Homework 6 Solution

Choose the best answer

1. Which statement is **true** of a typical Battle of sexes game?
   a. The game has a unique Nash equilibrium.
   b. A player may receive lower expected payoff in the mixed-strategy Nash equilibrium than some other equilibrium.
   c. The follower has an advantage in the sequential version.
   d. None of the above.

2. In a mixed-strategy Nash equilibrium, a player is willing to randomize because
   a. this confuses opponents.
   b. he or she is indifferent between the actions in equilibrium.
   c. the actions provide the same payoffs regardless of what the other player does.
   d. he or she does not know what the other player is doing.

3. In the mixed-strategy Nash equilibrium of the following game in which players randomize between B and C and do not play A at all, what is the probability that player 1 plays B?

		2		
		A	B	C
	A	4, 4	0, 5	0, 0
1	B	5, 0	1, 1	0, 0
	C	0, 0	0, 0	3, 3

   a. 3/4.
   b. 1/2.
   c. 1/4.
   d. 1/3.

4. The pure-strategy Nash equilibrium is social optimal in the situation of
   a. Prisoner dilemma.
   b. Battle of sexes.
   c. Cournot duopoly competition.
   d. None of above.

Analytical questions

1. Find all pure strategy Nash equilibrium of the following games

		2			
		C1	C2	C3	C4
	R1	2, 6	$\underline{5}$, 7	2, 5	6, $\underline{9}$
1	R2	5, 5	3, 2	$\underline{5}$, $\underline{6}$	2, 4
	R3	4, 4	4, $\underline{9}$	3, 4	$\underline{7}$, 2
	R4	$\underline{6}$, $\underline{8}$	4, 5	4, 2	6, 3

(R2, C3), (R4, C1).

2. Consider the following Battle of sexes game:

		Woman	
		Action $[c]$	Drama
Man	Action $[r]$	4, 2	1, 1
	Drama	0, 0	2, 4

Both the man and the woman would like to spend their time together at the cinema, but they have different preference on the type of movie.

a. Find all Nash equilibrium (including mixed strategy).

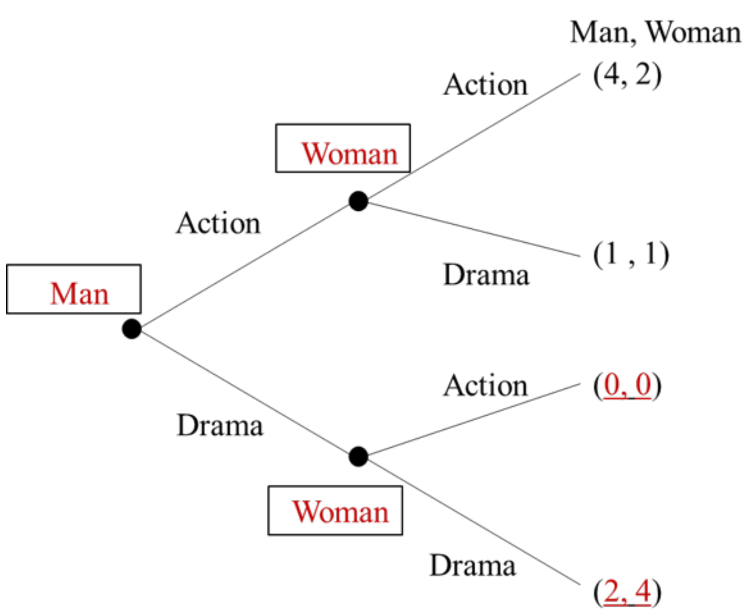
Clearly, there are two pure strategy NE (Action, Action), (Drama, Drama) Use indifference condition to find mixed strategy NE:

$$\pi_M(A) = \pi_M(D) \Rightarrow 4c + (1-c)1 = 2(1-c) \Rightarrow c^* = \frac{1}{5}$$

$$\pi_W(A) = \pi_W(D) \Rightarrow 2r = r + (1-r)4 \Rightarrow r^* = \frac{4}{5}$$

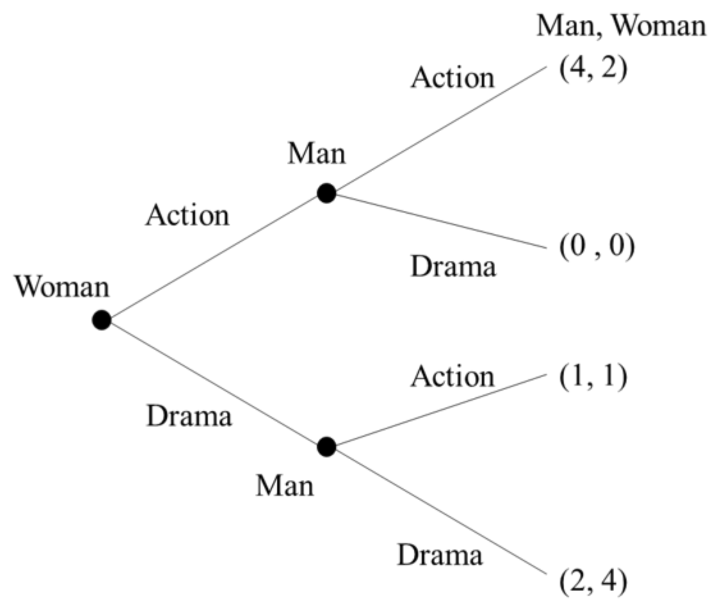
Therefore, the mixed strategy NE is $(r^* = \frac{4}{5},\ c^* = \frac{1}{5})$.

b. If the man arrive the cinema first and decides to purchase tickets before the woman arrives, what will be the equilibrium (SPE)? Finish the game tree (extensive form).



The equilibrium will be (Action, (Action, Drama)) because now the man can commit to play Action.

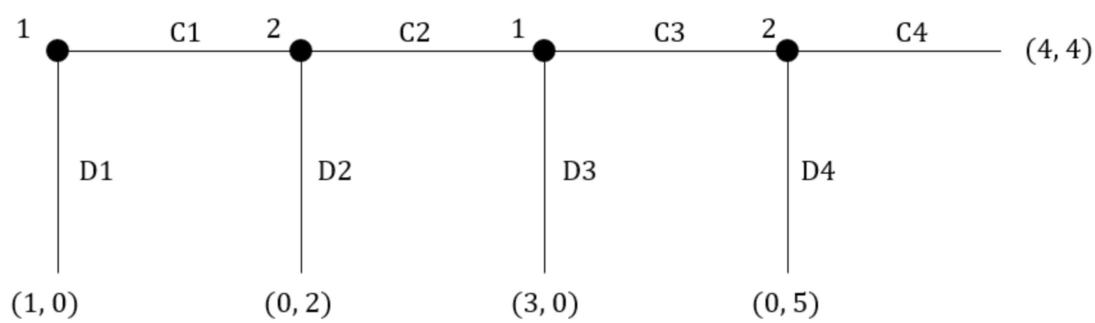
c. If the woman arrive the cinema first and decides to purchase tickets before the man arrives. Draw the complete extensive form of the game and find the subgame perfect equilibrium.



Use backward induction, the outcome of the game is (Drama, Drama)

The SPE is ((Action, Drama), Drama,).* (It means man play Action at the top decision node, and Drama at the bottom decision node.)

3. Consider the following game



a. Predict the outcome of this game.

By backward induction, we can easily get the outcome is D1.

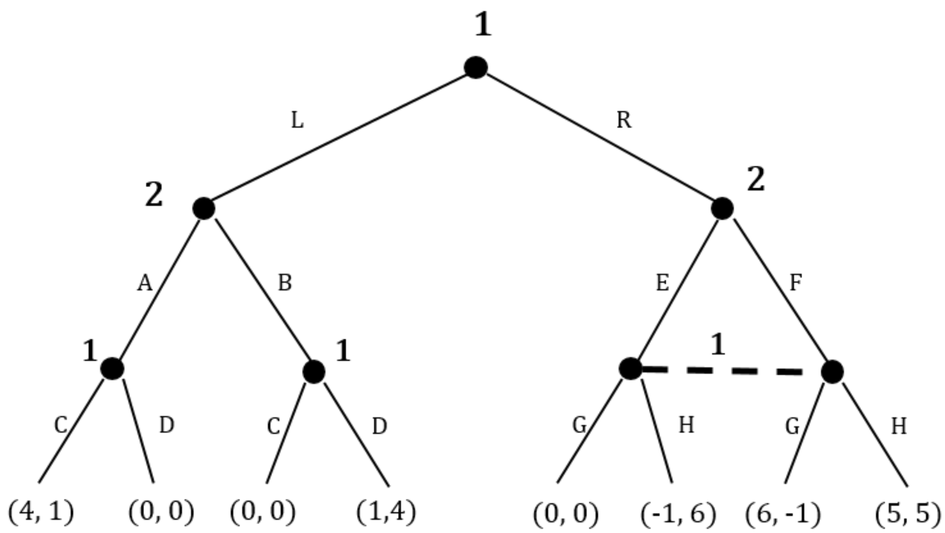
b. Can you revise only one payoff in the game so that the outcome becomes (C1,C2,C3,C4)?

Can make the payoff of D4 be $(0,\ x)$ with $x < 4$.

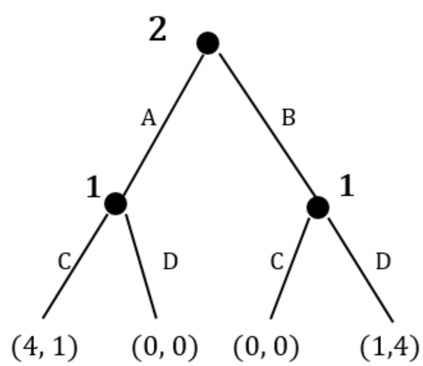
Or can make the payoff of C4 be $(4,\ y)$ with $y > 5$.

Student just need to come up with one number that make player 2 changes its choice at the last decision node.

4. Solve the following game following steps (a) (b) (c). Note that for each payoff (u_1, u_2), the left-hand side number is for player 1; and the right-hand side number is for player 2.

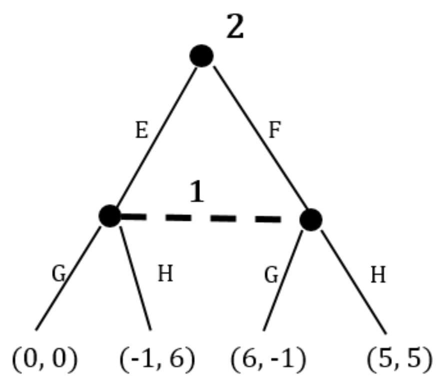


a. Predict the outcome if player 1 selects L by analyzing the following game.



By backward induction, the outcome is (B,D) (or (D,B)).

b. Predict the outcome if player 1 selects R by analyzing the following game. To do so, first turn this game into a normal form. Then, find the Nash equilibrium.



		2	
		E	F
1	G	0, 0	—, —
	H	—, —	—, —

		2	
		E	F
1	G	0, 0	6, -1
	H	-1, 6	5, 5

The unique Nash equilibrium is (G,E).

c. Predict the outcome of the entire sequential game.

If player 1 choose L, he gets 1. If player 1 chooses R, he gets 0. So player 1 will choose L.

The outcome of entire sequential game is (L,B,D) (or (L,D,B)).

5. Consider the following game in which two countries choose between Peace and War.

		Country 2	
		Peace	War
Country 1	Peace	2, 2	-3, x
	War	x, -3	-2, -2

a. Find all pure-strategy NE when $x = 1$.

		Country 2	
		Peace	War
Country 1	Peace	$\underline{2}$, $\underline{2}$	-3, 1
	War	1, -3	-$\underline{2}$, -$\underline{2}$

There are two pure-strategy NEs (Peace, Peace) and (War, War).

b. Find all pure-strategy NE when $x = 3$.

		Country 2	
		Peace	War
Country 1	Peace	2, 2	-3, $\underline{3}$
	War	$\underline{3}$, -3	-$\underline{2}$, -$\underline{2}$

There is one pure-strategy NE (War, War).

c. Continue with part (b) with $x = 3$. Suppose that the two countries play this game with infinitely number of times with discount factor $\delta \in (0, 1)$. What is the minimum discount factor δ that can induce the two countries to stay peace based on trigger strategies? Write down the trigger strategy for $t = 0, 1, 2.....$

For country 1, the trigger strategy is

$$\begin{cases} t = 0 & \text{Peace,} \\ t = 1, 2, ... & \begin{cases} \text{Peace} & \text{if (Peace,Peace) at } t-1, \\ \text{War} & \text{otherwise.} \end{cases} \end{cases}$$

In the "cooperative" outcome, $\pi^C = 2$. In the deviation outcome, $\pi^D = x = 3$. In the NE outcome, $\pi^{NE} = -2$

$$\delta \geq \frac{\pi^D - \pi^C}{\pi^D - \pi^{NE}} = \frac{3-2}{3-(-2)} \equiv \frac{1}{5} = \delta_{\min}.$$

d. Continue with part (c). If x increases, is it easier or harder to maintain peace between these two countries? Prove it.

The payoff from deviation is $\pi^D = x$. So when x increases, it is more attractive to deviate. Intuitively, it is harder to maintain peace.

Formally,

$$\delta_{\min} = \frac{\pi^D - \pi^C}{\pi^D - \pi^{NE}} \frac{x-2}{x-(-2)} \equiv \frac{x-2}{x+1} = 1 - \frac{3}{x+1},$$

which is increasing in x. So it is harder to have $\delta \geq \delta_{\min}$ when x becomes large.

6. A firm has hired a worker to implement a project at wage w. If the project is successful, it will earn the firm $v > 0$ dollars of revenue. The project will be successful if the worker *works* and unsuccessful if the worker *shirks*. The cost of effort for the worker is g dollars if he works and nothing if he shirks. The worker's effort is unobservable unless the firm chooses to monitor the worker, so the firm will have to pay the worker unless it monitors and the worker shirks. If the firm monitors the worker it incurs a monitoring cost of h dollars. In the first stage of the project, the firm chooses the wage $w \geq 0$ it will pay the worker. The worker chooses whether to work on the project or shirk, and the firm simultaneously chooses whether or not to monitor the worker. Let $w > g > h > 0$ and v is large enough.
a. Complete the normal form game expression and find all NE of this game.

		Firm	
		Monitor [m]	Not
Worker	Shirk [p]	$0, -h$	$w, \underline{-w}$
	Work	$\underline{w-g}, v-w-h$	$\underline{w-g}, \underline{v-w}$

Because $w > g$, Work is the best response to Monitor, and Shirk is the best response to Not, Not is the best response to Work, and Monitor is the best response to shirk. Therefore, if $w > g$, there are no pure strategy Nash equilibria and the unique Nash equilibrium of every subgame must be a mixed strategy Nash equilibrium. Let p denote the probability that the worker plays Shirk, and m denote the probability that the firm plays Monitor in a mixed strategy Nash equilibrium.

The firm's expected payoffs are:

$$\begin{aligned}\pi_F(\text{Monitor}) &= p(-h) + (1-p)(v-w-h) \\ \pi_F(\text{Not}) &= p(-w) + (1-p)(v-w)\end{aligned}$$

So, in a mixed strategy Nash equilibrium we must have:

$$p(-h) + (1-p)(v-w-h) = p(-w) + (1-p)(v-w) \Rightarrow p^* = h/w$$

The worker's expected payoffs are:

$$\begin{aligned}\pi_W(\text{Shirk}) &= (1-m)w \\ \pi_W(\text{Work}) &= m(w-g) + (1-m)(w-g)\end{aligned}$$

In a mixed strategy Nash equilibrium we must have:

$$(1-m)w = m(w-g) + (1-m)(w-g) \Rightarrow m^* = g/w.$$

The only Nash equilibrium is $(p^* = h/w, m^* = g/w)$.

b. What is the expected payoff of the firm in the equilibrium you obtain in (a)? If the firm can choose w before they player the game, what's the optimal w?

$$\begin{aligned}\pi_F(p^*, m^*) &= p^*(-w) + (1-p^*)(v-w) \\ &= \frac{h}{w}(-w) + (1-\frac{h}{w})(v-w) = v - \frac{vh}{w} - w\end{aligned}$$

$$\max_w \pi_F(w) = v - \frac{vh}{w} - w$$

$$\text{FOC. } \frac{vh}{w^2} - 1 = 0 \Rightarrow w^* = \sqrt{hv}.$$

8.2 Let α and $1-\alpha$ be the wife's probabilities, respectively, of playing ballet and boxing. The husband's expected payoff from ballet then is

$$(1)(\alpha)+(0)(1-\alpha)=\alpha$$

and from boxing is

$$(0)(\alpha)+(K)(1-\alpha)=K-K\alpha.$$

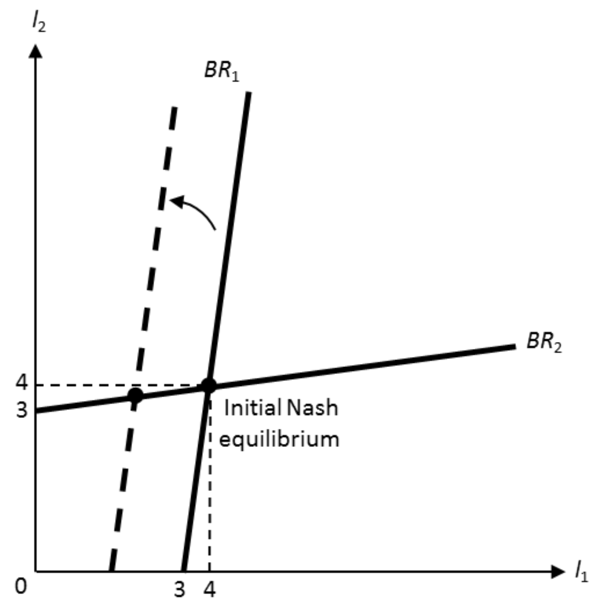
For the husband to be indifferent, and thus willing to randomize, these two expressions must be equal, implying $\alpha^*=K/(1+K)$. Similar calculations show the husband plays boxing with probability $K/(1+K)$ and ballet with the complementary probability. Substituting $K=2$ allows us to recover the mixed-strategy equilibrium found in Example 8.3.

8.4 a. Homeowner 1's objective function is

$$l_1(10-l_1+l_2/2)-4l_1.$$

Taking the first-order condition with respect to l_1 and rearranging yields the best-response function $l_1=3+l_2/4$. Symmetrically, Homeowner 2's best-response function is $l_2=3+l_1/4$. Solving simultaneously yields $l_1^*=l_2^*=4$.

b.



c. The change is indicated by the shift (following the arrow) in Homeowner 1's best-response function. In the new Nash equilibrium, 1 mows a lot less and 2 mows a little less.

8.6 a. Using the underlining algorithm or other method, one can verify that finking is still a dominant strategy for the players and that the Nash equilibrium is (fink, fink).

b. Cooperation on silent is best sustained using grim strategies as described in the text. In this cooperative equilibrium, each player earns present discounted value of 1 each period:

$$\begin{aligned} V^{\text{eq}} &= 1 + \delta(1) + \delta^2(1) + \cdots \\ &= 1\left(1 + \delta + \delta^2 + \cdots\right) \\ &= 1\left(\frac{1}{1-\delta}\right) \\ &= \frac{1}{1-\delta}. \end{aligned}$$

The most a player can earn from deviating is a present discounted value of

$$\begin{aligned} V^{\text{dev}} &= 3 + \delta(0) + \delta^2(0) + \cdots \\ &= 3. \end{aligned}$$

The player earns 3 in the deviation period from his/her surprise fink, but then players revert to the static Nash equilibrium of (fink, fink) from then on. Cooperation is sustainable if

$$\begin{aligned} & V^{\text{eq}} \geq V^{\text{dev}} \\ \Rightarrow \quad & \frac{1}{1-\delta} \geq 3 \\ \Rightarrow \quad & 1 \geq 3(1-\delta), \end{aligned}$$

implying $\delta \geq 2/3$.