Topic 2

Linear Models and Matrix Algebra

Outline

1. What is matrix?
2. Matrix operations
3. Special matrices
4. Inverse of a square matrix
5. Determinant of a square matrix
6. Linear equation system
7. Elementary operations
8. Rank of a matrix

1. What is a matrix?

- A matrix is a two dimensional rectangular array of numbers.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

- We say that the dimension (or size) of A is $m \times n$, denoted $A \in R^{m \times n}$
- If $m = n$, then A is a square matrix
- If $n = 1$, the $m \times 1$ matrix is called a column (m dimensional) vector
- If $m = 1$, the $1 \times n$ matrix is called a row (n dimensional) vector
- If $m = n = 1$, A is just a number, called scalar

Examples:

$m = 2, n = 3$:	2×3 matrix $A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 9 & 0 \end{pmatrix}$
$m = 2, n = 1$	Two-dimensional column vector $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$
$m = 1, n = 4$	Four-dimensional row vector (1,3,2,5)
$m = 1, n = 1$	Scalar number 3.14

2. Matrix operations

- Equality: $A = B$ iff $\dim(A) = \dim(B)$ and $a_{ij} = b_{ij}$ for all i and j

- Addition and subtraction: if $\dim(A) = \dim(B)$, then $A + B$ has a typical element $a_{ij}+b_{ij}$ while $A - B$ has a typical element $a_{ij} - b_{ij}$.

- **Example**: Let $A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 9 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 6 \\ 3 & 5 & 1 \end{pmatrix}$, then
$$A + B = \begin{pmatrix} 2 & 2 & 9 \\ 2 & 14 & 1 \end{pmatrix}, A - B = \begin{pmatrix} 2 & 0 & -3 \\ -4 & 4 & -1 \end{pmatrix}$$

- Scalar multiplication: For a scalar c, cA has typical element ca_{ij}.

- **Example**: Let $A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 9 & 0 \end{pmatrix}$, then $3A = \begin{pmatrix} 6 & 3 & 9 \\ -3 & 27 & 0 \end{pmatrix}$

Transpose of a matrix:

- For an $m \times n$ matrix A, the transpose A' (or A^T) is defined as an $n \times m$ matrix such that the $(i,j)th$ element of A' is equal the $(j,i)th$ element of A

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \text{ then } A' = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

- **Examples**: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, then, $A' = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

  $x = (1,2,3)$, then or $(1,2,3)' = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

- Properties of matrix transpose
  - $(A')' = A$
  - $(A + B)' = A' + B', (\alpha A)' = \alpha A'$ where a is a scalar

Matrix multiplication:

- if $A \in R^{m \times n}, B \in R^{n \times p}$, then A and B can be multiplied to produce a matrix of dimension of $m \times p$. Specifically, if

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots \\ b_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix}$$

then $C = AB$ is a matrix with $\dim(C) = m \times p$

$$C = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{pmatrix}, \quad \text{where } c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

- Note, for AB to be meaningful, the number of columns of A should be equal to the number of rows of B

Examples

- $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$=$\begin{pmatrix} 1\times 5+2\times 7 & 1\times 6+2\times 8 \\ 3\times 5+4\times 7 & 3\times 6+4\times 8 \end{pmatrix}$=$\begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$

- Let $A=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \end{pmatrix}$, $B=\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 5 \end{pmatrix}$, then

$$AB=\begin{pmatrix} 7 & 25 \\ 19 & 28 \end{pmatrix},\ BA=\begin{pmatrix} 9 & 12 & 3 \\ 19 & 26 & 9 \\ 20 & 25 & 0 \end{pmatrix}$$

- It is important to note that usually, $AB\neq BA$.

- Even if AB exists, BA may not be defined, for example,

$$A=\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B=\begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

Useful properties

- Transpose of product: $(AB)' = B'A'$
- Left distributive law: $A(B + C) = AB + AC$
- Right distributive law: $(A + B)C = AC + BC$
- Associative law: $(AB)C = A(BC) = ABC$,
- **Exercise**: Prove the above properties for 2×2 matrices A, B and C.

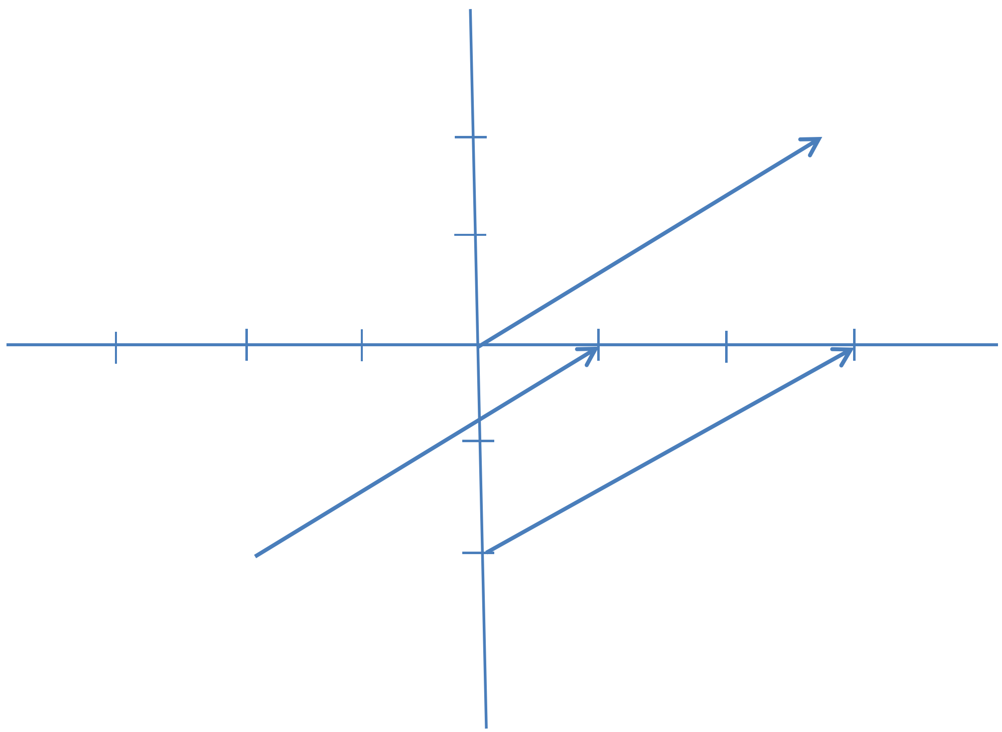
Exercises

1. Confirm the following three points by example
   - $AB \neq BA$ except in special cases
   - $AB = 0$ does not imply that A or B is 0
   - $AB = AC$ and $A \neq 0$ do not imply that $B = C$

2. Let A and B be $n \times n$ matrices. Show that
$$(A + B)(A - B) = A^2 - B^2$$
only when $AB = BA$

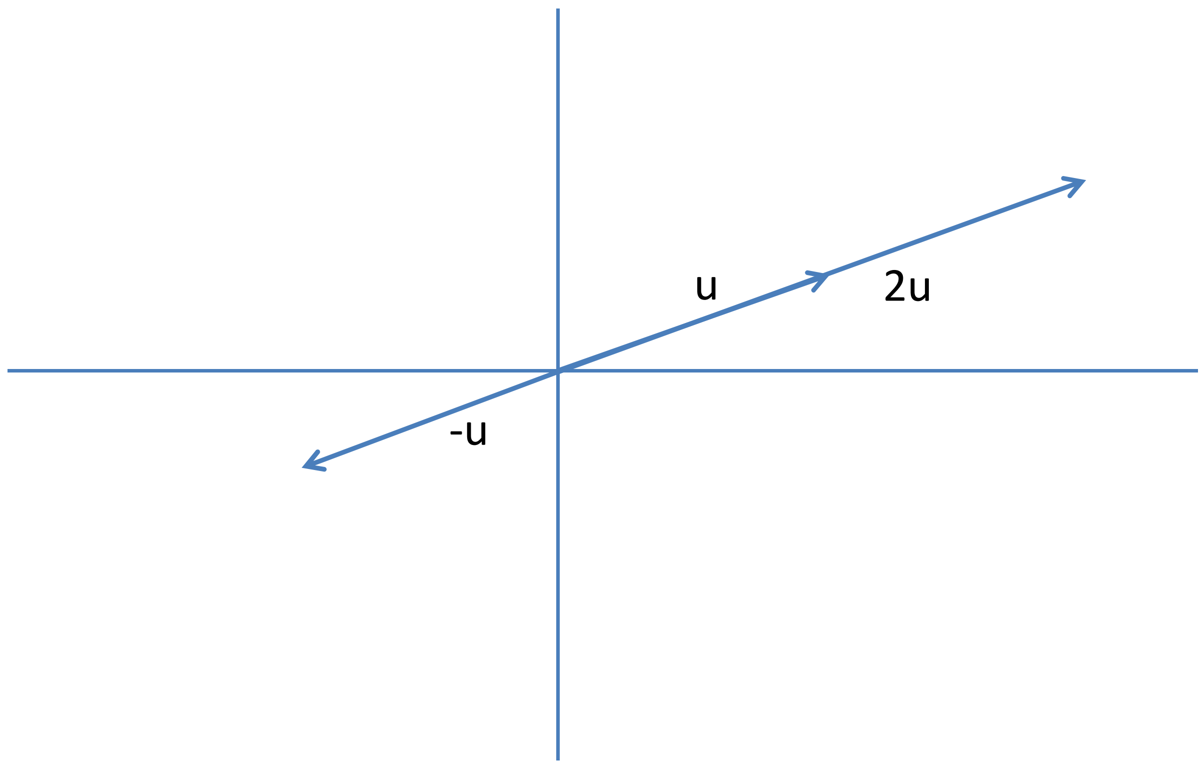
3. Special matrices

- Vectors: $x = (x_1, x_2, \ldots, x_n)'$, useful for modeling a wide variety of economic phenomenon because of because n-tuples of numbers have many useful interpretations
  - Special case: $n = 2$, the vector represents a particular location in the plane
  - Vector (3,2)



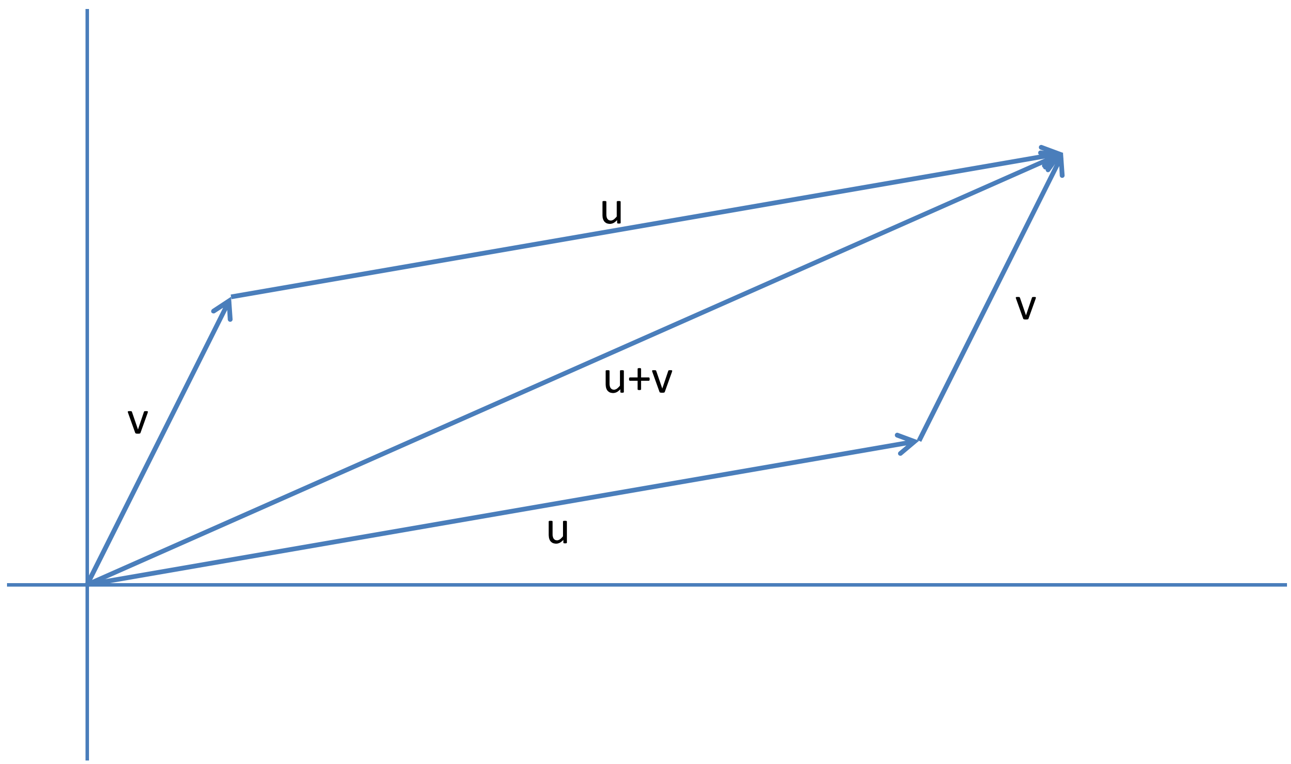
Vectors, *continued*

- Scalar multiplication: u = (2,1)'



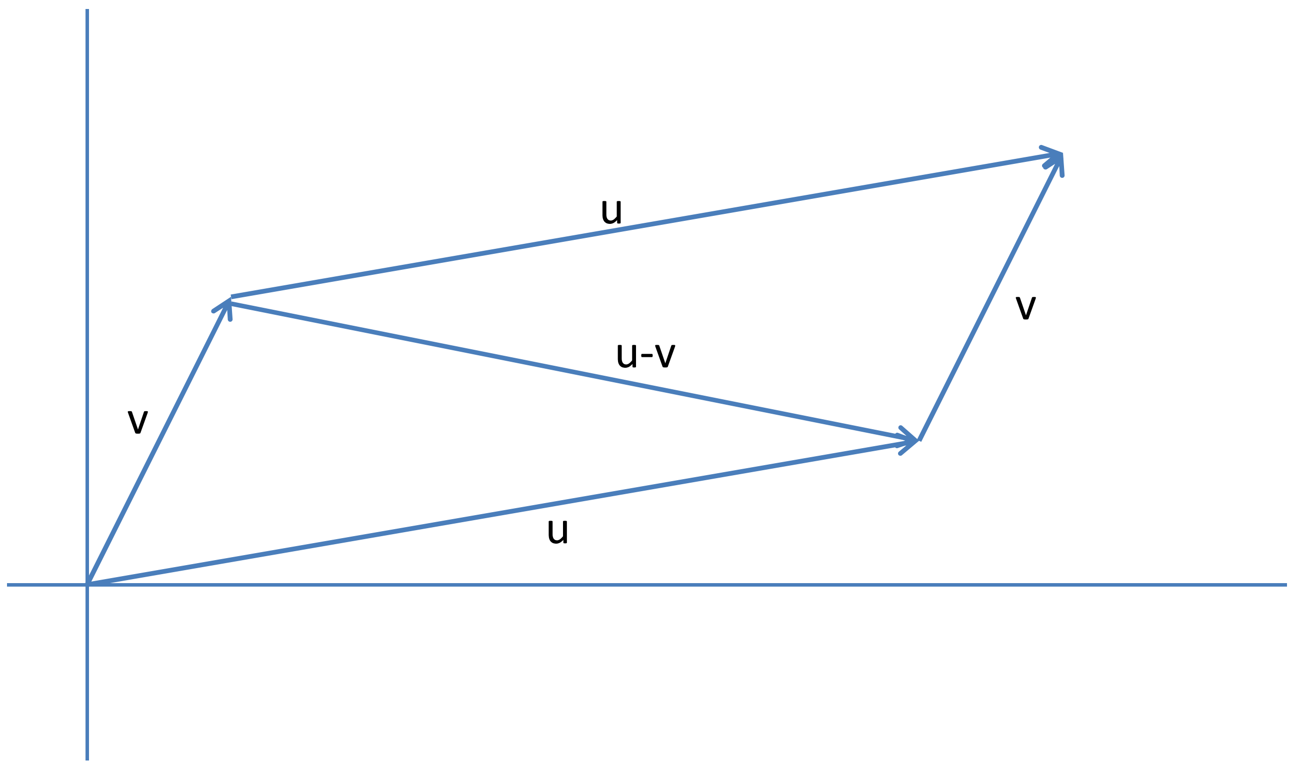
Vectors, *continued*

- Vector summation: (6,1)’ + (1,2)’ = (7,3)’



Vectors, *continued*

- Vector subtraction: (6,1)' - (1,2)' = (5,-1)'



Vectors, *continued*

- $u = (u_1, u_2, \dots, u_n)'$, v= $(v_1, v_2, \dots, v_n)'$

  then uv' is $n \times n$ matrix, and $u'v = \sum_{i=1}^{n} u_i v_i$ is a scalar, called

  the inner product of the two vectors.
- The length (or norm) of a vector: $\|u\| = \sqrt{u'u}$
- An n-dimensional vector can be viewed as a point in n-dimensional space.

Special matrices, *continued*

- Symmetrix metrix: If $A = A'$ (A must be a square matrix)
- For any matrix B, $B'B$ and BB' exist and are symmetric
- Diagonal matrix: A square matrix A is diagonal if $a_{ij} = 0$ for all $i \neq j$, denoted $A = diag(a_{11}, a_{22}, \dots, a_{nn})$
- For two diagonal matrices $B = diag(\beta_1, \beta_2, \dots, \beta_n)$ and $C = diag(\mu_1, \mu_2, \dots, \mu_n), BC = diag(\beta_1\mu_1, \beta_2\mu_2, \dots, \beta_n\mu_n)$
- For a diagonal matrix, it is particularly easy to compute powers: $D = diag(d_1, d_2, \dots, d_n)$, $\qquad D^k = DD \cdots D = diag({d_1}^k, {d_2}^k, \cdots {d_n}^k)$

- An Identity matrix $I_n \in R^{n\times n}$ is a diagonal matrix with all diagonal elements equal to 1.
- A zero matrix $O \in R^{m\times n}$ is a matrix (not necessarily square matrix) with all its elements equal to zero.
- For $A \in R^{m\times n}$
  - $O + A = A + O = A$
  - $I_m A = A I_n = A$
- Let e_i be the ith column of I_n, then

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

- If $A \in R^{m\times n}$, then $e_i'A = ith$ row of A, and $Ae_j = jth$ column of A. Note: $e_i \in R^m, e_j \in R^{\mathrm{n}}$
  - **Exercise**: Verify the above for 2×2 matrix

- **Example**: Compare the following 2×2 matrices:

Square matrix:	$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$
Symmetric matrix:	$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$
Diagonal matrix:	$\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$
Identity matrix:	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
Zero matrix:	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

- Idempotent matrix: If $A^2 \triangleq AA = A$ (A must be square matrix, why?)
  - **Examples**: I_n, $O_{n\times n}$ and $A = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$ are idempotent

- Upper triangular matrix: A square matrix A is upper triangular if $a_{ij} = 0$ for all $i > j$, i.e. all the elements below the diagonal are zero.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

- Lower triangular matrix: A square matrix A is lower triangular if $a_{ij} = 0$ for all $i < j$, i.e. all the elements above the diagonal are zero.

$$A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

4. Inverse of a square matrix

- For a scalar $a \neq 0$, if b satisfies $ab = ba = 1$, then $b = a^{-1}$

- If an $n \times n$ matrix A, if there exists a matrix B such that $AB = BA = I_n$, then B is the inverse of A, denoted $B = A^{-1}$. If A^{-1} exists, we say that A is non-singular or invertible. Otherwise, A is called singular.

Examples

- Since $I_n \, I_n = \, I_n, \; {I_n}^{-1} = I_n$
- Let O be $n \times n$ matrix of zeros, then for any $n \times n$ matrix B

  $$OB = BO = O$$

  so O^{-1} does not exist, O is singular
- Let $A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$ for $a_1, a_2 \neq 0$, then $A^{-1} = \begin{pmatrix} {a_1}^{-1} & 0 \\ 0 & {a_2}^{-1} \end{pmatrix}$

  If $a_1 = 0$ or $a_2 = 0$, then A^{-1} does not exist
- Let the diagonal matrix $B = diag(\beta_1, \beta_2, \dots, \beta_n)$ for $\beta_1, \beta_2, \dots, \beta_n \neq 0$, then $B^{-1} = diag({\beta_1}^{-1}, {\beta_2}^{-1}, \cdots, {\beta_n}^{-1})$. If any of the $\beta_i = 0$, then B^{-1} does not exist.
- For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, verify that if $ad - bc \neq 0$, $A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
- Verify that if $A^2 = 0$, then $(I - A)^{-1} = I + A$

Properties of matrix inversion (verify)

- $(A^{-1})^{-1} = A$
- $(A')^{-1} = (A^{-1})'$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

5. Determinant of a square matrix

- Determinant can be used to check whether a square matrix is invertible
- $|A|$ denote the determinant of a matrix
- If A is 1×1, i.e., $A = a$ is a scalar, then $|A| = a$
- If A is 2×2, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then $|A| = a_{11}a_{22} - a_{12}a_{21}$
- If A is 3×3, $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \qquad (2.1)$$

(complete expansion)

- For an $n \times n$ matrix A, the minor of a_{ij} is $M_{ij} = \left|A_{ij}\right|$ where A_{ij} is the $(n-1) \times (n-1)$ sub-matrix of A obtained by deleting the ith row and jth column of A

- The cofactor of a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$

- Note that for 3×3 matrix A, $|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ (expansion along the 1st row)

- **Exercise**: For 3×3 matrix A, work out the complete expansion along the 2nd row:

  $$a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

  and compare with (2.1).

- Laplace Expansion of $|A|$

$$|A| = \sum_{j=1}^{n} a_{ij} C_{ij} \qquad \text{[Expansion by the } ith \text{ row]}$$

$$= \sum_{i=1}^{n} a_{ij} C_{ij} \qquad \text{[Expansion by the } jth \text{ column]}$$

- **Exercise**: Write out the definition of the determinant of a 4×4 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

in terms of the determinants of certain of its 3×3 submatrices. How many terms are there in the complete expansion of the determinant of a 4×4 matrix?

- Example: Let $A = \begin{pmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, then

$$M_{11} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = 5 \times 9 - 6 \times 8 = -3; \quad M_{12} = \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = 36 - 42 = -6$$

$$M_{13} = \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = 32 - 35 = -3; \quad M_{21} = \begin{vmatrix} 1 & 3 \\ 8 & 9 \end{vmatrix} = 9 - 24 = -15$$

$$M_{22} = -3; \quad M_{23} = 9; \quad M_{31} = -9; \quad M_{32} = 0; \quad M_{33} = 6$$

thus

$$C_{11} = M_{11} = -3; \ C_{12} = -M_{12} = 6; \ C_{13} = M_{13} = -3; \ C_{21} = -M_{21} = 15$$

$$C_{22} = -3; \ C_{23} = -9; \ C_{31} = -9; \ C_{32} = 0; \ C_{33} = 6$$

therefore

$$\sum_{j=1}^{3} a_{1j}C_{1j} = (2)(-3)+(1)(6)+(3)(-3) = -9 \text{ (expansion by first row)}$$

$$\sum_{j=1}^{3} a_{2j}C_{2j} = (4)(15)+(5)(-3)+(6)(-9) = -9 \text{ (expansion by second row)}$$

$$\sum_{j=1}^{3} a_{3j}C_{3j} = (7)(-9)+(8)(0)+(9)(6) = -9 \text{ (expansion by third row)}$$

Similarly (check out)

$$\sum_{i=1}^{3} a_{i1}C_{i1} = \sum_{i=1}^{3} a_{i2}C_{i2} = \sum_{i=1}^{3} a_{i3}C_{i3} = -9 \text{ (expansion by columns)}$$

Furthermore, expansion by alien cofactor

$$\sum_{i=1}^{3} a_{i1}C_{i2} = (2)(6)+(4)(-3)+(7)(0) = 0$$

(expansion by 1st row, cofactor of 2nd row)

In general

$$\sum_{j=1}^{n} a_{ij}C_{i'j} = 0 \quad (i \neq i') \quad \text{[expansion by } ith \text{ row and cofactor of } i'th \text{ row]}$$

$$\sum_{i=1}^{n} a_{ij}C_{ij'} = 0 \quad (j \neq j') \quad \text{[expansion by } jth \text{ column and cofactor of } j'th \text{ column]}$$

- It is easy to verify that

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix} = \begin{pmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & |A| \end{pmatrix} = |A| I_n$$

- The adjoint of A is defined as

$$Adj(A) = C' = \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}$$

- then $A \cdot adj(A) = |A| I_n$, when $|A| \neq 0$, $A \cdot [adj(A)/|A|] = I_n$
- Theorem: A square matrix A is invertible

  $\Leftrightarrow |A| \neq 0$ and $A^{-1} = adj(A)/|A|$

- **Example**: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible when $|A| = ad - bc \neq 0$, and

$$C_{11} = d, C_{12} = -c,\ C_{21} = -b,\ C_{22} = a$$

therefore, $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

- **Example**: Derive

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{pmatrix}^{-1} = -\frac{1}{2}\begin{pmatrix} -7 & 4 & 5 \\ 4 & -2 & -4 \\ -1 & 0 & 1 \end{pmatrix}$$

Determinant of special matrices

- If a square matrix A contains one zero row/column, then $|A| = 0$
- $|I_n| = 1$
- Determinant of upper triangular/lower triangular matrix is the product of its diagonal elements. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \text{ and } B = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

then

$$|A| = |B| = a_{11}a_{22}\cdots a_{nn} = \det \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

6. Linear equation system

- The general linear system of m equations in n unknown variables can be written as

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \tag{2.2}$$

- In matrix form, (2.2) can be written as $Ax = b$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and } b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

- Here, A is called the coefficient matrix, $\hat{A} = (A, b)$ is called the augmented matrix

- **Example**: Consider the problem of solving the linear equations

$$\begin{cases} 2x_1 + x_2 = 5 \\ x_1 + 3x_2 = 10 \end{cases} \tag{2.3}$$

- In matrix form, it can be written as $Ax = b$, where

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad b = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$$

- the augmented matrix is $\hat{A} = \begin{pmatrix} 2 & 1 & 5 \\ 1 & 3 & 10 \end{pmatrix}$

- If A is a square matrix and A^{-1} exists, from

  $Ax = b$, we have $A^{-1}Ax = A^{-1}b$ or $x = A^{-1}b$ (solution)

- **Example** (application of inverse matrix), solve the simultaneous linear equations (2.3) using matrix inverse. Here

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ b = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$$

- Threfore $x = A^{-1}b$, where

$$A^{-1} = \frac{1}{2 \times 3 - 1} \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$$

$$A^{-1}\, b = \frac{1}{5} \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 10 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

- So the solution to the simultaneous equations is $x_1 = 1, x_2 = 3$.

- **Example**: The Keynesian model:

$$\begin{cases} Y = C + I_0 + G_0 \\ C = a + bY \end{cases}$$

Can be written in matrix notation as $Ax = b$, where

$$A = \begin{pmatrix} 1 & -1 \\ -b & 1 \end{pmatrix}, \quad x = \begin{pmatrix} Y \\ C \end{pmatrix}, \quad b = \begin{pmatrix} I_0 + G_0 \\ a \end{pmatrix}$$

the solution to the model is

$$x^* = \begin{pmatrix} Y^* \\ C^* \end{pmatrix} = A^{-1}b = \begin{pmatrix} 1 & -1 \\ -b & 1 \end{pmatrix}^{-1} \begin{pmatrix} I_0 + G_0 \\ a \end{pmatrix}$$

$$= \frac{1}{1-b}\begin{pmatrix} 1 & 1 \\ b & 1 \end{pmatrix}\begin{pmatrix} I_0 + G_0 \\ a \end{pmatrix} = \frac{1}{1-b}\begin{pmatrix} I_0 + G_0 + a \\ b(I_0 + G_0) + a \end{pmatrix}$$

- Cramer's rule: an alternative way of solving system of linear equations.
- For the equation system $Ax = b$, if $m = n$ and $|A| \neq 0$, then the solution is $x_j = \frac{|A_j|}{|A|}$, where A_j is obtained by replacing the jth column of A with b.
- **Example**: For the equation system (2.3) $Ax = b$ with

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \quad b = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$$

  - Apply Cramer's rule, the solution is

$$x_1^* = \frac{\begin{vmatrix} 5 & 1 \\ 10 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}} = \frac{15-10}{6-1} = 1; \quad x_2^* = \frac{\begin{vmatrix} 2 & 5 \\ 1 & 10 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}} = \frac{20-5}{6-1} = 3$$

- **Example**: Two-good equilibrium model:

$$\begin{cases} c_1 P_1 + c_2 P_2 = -c_0 \\ \gamma_1 P_1 + \gamma_2 P_2 = -\gamma_0 \end{cases}$$

- Apply Cramer's rule, the solution is

$$P_1^* = \frac{\begin{vmatrix} -c_0 & c_2 \\ -\gamma_0 & \gamma_2 \end{vmatrix}}{\begin{vmatrix} c_1 & c_2 \\ \gamma_1 & \gamma_2 \end{vmatrix}} = \frac{c_2\gamma_0 - c_0\gamma_2}{c_1\gamma_2 - c_2\gamma_1}, \quad P_2^* = \frac{\begin{vmatrix} c_1 & -c_0 \\ \gamma_1 & -\gamma_0 \end{vmatrix}}{\begin{vmatrix} c_1 & c_2 \\ \gamma_1 & \gamma_2 \end{vmatrix}} = \frac{c_0\gamma_1 - c_1\gamma_0}{c_1\gamma_2 - c_2\gamma_1}$$

- **Example**: The Keynesian model including

a market for goods:

$$\begin{cases} Y = C + I + G_0 \\ C = a + bY \\ I = c - di \end{cases}$$

and a market for money:

$$\begin{cases} M^d = kY - ei \\ M^s = M_0 \\ M^d = M^s \end{cases}$$

- where G_0, M_0 are exogenous variables, a, b, c, d, e, k are positive constants (parameters), the endogenous variables are Y, C, I, i. The two sets of equations imply

$$\begin{cases} Y - C - I = G_0 \\ -bY + C = a \\ I + di = c \\ kY - ei = M_0 \end{cases}$$

- in matrix form: $$\begin{pmatrix} 1 & -1 & -1 & 0 \\ -b & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ k & 0 & 0 & -e \end{pmatrix}\begin{pmatrix} Y \\ C \\ I \\ i \end{pmatrix} = \begin{pmatrix} G_0 \\ a \\ c \\ M_0 \end{pmatrix}$$

- since $$\begin{vmatrix} 1 & -1 & -1 & 0 \\ -b & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ k & 0 & 0 & -e \end{vmatrix} = -k\begin{vmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & d \end{vmatrix} - e\begin{vmatrix} 1 & -1 & -1 \\ -b & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= -kd\begin{vmatrix} -1 & -1 \\ 1 & 0 \end{vmatrix} - e\begin{vmatrix} 1 & -1 \\ -b & 1 \end{vmatrix} = -kd - e(1-b)$$

- Similarly $$\begin{vmatrix} G_0 & -1 & -1 & 0 \\ a & 1 & 0 & 0 \\ c & 0 & 1 & d \\ M_0 & 0 & 0 & -e \end{vmatrix} = -M_0 d - e(c + G_0 + a)$$

- Therefore $$Y^* = \frac{M_0 d + e(c + G_0 + a)}{kd + e(1-b)}$$

7. Elementary operations

- Elementary row operations
  1. interchange two rows of a matrix ($R_i \leftrightarrow R_j$)
  2. multiply a row by a non-zero scalar k ($R_i \to kR_i$)
  3. multiply a row by a non-zero scalar k and add it to another row ($R_i \to R_i + kR_j$)

- The simplified form of the matrix that we aim to arrive at is the reduced row echelon form, which satisfies the following four conditions:
  1. If there are any rows consisting only of zeros entries, then they appear at the bottom of the matrix
  2. The lower row starts with more zeros (called leading zeros) than the row above
  3. For each nonzero row, the leftmost nonzero entry is a 1 (known as the leading 1 of that row)
  4. Each column that contains a leading 1 has zero entries everywhere else.

- For example, the following matrices are in reduced row echelon form:

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 5 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

  whereas the following are not:

$$\begin{pmatrix} 1 & 5 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- A matrix has a unique reduced row echelon form

- Example: For the equation system (2.3)

$$\begin{cases} 2x_1 + x_2 = 5 \\ x_1 + 3x_2 = 10 \end{cases}$$

- apply elementary row operations to the augmented matrix:

$$\left(\begin{array}{cc|c} 2 & 1 & 5 \\ 1 & 3 & 10 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 3 & 10 \\ 2 & 1 & 5 \end{array}\right) \xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 3 & 10 \\ 0 & -5 & -15 \end{array}\right)$$

$$\xrightarrow{R_2 \to -R_2/5} \left(\begin{array}{cc|c} 1 & 3 & 10 \\ 0 & 1 & 3 \end{array}\right) \xrightarrow{R_1 \to R_1 - 3R_2} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 3 \end{array}\right)$$

which corresponds to the matrix equation:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \text{or } x_1 = 1,\ x_2 = 3$$

- **Example**: Apply elementary row operations to augmented matrices to systems of linear equations

$$I:\begin{cases}2x-y=7\\3x-4y=3\end{cases}\qquad II:\begin{cases}2x-y=7\\-4x+2y=-10\end{cases}\qquad III:\begin{cases}2x-y=7\\-4x+2y=-14\end{cases}$$

- **For system I**:

$$\left(\begin{array}{cc|c}2&-1&7\\3&-4&3\end{array}\right)\xrightarrow{R_1\leftrightarrow R_2}\left(\begin{array}{cc|c}3&-4&3\\2&-1&7\end{array}\right)\xrightarrow{R_1\to R_1-R_2}\left(\begin{array}{cc|c}1&-3&-4\\2&-1&7\end{array}\right)$$

$$\xrightarrow{R_2\to R_2-2R_1}\left(\begin{array}{cc|c}1&-3&-4\\0&5&15\end{array}\right)\xrightarrow{R_2\to R_2/5}\left(\begin{array}{cc|c}1&-3&-4\\0&1&3\end{array}\right)$$

$$\xrightarrow{R_1\to R_1-3R_2}\left(\begin{array}{cc|c}1&0&5\\0&1&3\end{array}\right)$$

which corresponds to the matrix equation:

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}5\\3\end{pmatrix},\quad \text{or } x_1=5,\ x_2=3$$

- **For system II**: $\left(\begin{array}{cc|c} 2 & -1 & 7 \\ -4 & 2 & -10 \end{array}\right) \xrightarrow{R_2 \to R_2 + 2R_1} \left(\begin{array}{cc|c} 2 & -1 & 7 \\ 0 & 0 & 4 \end{array}\right)$
- Which corresponds to the matrix equation $\begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$
- Or $\begin{cases} 2x - y = 7 \\ 0 = 4 \end{cases}$
- the second equation can never be hold, therefore, the system has no solution
- **For system III**,

$$\left(\begin{array}{cc|c} 2 & -1 & 7 \\ -4 & 2 & -14 \end{array}\right) \xrightarrow{R_2 \to R_2 + 2R_1} \left(\begin{array}{cc|c} 2 & -1 & 7 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 \to R_1/2} \left(\begin{array}{cc|c} 1 & -0.5 & 3.5 \\ 0 & 0 & 0 \end{array}\right)$$

- Or $x - 0.5y = 3.5$
- thus, the system has infinite many solutions: $t \in R$

$$\begin{cases} x = 0.5t + 3.5 \\ y = t \end{cases}$$

- A system of real linear equations either has no solutions, one (unique) solution, or infinitely many solutions.
- If the number of variables is more than the number of equations, then there must be either no solution, or infinite many solutions.

- **Exercise**. Solve the following systems of linear equations, by writing out an augmented matrix, and then using elementary row operations to transform it to reduced row echelon form:

$$\begin{cases} x_1 + x_2 + 2x_3 + 3x_4 = 7 \\ x_1 + 4x_2 - 2x_3 = -5 \\ 3x_1 + 9x_2 - 2x_3 + 3x_4 = -3 \end{cases}$$

- Elementary row operations can be used to find the inverse of a matrix

- The following statements are equivalent:
  1. A is invertible
  2. the reduced row echelon form of the matrix A is the identity matrix
  3. For $A, B \in R^{n \times n}$, if the reduced row echelon form of (A, I) is (I, B), then $B = A^{-1}$

- **Example**: For $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, find A^{-1}
  - Recall: $A^{-1} = \frac{1}{|A|} adj(A) = -\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$
  - Alternatively, use elementary row operations on $(A|I_2)$
  - $\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow{R_1 \to R_1 - R_2} \left(\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right)$
  - We obtain $A^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$

- **Exercise**: For $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 4 \\ 2 & -1 & 0 \end{pmatrix}$, find A^{-1} by performing elementary row operations on (A, I_3).

8. Rank of a matrix

- Consider the following data collected on monthly income in \$1000:

Husband's income	10	12	31	20	9	14
Wife's income	8	13	0	13	6	10
Couple's total income	18	25	31	33	15	24

- The data matrix is

$$A = \begin{pmatrix} 10 & 12 & 31 & 20 & 9 & 14 \\ 8 & 13 & 0 & 13 & 6 & 10 \\ 18 & 25 & 31 & 33 & 15 & 24 \end{pmatrix}$$

- For this data set, obviously, one row is redundant, since you can get the third row by adding up the first two rows. In a way, rank is the number of rows that are not redundant. For this data set, $rank(A) = 2$.

- A matrix is in row echelon form if each row has more leading zeros than the row above it.
- **Example**: the matrices

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 4 & 2 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 0 & 5 \\ 0 & 0 \end{pmatrix}$$

  are in row echelon form

- **Example**: the matrices

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 4 & 2 \\ 0 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 0 & 5 \\ 0 & 2 \end{pmatrix}$$

  are not in row echelon form

- The Rank of a matrix is the number of nonzero rows in its row echelon form.

- **Example**: Find the rank of the following matrix by performing elementary row operations:

$$A = \begin{pmatrix} 1 & 4 & 8 \\ 2 & 12 & 23 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 1 & 4 & 8 \\ 0 & 4 & 7 \end{pmatrix}$$

therefore, $rank(A) = 2$

- **Example**: find the rank of $A' = \begin{pmatrix} 1 & 2 \\ 4 & 12 \\ 8 & 23 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 4 \\ 0 & 7 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 4 \\ 0 & 0 \end{pmatrix}$

therefore, $rank(A') = 2$

- **Facts**: for $A \in R^{m \times n}$
  - $rank(A) = rank(A')$
  - $rank(A) \leq \min(m, n)$
  - $rank(A) \leq rank(\hat{A})$
- If $rank(A) < rank(\hat{A})$, then there is no solution to $Ax = b$
- **Example**: the following system $\begin{cases} x_1 + x_2 = 1 \\ 2x_1 + 2x_2 = 3 \end{cases}$ has no solution.
  - check that $rank(A) < rank(\hat{A})$
- If $rank(A) = rank(\hat{A}) = r$, then there is at least one solution.
  - if $r = n$, there is a unique solution; If $m = n$, then A is invertible and the unique solution is $x^* = A^{-1}b$.
  - If $r < n$, there are infinitely many solutions.

- **Example**: the following equation system

$$\begin{cases} x_1 + x_2 = 1 \\ 2x_1 + 2x_2 = 2 \end{cases}$$

Has infinitely many solutions

– note that $rank(A) = rank(\hat{A}) = 1 < 2$

- **Example**: the following system

$$\begin{cases} x_1 + x_2 = 1 \\ 2x_1 + 3x_2 = 2 \end{cases}$$

has a unique solution.

– note that $rank(A) = rank(\hat{A}) = 2$