2. Cost Functions

Cost minimization problem

$$\min_{k,l} wl + vk, \text{ s.t. } f(k,l) = q$$

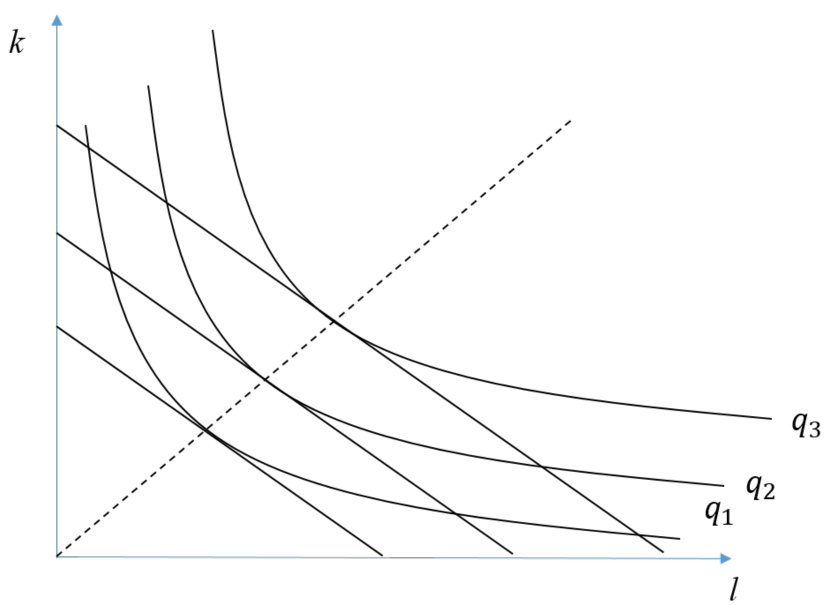
Equal-cost lines,

$$c_1 = wl + vk$$
$$vk = c_1 - wl$$
$$k = \frac{c_1}{v} - \frac{w}{v}l$$

The intercept is $\frac{c_1}{v}$.

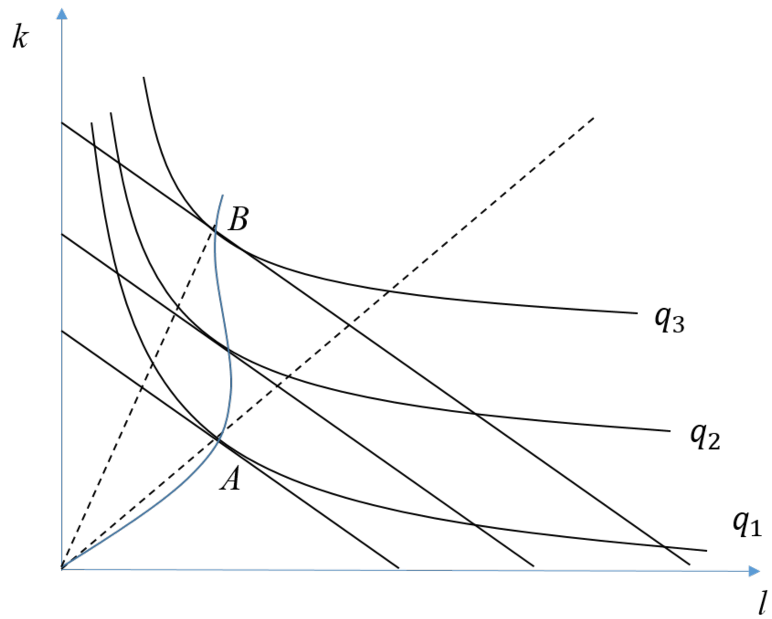
The slopt of equal-cost lines is $-\frac{w}{v}$, which is a input price ratio.

When technology advances, physical capital becomes cheaper, the equal-cost lines become steeper.



Homothetic production function

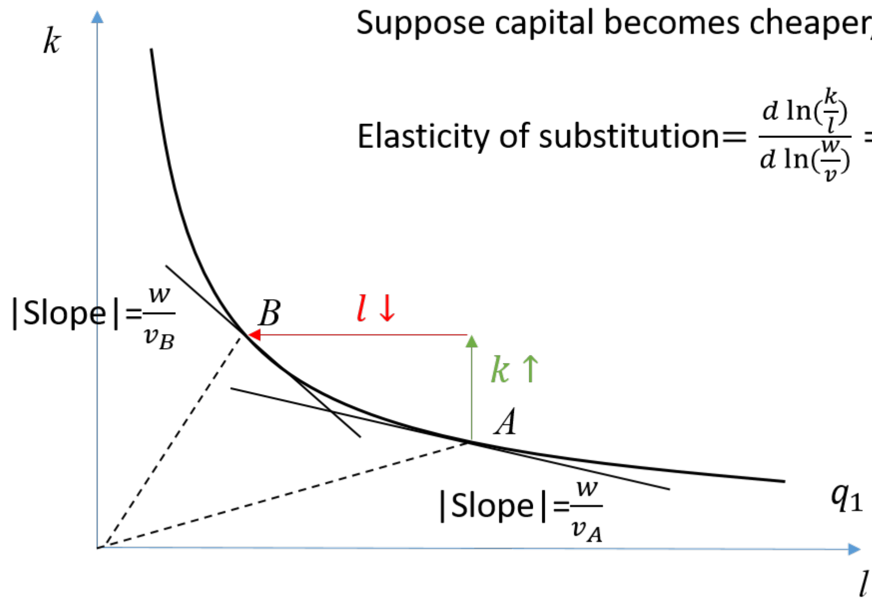
As q increases, $\frac{k^*}{l^*}$ (capital-labor ratio) does not change.



As q increases, $\frac{k^*}{l^*}$ may increase.
So economic growth may lead to less job.

Suppose capital becomes cheaper, v_A reduces to v_B, $v_B < v_A$

Elasticity of substitution$=\dfrac{d\ln(\frac{k}{l})}{d\ln(\frac{w}{v})}=\dfrac{\frac{k^*}{l^*}\text{ rises for }10\%}{\frac{w}{v}\text{ rises for }20\%}=\dfrac{1}{2}$



When input price ratio $(\frac{w}{v})$ changes, capital-labor ratio $(\frac{k^*}{l^*})$ also changes.
From A to B, capital becomes relatively cheaper, more capital is employed, and less labor is hired. Capital-labor ratio rises.

Example, Cost minimization for $f(k,l) = Ak^{\alpha}l^{\beta}$

$$\min_{k,l} wl + vk, \text{ s.t. } Ak^{\alpha}l^{\beta} = q$$

If the isoquant is "well-behaved" (decreasing and convex), we can use the tangency condition

$$RTS = \frac{f_l}{f_k} = \frac{A\beta k^{\alpha}l^{\beta-1}}{A\alpha k^{\alpha-1}l^{\beta}} = \frac{\beta k}{\alpha l} = \frac{w}{v}$$

Together with the constraint

$$Ak^{\alpha}l^{\beta} = q,$$

we can solve two unknowns k and l from two equations.

$$k = \frac{\alpha w}{\beta v}l$$

$$A(\frac{\alpha w}{\beta v}l)^{\alpha}l^{\beta} = q$$

$$l^{\alpha+\beta}A\frac{\alpha^{\alpha}w^{\alpha}}{\beta^{\alpha}v^{\alpha}} = q$$

$$l^{\alpha+\beta} = \frac{\beta^{\alpha}v^{\alpha}}{A\alpha^{\alpha}w^{\alpha}}q$$

$$l^* = l^c(v,w,q) = \left[\frac{\beta^{\alpha}v^{\alpha}}{A\alpha^{\alpha}w^{\alpha}}q\right]^{\frac{1}{\alpha+\beta}} = \frac{\beta^{\frac{\alpha}{\alpha+\beta}}v^{\frac{\alpha}{\alpha+\beta}}}{A^{\frac{1}{\alpha+\beta}}\alpha^{\frac{\alpha}{\alpha+\beta}}w^{\frac{\alpha}{\alpha+\beta}}}q^{\frac{1}{\alpha+\beta}}$$

$$\begin{aligned}
k^* = k^c(v,w,q) &= \frac{\alpha w}{\beta v}l^* = \frac{\alpha w}{\beta v}\frac{\beta^{\frac{\alpha}{\alpha+\beta}}v^{\frac{\alpha}{\alpha+\beta}}}{A^{\frac{1}{\alpha+\beta}}\alpha^{\frac{\alpha}{\alpha+\beta}}w^{\frac{\alpha}{\alpha+\beta}}}q^{\frac{1}{\alpha+\beta}} \\
&= \frac{\beta^{\frac{\alpha}{\alpha+\beta}-1}v^{\frac{\alpha}{\alpha+\beta}-1}}{A^{\frac{1}{\alpha+\beta}}\alpha^{\frac{\alpha}{\alpha+\beta}-1}w^{\frac{\alpha}{\alpha+\beta}-1}}q^{\frac{1}{\alpha+\beta}} \\
&= \frac{\alpha^{\frac{\beta}{\alpha+\beta}}w^{\frac{\beta}{\alpha+\beta}}}{A^{\frac{1}{\alpha+\beta}}\beta^{\frac{\beta}{\alpha+\beta}}v^{\frac{\beta}{\alpha+\beta}}}q^{\frac{1}{\alpha+\beta}}
\end{aligned}$$

The solution to the cost minimization problem are **conditional/contingent** input demand function

$$\begin{cases} l^c(v,w,q) = \frac{\beta^{\frac{\alpha}{\alpha+\beta}}v^{\frac{\alpha}{\alpha+\beta}}}{A^{\frac{1}{\alpha+\beta}}\alpha^{\frac{\alpha}{\alpha+\beta}}w^{\frac{\alpha}{\alpha+\beta}}}q^{\frac{1}{\alpha+\beta}} \\ k^c(v,w,q) = \frac{\alpha^{\frac{\beta}{\alpha+\beta}}w^{\frac{\beta}{\alpha+\beta}}}{A^{\frac{1}{\alpha+\beta}}\beta^{\frac{\beta}{\alpha+\beta}}v^{\frac{\beta}{\alpha+\beta}}}q^{\frac{1}{\alpha+\beta}} \end{cases}$$

Here q is fixed.

Later, we have $l^*(v,w,p)$ and $k^*(v,w,p)$ as solution to profit-max problem when q is chosen (not fixed). They are called input demand functions.

Cost function

Cost function is the value function of cost minmization problem. (The objective function, $wl+vk$, has not been optimized).

If we evaluated the objective function at the solution, this is the cost function

$$\begin{aligned} C(v,w,q) &= wl^* + vk^* \\ &= wl^c(v,w,q) + vk^c(v,w,q) \end{aligned}$$

For the Cobb-douglas example, the cost function is

$$\begin{aligned} C(v,w,q) &= w\frac{\beta^{\frac{\alpha}{\alpha+\beta}} v^{\frac{\alpha}{\alpha+\beta}}}{A^{\frac{1}{\alpha+\beta}} \alpha^{\frac{\alpha}{\alpha+\beta}} w^{\frac{\alpha}{\alpha+\beta}}} q^{\frac{1}{\alpha+\beta}} + v\frac{\alpha^{\frac{\beta}{\alpha+\beta}} w^{\frac{\beta}{\alpha+\beta}}}{A^{\frac{1}{\alpha+\beta}} \beta^{\frac{\beta}{\alpha+\beta}} v^{\frac{\beta}{\alpha+\beta}}} q^{\frac{1}{\alpha+\beta}} \\ &= \left\{ w\frac{\beta^{\frac{\alpha}{\alpha+\beta}} v^{\frac{\alpha}{\alpha+\beta}}}{A^{\frac{1}{\alpha+\beta}} \alpha^{\frac{\alpha}{\alpha+\beta}} w^{\frac{\alpha}{\alpha+\beta}}} + v\frac{\alpha^{\frac{\beta}{\alpha+\beta}} w^{\frac{\beta}{\alpha+\beta}}}{A^{\frac{1}{\alpha+\beta}} \beta^{\frac{\beta}{\alpha+\beta}} v^{\frac{\beta}{\alpha+\beta}}} \right\} q^{\frac{1}{\alpha+\beta}} \\ C(q) &= Bq^{\frac{1}{\alpha+\beta}} \end{aligned}$$

Average cost

$$AC(q) = \frac{C(q)}{q} = Bq^{\frac{1}{\alpha+\beta}-1} = Bq^{\frac{1-(\alpha+\beta)}{\alpha+\beta}}$$

Shape of cost function and average cost depend on parameter α and β.

If $\alpha+\beta=1$, then $C(q)=Bq$ (straight line), $AC(q)=B$, which is flat. This is the case of constant returns to scale.

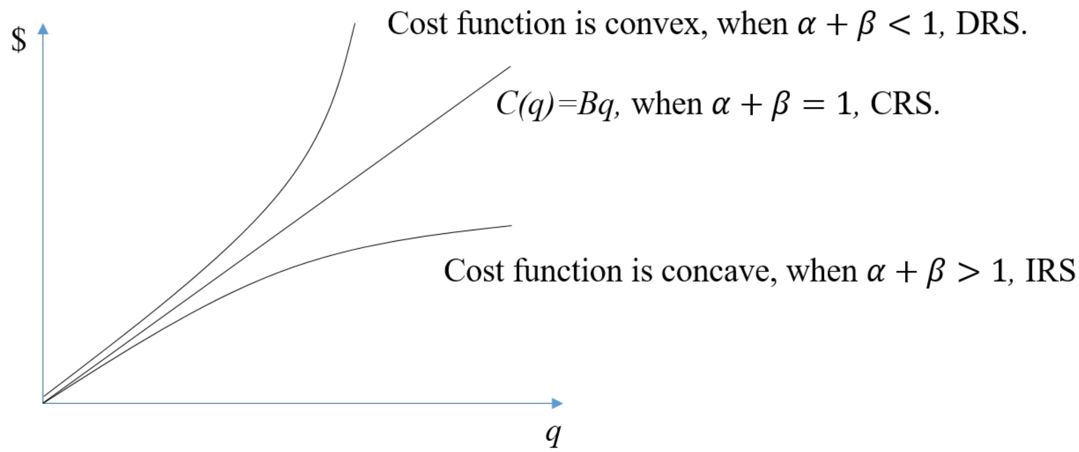
If $\alpha+\beta>1$, then $\frac{1-(\alpha+\beta)}{\alpha+\beta}<0$, $AC(q)$ decreases in q, $C(q)$ is concave. This is when the production function exhibit increasing returns to scale.

$$C'(q) = B\frac{1}{\alpha+\beta} q^{\frac{1}{\alpha+\beta}-1} > 0$$

$$C''(q) = B\frac{1}{\alpha+\beta} \underbrace{\left(\frac{1-(\alpha+\beta)}{\alpha+\beta}\right)}_{>0 \text{ or } <0} q^{\frac{1}{\alpha+\beta}-2}$$

($\alpha+\beta>1$, then $\frac{1-(\alpha+\beta)}{\alpha+\beta}<0$, then $C''(q)<0$.)

If $\alpha+\beta<1$, then $\frac{1-(\alpha+\beta)}{\alpha+\beta}>0$, $AC(q)$ increases in q, $C(q)$ is convex. This is when the production function exhibit decreasing returns to scale.



Lagrangian approach and evenlop theorem

We only consider equality constraint in our course, so we are only using a very simple version of Lagrangian method, which only requires first-order conditions (FOC).

In a more complicated problem, there may be several constraints with inequality. Some of the inequalities "binds" (holds in equality at the solution); some of the inequalities "slacks". In this case, we need Kuhn-Tucker conditions to find the solution.

$$\max_{x,y} f(x,y), \text{ s.t. } g(x,y) = b \Leftrightarrow g(x,y) - b = 0$$

$$\min_{x,y} f(x,y), \text{ s.t. } g(x,y) = b \Leftrightarrow g(x,y) - b = 0$$

$$\mathscr{L} = f(x,y) + \lambda(g(x,y) - b)$$

Or you can write

$$\mathscr{L} = f(x,y) + \lambda(b - g(x,y))$$

When there is inequality constraint, then the sign of λ matters. (λ is called Lagrangian multiplier.)

But if we only consider equality constraint, then we can ignore the sign of λ.

The solution is characterized by three FOCs

$$\begin{cases} \frac{\partial \mathscr{L}}{\partial x} = 0 \\ \frac{\partial \mathscr{L}}{\partial y} = 0 \\ \frac{\partial \mathscr{L}}{\partial \lambda} = 0 \end{cases}$$

For cost minimization problem

$$\min_{k,l} wl + vk, \text{ s.t. } f(k,l) - q = 0$$

$$\mathscr{L} = wl + vk + \lambda[f(k,l) - q]$$

$$\begin{cases} \frac{\partial \mathscr{L}}{\partial l} = w + \lambda f_l = 0 & \Rightarrow \frac{w}{f_l} = -\lambda \\ \frac{\partial \mathscr{L}}{\partial k} = v + \lambda f_k = 0 & \Rightarrow \frac{v}{f_k} = -\lambda \\ \frac{\partial \mathscr{L}}{\partial \lambda} = f(k,l) - q = 0 \end{cases}$$

From the first two FOCs

$$\frac{w}{f_l} = -\lambda = \frac{v}{f_k}$$

$$\Rightarrow \frac{w}{f_l} = \frac{v}{f_k} \Rightarrow \frac{w}{v} = \frac{f_l}{f_k}$$

The solution is determined by

$$\begin{cases} \frac{w}{v} = \frac{f_l}{f_k} \\ f(k,l) = q \end{cases}$$

The solution is

$$l^c(v,w,q),\ k^c(v,w,q)$$

The value function is

$$V(v,w,q) = f(l^c(v,w,q), k^c(v,w,q))$$

Envelop theorem: the derivative of the value function w.r.t. an exogenous variable is equal to the derivative of the Lagrangian function w.r.t. that variable.

$$\frac{\partial V(v,w,q)}{\partial v} = \frac{\partial \mathscr{L}}{\partial v}$$

$$\mathscr{L} = wl + vk + \lambda[f(k,l) - q]$$

$$\frac{\partial \mathscr{L}}{\partial v} = k = k^c(v,w,q)$$

$$\frac{\partial \mathscr{L}}{\partial w} = l = l^c(v,w,q)$$

Example,

$$C(v,w,q) = \left\{ w \frac{\beta^{\frac{\alpha}{\alpha+\beta}} v^{\frac{\alpha}{\alpha+\beta}}}{A^{\frac{1}{\alpha+\beta}} \alpha^{\frac{\alpha}{\alpha+\beta}} w^{\frac{\alpha}{\alpha+\beta}}} + v \frac{\alpha^{\frac{\beta}{\alpha+\beta}} w^{\frac{\beta}{\alpha+\beta}}}{A^{\frac{1}{\alpha+\beta}} \beta^{\frac{\beta}{\alpha+\beta}} v^{\frac{\beta}{\alpha+\beta}}} \right\} q^{\frac{1}{\alpha+\beta}}$$

$$\frac{\partial C(v,w,q)}{\partial v} = k^c(v,w,q) = \frac{\alpha^{\frac{\beta}{\alpha+\beta}} w^{\frac{\beta}{\alpha+\beta}}}{A^{\frac{1}{\alpha+\beta}} \beta^{\frac{\beta}{\alpha+\beta}} v^{\frac{\beta}{\alpha+\beta}}} q^{\frac{1}{\alpha+\beta}}$$

$$\frac{\partial C(v,w,q)}{\partial w} = l^c(v,w,q) = \frac{\beta^{\frac{\alpha}{\alpha+\beta}} v^{\frac{\alpha}{\alpha+\beta}}}{A^{\frac{1}{\alpha+\beta}} \alpha^{\frac{\alpha}{\alpha+\beta}} w^{\frac{\alpha}{\alpha+\beta}}} q^{\frac{1}{\alpha+\beta}}$$

These two equations are called Shepard's Lemma.

$\frac{\partial C(v,w,q)}{\partial v}$ and $\frac{\partial C(v,w,q)}{\partial w}$, are **comparative statics** properties.

Comparative statics means comparing two static situations due to change of some exogenous variable.

For example, v_1 changes to v_2, how will cost changes? We need to know $\frac{\partial C(v,w,q)}{\partial v}$.

Some examples

Cost minization of fixed proportion (Liontief) production function

$$\min_{k,l} wl + vk, \text{ s.t. } q = \min\{\alpha k, \beta l\}.$$

The isoquant of this production function is not differentiable, so we cannot use the tangency condition.

The solution is characterized by

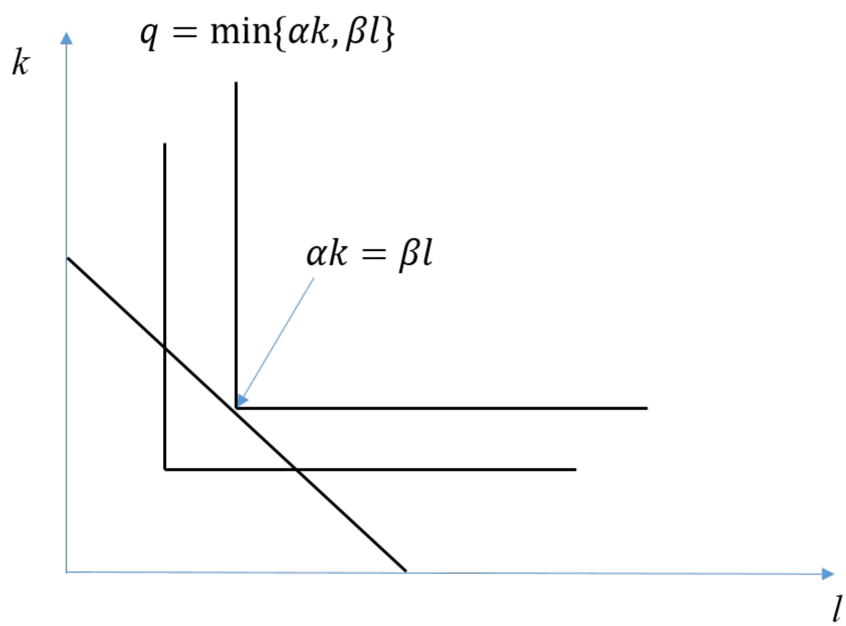
$$\alpha k = \beta l = q$$

Capital labor ratio (after cost minization), it is fixed for any output level q.

$$\frac{k}{l} = \frac{\beta}{\alpha}$$

The contingent input demand function

$$\begin{cases} k^c(w,v,q) = \frac{q}{\alpha} \\ l^c(w,v,q) = \frac{q}{\beta} \end{cases}$$



The cost function

$$\begin{aligned} C(w,v,q) &= wl^c(w,v,q) + vk^c(w,v,q) \\ &= w\frac{q}{\beta} + v\frac{q}{\alpha} = (\frac{w}{\beta} + \frac{v}{\alpha})q \end{aligned}$$

This function is linear in q, so it exhibit constant returns to scale.

Verify Shepard's lemma, check whether the theorem holds

$$\begin{aligned} \frac{\partial C(v,w,q)}{\partial v} &= \frac{q}{\alpha} = k^c(v,w,q) \\ \frac{\partial C(v,w,q)}{\partial w} &= \frac{q}{\beta} = l^c(v,w,q) \end{aligned}$$

$$C(w) = vk^* + l^*w$$

Short run cost

Example, $q = k_1^\alpha l^\beta$, in the short run, capital k is fixed at some level k_1.

The cost minimization problem becomes

$$\min_l vk_1 + wl, \ \ \text{.s.t. } f(k_1,l) = q$$

Because the constraint must be satified, so short-run labor demand can directly be solved

$$k_1^\alpha l^\beta = q$$

$$l^\beta = qk_1^{-\alpha}$$

$$\Rightarrow l^* = l^{SR}(v,w,q,k_1) = q^{\frac{1}{\beta}} k_1^{-\frac{\alpha}{\beta}}$$

Short run cost function

$$\begin{aligned} C^{SR}(v,w,q,k_1) &= vk_1 + wl^{SR}(v,w,q,k_1) \\ &= vk_1 + wk_1^{-\frac{\alpha}{\beta}} q^{\frac{1}{\beta}} \\ C(q) &= F + VC(q) \end{aligned}$$

For $\beta \in (0,1)$, this function is convex in q

$$\frac{\partial C^{SR}}{\partial q} = wk_1^{-\frac{\alpha}{\beta}} \frac{1}{\beta} q^{\frac{1}{\beta}-1} > 0$$

$$\frac{\partial^2 C^{SR}}{\partial q^2} = wk_1^{-\frac{\alpha}{\beta}} \frac{1}{\beta} \underbrace{(\frac{1}{\beta} - 1)}_{>0} q^{\frac{1}{\beta}-2} > 0$$

Let the marginal cost be

$$MC^{SR}(q) = \frac{\partial C^{SR}(q)}{\partial q}$$

$$\frac{\partial^2 C^{SR}}{\partial q^2} = \frac{\partial}{\partial q}\left(\frac{\partial C^{SR}(q)}{\partial q}\right) = \frac{\partial}{\partial q}\left(MC^{SR}(q)\right) > 0$$

So marginal cost increases in q.

(Second order derivative of cost function w.r.t. q is the slope of marginal cost. Note that marginal cost is the slope of the cost function.)

Cost functions

Example $C(q) = q^3 - 4q^2 + 6q + 18$

$$F = 18, \quad VC(q) = q^3 - 4q^2 + 6q$$

$$MC(q) = \frac{\partial C}{\partial q} = 3q^2 - 8q + 6$$

$$AC(q) = \frac{C(q)}{q} = q^2 - 4q + 6 + \frac{18}{q}$$

$$AVC(q) = q^2 - 4q + 6$$

Find short-run shut-down point, long-run zero profit point, and cost function.
Short-run shut-down point is the minimum of $AVC(q)$

$$AVC(q) = q^2 - 4q + 4 + 2 = (q-2)^2 + 2$$

$$q_1 = 2, \quad AVC(q_1) = 2$$

WE can also find the minimum by taking derivative or equating

$$MC(q) = AVC(q)$$

$$3q^2 - 8q + 6 = q^2 - 4q + 6$$

$$\Rightarrow 2q^2 = 4q$$

Both $q = 0$ and $q = 2$ are the solution.

In general, average cost has a **U-shape**.

$$AC(q) = \frac{F}{q} + \frac{VC(q)}{q}$$

When q is very small, then the average fixed cost ($\frac{F}{q}$) is very large.

For most industries (except for those with natural monopoly), the average cost will rise when q is large. There are two intrinsic reasons: First, input is scarce, so its price will rise when q becomes large. Second, when the firm becomes large, it will have multiple hierarchy. Every layer of hierarchy has principal-agent problem.

Find the minimum of

$$AC(q) = \frac{C(q)}{q} = q^2 - 4q + 6 + \frac{18}{q}$$

$$AC'(q) = 2q - 4 - \frac{18}{q^2} = 0$$

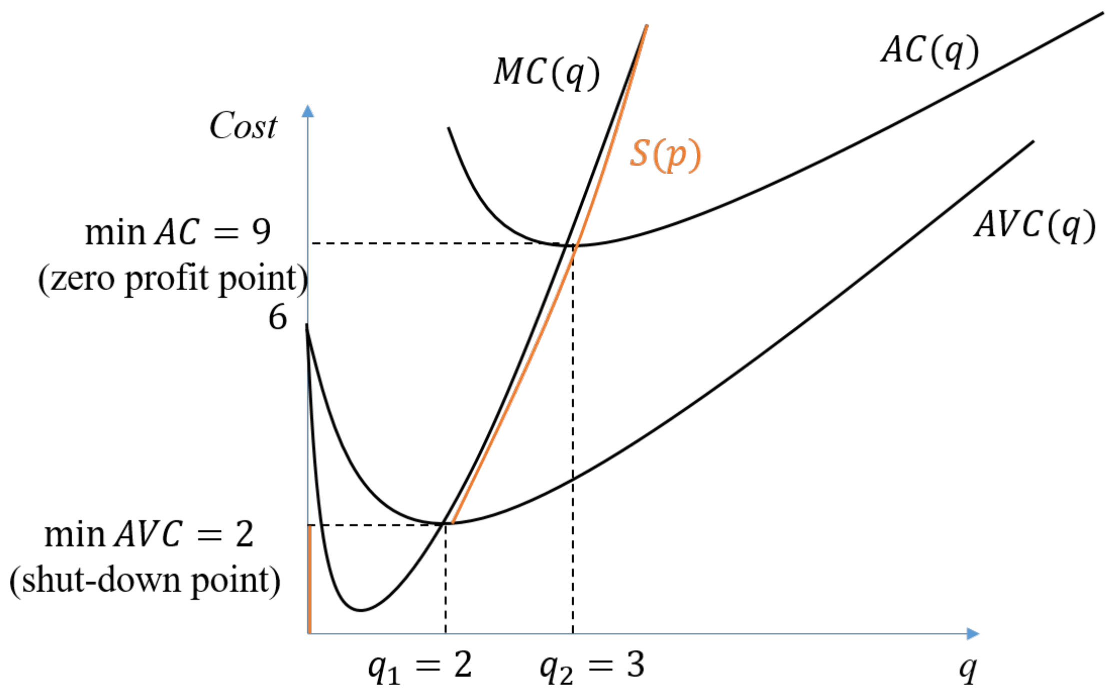
$$2q^3 - 4q^2 - 18 = 0$$

$$q^3 - 2q^2 - 9 = 0$$

The solution is $q_2 = 3$, the minimum of average cost is

$$AC(q_2) = 9 - 12 + 6 + 6 = 9.$$

The shut-down point is at $p_1 = 2$.
The zero-profit point is at $p_2 = 9$.



For a price-taking firm, price is fixed at p, the profit is

$$\pi(q) = p \times q - C(q)$$

$$\max_q \{p \times q - C(q)\}$$

The FOC is

$$p = C'(q^*) = MC(q^*) = g(q^*).$$

This determines the supply q^*.

Supply function is quantity as a function of p,

$$q^* = MC^{-1}(p) = g^{-1}(q^*).$$

(For example, if $p = MC(q) = 2q$, then $q = MC^{-1}(p) = S(p) = \frac{1}{2}p$.)

Note that, we need the price to be greater than the shut-down price, so the supply function (short-run) is

$$S(p) = \begin{cases} MC^{-1}(p) & \text{if } p \geq \min AVC(q) = p_1 \\ 0 & \text{if } p < \min AVC(q) = p_1 \end{cases}$$

For this example,

$$MC(q) = 3q^2 - 8q + 6 = p$$
$$3(q^2 - \frac{8}{3}q + \frac{16}{9}) + \frac{2}{3} = p$$
$$3(q - \frac{4}{3})^2 = p - \frac{2}{3}$$
$$(q - \frac{4}{3})^2 = \frac{p}{3} - \frac{2}{9}$$
$$q - \frac{4}{3} = \sqrt{\frac{p}{3} - \frac{2}{9}}$$
$$q = \sqrt{\frac{p}{3} - \frac{2}{9}} + \frac{4}{3}$$

So the supply function is

$$S(p) = \begin{cases} \sqrt{\frac{p}{3} - \frac{2}{9}} + \frac{4}{3} & \text{if } p \geq 2 \\ 0 & \text{if } p < 2 \end{cases}$$

Note: Homogeneous of degree 1, $f(x,y)$, then for $t > 0$

$$f(tx,ty) = t^r f(x,y), \quad r = 1$$

(Constant return to scale means production function is Homogeneous of degree 1.)

Homogeneous of degree r, $f(x,y)$, then for $t > 0$

$$f(tx,ty) = t^r f(x,y)$$