Homework 1 Solution

Choose the best answer

1. If more and more labor is employed while keeping all other inputs constant, the marginal physical productivity of labor will eventually
   a. increase.
   b. decrease.
   c. remain constant.
   d. cannot tell from the information provided.

2. For a fixed proportion production function, at the vertex of any of the (L shaped) isoquants the marginal productivity of either input is
   a. constant
   b. zero.
   c. negative.
   d. a value that cannot be determined.

3. Which production technology is the most flexible in replacing one input by another input in producing output q?
   a. Cobb-Douglas.
   b. Fixed-proportion.
   c. Linear.
   d. It depends on the level of q.

Analytical questions

1. A car production company's production function is $f(k, l) = \alpha k^{0.5} l^{0.5}$ where k represents units of capital, l represents units of labor and $\alpha > 0$ represents technology.

a. Calculate the marginal product of capital and marginal product of labor.

$$\begin{aligned} MP_k &= f_k = 0.5\alpha k^{-0.5} l^{0.5} \\ MP_l &= f_l = 0.5\alpha k^{0.5} l^{-0.5} \end{aligned}$$

b. In short run, capital is fixed. Show that the production function follows the law of diminishing return to labor.

$$\frac{\partial MP_l}{\partial l} = f_{ll} = -0.25\alpha k^{0.5} l^{-1.5} < 0$$

The marginal product of labor is decreasing as adding more labor, which means "diminishing return to labor"

c. In long run, capital can be adjusted. Determine this production function is constant, increasing or decreasing return to scale.

Let $q \equiv f(k, l) = \alpha k^{0.5} l^{0.5}$. Double all inputs,

$$f(2k, 2l) = \alpha(2k)^{0.5}(2l)^{0.5} = 2\alpha k^{0.5} l^{0.5} = 2q.$$

Output double exactly, therefore constant return to scale. (Replace 2 by $t > 1$ for a general proof).

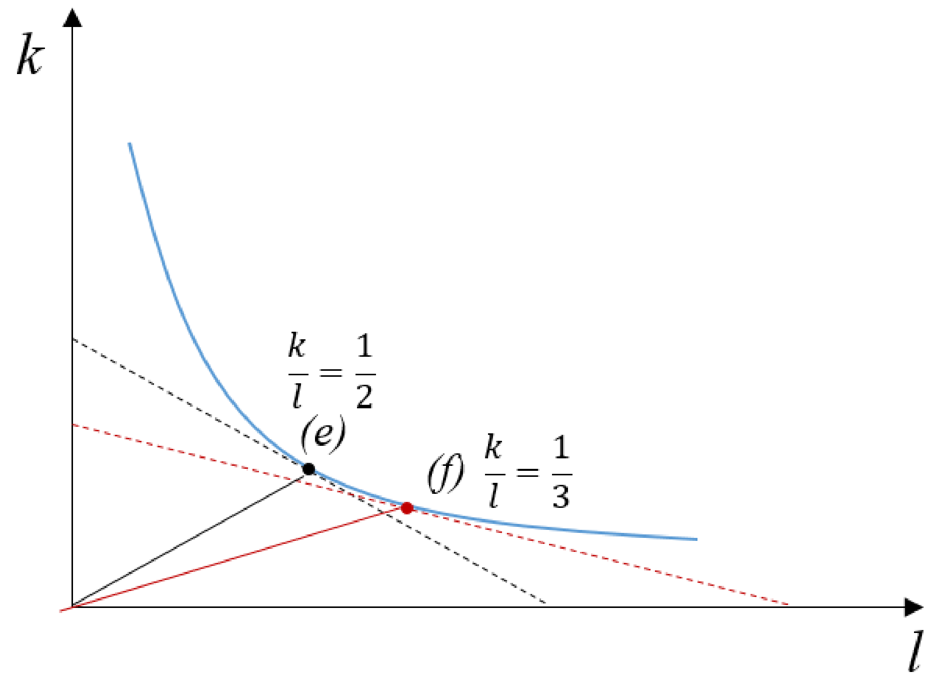
d. Compute the RTS and elasticity of substituton between k and l.

$$RTS = -\frac{MP_l}{MP_k} = -\frac{0.5\alpha k^{0.5} l^{-0.5}}{0.5\alpha k^{-0.5} l^{0.5}} = -\frac{k}{l}.$$

e. If the input prices for capital and labor is $v = 4$ and $w = 2$ respectively, what capital labor ratio will minimize cost?

f. Continue with (e), if the price of capital rises to $v' = 6$, will the company increase or decrease capital labor ratio?

g. Illustrate part (e) and (f) on a graph.



9.2 Given production function $q = kl - 0.8k^2 - 0.2l^2$.

a. When k = 10, total labor productivity is

$$TP_l = 10l - 0.2l^2 - 80,$$

and average labor productivity is

$$AP_l = \frac{q}{l} = 10 - 0.2l - \frac{80}{l}.$$

To find where AP_l reaches a maximum, take the first-order condition:

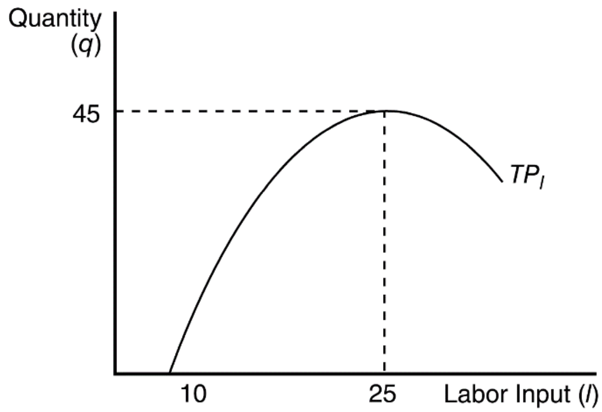
$$\frac{dAP_l}{dl} = \frac{80}{l} - 0.2 = 0.$$

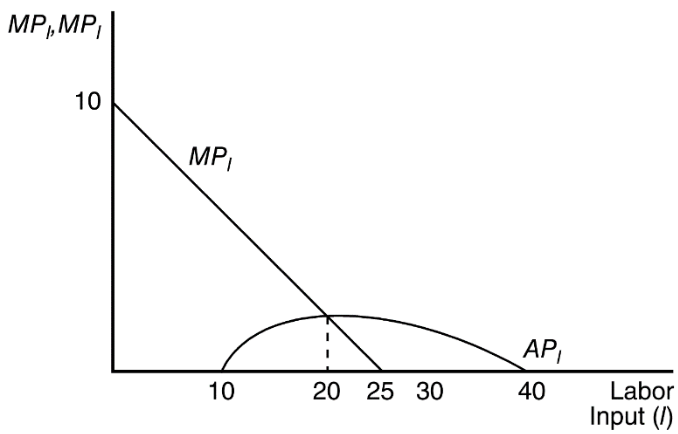
The maximum is at $l = 20$. When $l = 20$, $q = 40$. The graph is provided after part (b).

b. Marginal labor productivity is

$$MP_l = \frac{dq}{dl} = 10 - 0.4l.$$

To find where this is 0, set $MP_l = 10 - 0.4l = 0$, implying $l = 25$.





c. If $k = 20$,

$$TP_l = 20l - 0.2l^2 - 320 = q,$$

$$AP_l = 20 - 0.2l - \frac{320}{l},$$

$$MP_l = 20 - 0.4l.$$

AP_l reaches a maximum at $l = 40,\ \ q = 160$. At $l = 50,\ MP_l = 20 - 0.4l = 0$.

d. Doubling of k and l here multiplies output by 4 (compare parts (a) and (c)). Hence, the function exhibits increasing returns to scale.

9.3 Given production function $q = 0.1k^{0.2}l^{0.8}$.

a. Given Sam spends \$10,000 in total and equal amounts on both inputs, he spends \$5,000 on each. At the \$50 per hour, he uses inputs $k ≩ 00$, $l = 100$, and produces output $q = 10$. Total cost is 10,000 (by design).

b. We have

$$MP_k = \frac{\partial q}{\partial k} = 0.02\left(\frac{l}{k}\right)^{0.8},$$

$$MP_l = 0.08\left(\frac{k}{l}\right)^{0.2}.$$

Setting these equal yields $l/k = 4$. Substituting into the production function,

$$q = 10 = 0.1k^{0.2}(4k)^{0.8} = 0.303k.$$

Solving, $k \approx 33$ and $l \approx 132$. Total cost is 8,250.

c. The cost savings in part (b) is 1,750. We saw in part (b) that $8,250 used in the way Norm suggested produced 10 stools. Because the production function exhibits constant returns to scale, if the full $10,000 were spent to produce stools following Norm's suggestion, more stools can be produced in proportion:

$$\frac{10,000}{8,250} \times 10 = 12.12,$$

a little more than two extra stools.

9.6 a. We have

$$MP_k = \frac{\partial q}{\partial k} = \frac{1}{\rho}\left[k^\rho + l^\rho\right]^{\frac{1-\rho}{\rho}} \cdot \rho k^{\rho-1}$$

$$= q^{1-\rho} \cdot k^{\rho-1}$$

$$= \left(\frac{q}{k}\right)^{1-\rho}.$$

Similar manipulations yield

$$MP_l = \left(\frac{q}{l}\right)^{1-\rho}.$$

b. Using the results from part (a),

$$RTS = \frac{MP_l}{MP_k} = \left(\frac{k}{l}\right)^{1-\rho}.$$

Inverting,

$$\frac{k}{l} = RTS^{\frac{1}{1-\rho}},$$

in turn implying

$$\ln\left(\frac{k}{l}\right) = \frac{1}{1-\rho} \ln RTS.$$

From Equation 9.32,

$$\sigma = \frac{d \ln\left(\frac{k}{l}\right)}{d \ln RTS} = \frac{1}{1-\rho}.$$

c. Computing elasticities,

$$e_{q,k}=\frac{\partial q}{\partial k}\cdot\frac{k}{q}=\left(\frac{q}{k}\right)^{-\rho}=\frac{1}{1+(l/k)^{\rho}},$$

$$e_{q,l}=\left(\frac{q}{l}\right)^{-\rho}=\frac{1}{1+\left(k/l\right)^{\rho}}=\frac{1}{1+\left(l/k\right)^{-\rho}}.$$

Putting these over a common denominator yields $e_{q,k}+e_{q,l}=1$, which shows constant returns to scale.

d. The result follows directly from part (a) since

$$\sigma=\frac{1}{1-\rho}.$$

9.7 Given production function $f(k,l)=\beta_0+\beta_1\sqrt{kl}+\beta_2 k+\beta_3 l.$

a. For constant returns to scale, $f(tk,tl)=tf(k,l)$. But

$$\begin{aligned}f(tk,tl)&=\beta_0+\beta_1\sqrt{tk\cdot tl}+\beta_2 tk+\beta_3 tl\\&=\beta_0+t\left(\beta_1\sqrt{kl}+\beta_2 k+\beta_3 tl\right),\end{aligned}$$

while

$$tf(tk,tl)=t\beta_0+t\left(\beta_1\sqrt{kl}+\beta_2 k+\beta_3 tl\right).$$

For these two equations to be equal, $\beta_0=0$.

b. Assume $\beta_0=0$ to ensure constant returns to scale. Then

$$MP_l=0.5\beta_1\left(\frac{k}{l}\right)^{0.5}+\beta_3,$$

$$MP_k=0.5\beta_1\left(\frac{l}{k}\right)^{0.5}+\beta_2.$$

Both are homogeneous of degree zero with respect to (k,l) and exhibit diminishing marginal productivities.

c Footnote 6 provides the key formula in the special case of constant returns to scale:

$$\sigma = \frac{f_l f_k}{f \cdot f_{kl}}$$
$$= \frac{\left[(\beta_1/2)(k/l)^{1/2} + \beta_3\right]\left[(\beta_1/2)(l/k)^{1/2} + \beta_2\right]}{\left[\beta_1 (kl)^{1/2} + \beta_2 k + \beta_3 l\right]\left[(\beta_1/4)(kl)^{-1/2}\right]}.$$

For $\sigma = 0$, one of the factors in the numerator has to be 0 and the denominator should not be zero, which is impossible.

For $\sigma = 1$, the numerator has to equal the denominator. Expanding out the numerator gives

$$\frac{\beta_1^2}{4} + \frac{\beta_1\beta_3}{2}(l/k)^{1/2} + \frac{\beta_1\beta_2}{2}(k/l)^{1/2} + \beta_2\beta_3$$

and the denominator gives

$$\frac{\beta_1^2}{4} + \frac{\beta_1\beta_3}{4}(l/k)^{1/2} + \frac{\beta_1\beta_2}{4}(k/l)^{1/2}.$$

For these two expressions to be equal for all (k,l) requires

$$\beta_2 = \beta_3 = 0.$$

And β_1 should not be zero.

For $\sigma = \infty$, the denominator must be 0. This only holds for all (k,l) if the second factor is 0, that is,

$$(\beta_1/4)(kl)^{-1/2} = 0.$$

For this condition to hold for all (k,l) requires $\beta_1 = 0.$ and either β_2 or β_3 not equal to zero.