Topic 3

Differentials and general function models

Outline

1. Review of continuity and derivative
2. Partial differentiation
3. Comparative-static analysis
4. Differentials
5. Total differentials
6. Total derivatives
7. Homogeneous Functions
8. Derivatives of implicit functions

1. Review of continuity and derivative

- $f(x)$ is continuous at x_0, denoted

$$\lim_{x \to x_0} f(x) = f(x_0)$$

- **Example**: a continuous function

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

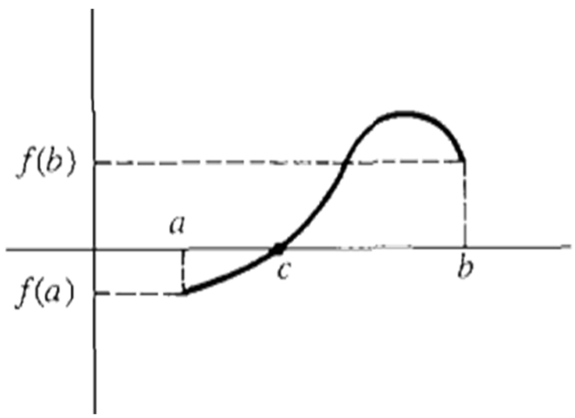
- **Example**: a discontinuous function

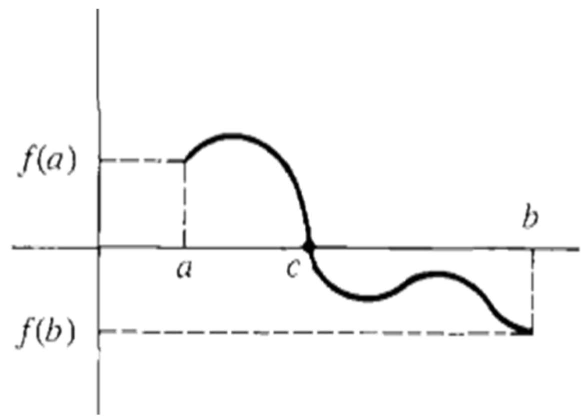
$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases} \qquad \text{note that } \begin{cases} \lim_{x \to +0} \operatorname{sgn}(x) = 1 \\ \lim_{x \to -0} \operatorname{sgn}(x) = -1 \end{cases}$$

so it is not continuous at 0

- Roughly speaking, a function is continuous if its graph can be drawn in a single stroke, without ever lifting the pen. There should be no "jumps".

- An important property of continuous functions is the intermediate value theorem: Suppose that the function $y = f(x)$ is continuous in $[a, b]$ and assume that $f(a)$ and $f(b)$ have different signs, then there is at least one root to the equation $f(x) = 0$ in (a, b), i.e., a number c with $a < c < b$ such that $f(c) = 0$





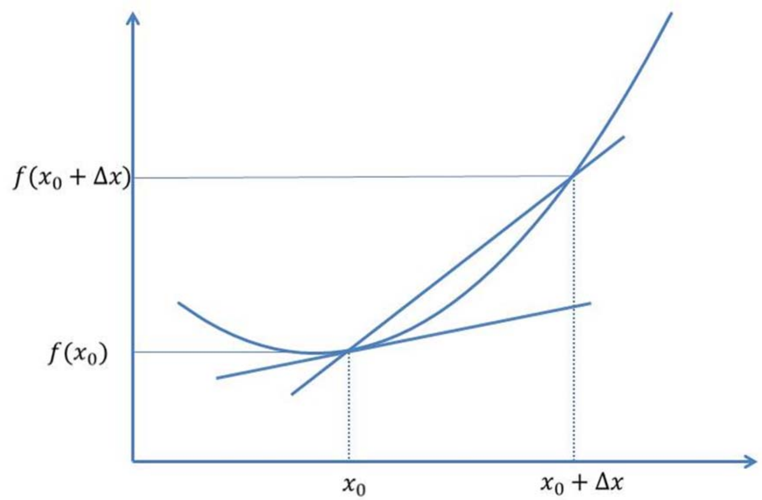
- If f is continuous and $f(a) > 0$, then $f(x) > 0$ for all x close to a.

Review of derivative

- An important topic in economics is the study of rates of change
- Consider a function $y = f(x)$ defined on D and $x_0 \in D$, suppose for a small Δx, $x_0 + \Delta x \in D$.
  - The rate of change of $f(x)$ per change of x is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

  - Geometrically, $\frac{\Delta y}{\Delta x}$ is the slope of the chord passing through the two points $(x_0, f(x_0))$ and $(x_0 + \Delta x, f(x_0 + \Delta x))$



  - As Δx becomes smaller and smaller, the slope of the chord approaches the slope of the line tangent to the curve at $(x_0, f(x_0))$, the limit

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \quad (\text{or } \left.\frac{df}{dx}\right|_{x=x_0}, \left.\frac{df}{dx}\right|_{x_0})$$

  is called the derivative of f at point x_0
  - The number $f'(x_0)$ gives the slope of the tangent to the curve $y = f(x)$ at the point $(x_0, f(x_0))$

Derivative of monotone functions

- f is increasing function on $(a, b) \Leftrightarrow f'(x) \geq 0$ for $x \in (a, b)$
- f is decreasing function on $(a, b) \Leftrightarrow f'(x) \leq 0$ for $x \in (a, b)$
- $f'(x) = 0$ for all $x \in (a, b) \Leftrightarrow f$ is constant for $x \in (a, b)$
- $f'(x) > 0$ for all $x \in (a, b) \Rightarrow f$ is strictly increasing for $x \in (a, b)$
- $f'(x) < 0$ for all $x \in (a, b) \Rightarrow f$ is strictly decreasing for $x \in (a, b)$

Rules of differentiation

- Constant function rule: if $f(x) = k$, then $f'(x) \equiv 0$
- Power function rule: If $f(x) = x^a$, where $a \in R$ is a constant, then
  $f'(x) = ax^{a-1}$
- Derivative of Exponential functions and logarithmic functions:
  $(e^x)' = e^x, \qquad (\ln x)' = (1/x)$
- Sum-difference rule: $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$
- Product rule: $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$
- Quotient rule: $[\frac{f(x)}{g(x)}]' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$
- The chain rule: The composition of two differentiable functions $y = f(u)$ and $u = g(x)$ is again differentiable and has the derivative

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \text{ or } \left[f\left(g(x)\right)\right]' = f'\left(g(x)\right) \cdot g'(x)$$

Higher order derivatives

- The derivative of a function f is often called the first order derivatives of function $y = f(x)$
- If f' is differentiable, the derivative of function f' is called the second order derivative of f, denoted y'', $f''(x)$, or $\frac{d^2}{dx^2}f(x)$
- Notation for third order derivative: y''', $f'''(x)$, $f^{(3)}(x)$,or $\frac{d^3}{dx^3}f(x)$
- nth order derivative: $f^{(\mathrm{n})}(x)$ or $\frac{d^n}{dx^n}f(x)$
- If $f^{(\mathrm{n})}(x)$ exists and is continuous, then we denote $f \in C^{\mathrm{n}}$

Examples: Higher order derivatives

- Example 1: $f(x) = x^5 + e^{2x}$, then
  $f'(x) = 5x^4 + 2e^{2x}, f''(x) = 20x^3 + 4e^{2x}, f'''(x) = 60x^2 + 8e^{2x}$

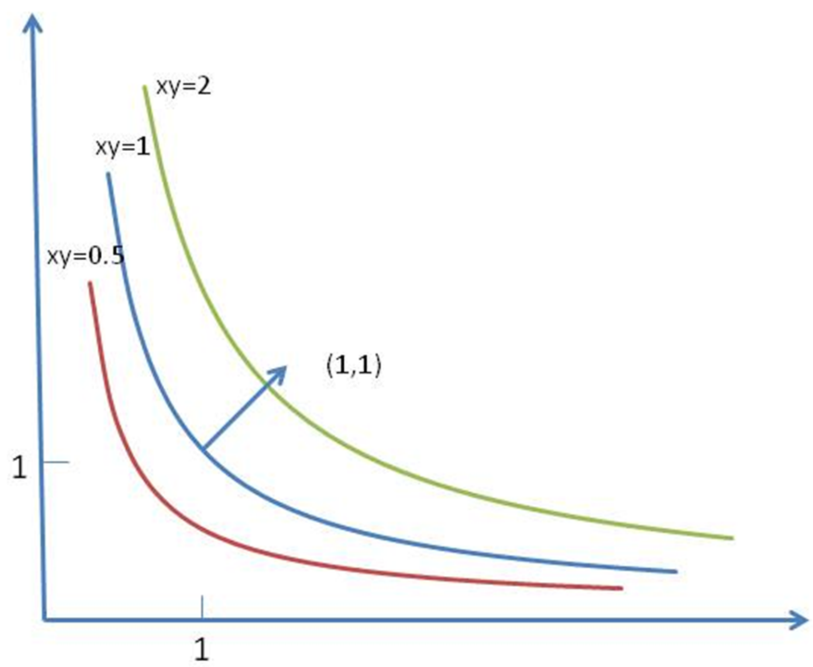
- Example 2: Let $f \in C^2$ and $g(x) = f(x^2)$, then
  $g'(x) = 2xf'(x^2),\ g''(x) = 2f'(x^2) + (2x)(2x)f''(x^2)$

2. Partial differentiation

- Consider the function $z = f(x, y) = x^3 + y^2$, suppose y is a constant, i.e.,$y = 1$, then z becomes a function of one variable x only, change of z wrt x is $\frac{dz}{dx} = 3x^2$
  - If x is a constant, then, z becomes a function of one variable y only, change of z wrt y is $\frac{dz}{dy} = 2y$
- When we consider functions of two (or more) variables, we use $\frac{\partial z}{\partial x}$ (or $\frac{\partial f}{\partial x}$ or ${f_1}', {f_x}'$ or f_1, f_x when there is no confusion) instead of $\frac{dz}{dx}$ for the derivative of z with respect to x when y is held fixed, called the partial derivative of z wrt x
  - In the same way, we use $\frac{\partial z}{\partial y}$ (or $\frac{\partial f}{\partial y}$ or ${f_2}', {f_y}'$ or f_2, f_y) instead of $\frac{dz}{dy}$ for the derivative of z with respect to y when x is held fixed, called the partial derivative of z wrt y
- The vector of first order partial derivatives $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})'$ is called the gradient vector

- For $f \in C^1$ (meaning ∇f is continuous), at any point (x, y) on the level curve at which $\nabla f \neq 0$, the gradient vector $\nabla f(x, y)$ has the property that it points to the direction in which f increases most rapidly, and it is perpendicular to the tangent line of level curve.

- **Example**: The function $f(x, y) = xy$ increase most rapidly in the direction $(1,1)$ at the point $(1,1)$



- Second order partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

- $f \in C^2$ if the second order partial derivatives are continuous

- For $f(x_1, x_2, \ldots, x_n)$ of n variables, Hessian matrix

$$f''(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1{}^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2{}^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n{}^2} \end{pmatrix}$$

- **Example**: Let $f(x, y) = x^3 + x^2y^2 + y^4$, then

$$\frac{\partial f}{\partial x} = 3x^2 + 2xy^2, \qquad \frac{\partial f}{\partial y} = 2x^2 y + 4y^3$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(3x^2 + 2xy^2) = 6x + 2y^2, \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(2x^2 y + 4y^3\right) = 2x^2 + 12y^2$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(2x^2 y + 4y^3\right) = 4xy, \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(3x^2 + 2xy^2\right) = 4xy = \frac{\partial^2 f}{\partial x \partial y}$$

- Note: $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$

- Hessian matrix: $f''(x) = \begin{pmatrix} 6x + 2y^2 & 4xy \\ 4xy & 2x^2 + 12y^2 \end{pmatrix}$ is a symmetric matrix

- **Example**: Let $f \in C^2$ and $g(x, y) = f(xy)$, then

$$\frac{\partial g}{\partial x} = yf\,'(xy), \quad \frac{\partial g}{\partial y} = xf\,'(xy)$$

$$\frac{\partial^2 g}{\partial^2 x} = y^2 f\,''(xy), \quad \frac{\partial^2 g}{\partial x \partial y} = f\,'(xy) + xyf\,''(xy), \quad \frac{\partial^2 g}{\partial^2 y} = x^2 f\,''(xy)$$

- **Example**: A function of two variables appearing in many economic models is
  $$q = f(K, L) = AK^{\alpha}L^{\beta}, \quad A, 0 < \alpha < 1, 0 < \beta < 1 \text{ are constants}$$
  where K =capital, L =labor, and q =number of units produced.
  - f is generally called a Cobb-Douglas function, it is most often used to describe certain production processes
  - The two partial derivatives $\frac{\partial q}{\partial K} = A\alpha K^{\alpha-1}L^{\beta}$ and $\frac{\partial q}{\partial L} = A\beta K^{\alpha}L^{\beta-1}$ are called the marginal product of capital and labor respectively
  - Differentiating a second time we obtain: $\frac{\partial^2 q}{\partial L^2} = A\beta(\beta - 1)K^{\alpha}L^{\beta-2} < 0$,or the marginal product of labor is decreasing (diminishing productivity of labor).
  - Verify similarly that there is diminishing productivity of capital.

3. Comparative-static analysis

- Comparative-static analysis deals with how the equilibrium value of an endogenous variables (variables solved from the model) change when there is a change of the exogenous variables (variables determined by forces external to the model) or parameters.
- **Example 1**: Single-good equilibrium model

$$\begin{cases} Q = a - bP & a,b > 0 \quad \text{[demand]} \\ Q = \text{-}c + dP & c,d > 0 \quad \text{[supply]} \end{cases}$$

with solution (P^*, Q^*), where

$$P^* = \frac{a+c}{b+d}, \quad Q^* = \frac{ad-bc}{b+d}$$

  – How will P^* and Q^* change when there is a change of the parameters?

$$\frac{\partial P^*}{\partial a} = \frac{1}{b+d} > 0, \quad \frac{\partial P^*}{\partial b} = -\frac{a+c}{(b+d)^2} < 0$$

$$\frac{\partial P^*}{\partial c} = \frac{1}{b+d} > 0, \quad \frac{\partial P^*}{\partial d} = -\frac{a+c}{(b+d)^2} < 0,$$

- **Example 2**: National income model with taxes:

$$\begin{cases} Y = C + I_0 + G_0 \\ C = \alpha + \beta Y^d \qquad (\alpha > 0, 0 < \beta < 1) \\ T = \gamma + \delta Y \qquad (\gamma > 0, 0 < \delta < 1) \end{cases}$$

Here $Y^d = Y - T$ is disposable income, β is marginal propensity to consume, δ is income tax rate.

- The equilibrium income is $Y^* = \dfrac{\alpha - \beta\gamma + I_0 + G_0}{1 - \beta + \beta\delta}$, thus $\dfrac{\partial Y^*}{\partial G_0} = \dfrac{1}{1 - \beta + \beta\delta} > 1$

  The term $\frac{1}{1-\beta+\beta\delta}$ is called the government-expenditure multiplier: \$1 million increase in government-expenditure will lead to more than \$1 million increase in national income. For example, if $\beta = 0.9, \delta = 0.3$, then $\frac{1}{1-\beta+\beta\delta} = 2.7$.

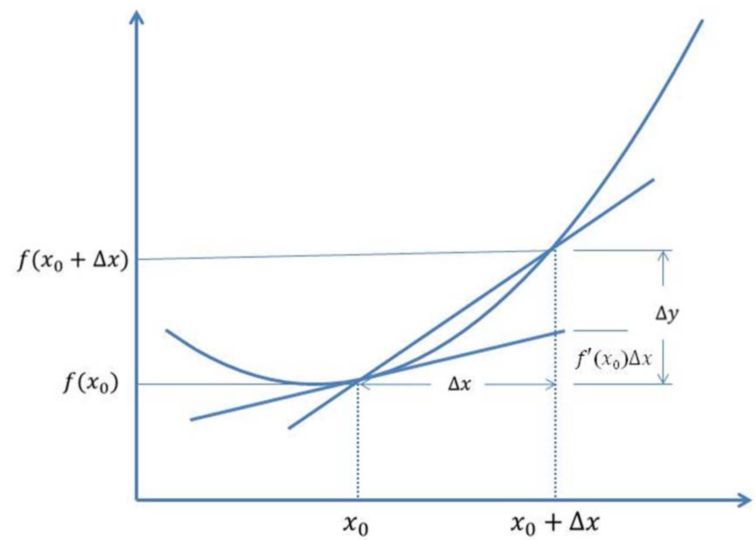
- In addition

$$\frac{\partial Y^*}{\partial \delta} = -\beta \frac{\alpha - \beta\gamma + I_0 + G_0}{(1 - \beta + \beta\delta)^2} = -\frac{\beta Y^*}{1 - \beta + \beta\delta} < 0$$

  or, an increase in the income tax rate δ will lower the equilibrium income Y^*.

4. Differentials

- Recall $f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, when Δx is very small, $f'(x_0) \approx \frac{\Delta y}{\Delta x}$, or equivalently, $\Delta y \approx f'(x_0)\Delta x$



- Example: for $y = f(x) = x^2$
at $x_0 = 1$, $f'(x_0) = 2x_0 = 2$

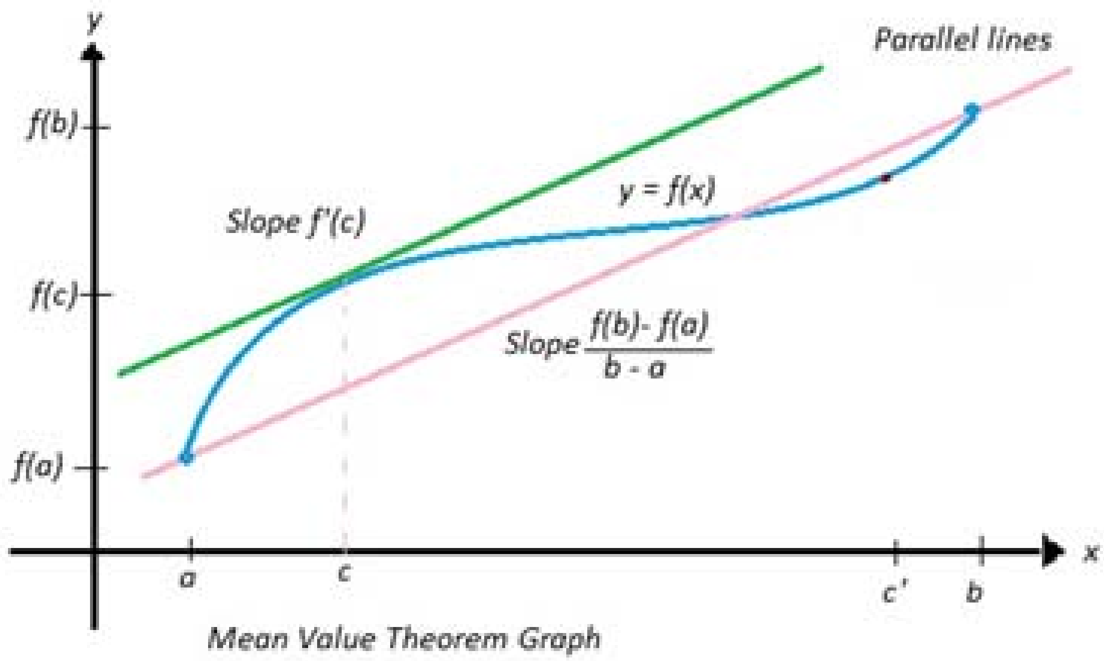
Δx	Δy	$f'(x_0)\Delta x$	Difference
0.1	0.21	0.2	0.01
0.01	0.0201	0.02	0.0001

when Δx is small, Δy is very close to $f'(x_0)\Delta x$

- Another way to understand the approximation is through the Mean Value Theorem.

- Mean Value Theorem: If $f \in C^1$, then there is at least one c between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



- Differential for one-variable function $y = f(x)$ at $x = x_0$:

  $$dy(or\ df) = f'(x_0)dx,$$

  here dx stands for a very small change in x

  - The definition of differential coincides with the notation $\frac{dy}{dx} = f'(x)$
  - Note: it is not meaningful to write $dy = f'(x)$

- Using the notation of differential, if $y = f(g(x))$, then by the chain rule $y' = f'(g(x))g'(x)$, thus $dy = f'(g(x))g'(x)dx = f'(u)du\ \ (u = g(x))$
- Example: $y = f(x^2)$, let $u = x^2$, $du = 2xdx$

  thus $dy = f'(u)du = f'\left(x^2\right)2xdx$, $\frac{dy}{dx} = 2xf'(x^2)$

Price (point) Elasticity of demand

- By what percentage the quantity demanded changes when the price increases by 1%? This number is independent of the units in which both quantities and prices are measured.
- Let $x = x(p)$ be the quantity demanded at price p, the elasticity of $x(p)$ with respect to p is

$$\varepsilon_{xp} = \frac{dx / x}{dp / p} = \frac{p}{x} \cdot \frac{dx}{dp} = \frac{d \ln x}{d \ln p}$$

- **Example**: Assume $x(p) = a - bp$, where $a, b > 0$ are constants,

$$\frac{dx}{dp} = -b, \;\; \varepsilon_{xp} = \frac{p}{x} \cdot \frac{dx}{dp} = -\frac{bp}{x} = -\frac{bp}{a - bp}$$

- **Example**: Assume $x(p) = Ap^b$ where $A > 0$ and b are constants.

$$\frac{dx}{dp} = Abp^{b-1}, \quad \varepsilon_{xp} = \frac{p}{x} \cdot \frac{dx}{dp} = \frac{p}{x} Abp^{b-1} = b$$

Alternatively,

$$\ln x = \ln(A) + b \ln p, \;\; \text{thus } \varepsilon_{xp} = \frac{d \ln x}{d \ln p} = b$$

5. Total differentials

- The concept of differential can be easily generalized to a function of two or more variables.
- Let $z = f(x, y)$ be a function of two variables.
  - When there is a small change (Δx) in x while holding y as constant, the change in z is
    $\Delta z \approx f_x{}'\Delta x$, in differential format: $dz = f_x{}'dx$
  - likewise, when there is small change in y while x is held constant,
    $\Delta z \approx f_y{}'\Delta y$, thus $dz = f_y{}'dy$
  - When there is a small change in both x and y, the change in z is
    $$\Delta z \approx f_x{}'\Delta x + f_y{}'\Delta y$$
  - Total differential for two-variable function $z = f(x, y)$:
    $$dz(or\ df) = f_x{}'dx + f_y{}'dy$$

- **Example**: Consider the Cobb-Douglas production function $Q = 4K^{\frac{1}{2}}L^{\frac{1}{2}}$
  when $K = K_0 = 100$ and $L = L_0 = 25$, output is $Q = 4(100)^{\frac{1}{2}}(25)^{\frac{1}{2}}$=200
- Compute the partial derivatives

$$\frac{\partial Q}{\partial K} = 2K^{-\frac{1}{2}}L^{\frac{1}{2}}, \ \frac{\partial Q}{\partial L} = 2K^{\frac{1}{2}}L^{-\frac{1}{2}}$$

- Thus, $\frac{\partial Q}{\partial K}(100,25) = 1, \ \frac{\partial Q}{\partial L}(100,25) = 4$

- if there is small change in K or L
- $$\Delta Q \approx \frac{\partial Q}{\partial K}(100,25)\cdot \Delta K + \frac{\partial Q}{\partial L}(100,25)\cdot \Delta L = 1\cdot \Delta K + 4\cdot \Delta L$$
  - If L is held constant, and K is increased by $\Delta K = 1$, Q will increase by approximately $1 \cdot \Delta K = 1$, thus $Q(101,25) \approx 200 + 1 = 201$ which is a good approximation to $Q(101,25) = 200.9975$
  - If K is held constant and $\Delta L = -1, \Delta Q \approx 4 \cdot \Delta L = -4$, thus $Q(100,24) \approx 200 - 4 = 196$, a good approximation to $Q(100,24) = 195.9592$
  - If there are small changes to both variables, $\Delta K = 1, \Delta L = -1$, then

$$Q \approx 200 + \left(\frac{\partial Q}{\partial K}\right)(100,25)\cdot \Delta K + \left(\frac{\partial Q}{\partial L}\right)(100,25)\cdot \Delta L = 197$$

which is good approximation to the exact value of $Q(101,24) = 196.9365$

- **Example**: Find the total differential of $f(x,y) = x^3 + x^2y^2 + y^4$,

$$\frac{\partial f}{\partial x} = 3x^2 + 2xy^2,\ \frac{\partial f}{\partial y} = 2x^2 y + 4y^3$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \left(3x^2 + 2xy^2\right)dx + \left(2x^2 y + 4y^3\right)dy$$

- **Example**: Find the total differential of $f(x,y) = \frac{x+y}{2x^2}$

$$\frac{\partial f}{\partial x} = \frac{1}{2}\frac{(1)\left(x^2\right) - (x+y)(2x)}{\left(x^2\right)^2} = -\frac{x+2y}{2x^3},\ \frac{\partial f}{\partial y} = \frac{1}{2x^2}$$

$$df = -\frac{x+2y}{2x^3}dx + \frac{1}{2x^2}dy$$

Rules of differentials

1. Constant-function rule: $dk = 0$
2. Power-function rule $d(cu^{\mathrm{n}}) = cnu^{\mathrm{n-1}}du$
3. sum-difference rule $d(u \pm v) = du \pm dv$
4. product rule $d(uv) = vdu + udv$
5. quotient rule $d\left(\frac{u}{v}\right) = \frac{1}{v^2}(vdu - udv)$

- **Example (revisit)**: Find the total differential of $f(x, y) = x^3 + x^2y^2 + y^4$,

$$\begin{aligned} df &= d(x^3) + d(x^2y^2) + d(y^4) \quad \text{[by Rule 3]} \\ &= d(x^3) + y^2 d(x^2) + x^2 dy^2 + d(y^4) \quad \text{[by Rule 4]} \\ &= 3x^2dx + 2xy^2dx + 2x^2ydy + 4y^3dy \quad \text{[by Rule 2]} \\ &= (3x^2 + 2xy^2)dx + (2x^2y + 4y^3)dy \end{aligned}$$

- **Example (revisit)**: Find the total differential of $f(x, y) = \frac{x+y}{2x^2}$

$$\begin{aligned} df &= \frac{1}{(2x^2)^2}\left(2x^2 d(x+y) - (x+y)d(2x^2)\right) \quad \text{[by Rule 5]} \\ &= \frac{1}{(2x^2)^2}\left(2x^2(dx+dy) - 4x(x+y)dx\right) \quad \text{[by Rules 2 and 3]} \\ &= \frac{1}{4x^4}\left((-2x^2 - 4xy)dx + 2x^2dy\right) \\ &= -\frac{x+2y}{2x^3}dx + \frac{1}{2x^2}dy \end{aligned}$$

6. Total derivatives

- *Case 1:* Consider function $y = f(x, w)$, where $x = g(w)$, this can be combined into a composite function $y = f(g(w), w)$
  - Then $dy = f_x{}'dx + f_w{}'dw = f_x{}'g'dw + f_w{}'dw$
  - the total derivative is $\frac{dy}{dw} = f_x' g' + f_w'$

- **Example**: $y = f(x, w) = 3x - w^2$, where $x = g(w) = e^{2w}$
  - $f_x' = 3, f_w' = -2w, g' = 2e^{2w}$
  - the total derivative is $\frac{dy}{dw} = f_x{}'g' + f_w{}' = (3)(2e^{2w}) - 2w = 6e^{2w} - 2w$
  - Check $y = 3e^{2w} - w^2$, $\frac{dy}{dw}$ can then be easily found

- *Case 2*: $y = f(x_1, x_2, w)$, where $x_1 = g(w), x_2 = h(w)$
  - Then
$$dy = f_1'dx_1 + f_2'dx_2 + f_w'dw$$
$$= f_1'g'dw + f_2'h'dw + f_w'dw$$
$$\frac{dy}{dw} = f_1'g' + f_2'h' + f_w'$$

- **Example**: Suppose $z = f(x, y, t) = xy + 2t$, where
  $x = g(t) = t^3, y = h(t) = t^2 - t$
  - $f_1' = y = t^2 - t, g' = 3t^2, f_2' = x = t^3, h' = 2t - 1, f_t' = 2,$
  - the total derivative is $\frac{dz}{dt} = f_1'g' + f_2'h' + f_t'$
$$= (t^2 - t)(3t^2) + t^3(2t - 1) + 2$$
  - Alternatively, we can get the same result by differentiating $z = t^3(t^2 - t) + 2t$
    $\frac{dz}{dt} = (3t^2)(t^2 - t) + t^3(2t - 1) + 2$

- *Case 3:* $z = f(x, y)$, where $x = g(s,t), y = h(s,t)$
  - The total differentials of z, x, y:

$$dz = f_x' dx + f_y' dy$$

$$dx = g_s' ds + g_t' dt, \;\; dy = h_s' ds + h_t' dt$$

  - Therefore

$$dz = f_x'(g_s' ds + g_t' dt) + f_y'(h_s' ds + h_t' dt)$$

$$= (f_x' g_s' + f_y' h_s')ds + (f_x' g_t' + f_y' h_t')dt$$

  - and

$$\frac{\partial z}{\partial s} = f_x' g_s' + f_y' h_s'$$

$$\frac{\partial z}{\partial t} = f_x' g_t' + f_y' h_t'$$

- *General version:* suppose $y = f(x_1, \dots, x_n)$ and $x_i = g_i(t_1, \dots, t_m)$ for $i = 1, \dots, n,$ then y is a function of $t_1, \dots, t_m$ and

$$\frac{\partial y}{\partial t_i} = f_1 \cdot \frac{\partial x_1}{\partial t_i} + f_2 \cdot \frac{\partial x_2}{\partial t_i} + \cdots + f_n \cdot \frac{\partial x_n}{\partial t_i} \quad \text{for } i = 1, 2, \dots, m$$

- **Example:** assume the following:

$$z = x^2 + xy^3, \;\; x = uv^2 + w^3, \;\; y = u + ve^w$$

- Then (let $f(x, y) = x^2 + xy^3$)

$$\frac{\partial z}{\partial u} = f_x \cdot \frac{\partial x}{\partial u} + f_y \cdot \frac{\partial y}{\partial u} = (2x + y^3)(v^2) + (3xy^2)(1)$$

$$\frac{\partial z}{\partial v} = f_x \cdot \frac{\partial x}{\partial v} + f_y \cdot \frac{\partial y}{\partial v} = (2x + y^3)(2uv) + (3xy^2)(e^w)$$

$$\frac{\partial z}{\partial w} = f_x \cdot \frac{\partial x}{\partial w} + f_y \cdot \frac{\partial y}{\partial w} = (2x + y^3)(3w^2) + (3xy^2)(ve^w)$$

Example: assume the following: $y = f(tx_1, tx_2, \ldots, tx_n)$

- Write $y = f(w_1, \ldots, w_n)$, where $w_i = tx_i$ for $i = 1, \ldots, n$, then

$$\begin{aligned} dy &= f_1' dw_1 + \cdots + f_n' dw_n \\ &= f_1'(t dx_1 + x_1 dt) + \cdots + f_n'(t dx_n + x_n dt) \\ &= t f_1' dx_1 + \cdots + t f_n' dx_n + (x_1 f_1' + \cdots + x_n f_n') dt \end{aligned}$$

- Thus $\frac{\partial y}{\partial x_1} = t f_1', \ldots, \frac{\partial y}{\partial x_n} = t f_n'$, and $\frac{\partial y}{\partial t} = x_1 f_1' + \cdots + x_n f_n'$

7. Homogeneous Functions

- Homogeneous functions arise naturally throughout economics.
- A function $f: R_+{}^{\mathrm{n}} \to R$ is homogeneous of degree k (Chiang 2005, P383) if

  $$f(\lambda x_1, \lambda x_2, \dots, \lambda x_n) = \lambda^k f(x_1, x_2, \dots, x_n),$$

  for all $(x_1, x_2, \dots, x_n) \in R^{\mathrm{n}}$
- When $k = 1, f$ is linearly homogeneous.
- When $k = 0$, f is zero homogeneous.
- **Examples**:
  - $x_1{}^2 x_2 + 3x_1 x_2{}^2 + x_2{}^3$ is homogeneous of degree 3
  - The Cobb-Douglas production function $f(K, L) = AK^\alpha L^\beta$ is homogeneous of degrees $k = \alpha + \beta$
  - The CES function $f(x, y) = (\alpha x^\rho + \beta y^\rho)^{1/\rho}$ is linearly homogeneous

- Production function $f(K, L) = AK^{\alpha}L^{\beta}$ is linearly homogeneous,
$$f(\lambda K, \lambda L) = \lambda f(K, L) \tag{3.1}$$
when α+β=1.
  - taking λ=2, equation (3.1) says that if the firm doubles all its inputs (K and L), it doubles its output too. For $\lambda = 3$, if the firm triples each input, it triples its corresponding output. Such a firm is said to exhibit constant returns to scale.
  - Suppose, on the other hand, $k = \alpha + \beta > 1$, if such a firm double the amount of each input, its output would rise by a factor of $2^k > 2$, its output would more than double. Such a firm is said to exhibit increasing returns to scale.
  - If $k = \alpha + \beta < 1$, such a firm is said to exhibit decreasing returns to scale.

- (Euler's Theorem) If f is linearly homogeneous, then

$$\sum_{i=1}^{n} x_i f_i(x_1, \dots, x_n) = f(x_1, \dots, x_n)$$

- **Example**: For the Cobb-Douglas production function

  $$f(K, L) = AK^{\alpha}L^{1-\alpha}$$

  we have $f_K(K,L) = A\alpha K^{\alpha-1}L^{1-\alpha}, \ f_L(K,L) = A(1-\alpha)K^{\alpha}L^{-\alpha}$

- thus

$$\begin{aligned} Kf_K(K,L) + Lf_L(K,L) &= A\alpha K^{\alpha}L^{1-\alpha} + A(1-\alpha)K^{\alpha}L^{1-\alpha} \\ &= AK^{\alpha}L^{1-\alpha} = f(K,L) \end{aligned}$$

- If $f(x_1, \dots, x_n)$ is homogeneous of degree k, then $f_i(x_1, \dots, x_n)$ is homogeneous of degree $k-1$

- **Example**: $f(x_1, x_2) = {x_1}^2 x_2 + 3x_1{x_2}^2 + {x_2}^3$ is homogeneous of degrees 3, $f_1(x_1, x_2) = 2x_1x_2 + 3{x_2}^2$ and $f_2(x_1, x_2) = {x_1}^2 + 6x_1x_2 + 3{x_2}^2$ are both homogeneous of degrees 2.

8. Derivatives of implicit functions

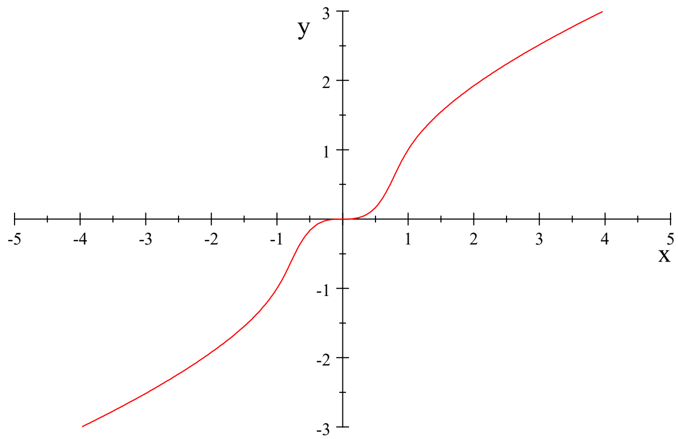
- **Example**: Consider the following relationship

$$y^5 + 3y = 4x^3$$

  – obviously, y is an increasing function of x, and it passes through $(1,1)$
  – How to find the slope of the tangent at (1,1)?
  – It is not easy to solve for y in this case, use implicit differentiation, you get

$$5y^4 y' + 3y' = 12x^2, \;\Rightarrow y' = \frac{12x^2}{5y^4 + 3}$$

  – At $(1,1), y' = \frac{12}{5+3} = \frac{3}{2}$



- An implicit function can be written in the form of $F(x, y) = 0$
- Previous example: $F(x, y) = y^5 + 3y - 4x^3$, $F(x, y) = 0$ defines y as a function of x.
- An implicit function does not automatically defines y as a function of x.

Example: Consider the implicit function $F(x, y) = 0$ where

$$F(x, y) = x^2 + y^2 - 25 \tag{3.2}$$

- when $|x| > 5$, there is no y which satisfy (3.2)
- We start with a specific solution (x_0, y_0) of the implicit equation $F(x, y) = 0$ and ask if we vary x a little from x_0, can we find a y near y_0 that satisfies the equation
- For example, $(x_0, y_0) = (3,4)$ satisfies $F(x, y) = 0$, and vary x a little, we can find a unique $y = \sqrt{25 - x^2}$ near $y = 4$ that corresponds to the new x
- $(x_0, y_0) = (3, -4)$ also satisfies $F(x, y) = 0$, and vary x a little, we can find a unique $y = -\sqrt{25 - x^2}$ near $y = -4$ that corresponds to the new x
- However, for $(x_0, y_0) = (5,0)$, if we increase x a little, e.g, $x = 5.001$, there is no corresponding y so that $(5.001, y)$ solves $F(x, y) = 0$; if we decrease x a little to $x_1 = 4.999$, there are two y's near $y = 0$ which satisfy $F(x_1, y) = 0$, namely $y = \sqrt{25 - 4.999^2}$ and $y = -\sqrt{25 - 4.999^2}$

- For a given implicit function $F(x, y) = 0$ and a specified solution point (x_0, y_0), we want to know the answers to the following two questions:

1. Does $F(x, y) = 0$ determine y as a continuous function of x for x near x_0 and y near y_0?
2. If so, how do changes in x affect the corresponding y's? in other words, what is dy/dx?

- Implicit function Theorem: If the function $F(x,y) \in C^1$, suppose that $F(x_0,y_0)=0$ and $F_y{}'(x_0,y_0) \neq 0$, then equation $F(x,y)=0$ defines y as a continuously differentiable function of x: $y=f(x)$ for (x,y) close to (x_0,y_0).
  - For $F(x,y)=y^5+3y-4x^3$, $\frac{\partial F(x,y)}{\partial y}=5y^4+3>0$ for all (x,y), $F(x,y)=0$ defines y as a differentiable function of x for all $x \in R$.
  - For $F(x,y)=x^2+y^2-25$ and $(x_0,y_0)=(3,4)$, $F_y{}'(x_0,y_0)=2y_0=8\neq 0$, $F(x,y)=0$ defines y as a differentiable function of x for x close to x_0.
    However, when $y_0=0$, ($x_0=5$ or -5), you can not find $y=f(x)$ for x close to x_0.

- When $F(x,y)=0$ defines y as a differentiable function of x, differentiate the equation gives

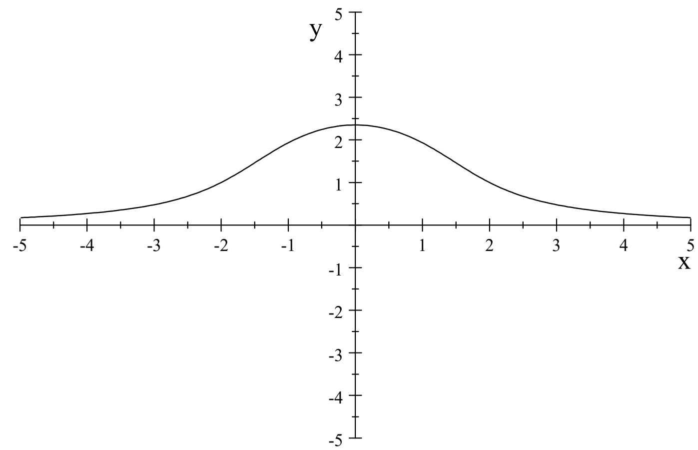
$$F_x{}'(x,y)+F_y{}'(x,y)y'=0$$

- or

$$y'=\frac{dy}{dx}=-\frac{F_x{}'(x,y)}{F_y{}'(x,y)}$$

- **Example**: $F(x, y) = y^3 + 3x^2y - 13$,
  - Let (x_0, y_0) be a point on $F(x, y) = 0$,
  - $F_y\,(x_0, y_0) = 3{y_0}^2 + 3{x_0}^2 \neq 0$
  - By the implicit function theorem, $F(x, y) = 0$ defines y as a differentiable function of x.

    Take the derivative wrt x:



$$3y^2y' + 3(2xy + x^2y') = 0,$$

$$y' = -\frac{2xy}{x^2+y^2}$$

  - $F(x, y) = 0$ passes through the points $(2,1)$ and $(-2,1)$

    using above formula for y', we get:

    at $(2,1)$, $y' = -(4/5)$, and at $(-2,1)$, $y' = (4/5)$

- **Example** (Linear supply and demand model) Suppose that a tax of t per unit is imposed on consumers, then

$$\begin{cases} Q^d = a - b(P+t) \\ Q^s = -c + dP \end{cases}$$

Here a, b, c and d are positive parameters

- The equilibrium price (P^*) is determined by equating the supply and demand

$$a - b(P^* + t) = -c + dP^* \Rightarrow P^* = \frac{a + c - bt}{b + d}$$

- Differentiate wrt t:

$$\frac{dP^*}{dt} = -\frac{b}{b+d} < 0$$

i.e., the price received by the producer will go down if the tax rate t increases

- On the other hand

$$\frac{d(P^*+t)}{dt} = \frac{dP^*}{dt} + 1 = \frac{d}{b+d}$$

$$\Rightarrow 0 < \frac{d(P^*+t)}{dt} < 1$$

i.e., the consumer price increases, but by less than the increase in the tax

- **Example** (non-linear supply and demand model): Assume that

$$Q^d = f(P+t), Q^s = g(P)$$

where f and g are differentiable functions with $f' < 0$ and $g' > 0$

  - The equilibrium price (P^*) now satisfies

$$f(P^* + t) = g(P^*), \text{or } F(P^*, t) = 0, \tag{3.3}$$

where $F(P,t) = f(P+t) - g(P)$

  - Since $\frac{\partial F}{\partial P} = f'(P+t) - g'(P) < 0$, so $F(P^*, t) = 0$ defines P^* implicitly as a differentiable function of t
  - Differentiate the equilibrium condition (3.3) with respect to t gives

$$f'(P^* + t)\left(\left(\frac{dP^*}{dt}\right) + 1\right) = g'(P^*)\left(\frac{dP^*}{dt}\right)$$

$$\frac{dP^*}{dt} = \frac{f'(P^*+t)}{g'(P^*) - f'(P^*+t)} < 0$$

  - Moreover $\frac{d(P^*+t)}{dt} = \frac{g'(P^*)}{g'(P^*) - f'(P^*+t)}$
  - Thus, $0 < \frac{d(P^*+t)}{dt} < 1$

- Generalization of the implicit function theorem to multi-variable: If the function $F(x_1, x_2, \dots, x_m, y)$ is continuously differentiable. Suppose further that $F({x_1}^0, {x_2}^0, \dots, {x_m}^0, y^0) = 0, F_y{}'({x_1}^0, {x_2}^0, \dots, {x_m}^0, y^0) \neq 0$, then equation $F(x_1, x_2, \dots, x_m, y) = 0$ defines y as a continuously differentiable function of $x_1, x_2, \dots, x_m$: $y = f(x_1, x_2, \dots, x_m)$ for $(x_1, x_2, \dots, x_m, y)$ close to $\left({x_1}^0, {x_2}^0, \dots, {x_m}^0, y^0\right)$.
- Since $F(x_1, x_2, \dots, x_m, y) = 0$, take partial derivative wrt x_i,
- $$F_{x_i} + F_y \frac{\partial y}{\partial x_i} = 0$$

  therefore

$$f_{x_i} = \frac{\partial y}{\partial x_i} = -F_{x_i} / F_y \quad \text{for } i = 1,2, \dots, m$$

- **Example**: Let $F(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$. F has continuous partial derivatives and $\frac{\partial F}{\partial z} = F_z{}' = 2z \neq 0$ when $z \neq 0$
- From the implicit function Theorem, the equation $F(x, y, z) = 0$ defines z as a continuous function of (x, y): $z = f(x, y)$.
- Take partial derivatives wrt x and y respectively

$$\begin{cases} 2x + 2z\dfrac{\partial z}{\partial x} = 0 \\ 2y + 2z\dfrac{\partial z}{\partial y} = 0 \end{cases}$$

- therefore

$$\frac{\partial z}{\partial x} = -\frac{x}{z}, \frac{\partial z}{\partial y} = -\frac{y}{z}$$

- Generalization of the implicit function theorem to multi-variable, multi-function:
If the functions $F^1(x_1,\dots,x_m,y_1,\dots,y_n),\dots,F^n(x_1,\dots,x_m,y_1,\dots,y_n)$
are continuously differentiable. Suppose further that

$$\begin{cases} F^1(x_1{}^0,\dots,x_m{}^0,y_1{}^0,\dots,y_n{}^0)=0 \\ F^2(x_1{}^0,\dots,x_m{}^0,y_1{}^0,\dots,y_n{}^0)=0 \\ \vdots \\ F^n(x_1{}^0,\dots,x_m{}^0,y_1{}^0,\dots,y_n{}^0)=0 \end{cases} \quad \text{and} \quad |J|=\left|\frac{\partial(F^1,\dots,F^n)}{\partial(y_1,\dots,y_n)}\right|=\begin{vmatrix} \frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_n} \\ \frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} & \cdots & \frac{\partial F^2}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial F^n}{\partial y_1} & \frac{\partial F^n}{\partial y_2} & \cdots & \frac{\partial F^n}{\partial y_n} \end{vmatrix}\neq 0$$

at $(x_1{}^0,\dots,x_m{}^0,y_1{}^0,\dots,y_n{}^0)$, then equation

$$F^1(x_1,\dots,x_m,y_1,\dots,y_n)=0,\dots,F^n(x_1,\dots,x_m,y_1,\dots,y_n)=0$$

defines $(y_1,\dots,y_n)$ as a continuously differentiable functions of $(x_1,x_2,\dots,x_m)$:

$$\begin{cases} y_1 = f_1(x_1,\dots,x_m) \\ \quad\vdots \\ y_n = f_n(x_1,\dots,x_m) \end{cases}$$

for $(x_1,\dots,x_m,y_1,\dots,y_n)$ close to $(x_1{}^0,\dots,x_m{}^0,y_1{}^0,\dots y_n{}^0)$.

- **Example**: Consider the system of non-linear equations

$$\begin{cases} u^2 + v = xy \\ uv = -x^2 + y^2 \end{cases} \tag{3.4}$$

when does the equations define u, v as differential functions of x and y? Find the partial derivative of u, v wrt x and y.

- Let
$$\begin{cases} F^1(x,y,u,v) = u^2 + v - xy \\ F^2(x,y,u,v) = uv + x^2 - y^2 \end{cases}$$
then the determinant of Jacobian matrix is $|J| = \left|\dfrac{\partial(F^1, F^2)}{\partial(u,v)}\right| = \begin{vmatrix} 2u & 1 \\ v & u \end{vmatrix} = 2u^2 - v$

- From the implicit function theorem, when $2u^2 - v \neq 0$, the two equations define u, v as differential functions of x and y
- To find the partial derivatives $\partial u/\partial x$ and $\partial v/\partial x$, take partial derivatives wrt x in (3.4)

$$\begin{cases} 2u\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial x} = y \\ v\dfrac{\partial u}{\partial x} + u\dfrac{\partial v}{\partial x} = -2x \end{cases} \qquad \text{or} \begin{pmatrix} 2u & 1 \\ v & u \end{pmatrix} \begin{pmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial x} \end{pmatrix} = \begin{pmatrix} y \\ -2x \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial x} \end{pmatrix} = \begin{pmatrix} 2u & 1 \\ v & u \end{pmatrix}^{-1} \begin{pmatrix} y \\ -2x \end{pmatrix} = \frac{1}{|J|}\begin{pmatrix} u & -1 \\ -v & 2u \end{pmatrix}\begin{pmatrix} y \\ -2x \end{pmatrix} = \frac{1}{|J|}\begin{pmatrix} yu + 2x \\ -yv - 4xu \end{pmatrix}$$