# Derivative Securities (FINA 3203)
# Solutions to Problem Set 5

**Question 1: European and American Puts (4/10)**

SOLUTION:

(i) The Black-Scholes-Merton price of a European call is

$$C_t = S_t \mathcal{N}(d_1) - e^{-r(T-t)} K \mathcal{N}(d_2) \tag{1}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

Substituting in the values given,

$$d_1 = 0.28146 \quad \text{and} \quad d_2 = 0.064954. \tag{2}$$

So,

$$\begin{aligned} C_t &= 100 \times \mathcal{N}(0.28146) - 100 \times e^{-.05\times.75} \mathcal{N}(0.064954) \\ &= 100 \times 0.61082 - 100 \times e^{-.05\times.75} \times 0.52589 \\ &= 10.429. \end{aligned}$$

So by put-call parity,

$$\begin{aligned} P_t &= C_t + e^{-r(T-t)} K - S_t \\ &= 10.429 + 100 \times e^{-.05\times.75} - 100 \\ &= 6.7484. \end{aligned}$$

(ii) In a three-stage tree, the step size is $h = .75/3 = .25$, and so the up and down movements are

$$\begin{aligned} u &= \exp\left(\sigma\sqrt{h}\right) - 1 = \exp\left(0.25 \times \sqrt{0.25}\right) = 1.1331 - 1 = 0.1331 \\ d &= \exp\left(-\sigma\sqrt{h}\right) - 1 = \exp\left(-0.25 \times \sqrt{0.25}\right) = 0.8825 - 1 = -0.1175. \end{aligned}$$

and so the risk-neutral probability of an up move is

$$q = \frac{e^{rh} - (1+d)}{u-d} = \frac{e^{0.05\times0.25} - 0.8825}{1.1331 - 0.8825} = 0.5191.$$

The stock price for the whole tree are:

- after 1 step, the stock price is 113.31 and 88.25 at the u and d nodes respectively.
- after 2 steps, the stock price is 128.39, 100, and 77.88 at the $uu$, $ud$ and $dd$ nodes respectively.
- after 3 steps, the stock price is 145.48, 113.31, 88.25, and 68.73 at the $uuu$, $uud$, and $udd$ and $ddd$ nodes respectively.

The risk neutral probability of ending at the udd node is $3q(1-q)^2 = 3\times0.5191\times(1-0.5191)^2 = 0.36015$ (there are three ways to get there: up, down, down; down, up, down; down, down, up). The risk neutral probability of ending at the $ddd$ node is $(1-q)^3 = (1-0.5191)^3 = 0.11122$. So the European put price is:

$$e^{-.75..05} \times (0.36015 \times (100 - 88.25) + 0.11122 \times (100 - 68.73)) = \$7.43.$$

(iii) Since the put price is below its no-arbitrage value, you should buy the put. To hedge, you need to sell a synthetic put. This requires you to work out the $\Delta$ of the put at each stage in the tree. For this calculation, you need to know the put value at each stage in the tree. Working backwards:

- after 3 steps, the put value is 0, 0, $100 - 88.25 = 11.75$, and $100 - 68.73 = 31.27$ at the $uuu$, $uud$, and $udd$ and $ddd$ nodes respectively.
- after 2 steps, the put value is

$$\begin{aligned} e^{-.05\times.25}(0.5191 \times 0 + (1-0.5191)\times 0) &= 0 \\ e^{-.05\times.25}(0.5191 \times 0 + (1-0.5191)\times 11.75) &= 5.58 \\ e^{-.05\times.25}(0.5191 \times 11.75 + (1-0.5191)\times 31.27) &= 20.88 \end{aligned} \tag{3}$$

  at the $uu$, $ud$ and $dd$ nodes respectively.
- after 1 step, the put value is

$$\begin{aligned} e^{-.05\times.25}(0.5191 \times 0 + (1-0.5191)\times 5.58) &= 2.65 \\ e^{-.05\times.25}(0.5191 \times 5.58 + (1-0.5191)\times 20.88) &= 12.78 \end{aligned}$$

  at the u and d nodes respectively. With the prices in hand, we can calculate the deltas:
- initially, the delta is

$$\frac{2.65 - 12.78}{113.31 - 88.25} = -.404$$

  (the numerator is the di.erence in put prices at step 1; the denominator is the di.erence in stock prices)

– after one step, the delta is

$$\frac{0 - 5.58}{128.39 - 100} = -0.197$$
$$\frac{5.58 - 20.88}{100 - 77.88} = -0.692$$

at the $u$ and $d$ nodes respectively.

– after two steps, the delta is

$$\frac{0 - 0}{145.48 - 113.31} = 0$$
$$\frac{0 - 11.75}{113.31 - 88.25} = -0.469$$
$$\frac{11.75 - 31.27}{88.25 - 68.73} = -1.$$

at the $uu$, $ud$ and $dd$ nodes respectively.

The arbitrage strategy is thus: buy the put, and at all nodes hold $-\Delta$ of the underlying stock. Note that $\Delta$ varies from node-to-node, so you must constantly update your hedge.

(iv) Working backwards through the tree:

– after 3 steps, the put value is 0, 0, $100 - 88.25 = 11.75$, and $100 - 68.73 = 31.27$ at the $uuu$, $uud$, and $udd$ and $ddd$ nodes respectively.

– after 2 steps, the put value is

$$\max\left\{100 - 128.39, e^{-.05\times.25}(0.5191 \times 0 + (1 - 0.5191) \times 0)\right\} = 0$$
$$\max\left\{100 - 100, e^{-.05\times.25}(0.5191 \times 0 + (1 - 0.5191) \times 11.75)\right\} = 5.58$$
$$\max\left\{100 - 77.88, e^{-.05\times.25}(0.5191 \times 11.75 + (1 - 0.5191) \times 31.27)\right\} = 22.12$$

at the $uu$, $ud$ and $dd$ nodes respectively.

– after 1 step, the put value is

$$\max\left\{100 - 113.31, e^{-.05\times.25}(0.5191 \times 0 + (1 - 0.5191) \times 5.58)\right\} = 2.65$$
$$\max\left\{100 - 88.25, e^{-.05\times.25}(0.5191 \times 5.58 + (1 - 0.5191) \times 22.12)\right\} = 13.37$$

at the u and d nodes respectively.

The price of the American put is thus

$$\max\{100 - 100, e - .05..25(0.5191.2.65 + (1 - 0.5191).13.37)\} = 7.71.$$

(v) From part (iv), you should exercise early only at the $dd$ node of the tree, i.e., only after six months, and only if the stock price declines twice.

**Question 2: Binomial Option Pricing (3/10)**

SOLUTION:

(i) It is straightforward to derive the option prices using the Excel file on Canvas:

Table 1: Excel Binomial Tree Model

| Number of periods $N$ | Period length $h$ | European Call | European Put |
|---|---|---|---|
| 5 | 0.2 | 9.46 | 7.10 |
| 10 | 0.1 | 8.88 | 6.52 |
| 50 | 0.02 | 9.04 | 6.68 |
| 100 | 0.01 | 9.06 | 6.69 |

(ii.a) It is straightforward to derive the option prices using the Excel file on Canvas:

Table 2: Excel Binomial Tree Model

| Number of periods $N$ | Period length $h$ | American Call | American Put |
|---|---|---|---|
| 5 | 0.2 | 9.46 | 7.306 |
| 10 | 0.1 | 8.88 | 6.831 |
| 50 | 0.02 | 9.04 | 6.918 |
| 100 | 0.01 | 9.06 | 6.929 |

(ii.b) The American put is more valuable than the European put.

Intuitively, if you keep the put alive, then you might end up with a large profit if the option moves deep in the money whereas your losses are bounded by zero if the put moves out of the money (you don't exercise if the option finishes out of the money). The loss of dividend payments is zero since the underlying stock does not pay dividends.

On the other hand, if you exercise the put early, then you immediately receive the strike for the asset and thus save interest.

Hence, early exercise is a tradeoff involving the time value of money on the strike price and the value of the insured short position in the underlying.

(iii) It is never optimal to exercise an American call on a non-dividend paying stock early because you have to pay the strike price for the asset early and you give up an insured long position in the underlying (again your losses are bounded by zero if the option finishes in the money in contrast to your long stock position after exercising.)

More formally, a put is worth at least zero and hence from put-call parity we have that

$$C_t \geq S_t - \mathrm{PV}_t(K) \geq S_t - K.$$

$S - K$ is the value of the American Call if you exercise early and hence the call is more valuable when kept alive.

### Question 3: Implied Volatility and Put-Call Parity (3/10)

SOLUTION:

(i) The Black-Scholes-Merton implied volatility is 36.00%.

(ii) It follows from put-call parity that the implied volatility of the call is equal to the implied volatility of the put to avoid arbitrage. So yes, the put is priced too high since its implied volatility is not the same as for the call. The arbitrage is to sell the put (its price is $P = 10.689$) and go long the call ($-C = -14.087$), short the stock ($S = 100$) and long $\mathrm{PV}(K) = -96.319$. The net cash flow from the replicating strategy is $-10.406$. The sale of the put provides income 10.689 and arbitrage profits are 0.283.

### Question 4: Greeks for Black-Scholes-Merton Model (Optional)

SOLUTION:

(i) The price of a European call option on a non-dividend paying stock is

$$C_t = S_t \mathcal{N}(d_1) - K e^{-r(T-t)} \mathcal{N}(d_2) \tag{4}$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

The Delta ($\Delta$) of the European call option is

$$\begin{aligned}\Delta_t = \frac{\partial C_t}{\partial S_t} &= \mathcal{N}(d_1) + S_t \phi(d_1) \frac{1/S_t}{\sigma\sqrt{T-t}} - K e^{-r(T-t)} \phi(d_2) \frac{1/S_t}{\sigma\sqrt{T-t}} \\ &= \mathcal{N}(d_1) + \frac{1}{\sigma\sqrt{T-t}} \left[\phi(d_1) - \frac{K}{S_t} e^{-r(T-t)} \phi(d_2)\right], \end{aligned} \tag{5}$$

where $\phi(x)$ is the probability density function (PDF) of standard normal distribution:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}. \tag{6}$$

Using the PDF function of standard normal, it follows that

$$\begin{aligned}
&\frac{K}{S_t} e^{r(T-t)} \phi(d_2) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left\{ -\ln\left(\frac{S_t}{K}\right) - r(T-t) - \frac{\left(d_1 - \sigma\sqrt{T-t}\right)^2}{2} \right\} \\
&= \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{d_1^2}{2} - \ln\left(\frac{S_t}{K}\right) - r(T-t) - \frac{-2d_1\sigma\sqrt{T-t} + \sigma^2(T-t)}{2} \right\} \\
&= \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{d_1^2}{2} - \ln\left(\frac{S_t}{K}\right) - r(T-t) - \frac{-2\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\sqrt{T-t}\right] + \sigma^2(T-t)}{2} \right\} \\
&= \frac{1}{\sqrt{2\pi}} \exp\left( -\frac{d_1^2}{2} \right) \\
&= \phi(d_1).
\end{aligned} \tag{7}$$

Thus, according to (5) and (7), it follows that

$$\Delta_t = \mathcal{N}(d_1).$$

The Gamma ($\Gamma_t$) of the European call option is

$$\Gamma_t = \frac{\partial \Delta_t}{\partial S_t} = \frac{\phi(d_1)}{S_t \sigma \sqrt{T-t}}. \tag{8}$$

The Delta ($\Delta_t$) and the Gamma ($\Gamma_t$) can be obtained by plugging the numbers.

(ii) The Vega of European call options is

$$\mathcal{V}_t = \frac{\partial C_t}{\partial \sigma} = S_t \phi(d_1) \frac{\partial d_1}{\partial \sigma} - K e^{-r(T-t)} \phi(d_2) \frac{\partial d_2}{\partial \sigma}. \tag{9}$$

First, we note that

$$\phi(d_2) = \phi(d_1) \frac{S_t}{K} e^{r(T-t)}. \tag{10}$$

Second, we also have

$$\frac{\partial d_1}{\partial \sigma} = \frac{\sigma^2(T-t)^{3/2} - \left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right](T-t)^{1/2}}{\sigma^2(T-t)} \tag{11}$$

$$\frac{\partial d_2}{\partial \sigma} = \frac{-\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right](T-t)^{1/2}}{\sigma^2(T-t)} \tag{12}$$

We plug (10), (11), and (12) into (8), and it leads to

$$\mathcal{V}_t = \phi(d_1) S_t \sqrt{T - t}. \tag{13}$$

The Vega ($\mathcal{V}_t$) can be obtained by plugging the numbers.