ECON3113
Microeconomic Theory I

Tutorial #5:

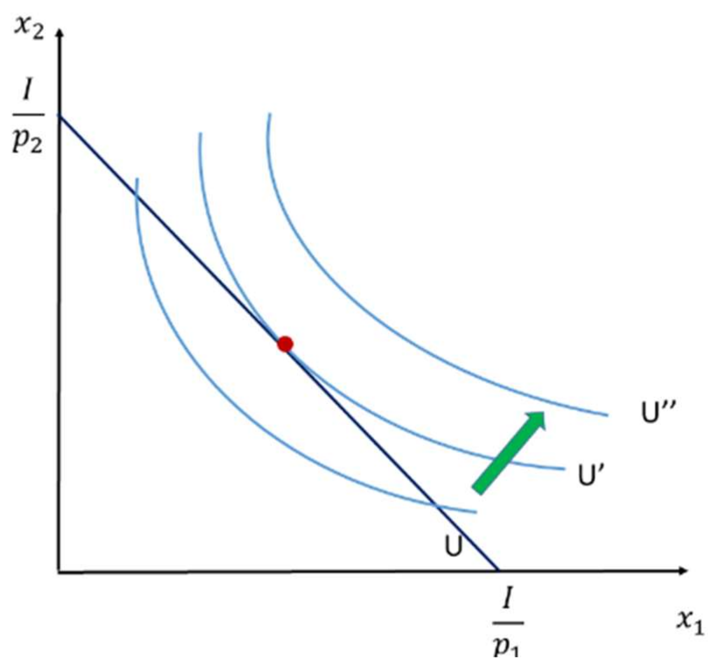
The theory of utility maximisation – the framework and examples

Today's tutorial

- De-brief on the first online quiz
  - What to expect in the next quiz (which will be next week)
- Constrained utility optimization
  - The framework
  - Examples with different utility functions (taken from Nicholson & Snyder, Ch.3 & Ch.4)
    - Constant Elasticity of Substitution (CES) utility function
    - Quasi-Linear utility function
    - Stone Geary utility function

Constrained utility maximisation: the framework

- Superposing the budget line with the indifference curves.
- Look for the bundle/point in the budget set that lies on the indifference curve with the highest utility.



- We have:
  - $U(x, y)$
  - $I = P_x x + P_y y$
- Affordable bundles on or inside the budget constraint
- Tangency at: $MRS = \frac{P_x}{P_y}$

- Note: Limitations of this approach in lecture notes:
  - Corner solutions
  - Tangency not always optimal

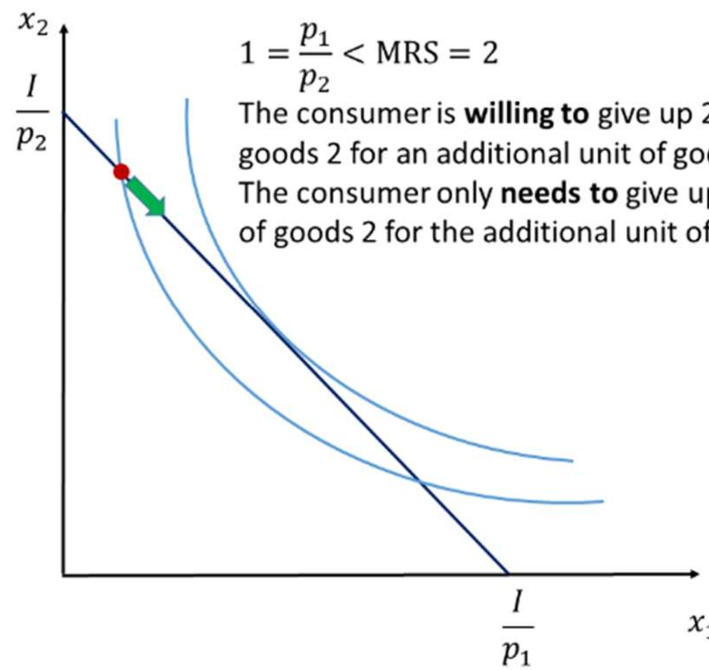
Constrained utility maximisation: the framework

- Given utility function $U(x, y)$ find the marginal rate of substitution on an indifference curve $\bar{U}$:
- MRS $=$

Constrained utility maximisation: the framework

- Intuition of why the tangency condition works

- What bundle would make the consumer willing to stay put?
- Start with any bundle $(x_1, x_2) > (0, 0)$. If she wants to increase his consumption of goods 1 by one unit,
  - the amount of goods 2 she is *willing to* give up is MRS;
  - the amount of goods 2 she *has to* give up is $p_1 \times \frac{1}{p_2}$.
- She wants to consume more of goods 1 if $\frac{p_1}{p_2} < MRS$.



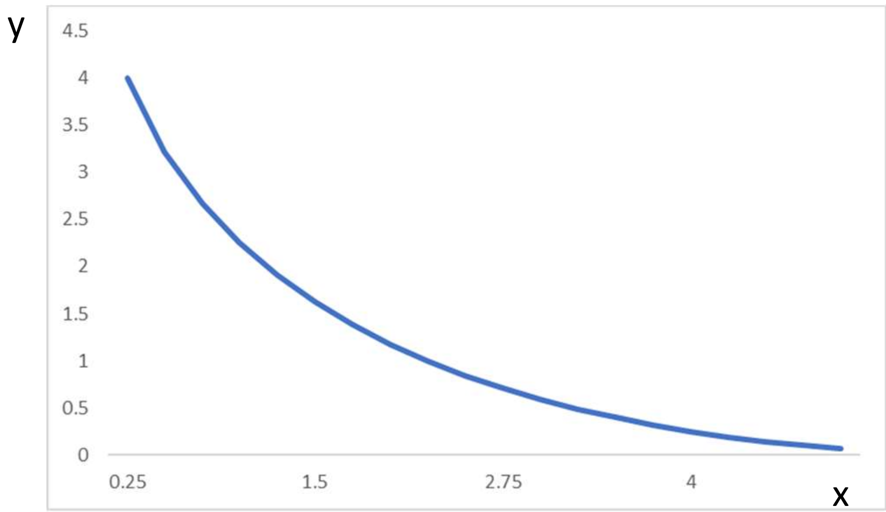
$$1 = \frac{p_1}{p_2} < \text{MRS} = 2$$

The consumer is **willing to** give up 2 units of goods 2 for an additional unit of goods 1. The consumer only **needs to** give up 1 unit of goods 2 for the additional unit of goods 1.

The Constant Elasiticity of Substitution (CES) utility function

- Given by:
  - $U(x, y) = \left(ax^{\delta} + (1-a)y^{\delta}\right)^{\frac{1}{\delta}}$
  - Or:
  - $U(x, y) = \left(x^{\delta} + y^{\delta}\right)^{\frac{1}{\delta}}$
  - $\delta \leq 1, \delta \neq 0$ in both cases
- Some special cases:
  - $\delta = 1$: perfect substitutes ie $U(x, y) = ax + (1-a)y$
  - $\delta \to 0$: Cobb-Douglas ie $U(x, y) = x^{a}y^{1-a}$
  - $\delta \to -\infty$: perfect complements ie $U(x, y) = \min[ax, (1-a)y]$
- A monotonic transformation $U^* = \frac{U^{\delta}}{\delta}$ is often used

Exercise: N&S ch.3 q.3.12



- Example with:
  - $\alpha = 2, \beta = 2$
  - $\delta = 0.5$

- (a) Show that the MRS for CES function:
  - $U(x, y) = \alpha \frac{x^\delta}{\delta} + \beta \frac{y^\delta}{\delta}$

  depends only on the ratio $\frac{x}{y}$ and not on the amounts of the goods consumed.
  - Note that this is a characteristic of homothetic preferences
- How does the MRS depend on the ratio $\frac{y}{x}$?

Exercise: N&S ch.3 q.3.12

- We have $U(x, y) = \alpha \frac{x^\delta}{\delta} + \beta \frac{y^\delta}{\delta}$
- To find the MRS:

Exercise: N&S ch.3 q.3.12

(b) Show that these results are consistent with (i) the Cobb Douglas function and (ii) perfect substitutes function

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Exercise: N&S ch.3 q.3.12

(c) Show that the MRS is strictly diminishing for all values of $\delta < 1$

Exercise: N&S ch.3 q.3.12

(d) Show that if $x = y$ the MRS of this function depends only on the relative sizes of α and β

Exercise: N&S ch.3 q.3.12

(e) Calculate the MRS for this function when $\frac{y}{x} = 0.9$ and 1.1 for the two cases $\delta = 0.5$ and $\delta = -1$. What do you conclude about the extent to which the MRS changes in the vicinity of $x = y$? How would you interpret this geometrically?

We have:

y/x	δ	MRS
0.9	0.5	$0.95\frac{\alpha}{\beta}$
1.1	0.5	$1.05\frac{\alpha}{\beta}$
0.9	-1	$0.81\frac{\alpha}{\beta}$
1.1	-1	$1.21\frac{\alpha}{\beta}$

Exercise: N&S ch.3 q.3.12

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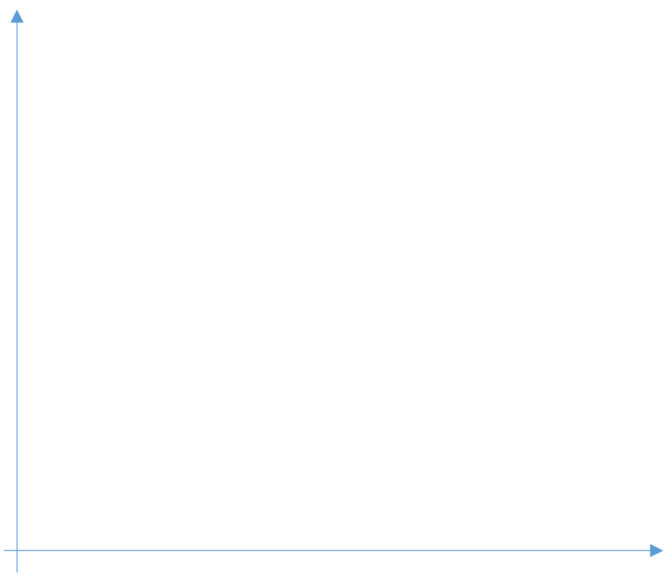

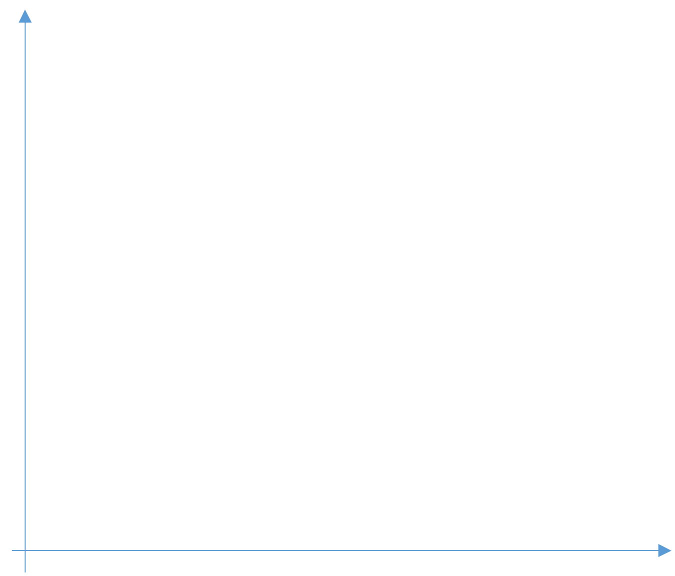

The Quasi-Linear utility function

- Suppose the consumer has a quasi-linear utility function:

$$u(x_1, x_2) = v(x_1) + x_2,$$

for some strictly increasing and strictly concave function v.

- The MRS is given by

$$MRS = \frac{\partial u / \partial x_1}{\partial u / \partial x_2} = v'(x_1),$$

so it is strictly decreasing in x_1 but independent of x_2.

  - Strict concavity of v implies DMRS.

- A function f is strictly concave in x and y if for every $\alpha \in [0.1]$:

$$f((1-\alpha)x + \alpha y) > (1-\alpha)f(x) + \alpha f(y)$$

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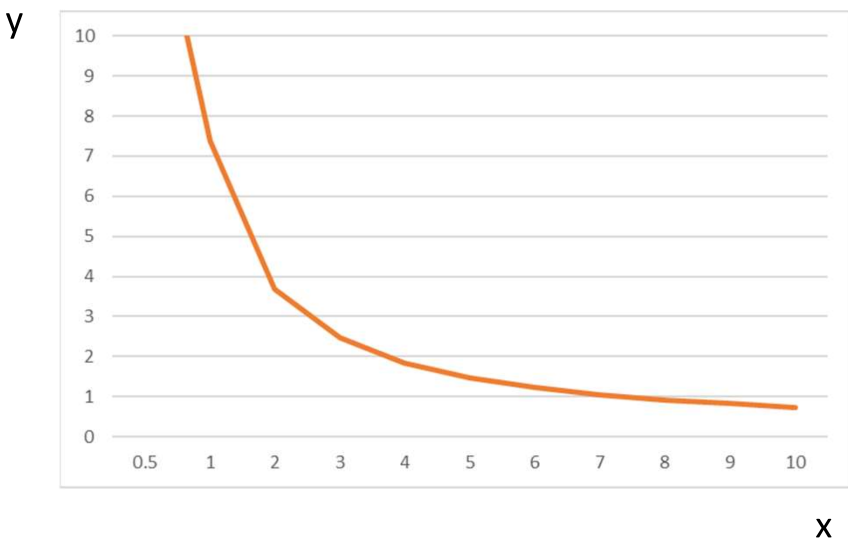
Exercise: N&S ch.3 q.3.13

Consider the function $U(x, y) = x + \ln(y)$

(a) Find the MRS of the function, and interpret the result



Exercise: N&S ch.3 q.3.13

Consider the function $U(x, y) = x + \ln(y)$

(b) A function f is quasi- concave if the following condition holds:

- $f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2 < 0$

- Show that this function is quasi-concave

Exercise: N&S ch.3 q.3.13

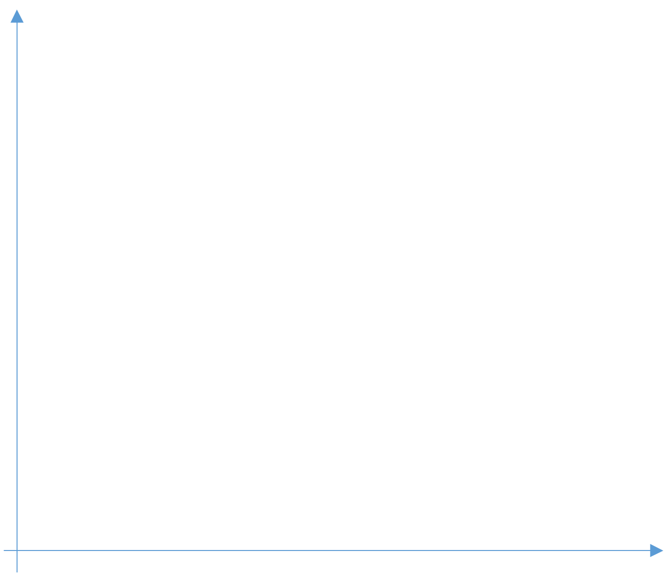
Consider the function $U(x, y) = x + \ln(y)$

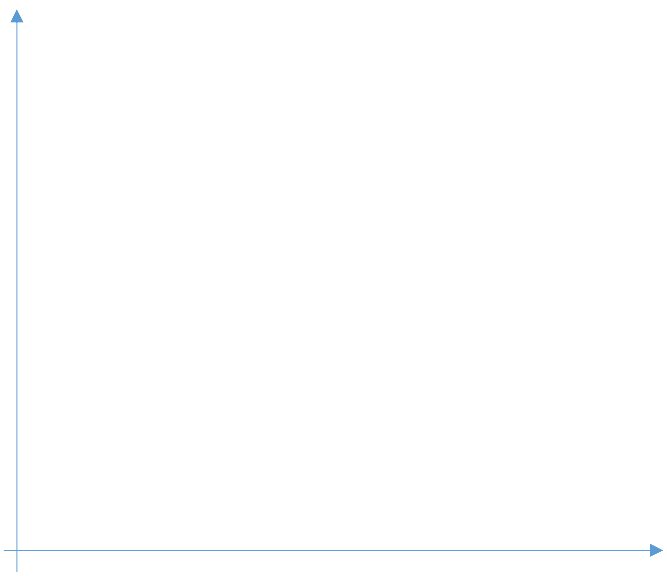
(c) Find the equation for an indifference curve for this function

Exercise: N&S ch.3 q.3.13

Consider the function $U(x, y) = x + \ln(y)$

(d) Compare the marginal utilities of x and y. How do you interpret these functions? How might consumers choose between x and y as they try to increase their utility by, for example, consuming more when their income increases?





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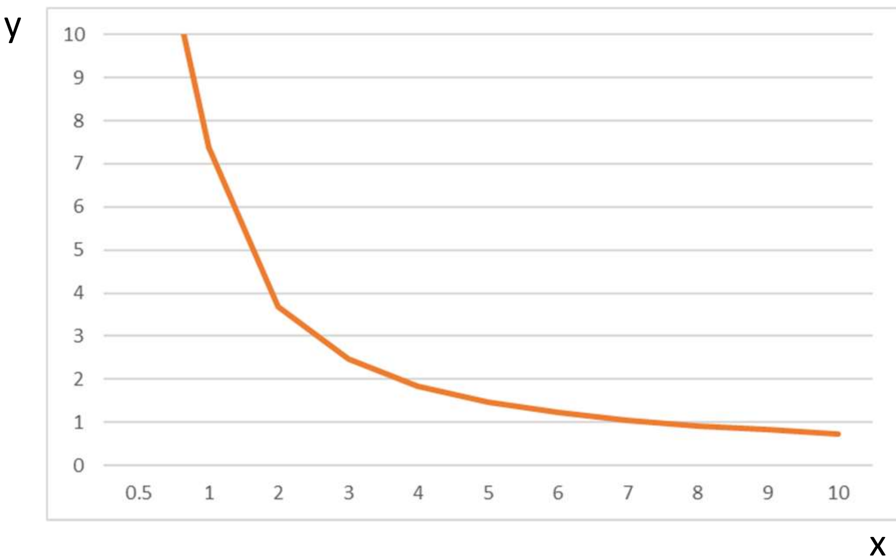

- When income increases, do consumers buy more of both x and y ?

Exercise: N&S ch.3 q.3.13

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- Let's look at the demand curves for x and y:

- Demand curve for y:

- Demand curve for x:

Exercise: N&S ch.3 q.3.13

Consider the function $U(x, y) = x + \ln(y)$

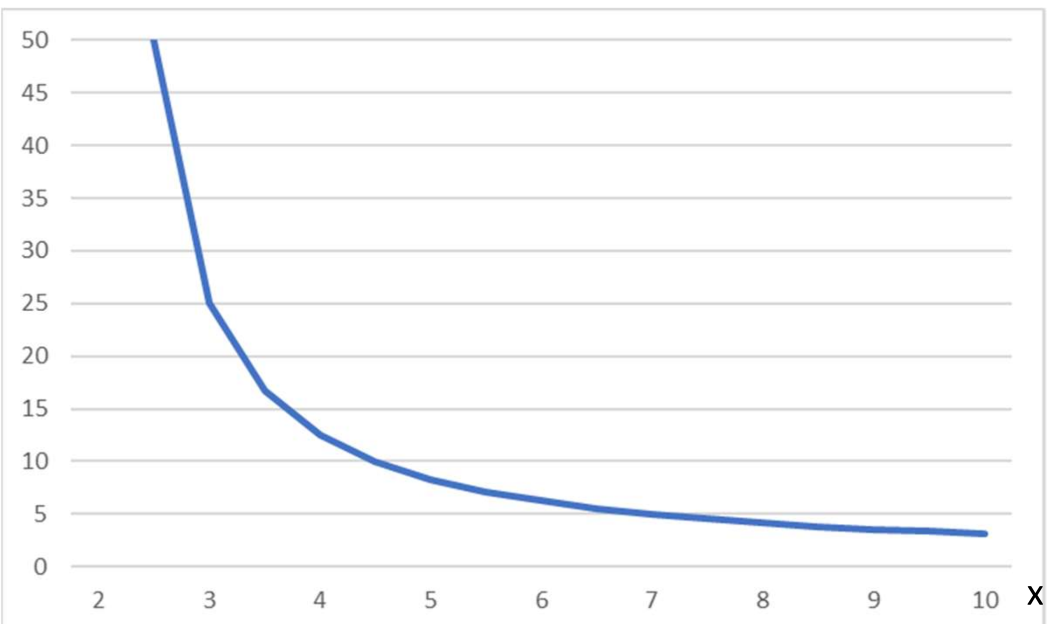
(d) Compare the marginal utilities of x and y. How do you interpret these functions? How might consumers choose between x and y as they try to increase their utility by, for example, consuming more when their income increases?

- With these demand curves, what happens to demand for x and y as income, I, increases?
- Demand for y:
- Demand for x:
- Are x and y:
- 1) Substitutes or complements?
- 2) Normal or inferior goods

Exercise: N&S ch.4 q.4.12

Suppose that individuals require a certain amount of food, x, to stay alive. Let this amount be given by x_0. Once x_0 is purchased, individuals obtain utility from food and other goods, y, of the form:

- $U(x,y) = (x - x_0)^{\alpha} y^{\beta}$ $\qquad \alpha + \beta = 1$



Example:

- $\alpha = \beta = 0.5$
- $x_0 = 2$
- $\bar{U} = 5$
- Note: U not defined for $\mathrm{x} < x_0$

$MU_x =$

$MU_y =$

$MRS =$

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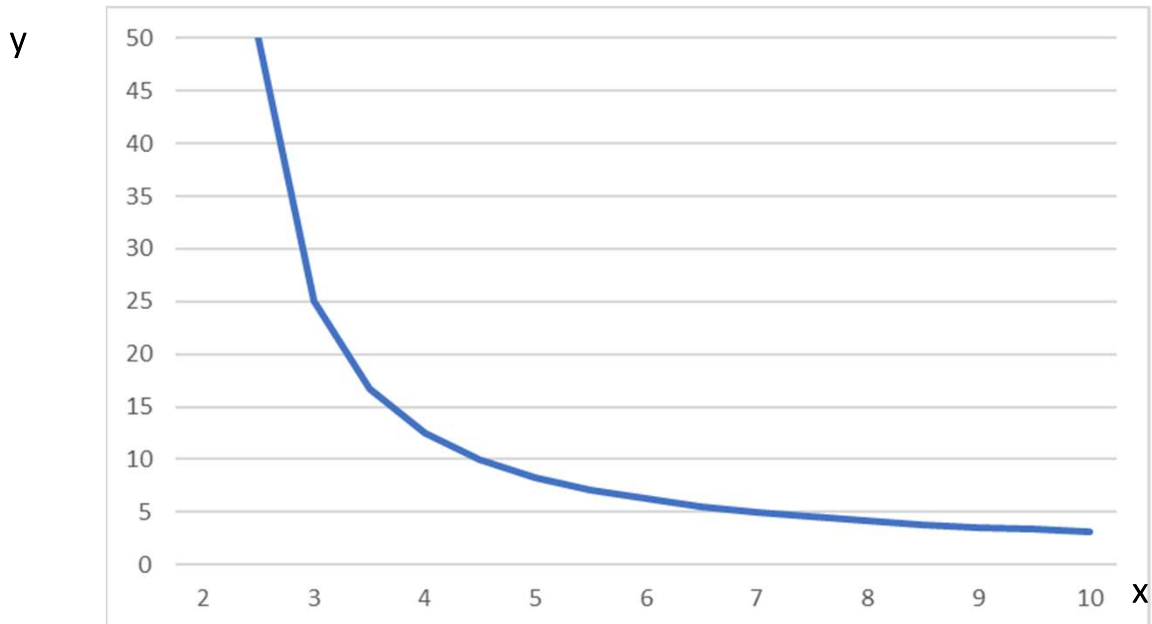


$$MRS = \frac{\alpha}{\beta} \frac{y}{(x - x_0)}$$

- The greater is x_0:

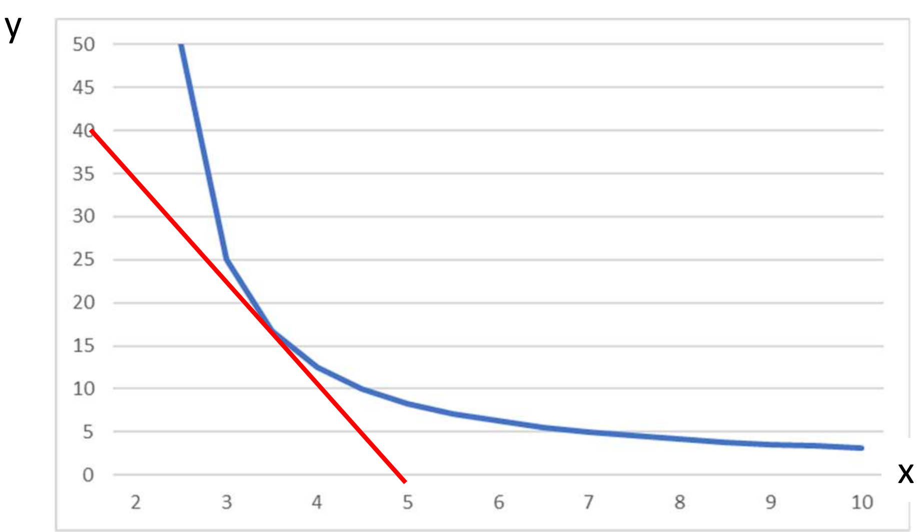
- The greater is α compared to β:

Exercise: N&S ch.4 q.4.12

Suppose that individuals require a certain amount of food, x, to stay alive. Let this amount be given by x_0. Once x_0 is purchased, individuals obtain utility from food and other goods, y, of the form:

- $U(x, y) = (x - x_0)^{\alpha} y^{\beta}$ $\qquad \alpha + \beta = 1$

(a) Show that if $I > p_x x_0$, then the individual will maximise utility by spending $\alpha(I - p_x x_0) + p_x x_0$ on good x and $\beta(I - p_x x_0)$ on good y.



- What does it mean if if $I < p_x x_0$?

- It means that the consumer is not alive!

Exercise: N&S ch.4 q.4.12

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- At maximum utility:

Exercise: N&S ch.4 q.4.12

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Step 1: Find y in terms of the other variables and parameters

Exercise: N&S ch.4 q.4.12

- We have:
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Step 2: Substitute into the budget constraint and simplify

Exercise: N&S ch.4 q.4.12

- We have:
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Step 3: Find the required expression for $p_y y$

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Step 4: Interpret the result

We have:

1) $p_x x = \alpha(I - p_x x_0) + p_x x_0$
2) $p_y y = \beta(I - p_x x_0)$

Exercise: N&S ch.4 q.4.12

- We have:
- $U(x, y) = (x - x_0)^\alpha y^\beta$ $\qquad\qquad \alpha + \beta = 1$

(b) How do the ratios $\frac{p_x x}{I}, \frac{p_y y}{I}$ change as income increases in this problem?

We have:

1) $$\frac{p_x x}{I} = \frac{\alpha(I - p_x x_0) + p_x x_0}{I}$$

2) $$\frac{p_y y}{I} = \frac{\beta(I - p_x x_0)}{I}$$

Exercise: N&S ch.4 q.4.12

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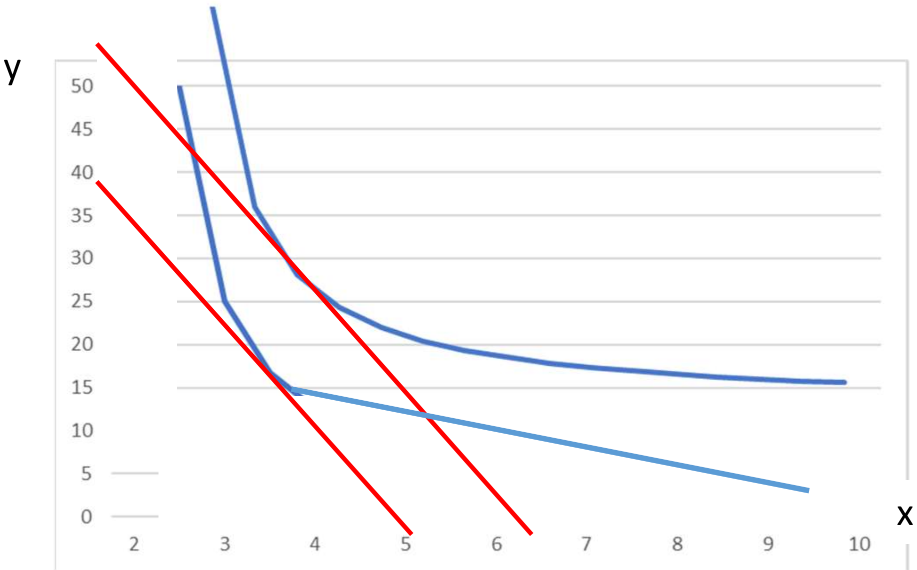

- Are x and y normal or inferior goods?