Final Exam Fall 2017 – solution

1. Three (unrelated) questions on **Concavity/convexity and quasi-concavity/convexity**

   1a. *Solution:* Bordered Hessian:

$$B = \begin{pmatrix} 0 & y^2 & 2xy \\ y^2 & 0 & 2y \\ 2xy & 2y & 2x \end{pmatrix}$$

   its leading principal minors satisfies

$$\begin{aligned} b_2 &= -y^4 < 0 \\ b_3 &= 6xy^4 > 0 \end{aligned}$$

   1b. *Solution:* None of the above.

   The upper level set $\{(x, y) \in R^2 : f(x, y) \geq t\}$ is not convex when $t < 0$, thus function not quasi-concave, thus not concave

   The lower level set $\{(x, y) \in R^2 : f(x, y) \leq t\}$ is not convex when $t > 0$, thus function not quasi-convex, thus not convex

   1c. *Solution:* Yes, a strictly concave monotone function such as $f(x) = -e^x \,(x \in R)$ or $f(x) = -x^2 \,(x \in R_+)$ are both strictly concave and strictly quasi-convex

2. Three (unrelated) questions on **Optimization**

   2a. Consider the following function defined for $x \in R^2$ by

$$f(x) = x_1^2 (1 + x_2)^3 + x_2^2$$

   i. *Solution:* FOC

$$\begin{aligned} f_{x_1} &= 2x_1 (1 + x_2)^3 = 0 \\ f_{x_2} &= 3x_1^2 (1 + x_2)^2 + 2x_2 = 0 \end{aligned}$$

   only solution: $x^* = (0, 0)$

   ii. *Solution:* Hessian matrix:

$$f''(x) = \begin{pmatrix} 2(1 + x_2)^3 & 6x_1 (1 + x_2)^2 \\ 6x_1 (1 + x_2)^2 & 6x_1^2 (1 + x_2) + 2 \end{pmatrix}$$

   thus

$$f'(x^*) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} > 0$$

   thus x^* is a local minimum.

iii. *Solution:* x^* is not global minimum For example

$$f(1,-4) = -3^3 + 4^2 < 0 = f(0,0)$$

In fact, fixed $x_1 \neq 0$, $f(x) \to -\infty$ when $x_2 \to -\infty$, and for fixed $x_2 < -1$, $f(x) \to -\infty$ when $x_1 \to \infty$.

2b. Consider the following problem:

$$\begin{cases} F(p_1, p_2, I) = \max_{x_1, x_2 > 0} \{x_1^2 x_2\} \\ s.t. \;\; p_1 x_1 + p_2 x_2 = I \end{cases}$$

i. *Solution:*

$$\begin{array}{c} F(p_1, p_2, I) = \max_{x_1, x_2 > 0} \{x_1^2 x_2\} \\ s.t. \;\; p_1 x_1 + p_2 x_2 = I \end{array}$$

Lagrange function

$$L(x, \lambda) = x_1^2 x_2 + \lambda (I - p_1 x_1 - p_2 x_2)$$

FOC:

$$\begin{cases} 2x_1 x_2 - \lambda p_1 = 0 \\ x_1^2 - \lambda p_2 = 0 \end{cases} \implies \begin{cases} x_1 = \sqrt{\lambda p_2} \\ x_2 = \frac{\sqrt{\lambda} p_1}{2\sqrt{p_2}} \end{cases}$$

sub this into the equality contraint: $\implies \sqrt{\lambda} p_1 \sqrt{p_2} + \frac{1}{2}\sqrt{\lambda} p_1 \sqrt{p_2} = I$

$$\lambda^* = \left(\frac{2I}{3p_1\sqrt{p_2}}\right)^2; x_1^* = \frac{2I}{3p_1}; x_2^* = \frac{2I}{3p_1\sqrt{p_2}} \frac{p_1}{2\sqrt{p_2}} = \frac{I}{3p_2}$$

ii. *Solution:* Bordered Hessian:

$$\begin{aligned} B &= \begin{vmatrix} 0 & -p_1 & -p_2 \\ -p_1 & 2x_2^* & 2x_1^* \\ -p_2 & 2x_1^* & 0 \end{vmatrix} = 4p_1 p_2 x_1^* - 2p_2^2 x_2^* \\ &= (4p_1 p_2) \frac{2I}{3p_1} - 2p_2^2 \left(\frac{I}{3p_2}\right) = 2p_2 I > 0 \end{aligned}$$

thus x^* is local maximum

iii. *Solution:* objective function $f(x) = x_1^2 x_2$ and $\lambda^* (I - p_1 x_1 - p_2 x_2)$ are quasi-concave and $f'(x^*) = \left(2x_1^* x_2^*, (x_1^*)^2\right) \neq 0$, global maximum follows from Sufficient condition #2.

iv. *Solution:* the Lagrange function

$$L(x, \lambda, p_1, p_2, I) = x_1^2 x_2 + \lambda (I - p_1 x_1 - p_2 x_2)$$

thus

$$\frac{\partial L}{\partial p_1} = -\lambda x_1, \frac{\partial L}{\partial p_2} = -\lambda x_2, \frac{\partial L}{\partial I} = \lambda$$

from Envelope Theorem

$$\frac{\partial F}{\partial p_1} = -\lambda^* x_1^*, \frac{\partial F}{\partial p_2} = -\lambda^* x_2^*, \frac{\partial F}{\partial I} = \lambda^*$$

where λ^*, x_1^*, x_2^* can be found in part (i).

2c.

i. *Solution:* Convert to maximization problem:

$$G(w,p,q,a) = \max_{x_1>0, x_2>0} \{-wx_1 - px_2\}$$

$$\text{subject to } x_1 \geq a \text{ and } \sqrt{x_1 x_2} \geq q$$

Lagrange function:

$$L(x_1, x_2, \lambda, \mu) = -wx_1 - px_2 + \lambda(x_1 - a) + \mu\left(\sqrt{x_1 x_2} - q\right)$$

$$FOC \quad : \quad \begin{cases} -w + \lambda + \frac{\mu}{2}\sqrt{\frac{x_2}{x_1}} = 0 \\ -p + \frac{\mu}{2}\sqrt{\frac{x_1}{x_2}} = 0 \end{cases}$$

$$KTC \quad : \quad \begin{cases} \lambda(x_1 - a) = 0 \\ \mu\left(\sqrt{x_1 x_2} - q\right) = 0 \end{cases}$$

obviously, $\mu \neq 0$, thus $\sqrt{x_1 x_2} - q = 0$

Case 1: $\lambda = 0$, then

$$\begin{cases} -w + \frac{\mu}{2}\sqrt{\frac{x_2}{x_1}} = 0 \\ -p + \frac{\mu}{2}\sqrt{\frac{x_1}{x_2}} = 0 \\ \sqrt{x_1 x_2} - q = 0 \end{cases} \Longrightarrow \begin{cases} \frac{x_2}{x_1} = \frac{w}{p} \\ \sqrt{x_1 x_2} = q \end{cases} \Longrightarrow \begin{cases} x_1^* = q\sqrt{\frac{p}{w}} \\ x_2^* = q\sqrt{\frac{w}{p}} \\ \lambda^* = 0 \\ \mu^* = 2\sqrt{pw} \end{cases}$$

solution if $q\sqrt{\frac{p}{w}} \geq a$, or $p\left(\frac{q}{a}\right)^2 \geq w$

Case 2: $\lambda > 0$, then

$$\begin{cases} x_1^* = a \\ x_2^* = \frac{q^2}{a} \\ \mu^* = 2p\sqrt{\frac{x_2^*}{x_1^*}} = \frac{2pq}{a} \\ \lambda^* = w - \frac{pq}{a} \times \left(\frac{q}{a}\right) = w - p\left(\frac{q}{a}\right)^2 \end{cases}$$

solution if $w - p\left(\frac{q}{a}\right)^2 > 0$

Summary:

$$\text{Case 1: If } w-p\left(\frac{q}{a}\right)^2 \le 0 \quad \text{Solution: } \begin{cases} x_1^* = q\sqrt{\frac{p}{w}} \\ x_2^* = q\sqrt{\frac{w}{p}} \\ \lambda^* = 0 \\ \mu^* = 2\sqrt{pw} \end{cases}$$

$$\text{Case 2: If } w-p\left(\frac{q}{a}\right)^2 > 0 \quad \text{Solution: } \begin{cases} x_1^* = a \\ x_2^* = \frac{q^2}{a} \\ \mu^* = \frac{2pq}{a} \\ \lambda^* = w-p\left(\frac{q}{a}\right)^2 \end{cases}$$

ii.

$$L(x_1,x_2,\lambda^*,\mu^*) = -wx_1 - px_2 + \lambda^*(x_1-a) + \mu^*\left(\sqrt{x_1x_2}-q\right)$$

Since for both cases, $\mu^* > 0$ and

$$L_x''(x_1,x_2,\lambda^*,\mu^*) = \mu^* \begin{pmatrix} -\frac{\sqrt{x_2}}{4x_1\sqrt{x_1}} & \frac{1}{4\sqrt{x_1x_2}} \\ \frac{1}{4\sqrt{x_1x_2}} & -\frac{\sqrt{x_1}}{4x_2\sqrt{x_2}} \end{pmatrix} \le 0$$

thus $L(x_1,x_2,\lambda^*,\mu^*)$ is concave function, therefore, in both cases, the solution is the global maximum.

3. Two (unrelated) questions on **Definiteness of matrices**

3a. Is the following matrix positive definite?

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Solution: $A > 0$,if and only if all leading principal minors $b_i > 0$, since

$$\begin{aligned} b_1 &= 1 > 0 \\ b_2 &= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 > 0 \end{aligned}$$

$$\begin{aligned} b_3 &= \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 1 > 0 \\ b_4 &= |A| = -\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -1 < 0 \end{aligned}$$

thus A is not positive definite

3b. Determine the value(s) of a for which the following matrix is positive definite, positive semidefinite, negative definite, negative semidefinite, or indefinite (There may be no values of a for which the matrix satisfies some of these conditions.)

$$A = \begin{pmatrix} a & 1 & -2 \\ 1 & -1 & 0 \\ -2 & 0 & -2 \end{pmatrix}$$

Solution: A can only be candidate for $A < 0, A \leq 0$ or indefinite

Case 1: $A < 0$ iff

$$\begin{aligned} b_1 &= a < 0 \\ b_2 &= \begin{vmatrix} a & 1 \\ 1 & -1 \end{vmatrix} = -a - 1 > 0, \text{ or } a < -1 \\ b_3 &= |A| = 2a + 6 < 0, \text{ or } a < -3 \end{aligned}$$

summary: $A < 0$ iff $a < -3$

Case 2: $A \leq 0$ iff

$$\begin{aligned} a &\leq 0 \\ \begin{vmatrix} a & 1 \\ 1 & -1 \end{vmatrix} &= -a - 1 \geq 0, \text{ or } a \leq -1 \\ \begin{vmatrix} -1 & 0 \\ 0 & -2 \end{vmatrix} &= 2 \geq 0 \\ \begin{vmatrix} a & -2 \\ -2 & -2 \end{vmatrix} &= -2a - 4 \geq 0, \text{ or } a \leq -2 \\ |A| &= 2a + 6 \leq 0, \text{ or } a \leq -3 \end{aligned}$$

summary: $A \leq 0$ iff $a \leq -3$

In conclusion: $A < 0$ when $a < -3$; $A \leq 0$ when $a \leq -3$, and A is indefinite when $a > -3$