ECON 3113 Microeconomic Theory I
Lecture 3: Structural Properties of Preferences and Utility Functions

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Review and Roadmap

- We have seen that (complete transitive) preference relation and utility functions are close cousins:

$$x \succsim y \text{ if and only if } u(x) \geq u(y).$$

- We have also seen that coherent choice behaviors can be fully explained/rationalized by a utility function.
- Specialize to consumer's problem: buying a bundle of goods given the respective prices and expendable income.
- If we add structures to the utility function, what can we say about the implied choice behaviors?
- If we add structures to the choice behaviors, what can we say about the utility function? (save for later...)

We will be looking into these properties of preferences...

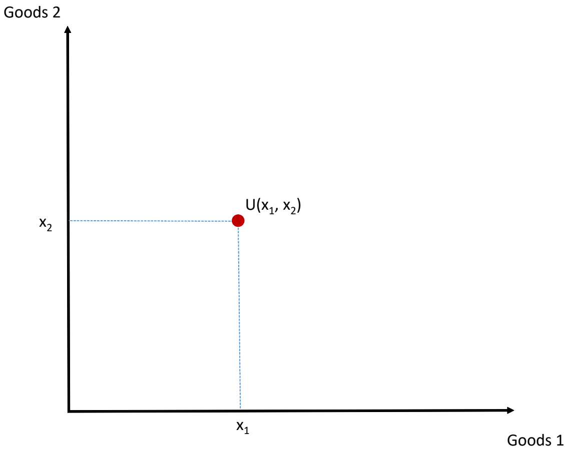
- Monotonicity: more is better
- Continuity: no jumps
- Convexity: balanced consumption is better than extremes

Basics

- There are n (infinitely divisible) goods available for consumption.
- The consumption set is $X = \mathbb{R}^n_+$, the set of all nonnegative n-dimensional lists/vectors.
- A generic consumption bundle is $x = (x_1, x_2, ..., x_n)$, where $x_i \geq 0$ represents the quantity of goods i in the bundle.
- We write $x \geq y$ if $x_i \geq y_i$ for every goods i.
- We write $x \neq y$ if $x_i \neq y_i$ for at least one goods i.
- Preferences $\succsim$ and utility function u are defined over X.
    - We will maintain the assumption that the consumer's preference is complete and transitive.

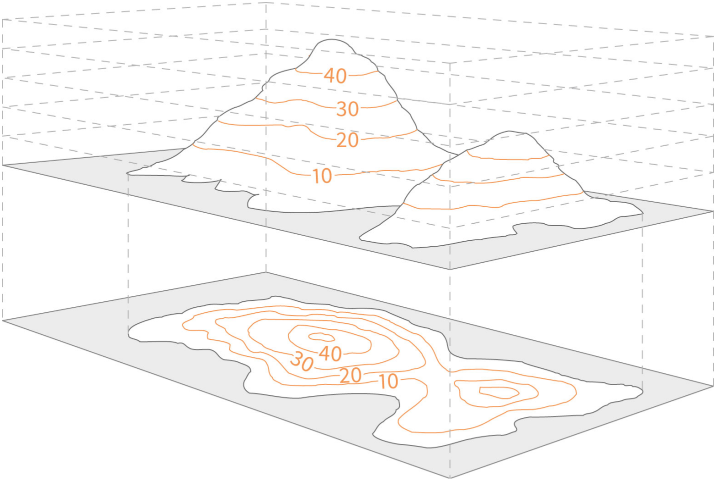
Indifference Curve Diagram

- If $X = \mathbb{R}^2_+$, indifference curve diagrams are often tremendously helpful in gaining intuition.
- The commodity space:



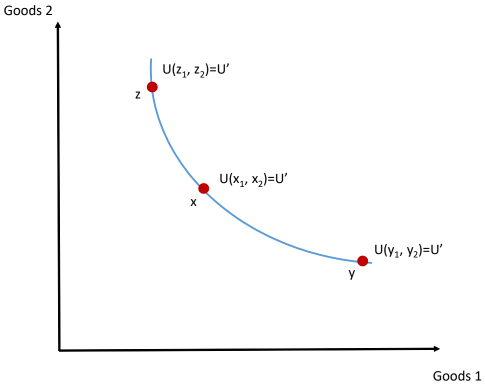
Indifference Curves are Contour Lines

- An indifference curve connects all bundles that the consumer finds indifferent to (derives the same utility level).



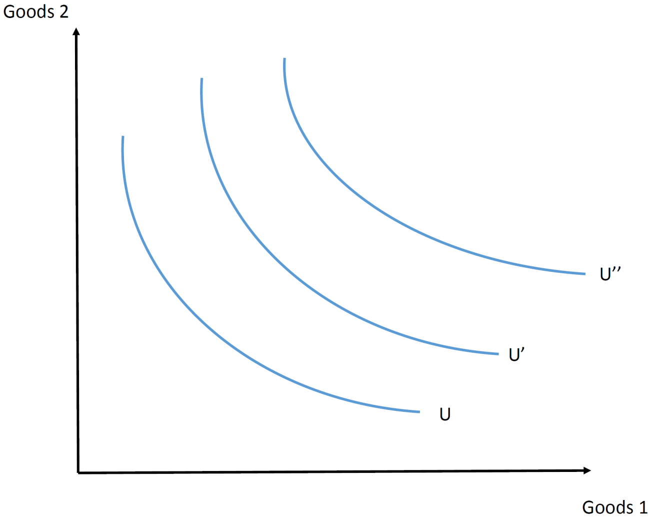
Indifference Curve Diagram

- Because of the completeness of preference, **every point in the commodity space sits on some indifference curve**.

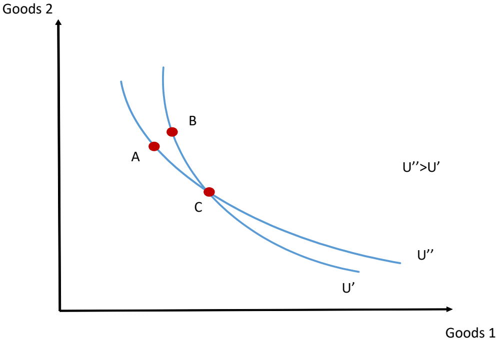


Indifference Curve Diagram

- Fixing a utility representation of the preference, **every indifference curve has a distinct utility level**.



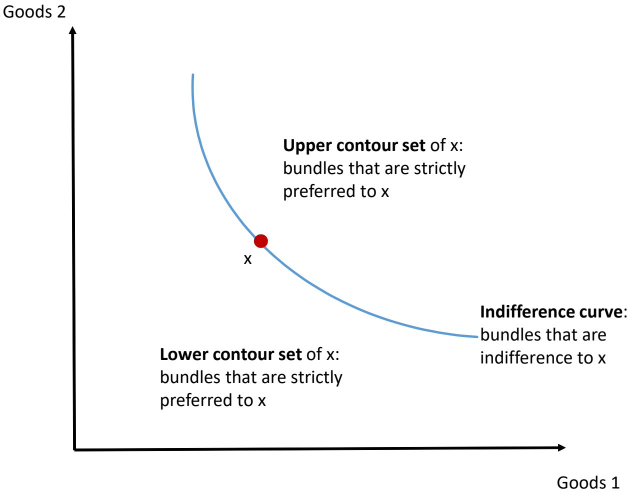
Indifference Curves Do Not Cross



- Point C has two utility levels?
- As $A \sim C$ and $B \sim C$, transitivity implies $A \sim B$, so A and B must be on the same indifference curve. But the diagram shows $A \succ B$, which can't be right.

Indifference Curves Diagram

- An indifference curve splits the commodity plane into three regions.



Monotonicity

- If the consumer prefers more to less, we say her preference is monotone.

Definition

Preference relation $\succsim$ is **monotone** if $x \succsim y$ for any two bundles x and y such that $x \geq y$.
It is **strictly monotone** if $x \succ y$ whenever $x \geq y$ and $x \neq y$.

Monotonicity

- And we say she has a nondecreasing utility function.

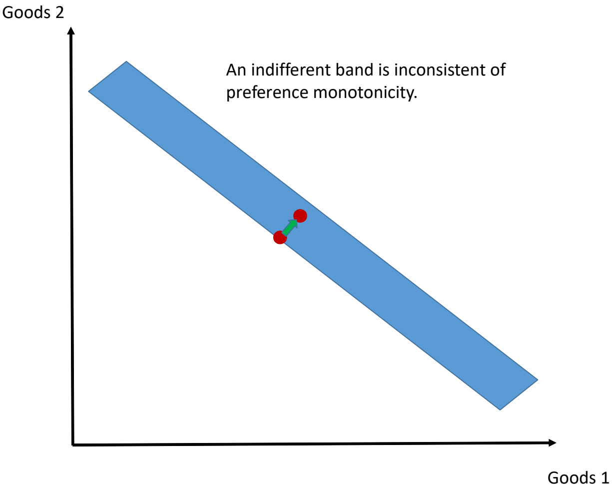
Definition

A utility function u is **nondecreasing** if $u(x) \geq u(y)$ for any two bundles x and y such that $x \geq y$.
It is **strictly increasing** if $u(x) > u(y)$ whenever $x \geq y$ and $x \neq y$.

- If preference relation $\succsim$ can be represented by utility function u, then
  - $\succsim$ is monotone if and only if u is nondecreasing;
  - $\succsim$ is strictly monotone if and only if u is strictly increasing.

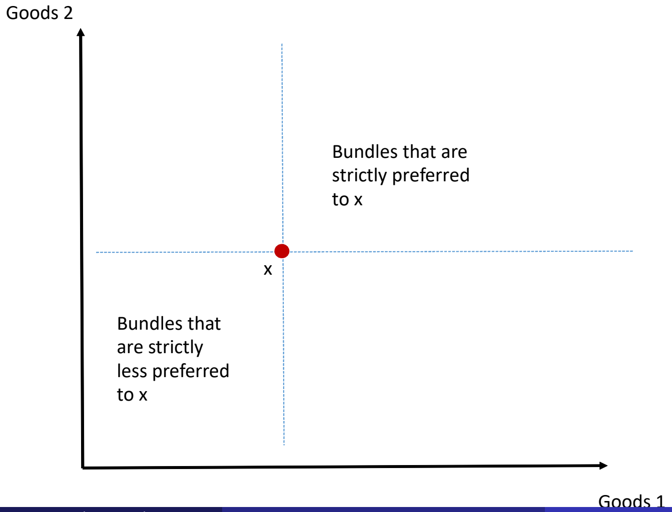
Indifference Curves Diagram: Monotonicity

- If the preference is strictly monotone, its indifference curves have no "width".



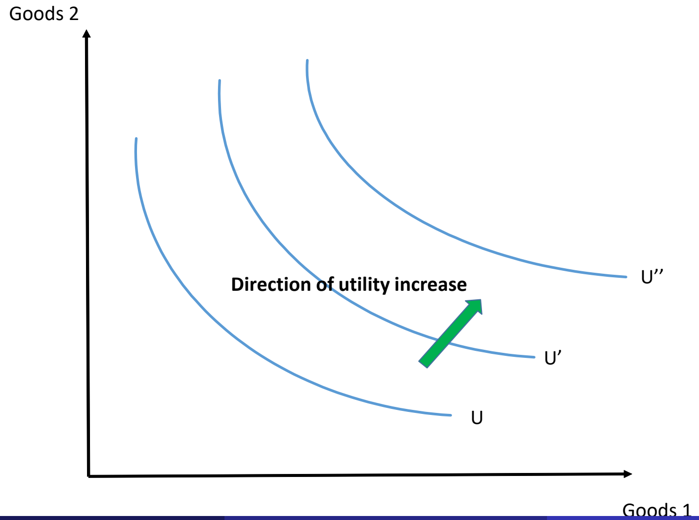
Indifference Curves Diagram: Monotonicity

- If the preference is (strictly) monotone, its indifference curves are (strictly) downward sloping.



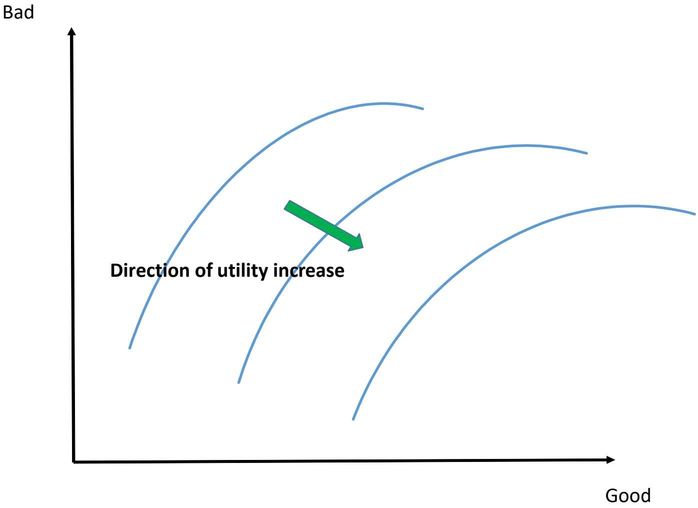
Indifference Curves Diagram: Monotonicity

- If the preference is (strictly) monotone, utility (strictly) increases in the northeast direction.



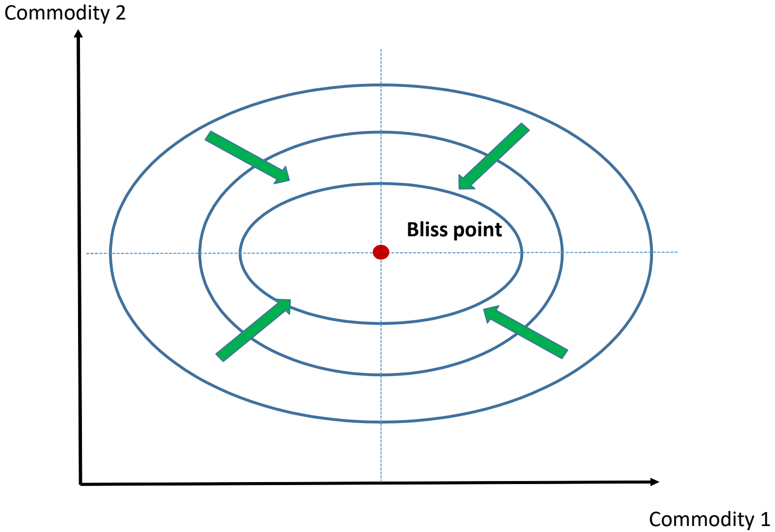
Nonmonotone Preference

- A commodity can be "economic bad."



Nonmonotone Preference

- Whether a commodity is a good or bad can depend on its quantity.



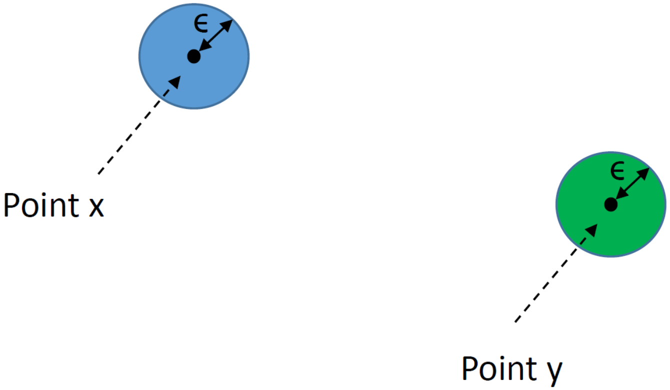
Continuity

- If the consumer's preferences do not show abrupt changes/jumps, then her preference is continuous.

Definition

A complete transitive preference $\succsim$ is **continuous** if, for every pair of alternatives x and y from X with $x \succ y$, we can always find two small balls, one containing x and one containing y, such that any alternative in the former ball is strictly preferred to any alternative in the latter ball.

Preference: Continuity

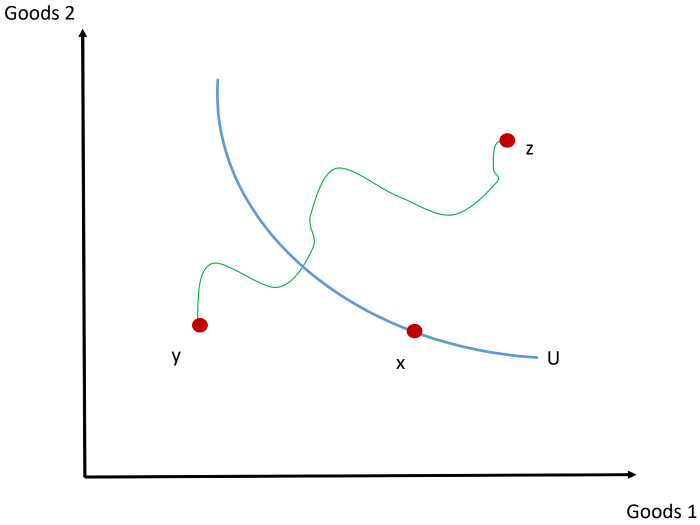


- If $x \succ y$, we can find small balls such that anything in the blue ball is strictly preferred to everything in the green ball.

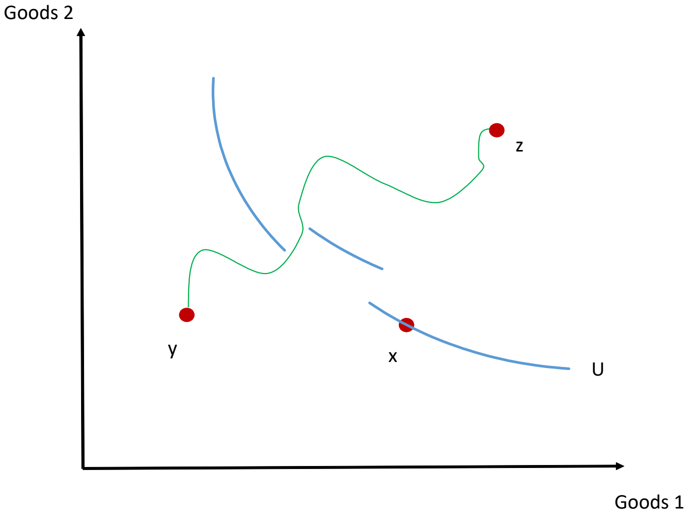
Indifference Curve Diagram: Continuity

- Continuity of preferences implies that indifference curves are continuous and do not "run out" (except at the boundary of the commodity space).
- Suppose $z \succ x \succ y$. Take a continuous path from y to z. We must cross the indifference curve of x.
    - If not, then there must be at least one "switching point" w and let's say $w \prec x$. Right beyond w, bundles (on the continuous path) are strictly preferred to x.
    - But this contradicts the continuity of preference, which requires that we can find a small ball containing w such that everything in the ball is strictly less preferred to x.

Indifference Curve Diagram: Continuity



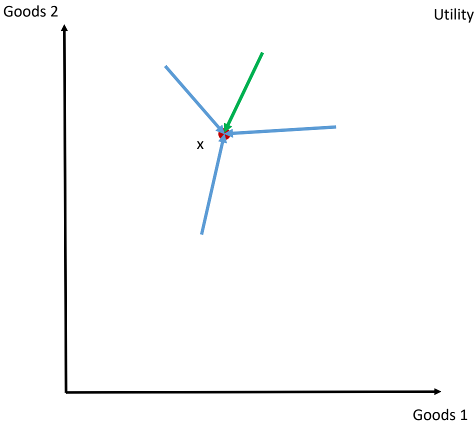
Indifference Curve Diagram: Discontinuity

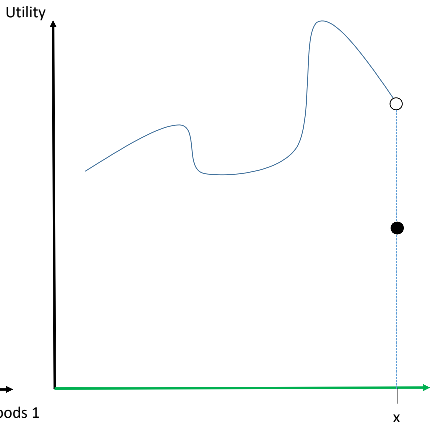


Continuous Functions

- Suppose $u(\cdot)$ is a utility function that has a jump at $x \in X$.
- It means that we can find a path approaching x such that the plot of utility against the path has a jump at x.
- A utility function is continuous if there is no jump at any $x \in X$.

Jump



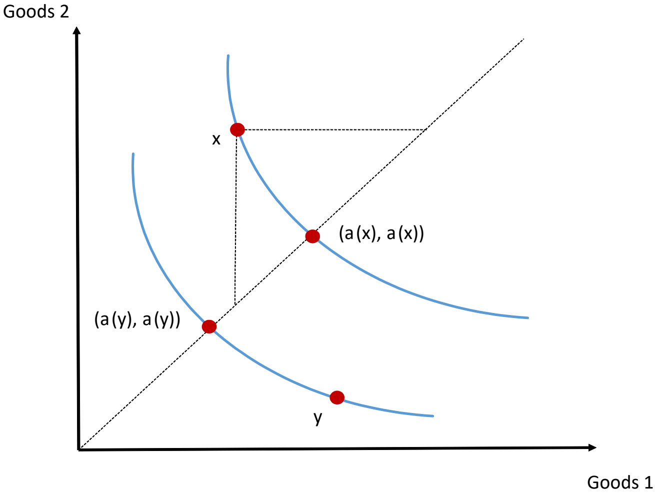


Utility Representation of Continuous Preference

Theorem (Debreu's Theorem)

If a preference relation is complete, transitive and continuous, then there exists a continuous utility function representing it. Conversely, if the utility function is continuous, then the implied preference relation is complete, transitive and continuous.

Sketching the Proof of Debreu's Theorem



Sketching the Proof of Debreu's Theorem

- Though not necessary, the proof is easier if we assume the preference is strictly monotone.
- Every point on the diagonal takes the form (α, α), where α is some real number.
- Let x be some bundle. By continuity, there is a bundle along the diagonal that is indifferent to x.
    - The indifferent curve through x must cut the diagonal at some point, say $(\alpha(x), \alpha(x))$.
    - Strict monotonicity implies that the intersection occurs at one and only one point.
- Take another bundle y and there is some diagonal bundle $(\alpha(y), \alpha(y))$ that is indifferent to y.

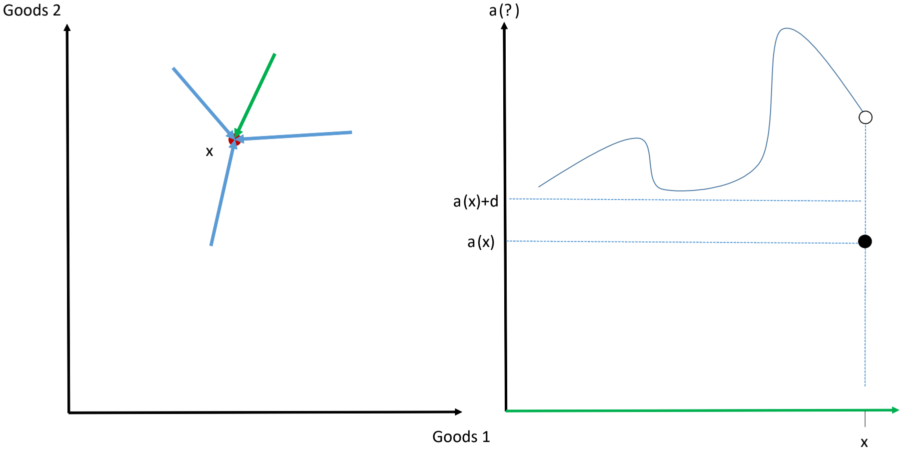
Sketching the Proof of Debreu's Theorem

- Suppose $\alpha(x) > \alpha(y)$ as in the diagram. Then strict monotonicity implies that $(\alpha(x), \alpha(x)) \succ (\alpha(y), \alpha(y))$.
- As $x \sim (\alpha(x), \alpha(x))$ and $y \sim (\alpha(y), \alpha(y))$, transitivity implies $x \succ y$.
- Therefore, the function $\alpha(\cdot)$ gives us a utility measure:
  - we have seem above that whenever $\alpha(x) > \alpha(y)$, we know $x \succ y$.
  - conversely, if $x \succ y$, we know from transitivity that $(\alpha(x), \alpha(x)) \succ (\alpha(y), \alpha(y))$ and from strict monotonicity that $\alpha(x) > \alpha(y)$.
- It remains to show that this utility function $\alpha(\cdot)$ we create is continuous.

Sketching the Proof of Debreu's Theorem

- Suppose the utility function α has a jump at some x.
- This means we can find a path approaching x such that the plot of α against the path has a jump at x. Say it jumps down.
    - The path may start really close to x.
- Along this path, all bundles have utilities strictly exceeding $\alpha(x) + \delta$.
- On the other hand, as $x \sim (\alpha(x), \alpha(x))$, we know $x \prec (\alpha(x) + \delta, \alpha(x) + \delta)$.
- As preference is continuous, we can find a small ball around x such that all bundles within the ball are strictly less preferred than $(\alpha(x) + \delta, \alpha(x) + \delta)$, which means that these bundles must then have utilities strictly less than $\alpha(x) + \delta$.
- But then the path we identified must eventually enter this ball; a contradiction.

Sketching the Proof of Debreu's Theorem



Convexity

- If the consumer prefers a balanced bundle to extreme bundles, we say her preference is convex.
- A lady decides between two suitors: Mr. Left and Mr. Right.
- The lady cares about the partner's intelligence and sense of responsibility.
- After some trial periods, the lady concludes that the suitors' "scores" are as follows.

	Intelligence	Sense of responsibility
Mr. Left	90	30
Mr. Right	10	90

- The lady struggles to decide which guy to pick/how to rank them.

Convexity

- Now suppose a third suitor emerges, and his scores are

	Intelligence	Sense of responsibility
Mr. Middle	50	60

- Mr Middle represents a balanced bundle.
- If the lady has a convex preference, then she would probably go for Mr. Middle.

Convexity

- If she has a convex preference and is indifferent between (x_1, x_2) and (y_1, y_2), she would prefer $\left(\frac{1}{2}(x_1 + y_1), \frac{1}{2}(x_2 + y_2)\right)$ to either (x_1, x_2) or (y_1, y_2). More generally,

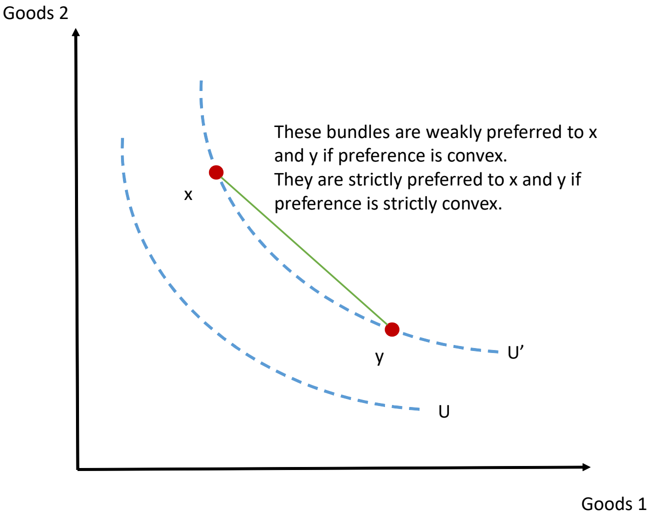
Definition

Preference relation $\succsim$ is **convex** if for every pair of bundles x and y such that $x \sim y$ and for every $a \in (0, 1)$, $ax + (1 - a) y \succsim y$.
It is **strictly convex** if for every pair of bundles x and y such that $x \sim y$ and for every $a \in (0, 1)$, $ax + (1 - a) y \succ y$.

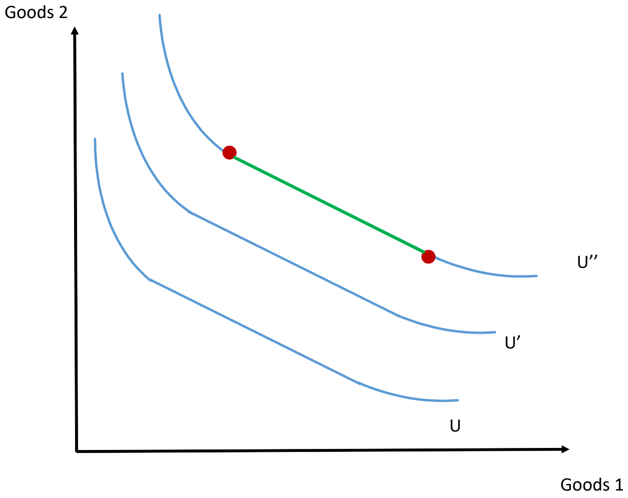
Indifference Curve Diagram: Convexity

- Take a pair of bundles x and y that are on the same higher indifference curve.
- Consider moving along a line segment from y to x.
- Convexity of preference means that we are never worse off, at any point along the line segment, than the two end points.
- Strict convexity means that we are strictly better off, at any point along the line segment, than the two end points.

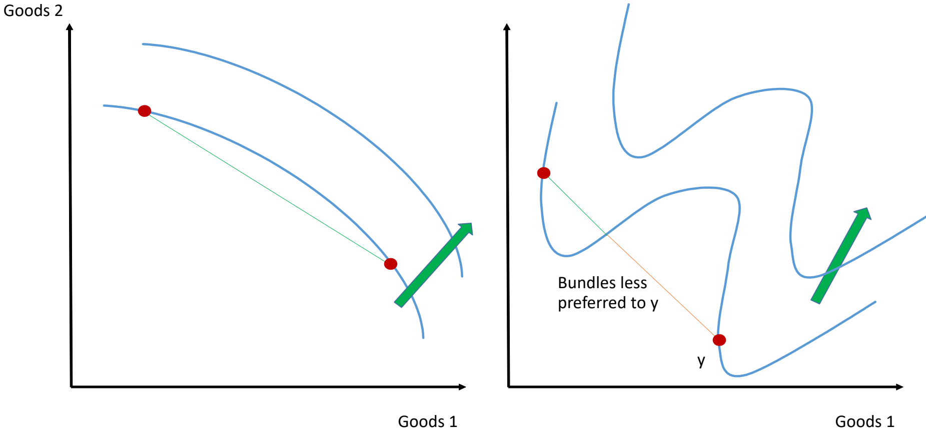
Indifference Curve Diagram: Convexity



Preference that is Convex but Not Strictly Convex



Non-Convex Preferences

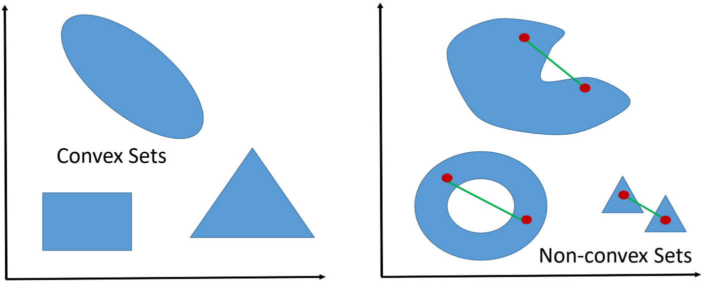


Convex Sets

- Convex preferences are tightly connected to the notion of convex sets in math.

Definition

A set S is convex if the straight line connecting any two points in S lies completely in S. That is, for any pair $x, y \in S$, we have $ax + (1-a)\,y \in S$ for any $a \in [0,1]$.



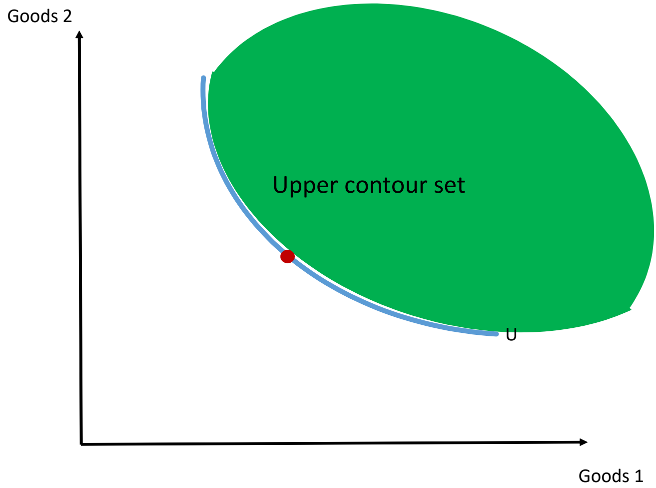
Convex Preferences have Convex Upper Contour Sets

- The upper contour set of bundle x is defined as $\{z \in X : z \succsim x\}$.
- The convexity of preference can be equivalently defined as the convexity of upper contour sets.

Definition

Preference relation $\succsim$ is **convex** if the upper contour set of every bundle is convex.

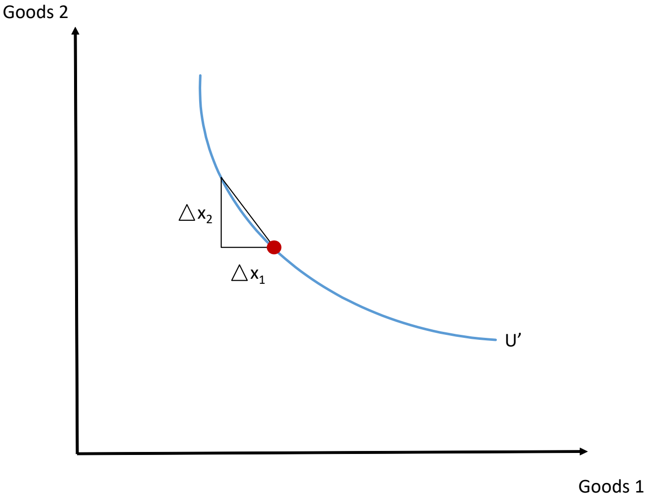
Convex Upper Contour Sets



Marginal Rate of Substitution

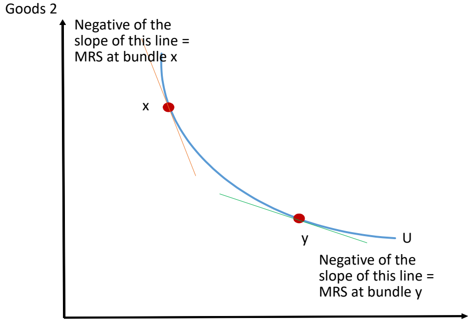
- Start with some initial consumption bundle, say (x_1, x_2).
- Now, let's take away $\triangle_1$ units of good 1 from the bundle, how many units of goods 2 do we need to compensate the consumer to keep him indifferent?
- That is, what is the value of $\triangle_2$ so that $(x_1, x_2) \sim (x_1 - \triangle_1, x_2 + \triangle_2)$?
- This ratio $\triangle_2 / \triangle_1$ is the rate at which the consumer is willing to **substitute goods** 2 **for goods** 1**.**
- If we take $\triangle_1$ to be extremely small, the ratio $\triangle_2 / \triangle_1$ is called the **marginal rate of substitution (MRS)**.
- The value of MRS, in general, depends on the bundle we evaluate it.

Marginal Rate of Substitution



Marginal Rate of Substitution

- In words, MRS is the maximum amount of goods 2 that the consumer is willing to trade for an extra unit of goods 1.
  - Flipping goods 1 and 2, the max. amount of goods 1 she is willing to trade for an extra unit of goods 2 is given by 1/MRS.
- Graphically, MRS at a bundle (x_1, x_2) can be read off by the slope of the indifference curve passing through (x_1, x_2).
  - Note the difference in sign.



Marginal Utility

- Given a utility function u and let's start with some initial consumption bundle, say (x_1, x_2).
- By how much does her utility change if we increase her goods 1 by an extremely small amount $\triangle_1$?

$$MU_1 = \frac{\triangle u}{\triangle_1} = \frac{u(x_1 + \triangle_1, x_2) - u(x_1, x_2)}{\triangle_1}.$$

- The rate of change in the consumer's utility with respect to goods 1 is called the marginal utility (MU).
  - Again, it depends on the bundle we evaluate it.
  - It also depends on the specification of utility function u.

Marginal Rate of Substitution

- Start with some consumption bundle, say (x_1, x_2), and we take away $\triangle_1$ units of goods 1 and add $\triangle_2$ units of goods 2 to the bundle in such a way that the utility is preserved.
- How many units of goods 2 is needed for compensating each unit of goods 1?

$$
\begin{aligned}
& 0 = -MU_1 \times \triangle_1 + MU_2 \times \triangle_2 \\
\Leftrightarrow \quad & \frac{\triangle_2}{\triangle_1} = \frac{MU_1}{MU_2}.
\end{aligned}
$$

- Therefore, MRS at a bundle (x_1, x_2) is equal to the ratio MU_1 / MU_2 evaluated at the bundle.

Marginal Rate of Substitution

- Formally, MU_1 is the partial derivative of the utility function u with respect to x_1. In notation,

$$MU_1(x) = \frac{\partial}{\partial x_1} u(x) = u_1(x_1, x_2).$$

- Likewise,

$$MU_2(x) = \frac{\partial}{\partial x_2} u(x) = u_2(x_1, x_2).$$

- Therefore, the marginal rate of substitution at bundle x is

$$MRS(x) = \frac{MU_1(x)}{MU_2(x)} = \frac{u_1(x_1, x_2)}{u_2(x_1, x_2)}.$$

Marginal Rate of Substitution

- Recall any monotone transformation of utility function preserves the preference relation.
- Recall also that MRS is a property of the preference relation.
- Therefore, MRS should be invariant to any monotone transformation of utility function.
- Formally, let f be a strictly increasing function. Then $u(x)$ and $f(u(x))$ are equivalent. Using $f(u(x))$ to compute the MRS gives:

$$MRS(x) = \frac{\frac{\partial}{\partial x_1} f(u(x))}{\frac{\partial}{\partial x_2} f(u(x))} = \frac{f'(u(x)) \frac{\partial}{\partial x_1} u(x)}{f'(u(x)) \frac{\partial}{\partial x_2} u(x)} = \frac{u_1(x)}{u_2(x)},$$

where we have invoked the chain rule of differentiation.

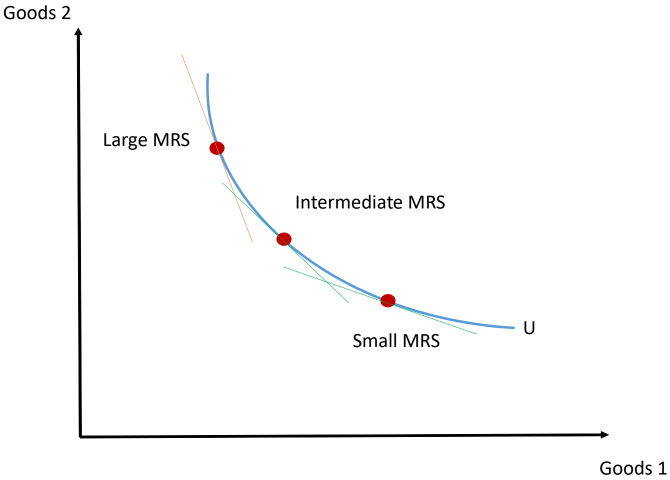
Diminishing Marginal Rate of Substitution

- A consumer's preference satisfies **diminishing marginal rate of substitution (DMRS)** if MRS is decreasing in the quantity of good 1 in the bundle (holding utility constant).
- This is intuitive: when you have little goods 1, you really want them and are willing to give up a lot of other things (goods 2) to get them.
- When you have lots of goods 1, you don't really want them anymore and are willing to give up little other things (goods 2) to get them.

Diminishing Marginal Rate of Substitution

- DMRS means that the negative of the slope of every indifference curve is decreasing as x_1 increases.
- Mathematic fact: a function with increasing derivative is strictly convex.
- DMRS implies indifference curves are strictly convex, which in turn implies that the preference is strictly convex.

Diminishing Marginal Rate of Substitution



Quasi-concavity of Utility Function

- A function $f : X \to \mathbb{R}$ is **quasi-concave** if for all $x, y \in X$ and $a \in (0,1)$, $f\left(ax + (1-a)\,y\right) \geq \min\left\{f\left(x\right), f\left(y\right)\right\}$. It is strictly quasi-concave if the inequality is strict.
- A preference is (strictly) convex if and only if its utility function is (strictly) quasi-concave.

Example: Cobb-Douglas

- A general Cobb-Douglas utility function is

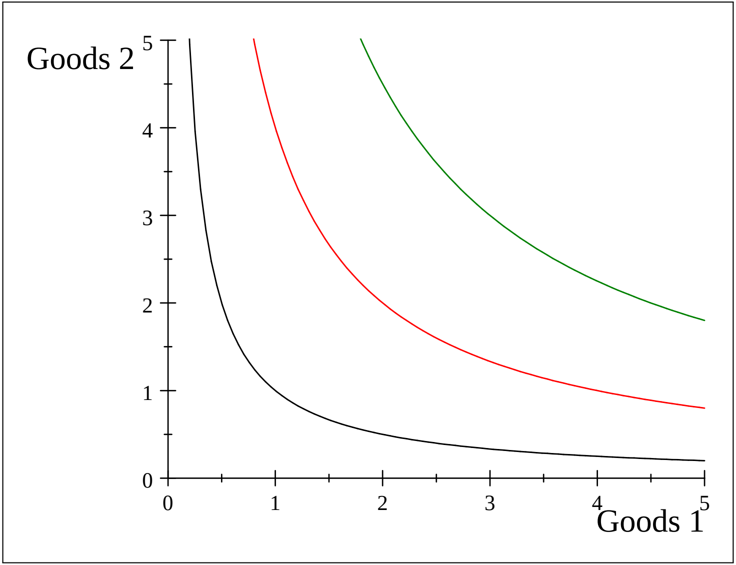
$$u(x_1, x_2, ..., x_n) = x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n},$$

  for some positive constants α_1, α_2,...,α_n.
- With two goods, a Cobb-Douglas utility function is simply

$$u(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2}.$$

- It is clear that Cobb-Douglas utility function is increasing and strictly so in the interior of the commondity space (i.e., $x_1, x_2 > 0$).

Example: Cobb-Douglas



Example: Cobb-Douglas

- The marginal rate of substitution is

$$MRS = \frac{\partial u / \partial x_1}{\partial u / \partial x_2} = \frac{\alpha_1 x_1^{\alpha_1 - 1} x_2^{\alpha_2}}{\alpha_2 x_1^{\alpha_1} x_2^{\alpha_2 - 1}} = \frac{\alpha_1}{\alpha_2} \frac{x_2}{x_1}.$$

- Applying the transformation $f(z) = z^{\frac{1}{\alpha_1 + \alpha_2}}$:

$$U(x_1, x_2) = f(u(x_1, x_2)) = \left(x_1^{\alpha_1} x_2^{\alpha_2}\right)^{\frac{1}{\alpha_1 + \alpha_2}} = x_1^{\frac{\alpha_1}{\alpha_1 + \alpha_2}} x_2^{\frac{\alpha_2}{\alpha_1 + \alpha_2}} = x_1^{\beta} x_2^{1-\beta},$$

where $\beta = \frac{\alpha_1}{\alpha_1 + \alpha_2}$.

- The MRS of U:

$$MRS_U = \frac{\partial U / \partial x_1}{\partial U / \partial x_2} = \frac{\beta}{1 - \beta} \frac{x_2}{x_1} = \frac{\frac{\alpha_1}{\alpha_1 + \alpha_2}}{1 - \frac{\alpha_1}{\alpha_1 + \alpha_2}} \frac{x_2}{x_1} = \frac{\alpha_1}{\alpha_2} \frac{x_2}{x_1}.$$

- For this reason, we often assume the coefficients α_1 and α_2 add up to 1.

Example: Cobb-Douglas

- To determine convexity property of preference, we ask whether MRS decreases in x_1 along the indifference curves.
- Fix an indifference curve with utility $\bar{u}$:

$$u\left(x_1, x_2\right) = x_1^{\alpha_1} x_2^{\alpha_2} = \bar{u}.$$

- We can rewrite this indifference curve into

$$x_2 = \left(\bar{u} x_1^{-\alpha_1}\right)^{\frac{1}{\alpha_2}}.$$

- The MRS of a bundle on the indifference curve as a function of x_1:

$$MRS = \frac{\alpha_1}{\alpha_2}\frac{x_2}{x_1} = \frac{\alpha_1}{\alpha_2}\frac{\left(\bar{u} x_1^{-\alpha_1}\right)^{\frac{1}{\alpha_2}}}{x_1} = \frac{\alpha_1}{\alpha_2}\bar{u}^{\frac{1}{\alpha_2}} x_1^{-\left(\frac{\alpha_1}{\alpha_2}+1\right)},$$

  which is clearly decreasing in x_1, so Cobb-Douglas preference is strictly convex.

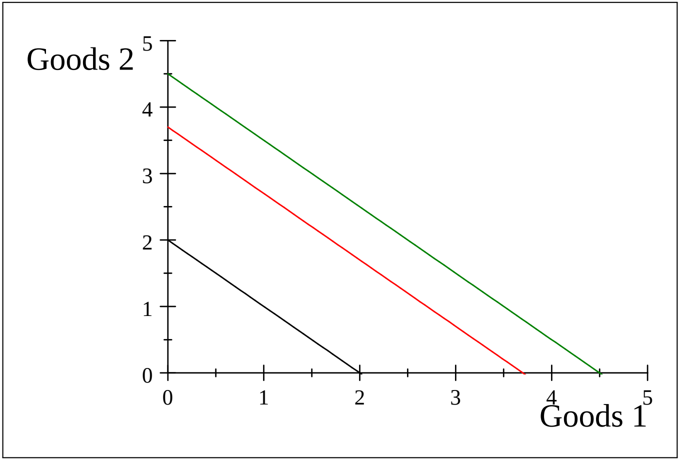
Example: Perfect Substitutes

- If two goods are perfect substitutes, the utility function is given by

$$u(x_1, x_2) = x_1 + x_2.$$

- E.g., Coke and Pepsi, different brands of toilet papers (of similar quality).
- As the utility is strictly increasing, the preference is strictly monotone.
- The MRS is constant at 1; the preference is convex but not strictly convex.

Example: Perfect Substitutes



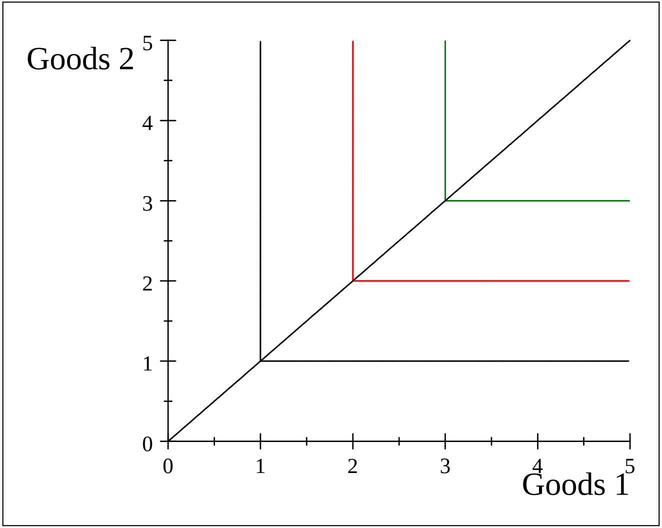
Example: Perfect Complements

- If two goods are perfect complements, the utility function is given by

$$u(x_1, x_2) = \min\{x_1, x_2\}.$$

- E.g., bread and jam, phone and OS
- The MRS is 0 below the 45-degree line, ∞ above the 45-degree line, and undefined along the 45-degree line.
    - The preference is convex but not strictly convex.

Example: Perfect Complements



Example: Quasi-Linearity

- A general quasi-linear utility function is

$$u(x_1, x_2) = v(x_1) + x_2,$$

for some strictly increasing and strictly concave function v.

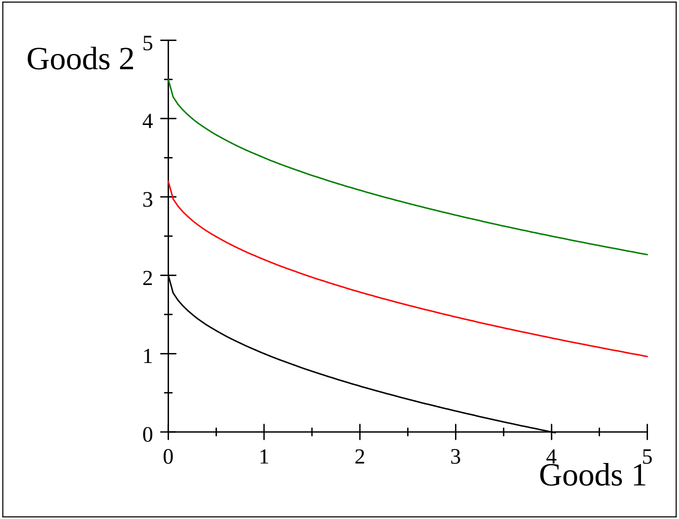
- The preference is clearly strictly monotone.
- The MRS is

$$MRS = \frac{\partial u / \partial x_1}{\partial u / \partial x_2} = \frac{v'(x_1)}{1} = v'(x_1),$$

which depends only on x_1.

- The indifference curves are parallel vertical shifts of one another.
- Moving along an indifference curve, the MRS is decreasing in x_1 because v is strictly concave.
  - The preferene is thus strictly convex and satisfies DMRS.

Example: Quasi-Linearity



Takeaways

1. Monotonicity of preference (more is better) implies
   - ICs have no width;
   - ICs are downward sloping.
2. Adding continuity (no jumps), we get
   - ICs are continuous;
   - utility function is continuous.
3. Adding further convexity (balanced consumption is better than extremes), we get
   - ICs are convex;
   - Diminishing Marginal Rate of Substitution (with strict convexity).