6. Introduction to Game Theory

Static Games

(1) Prisoner delimma

		2 Defect	Cooperate
1	Defect	1, 1	3, 0
	Cooperate	0, 3	2, 2

$$u_1(s_1, s_2)$$

$$u_1(D,D) = 1 > u_1(C,D) = 0$$

$$s_1^* = D = BR_1(s_2 = D)$$

$$u_1(D,C) = 3 > u_1(C,C) = 2$$

$$s_1^* = D = BR_1(s_2 = C)$$

Choosing $s_1^* = D$ is a dominant strategy for player 1.

Mutual best response means

$$\begin{cases} s_1^* = D = BR_1(s_2 = D) \\ s_2^* = D = BR_2(s_1 = D) \end{cases} \Rightarrow (D,D) \text{ is NE}$$

The efficient outcome of this game is (C,C) because it maximizes the sum of payoff (social welfare).

This is different from equilibrium outcome (D,D).

(2) Voting

A

		2 A	B	C
	A			
1	B			
	C			

B

		2 A	B	C
	A			
1	B			
	C			

C

		2 A	B	C
	A			
1	B			
	C			

If all players vote based on their preferences:
If candidcates A and B appear on the ballot, A will win. $A > B$
If candidcates B and C appear on the ballot, B will win. $B > C$
If candidcates A and C appear on the ballot, C will win. $C > A$
The collective preference under majority rule is not transitive.

(3) Mixed strategy NE

		2	
		Opera [q]	Boxing
1	Opera [p]	2, 1	0, 0
	Boxing	0, 0	1, 2

Player 1's expected payoff from choosing O is

$$\begin{aligned} Eu_1(O) &= qu_1(O,O) + (1-q)u_1(O,B) \\ &= q \times 2 + (1-q) \times 0 = 2q \end{aligned}$$

Player 1's expected payoff from choosing B is

$$\begin{aligned} Eu_1(B) &= qu_1(B,O) + (1-q)u_1(B,B) \\ &= q \times 0 + (1-q) \times 1 = 1-q \end{aligned}$$

Player 1 chooses O if $Eu_1(O) > Eu_1(B)$, this happens when $q > \frac{1}{3}$;
Player 1 chooses B if $Eu_1(O) < Eu_1(B)$, this happens when $q < \frac{1}{3}$;
Player 1 is indifferent between B and O if $Eu_1(O) = Eu_1(B)$, when $q = q^* = \frac{1}{3}$.

$$Eu_1(O) = Eu_1(B) \Leftrightarrow 2q = 1-q \Rightarrow q^* = \frac{1}{3}$$

These are best responses of player 1 to player 2's choice probability. We treat $q \in [0,1]$ as player 2's strategy space.

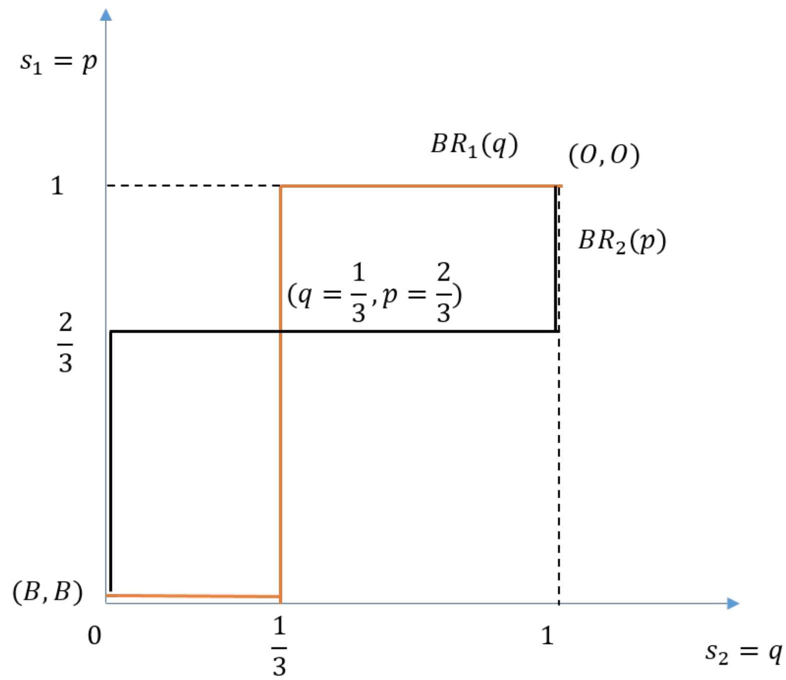
$$s_1 = p = BR_1(q) = \begin{cases} B\ (p=0), & q < \frac{1}{3} \\ p \in [0,1] & q = \frac{1}{3} \\ O\ (p=1), & q > \frac{1}{3} \end{cases}$$

(Player 1 choose O ($p = 1$) when player 2 chooses O with sufficiently large probabiliy $q > \frac{1}{3}$).

Player 2's expected payoff

$$\begin{aligned} Eu_2(O) &= pu_2(O,O) + (1-p)u_2(B,O) \\ &= p \times 1 + (1-p) \times 0 = p \end{aligned}$$

$$\begin{aligned} Eu_2(B) &= pu_2(O,B) + (1-p)u_2(B,B) \\ &= p \times 0 + (1-p) \times 2 = 2(1-p) \end{aligned}$$



Indifference condition:

$$Eu_2(O) = Eu_2(B) \Leftrightarrow p = 2(1-p) \Rightarrow p^* = \frac{2}{3}$$

When $p < \frac{2}{3}$, $Eu_2(O) < Eu_2(B)$, player 2 chooses B, $q = 0$;
When $p > \frac{2}{3}$, $Eu_2(O) > Eu_2(B)$, player 2 chooses O, $q = 1$;
When $p = \frac{2}{3}$, $Eu_2(O) = Eu_2(B)$, player 2 is indifferent, $q \in [0,1]$.

$$q = BR_2(p) = \begin{cases} q = 0 & p < \frac{2}{3} \\ q \in [0,1] & p = \frac{2}{3} \\ q = 1 & p > \frac{2}{3} \end{cases}$$

There three intersection of two best response curves.

$$\begin{cases} p = BR_1(q) \\ q = BR_2(p) \end{cases}$$

(i) $q = 0, p = 0$, this represents (B, B).
(ii) $q = 1$, $p = 1$, this represents (O, O).
(iii) $q = \frac{1}{3}$, $p = \frac{2}{3}$, this is a mixed strategy NE, $(p = \frac{2}{3}, q = \frac{1}{3})$.
There are three NEs, (B, B), (O, O), and $(p = \frac{2}{3}, q = \frac{1}{3})$.

(4)

		2	
		H [q]	D
1	H [p]	-2, -2	4, 0
	D	0, 4	2, 2

Use indifference conditions to find mixed strategy NE.

$$Eu_1(H) = Eu_1(D)$$
$$Eu_1(H) = q \times (-2) + (1-q) \times 4 = -2q + 4 - 4q = 4 - 6q$$
$$Eu_1(D) = q \times 0 + (1-q) \times 2 = 2 - 2q$$

$$4 - 6q = 2 - 2q$$
$$4q = 2 \Rightarrow q^* = \frac{1}{2}$$

Player 2's indifference condition

$$Eu_2(H) = Eu_2(D)$$

$$p(-2) + (1-p)4 = p \times 0 + (1-p)2$$
$$p^* = \frac{1}{2}$$

.

There are three NEs of this game,

$$(D,H), (H,D), \text{ and } (p = \frac{1}{2}, q = \frac{1}{2}).$$

(5) Example 8.5 (tragedy of common/Cournot duopoly)
The per-unit value of fish depends on the total quantity of supply, $q_1 + q_2$,

$$v(q_1, q_2) = 120 - (q_1 + q_2).$$

Each fishery wants to maximize revenue.
For fishery 1, it solves

$$\max_{q_1} \pi_1(q_1, q_2) = v(q_1, q_2) \times q_1$$

For fisher 2, it solves

$$\max_{q_2} \pi_2(q_1, q_2) = v(q_1, q_2) \times q_2$$

We can think of it as player 2's expansion of supply causes a negative externality to player 1.

For fisher 1, taking q_2 as given, he solves

$$q_1 = BR_1(q_2) = \arg\max_{q_1} \pi_1(q_1|q_2)$$

$$\begin{aligned}\max_{q_1} \pi_1(q_1|q_2) = v(q_1, q_2) \times q_1 &= (120 - q_1 - q_2)q_1 \\ &= 120q_1 - q_1^2 - q_2 q_1\end{aligned}$$

The solution of this maximization problem is characterized by the FOC

$$\frac{\partial \pi_1}{\partial q_1} = 120 - 2q_1 - q_2 = 0$$

Write q_1 as a function of q_2, this is fishery 1's best reponse

$$2q_1 = 120 - q_2$$

$$q_1 = 60 - \frac{1}{2}q_2 = BR_1(q_2)$$

For fishery 2, he chooses q_2 taking q_1 as given

$$\begin{aligned}\max_{q_2} \pi_2(q_2|q_1) &= (120 - q_1 - q_2) \times q_2 \\ &= 120q_2 - q_1 q_2 - q_2^2\end{aligned}$$

The solution is characterized by the FOC

$$\frac{\partial \pi_2}{\partial q_2} = 120 - q_1 - 2q_2 = 0$$

$$q_2 = 60 - \frac{1}{2}q_1 = BR_2(q_1)$$

The equilibrium of this game will be determined by mutual best responses

$$\begin{cases} q_1 = BR_1(q_2) = 60 - \frac{1}{2}q_2 \\ q_2 = BR_2(q_1) = 60 - \frac{1}{2}q_1 \end{cases}$$

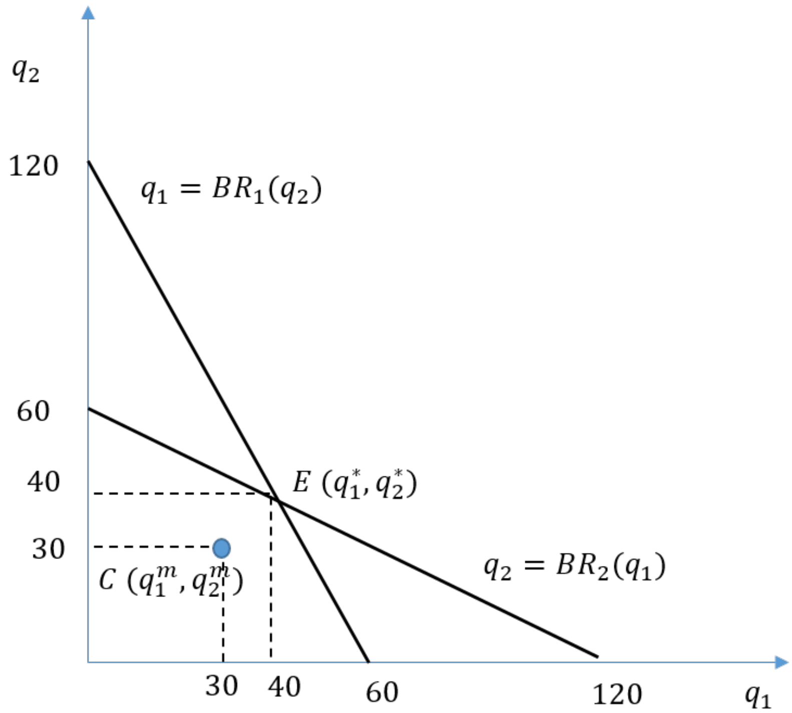
$$q_1 = 60 - \frac{1}{2}(60 - \frac{1}{2}q_1) = 60 - 30 + \frac{1}{4}q_1$$

$$\frac{3}{4}q_1 = 30 \Rightarrow q_1^* = 40$$

$$q_2^* = 40$$

So the NE of this game is $(q_1^* = 40, q_2^* = 40)$.

Is this NE the efficient outcome (maximize joint profit)?



$$\pi_1^{NE}(q_1^*, q_2^*) = (120 - 40 - 40) \times 40 = 1600$$
$$\pi_1^{NE}(q_1^*, q_2^*) = 1600$$

To find the efficient outcome, think about the monopoly problem (one firm owns both fisherys)

$$\begin{aligned}
\Pi(q_1, q_2) &= \pi_1(q_1, q_2) + \pi_2(q_1 + q_2) \\
&= v(q_1, q_2) \times (q_1 + q_2) \\
&= (120 - q_1 - q_2)(q_1 + q_2) \\
&= (120 - Q)Q
\end{aligned}$$

$$\Rightarrow Q_M = 60 = q_1^M + q_2^M$$

(The most natural way to divide the market is $q_1^M = q^M = 30$.)

$$\Pi^M = (120 - 60) \times 60 = 3600$$

Note that
$$\Pi^M = 3600 > \pi_1^{NE}(q_1^*, q_2^*) + \pi_2^{NE}(q_1^*, q_2^*) = 3200$$

So in the efficient outcome, the joint profit is larger than the joint profit in NE. (Decentralized decisions is less efficient than a centralized one).

Collaboration (collusion) can improve the joint profit.

(6) Dividing the pizza

Player 1's maximization problem is

$$\max_{s_1} \pi_1(s_1,s_2) = \begin{cases} s_1 & s_1+s_2 \le 1 \\ 0 & s_1+s_2 > 1 \end{cases}$$

Player 2's maximization problem is

$$\max_{s_2} \pi_2(s_1,s_2) = \begin{cases} s_2 & s_1+s_2 \le 1 \\ 0 & s_1+s_2 > 1 \end{cases}$$

Best reponses

$$s_1 = BR_1(s_2) = 1 - s_2$$

$$s_2 = BR_2(s_1) = 1 - s_1$$

NE by mutual best reponses

$$\Rightarrow \begin{cases} s_1 = 1 - s_2 \\ s_2 = 1 - s_1 \end{cases} \quad \Rightarrow s_1^* + s_2^* = 1$$

The NE is $\{(s_1^*, s_2^*) : s_1^* + s_2^* = 1\}$.

(7) Clearning the room (public good provision)

$$u_i(s_1, s_2, ..., s_n) = \sum_{j=1}^{n} s_j - c(s_i)$$

Player i's maximization problem

$$\max_{s_i} u_i(s_1, s_2, ..., s_n) = s_1 + s_2 + \cdots + s_n - c(s_i)$$

Finding BR (by FOC)

$$\frac{\partial u_i}{\partial s_i} = 1 - c'(s_i) = 0$$

When $c(s_i) = \frac{1}{2}s_i$,

$$\frac{\partial u_i}{\partial s_i} = 1 - \frac{1}{2} = \frac{1}{2} > 0.$$

It means the marginal benefit of contribution is larger than the marginal cost. Every one will contribution all five hours. NE (5,5,5,...)

When $c(s_i) = 2s_i$,

$$\frac{\partial u_i}{\partial s_i} = 1 - 2 = -1 < 0.$$

The marginal benefit of contributing is less than marginal cost. Every one will contribute 0 hour. NE (0,0,0...)

When $c(s_i) = s_i^2$,

$$\frac{\partial u_i}{\partial s_i} = 1 - 2s_i = 0$$

$$s_i^* = \frac{1}{2}$$

Then everyone contribute $s_i^* = \frac{1}{2}$. $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, ...)$

Sequential Games

(1) Battle of sexes in sequence

		2	
		Opera [q]	Boxing
1	Opera [p]	2, 1	0, 0
	Boxing	0, 0	1, 2

If they move simultaneously, then three NEs {O,O}, {B,B}, $\{p^*, q^*\}$.

Let player 1 move first, the outcome become $\{O, O\}$ (SPE) $\{O, OB\}$.

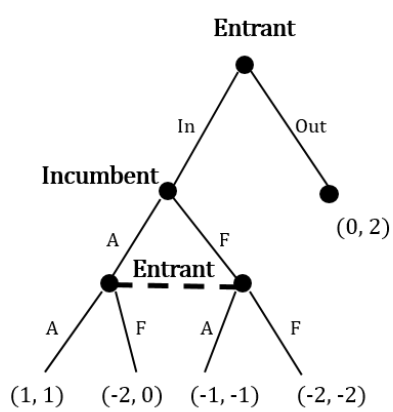
Strategy space of player 2: $\{OO, OB, BO, BB\} = \{O, B\} \times \{O, B\}$

Each element is a combination of strategy at two decision nodes. The left-hand side element is strategy at upper node, the right-hand side element is strategy at lower node.

		2			
		OO	OB	BO	BB
1	O	$\underline{2}, \underline{1}$	$\underline{2}, \underline{1}$	$\underline{0}, 0$	0, 0
	B	0, 0	1, $\underline{2}$	$\underline{0}, 0$	$\underline{1}, \underline{2}$

NEs: $\{O, OO\}, \{O, OB\}^*, \{B, BB\}$

(2) Incumbent versus Entrant

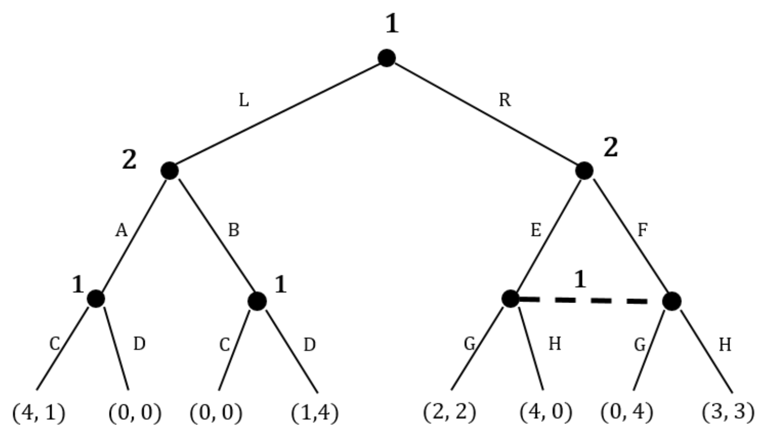


		2 (Incumbent)	
		A	F
1 (Entrant)	A	$\underline{1}$, $\underline{1}$*	$\underline{-1}$, -1
	F	-2, $\underline{0}$	-2, -2

NE of the left-hand side **subgame** is (A,A).
It means that if the Entrant chooses In, it will gain a payoff of 1.
If the Entrant stays Out, it will gain a payoff of 0.
Therefore, the Entrant will choose "In".

Outcome of the game is $\{\overbrace{(In,A)}^{\text{Entrant}},A\}$
SPE of the game is $\{(In,A),A\}$.
We treat decision nodes connected by information set as one node.

(3)



The right-hand side subgame

		2	
		E $[q]$	F
1	C $[p]$	4, 1	0, 0
	D	0, 0	1, 4

2 pure strategy NE $\{C,E\}$ $(u_1 = 4)$; $\{D,F\}$ $(u_1 = 1)$
1 mixed strategy NE $\{p^* = \frac{4}{5}, q^* = \frac{1}{5}\}$

$$\begin{cases} u_1 = u_1(C) = 4q = u_1(D) = (1-q) \\ u_2 = u_2(E) = p = u_2(F) = 4(1-p) \end{cases} \quad p^* = \frac{4}{5}, q^* = \frac{1}{5}$$

Player 1's payoff at the mixed strategy NE is

$$u_1 = 4q^* = \frac{4}{5}.$$

(i) When the right-hand side subgame leads to NE $\{C,E\}$, the outcome is $\{R;(C,E)\}$.
(ii) When the right-hand side subgame leads to NE $\{D,F\}$, the outcome is $\{L,F,D\}$ or $\{R,(D,F)\}$.
(iii) When the right-hand side subgame leads to NE $\{p^* = \frac{4}{5}, q^* = \frac{1}{5}\}$, the outcome is $\{L,F,D\}$.

Three possible SPEs

(i) $\{ \overbrace{(R,C,D,C)}^{\text{4 nodes of player 1}} ; \overbrace{(F,E)}^{\text{2 nodes of player 2}} \}$
(ii) $\{(L \text{ or } R, C, D, D); (F,F)\}$
(iii) $\{(L,C,D,p^* = \frac{4}{5}); (F, q^* = \frac{1}{5})\}$

Repeated Games

Stage Game:

		2 D	**2** C
1	D	$\underline{1}, \underline{1}$	$\underline{5}, 0$
1	C	$0, \underline{5}$	$4, 4$

$, t = 1,2,3,\ldots$

Player 1, given that player 2 chooses the trigger strategy, can choose from C or D at $t = 1$.
If player 1 chooses C, then:

t	1	2	3	
s_1	C	C	C	$\cdots$
s_2	C	C	C	$\cdots$

If player 1 chooses D, then

t	1	2	3	4	
s_1	D	D	D	D	$\cdots$
s_2	C	D	D	D	$\cdots$

Consider time $t = 4$. From player 1's perspective, he faces the same problem as in $t = 1$.

t	1	2	3	4	5	6	
s_1	C	C	C	C	C	C	$\cdots$
s_2	C	C	C	C	C	C	$\cdots$

t	1	2	3	4	5	6	
s_1	C	C	C	D	D	D	$\cdots$
s_2	C	C	C	C	D	D	$\cdots$

Due to the nature of infinitely repeated game, players face the same problem at each point of time. We can focus on whether a player wants to deviate at the first period.

To consider whether player 1 wants to deviate at $t = 1$, we need to compare two present values of payoff flows.

t	1	2	3	4	
s_1	D	D	D	D	$\cdots$
s_2	C	D	D	D	$\cdots$

$$\begin{aligned}\Pi(D) &= \pi_1(D,C)+\delta\pi_2(D,D)+\delta^2\pi_3(D,D)+\cdots\\ &= 5+\delta\times 1+\delta^2\times 1+\delta^3\times 1+\cdots\\ &= 5+\delta+\delta^2+\delta^3+\cdots\\ &= 5+\frac{\delta}{1-\delta}\end{aligned}$$

Note that, for $\delta\in(0,1)$

$$\delta+\delta^2+\delta^3+\cdots=\frac{\delta}{1-\delta}.$$

$$\begin{aligned}S &= \delta+\delta^2+\delta^3+\cdots\\ \delta S &= \delta^2+\delta^3+\delta^4+\cdots\end{aligned}$$

$$\begin{aligned}S-\delta S &= \delta+\delta^2-\delta^2+\delta^3-\delta^3+\cdots\\ (1-\delta)S &= \delta\\ S &= \frac{\delta}{1-\delta}\end{aligned}$$

$$1+\delta+\delta^2+\delta^3+\cdots=\frac{1}{1-\delta}.$$

t	1	2	3	
s_1	C	C	C	$\cdots$
s_2	C	C	C	$\cdots$

$$\begin{aligned}\Pi(C) &= \pi_1(C,C)+\delta\pi_2(C,C)+\delta^2\pi_3(C,C)+\cdots\\ &= 4+\delta 4+\delta^2 4+\cdots\\ &= 4(1+\delta+\delta^2+\cdots)\\ &= \frac{4}{1-\delta}\end{aligned}$$

Player 1 does not want to deviate when $\Pi(C)>\Pi(D)$

$$\frac{4}{1-\delta}>5+\frac{\delta}{1-\delta}$$

$$
\begin{aligned}
4 &> 5(1-\delta)+\delta \\
4 &> 5-5\delta+\delta \\
4\delta &> 1 \\
\delta &> \frac{1}{4} \equiv \delta_{\min}
\end{aligned}
$$

General formula to obtain threshold discount factor

$$
\begin{gathered}
\Pi^C = \frac{\pi^C}{1-\delta} > \Pi^D = \pi^D + \frac{\delta}{1-\delta}\pi^{NE} \\
\pi^C > \pi^D(1-\delta) + \delta\pi^{NE} \\
\pi^C > \pi^D - \delta\pi^D + \delta\pi^{NE} \\
\delta(\pi^D - \pi^{NE}) > \pi^D - \pi^C \\
\delta > \frac{\pi^D - \pi^C}{\pi^D - \pi^{NE}} \equiv \delta_{\min}
\end{gathered}
$$

		2	
		D	*C*
1	*D*	$\underline{1}, \underline{1}$	$\underline{3}, 0$
	C	$0, \underline{3}$	$2, 2$

$$
\delta > \frac{3-2}{3-1} = \frac{1}{2} \equiv \delta_{\min}
$$