2.6.

(a)

$$E(Y) = \mu_Y = 0 \times P(Y = 0) + 1 \times P(Y = 1)$$
$$= 0 \times 0.12 + 1 \times 0.88 = 0.88$$

(b)

Unemployment Rate = $P(Y = 0) = 1 - P(Y = 1) = 1 - E(Y) = 1 - 0.88 = 0.12$

(c) Calculate the conditional probabilities first:

$$P(Y = 0|X = 0) = \frac{P(X = 0, Y = 0)}{P(X = 0)} = \frac{0.078}{0.751} = 0.104$$
$$P(Y = 1|X = 0) = \frac{P(X = 0, Y = 1)}{P(X = 0)} = \frac{0.673}{0.751} = 0.896$$

$$P(Y = 0|X = 1) = \frac{P(X = 1, Y = 0)}{P(X = 1)} = \frac{0.042}{0.249} = 0.169$$

$$P(Y = 1|X = 1) = \frac{P(X = 1, Y = 1)}{P(X = 1)} = \frac{0.207}{0.249} = 0.831$$

The conditional expectations are

$$E(Y|X = 1) = 0 \times P(Y = 0|X = 1) + 1 \times P(Y = 1|X = 1)$$
$$= 0 \times 0.169 + 1 \times 0.831 = 0.831$$
$$E(Y|X = 0) = 0 \times P(Y = 0|X = 0) + 1 \times P(Y = 1|X = 0)$$
$$= 0 \times 0.104 + 1 \times 0.896 = 0.896$$

(d) Use the solution to part (b),

Unemployment rate for college graduates = 1 – $E(Y|X{=}1)$ = 1−0.831 = 0.169

Unemployment rate for non-college graduates = 1 – $E(Y|X{=}0)$ = 1−0.896 = 0.104

(e) The probability that a randomly selected worker who is reported being unemployed is a college graduate is

$$P(X=1|Y=0)=\frac{P(X=1,Y=0)}{P(Y=0)}=\frac{0.042}{0.12}=0.35$$

The probability that this worker is a non-college graduate is

$$P(X=0|Y=0)=1-P(X=1|Y=0)=1-0.35=0.65$$

(f) Educational achievement and employment status are not independent because they do not satisfy that, for all values of x and y,

$$\Pr(X=x|Y=y)=\Pr(X=x).$$

For example, from part (e) $\Pr(X=0|Y=0)$ = 0.65, while from the table $\Pr(X = 0) = 0.751$.

2.14. The central limit theorem suggests that when the sample size (n) is large, the distribution of the sample average $(\bar{Y})$ is approximately $N(\mu_Y, \sigma_{\bar{Y}}^2)$ with $\sigma_{\bar{Y}}^2 = \frac{\sigma_Y^2}{n}$.

Given $\mu_Y = 50$, $\sigma_Y^2 = 21$,

(a) $n = 50$, $\sigma_{\bar{Y}}^2 = \frac{\sigma_Y^2}{n} = \frac{21}{50} = 0.42$, and

$$P(\bar{Y} \leq 51) = P\left(\frac{\bar{Y}-50}{\sqrt{0.42}} \leq \frac{51-50}{\sqrt{0.42}}\right) \approx \Phi(1.543) = 0.9386$$

(b) $n = 150$, $\sigma_{\bar{Y}}^2 = \frac{\sigma_Y^2}{n} = \frac{21}{150} = 0.14$, and

$$P(\bar{Y} > 49) = 1 - P(\bar{Y} \leq 49) = 1 - P\left(\frac{\bar{Y} - 50}{\sqrt{0.14}} \leq \frac{49 - 50}{\sqrt{0.14}}\right) \approx 1 - \Phi(-2.673)$$
$$= \Phi(2.673) = 0.9962$$

(c) $n = 45$, $\sigma_{\bar{Y}}^2 = \frac{\sigma_Y^2}{n} = \frac{21}{45} = 0.4667$, and

$$P(50.5 \leq \bar{Y} \leq 51) = P\left(\frac{50.5 - 50}{\sqrt{0.4667}} \leq \frac{\bar{Y} - 50}{\sqrt{0.4667}} \leq \frac{51 - 50}{\sqrt{0.4667}}\right) \approx \Phi(1.464) - \Phi(0.732)$$
$$= 0.9284 - 0.7679 = 0.1605$$

2.15. (a)

$$P(19.6 \le \bar{Y} \le 20.4) = P\left(\frac{19.6-20}{\sqrt{4/n}} \le \frac{\bar{Y}-20}{\sqrt{4/n}} \le \frac{20.4-20}{\sqrt{4/n}}\right)$$

$$= P(\frac{19.6-20}{\sqrt{4/n}} \le Z \le \frac{20.4-20}{\sqrt{4/n}})$$

where $Z \sim N(0, 1)$. Thus,

(i) $n = 25$; $P\left(\frac{19.6-20}{\sqrt{4/n}} \le Z \le \frac{20.4-20}{\sqrt{4/n}}\right) = P(-1 \le Z \le 1) = 0.6826$

(ii) $n = 100$; $P\left(\frac{19.6-20}{\sqrt{4/n}} \le Z \le \frac{20.4-20}{\sqrt{4/n}}\right) = P(-2 \le Z \le 2) = 0.9544$

(iii) $n = 800$; $P\left(\frac{19.6-20}{\sqrt{4/n}} \le Z \le \frac{20.4-20}{\sqrt{4/n}}\right) = P(-5.657 \le Z \le 5.657) = 1$

(b)

$$P(20 - c \le \bar{Y} \le 20 + c) = P(\frac{-c}{\sqrt{4/n}} \le \frac{\bar{Y}-20}{\sqrt{4/n}} \le \frac{c}{\sqrt{4/n}})$$

$$= P(\frac{-c}{\sqrt{4/n}} \le Z \le \frac{c}{\sqrt{4/n}})$$

As n get large $\frac{c}{\sqrt{4/n}}$ gets large, and the probability converges to 1.

(c) This follows from (b) and the definition of convergence in probability given in Key Concept 2.6.

2.19. (a) $\Pr(Y=y_j)=\sum_{i=1}^{l}\Pr(X=x_i, Y=y_j)$

$$=\sum_{i=1}^{l}\Pr(Y=y_j|X=x_i)\Pr(X=x_i)$$

(b) $E(Y)=\sum_{j=1}^{k}y_j\Pr(Y=y_j)=\sum_{j=1}^{k}y_j\sum_{i=1}^{l}\Pr(Y=y_j|X=x_i)\Pr(X=x_i)$

$$=\sum_{i=1}^{l}\left(\sum_{j=1}^{k}y_j\Pr(Y=y_j|X=x_i)\right)\Pr(X=x_i)$$

$$=\sum_{i=1}^{l}E(Y|X=x_i)\Pr(X=x_i).$$

(c) When X and Y are independent,

$$\Pr(X=x_i, Y=y_j)=\Pr(X=x_i)\Pr(Y=y_j),$$

so

$$\sigma_{XY}=E[(X-\mu_X)(Y-\mu_Y)]$$

$$=\sum_{i=1}^{l}\sum_{j=1}^{k}(x_i-\mu_X)(y_j-\mu_Y)\Pr(X=x_i, Y=y_j)$$

$$=\sum_{i=1}^{l}\sum_{j=1}^{k}(x_i-\mu_X)(y_j-\mu_Y)\Pr(X=x_i)\Pr(Y=y_j)$$

$$=\left(\sum_{i=1}^{l}(x_i-\mu_X)\Pr(X=x_i)\right)\left(\sum_{j=1}^{k}(y_j-\mu_Y)\Pr(Y=y_j\right)$$

$$=E(X-\mu_X)E(Y-\mu_Y)=0\times 0=0,$$

$$\mathrm{corr}(X,Y)=\frac{\sigma_{XY}}{\sigma_X\sigma_Y}=\frac{0}{\sigma_X\sigma_Y}=0.$$

2.23. X and Z are two independently distributed standard normal random variables, so

$$\mu_X=\mu_Z=0, \sigma_X^2=\sigma_Z^2=1, \sigma_{XZ}=0.$$

(a) Because of the independence between X and Z, $\Pr(Z=z|X=x)=\Pr(Z=z)$, and $E(Z|X)=E(Z)=0.$ Thus $E(Y|X)=E(X^2+Z|X)=E(X^2|X)+E(Z|X)=X^2+0=X^2.$

(b) $E(X^2)=\sigma_X^2+\mu_X^2=1$, and $\mu_Y=E(X^2+Z)=E(X^2)+\mu_Z=1+0=1.$

(c) $E(XY)=E(X^3+ZX)=E(X^3)+E(ZX).$ Using the fact that the odd moments of a standard normal random variable are all zero, we have $E(X^3)=0.$ Using the independence between X and Z, we have $E(ZX)=\mu_Z\mu_X=0.$ Thus $E(XY)=E(X^3)+E(ZX)=0.$

(d) $$\begin{aligned}\text{cov}(XY) &= E[(X-\mu_X)(Y-\mu_Y)] = E[(X-0)(Y-1)] \\ &= E(XY - X) = E(XY) - E(X) \\ &= 0 - 0 = 0.\end{aligned}$$

$$\text{corr}(X, Y) = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{0}{\sigma_X \sigma_Y} = 0.$$

2.27

(a) $E(u) = E[E(u|X)] = E[E(Y - \hat{Y})|X] = E[E(Y|X) - E(Y|X)] = 0.$

(b) $E(uX) = E[E(uX|X)] = E[XE(u|X)] = E[X\times 0] = 0$

(c) Using the hint: $v = (Y - \hat{Y}) - h(X) = u - h(X)$, so that $E(v^2) = E[u^2] + E[h(X)^2] - 2\times E[u\times h(X)]$. Using an argument like that in (b), $E[u\times h(X)] = 0$. Thus, $E(v^2) = E(u^2) + E[h(X)^2]$, and the result follows by recognizing that $E[h(X)^2] \geq 0$ because $h(x)^2 \geq 0$ for any value of x.