Topic 6

Constrained optimization

Outline

1. Optimization with equality constraints
2. Optimization with inequality constraints
3. General optimization Problem
4. Global maximization for general optimization problem
5. Envelope Theorem for maximization problem with constraints

1. Optimization with equality constraints

- In some optimization problems, the variables to be chosen are often required to satisfy certain constraints.
- For example, while a consumer's objective is to maximize utility, he/she has a budget constraint $p_1x_1 + p_2x_2 = I$

Example 1: Consider the 2-variable utility maximization problem

$$u(p_1, p_2, I) = \max_{x_1, x_2 \geq 0} (x_1 x_2)$$

subject to $p_1 x_1 + p_2 x_2 = I$

- From the budget constraint $x_2 = \frac{I - p_1 x_1}{p_2}$
- Let $f(x_1) = x_1 \left(\frac{I - p_1 x_1}{p_2} \right) = \frac{I}{p_2} x_1 - \frac{p_1}{p_2} {x_1}^2$
- The 2-variable maximization problem becomes a single variable maximization problem: $\max_{x_1} f(x_1)$
- FOC: $f'(x_1) = \frac{I}{p_2} - 2\frac{p_1}{p_2} x_1 = 0$, stationary point $x_1^* = \frac{I}{2p_1}$
- SOSC (sufficient condition for $x_1{}^*$ to be maximizer) $f''(x_1) = -2\frac{p_1}{p_2} < 0$
- x_1^* maximizes $f(x_1)$, from the budget constraint, $x_2^* = \frac{I}{2p_2}$

- Consider the 2-variable utility maximization problem with a linear constraint

$$\max_{x_1, x_2 \geq 0} u(x_1, x_2) \quad s.t. \quad p_1 x_1 + p_2 x_2 = I$$

  - using the budget constraint,

    $x_2 = \frac{I - p_1 x_1}{p_2} = h(x_1)$

  - The original problem becomes a single variable maximization problem $\max_{x_1} k(x_1)$ where $k(x_1) = u(x_1, h(x_1))$
  - FOC: $k'(x_1) = u_1' + u_2' h'(x_1) = u_1' - u_2' \frac{p_1}{p_2} = 0$
  - SOSC: $k''(x_1{}^*) < 0$, where

$$k''(x_1) = u_{11}'' + u_{12}''\left(-\frac{p_1}{p_2}\right) + \left(u_{21}'' + u_{22}''\left(-\frac{p_1}{p_2}\right)\right)\left(-\frac{p_1}{p_2}\right)$$

$$= u_{11}'' - 2\frac{p_1}{p_2}u_{12}'' + \left(\frac{p_1}{p_2}\right)^2 u_{22}'' = -\left(\frac{1}{p_2}\right)^2 \begin{vmatrix} 0 & p_1 & p_2 \\ p_1 & u_{11}'' & u_{12}'' \\ p_2 & u_{12}'' & u_{22}'' \end{vmatrix} < 0$$

– note the FOC is the same as $\frac{u_1'}{p_1} = \frac{u_2'}{p_2}$

– denote this common ratio as λ, then $u_1' - p_1\lambda = 0, u_2' - p_2\lambda = 0$

– Alternatively, define Lagrangian

$$L(x_1, x_2, \lambda) = u(x_1, x_2) + \lambda(I - p_1x_1 - p_2x_2)$$

– FOC: $\begin{cases} L_1' = u_1' - \lambda p_1 = 0 \\ L_2' = u_2' - \lambda p_2 = 0 \\ L_\lambda' = I - p_1x_1 - p_2x_2 = 0 \end{cases}$

– SOSC: $\begin{vmatrix} 0 & p_1 & p_2 \\ p_1 & u_{11}''(x^*) & u_{12}''(x^*) \\ p_2 & u_{12}''(x^*) & u_{22}''(x^*) \end{vmatrix} > 0$

- (Lagrange Theorem) For an optimization problem with two choice variables and one equality constraint (see supplementary notes):

$$\max(\min) f(x_1, x_2) \qquad s.t.\ g(x_1, x_2) = 0 \tag{6.1}$$

where f and g are C^2 functions.

- Define the Lagrangian function $L(x, \lambda) = f(x) + \lambda \cdot g(x)$
- and bordered Hessian matrix $B = \begin{pmatrix} 0 & g'(x^*) \\ \left[g'(x^*)\right]^T & L_x''(x^*, \lambda^*) \end{pmatrix}$

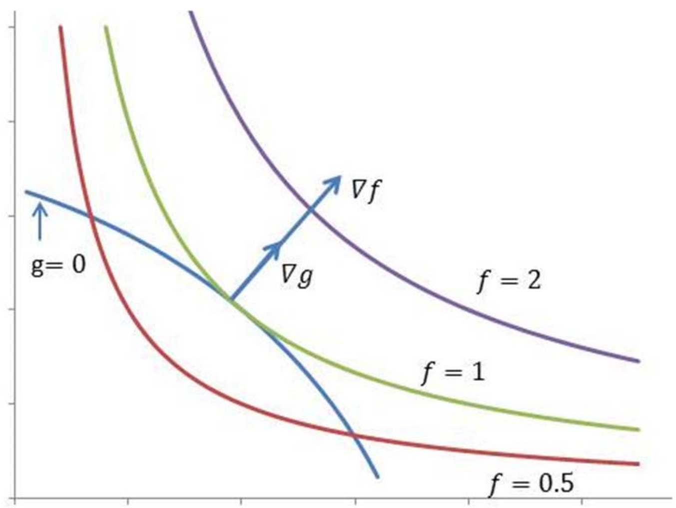
1. If x^* solves (6.1) and if the derivative $g'(x^*) \neq 0$, then there exists λ^* such that FOC: $L_x'(x^*, \lambda^*) = 0$
2. If the FOC is satisfied for some pair (x^*, λ^*) satisfying $g(x^*) = 0$ and SOSC: $|B| > 0$ then x^* is a unique local maximum.
3. If the FOC is satisfied for some pair (x^*, λ^*) satisfying $g(x^*) = 0$ and SOSC: $|B| < 0$ then x^* is a unique local minimum.

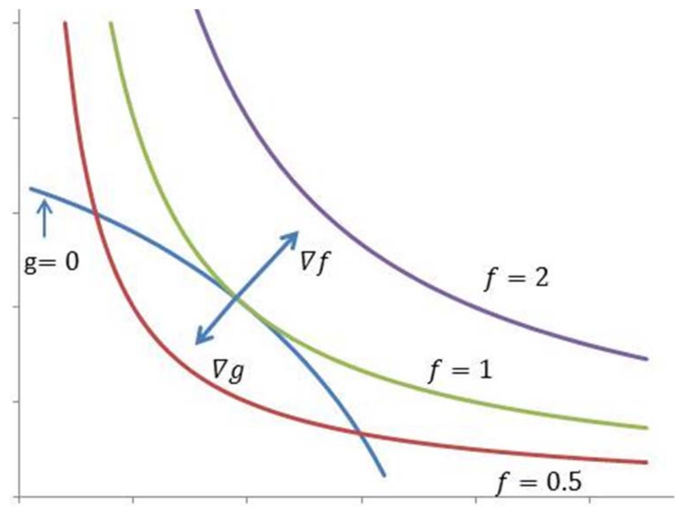
- **Example 1 (revisit)** define Lagrangian function

$$L(x_1, x_2, \lambda) = x_1 x_2 + \lambda(I - p_1 x_1 - p_2 x_2)$$

  - FOC: $\begin{cases} L_1' = x_2 - \lambda p_1 = 0 \\ L_2' = x_1 - \lambda p_2 = 0 \\ L_\lambda' = I - p_1 x_1 - p_2 x_2 = 0 \end{cases}$
  - solution: $(x_1{}^*, x_2{}^*, \lambda^*) = (\frac{I}{2p_1}, \frac{I}{2p_2}, \frac{I}{2p_1 p_2})$
  - Bordered Hessian matrix: $B = \begin{pmatrix} 0 & -p_1 & -p_2 \\ -p_1 & 0 & 1 \\ -p_2 & 1 & 0 \end{pmatrix}$
  - SOSC: $|B| = 2p_1 p_2 > 0$
  - From Lagrange Theorem, $(x_1{}^*, x_2{}^*)$ is constrained local maximum
  - Note: $g'(x) = (-p_1, -p_2) \neq 0$

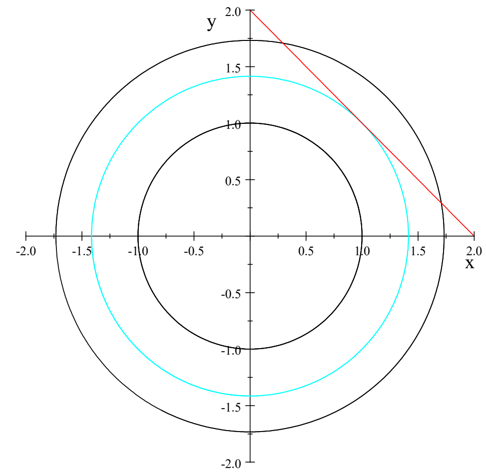
- Remarks on Lagrange Theorem for objective function of two variables with one constraint:
  - At $(x_1{}^*, x_2{}^*)$, the level curve of f and the curve $g = 0$ are tangent to each other
  - Since the gradient vectors $(f_1{}', f_2{}')$ and $(g_1{}', g_2{}')$ is perpendicular to the level curves, they much line up at $(x_1{}^*, x_2{}^*)$
  - They point in the same direction or in opposite directions





  - In either case, the gradients are scalar multiples of each other, we write this multiplier as $-\lambda^*$, so $(f_1', f_2') = -\lambda^*(g_1{}', g_2{}')$, this is exactly the FOC

- **Example 2**: $\min f(x_1, x_2) = {x_1}^2 + {x_2}^2$ subject to $x_1 + x_2 = 2$



  – Constraint: $g(x_1, x_2) = 2 - x_1 - x_2 = 0$, g satisfies $g'(x_1, x_2) = (-1, -1) \neq 0$

  – Lagrange function
  $L(x_1, x_2, \lambda) = {x_1}^2 + {x_2}^2 + \lambda(2 - x_1 - x_2)$

  – FOC: $\begin{cases} L_1' = 2x_1 - \lambda = 0 \\ L_2' = 2x_2 - \lambda = 0 \\ L_\lambda' = 2 - x_1 - x_2 = 0 \end{cases} \Rightarrow \lambda^* = 2,\ x_1^* = x_2^* = 1$

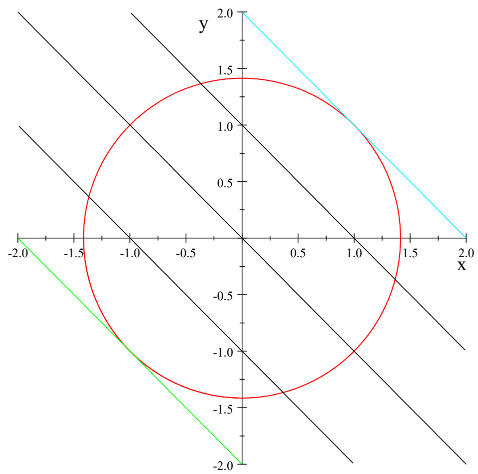
  – SOSC: $|B| = \begin{vmatrix} 0 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{vmatrix} = -4 < 0$

  – (1,1) is the constrained local minimum.

- **Example 3**: Solve the following:

$$\max(\min) f(x_1, x_2) = x_1 + x_2 \quad \text{subject to } {x_1}^2 + {x_2}^2 = 2$$

  - Constraint $g(x_1, x_2) = 2 - {x_1}^2 - {x_2}^2 = 0$
    satisfies $g'(x_1, x_2) = (-2x_1, -2x_2) \neq 0$ (why?)
  - Lagrange function $L(x_1, x_2, \lambda) = x_1 + x_2 + \lambda(2 - {x_1}^2 - {x_2}^2)$
  - FOC: $$\begin{cases} L_1' = 1 - 2\lambda x_1 = 0 \\ L_2' = 1 - 2\lambda x_2 = 0 \\ L_\lambda' = 2 - x_1^2 - x_2^2 = 0 \end{cases}$$
  - two solutions: $$\begin{array}{l} \text{Solution 1: } \lambda^* = 1/2,\ x_1^* = x_2^* = 1 \\ \text{Solution 2: } \lambda^* = -1/2,\ x_1^* = x_2^* = -1 \end{array}$$



  - SOSC: $$|B| = \begin{vmatrix} 0 & -2x_1^* & -2x_2^* \\ -2x_1^* & -2\lambda^* & 0 \\ -2x_2^* & 0 & -2\lambda^* \end{vmatrix} = 8\lambda^* \left(\left(x_1^*\right)^2 + \left(x_2^*\right)^2 \right)$$

    Solution 1: $|B| > 0$; Solution 2: $|B| < 0$
  - $({x_1}^*, {x_2}^*) = (1,1)$ is the constrained local maximum while $({x_1}^*, {x_2}^*) = (-1,-1)$ is the constrained local minimum.

- **Example 4**: For $a \in (0,1)$, consider

$$F(p_1, p_2, I) = \max_{x_1, x_2 \geq 0} \left\{ x_1{}^a + x_2{}^a \right\} \text{ s.t. } p_1 x_1 + p_2 x_2 = I$$

  - Define the Lagrange function $L(x_1, x_2, \lambda) = x_1{}^a + x_2{}^a + \lambda(I - p_1 x_1 - p_2 x_2)$
  - FOC: $$L_1' = a x_1{}^{a-1} - \lambda p_1 = 0; L_2' = a x_2{}^{a-1} - \lambda p_2 = 0$$ $$L_\lambda' = I - p_1 x_1 - p_2 x_2 = 0$$
  - Solution: $$x_1{}^* = \frac{p_1{}^{\frac{1}{a-1}} I}{p_1{}^{\frac{a}{a-1}} + p_2{}^{\frac{a}{a-1}}}; \quad x_2{}^* = \frac{p_2{}^{\frac{1}{a-1}} I}{p_1{}^{\frac{a}{a-1}} + p_2{}^{\frac{a}{a-1}}}$$
  - SOSC: $$|B| = \begin{vmatrix} 0 & -p_1 & -p_2 \\ -p_1 & a(a-1)(x_1{}^*)^{a-2} & 0 \\ -p_2 & 0 & a(a-1)(x_2{}^*)^{a-2} \end{vmatrix} = a(1-a)\left(p_1{}^2 (x_2{}^*)^{a-2} + p_2{}^2 (x_1{}^*)^{a-2} \right) > 0$$
  - x^* is local maximum point

- (Lagrange Theorem for general problem) For an optimization problem with n choice variables ($x \in R^n$) and m equality constraints:

$$\max_{x \in R^n} f(x) \quad s.t.\ G(x) = 0 \tag{6.2}$$

  - where $f: R^n \to R$ and $G: R^n \to R^m$ $(m < n)$ are C^2 functions.
  - Define the Lagrange function $L(x, \lambda) = f(x) + \lambda \cdot G(x)$

1. If x^* solves (6.2) and if the derivative $G'(x^*)$ has full rank, then there exists $\lambda^* \in R^m$ such that FOC is satisfied: $L_x(x^*, \lambda^*) = 0$

   Define the bordered Hessian matrix $B = \begin{pmatrix} 0 & G'(x^*) \\ [G'(x^*)]^T & L_x''(x^*, \lambda^*) \end{pmatrix}$

   with its leading principal minors $b_1, b_2, \ldots, b_{m+n}$

2. If the FOC is satisfied for some pair (x^*, λ^*) satisfying $G(x^*) = 0$ and SOSC: $(-1)^{m+k} b_k > 0, \forall k = 2m+1, \ldots, m+n$, then x^* is a unique local maximum.

3. If the FOC is satisfied for some pair (x^*, λ^*) satisfying $G(x^*) = 0$ and SOSC: $(-1)^{m} b_k > 0, \forall k = 2m+1, \ldots, m+n$, then x^* is a unique local minimum.

- Note: For $G: R^{\mathrm{n}} \to R^m$,

$$G(x) = \begin{pmatrix} G_1(x) \\ G_2(x) \\ \vdots \\ G_m(x) \end{pmatrix} = \begin{pmatrix} G_1(x_1, \ldots, x_n) \\ G_2(x_1, \ldots, x_n) \\ \vdots \\ G_m(x_1, \ldots, x_n) \end{pmatrix} \text{ is m-dimensional vector}$$

$$G'(x) = \begin{pmatrix} \frac{\partial G_1(x)}{\partial x_1} & \frac{\partial G_1(x)}{\partial x_2} & \cdots & \frac{\partial G_1(x)}{\partial x_n} \\ \frac{\partial G_2(x)}{\partial x_1} & \frac{\partial G_2(x)}{\partial x_2} & \cdots & \frac{\partial G_2(x)}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial G_m(x)}{\partial x_1} & \frac{\partial G_m(x)}{\partial x_2} & \cdots & \frac{\partial G_m(x)}{\partial x_n} \end{pmatrix} \text{ is } m \times n \text{ Jacobian matrix}$$

- **Example 5**: Solve the problem

$$max\,(min)\left\{x_1^2 + x_2^2 + x_3^2\right\}$$
$$\text{s.t. } x_1 + 2x_2 + x_3 = 30 \text{ and } 2x_1 - x_2 - 3x_3 = 10$$

  - The constraints can be written as $G(x) = 0$, where

$$G(x) = \begin{pmatrix} x_1 + 2x_2 + x_3 - 30 \\ 2x_1 - x_2 - 3x_3 - 10 \end{pmatrix}$$

  - The Jacobian matrix of the constraint functions is $G'(x) = \begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & -3 \end{pmatrix}$

    its rank is 2
  - Define the Lagrange function

$$L(x, \lambda, \mu) = {x_1}^2 + {x_2}^2 + {x_3}^2 + \lambda(x_1 + 2x_2 + x_3 - 30) + \mu(2x_1 - x_2 - 3x_3 - 10)$$

  - FOC:

$$\frac{\partial L}{\partial x_1} = 2x_1 + \lambda + 2\mu = 0,\ \frac{\partial L}{\partial x_2} = 2x_2 + 2\lambda - \mu = 0,\ \frac{\partial L}{\partial x_3} = 2x_3 + \lambda - 3\mu = 0$$
$$\frac{\partial L}{\partial \lambda} = x_1 + 2x_2 + x_3 - 30 = 0,\ \frac{\partial L}{\partial \mu} = 2x_1 - x_2 - 3x_3 - 10 = 0$$

its solution: $x_1^* = 10,\ x_2^* = 10,\ x_3^* = 0,\ \lambda^* = -12,\ \mu^* = -4$

- **Example 5 (continued)**
  - Verify SOSC: Let the bordered Hessian be

$$B = \begin{pmatrix} 0 & G'(x^*) \\ \left[G'(x^*)\right]^T & L_x''(x^*, \lambda^*) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & -1 & -3 \\ 1 & 2 & 2 & 0 & 0 \\ 2 & -1 & 0 & 2 & 0 \\ 1 & -3 & 0 & 0 & 2 \end{pmatrix}$$

- since $m = 2, n = 3, k = 2m + 1 = m + n = 5$, condition for local maximum is $b_5 = |B| < 0$, and condition for local minimum is $b_5 > 0$.

- |B|=150, therefore, $x^* = (10,10,0)$ is a local minimum

- **Example 6**: Consider the problem

$$max(min)\left(x_1^{\,2}x_2^{\,2}x_3^{\,2}\right) \;\; s.t.\; x_1^{\,2}+x_2^{\,2}+x_3^{\,2}=3$$

  - Constraint can be written as $G(x) = 3 - x_1{}^2 - x_2{}^2 - x_3{}^2 = 0$, which satisfies $G'(x) = (-2x_1, -2x_2, -2x_3) \neq 0$ for any solution.
  - Define Lagrange function $L(x,\lambda) = x_1{}^2 x_2{}^2 x_3{}^2 + \lambda(3 - x_1{}^2 - x_2{}^2 - x_3{}^2)$
  - FOC:

$$\begin{aligned} L_1' &= 2x_1x_2^2x_3^2 - 2\lambda x_1 = 0 \\ L_2' &= 2x_1^2x_2x_3^2 - 2\lambda x_2 = 0 \\ L_3' &= 2x_1^2x_2^2x_3 - 2\lambda x_3 = 0 \\ L_\lambda' &= 3 - x_1^2 - x_2^2 - x_3^2 = 0 \end{aligned}$$

  - Solutions: $\lambda^* = 1,\;\; x_1^{\,*} = \pm 1,\, x_2^{\,*} = \pm 1,\, x_3^{\,*} = \pm 1$

    or $\lambda^* = 0$ and any one (or two) of the $x_i{}^* = 0$ for $i = 1,2,3$ and the rest satisfying

$$(x_1^*)^2 + (x_2^*)^2 + (x_3^*)^2 = 3$$

- **Example 6 (continued)**:

  – Verify SOSC: The bordered Hessian for this problem is

$$B = \begin{pmatrix} 0 & G'(x^*) \\ \left[G'(x^*)\right]^T & L_x''(x^*, \lambda^*) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -2x_1^* & -2x_2^* & -2x_3^* \\ -2x_1^* & 2(x_2^* x_3^*)^2 - 2\lambda^* & 4x_1^* x_2^* (x_3^*)^2 & 4x_1^* (x_2^*)^2 x_3^* \\ -2x_2^* & 4x_1^* x_2^* (x_3^*)^2 & 2(x_1^* x_3^*)^2 - 2\lambda^* & 4(x_1^*)^2 x_2^* x_3^* \\ -2x_3^* & 4x_1^* (x_2^*)^2 x_3^* & 4(x_1^*)^2 x_2^* x_3^* & 2(x_1^* x_2^*)^2 - 2\lambda^* \end{pmatrix}$$

  – since $m = 1, n = 3, 2m + 1 = 3, m + n = 4, k = 3$ or 4, SOSC for

Local maximum:	$(-1)^{1+3} b_3 > 0 \; and \; (-1)^{1+4} b_4 > 0, \;\; i.e., b_3 > 0, b_4 < 0$
Local minimum:	$(-1)^1 b_3 > 0 \; and \; (-1)^1 b_4 > 0, \;\; i.e., b_3 < 0, b_4 < 0$

- **Example 6 (continued)**:

  – At $\lambda^* = x_1^* = x_2^* = x_3^* = 1,\ B = \begin{pmatrix} 0 & -2 & -2 & -2 \\ -2 & 0 & 4 & 4 \\ -2 & 4 & 0 & 4 \\ -2 & 4 & 4 & 0 \end{pmatrix}$

  and $b_3 = \begin{vmatrix} 0 & -2 & -2 \\ -2 & 0 & 4 \\ -2 & 4 & 0 \end{vmatrix} = 32 > 0,\ b_4 = |B| = -192 < 0$

  thus (1,1,1) is a local maximum.

  (In fact, all eight solutions corresponding to $\lambda^* = 1$ are local maximum)

  – At $\lambda^* = 0, x_1^* = x_2^* = 0, x_3^* = \sqrt{3},\ B = \begin{pmatrix} 0 & 0 & 0 & -2\sqrt{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2\sqrt{3} & 0 & 0 & 0 \end{pmatrix}$

  – $b_3 = b_4 = 0$, the sufficient condition for local minimum is not satisfied (anything wrong?)

2. Optimization with inequality constraints

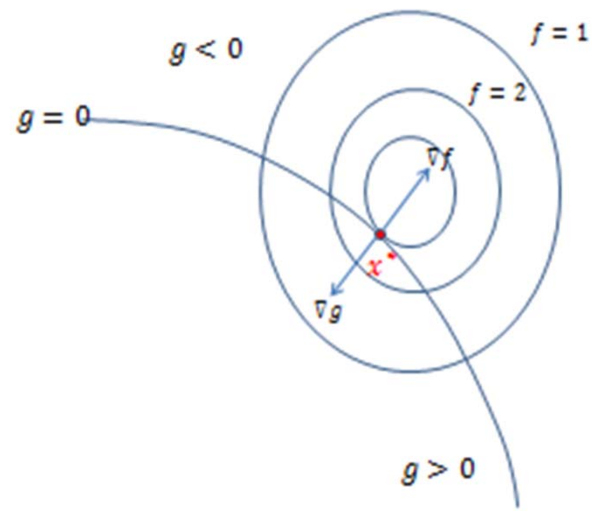
- We are more interested in inequality constraints such as the following problem:

$$u(p_1, p_2, I) = \max_{x_1, x_2}\{x_1 x_2\} \quad s.t. \;\; p_1 x_1 + p_2 x_2 \leq I$$

- A general two-variable optimization problem with inequality constraints:

$$\max_{x_1, x_2} f(x_1, x_2) \quad s.t. \;\; g(x_1, x_2) \geq 0 \qquad (6.3)$$

- In the following graph, the region to the left and below the curve $g = 0$ is the constraint set $g \geq 0$



- If at solution point x^*, $g(x^*) = 0$, we say that the constraint is binding at x^*

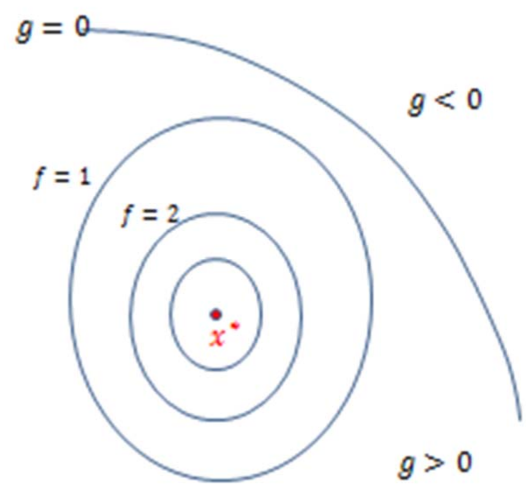
- At x^*, the level curve of f and g are tangent to each other, so $\nabla f(x^*)$ and $\nabla g(x^*)$ line up but point in different direction, there is a $\lambda > 0$ such that

$$\nabla f(x^*) = -\lambda \nabla g(x^*) \tag{6.4}$$

- Form the Lagrange function

$$L(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda g(x_1, x_2)$$

- (6.4) corresponds to the FOCs for L: $L'_{x_1} = 0$ and $L'_{x_2} = 0$
- If the maximum of f occurs at a point where $g(x_1, x_2) > 0$, then the constraint is not binding, therefore, x^* is an interior point satisfying
  $f'_{x_1}(x^*) = f'_{x_2}(x^*) = 0$



- we can still use the Lagrange function and set ${L_{x_1}}'$ and ${L_{x_2}}'$ to 0, provided that we set $\lambda = 0$ if $g(x_1, x_2) > 0$.

- (Kuhn-Tucker Theorem): Suppose that f and g are C^2 functions on R^2 and that $x^* = (x_1{}^*, x_2{}^*)$ maximizes f on the constraint set $g(x) \geq 0$. $g'(x^*) \neq 0 \; if \; g(x^*) = 0$.
  - There is a multiplier $\lambda^* \geq 0$ such that the following holds

$$\text{FOC: } L_{x_1}'(x^*, \lambda^*) = 0, \; L_{x_2}'(x^*, \lambda^*) = 0$$

$$\text{KTC: } \lambda^* g(x^*) = 0$$

  - SOSC for local maximum:
    (a) If $\lambda^* > 0, g(x^*) = 0$, SOSC is

$$|B| = \begin{vmatrix} 0 & g'(x^*) \\ [g'(x^*)]^T & L_x''(x^*, \lambda^*) \end{vmatrix} > 0$$

    (b) If $\lambda^* = 0$, $g(x^*) > 0$, SOSC is $f''(x^*) < 0$

- **Example 7**: For $a > 0, b > 0$, consider

$$F(a,b) = \max_{x_1,x_2}\{x_1 + x_2\} \qquad s.t. \quad a{x_1}^2 + b{x_2}^2 \le 1$$

  - The Lagrange function is $L(x_1, x_2, \lambda) = x_1 + x_2 + \lambda(1 - a{x_1}^2 - b{x_2}^2)$

$$\text{FOC: } 1 - 2\lambda a x_1 = 0,\ 1 - 2\lambda b x_2 = 0$$
$$\text{KTC: } \lambda(1 - a{x_1}^2 - b{x_2}^2) = 0$$

  - From FOC: $\lambda \neq 0$, by KTC, the constraint is binding, and $x_1 = \dfrac{1}{2a\lambda},\ x_2 = \dfrac{1}{2b\lambda}$
  - substitute this into $1 - a{x_1}^2 - b{x_2}^2 = 0$, we get

$$\lambda^* = \frac{1}{2}\sqrt{a^{-1} + b^{-1}} \ \text{(since } \lambda^* \ge 0\text{)}; \ {x_1}^* = \frac{1}{a\sqrt{a^{-1} + b^{-1}}},\ {x_2}^* = \frac{1}{b\sqrt{a^{-1} + b^{-1}}}$$

  - SOSC: Bordered Hessian

$$|B| = \begin{vmatrix} 0 & -2a{x_1}^* & -2b{x_2}^* \\ -2a{x_1}^* & -2a\lambda^* & 0 \\ -2b{x_2}^* & 0 & -2b\lambda^* \end{vmatrix} = 8ab\lambda^*\left(a\left({x_1}^*\right)^2 + b\left({x_2}^*\right)^2\right) > 0$$

  - x^* is local maximum

- **Example 8**: Given $\alpha, \beta > 0$, consider

$$F(p_1, p_2, I) = \max_{x_1, x_2 \geq 0} \left\{ x_1^{\alpha} x_2^{\beta} \right\} \quad s.t. \;\; p_1 x_1 + p_2 x_2 \leq I$$

  – The Lagrangian function is

$$L(x_1, x_2, \lambda) = x_1^{\alpha} x_2^{\beta} + \lambda(I - p_1 x_1 - p_2 x_2)$$

$$\text{FOC:} \;\; \alpha x_1^{\alpha-1} x_2^{\beta} - \lambda p_1 = 0, \;\; \beta x_1^{\alpha} x_2^{\beta-1} - \lambda p_2 = 0$$
$$\text{KTC:} \;\; \lambda(I - p_1 x_1 - p_2 x_2) = 0$$

  – From FOC, $\lambda \neq 0$, by KTC, the constraint is binding, and

$$x_1^* = \frac{\alpha I}{(\alpha + \beta) p_1}, \; x_2^* = \frac{\beta I}{(\alpha + \beta) p_2}$$

  – SOSC: check |B|>0, x^* is local maximum.

3. General optimization Problem

- In many applications, there will be both equality and/or inequality constraints. Given functions $g_i, h_j: R^{\mathrm{n}} \to R$ for $i = 1,2, \ldots, m$ and $j = 1,2, \ldots, k$, we encounter constraints of the form:

$$g_1(x) \geq 0, \ldots, g_m(x) \geq 0 \text{ and } h_1(x) = 0, \ldots h_k(x) = 0$$

- an $x \in R^{\mathrm{n}}$ satisfying the above constraints is said to be admissible.

- (Kuhn-Tucker Theorem) given C^2 functions $f, g_i, h_j: R^{\mathrm{n}} \rightarrow R$ for $i = 1,2,\dots,m$ and $j = 1,2,\dots,k (k < n),$. Let $G = (g_1,\dots,g_m)^T$, and $\mathrm{H} = (h_1,\dots,h_k)^T$. Consider

$$\max_x \{f(x)\} \;\; s.t. \;\; G(x) \geq 0 \text{ and } H(x) = 0 \tag{6.5}$$

  – Suppose that x^* is a solution. Let $I(x^*) = \{i | g_i(x^*) = 0\}$ and $g_i'(x^*)$ for $i \in I(x^*)$ together with all $h_j'(x^*)$ are linearly independent (the full rank condition), $L(x, \lambda, \mu) = f(x) + \lambda \cdot G(x) + \mu \cdot H(x)$ for $\lambda \in R^m$ and $\mu \in R^k$

  – There exists $\lambda^* \in R^m (\lambda \geq 0)$ and $\mu^* \in R^k$ such that

$$\text{FOC: } L_x'(x^*, \lambda^*, \mu^*) = 0$$

$$\text{KTC: } \lambda^* G(x^*) = 0$$

  – SOSC for local maximum: If $\lambda_i{}^* > 0$ for any $i \in I(x^*)$ and (x^*, λ^*, μ^*) satisfies the SOSC for the following problem

$$\max_x \{f(x)\} \;\; s.t. \;\; g_i(x) = 0 \text{ for } i \in I(x^*) \text{ and } H(x) = 0$$

  – then x^* is the unique local maximum

- **Example 9**: Solve the following maximization problem:

$$\max\{xy\} \ \ s.t. \begin{cases} x+y \geq -1 \\ x+y \leq 2 \end{cases}$$

  - The Lagrange function is $L(x,y,\lambda,\mu) = xy + \lambda(x+y+1) + \mu(2-x-y)$

$$\text{FOC: } \begin{cases} L_x = y+\lambda-\mu = 0 \\ L_y = x+\lambda-\mu = 0 \end{cases}$$

$$\text{KTC: } \begin{cases} \lambda(x+y+1) = 0 \\ \mu(2-x-y) = 0 \end{cases}$$

where λ,μ≥0

  - Case 1: $\lambda = \mu = 0 \Rightarrow x = y = 0$
    No binding constraints, SOSC for local maximum is $f''(x^*, y^*) < 0$. Since $f''(x,y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is indefinite, $(0{,}0)$ is not local maximum

- **Example 9 (continued)**:

  - Case 2: $\lambda = 0, \mu > 0 \Rightarrow$ μ=x=y=1
    One binding constraint: $2 - x - y = 0$, $m = 1, n = 2$, the bordered Hessian
    $$|B| = \begin{vmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{vmatrix} = 2 > 0$$, (1,1) is local maximum.
  - Case 3: $\lambda > 0, \mu = 0 \Rightarrow \lambda = \frac{1}{2}, x = y = -\frac{1}{2}$
    One binding constraint: $x + y + 1 = 0$, the bordered Hessian
    $$|B| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$=2>0, $(-\frac{1}{2}, -\frac{1}{2})$ is a local maximum
  - Case 4: $\lambda > 0, \mu > 0 \Rightarrow$ no solution

- **Example 10**: Consider the following problem:

$$min\left\{x^2+2y^2+3z^2\right\} \quad s.t.\ \ 3x+2y+z\geq 17$$

  - This problem is equivalent to the maximization problem:

$$\max\left\{-x^2-2y^2-3z^2\right\} \quad s.t.\ \ 3x+2y+z\geq 17$$

  - The Lagrange function $L=-x^2-2y^2-3z^2+\lambda(3x+2y+z-17)$

$$\text{FOC:}\ \begin{cases} -2x+3\lambda=0 \\ -4y+2\lambda=0 \\ -6z+\lambda=0 \end{cases}$$

$$\text{KTC:}\ \lambda(3x+2y+z-17)=0$$

- **Example 10 (continued)**:

  - Case 1: $\lambda = 0$, from FOC, $x = y = z = 0$, which violates the constraints
  - Case 2: $\lambda > 0$, then

$$\begin{cases} -2x + 3\lambda = 0 \\ -4y + 2\lambda = 0 \\ -6z + \lambda = 0 \\ 3x + 2y + z - 17 = 0 \end{cases} \Rightarrow \begin{cases} \lambda = 3 \\ x = 9/2 \\ y = 3/2 \\ z = 1/2 \end{cases}$$

  - SOSC: the bordered Hessian matrix is

$$B = \begin{pmatrix} 0 & 3 & 2 & 1 \\ 3 & -2 & 0 & 0 \\ 2 & 0 & -4 & 0 \\ 1 & 0 & 0 & -6 \end{pmatrix}$$

in this case, $m = 1, n = 3$, the SOSC for local maximum is satisfied: $b_3 = 44 > 0$, $b_4 = |B| = -272 < 0$. Thus we found a local maximizer.

4. Global maximization for general optimization problem

- (Global maximum theorem) For general problem (6.5), suppose (x^*, λ^*, μ^*) satisfies FOC and KTC
  - (Sufficient condition #1) If $L(x, \lambda^*, \mu^*)$ is a concave function in x, then x^* is a global constrained maximum point
  - (Sufficient condition #2) If following two conditions hold:

    (a): f, ${\lambda_i}^* g_i(\cdot)$ and ${\mu_j}^* h_j(\cdot)$ are quasi-concave for all i and j; and

    (b): $f'(x^*) \neq 0$ or f concave

    then x^* is a global constrained maximum point.
- The above global maximum theorem is very useful. Since the quasi-concavity requirement is often satisfied in economics problems.
- From Optimal Value Theorem, we have sufficient condition #3: f is continuous and the constraint set (feasible set) $\{x \in R^{\mathrm{n}}: G(x) \geq 0, H(x) = 0\}$ is compact, then the best of local solutions is the global solution.

- **Example 1 (revisit)**: For the utility maximization problem

$$u(p_1, p_2, I) = \max_{x_1, x_2 \geq 0} (x_1 x_2)$$

subject to $p_1 x_1 + p_2 x_2 = I$

  - Recall the Lagrange function $L(x_1, x_2, \lambda) = x_1 x_2 + \lambda(I - p_1 x_1 - p_2 x_2)$

    And solution to FOC: ${x_1}^* = \frac{I}{2p_1}$, ${x_2}^* = \frac{I}{2p_2}$, $\lambda^* = \frac{I}{2p_1 p_2}$

  - $L(x_1, x_2, \lambda^*)$ is not concave or convex, try Sufficient condition #2,
    (a) f and $\lambda^* g$ are quasi concave; (b) $f'_{x_i}(x^*) \neq 0$

- **Example 3 (revisit)**: Problem:

$$\max(\min) f(x_1, x_2) = x_1 + x_2 \quad \text{subject to } {x_1}^2 + {x_2}^2 = 2$$

  - Recall Lagrange function $L(x_1, x_2, \lambda) = x_1 + x_2 + \lambda(2 - {x_1}^2 - {x_2}^2)$

  - two solutions:

$$\text{Solution 1: } \lambda^* = 1/2,\ x_1^* = x_2^* = 1$$

$$\text{Solution 2: } \lambda^* = -1/2,\ x_1^* = x_2^* = -1$$

  - Since $L(x_1, x_2, \lambda^*)$ is concave for solution 1 and convex for solution 2, it follows from sufficient condition #1 that solution 1 is a global maximum and solution 2 is a global minimum

  - Alternatively, since the constraint set $\{x \in R^2 : {x_1}^2 + {x_2}^2 = 2\}$ is a compact set, from sufficient condition #3 the only local maximum $(1,1)$ is global maximum and the only local minimum $(-1, -1)$ is global minimum.

- **Example 4 (revisit)**: For $a \in (0,1)$, the problem:

$$F(p_1, p_2, I) = \max_{x_1, x_2 \geq 0} \left\{ x_1^{\ a} + x_2^{\ a} \right\} \text{ s.t. } p_1 x_1 + p_2 x_2 = I$$

  – x^* is a global maximum since sufficient condition #1 is satisfied

- **Example 5 (revisit)**:

$$\begin{aligned} &max\left(min\right)\left\{x_1^{\ 2} + x_2^{\ 2} + x_3^{\ 2}\right\} \\ &\text{s.t. } x_1 + 2x_2 + x_3 = 30 \text{ and } 2x_1 - x_2 - 3x_3 = 10 \end{aligned}$$

  – x^* is a global minimum since sufficient condition #1 is satisfied

- **Example 6 (revisit)**: the problem

$$max(min)\left(x_1{}^2 x_2{}^2 x_3{}^2\right) \;\; s.t. \; x_1{}^2 + x_2{}^2 + x_3{}^2 = 3$$

  - Solutions: $\lambda^* = 1, \;\; x_1{}^* = \pm 1, \, x_2{}^* = \pm 1, \, x_3{}^* = \pm 1$ (global maximum?)

    or $\lambda^* = 0$ and any one (or two) of the $x_i{}^* = 0$ for $i = 1,2,3$ and the rest satisfying the equality constraint (global minimum?)

- **Example 7 (revisit)**: the problem

$$F(a,b) = \max\{x_1 + x_2\} \qquad s.t. \quad ax_1{}^2 + bx_2{}^2 \le 1$$

Is x^* global maximum?

- **Example 8 (revisit)**: the problem

$$F(p_1, p_2, I) = \max_{x_1, x_2 \ge 0}\left\{x_1{}^\alpha x_2{}^\beta\right\} \quad s.t. \;\; p_1 x_1 + p_2 x_2 \le I$$

Is x^* global maximum?

- **Example 9 (revisit)**: the problem

$$\max\{xy\} \ \ s.t. \begin{cases} x+y \geq -1 \\ x+y \leq 2 \end{cases}$$

  - two local maximums: $(1,1)$ and $(-1/2,-1/2)$, is $(1,1)$ global maximum? Note that none of the sufficient conditions can be applied here.

5. Envelope Theorem for maximization problem with constraints

- **Example 1 (revisit)**: For the utility maximization problem

$$u(p_1, p_2, I) = \max_{x_1, x_2 \geq 0} (x_1 x_2)$$

subject to $p_1 x_1 + p_2 x_2 = I$

  - Recall the Lagrange function $L(x_1, x_2, \lambda) = x_1 x_2 + \lambda(I - p_1 x_1 - p_2 x_2)$

    And solution to FOC: ${x_1}^* = \frac{I}{2p_1}$, ${x_2}^* = \frac{I}{2p_2}$, $\lambda^* = \frac{I}{2p_1 p_2}$

  - Thus, $u(p_1, p_2, I) = x_1^* x_2^* = \dfrac{I^2}{4 p_1 p_2}$

  - Taking partial derivatives of u:

$$\frac{\partial u}{\partial p_1} = -\frac{I^2}{4 {p_1}^2 p_2}, \quad \frac{\partial u}{\partial p_2} = -\frac{I^2}{4 p_1 {p_2}^2}, \quad \frac{\partial u}{\partial I} = \frac{I}{2 p_1 p_2}$$

- An alternative way of finding the partial derivatives is to apply the Envelope Theorem
- Given $A \subset R^l$ and C^1 functions $f, g_i, h_j\colon R^{\mathrm{n}} \times A \to R$, let $G = (g_1, \dots, g_m)^T, H = (h_1, \dots, h_k)^T$, and the Lagrange function

$$L(x, a, \lambda, \mu) = f(x, a) + \lambda \cdot G(x, a) + \mu \cdot H(x, a)$$

- if $x^*(a)$ is the solution of the following problem:

$$F(a) = \max_{x \in R^n} \{f(x, a)\} \quad \text{s.t.} \begin{cases} G(x, a) \geq 0 \\ H(x, a) = 0 \end{cases}$$

- and $\lambda^*(a)$ and $\mu^*(a)$ are the corresponding Lagrange multipliers, then

$$\frac{\partial F(a)}{\partial a_i} = \left.\frac{\partial L(x, a, \lambda, \mu)}{\partial a_i}\right|_{x = x^*(a), \lambda = \lambda^*(a), \mu = \mu^*(a)}$$

for $i = 1, 2, \dots, l$

- **Example 1 (revisit)**: The problem

  $$u(p_1, p_2, I) = \max_{x_1, x_2 \geq 0} (x_1 x_2)$$

  subject to $p_1 x_1 + p_2 x_2 = I$

  – Recall the Lagrange function $L(x_1, x_2, \lambda) = x_1 x_2 + \lambda(I - p_1 x_1 - p_2 x_2)$

  And solution to FOC: ${x_1}^* = \frac{I}{2p_1}$, ${x_2}^* = \frac{I}{2p_2}$, $\lambda^* = \frac{I}{2p_1 p_2}$

  – Note $n = 2, l = 3,$ the three parameters are p_1, p_2 and I

  – Since $\frac{\partial L}{\partial p_1} = -\lambda x_1,\ \frac{\partial L}{\partial p_2} = -\lambda x_2,\ \frac{\partial L}{\partial I} = \lambda$

  – Apply the Envelope theorem:

$$\frac{\partial u}{\partial p_1} = \left.\frac{\partial L}{\partial p_1}\right|_{x=x^*, \lambda=\lambda^*} = -\lambda^* {x_1}^* = -\frac{I^2}{4{p_1}^2 p_2}$$

$$\frac{\partial u}{\partial p_2} = \left.\frac{\partial L}{\partial p_2}\right|_{x=x^*, \lambda=\lambda^*} = -\lambda^* {x_2}^* = -\frac{I^2}{4p_1 {p_2}^2}$$

$$\frac{\partial u}{\partial I} = \left.\frac{\partial L}{\partial I}\right|_{x=x^*, \lambda=\lambda^*} = \lambda^* = \frac{I}{2p_1 p_2}$$

- **Example 5 (revisit)**:
  –
$$\max(\min)\left\{x_1^{\,2}+x_2^{\,2}+x_3^{\,2}\right\}$$
$$\text{s.t. } x_1+2x_2+x_3=30 \text{ and } 2x_1-x_2-3x_3=10$$

Suppose we change the two constraints to $x_1+2x_2+x_3=31$ and $2x_1-x_2-3x_3=9$. Estimate the corresponding change in the value function by applying the envelope theorem. Find also the new exact value of the value function.

– Recall $L(x,\lambda,\mu)=x_1{}^2+x_2{}^2+x_3{}^2+\lambda(x_1+2x_2+x_3-30)+\mu(2x_1-x_2-3x_3-10)$

solution (global minimum): $x_1^{\,*}=10,\ x_2^{\,*}=10,\ x_3^{\,*}=0,\ \lambda^*=-12,\ \mu^*=-4$

– For $a=(a_1,a_2)$, the value function is

$$F(a_1,a_2)=\min\left\{x_1^{\,2}+x_2^{\,2}+x_3^{\,2}\right\}$$
$$\text{s.t. } x_1+2x_2+x_3=a_1 \text{ and } 2x_1-x_2-3x_3=a_2$$

with $F(30,10)=\left(x_1^{\,*}\right)^2+\left(x_2^{\,*}\right)^2+\left(x_3^{\,*}\right)^2=200$

- **Example 5 (continued)**:
  - The Lagrange function as a function of a is

$$L(x, \lambda, \mu, a_1, a_2) = {x_1}^2 + {x_2}^2 + {x_3}^2 + \lambda(x_1 + 2x_2 + x_3 - a_1) + \mu(2x_1 - x_2 - 3x_3 - a_2)$$

with $\dfrac{\partial L}{\partial a_1} = -\lambda, \ \dfrac{\partial L}{\partial a_2} = -\mu$

  - From the Envelope Theorem

$$\frac{\partial F(30,10)}{\partial a_1} = -\lambda^* = 12, \ \frac{\partial F(30,10)}{\partial a_2} = -\mu^* = 4$$

$$\Delta F \approx \frac{\partial F(30,10)}{\partial a_1}\Delta a_1 + \frac{\partial F(30,10)}{\partial a_2}\Delta a_2 = (12)(1) + (4)(-1) = 8$$

$$F(31,9) = F(30,10) + \Delta F \approx 208$$

  - Solve the problem with the new constraints gives $x^* = \left(\frac{151}{15}, \frac{31}{3}, \frac{4}{15}\right)$, thus

$$F(31,9) = \left(\frac{151}{15}\right)^2 + \left(\frac{31}{3}\right)^2 + \left(\frac{4}{15}\right)^2 \approx 208.19.$$

- **Example 11**: Consider the following problem

$$u^*(p_1,\dots,p_n,I)=\max_{x\geq 0}\{u(x_1,\dots,x_n)\}\ \ s.t.\ \ p_1x_1+\cdots+p_nx_n\leq I$$

where $u_i' > 0$.

– Construct Lagrange function $L(x,\lambda,p_1,\dots,p_n,I)=u(x)+\lambda(I-p_1x_1-\cdots-p_nx_n)$

$$\text{FOC: } u_i'(x)=\lambda p_i \quad \text{for } i=1,2,\dots,n$$

$$\text{KTC: } \lambda(I-p_1x_1-\cdots-p_nx_n)=0$$

– Since $u_i' > 0$, $\lambda^* > 0$ and x^* satisfy

$$u_i'(x)=\lambda p_i \quad \text{for } i=1,2,\dots,n$$

$$I-p_1x_1-\cdots-p_nx_n=0$$

– By the Envelope Theorem, we have

$$\frac{\partial u^*}{\partial p_i}=\left.\frac{\partial L}{\partial p_i}\right|_{x=x^*,\lambda=\lambda^*}=-\lambda^* x_i^*,\quad \frac{\partial u^*}{\partial I}=\left.\frac{\partial L}{\partial I}\right|_{x=x^*,\lambda=\lambda^*}=\lambda^*$$

– implying

$$x_i^*=-\frac{\frac{\partial u^*}{\partial p_i}}{\frac{\partial u^*}{\partial I}}\ \text{(Roy's identity)}$$