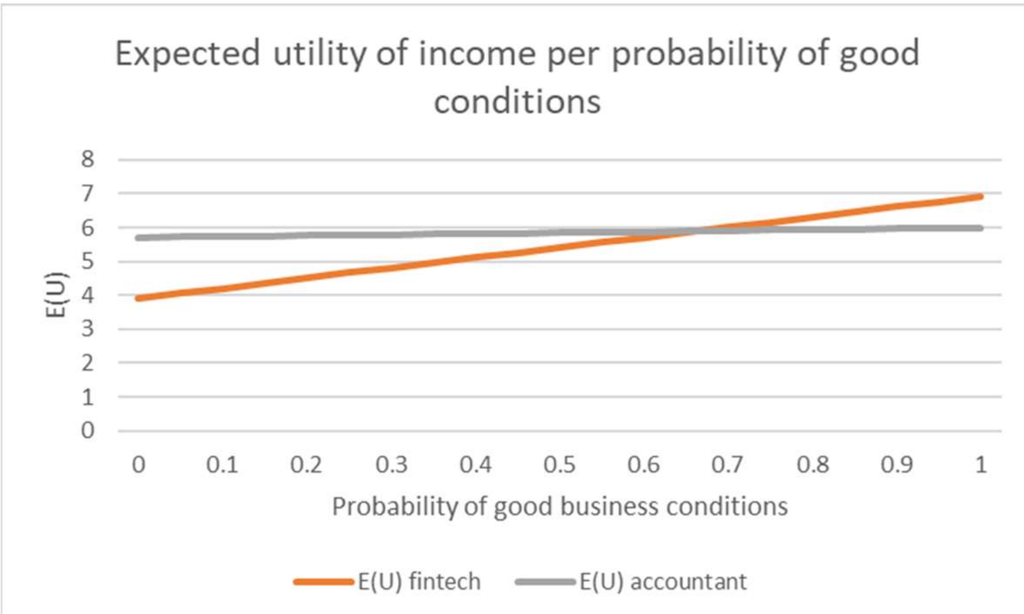

- At $p \approx 0.66$ expected utilities are equal
- So if you become a bit more optimistic about the future, you can work in fintech