Final Exam Fall 2017
Dec. 14, 2017

- **Answer all the following questions, full work must be shown.**
- **Calculators are not allowed.**
- **Duration of exam: 2.5 hours**

1. *[30 marks, 10 marks each]* Three (unrelated) questions on **Concavity/convexity and quasi-concavity/convexity**

   1a. Use the bordered Hessian to show that $f(x,y) = xy^2$ for $(x,y) \in R^2_{++}$ is strictly quasi-concave.

   1b. Is the following function $f(x,y) = \frac{y}{1+x^2}$ for $(x,y) \in R^2$ concave, convex, quasi-concave, or quasi-convex?

   1c. Are there functions which are both strictly concave and strictly quasi-convex? If your answer is "yes", give an example of a function f with such property, explain why the function you choose is strictly concave/quasi-convex. If your answer is "no", explain why.

2. *[50 marks]* Three (unrelated) questions on **Optimization**

   2a. *[10 marks = 4 + 4 + 2]* Consider the following function defined for $x \in R^2$ by

$$f(x) = x_1^2 (1 + x_2)^3 + x_2^2$$

   i. Find the stationary point(s) x^*
   ii. Is x^* local maximum, local minimum, or saddle points?
   iii. Is x^* global maximum or global minimum?

   2b. *[20 marks = 5+5+5+5]* Consider the following problem:

$$\begin{cases} F(p_1, p_2, I) = \max_{x_1>0, x_2>0} \{x_1^2 x_2\} \\ s.t. \;\; p_1 x_1 + p_2 x_2 = I \end{cases}$$

   i. Find the solution satisfying the first order conditions.
   ii. Claim that it is a local maximum by checking the property of bordered Hessian matrix
   iii. Claim that it is a global maximum.
   iv. Find $\frac{\partial F}{\partial p_1}, \frac{\partial F}{\partial p_2}$ and $\frac{\partial F}{\partial I}$

2c. *[20 marks =15+5]* By using x_1 and x_2 units of two inputs, a firm produces $\sqrt{x_1 x_2}$ units of product. The input factor costs are w and p per unit, respectively. The firm wants to minimize the costs of producing at least q units, but it is required to use at least a units of the first input. Here, w, p, q, a are positive constants. The optimization problem is

$$F(w,p,q,a) = \min_{x_1>0, x_2>0} \{wx_1 + px_2\}$$

$$\text{subject to } x_1 \geq a \text{ and } \sqrt{x_1 x_2} \geq q$$

i. Find all solutions satisfying the first order conditions and Kuhn Tucker conditions.

ii. Find the global maximum by checking sufficient conditions

3. *[20 marks]* Two (unrelated) questions on **Definiteness of matrices**

3a. *[5 marks]*Is the following matrix positive definite?

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

3b. *[15 marks]*Determine the value(s) of a for which the following matrix is positive definite, positive semidefinite, negative definite, negative semidefinite, or indefinite (There may be no values of a for which the matrix satisfies some of these conditions.)

$$\begin{pmatrix} a & 1 & -2 \\ 1 & -1 & 0 \\ -2 & 0 & -2 \end{pmatrix}$$