1. $A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 0 & 2 & 3 \end{pmatrix}$

   (a) expand using first row: $|A| = \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} + 2 \begin{vmatrix} -1 & 1 \\ 0 & 2 \end{vmatrix} = 3 + 2 \times (-2) = -1$

   (b) A is invertible since $|A| \neq 0$

   (c) rank$(A) = 3$, for invertible matrix, its reduced row echelon form is identity matrix, with rank $=$ n $= 3$

   (d) Use elementary row operations to solve $Ax = b$, where $b = (2, 1, 0)$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ -1 & 1 & 0 & 1 \\ 0 & 2 & 3 & 0 \end{array}\right) \xrightarrow{R_2 \to R_1 + R_2} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 3 & 0 \end{array}\right) \xrightarrow{R_3 \to R_3 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & -6 \end{array}\right)$$

$$\xrightarrow{R_3 \to -R_3} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 6 \end{array}\right) \xrightarrow[R_2 \to R_2 - 2R_3]{R_1 \to R_1 - 2R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & 6 \end{array}\right)$$

   solution: $x = (-10, -9, 6)^T$

2. $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & \alpha \end{pmatrix}$ and $b = \begin{pmatrix} -1 \\ \beta \\ 1 \end{pmatrix}$

   (a) $|A| = \alpha - 2$, when $\alpha \neq 2$, A is invertible, there is unique solution

   (b) when $\alpha = 2$, perform elementary row operations to the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & \beta \\ -1 & -1 & 2 & 1 \end{array}\right) \xrightarrow{R_3 \to R_1 + R_3} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & \beta \\ 0 & -1 & 1 & 0 \end{array}\right) \xrightarrow{R_3 \to R_2 + R_3} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & \beta \\ 0 & 0 & 0 & \beta \end{array}\right)$$

   if $\beta \neq 0$, then there is no solution

   (c) when $\beta = 0$, there are infinitely many solutions (rank $(A) = rank\,(A|b)$)

3. Let $x = (x_1, x_2)'$, $A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{pmatrix}$, and $f(x) = (x_1, x_2, 1)\, A \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}$

   (a) $f(x) = (x_1 - 2, x_2, -2x_1) \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} = x_1 (x_1 - 2) + x_2^2 - 2x_1 = x_1^2 + x_2^2 - 4x_1$

(b) f is not homogeneous, since you can not find k so that $f(\lambda x) = \lambda^k f(x)$

(c) $f'(x) = \begin{pmatrix} 2x_1 - 4 \\ 2x_2 \end{pmatrix}, f''(x) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

(d) $f(x) = 0 \implies (x_1 - 2)^2 + x_2^2 = 2^2$, circle with center $(2, 0)$ and radius 2

(e) $f(2, -2) = 2^2 + (-2)^2 - 4 \times 2 = 0$, $(2, -2)$ is a point on the level curve in (d). $f'(2, -2) = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$

(vector pointing downwards)

(f) At the center of circle, $f(2, 0) = -4 < 0$, that means that $f(x) < 0$ of x is inside the circle, $f(x) = 0$ for x on the circle and $f(x) > 0$ for x outside the circle. So $A = \{x \in R^2 : f(x) \geq 0\}$ include all points on or outside the circle in (d)

4. Define:

$$\begin{cases} F^1(x, y, z, w) = x + y^2 + z^3 + e^z w^2 - 2 \\ F^2(x, y, z, w) = e^{2x} - y + xz^2 + w\ln(w) \end{cases}$$

(a) $F^1(0, 1, 0, 1) = 0, F^2(0, 1, 0, 1) = 0$

(b) Jacobian matrix

$$J = \frac{\partial(F^1, F^2)}{\partial(z, w)} = \begin{pmatrix} 3z^2 + e^z w^2 & 2e^z w \\ 2xz & \ln(w) + 1 \end{pmatrix}$$

at $(0, 1, 0, 1)$

$$J = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, |J| = 1 \neq 0$$

From implicit function Theorem, Equations (1) implicitly define (z, w) as differentiable function of (x, y) for (x, y, z, w) close to $(0, 1, 0, 1)$

(c) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial w}{\partial x}$ at the point $(0, 1, 0, 1)$. Take partial derivative with respect to x in Equations (1)

$$\begin{cases} 1 + 3z^2 \frac{\partial z}{\partial x} + e^z w^2 \frac{\partial z}{\partial x} + 2e^z w \frac{\partial w}{\partial x} = 0 \\ 2e^{2x} + z^2 + 2xz\frac{\partial z}{\partial x} + (\ln(w) + 1)\frac{\partial w}{\partial x} = 0 \end{cases}$$

at $(0, 1, 0, 1)$

$$\begin{cases} 1 + \frac{\partial z}{\partial x} + 2\frac{\partial w}{\partial x} = 0 \\ 2 + \frac{\partial w}{\partial x} = 0 \end{cases}$$

5. $g(x) = f(2x, e^x)$,

$$\begin{aligned} g'(x) &= 2f_1' + e^x f_2' \\ g''(x) &= 2\left(2f_{11}'' + e^x f_{12}''\right) + e^x f_2' + e^x\left(2f_{12}'' + e^x f_{22}''\right) \\ &= 4f_{11}'' + 4e^x f_{12}'' + e^{2x} f_{22}'' + e^x f_2' \end{aligned}$$