Exercise 6 solution

2. FOC: $\begin{cases} f_x = -e^{-x}\left(2x^2+y^3-3y\right)+e^{-x}\left(4x\right) = e^{-x}\left(-2x^2+4x-y^3+3y\right)=0 \\ f_y = e^{-x}\left(3y^2-3\right)=0 \implies y=\pm 1 \end{cases}$

For $y=1, y^3-3y=-2, -2x^2+2+4x=0, x^2-2x-1=0, x=\frac{2\pm\sqrt{8}}{2}=1\pm\sqrt{2}$

For $y=-1, y^3-3y=2, -2x^2-2+4x=0, x^2-2x+1=0, x=1$

Stationary points:

$(1,-1), \left(1+\sqrt{2},1\right), \left(1-\sqrt{2},1\right)$

$f_{xx}=-e^{-x}\left(-2x^2+4x-y^3+3y\right)+e^{-x}\left(-4x+4\right)$

$=e^{-x}\left(2x^2-8x+y^3-3y+4\right)$

$f_{xy}=-e^{-x}\left(3y^2-3\right); f_{yy}=6ye^{-x}$

at $(1,-1), f_{xx}=e^{-1}\left(-2+2\right)=0; f_{xy}=0, f_{yy}=-6e^{-1},\ f''=\begin{pmatrix} 0 & 0 \\ 0 & -6e^{-1} \end{pmatrix}\leq 0$

when $y=1, f''=2e^{-x}\begin{pmatrix} x^2-4x+1 & 0 \\ 0 & 3 \end{pmatrix}$

$\left(1+\sqrt{2}\right)^2-4\left(1+\sqrt{2}\right)+1=-2\sqrt{2}$

$\left(1-\sqrt{2}\right)^2-4\left(1-\sqrt{2}\right)+1=2\sqrt{2}$

$\left(1+\sqrt{2},1\right)$ is saddle point since f'' indefinite

$\left(1-\sqrt{2},1\right)$ is local minimum since $f''>0$

Summary:

$\left(1+\sqrt{2},1\right)$	saddle point
$\left(1-\sqrt{2},1\right)$	local minimum
$(1,-1)$	can not conclude for sure whether it is local minimum or maximum.