8. Externalities and Public Goods

Externality

Solve for the Pareto efficient allocation (given $\bar{u}_2$), the social planner's problem is

$$\max_{x_1,x_2,y_1,y_2} u_1(x_1,y_1,x_2), \text{ s.t. } \begin{cases} u_2(x_2,y_2) \geq \bar{u}_2 \\ x_1+y_1+x_2+y_2 = w_1+w_2 \end{cases}$$

Lagrangian approach (with equality constraint)

$$\mathscr{L} = u_1(x_1,y_1,x_2) + \lambda(u_2(x_2,y_2)-\bar{u}_2) + \mu(w_1+w_2-x_1-y_1-x_2-y_2)$$

The solution to this constraint maximization problem is determined by the following FOCs

$$\begin{cases} \frac{\partial \mathscr{L}}{\partial x_1} = 0 & \frac{\partial u_1}{\partial x_1} - \mu = 0 \\ \frac{\partial \mathscr{L}}{\partial x_2} = 0 & \frac{\partial u_1}{\partial x_2} + \lambda \frac{\partial u_2}{\partial x_2} - \mu = 0 \\ \frac{\partial \mathscr{L}}{\partial y_1} = 0 & \frac{\partial u_1}{\partial y_1} - \mu = 0 \\ \frac{\partial \mathscr{L}}{\partial y_2} = 0 & \lambda \frac{\partial u_2}{\partial y_2} - \mu = 0 \\ \frac{\partial \mathscr{L}}{\partial \lambda} = 0 & u_2(x_2,y_2) - \bar{u}_2 = 0 \\ \frac{\partial \mathscr{L}}{\partial \mu} = 0 & w_1+w_2-x_1-y_1-x_2-y_2 = 0 \end{cases}$$

The first and third condition together yield

$$MRS_1 = \frac{\frac{\partial u_1}{\partial x_1}}{\frac{\partial u_1}{\partial y_1}} = 1,$$

which is the same as the competitive allocation.

First, obtain λ from condition 1 and 4

$$\frac{\partial u_1}{\partial x_1} = \mu$$

$$\lambda \frac{\partial u_2}{\partial y_2} = \mu \Rightarrow \lambda \frac{\partial u_2}{\partial y_2} = \frac{\partial u_1}{\partial x_1} \Rightarrow \lambda = \frac{\frac{\partial u_1}{\partial x_1}}{\frac{\partial u_2}{\partial y_2}}$$

The second yields

$$\frac{\partial u_1}{\partial x_2} + \lambda \frac{\partial u_2}{\partial x_2} = \mu$$

$$\frac{\partial u_1}{\partial x_2} + \frac{\frac{\partial u_1}{\partial x_1}}{\frac{\partial u_2}{\partial y_2}} \frac{\partial u_2}{\partial x_2} = \frac{\partial u_1}{\partial x_1}$$

Divide both sides by $\frac{\partial u_1}{\partial x_1}$

$$\frac{\frac{\partial u_1}{\partial x_2}}{\frac{\partial u_1}{\partial x_1}} + \frac{\frac{\partial u_2}{\partial x_2}}{\frac{\partial u_2}{\partial y_2}} = 1$$

$$MRS_2^{PO} = \frac{\frac{\partial u_2}{\partial x_2}}{\frac{\partial u_2}{\partial y_2}} = 1 - \overbrace{\frac{\frac{\partial u_1}{\partial x_2}}{\frac{\partial u_1}{\partial x_1}}}^{<0} > 1$$

Compared to the case in competitive allocation (NE) with

$$MRS_2^{NE} = \frac{\frac{\partial u_2}{\partial x_2}}{\frac{\partial u_2}{\partial y_2}} = 1$$

Therefore, $MRS_2^{PO} > MRS_2^{NE}$. Because MRS_2 is a decreasing function of x_2. So $MRS_2^{PO} > MRS_2^{NE}$ implies $x_2^{PO} < x_2^{NE}$.

Without a social planner, consumer 2 will choose x_2 that is higher than the efficient level.

Example, fishery versus steel firm

(1) Competitive equilibrium/allocation (NE)

$$\max_{s,x} \pi_s(s,x) = p_s s - c_s(s,x)$$

$$\frac{\partial \pi_s}{\partial s} = p_s - \frac{\partial c_s}{\partial s} = 0 \Rightarrow \frac{\partial c_s}{\partial s} = p_s$$

$$\frac{\partial \pi_s}{\partial x} = -\frac{\partial c_s}{\partial x} \geq 0$$

Because pollution saves cost, so $\frac{\partial c_s}{\partial x} \leq 0$. Polluting more increases profit, the firm will pollute as much as possible as long as it saves cost (according to the regulation).

If there is regulation that restrict $x \leq \bar{x}$, then firm will pollute up to $\bar{x}$ level.

$$\max_{f} \pi_f(f,x) = p_f f - c_f(f,x)$$

$$\frac{\partial \pi_f}{\partial f} = p_f - \frac{\partial c_f}{\partial f} = 0 \Rightarrow \frac{\partial c_f}{\partial f} = p_f$$

The competitive equilibrium is determined by three FOCs

$$\frac{\partial c_s}{\partial s} = p_s, \quad \frac{\partial c_f}{\partial f} = p_f, \quad \frac{\partial c_s}{\partial x} = 0$$

(2) Efficient allocation (solution of a benevolent social planner)

Find efficient allocation by think about the problem after merger, which will internalize all externalities.

$$\max_{s,x,f} \Pi(s,f,x) = \pi_s(s,x) + \pi_f(f,x)$$
$$= p_s s - c_s(s,x) + p_f f - c_f(f,x)$$

$$\frac{\partial \Pi_s}{\partial s} = p_s - \frac{\partial c_s}{\partial s} = 0 \Rightarrow \frac{\partial c_s}{\partial s} = p_s$$
$$\frac{\partial \Pi_f}{\partial f} = p_f - \frac{\partial c_f}{\partial f} = 0 \Rightarrow \frac{\partial c_f}{\partial f} = p_f$$
$$\frac{\partial \Pi}{\partial x} = -\frac{\partial c_s}{\partial x} - \frac{\partial c_f}{\partial x} = 0 \Rightarrow \frac{\partial c_f}{\partial x} = -\frac{\partial c_s}{\partial x} > 0$$

The marginal damange caused by pollution to the fishery needs to be equal to the marginal cost saving benefit to the steel firm.

Because the optimal allocation require $\frac{\partial c_s}{\partial x} < 0$, and $\frac{\partial c_s}{\partial x}$ decreases in x.

Therefore, the efficient allocation implies a lower x than the competitive allocation.

(3) Get the optimal level of pollution by Pigouvian tax

Charge a quantity tax for polution. The steel firm's problem become:

$$\max_{s,x} \pi_s(s,x) = p_s s - c_s(s,x) - tx$$

The FOC w.r.t. x becomes

$$\frac{\partial \pi_s}{\partial x} = -\frac{\partial c_s}{\partial x} - t = 0$$
$$-\frac{\partial c_s}{\partial x} = t$$

(Marginal benefit from cost saving is equal to the tax.)

If the society has only the fishery and the steel firm, the optimal level of tax should be the marginal damage of pollution to the fishery

$$-\frac{\partial c_s}{\partial x} = t = \frac{\partial c_f}{\partial x}.$$

This FOC is the same as the one we find in the merger problem ($\frac{\partial c_f}{\partial x} = -\frac{\partial c_s}{\partial x}$).

(4) Get the optimal level of pollution by assigning property right.

If the fishery owns the river, the steel firm needs to pay the fishery to pollute.

$$\max_{s,x} \pi_s(s,x) = p_s s - c_s(s,x) - px$$

The FOC w.r.t. x becomes

$$\frac{\partial \pi_s}{\partial x} = -\frac{\partial c_s}{\partial x} - p = 0 \Rightarrow -\frac{\partial c_s}{\partial x} = p$$

$$\max_{f} \pi_f(f,x) = p_f f - c_f(f,x) + px$$

$$\frac{\partial \pi_f}{\partial x} = -\frac{\partial c_f}{\partial x} + p = 0 \Rightarrow \frac{\partial c_f}{\partial x} = p$$

$$\Rightarrow -\frac{\partial c_s}{\partial x} = p = \frac{\partial c_f}{\partial x}$$

The equilibrium price will induce the efficient level of pollution.

Public goods

(1) Competitive allocation (Non-cooperative/Nash equilibrium)

$$x_1^{NE}, x_2^{NE}, y_1^{NE}, y_2^{NE}$$

The public good is $X = x_1 + x_2$.

Each individual decides how much money to keep as private spending, while the rest is contribution to the public good.

$$\max_{x_1,y_1} u_1(X,y_1) = u_1(x_1+x_2,y_1), \quad \text{s.t. } p_X x_1 + y_1 = w_1$$

p_X is the price of public good. (p_y is normalized to 1.)

$$MRS_1 = \frac{\frac{\partial u_1}{\partial X}}{\frac{\partial u_1}{\partial y_1}} = \frac{\frac{\partial u_1}{\partial x_1}}{\frac{\partial u_1}{\partial y_1}} = \frac{p_X}{p_y} = p_X$$

e.g.

$$u_i(X,y_i) = X^{\frac{1}{2}} y_i^{\frac{1}{2}} = (x_1+x_2)^{\frac{1}{2}} y_i^{\frac{1}{2}}$$

Set $p_X = 10$, $w_1 = w_2 = 300$

$$MRS_1 = \frac{\frac{1}{2} X^{-\frac{1}{2}} y_1^{\frac{1}{2}}}{X^{\frac{1}{2}} \frac{1}{2} y_1^{-\frac{1}{2}}} = \frac{y_1}{X} = \frac{y_1}{x_1+x_2} = p_X = 10$$

$$\Rightarrow y_1 = 10x_1 + 10x_2$$

Together with the budget constraint of individual 1

$$p_X x_1 + y_1 = 300$$

$$10x_1 + 10x_1 + 10x_2 = 300$$

$$2x_1 + x_2 = 30$$

$$x_1 = BR_1(x_2) = 15 - \frac{1}{2} x_2.$$

For individual 2

$$\max_{x_2,y_2} u_2(X,y_2) = u_2(x_1+x_2,y_2), \quad \text{s.t. } p_X x_2 + y_2 = w_2$$

$$MRS_2 = \frac{\frac{\partial u_2}{\partial X}}{\frac{\partial u_2}{\partial y_2}} = \frac{y_2}{x_1 + x_2} = 10$$
$$y_2 = 10x_1 + 10x_2$$

Plug in budget constraint

$$10x_2 + 10x_1 + 10x_2 = 300$$
$$x_1 + 2x_2 = 30$$
$$x_2 = BR_2(x_1) = 15 - \frac{1}{2}x_1.$$

The equilibrium is determined by mutual best responses

$$\begin{cases} x_1 = BR_1(x_2) \\ x_2 = BR_2(x_1) \end{cases}$$

$$x_2 = 15 - \frac{1}{2}(15 - \frac{1}{2}x_2)$$
$$= 15 - \frac{15}{2} + \frac{1}{4}x_2$$

$$\frac{3}{4}x_2 = \frac{15}{2}$$
$$x_2^{NE} = 10$$

$$x_1^{NE} = 10$$
$$X^{NE} = x_1^{NE} + x_2^{NE} = 20$$

Competitive allocation

$$x_1^{NE} = 10, x_2^{NE} = 10, y_1^{NE} = 200, y_2^{NE} = 200$$

Utility levels, $u_1(X, y_1)$

$$u_1^{NE} = u_1(20, 200) = 20^{\frac{1}{2}} 200^{\frac{1}{2}} = 20 \times \sqrt{10} = u_2^{NE}$$

(2) Pareto efficient/optimal allocation
Suppose there is a benevolent social planner

$$\max_{y_1, y_2, X} u_1(X, y_1)$$
$$\text{s.t.} u_2(X, y_2) \geq \bar{u}_2$$
$$\text{(resource constraint)} p_X X + y_1 + y_2 = w_1 + w_2$$

$$\mathscr{L} = u_1(X, y_1) + \lambda(u_2(X, y_2) - \bar{u}_2) + \mu(w_1 + w_2 - p_X X - y_1 - y_2)$$

FOCs

$$\begin{cases} \frac{\partial \mathscr{L}}{\partial y_1} = \frac{\partial u_1}{\partial y_1} - \mu = 0 \\ \frac{\partial \mathscr{L}}{\partial y_2} = \lambda \frac{\partial u_2}{\partial y_2} - \mu = 0 \\ \frac{\partial \mathscr{L}}{\partial X} = \frac{\partial u_1}{\partial X} + \lambda \frac{\partial u_2}{\partial X} - p_X \mu = 0 \end{cases}$$

By the first condition

$$\mu = \frac{\partial u_1}{\partial y_1}$$

By the second condition

$$\lambda \frac{\partial u_2}{\partial y_2} - \mu = \lambda \frac{\partial u_2}{\partial y_2} - \frac{\partial u_1}{\partial y_1} = 0$$

$$\lambda = \frac{\frac{\partial u_1}{\partial y_1}}{\frac{\partial u_2}{\partial y_2}}$$

Plug these two conditions into the third FOC

$$\frac{\partial u_1}{\partial X} + \lambda \frac{\partial u_2}{\partial X} - p_X \mu = 0$$

$$\frac{\partial u_1}{\partial X} + \frac{\frac{\partial u_1}{\partial y_1}}{\frac{\partial u_2}{\partial y_2}} \frac{\partial u_2}{\partial X} - p_X \frac{\partial u_1}{\partial y_1} = 0$$

Divide both sides by $\frac{\partial u_1}{\partial y_1}$

$$\frac{\frac{\partial u_1}{\partial X}}{\frac{\partial u_1}{\partial y_1}} + \frac{1}{\frac{\partial u_2}{\partial y_2}} \frac{\partial u_2}{\partial X} - p_X 1 = 0$$

$$MRS_1 + MRS_2 = \frac{\frac{\partial u_1}{\partial X}}{\frac{\partial u_1}{\partial y_1}} + \frac{\frac{\partial u_2}{\partial X}}{\frac{\partial u_2}{\partial y_2}} = p_X$$

$$MRS_1^{PO} = p_X - MRS_2^{PO}$$

Note that this is different from the condition that determine competitive allocation

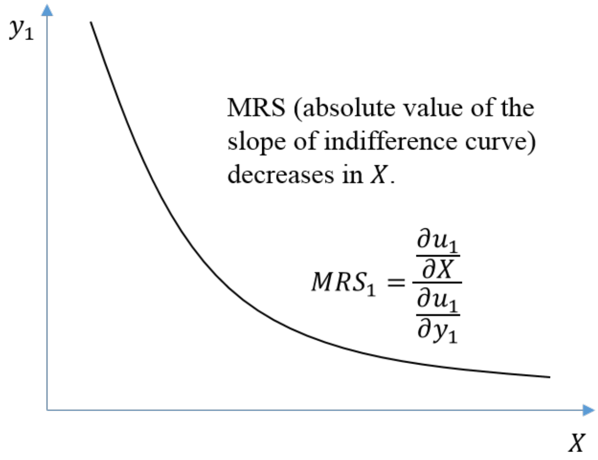
$$MRS_1^{NE} = p_X, \;\; MRS_2^{NE} = p_X$$

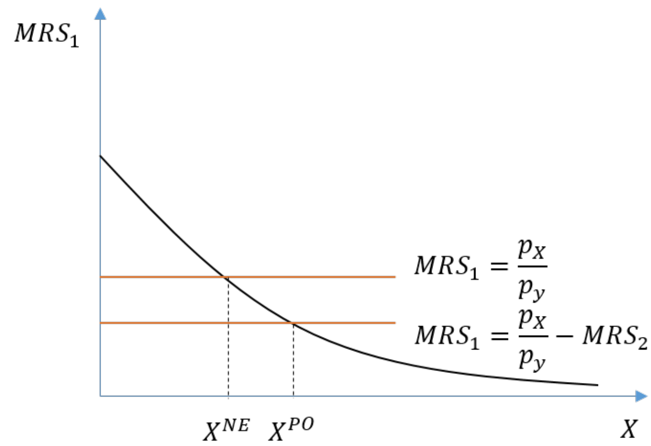
Here, we use the definition that $MRS_1 = \frac{\frac{\partial u_1}{\partial X}}{\frac{\partial u_1}{\partial y_1}} > 0$, $MRS_2 = \frac{\frac{\partial u_2}{\partial X}}{\frac{\partial u_2}{\partial y_2}} > 0$.

Note that, MRS_1 and MRS_2 are both decreasing function of X. As X increases, the marginal utility $\frac{\partial u_i}{\partial X}$ decreases. Therefore,

$$MRS_1^{PO} = p_X - \overbrace{MRS_2^{PO}}^{>0} < MRS_1^{NE},$$

which implies $X^{PO} > X^{NE}$.





In the example, solve for the allocation

$$MRS_1 + MRS_2 = p_X = 10$$

$$\frac{y_1}{x_1+x_2} + \frac{y_2}{x_1+x_2} = \frac{y_1}{X} + \frac{y_2}{X} = \frac{y_1+y_2}{X} = 10$$

$$y_1 + y_2 = 10X$$

Together with the resource constraint

$$y_1 + y_2 + p_X X = w_1 + w_2 = 600$$

$$10X + 10X = 600$$

$$X^{PO} = 30 > X^{NE} = 20.$$

The Pareto optimal allocation depends on how two individual share the public good spending. Suppose they share the spending evenly,

$$X^{PO} = 30, x_1^{PO} = 15, x_2^{PO} = 15, y_1^{PO} = 150, y_2^{PO} = 150$$

Under Pareto optimal allocation, individual utility is

$$u_1^{PO} = u_1(30,150) = 30^{\frac{1}{2}} \times 150^{\frac{1}{2}} = 30 \times \sqrt{5} = u_2^{PO} = 67.08$$

Compared with NE utillity levels $u_1^{NE} = u_2^{NE} = 20 \times \sqrt{10} = 63.25$, both individuals are better off.

However, under an unequal division of public spending, some individual can be worse off under Pareto optimal allocation.

$$X^{PO} = 30, x_1^{PO} = 5, x_2^{PO} = 25, y_1^{PO} = 250, y_2^{PO} = 50$$

$$u_2^{PO} = u_2(30,50) = 30^{\frac{1}{2}} 50^{\frac{1}{2}} = 38.73 < u_2^{NE}$$

Pareto optimal allocation is determined by

$$\begin{cases} MRS_1 + MRS_2 = \frac{p_X}{p_y} \\ p_X X + p_y y_1 + p_y y_2 = w_1 + w_2 \\ u_2(X, y_2) = \bar{u}_2 \end{cases}$$

The last condition determines allocation across individuals.

(3) Social welfare function

Social welfare function specify how to aggregate individual utility/welfare. It involves value judgment.

Once there is a social welfare function, the social planner can solve

$$\max_{y_1, y_2, X} W(u_1(X, y_1), u_2(X, y_2))$$

$$\text{s.t.} p_X X + p_y y_1 + p_y y_2 = w_1 + w_2$$

For example,

$$W(u_1, u_2) = u_1 + u_2$$

This social welfare function maximize the summation of payoffs. It means the allocation that induces $(10, 1)$ is better than an allocation $(5, 5)$.

$$W(u_1, u_2) = \min\{u_1, u_2\}$$

Then, the allocation that induces $(5, 5)$ is better than $(10, 1)$. This social welfare function try to maximize the individual that is least well off. It emphasize equality.

$$W(u_1, u_2) = \alpha(u_1 + u_2) + (1 - \alpha) \min\{u_1, u_2\}$$

(4) Production of public goods

Public goods can be produced by the following "production function"

$$G = f(x_1 + x_2)$$

$$f'() > 0, f''() < 0$$

Voluntary contribution, decentralized/noncooperative outcome

$$\max_{x_1} u_1(G, y_1) = u_1(f(x_1 + x_2), w_1 - x_1)$$

where $y_1 = w_1 - x_1$ is the private spending of individual 1.

$$\max_{x_2} u_2(G, y_2) = u_2(f(x_1 + x_2), w_2 - x_2)$$

FOCs for unconstraint max problem

$$\frac{du_1}{dx_1} = \frac{\partial u_1}{\partial G} f'(x_1 + x_2) + \frac{\partial u_1}{\partial y_1}(-1) = 0$$

$$\frac{du_2}{dx_2} = \frac{\partial u_2}{\partial G} f'(x_1+x_2) + \frac{\partial u_2}{\partial y_2}(-1) = 0$$

$$\Rightarrow \frac{\partial u_1}{\partial G} f'(x_1+x_2) = \frac{\partial u_1}{\partial y_1} \Rightarrow \frac{\frac{\partial u_1}{\partial G}}{\frac{\partial u_1}{\partial y_1}} = MRS_1 = \frac{1}{f'(x_1+x_2)}$$

$$\frac{\frac{\partial u_2}{\partial G}}{\frac{\partial u_2}{\partial y_2}} = MRS_2 = \frac{1}{f'(x_1+x_2)}$$

MRS indicates the substitution between private spending and public spending.

Let us now think about the social optimal level of public goods

$$\max_{x_1,x_2} u_1(f(x_1+x_2), w_1 - x_1)$$
$$\text{s.t. } u_2(f(x_1+x_2), w_2 - x_2) \geq \bar{u}_2$$

For this constraint max problem, use Lagrangian

$$\mathscr{L} = u_1(f(x_1+x_2), w_1 - x_1) + \lambda\left[u_2(f(x_1+x_2), w_2 - x_2) - \bar{u}_2\right]$$

$$\frac{\partial \mathscr{L}}{\partial x_1} = \frac{\partial u_1}{\partial G} f'(x_1+x_2) + \frac{\partial u_1}{\partial y_1}(-1) + \lambda \frac{\partial u_2}{\partial G} f'(x_1+x_2) = 0$$

$$\Rightarrow \frac{\partial u_1}{\partial G} f'(x_1+x_2) = \frac{\partial u_1}{\partial y_1} - \lambda \frac{\partial u_2}{\partial G} f'(x_1+x_2)$$

$$\frac{\partial \mathscr{L}}{\partial x_2} = \frac{\partial u_1}{\partial G} f'(x_1+x_2) + \lambda \frac{\partial u_2}{\partial G} f'(x_1+x_2) + \lambda \frac{\partial u_2}{\partial y_2}(-1) = 0$$

$$\Rightarrow \frac{\partial u_1}{\partial G} f'(x_1+x_2) = \lambda \frac{\partial u_2}{\partial y_2} - \lambda \frac{\partial u_2}{\partial G} f'(x_1+x_2)$$

Therefore

$$\frac{\partial u_1}{\partial y_1} - \lambda \frac{\partial u_2}{\partial G} f'(x_1+x_2) = \lambda \frac{\partial u_2}{\partial y_2} - \lambda \frac{\partial u_2}{\partial G} f'(x_1+x_2)$$

$$\frac{\partial u_1}{\partial y_1} = \lambda \frac{\partial u_2}{\partial y_2}$$

$$\lambda = \frac{\frac{\partial u_1}{\partial y_1}}{\frac{\partial u_2}{\partial y_2}}$$

(Substitution between private spending of two individuals.) Plug into FOCs

$$\frac{\partial u_1}{\partial G} f'(x_1+x_2) = \frac{\partial u_1}{\partial y_1} - \frac{\frac{\partial u_1}{\partial y_1}}{\frac{\partial u_2}{\partial y_2}} \frac{\partial u_2}{\partial G} f'(x_1+x_2)$$

$$\frac{\partial u_1}{\partial G} f'(x_1+x_2) + \frac{\frac{\partial u_1}{\partial y_1}}{\frac{\partial u_2}{\partial y_2}} \frac{\partial u_2}{\partial G} f'(x_1+x_2) = \frac{\partial u_1}{\partial y_1}$$

Divide both sides by $\frac{\partial u_1}{\partial y_1} \times f'(x_1+x_2)$

$$\frac{\frac{\partial u_1}{\partial G}}{\frac{\partial u_1}{\partial y_1}} + \frac{\frac{\partial u_2}{\partial G}}{\frac{\partial u_2}{\partial y_2}} = MRS_1 + MRS_2 = \frac{1}{f'(x_1+x_2)}$$

This condition determines the efficient level of public good spending.

Note that this is different from the non-cooperative level determined by

$$MRS_1 = \frac{1}{f'(x_1+x_2)} \text{ and } MRS_2 = \frac{1}{f'(x_1+x_2)}$$

Technical skills:

(i) Solve constraint max problem by Lagrangian.

(ii) Find decentralized/noncooperative allocation/equilibrium. Find efficient/centralized allocation.

(5)* Lindahl pricing (a mechanism of sharing public good cost proportionally that induce the efficient allocation)

The target public good level is $G = f(x_1+x_2)$, total spending is $X = x_1+x_2 = f^{-1}(G)$.

Individual 1 is asked to pay α_1 proportion of the cost $x_1 = \alpha_1 X = \alpha_1 f^{-1}(G)$.

Individual 2 is asked to pay α_2 proportion of the cost $x_2 = \alpha_2 X = \alpha_2 f^{-1}(G)$.

Let individual 1 to report what is the optimal public good level G. The more public spending, the more tax need to be paid.

$$\max_G u_1(G, w_1 - \alpha_1 f^{-1}(G))$$

FOC

$$\frac{\partial u_1}{\partial G} + \frac{\partial u_1}{\partial y_1}\left(-\frac{\alpha_1}{f'(G)}\right) = 0 \Rightarrow \frac{\frac{\partial u_1}{\partial G}}{\frac{\partial u_1}{\partial y_1}} = MRS_1 = \frac{\alpha_1}{f'(G)}$$

For individual 2

$$\max_G u_2(G, w_2 - \alpha_2 f^{-1}(G))$$

$$\frac{\frac{\partial u_2}{\partial G}}{\frac{\partial u_2}{\partial y_2}} = MRS_2 = \frac{\alpha_2}{f'(G)}$$

Note that when $\alpha_1 + \alpha_2 = 1$

$$MRS_1 + MRS_2 = \frac{\alpha_1}{f'(G)} + \frac{\alpha_2}{f'(G)} = \frac{\alpha_1+\alpha_2}{f'(G)} = \frac{1}{f'(G)}.$$

This implied the efficient level of public goods provision.

This mechanism can induce efficient allocation only when people report their public good demand truthfully.