Topic 5:

Unconstrained optimization

Outline

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1. Local optimization, first order necessary condition

- Consider a two-variable differentiable function $z = f(x, y)$ defined on S, (x_0, y_0) is an interior point of S
- (x_0, y_0) is said to be local maximum point of f if $f(x, y) \leq f(x_0, y_0)$ for all pairs of (x, y) in S that lie close to (x_0, y_0).
- (x_0, y_0) is said to be local minimum point of f if $f(x, y) \geq f(x_0, y_0)$ for all pairs of (x, y) in S that lie close to (x_0, y_0).
- Let $y = y_0$, if (x_0, y_0) is a local maximum point of f, then $g(x) = f(x, y_0)$ will reach its maximum at $x = x_0$, so $g'(x_0) = f_1'(x_0, y_0) = 0$

- First-order necessary condition for interior extreme point: (x_0, y_0) is a local extreme point of f, then (x_0, y_0) is a stationary point, satisfying the FOC:

$$f_1'(x_0, y_0) = 0,\ f_2'(x_0, y_0) = 0, \text{ or } f'(x_0, y_0) = 0$$

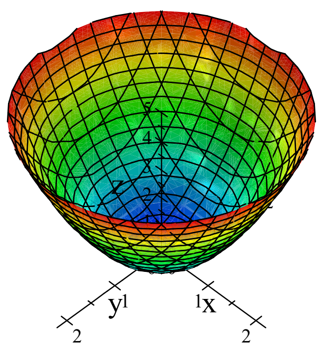
- A saddle point (x_0, y_0) is a stationary point with the property that there exist points (x, y) close to (x_0, y_0) with $f(x, y) < f(x_0, y_0)$, and there also exist such points with $f(x, y) > f(x_0, y_0)$

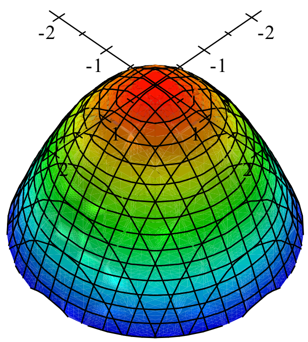
- **Example**: For the following functions, obviously $(0,0)$ is the stationary point

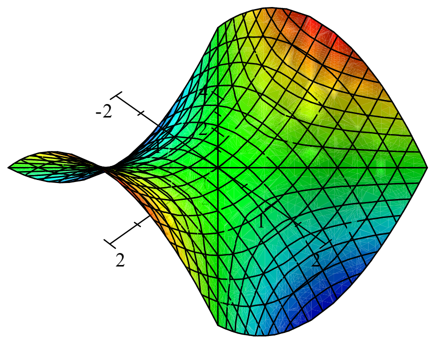
$$f(x,y) = x^2 + y^2 \qquad \text{[minimum point]}$$

$$f(x,y) = -x^2 - y^2 \qquad \text{[maximum point]}$$

$$f(x,y) = x^2 - y^2 \qquad \text{[saddle point]}$$







- Recall: for one-variable function $f(x)$, if x_0 is a stationary point, then sufficient condition for x_0 to be extreme points is

  x_0 is local maximum $\qquad f''(x_0) < 0$

  x_0 is local minimum $\qquad f''(x_0) > 0$

- Recall justification of second derivative test: For $f \in C^2$:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(\xi)}{2}(x - x_0)^2$$

  where ξ is between x and x_0.
  - If $f'(x_0) = 0$ and $f''(x_0) < 0$, then $f''(\xi) < 0$ when x is close to x_0, therefore $f(x) < f(x_0)$: x_0 is a local maximum.
  - If $f'(x_0) = 0$ and $f''(x_0) > 0$, then $f''(\xi) > 0$ when x is close to x_0. therefore $f(x) > f(x_0)$: x_0 is a local minimum.

- Extension of Taylor expansion from one variable to multi-variable: for $x \in R^{\mathrm{n}}$

$$f'(x) = (f_{x_1}'(x), f_{x_2}'(x), \cdots, f_{x_n}'(x))^T = \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \cdots, \frac{\partial f(x)}{\partial x_n} \right)^T$$

$$f''(x) = \begin{pmatrix} \frac{\partial^2 f(x)}{\partial x_1^{\ 2}} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^{\ 2}} & \cdots & \frac{\partial^2 f(x)}{\partial x_2 \partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \frac{\partial^2 f(x)}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_n^{\ 2}} \end{pmatrix}$$

note that $f'(x)$ is a $n \times 1$ vector and $f''(x)$ is $n \times n$ symmetric matrix.

- Taylor expansion for $f \in C^2$(f: function of n variables):

$$f(x) = f(x_0) + f'(x_0) \cdot (x - x_0) + (1/2)(x - x_0)' f''(\xi)(x - x_0)$$

  where ξ is between x and x_0
- x_0 is a stationary point if $f'(x_0) = 0$.
- whether x_0 is local maximum or minimum depends on whether $(x - x_0)' f''(x_0)(x - x_0)$ is > 0 *or* < 0.
- Let $h = x - x_0$. We need to learn when $f(h) = h' A h > 0$ or < 0 for a symmetric matrix A.

2. Positive (negative) definite matrix

- Let $x \in R^{\mathrm{n}}$ and A be symmetric matrix,
$$f(x) = x'Ax$$
is said to be of quadratic form.

- **Examples**: for symmetric matrices

$$A_1 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},\ A_2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},\ A_3 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix},\ A_4 = \begin{pmatrix} -2 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & -2 \end{pmatrix}$$

$$f_1(x) = f_1(x_1, x_2) = x' A_1 x = 2x_1^2 - 2x_1x_2 + 2x_2^2$$
$$f_2(x) = f_2(x_1, x_2) = x' A_2 x = x_1^2 - 2x_1x_2 + x_2^2$$
$$f_3(x) = f_3(x_1, x_2) = x' A_3 x = x_1x_2$$
$$f_4(x) = f_4(x_1, x_2, x_3) = x' A_4 x = -2x_1^2 - 3x_2^2 - 2x_3^2 + 2x_1x_3$$

Definition of definite matrices

- A symmetric matrix $A \in R^{n \times n}$ is

Positive semi-definite ($A \geq 0$)	if	$x'Ax \geq 0$ for any $x \in R^{\mathrm{n}}$
Positive definite ($A > 0$)	if	$x'Ax > 0$ for any $x \in R^{\mathrm{n}}, x \neq 0$
Negative semi-definite ($A \leq 0$)	if	$x'Ax \leq 0$ for any $x \in R^{\mathrm{n}}$
Negative definite ($A < 0$)	if	$x'Ax < 0$ for any $x \in R^{\mathrm{n}}, x \neq 0$
Indefinite	If	$x'Ax > 0$ for some x and < 0 for some other x

Examples (revisit) :

$$A_1 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},\ A_2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},\ A_3 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix},\ A_4 = \begin{pmatrix} -2 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & -2 \end{pmatrix}$$

$$f_1(x) = 2x_1^{\ 2} - 2x_1x_2 + 2x_2^{\ 2} = (x_1 - x_2)^2 + x_1^{\ 2} + x_2^{\ 2} > 0 \text{ for } x \neq 0$$

$$f_2(x) = x_1^{\ 2} - 2x_1x_2 + x_2^{\ 2} = (x_1 - x_2)^2 \geq 0$$

$$f_3(x) = x_1x_2 > 0 \text{ for some } x, \text{ and } < 0 \text{ for some other } x$$

$$f_4(x) = -2x_1^{\ 2} - 3x_2^{\ 2} - 2x_3^{\ 2} + 2x_1x_3 = -x_1^{\ 2} - 3x_2^{\ 2} - x_3^{\ 2} - (x_1 - x_3)^2 < 0 \text{ for } x \neq 0$$

$$A_1 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} > 0,\ A_2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \geq 0,$$

$$A_3 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \text{ is indefinite, } A_4 = \begin{pmatrix} -2 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & -2 \end{pmatrix} < 0$$

- Note: $A \leq 0$ iff $-A \geq 0$, and $A < 0$ iff $-A > 0$
- If $A \geq 0$, then $a_{ii} \geq 0$ for all i; If $A > 0$, then $a_{ii} > 0$ for all i.
- If $A \leq 0$, then $a_{ii} \leq 0$ for all i; If $A < 0$, then $a_{ii} < 0$ for all i.

- **Exercise**: is

$$A = \begin{pmatrix} 0 & 0 & 3 \\ 0 & -1 & 2 \\ 3 & 2 & 5 \end{pmatrix} \geq 0?$$

- **Example**: Let $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, then
  - $A > 0 \iff a > 0, ac - b^2 > 0$
  - $A \geq 0 \iff a \geq 0, c \geq 0, ac - b^2 \geq 0$
  - $A < 0 \iff a < 0, ac - b^2 > 0$
  - $A \leq 0 \iff a \leq 0, c \leq 0, ac - b^2 \geq 0$
  - A is indefinite $\iff$ $ac - b^2 < 0$

- Given a matrix $A = (a_{ij})_{n\times n}$, for $i_1 < i_2 < \cdots < i_k \in \{1,2,\cdots,n\}$, define a k-dimensional principal minor as

$$d_{\{i_1,\ldots,i_k\}} = \begin{vmatrix} a_{i_1,i_1} & a_{i_1,i_2} & \cdots & a_{i_1,i_k} \\ a_{i_2,i_1} & a_{i_2,i_2} & \cdots & a_{i_2,i_k} \\ \vdots & \vdots & & \vdots \\ a_{i_k,i_1} & a_{i_k,i_2} & \cdots & a_{i_k,i_k} \end{vmatrix}$$

- In particular, denote the leading principal minors as

$$d_1 = d_{\{1\}},\ \ d_2 = d_{\{1,2\}}, \ldots, d_n = d_{\{1,2,\ldots,\mathrm{n}\}}$$

- **Example**: Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

- The 1-dimensional principal minors are:

$$d_{\{1\}} = 1, d_{\{2\}} = 5, d_{\{3\}} = 9$$

- The 2-dimensional principal minors are:

$$d_{\{1,2\}} = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}, d_{\{1,3\}} = \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix}, d_{\{2,3\}} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}$$

- The 3-dimensional principal minor is $d_{\{1,2,3\}} = |A|$
- The leading principal minors are

$$d_1 = d_{\{1\}} = 1, d_2 = d_{\{1,2\}} = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}, d_3 = d_{\{1,2,3\}} = |A|$$

- **Example**: For $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$
- The 1-dimensional principal minors are $d_{\{1\}} = a, d_{\{2\}} = c$
- The 2-dimensional principal minors are: $d_{\{1,2\}} = |A| = ac - b^2$
- The leading principal minors are $d_1 = d_{\{1\}} = a,\ d_2 = d_{\{1,2\}} = ac - b^2$

- Compare
    1. $A > 0 \iff a > 0, ac - b^2 > 0$ (all leading principal minors > 0)
    2. $A \geq 0 \iff a \geq 0, c \geq 0, ac - b^2 \geq 0$ (all principal minors ≥ 0)
    3. $A < 0 \iff a < 0, ac - b^2 > 0$ (all leading principal minors < 0 if odd dimension, > 0 if even dimension)
    4. $A \leq 0 \iff a \leq 0, c \leq 0, ac - b^2 \geq 0$ (all principal minors, ≤ 0 if odd dimension, ≥ 0 if even dimension)

- **Theorem**: For a symmetric matrix A
  1. $A > 0 \Leftrightarrow d_k > 0$ for all k
  2. $A < 0 \Leftrightarrow (-1)^k d_k > 0$ for all k
  3. $A \geq 0 \Leftrightarrow d_{\{i_1, i_2, \ldots, i_k\}} \geq 0$ for all $\{i_1, i_2, \ldots, i_k\} \in \{1,2, \ldots, n\}$ with $i_1 < i_2 < \cdots < i_k$
  4. $A \leq 0 \Leftrightarrow (-1)^k d_{\{i_1, i_2, \ldots, i_k\}} \geq 0$ for all $\{i_1, i_2, \ldots, i_k\} \in \{1,2, \ldots, n\}$ with $i_1 < i_2 < \cdots < i_k$

- **Example** (revisit): use the above theorem to verify that

$$A_1 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} > 0, \ A_2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \geq 0,$$

$$A_3 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \text{ is indefinite, } A_4 = \begin{pmatrix} -2 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & -2 \end{pmatrix} < 0$$

- For $A_1, d_1 = 2 > 0, d_2 = |A_1| = 3 > 0$, thus $A_1 > 0$
- For $A_2, d_{\{1\}} > 0, d_{\{2\}} > 0, d_{\{1,2\}} = 0$, thus $A_2 \geq 0$
- For $A_3, d_{\{1,2\}} < 0$, thus A_3 is indefinite
- For $A_4, d_1 = -2 < 0, d_2 = \begin{vmatrix} -2 & 0 \\ 0 & -3 \end{vmatrix} = 6 > 0,\ d_3 = |A_4| < 0$, thus $A_4 < 0$

- **Exercise**: Use the above theorem to check the following matrices for definiteness

$$A = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 4 \end{pmatrix}$$

3. Second order sufficient conditions for local extreme points

- **Example**: $f(x,y) = -x^2 + xy - y^2 = (x,y)\begin{pmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = v'Av,$

  the matrix $A = \begin{pmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & -1 \end{pmatrix}$ is negative definite, therefore, $v'Av < 0$ if $v \neq 0$. In summary, $f(0,0) = 0$, and $f(x,y) < 0$ if $(x,y) \neq (0,0)$, the stationary point $(0,0)$ is (unique) local (global) maximum.

- **Example**: $f(x,y) = -x^2 + 2xy - y^2 = (x,y)\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, the matrix $A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ is negative semi-definite, (0,0) is a local maximum, but not the unique maximum point.

- **Example**: $f(x,y) = ax^2 + 2bxy + cy^2 = (x,y)A\begin{pmatrix} x \\ y \end{pmatrix}$ with $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$
  - If $A < 0$, then $(0,0)$ is unique maximum point
  - If $A > 0$, then $(0,0)$ is unique minimum point
  - Note: $(x_0, y_0) = (0,0)$ is the stationary point of f
  - $f''(x,y) = 2A$

- Second-order sufficient condition for local extreme points for general function of two variables $f(x,y)$
  - (x_0, y_0) is a stationary point of f
  - A sufficient condition for (x_0, y_0) to be local maximum point is $f''(x^0, y^0) < 0$
  - A sufficient condition for (x_0, y_0) to be local minimum point is $f''(x^0, y^0) > 0$
  - A necessary condition for (x_0, y_0) to be local maximum point is $f''(x^0, y^0) \leq 0$
  - A necessary condition for (x_0, y_0) to be local minimum point is $f''(x^0, y^0) \geq 0$
  - If $f''(x_0, y_0)$ is indefinite, then, (x_0, y_0) must is a saddle point

- **Example**: $f(x,y) = x^3 - x^2 - y^2 + 8$, find local extreme points
  - FOC: $\begin{cases} f_1'(x,y) = 3x^2 - 2x = 0 \\ f_2'(x,y) = -2y = 0 \end{cases}$
  - Stationary points: $(0,0)$ and $(\frac{2}{3}, 0)$
  - Hessian matrix: $f''(x,y) = \begin{pmatrix} 6x-2 & 0 \\ 0 & -2 \end{pmatrix}$
  - Thus,

$$f''(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} < 0, \quad f''(\frac{2}{3}, 0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \text{ is indefinite}$$

  - Therefore, $(0,0)$ is local maximum point and $(2/3,0)$ is a saddle point

- **Exercise**: Find local extreme point(s) of

$$f(x,y) = x^2 + y - xy - y^3$$

- **Exercise**: Find local extreme point(s) of

$$f(x,y) = x + 2ey - e^x - e^{2y}$$

- Extension to n-variable function: Let $A \in R^{\mathrm{n}}$, for $f: R^{\mathrm{n}} \rightarrow R$ twice continuously differentiable
- $x^* \in R^{\mathrm{n}}$ is a stationary point if it satisfies $f'(x^*) = 0$, or equivalently $\frac{\partial f(x^*)}{\partial x_i} = 0$ for $i = 1,2, \dots, n$
- Second-order sufficient condition for local maximum/minimum
  - A sufficient condition for x^* to be local maximum is $f''(x^*) < 0$
  - A sufficient condition for x^* to be local minimum is $f''(x^*) > 0$
  - A necessary condition for x^* to be local maximum is $f''(x^*) \leq 0$
  - A necessary condition for x^* to be local minimum is $f''(x^*) \geq 0$
  - If $f''(x^*)$ is indefinite, then, x^* must is a saddle point

- **Example**: $x \in R^3, f(x) = {x_1}^3 + {x_2}^2 + 2{x_3}^2 - 2x_2x_3 - 3x_1 + 10$, find local maximum/minimum point. -

- FOC: $\begin{cases} f_1'(x) = \dfrac{\partial f}{\partial x_1} = 3x_1^2 - 3 = 0 \\ f_2'(x) = \dfrac{\partial f}{\partial x_2} = 2x_2 - 2x_3 = 0 \\ f_3'(x) = \dfrac{\partial f}{\partial x_3} = 4x_3 - 2x_2 = 0 \end{cases}$

- Stationary points: $c_1 = (1,0,0)$ and $c_2 = (-1,0,0)$

- Hessian matrix: $f''(x) = \begin{pmatrix} 6x_1 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 4 \end{pmatrix}$

$$f''(c_1) = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 4 \end{pmatrix} > 0; \; f''(c_2) = \begin{pmatrix} -6 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 4 \end{pmatrix} \text{ is indefinite}$$

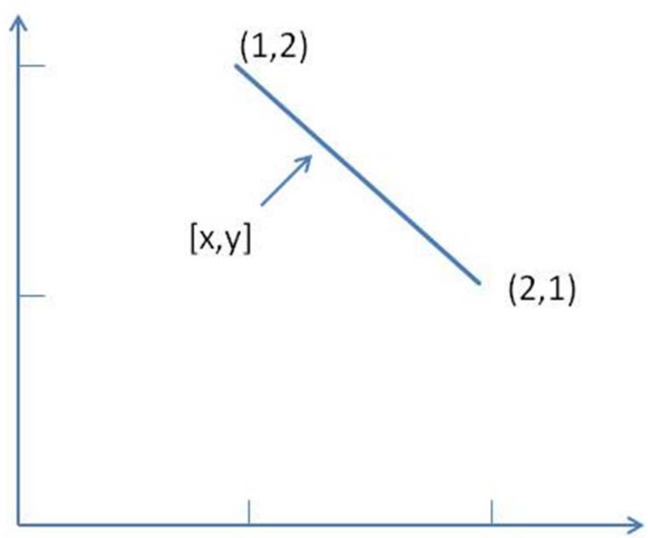
- c_1 is a local minimum point. c_2 is a saddle point.

4. Concavity

- Concavity is a sufficient condition for a stationary point to be a global maximum.
- Given any two points $x, y \in R^{\mathrm{n}}$, define the intervals:

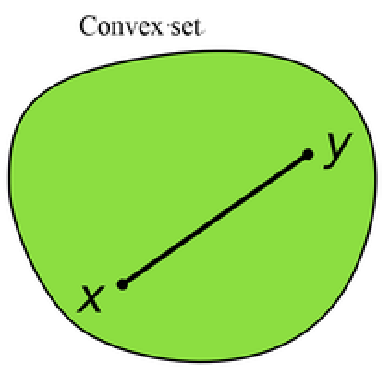
  Closed interval: $[x, y] = \{z | z = \lambda x + (1-\lambda)y, \lambda \in [0,1]\}$

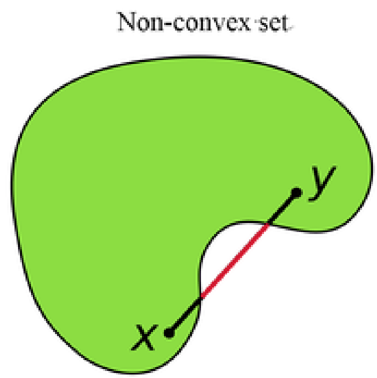
  Open interval: $(x, y) = \{z | z = \lambda x + (1-\lambda)y, \lambda \in (0,1)\}$
- **Example**: $n = 2, x = (1,2), y = (2,1), [x, y]$ is the line connecting the two points:



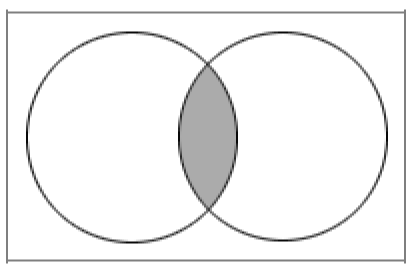
- A set $S \subset R^{\mathrm{n}}$ is a convex set if

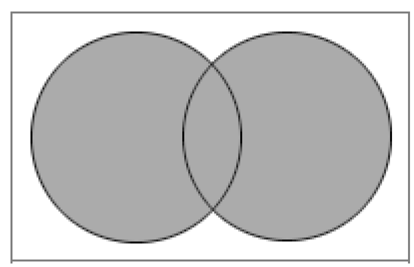
  $x, y \in S \Rightarrow [x, y] \subset S$





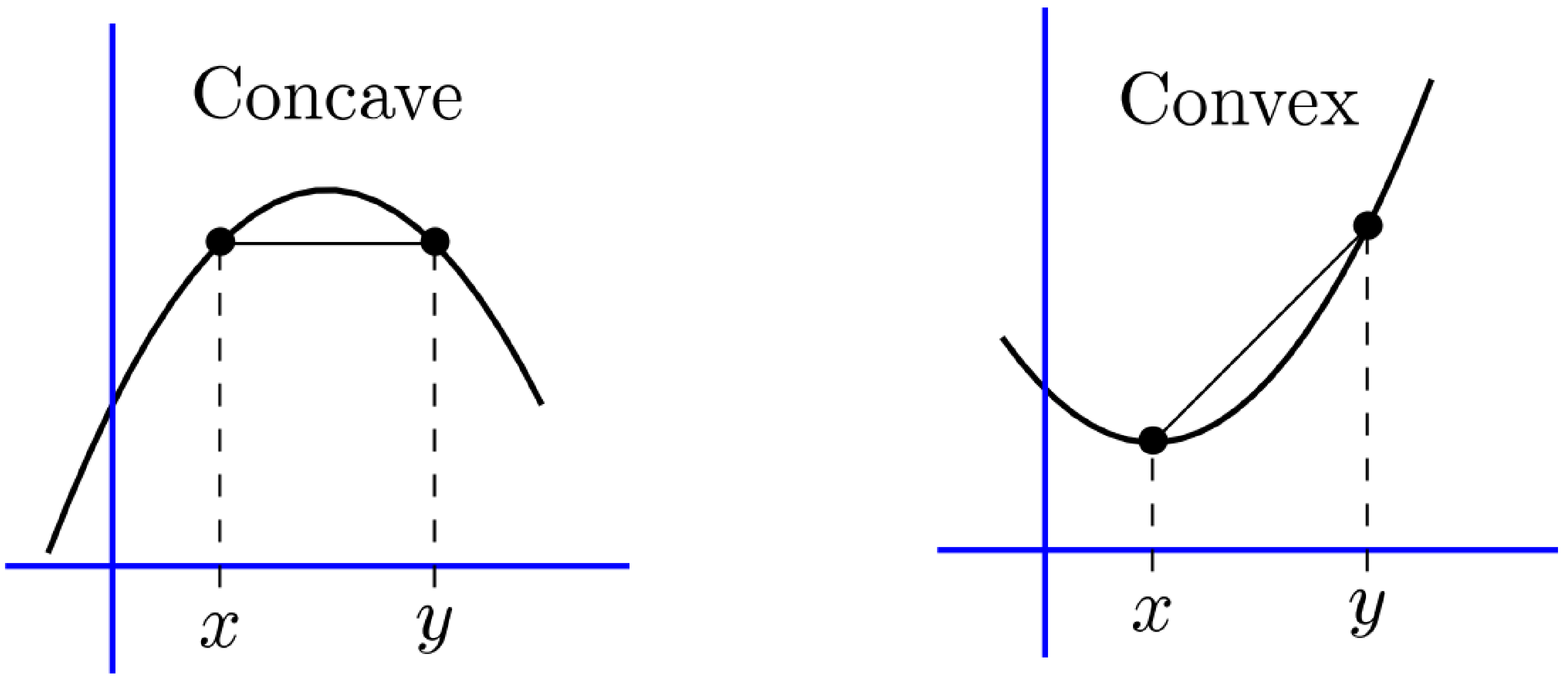
- If S and T are convex sets, then $S \cap T$ is a convex set (not true for $S \cup T$)





Concave/Convex functions

- Given a convex set $S \subset R^{\mathrm{n}}, f: S \rightarrow R$
  - f is concave if $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y),\ \forall \lambda \in (0,1)$, and $x, y \in S$
  - f is strictly concave if $f(\lambda x + (1-\lambda)y) > \lambda f(x) + (1-\lambda)f(y)$,
    $\forall \lambda \in (0,1)$, and $x, y \in S$
  - f is convex if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y), \forall \lambda \in (0,1)$, and $x, y \in S$
  - f is strictly convex if $f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y)$,
    $\forall \lambda \in (0,1)$, and $x, y \in S$



Example

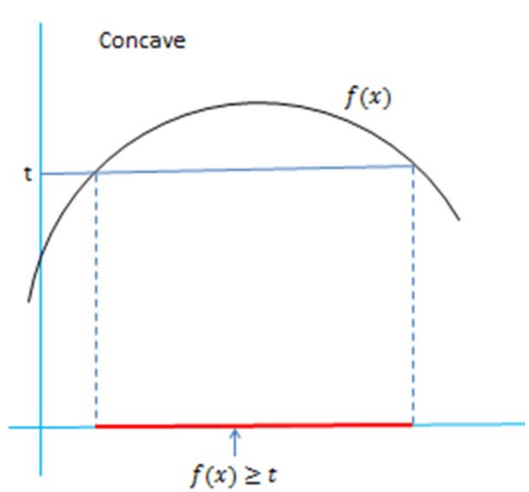
Use definition to argue that $f(x) = x_1^{\ 2} + x_2^{\ 2}$ defined on R^2 is a convex function

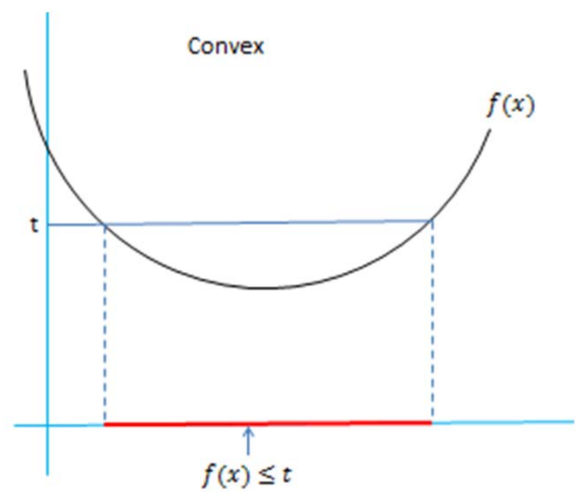
- Let $x, y \in R^2$, $\lambda \in (0,1)$

$$
\begin{aligned}
f(\lambda x + (1-\lambda)y) &= \left(\lambda x_1 + (1-\lambda)y_1\right)^2 + \left(\lambda x_2 + (1-\lambda)y_2\right)^2 \\
&= \lambda^2 (x_1^{\ 2} + x_2^{\ 2}) + 2\lambda(1-\lambda)(x_1 y_1 + x_2 y_2) + (1-\lambda)^2 (y_1^{\ 2} + y_2^{\ 2}) \\
&= \lambda^2 f(x) + 2\lambda(1-\lambda)(x_1 y_1 + x_2 y_2) + (1-\lambda)^2 f(y) \\
&\le \lambda^2 f(x) + \lambda(1-\lambda)(x_1^{\ 2} + y_1^{\ 2} + x_2^{\ 2} + y_2^{\ 2}) + (1-\lambda)^2 f(y) \\
&= (\lambda^2 + \lambda(1-\lambda)) f(x) + (\lambda(1-\lambda) + (1-\lambda)^2) f(\mathrm{y}) \\
&= \lambda f(x) + (1-\lambda) f(y)
\end{aligned}
$$

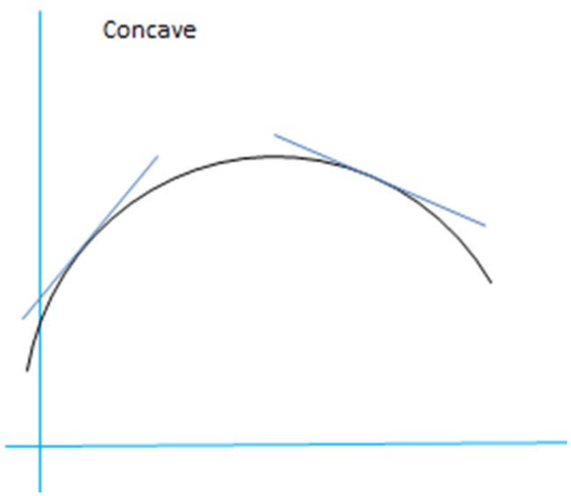
Properties

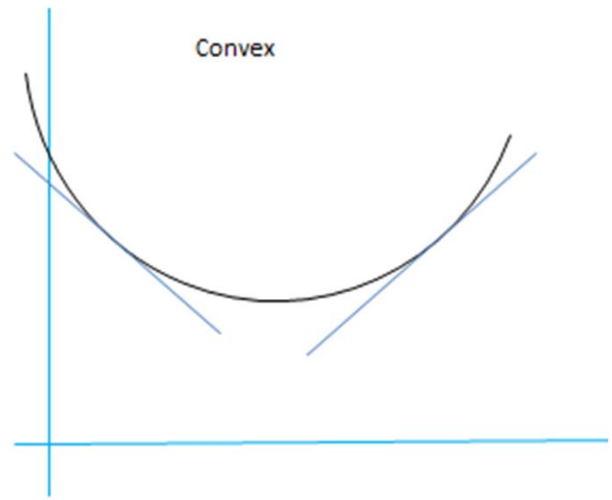
- f is concave iff $-f$ is convex;
- f is strictly concave iff $-f$ is strictly convex.
- A linear function is both concave and convex
- $f: S \to R$ is concave, then the upper level set $\{x \in S | f(x) \geq t\}$ is convex, $\forall t \in R$
- $f: S \to R$ is convex, then the lower level set $\{x \in S | f(x) \leq t\}$ is convex, $\forall t \in R$





- **Theorem** (First-order characterization of concave(convex) functions):
Let $f: S \rightarrow R$ be C^1 function defined on an open, convex set S, then
  1. f is concave $\Leftrightarrow$ $f(v) \leq f(u) + \nabla f(u) \cdot (v-u)$ for all $u, v \in S$. In other words, the curve is always below any tangent plane
  2. f is strictly concave $\Leftrightarrow f(v) < f(u) + \nabla f(u) \cdot (v-u)$ for all $u, v \in S$ and $u \neq v$.
  3. f is convex $\Leftrightarrow f(v) \geq f(u) + \nabla f(u) \cdot (v-u)$ for all $u, v \in S$. In other words, the curve is always above any tangent plane
  4. f is strictly convex $\Leftrightarrow f(v) > f(u) + \nabla f(u) \cdot (v-u)$ for all $u, v \in S$ and $u \neq v$.





Example

- Consider the function f defined on R^2

$$f(x) = x_1^{\,2} + x_2^{\,2}$$

- For $u, v \in R^2$

$$\nabla f(u) = (2u_1, 2u_2)^T$$

$$\big[f(v) - f(u)\big] - \nabla f(u) \cdot (v - u)$$

$$= \big[(v_1^{\,2} + v_2^{\,2}) - (u_1^{\,2} + u_2^{\,2})\big] - (2u_1, 2u_2)\begin{pmatrix} v_1 - u_1 \\ v_2 - u_2 \end{pmatrix}$$

$$= v_1^{\,2} + v_2^{\,2} - u_1^{\,2} - u_2^{\,2} - (2u_1v_1 - 2u_1^{\,2} + 2u_2v_2 - 2u_2^{\,2})$$

$$= (u_1 - v_1)^2 + (u_2 - v_2)^2 = \|u - v\|^2 > 0 \text{ for } u \neq v$$

thus f is strictly convex

- **Theorem**: Let $S \subset R^{\mathrm{n}}$ be a convex set, and $f: S \to R$ is twice differentiable ($f \in C^2$), then
  1. f is convex $\Leftrightarrow$ $\mathrm{f}''(x) \geq 0$ for all $x \in S$
  2. f is concave $\Leftrightarrow$ $f''(x) \leq 0$ for all $x \in S$
  3. $f''(x) > 0$ for all $x \in S \Rightarrow$ f is strictly convex
  4. $f''(x) < 0$ for all $x \in S \Rightarrow$ f is strictly concave

- **Note**, "f is strictly convex" does not necessarily imply that $f''(x) > 0$. Example: $f(x) = x^4$

- **Example:** function $f(x, y) = xy$

$$f''(x,y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

  is indefinite and f is neither concave nor convex

- **Example**: Discuss the concavity of $f(x, y) = ax^2 + 2bxy + cy^2$
  - Note $f''(x,y) = 2\begin{pmatrix} a & b \\ b & c \end{pmatrix}$

1. f is convex$\Leftrightarrow \begin{pmatrix} a & b \\ b & c \end{pmatrix} \geq 0 \Leftrightarrow a \geq 0, c \geq 0, ac - b^2 \geq 0$
2. f is concave $\Leftrightarrow \begin{pmatrix} a & b \\ b & c \end{pmatrix} \leq 0 \Leftrightarrow a \leq 0, c \leq 0, ac - b^2 \geq 0$
3. $\begin{pmatrix} a & b \\ b & c \end{pmatrix} > 0 \Leftrightarrow a > 0, ac - b^2 > 0$ $\Rightarrow f$ is strictly convex
4. $\begin{pmatrix} a & b \\ b & c \end{pmatrix} < 0 \Leftrightarrow a < 0, ac - b^2 > 0$ $\Rightarrow f$ is strictly concave

- **Example**: For $f(x, y) = x^\alpha + y^\beta$ defined on $R_{++}{}^2 (x > 0, y > 0)$ for $\alpha, \beta \geq 0$

$$f_x = \alpha x^{\alpha-1}, \quad f_y = \beta y^{\beta-1}$$

$$f_{xx} = \alpha(\alpha-1)x^{\alpha-2}, \quad f_{xy} = 0, \quad f_{yy} = \beta(\beta-1)\,\mathrm{y}^{\beta-2}$$

$$f''(x, y) = \begin{pmatrix} \alpha(\alpha-1)x^{\alpha-2} & 0 \\ 0 & \beta(\beta-1)\,\mathrm{y}^{\beta-2} \end{pmatrix}$$

$$f \text{ is } \begin{cases} \text{concave} & \text{if } 0 \leq \alpha, \beta \leq 1 \\ \text{strictly concave} & \text{if } 0 < \alpha, \beta < 1 \end{cases}$$

- **Example**: For Cobb-Douglas function $f(x, y) = x^{\alpha} y^{\beta}$ defined on ${R_{++}}^2$ for $\alpha, \beta \geq 0$ Since

$$f_x = \alpha x^{\alpha-1} y^{\beta}, \quad f_y = \beta x^{\alpha} y^{\beta-1}$$

$$f_{xx} = \alpha(\alpha-1) x^{\alpha-2} y^{\beta}, \quad f_{xy} = \alpha\beta x^{\alpha-1} y^{\beta-1}, \quad f_{yy} = \beta(\beta-1) x^{\alpha} y^{\beta-2}$$

$$f''(x, y) = \begin{pmatrix} \alpha(\alpha-1) x^{\alpha-2} y^{\beta} & \alpha\beta x^{\alpha-1} y^{\beta-1} \\ \alpha\beta x^{\alpha-1} y^{\beta-1} & \beta(\beta-1) x^{\alpha} y^{\beta-2} \end{pmatrix}$$

$$f \text{ is } \begin{cases} \text{concave} & \text{if } \alpha, \beta \geq 0, \alpha + \beta \leq 1 \\ \text{strictly concave} & \text{if } \alpha, \beta > 0, \alpha + \beta < 1 \end{cases}$$

- **Example**: Let $f(x, y) = -x^2 - y^2$,

$$f''(x,y) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} < 0$$

- $f(x, y)$ is strictly concave on R^2.

- Let $g(x,y) = e^{-x^2-y^2} = e^{f(x,y)}$ then

$$g''(x,y) = \begin{pmatrix} 2(2x^2-1)g & 4xyg \\ 4xyg & 2(2y^2-1)g \end{pmatrix} \le 0 \text{ only when } x^2 + y^2 \le \frac{1}{2}$$

- $g(x, y)$ is not concave on R^2
- Concavity is not preserved under monotone transformaton

5. Quasi-concavity

- One problem with concavity and convexity is that a monotone transformation of a concave (or convex) function need not be a concave (convex).

- A weaker condition to describe a function is quasiconcavity (quasiconvexity)

- Let $S \subset R^{\mathrm{n}}$ be a convex set, $f: S \to R$
  - f is quasi-concave if

    $f(y) \geq f(x) \quad f(z) \geq f(x)$, for all $x, y \in S, z \in (x, y)$
  - f is strictly quasi-concave if

    $f(y) \geq f(x) \quad f(z) > f(x)$, for all $x, y \in S, z \in (x, y)$
  - f is quasi-convex if

    $f(y) \leq f(x) \quad f(z) \leq f(x)$, for all $x, y \in S, z \in (x, y)$
  - f is strictly quasi-convex if

    $f(y) \leq f(x) \quad f(z) < f(x)$, for all $x, y \in S, z \in (x, y)$

- f is quasi-convex (strictly quasi-convex) if $-f$ is quasi-concave (strictly quasi-concave)

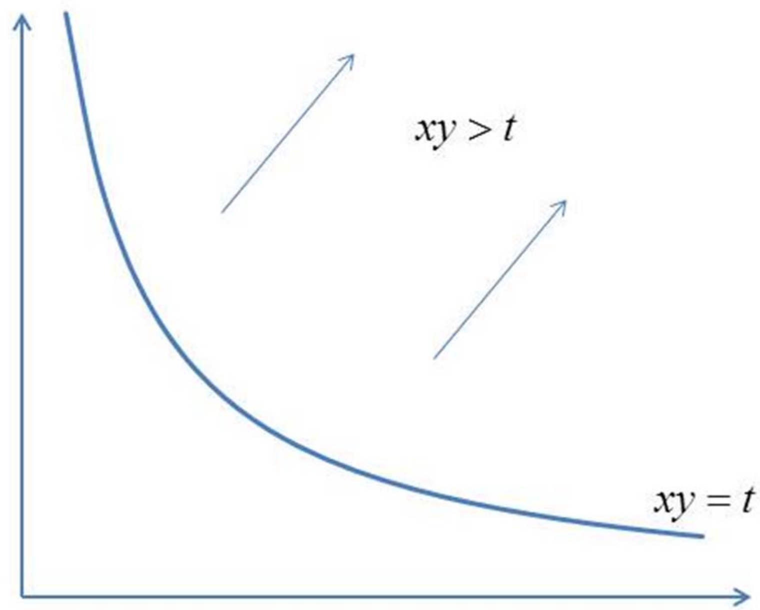
- $f: S \to R$ is quasi-concave iff the upper level set

  $L_f(t) = \{x \in S | f(x) \geq t\}$ is convex, $\forall t \in R$

- $f: S \to R$ is quasi-convex iff lower level set

  $L_f(t) = \{x \in S | f(x) \leq t\}$ is convex, $\forall t \in R$

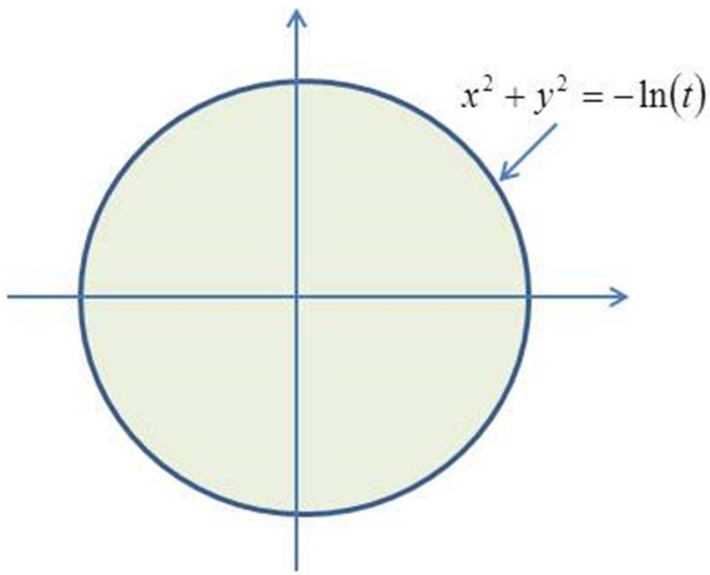
- **Example**: Show that $f(x, y) = xy$ is quasi-concave on ${R_+}^2$.
  - $\forall\, t \in R$, if $t > 0$, then the upper level set is a convex set



  - If $t \leq 0$, then the upper level set is ${R_+}^2$, obviously it is convex.

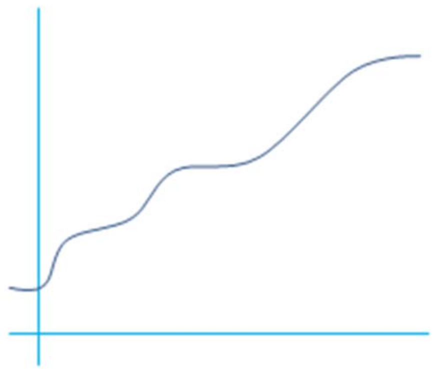
- **Example**: Show that $f(x,y) = e^{-x^2-y^2}$ is quasi-concave on R^2
  - $\forall\, t \in R$, if $0 < t \leq 1$, then the upper level set is a convex set

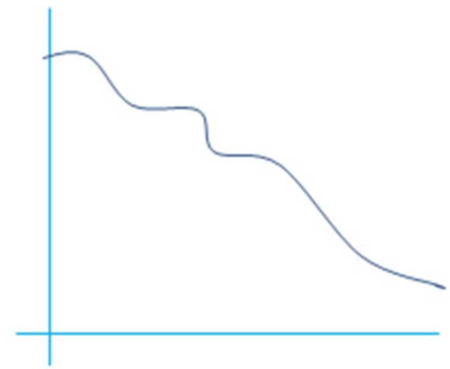
$$\{(x,y):\ f(x,y) \geq t\} = \{(x,y): e^{-x^2-y^2} \geq t\}$$
$$= \{(x,y): x^2 + y^2 \leq \ln(\mathrm{t})\}$$



  - if $t \leq 0$, then the upper level set $=\ R^2$, obviously it is convex
  - If $t > 1$, then the upper level set $= \emptyset$ (empty set, ignore)

- Monotone functions defined on R are both quasi-concave and quasi-convex





- Concave functions are quasi-concave; convex functions are quasi-convex
- Strictly Concave functions are strictly quasi-concave; strictly convex functions are strictly quasi-convex
- Summary:

$$\begin{array}{ccc} \text{strict concavity} & \Rightarrow & \text{concavity} \\ \Downarrow & & \Downarrow \\ \text{strict quasi-concavity} & \Rightarrow & \text{quasi-concavity} \end{array}$$

- Quasi-concavity does not necessarily imply concavity, an example: $f(x) = \exp(-x^2 - y^2)$

- (Quasiconcavity is preserved under monotone transformation) If f is quasi-concave (quasi-convex) and H is strictly increasing, then $H(f(\cdot))$ is quasi-concave (quasi-convex)

- **Example** (revisit): Show that $g(x,y)=e^{-x^2-y^2}$ is quasi-concave on R^2
  - The function $f(x,y) = -(x^2+y^2)$ is concave (thus quasi-concave) on R^2, and $H(z) = e^z$ is strictly increasing, thus $g(x,y) = H(f(x,y))$ is quasi-concave

- **Theorem**: Given second order differentiable $f: S \to R$, the **bordered Hessian matrix**

$$B_f(x) = \begin{pmatrix} 0 & f_1 & \cdots & f_n \\ f_1 & f_{11} & \cdots & f_{1n} \\ \vdots & \vdots & & \vdots \\ f_n & f_{n1} & \cdots & f_{nn} \end{pmatrix} \quad \text{where} \quad \begin{array}{l} f_i = \dfrac{\partial f(x)}{\partial x_i} \\ f_{ij} = \dfrac{\partial^2 f(x)}{\partial x_i \partial x_j} \end{array}$$

for $i, j = 1,2, \dots, n$, and its leading principal minors are $b_1(x), b_2(x), \dots, b_{n+1}(x)$, then

1. f is quasi-convex$\Rightarrow b_k(x) \leq 0$ for $\forall k \geq 2$ and $\forall x \in S$
2. f is quasi-concave$\Rightarrow (-1)^k b_k(x) \leq 0$ for $\forall k \geq 2$ and $\forall x \in S$
3. $b_k(x) < 0$ for $\forall k \geq 2$ and $\forall x \in S \Rightarrow f$ is strictly quasi-convex
4. $\Rightarrow (-1)^k b_k(x) < 0$ for $\forall k \geq 2$ and $\forall x \in S \Rightarrow f$ is strictly quasi-concave

- **Example**: Apply the theorem to how that $f(x, y) = xy$ is strictly quasi-concave on R^2_{++}.
  - Bordered Hessian matrix:

$$B_f(x,y) = \begin{pmatrix} 0 & y & x \\ y & 0 & 1 \\ x & 1 & 0 \end{pmatrix}$$

  - $(-1)^2 b_2(x,y) = -y^2 < 0$;

$$(-1)^3 b_3(x,y) = -\begin{vmatrix} 0 & y & x \\ y & 0 & 1 \\ x & 1 & 0 \end{vmatrix} = -2xy < 0$$

  - therefore, $f(x,y)$ is strictly quasi-concave on R^2_{++}

- **Exercise**: $f(x,y) = x^{\alpha} + y^{\beta}$ defined on R^2_{++} $(x > 0, y > 0)$ for $\alpha, \beta \geq 0$ is strictly quasi-concave if $0 < \alpha, \beta \leq 1$ and $\alpha \neq 1 \; or \; \beta \neq 1$
  - Recall: f is strictly concave if $0 < \alpha, \beta < 1$

- **Exercise**: Cobb-Douglas function $f(x,y) = x^{\alpha} y^{\beta}$ defined on R^2_{++} for $\alpha, \beta \geq 0$ is strictly quasi-concave if $\alpha, \beta > 0$
  - Recall: f is strictly concave if $\alpha, \beta > 0, \alpha + \beta < 1$

6. Global optimization

- Recall: A set $A \subset R^{\mathrm{n}}$ is a compact set if it is closed and bounded.
- (Existence of global maximum: Optimal Value Theorem). Given function $f: X \to R$, where $X \subset R^{\mathrm{n}}$, if f is continuous and X is compact, then f has at least one minimum point and one maximum point
  - If f is not continuous, e.g.

$$f(x) = \begin{cases} x+1 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \\ x-1 & \text{if } x > 1 \end{cases}$$

    does not have a minimum or maximum on $[0,2]$ (compact set).
  - If X is not compact, e.g., $f(x) = x$ for $x \in (0,1)$, f does not have minimum or maximum.

Sufficient conditions for global maximum

- Let $f: X \to R$, where $X \subset R^{\mathrm{n}}$ is convex set, consider problem

  $$max_{x \in X} f(x)$$

  1. Sufficient condition #1: If f is concave on X, any stationary point $x^* \in X$ is a global maximum point
  2. Sufficient condition #2.1: If f is quasi-concave, a local maximum x^* satisfing $f''(x^*) < 0$ is a global maximum
  3. Sufficient condition #2.2: If f is strictly quasi-concave, a local maximum x^* is a global maximum

Sufficient conditions for global minimum

- Let $f: X \to R$, where $X \subset R^{\mathrm{n}}$ is convex set, consider problem

  $$min_{x \in X} f(x)$$

  1. Sufficient condition #1: If f is convex on X, any stationary point $x^* \in X$ is a global minimum point
  2. Sufficient condition #2.1: If f is quasi-convex, a local minimum x^* satisfing $f''(x^*) > 0$ is a global minimum
  3. Sufficient condition #2.2: If f is strictly quasi-convex, a local minimum x^* is a global minimum

Notes:

- When x^* is a corner solution, it may not satisfy FOC. (consider $f(x) = x$, and $X = [0,1]$)
- FOC and strict quasi-concavity together are not sufficient to ensure optimality (consider $f(x) = x^3$)
- A quasi-concave can go up and down, but it can go up and down at most once. i.e., a quasi-concave function can have at most one hump, therefore, a local maximum must be a global maximum.

- **Example**: Quadratic function with non-zero stationary point:

$$f(x, y) = -2x^2 - 2xy - 2y^2 + 36x + 42y - 158$$

find extreme points.

- FOC: $\begin{cases} f_1' = -4x - 2y + 36 = 0 \\ f_2' = -2x - 4y + 42 = 0 \end{cases}$

- Stationary point: $(x_0, y_0) = (5,8)$

- Hessian matrix $f''(x, y) = \begin{pmatrix} -4 & -2 \\ -2 & -4 \end{pmatrix} < 0, f$ is strictly concave function

- Thus (5,8) is the unique maximum point of f.

- **Example**: Find the extreme point(s) of

$$f(x,y) = x + 2ey - e^x - e^{2y}$$

- FOC: $\begin{cases} f_x = 1 - e^x = 0 \\ f_y = 2e - 2e^{2y} = 0 \end{cases}$
- Stationary point $(0, \frac{1}{2})$
- Hessian matrix: $f''(x,y) = \begin{pmatrix} -e^x & 0 \\ 0 & -4e^{2y} \end{pmatrix}$<0
- f is strictly concave
- $(0,1/2)$ is a (unique) global maximum point.

- **Example**: Find extreme point(s) of

$$f(x,y) = \exp(-x^2 - y^2)$$

- FOC: $\begin{cases} f_x = -2xe^{-x^2-y^2} = 0 \\ f_y = -2ye^{-x^2-y^2} = 0 \end{cases}$
- stationary point: (0,0).
- (0,0) is a unique maximum point.
- Is f concave?
- Is f quasi-concave?

7. Economic applications

- **Example**: (problem of a multiproduct firm, P331 Example 1) -
  - A two-product firm under circumstances of pure competition, prices of two products: p_1 and p_2 are given.
  - Revenue of firm: $R = p_1Q_1 + p_2Q_2$ where Q_1, Q_2 are the output level of the two products
  - Cost function: $C = 2{Q_1}^2 + Q_1Q_2 + 2{Q_2}^2$
  - Profit: $\pi(Q_1, Q_2) = p_1Q_1 + p_2Q_2 - (2{Q_1}^2 + Q_1Q_2 + 2{Q_2}^2)$
  - FOC: $\begin{cases} \dfrac{\partial \pi}{\partial Q_1} = p_1 - 4Q_1 - Q_2 = 0 \\ \dfrac{\partial \pi}{\partial Q_2} = p_2 - Q_1 - 4Q_2 = 0 \end{cases}$
  - stationary point: $({Q_1}^*, {Q_2}^*) = (\frac{4p_1 - p_2}{15}, \frac{4p_2 - p_1}{15})$
  - Hessian matrix: $\begin{pmatrix} -4 & -1 \\ -1 & -4 \end{pmatrix} < 0$, $\pi(Q_1, Q_2)$ is strictly concave
  - $({Q_1}^*, {Q_2}^*)$ is unique maximum point.

- **Example**: (Monopoly price discrimination, P336 Example 4)
  - A monopoly sells its product in three separable markets.
  - Inverse market demands: $p_1 = 63 - 4Q_1, p_2 = 105 - 5Q_2, p_3 = 75 - 6Q_3$
  - Cost function: $C(Q) = 20 + 15Q$ where $Q = Q_1 + Q_2 + Q_3$
  - Profit function:

$$\begin{aligned}\pi(Q_1, Q_2, Q_3) &= p_1Q_1 + p_2Q_2 + p_3Q_3 - C(Q) \\ &= (63 - 4Q_1)Q_1 + (105 - 5Q_2)Q_2 + (75 - 6Q_3)Q_3 - \left[20 + 15(Q_1 + Q_2 + Q_3)\right]\end{aligned}$$

  - FOC: $\begin{cases} \dfrac{\partial \pi}{\partial Q_1} = 63 - 8Q_1 - 15 = 0 \\ \dfrac{\partial \pi}{\partial Q_2} = 105 - 10Q_2 - 15 = 0 \\ \dfrac{\partial \pi}{\partial Q_3} = 75 - 12Q_3 - 15 = 0 \end{cases}$

- Stationary point: $(Q_1{}^*, Q_2{}^*, Q_2{}^*) = (6,9,5)$

- Hessian matrix: $\begin{pmatrix} -8 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & -12 \end{pmatrix}$, $\pi(\cdot)$ is concave function

- therefore, $(Q_1{}^*, Q_2{}^*, Q_2{}^*)$ is a unique maximum of $\pi(Q_1, Q_2, Q_3)$

- The maximum profit: $\pi(Q_1{}^*, Q_2{}^*, Q_2{}^*) = 679$

- From inverse demands, the price that the firm charges in the three markets are:
  $p_1{}^* = 63 - 4Q_1{}^* = 39$, $p_2{}^* = 105 - 5Q_2{}^* = 60$, $p_3{}^* = 75 - 6Q_3{}^* = 45$

- **Example**: (input decisions of a firm, P337 Example 5)
  - A competitive firm has the following profit function: $\pi = R - C = pQ - wL - rK$
  - where
  -

$p =$ price (exogenous variable due to competitive market)
$L =$ labor
$K =$ capital
$Q =$ output $= Q(K, L) = K^{\alpha} L^{\alpha}$ where $0 < \alpha < 1/2$
$w, r =$ prices for labor and capital respectively, exogenous variables

  - Profit function: $\pi(K, L) = pK^{\alpha}L^{\alpha} - rK - wL$
  - FOC $\begin{cases} \pi_K = \alpha p K^{\alpha-1} L^{\alpha} - r = 0 \\ \pi_L = \alpha p K^{\alpha} L^{\alpha-1} - w = 0 \end{cases}$ or $\begin{cases} \alpha p K^{\alpha-1} L^{\alpha} = r \\ \alpha p K^{\alpha} L^{\alpha-1} = w \end{cases} \Rightarrow \dfrac{K}{L} = \dfrac{w}{r}$
  - Stationary points: $(K^*, L^*) = \left((p\alpha r^{\alpha-1} w^{-\alpha})^{\frac{1}{1-2\alpha}}, (p\alpha w^{\alpha-1} r^{-\alpha})^{\frac{1}{1-2\alpha}} \right)$

– Hessian matrix:

$$\pi''(K,L)=\begin{pmatrix}\dfrac{\partial^2\pi}{\partial K^2} & \dfrac{\partial^2\pi}{\partial K\partial L}\\ \dfrac{\partial^2\pi}{\partial L\partial K} & \dfrac{\partial^2\pi}{\partial L^2}\end{pmatrix}=\begin{pmatrix}p\alpha(\alpha-1)K^{\alpha-2}L^{\alpha} & p\alpha^2K^{\alpha-1}L^{\alpha-1}\\ p\alpha^2K^{\alpha-1}L^{\alpha-1} & p\alpha(\alpha-1)K^{\alpha}L^{\alpha-2}\end{pmatrix}$$

since $\quad p\alpha(\alpha-1)K^{\alpha-2}L^{\alpha}<0$

$$\begin{aligned}\left|\pi''(K,L)\right| &= p^2\alpha^2(\alpha-1)^2K^{2\alpha-2}L^{2\alpha-2}-\left(p\alpha^2K^{\alpha-1}L^{\alpha-1}\right)^2\\ &= p^2\alpha^2(1-2\alpha)K^{2\alpha-2}L^{2\alpha-2}>0\end{aligned}$$

– $\pi(K,L)$ is negative definite for all $K>0, L>0$, (K^*,L^*) is the unique maximum point.

8. Envelope Theorem

- **Example**: (problem of a one-product firm under pure competition):
  - The price of product p is given and the cost function $C(Q) = Q^2$, where Q is the output level of the product. The profit function with parameter p is $f(Q, p) = pQ - Q^2$ find Q that maximizes the profit function.
  - FOC: $\frac{\partial f}{\partial Q} = p - 2Q = 0$
  - Stationary point: $Q^* = p/2$
  - SOC: $\frac{\partial^2 f}{\partial Q^2} = -2 < 0$, f is strictly concave, Q^* maximizes the π for a given p
  - Note in this example the optimal value of the choice variable $Q^*(p) = p/2$ is a function of the parameter problem (p)
  - Once the optimal value of the choice variable has been substituted into the profit function, the objective function indirectly becomes a function of the parameter.

$$f^*(p) = pQ^*(p) - [Q^*(p)]^2 = \frac{p^2}{4}$$

  - The maximum value function $f^*(p)$ is referred to as the indirect profit function; $f(p, Q)$ is referred to as the direct profit function
  - We can easily work out $\frac{d}{dp} f^*(p) = \frac{p}{2}$

- Note that evaluating $f^*(p)$ requires a two-step procedure for general objective function $f(x, p)$ with parameter p
  - First, given p, find the value of $x^*(p)$ that solves the problem
  - Second, substitute this value of $x^*(p)$ into the objective function to obtain $f^*(p) = f(x^*(p), p)$
  - We want to take the derivative of f^* with respect to p

- Envelop Theorem for maximization problem without constraints: Given a differentiable function $f(x,a): X \times A \rightarrow R$, where $X \subset R^{\mathrm{n}}$ and $A \subset R^k$, if $x^*(a)$ is an interior optimal point of

$$f^*(a) = \max_{x \in X} f(x,a)$$

then

$$\frac{\partial f^*(a)}{\partial a_i} = \left.\frac{\partial f(x,a)}{\partial a_i}\right|_{x=x^*(a)}$$

for $i = 1,2,\ldots,k$

- Note the key advantage of the Envelope Theorem is that we can find the derivative of $f^*(p)$ without actually solving for $f^*(p)$.

- **Example** (revisit): (problem of a one-product firm under pure competition): Objective function $f(Q, p) = pQ - Q^2$
- Recall: $Q^*(p) = \frac{p}{2}$
- $\frac{\partial f(Q,p)}{\partial p} = Q$
- Apply the envelop theorem (for the case $n = k = 1$)

$$\frac{df^*(p)}{dp} = \left.\frac{\partial f(Q,p)}{\partial p}\right|_{Q=Q^*(p)} = Q^*(p) = \frac{p}{2}$$

- **Example**: (problem of a two-product firm, P331 Example 1, revisit)
  - Profit: $\pi(Q_1, Q_2, p_1, p_2) = p_1 Q_1 + p_2 Q_2 - (2{Q_1}^2 + Q_1 Q_2 + 2{Q_2}^2)$
  - Recall: for given p_1, p_2, unique maximum point:

$$(Q_1^*(p_1, p_2), Q_2^*(p_1, p_2)) = \left(\frac{4p_1 - p_2}{15}, \frac{4p_2 - p_1}{15} \right)$$

  - Maximum profit as function of p_1, p_2:

$$\pi^*(p_1, p_2) = \pi(Q_1^*(p_1, p_2), Q_2^*(p_1, p_2), p_1, p_2)$$

  - Since $\frac{\partial \pi}{\partial p_1} = Q_1, \frac{\partial \pi}{\partial p_2} = Q_2$
  - From the envelop theorem

$$\frac{\partial \pi^*(p_1, p_2)}{\partial p_1} = Q_1^*(p_1, p_2) = \frac{4p_1 - p_2}{15}$$

$$\frac{\partial \pi^*(p_1, p_2)}{\partial p_2} = Q_2^*(p_1, p_2) = \frac{4p_2 - p_1}{15}$$

- **Example**: (revisit, input decisions of a firm, P337 Example 5)
  - Profit function:

    $$\pi(K,L,p,r,w) = pQ(K,L) - rK - wL = pK^{\alpha}L^{\alpha} - rK - wL$$

    Where p, r, w are parameters
  - K, L that maximizes profit: $(K^*, L^*) = \left((p\alpha r^{\alpha-1} w^{-\alpha})^{\frac{1}{1-2\alpha}}, (p\alpha w^{\alpha-1} r^{-\alpha})^{\frac{1}{1-2\alpha}} \right)$
  - Maximized profit: $\pi^*(p,r,w) = \pi(K^*, L^*, p, r, w)$
  - Since $\frac{\partial \pi}{\partial p} = K^{\alpha}L^{\alpha}, \ \frac{\partial \pi}{\partial r} = -K, \ \frac{\partial \pi}{\partial w} = -L$
  - Apply the envelop theorem gives the following Hotelling's lemma:

    $$\frac{\partial \pi^*(p,r,w)}{\partial p} = (K^*)^{\alpha}(K^*)^{\alpha} = Q(K^*, L^*)$$

    $$\frac{\partial \pi^*(p,r,w)}{\partial r} = -K^*, \ \frac{\partial \pi^*(p,r,w)}{\partial w} = -L^*$$
  - Note it is extremely tedious to work out $\pi^*(p,r,w)$ and then take partial derivatives

    $$\frac{\partial \pi^*}{\partial p}, \frac{\partial \pi^*}{\partial r} \text{ and } \frac{\partial \pi^*}{\partial w}$$