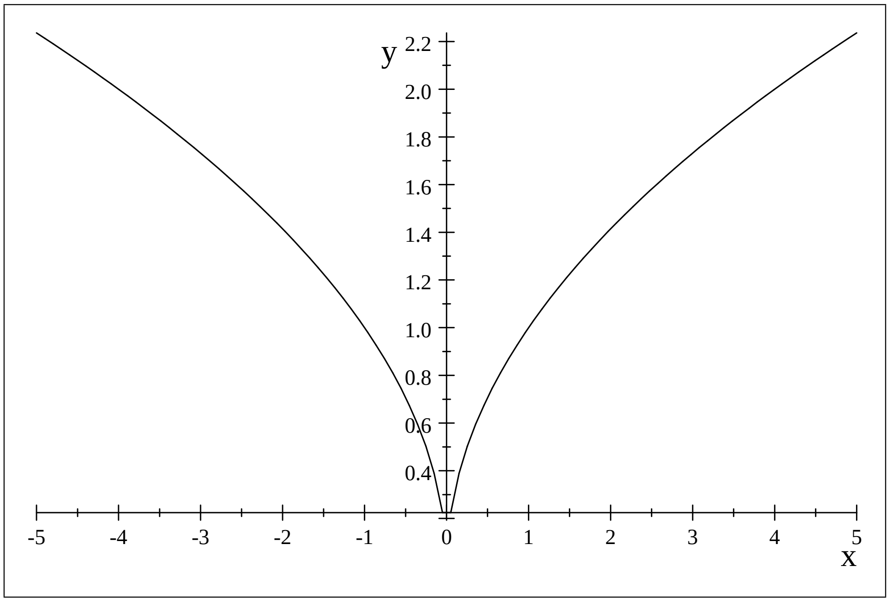
Solution to Final Exam Fall 2018

Question (1a)

i.$y = \sqrt{|x|}$



ii. f is not concave since pick two points (for example $(1,1), (-1,1)$) on the curve, the line connecting the two points is above the curve

f is not convex since pick two points (for example $(0,0), (1,1)$), the line connecting the two points is below the curve

iii. f is quasi-convex (either use definition to argue, or the lower level set is a convex set)

Question (1b)

$$g(x) = \ln f(x) = \ln x_1 + \cdots + \ln(x_n)$$

$$g''(x) < 0, \text{ thus } g \text{ is concave } \implies g \text{ is quasi-concave}$$

$$h(u) = e^u \text{ is strictly increasing function}$$

thus $f(x) = h(g(x))$ is quasi-concave

alternatively, use bordered hessian matrix to argue that the function is strictly quasi-concave

Question (1c):

For $x \in R^3$, any function with hessian matrix <0 is strictly concave function. An example:

$$f(x) = -x_1^2 - x_2^2 - x_3^2$$

$$f''(x) = \begin{pmatrix} -2 & & \\ & -2 & \\ & & -2 \end{pmatrix} < 0$$

Question (2a)

Let $F(K,L) = K^{\frac{1}{3}}L^{\frac{1}{3}}$ be a firm's production function, where K and L denotes capital and labor respectively. If the price of output is p and the cost of capital and labor is r and w respectively. The firm's profit is given by

$$\pi(K,L,p,r,w) = pK^{\frac{1}{3}}L^{\frac{1}{3}} - rK - wL$$

i. Find the solution to the problem

$$\pi^*(p,r,w) = \max_{K,L>0} \pi(K,L,p,r,w)$$

satisfying first order conditions

Sol: FOC:

$$\begin{aligned}\pi_K &= \frac{1}{3}pK^{-\frac{2}{3}}L^{\frac{1}{3}} - r = 0, \pi_L = \frac{1}{3}pK^{\frac{1}{3}}L^{-\frac{2}{3}} - w = 0 \\ &\Longrightarrow \frac{K}{L} = \frac{w}{r},\ K = \frac{p^3}{27wr^2}, L = \frac{p^3}{27w^2r}\end{aligned}$$

Solution: $(K^*, L^*) = \left(\frac{p^3}{27wr^2}, \frac{p^3}{27w^2r}\right)$

ii. Claim that the solution is in (i) is a global maximum

Sol: The Hessian matrix of π as function of (K,L) is

$$\begin{pmatrix} \pi_{KK} & \pi_{KL} \\ \pi_{KL} & \pi_{LL} \end{pmatrix} = \begin{pmatrix} -\frac{2}{9}pK^{-\frac{5}{3}}L^{\frac{1}{3}} & \frac{1}{9}pK^{-\frac{2}{3}}L^{-\frac{2}{3}} \\ \frac{1}{9}pK^{-\frac{2}{3}}L^{-\frac{2}{3}} & -\frac{2}{9}pK^{\frac{1}{3}}L^{-\frac{5}{3}} \end{pmatrix} < 0,$$

thus the function is concave, from Sufficient condition #1, (K^*, L^*) is global maximum.

iii. Find $\frac{\partial \pi^*}{\partial p}, \frac{\partial \pi^*}{\partial r}, \frac{\partial \pi^*}{\partial w}$

Sol: Apply the Envelope Theorem, since

$$\frac{\partial \pi}{\partial p} = K^{\frac{1}{3}}L^{\frac{1}{3}}, \frac{\partial \pi}{\partial r} = -K, \frac{\partial \pi}{\partial w} = -L$$

thus

$$\frac{\partial \pi^*}{\partial p} = (K^*)^{\frac{1}{3}}(L^*)^{\frac{1}{3}}, \frac{\partial \pi^*}{\partial r} = -K^*, \frac{\partial \pi^*}{\partial w} = -L^*$$

where, K^*, L^* can be found in the solution of (i)

Question (2b)

Consider the following problem:

$$\begin{cases} F(p_1,p_2,I) = \max_{x_1>0,x_2>0}\{\ln(x_1) + \ln(x_2)\} \\ \qquad s.t.\ \ p_1x_1 + p_2x_2 = I \end{cases}$$

i. denote $g(x) = I - p_1x_1 - p_2x_2$

Lagrange function:

$$L(x,\lambda) = \ln(x_1) + \ln(x_2) + \lambda(I - p_1x_1 - p_2x_2)$$

FOC:

$$\begin{cases} L_{x_1} = \frac{1}{x_1} - \lambda p_1 = 0 \\ L_{x_2} = \frac{1}{x_2} - \lambda p_2 = 0 \end{cases} \implies p_1x_1 = p_2x_2 = \frac{1}{\lambda} = \frac{I}{2}$$

thus $(x^*, \lambda^*) = \left(\frac{I}{2p_1}, \frac{I}{2p_2}, \frac{2}{I}\right)$

ii. Bordered Hessian matrix:

$$B = \begin{pmatrix} 0 & -p_1 & -p_2 \\ -p_1 & -\frac{1}{(x_1^*)^2} & 0 \\ -p_2 & 0 & -\frac{1}{(x_2^*)^2} \end{pmatrix}$$

with

$$|B| = \frac{p_2^2}{(x_1^*)^2} + \frac{p_1^2}{(x_2^*)^2} > 0$$

thus x^* is local maximum.

iii. Since

$$L''(x,\lambda^*) = \begin{pmatrix} -\frac{1}{x_1^2} & 0 \\ 0 & -\frac{1}{x_2^2} \end{pmatrix} < 0,$$

$L(x,\lambda^*)$ is concave in x, thus global result follows from Sufficient condition #1.

Question (2c)

i. The problem:

$$\max\left\{x_1^2 + x_2^2 + x_3^2\right\}$$
$$\text{subject to } 2x_1^2 + x_2^2 + x_3^2 \le 10 \text{ and } x_1 + x_2 + x_3 = 0$$

Lagrange function:

$$L(x,\lambda,\mu) = x_1^2 + x_2^2 + x_3^2 + \lambda\left(10 - 2x_1^2 - x_2^2 - x_3^2\right) + \mu(x_1 + x_2 + x_3)$$

FOC:

$$\begin{cases} 2x_1 - 4\lambda x_1 + \mu = 0 \\ 2x_2 - 2\lambda x_2 + \mu = 0 \\ 2x_3 - 2\lambda x_3 + \mu = 0 \end{cases}$$

$$\implies \begin{cases} 2x_1(2\lambda - 1) = \mu & (1) \\ 2x_2(\lambda - 1) = \mu & (2) \\ 2x_3(\lambda - 1) = \mu & (3) \end{cases}$$

KTC:

$$\lambda\left(6 - 2x_1^2 - x_2^2 - x_3^2\right) = 0, \quad \lambda \ge 0$$

Case 1: $\lambda = 0 \implies$ From FOC: $x_1 = x_2 = x_3 = -\mu/2$,

$$\text{Solution 1:} \quad \begin{cases} x_1^* = x_2^* = x_3^* = 0 \\ \lambda^* = \mu^* = 0 \end{cases}$$

Case 2: $\lambda > 0$,
Case 2.1: if $\lambda \neq \frac{1}{2}$ and $\lambda \neq 1$

$$\begin{cases} x_1 = \frac{\mu}{2(2\lambda-1)} \\ x_2 = x_3 = \frac{\mu}{2(\lambda-1)} \\ 2x_1^2 + x_2^2 + x_3^2 = 10 & (4) \\ x_1 + x_2 + x_3 = 0 & (5) \end{cases}$$

$$\implies \quad x_1 = -2x_2$$

$$\frac{\mu}{2\left(2\lambda - 1\right)} = -\frac{\mu}{\left(\lambda - 1\right)}$$

$\implies \mu = 0$ (rejected, since $x_i = 0$ does not satisfy binding constraint) ,or

$$\left(\lambda - 1\right) = -2\left(2\lambda - 1\right), \lambda = \frac{3}{5}$$

$$\implies x_2 = x_3 = -\frac{5\mu}{4}, x_1 = -2x_2 = \frac{5\mu}{2}$$

sub into binding constraint:

$$2\left(-2x_2\right)^2 + 2x_2^2 = 10$$

solutions:

$$\text{Solution 2:} \quad \begin{cases} x_1^* = -2 \\ x_2^* = x_3^* = 1 \\ \lambda^* = \frac{3}{5} \\ \mu^* = -\frac{4}{5} \end{cases}$$

$$\text{Solution 3:} \quad \begin{cases} x_1^* = 2 \\ x_2^* = x_3^* = -1 \\ \lambda^* = \frac{3}{5} \\ \mu^* = \frac{4}{5} \end{cases}$$

Case 2.2: $\lambda = \frac{1}{2} \implies$ by (1), (2), (3),$\mu = 0, x_2 = x_3 = 0$ which does not satisfy (4) and (5)
Case 2.3: $\lambda = 1 \implies$ by (1), (2), (3), $\mu = 0$, $x_1 = 0$, by (4) and (5)

$$\begin{cases} x_2^2 + x_3^2 = 10 \\ x_2 + x_3 = 0 \end{cases}$$

solutions:

$$\text{Solution 4:} \qquad \begin{cases} x_1^* = 0 \\ x_2^* = -x_3^* = \sqrt{5} \\ \lambda^* = 1 \\ \mu^* = 0 \end{cases}$$

$$\text{Solution 5} \quad : \qquad \begin{cases} x_1^* = 0 \\ x_2^* = -x_3^* = -\sqrt{5} \\ \lambda^* = 1 \\ \mu^* = 0 \end{cases}$$

ii.

$$L(x, \lambda^*, \mu^*) = x_1^2 + x_2^2 + x_3^2 + \lambda^* \left(10 - 2x_1^2 - x_2^2 - x_3^2\right) + \mu^* (x_1 + x_2 + x_3)$$

For solutions 4 and 5,

$$L(x, \lambda^*, \mu^*) = -x_1^2 + \mu^* (x_1 + x_2 + x_3)$$

is a concave function, By sufficient condition #1,

Solutions 4 and 5 and global maximum and the maximized value is 10.

iii. Let $a = (a_1, a_2)$

$$\begin{aligned} G(a) &= \max \left\{x_1^2 + x_2^2 + x_3^2\right\} \\ & \quad \text{subject to } 2x_1^2 + x_2^2 + x_3^2 \leq a_1 \text{ and } x_1 + x_2 + x_3 = a_2 \end{aligned}$$

then $G(10, 0) = 0$. Lagrange function:

$$L(x, \lambda^*, \mu^*) = x_1^2 + x_2^2 + x_3^2 + \lambda \left(a_1 - 2x_1^2 - x_2^2 - x_3^2\right) + \mu (x_1 + x_2 + x_3 - a_2)$$

By the envelope theorem:

$$\frac{\partial G(10, 0)}{\partial a_1} = \lambda^* = 1, \frac{\partial G(10, 0)}{\partial a_2} = \mu^* = 0,$$

therefore

$$\Delta G = \Delta a_1 = -0.1$$

$$G(9.9, 0.01) \approx 9.9$$

Question (3a)

A is indefinite since the diagonal elements are of different sign.

Question (3b)

$$A = \begin{pmatrix} a & 1 & b \\ 1 & -1 & 0 \\ b & 0 & -2 \end{pmatrix}$$

it is impossible for A to be positive or positive semi-definite since there are negative diagonal elements.

$A < 0$ iff $a < 0, \begin{vmatrix} a & 1 \\ 1 & -1 \end{vmatrix} = -a - 1 > 0$ and

$$|A| = 2a + b^2 + 2 < 0$$

thus $A < 0$ iff $a < -1$ and $b^2 < -2(a+1)$

$A \leq 0$ iff $a \leq 0$,

$$\begin{aligned} \begin{vmatrix} a & 1 \\ 1 & -1 \end{vmatrix} &\geq 0, \begin{vmatrix} a & b \\ b & -2 \end{vmatrix} \geq 0, \begin{vmatrix} -1 & 0 \\ 0 & -2 \end{vmatrix} \geq 0 \\ \implies & -a - 1 \geq 0 \ \text{ and } \ -2a - b^2 \geq 0 \end{aligned}$$

and

$$|A| = 2a + b^2 + 2 \leq 0$$

thus $A \leq 0$ iff $a \leq -1$ and $b^2 \leq -2(a+1)$

A is indefinite if either $a > -1$ or $b^2 > -2(a+1)$