Exercise 05

Question 2:

(a) $f(x) = x_1^2 + x_1x_2 + x_2x_3 + x_3^2 + 2x_1 - x_2$

$$f'(x) = \begin{pmatrix} 2x_1 + x_2 + 2 \\ x_1 + x_3 - 1 \\ x_2 + 2x_3 \end{pmatrix}, f''(x) = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 \end{pmatrix}$$

Stationary points satisfy$f'(x) = 0$, or

$$\begin{cases} 2x_1 + x_2 = -2 \\ x_1 + x_3 = 1 \\ x_2 + 2x_3 = 0 \end{cases}$$

Perform elementary row operation to augmented matrix:

$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & -2 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & -2 \\ 0 & 1 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -4 \\ 0 & 1 & 2 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 4 & 4 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Stationary point: $x_0 = (0, -2, 1)^T$

(b) $f(x) = e^{x_1}\left(x_1^2 + 2x_1x_2 - x_2^2 - 6\right)$

$$\begin{aligned}
f_1' &= e^{x_1}\left(x_1^2 + 2x_1x_2 - x_2^2 - 6\right) + e^{x_1}(2x_1 + 2x_2) \\
&= e^{x_1}\left(x_1^2 + 2x_1x_2 - x_2^2 + 2x_1 + 2x_2 - 6\right) \\
f_2' &= e^{x_1}(2x_1 - 2x_2) = 2e^{x_1}(x_1 - x_2) \\
f_{11}'' &= e^{x_1}\left(x_1^2 + 2x_1x_2 - x_2^2 + 2x_1 + 2x_2 - 6\right) + e^{x_1}(2x_1 + 2x_2 + 1) \\
&= e^{x_1}\left(x_1^2 + 2x_1x_2 - x_2^2 + 4x_1 + 4x_2 - 5\right) \\
f_{12}'' &= 2e^{x_1}(x_1 - x_2) + 2e^{x_1} = 2e^{x_1}(x_1 - x_2 + 1) \\
f_{22}'' &= -2e^{x_1}
\end{aligned}$$

Thus

$$\begin{aligned}
f'(x) &= \begin{pmatrix} e^{x_1}\left(x_1^2 + 2x_1x_2 - x_2^2 + 2x_1 + 2x_2 - 6\right) \\ 2e^{x_1}(x_1 - x_2) \end{pmatrix} \\
f''(x) &= \begin{pmatrix} e^{x_1}\left(x_1^2 + 2x_1x_2 - x_2^2 + 4x_1 + 4x_2 - 5\right) & 2e^{x_1}(x_1 - x_2 + 1) \\ 2e^{x_1}(x_1 - x_2 + 1) & -2e^{x_1} \end{pmatrix}
\end{aligned}$$

Stationary points satisfy:

$$\begin{cases} x_1^2 + 2x_1x_2 - x_2^2 + 2x_1 + 2x_2 - 6 = 0 \\ x_1 - x_2 = 0 \end{cases}$$

stationary points are: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ -3 \end{pmatrix}$

3. $A = \begin{pmatrix} -2 & -1 & 0 \\ -1 & -6 & 4 \\ 0 & 4 & -3 \end{pmatrix}$

Since $d_1 = -2 < 0, d_2 = \begin{vmatrix} -2 & -1 \\ -1 & -6 \end{vmatrix} = 11 > 0,\ d_3 = |A| = -1 < 0$

$\Longrightarrow\ A < 0$

4. Since some of the diagonal elements < 0 and some > 0,

$A = \begin{pmatrix} a & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 2 \end{pmatrix}$ is indefinite for any $a \in R$