1. Production Function

Production function

Two input production function

$$q = f(k, l)$$

Marginal product

$$MP_k = \frac{\partial f}{\partial k}, \quad MP_l = \frac{\partial f}{\partial l}$$

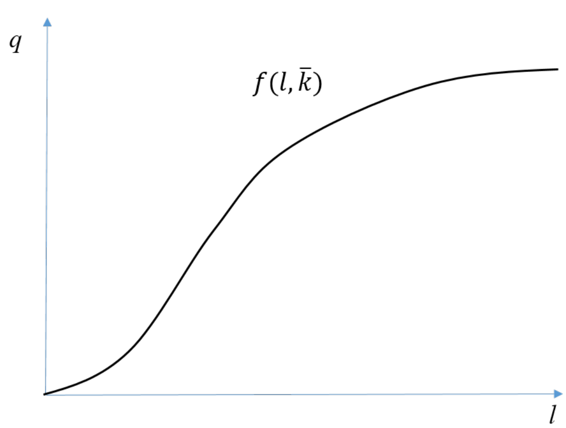
Diminishing marignal product means MP_k decreases in k (MP_l decreases in l)

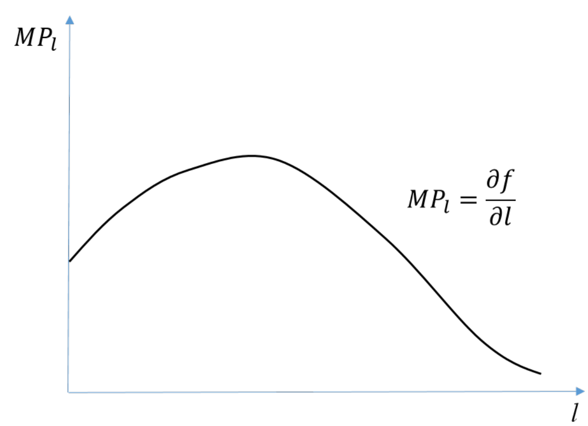
$$\frac{\partial MP_k}{\partial k} < 0 \Leftrightarrow \frac{\partial^2 f}{\partial k^2} = f_{kk} < 0$$

$$\frac{\partial MP_l}{\partial l} < 0 \Leftrightarrow \frac{\partial^2 f}{\partial l^2} = f_{ll} < 0$$

If k and l are discrete

$$MP_k = f(k, l) - f(k-1, l)$$





Example 9.1

$$f(k, l) = 600k^2l^2 - k^3l^3$$

$$MP_k = \frac{\partial f}{\partial k} = 600l^2 \times 2k - l^3 \times 3k^2$$

$$MP_l = \frac{\partial f}{\partial l} = 600k^2 \times 2l - k^3 \times 3l^2$$

Example, Cobb-douglas production function

$$f(k,l) = Ak^\alpha l^\beta$$

What parameter value (α, β) can insure diminishing marginal product?

$$MP_k = \frac{\partial f}{\partial k} = Al^\beta \times \alpha k^{\alpha-1} = A\alpha k^{\alpha-1} l^\beta \geq 0$$

$$MP_l = \frac{\partial f}{\partial l} = Ak^\alpha \times \beta l^{\beta-1} = A\beta k^\alpha l^{\beta-1} \geq 0$$

$$\frac{\partial^2 f}{\partial k^2} = \frac{\partial MP_k}{\partial k} = Al^\beta \alpha(\alpha-1)k^{\alpha-2} < 0 \Leftrightarrow \alpha \in (0,1)$$

(If $\alpha > 1$, then the marginal product will be increasing. This is not very realistic.)

$$\frac{\partial^2 f}{\partial l^2} = \frac{\partial MP_l}{\partial l} = Ak^\alpha \beta(\beta-1)l^{\beta-2} < 0 \Leftrightarrow \beta \in (0,1)$$

RTS and σ

Implicit function theorem

$$\bar{q} = f(k,l)$$

This identity (isoquant) implies an implicit function of $k(l)$. The slopt of this impicit function can be derived in the following way ($\frac{dk}{dl}$).

Take total differentiation of the function

$$0 = \frac{\partial f}{\partial k} dk + \frac{\partial f}{\partial l} dl$$

$$\frac{\partial f}{\partial k} dk = -\frac{\partial f}{\partial l} dl$$

$$\frac{dk}{dl} = -\frac{\frac{\partial f}{\partial l}}{\frac{\partial f}{\partial k}} < 0$$

$$RTS = -\frac{dk}{dl} = \frac{\frac{\partial f}{\partial l}}{\frac{\partial f}{\partial k}} = \frac{MP_l}{MP_k} > 0$$

Example, $f(k,l) = Ak^\alpha l^\beta$

$$RTS = \frac{MP_l}{MP_k} = \frac{A\beta k^\alpha l^{\beta-1}}{A\alpha k^{\alpha-1} l^\beta} = \frac{\beta k}{\alpha l}$$

Suppose $\alpha = \beta = 1$, $RTS = k/l$. At the point $k = 3$, $l = 2$, $RTS = 3/2$, $\frac{dk}{dl} = -\frac{3}{2}$.

Elasticity between X and Y

$$e_{X,Y} = \frac{\text{\% change of } X}{\text{\% change of } Y} = \frac{\frac{dX}{X}}{\frac{dY}{Y}}$$

Elasticity of substitution

$$\sigma = \frac{\text{\% change of } \frac{k}{l}}{\text{\% change of } RTS}$$

Example, $f(k,l) = Ak^{\alpha}l^{\beta}$. Let $z = \frac{k}{l}$

$$\begin{aligned}
\sigma &= \frac{\text{\% change of } \frac{k}{l}}{\text{\% change of } RTS} = \frac{\frac{d\frac{k}{l}}{\frac{k}{l}}}{\frac{dRTS}{RTS}} = \frac{1}{\frac{dRTS}{d\frac{k}{l}}}\frac{RTS}{\frac{k}{l}} \\
&= \frac{1}{\frac{d(\frac{\beta k}{\alpha l})}{d\frac{k}{l}}}\frac{\frac{\beta k}{\alpha l}}{\frac{k}{l}} = \frac{1}{\frac{d(\frac{\beta}{\alpha}z)}{dz}}\frac{\frac{\beta}{\alpha}z}{z} = \frac{1}{\frac{\beta}{\alpha}}\frac{\frac{\beta}{\alpha}z}{z} = 1
\end{aligned}$$

CES production function

CES refers to constant elasticity of substitution

$$f(k,l) = (k^{\rho} + l^{\rho})^{\frac{\gamma}{\rho}}$$

Marginal product

$$\begin{aligned}
MP_k = f_k &= \frac{\gamma}{\rho}(k^{\rho} + l^{\rho})^{\frac{\gamma}{\rho}-1}\rho k^{\rho-1} = \gamma(k^{\rho} + l^{\rho})^{\frac{\gamma}{\rho}-1}k^{\rho-1} \\
MP_l = f_l &= \frac{\gamma}{\rho}(k^{\rho} + l^{\rho})^{\frac{\gamma}{\rho}-1}\rho l^{\rho-1} = \gamma(k^{\rho} + l^{\rho})^{\frac{\gamma}{\rho}-1}l^{\rho-1}
\end{aligned}$$

RTS

$$RTS = \frac{MP_l}{MP_k} = \frac{\gamma(k^{\rho} + l^{\rho})^{\frac{\gamma}{\rho}-1}l^{\rho-1}}{\gamma(k^{\rho} + l^{\rho})^{\frac{\gamma}{\rho}-1}k^{\rho-1}} = \left(\frac{l}{k}\right)^{\rho-1} = \left(\frac{k}{l}\right)^{1-\rho}$$

Elasticity of substitution (let $\frac{k}{l} = z$)

$$\begin{aligned}
\sigma &= \frac{1}{\frac{dRTS}{d\frac{k}{l}}}\frac{RTS}{\frac{k}{l}} = \frac{1}{\frac{d\left(\frac{k}{l}\right)^{1-\rho}}{d\frac{k}{l}}}\frac{\left(\frac{k}{l}\right)^{1-\rho}}{\frac{k}{l}} \\
&= \frac{1}{\frac{d(z^{1-\rho})}{dz}}\frac{z^{1-\rho}}{z} = \frac{1}{(1-\rho)z^{-\rho}}\frac{z^{1-\rho}}{z} = \frac{1}{1-\rho}
\end{aligned}$$

Example 9.3

$$q = f(k,l) = k + l + 2\sqrt{kl}$$

Find its RTS and σ

$$\begin{aligned} q &= (k^{\frac{1}{2}})^2 + 2k^{\frac{1}{2}}l^{\frac{1}{2}} + (l^{\frac{1}{2}})^2 \\ &= (k^{\frac{1}{2}} + l^{\frac{1}{2}})^2 \end{aligned}$$

This is a CES production function where $\rho = \frac{1}{2}$, $\frac{\gamma}{\rho} = 2$, so $\gamma = 1$.

$$RTS = \left(\frac{k}{l}\right)^{\frac{1}{2}}$$

$$\sigma = \frac{1}{1 - \frac{1}{2}} = 2$$

Return to scale

For $t > 1$

Increasing return to scale (economy of scale) $f(tk,tl) > tf(k,l)$;

Constant return to scale $f(tk,tl) = tf(k,l)$;

Decreasing return to scale $f(tk,tl) < tf(k,l)$.

Diminishing marginal product is about increasing one input, while all other inputs remain constant.

Return to scale is about increasing all input by the same proportion.

Example 1, $f(k,l) = Ak^{\alpha}l^{\beta}$

$$f(tk,tl) = A(tk)^{\alpha}(tl)^{\beta} = At^{\alpha+\beta}k^{\alpha}l^{\beta} = t^{\alpha+\beta}f(k,l)$$

If $\alpha + \beta > 1$,IRS; if $\alpha + \beta = 1$, CRS; if $\alpha + \beta < 1$, DRS.

Example 2, $f(k,l) = (k^{\rho} + l^{\rho})^{\frac{\gamma}{\rho}}$

$$f(tk,tl) = ((tk)^{\rho} + (tl)^{\rho})^{\frac{\gamma}{\rho}} = (t^{\rho}[k^{\rho} + l^{\rho}])^{\frac{\gamma}{\rho}} = t^{\gamma}(k^{\rho} + l^{\rho})^{\frac{\gamma}{\rho}}$$

If $\gamma > 1$,IRS; if $\gamma = 1$, CRS; if $\gamma < 1$, DRS.