ECON 3113 Microeconomic Theory I
Lecture 11: Risk Preference

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April 2020

Why may expected value not be a good criterion in evaluating a lottery?

- **St. Petersburg Paradox**
- Consider the following lottery:

Prize	1	2	4	8	16	32	....
Probability	1/2	1/4	1/8	1/16	1/32	1/64	....

- The expected value of this lottery is

$$
\begin{aligned}
& \frac{1}{2} \times 1 + \frac{1}{4} \times 2 + \frac{1}{8} \times 4 + \frac{1}{16} \times 8 + \frac{1}{32} \times 16 + \frac{1}{64} \times 32 + \ldots. \\
= \; & \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots. \\
= \; & \infty.
\end{aligned}
$$

- Would anyone be willing to pay 1 million to play this lottery?

"Resolution" to St. Petersburg Paradox

- One dollar given to a rich man is worth less than a dollar given to a poor man —— the marginal utility of money is decreasing!
- Suppose $u(w) = \sqrt{w}$. Then the **expected utility** is

$$
\begin{aligned}
& \frac{1}{2}u(1) + \frac{1}{4}u(2) + \frac{1}{8}u(4) + \frac{1}{16}u(8) + \frac{1}{32}u(16) + ... \\
= \quad & \frac{1}{2} \times \sqrt{1} + \frac{1}{4} \times \sqrt{2} + \frac{1}{8} \times \sqrt{4} + \frac{1}{16} \times \sqrt{8} + \frac{1}{32} \times \sqrt{16} + ... \\
= \quad & \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{16}} + ...\right) \\
= \quad & \frac{1}{2}\frac{1}{1 - \frac{1}{\sqrt{2}}} \approx 1.7071.
\end{aligned}
$$

"Resolution" to St. Petersburg Paradox

- The resolution above is only partial. This lottery will make the problem re-emerges:

Prize	1^2	2^2	4^2	8^2	16^2	32^2	....
Probability	$1/2$	$1/4$	$1/8$	$1/16$	$1/32$	$1/64$	....

- To fully resolve St. Petersburg paradox, we need an upper bound on the utility value.
- Alternatively, there is no St. Petersburg paradox is the state space is finite.

Von Neumann-Morgenstern Utility

- This partial resolution is insightful because it shows that in some cases, expected utility can be a more useful concept than expected value.
- This idea was developed by John Von-Neumann and Oscar Morgenstern using the axiomatic approach.
- The alternatives (objects to be chosen) here are lotteries.
    - A lottery is a description of state space, the prize of each state, and the probability of each state.
- The consumer/individual has a complete and transitive preference $\succsim$ over lotteries.
- Von-Neumann and Morgenstern: If this preference $\succsim$ has some reasonable properties (axioms), then it can be represented using expected utility.

Lottery

- Fix the set of possible prizes x_1, x_2, ..., x_n.
  - For simplicity, we can name the states by the prizes they provide.
- A typical lottery thus takes the form

Prize	x_1	x_2	...	x_n
Probability	p_1	p_2	...	p_n

- With a fixed state space/ prize space, a lottery is a probability function over the set of all possible prizes.
  - A typical lottery is denoted by $L = (p_1, p_2, ..., p_n)$.
  - Being a probability function, $p_1 + p_2 + ... + p_n = 1$.

Some Special Lotteries

- A *degenerate* lottery assigns all probability to a single prize:

$$L_i = \left(0, 0, ...0, \underbrace{1}_{\text{i-th position}}, 0, ..., 0\right).$$

- Given any two lotteries, L and L', and some $\alpha \in [0, 1]$, a **compound lottery** $\alpha L + (1-\alpha) L'$ is a two-step lottery.
  - The first draw decides what lotteries to get: lottery L with probability α and lottery L' with probability $1-\alpha$.
  - The second draw decides the prize with probability determined by the lottery drawn in the first stage.

Consequentialist

- Any compound lottery can be reduced to a simple lottery.
- With lotteries $L = (p_1, p_2, ..., p_n)$ and $L' = (p'_1, p'_2, ..., p'_n)$ and $\alpha \in [0, 1]$, the compound lottery $\alpha L + (1-\alpha) L'$ has prize x_i realizing with probability $\alpha p_i + (1-\alpha) p'_i$:

Prize	x_1	...	x_n
L	p_1	...	p_n
L'	p'_1	...	p'_n
$\alpha L + (1-\alpha) L'$	$\alpha p_1 + (1-\alpha) p'_1$	...	$\alpha p_n + (1-\alpha) p'_n$

- The individual is a **consequentialist**: he views the compound lottery $\alpha L + (1-\alpha) L'$ and the reduced lottery $(\alpha p_1 + (1-\alpha) p'_1, ..., \alpha p_n + (1-\alpha) p'_n)$ as identical objects.

Axioms

Definition

Preference $\succsim$ over lotteries satisfies the **independence axiom** if for any three lotteries L, L', and L'', and any $\alpha \in [0,1]$,

$$L \succsim L' \Rightarrow \alpha L + (1-\alpha) L'' \succsim \alpha L' + (1-\alpha) L''.$$

Definition

Preference $\succsim$ over lotteries satisfies the **continuity axiom** if for any three lotteries such that $L'' \succsim L \succsim L'$, there is a $\alpha \in [0,1]$ such that $L \sim \alpha L' + (1-\alpha) L''$.

Challenging the Independence Axiom: Allais Paradox

- Which lottery L_1 or L_1' would you prefer?

Prize	$1.1M$	$1M$	0
Lottery L_1	0	1	0
Lottery L_1'	0.98	0	0.02

- Which lottery L_2 or L_2' would you prefer?

Prize	$1.1M$	$1M$	0
Lottery L_2	0	0.5	0.5
Lottery L_2'	0.49	0	0.51

Challenging the Independence Axiom: Allais Paradox

- If you have $L_1 \succ L_1'$ and $L_2 \prec L_2'$, then your preference violates the independence axiom.

$$L_2 = 0.5L_1 + 0.5L_0 \text{ and } L_2' = 0.5L_1' + 0.5L_0,$$

where L_0 is the degenerate lottery of zero prize:

Prize	$1.1M$	$1M$	0
Lottery L_0	0	0	1

Von Neumann-Morgenstern Theorem

Theorem

If a complete and transitive preference $\succsim$ over lotteries satisfies the independence axiom and the continuity axiom, then it can be represented by some utility function $u(x)$ over prizes, that is, for any pair of lotteries $L = (p_1, p_2, ..., p_n)$ and $L' = (p'_1, p'_2, ..., p'_n)$,

$$L \succsim L' \Leftrightarrow \sum_{i=1}^{n} p_i u(x_i) \geq \sum_{i=1}^{n} p'_i u(x_i).$$

- Function $u(x)$ is called the **von Neumann-Morgenstern utility function**.
- The **expected utility** of lottery $L = (p_1, p_2, ..., p_n)$ is

$$E_L[u(x)] = p_1 u(x_1) + p_2 u(x_2) + ... + p_n u(x_n) = \sum_{i=1}^{n} p_i u(x_i).$$

Proof (Optional)

- Let L_n and L_1 be the most and the least preferred degenerate lottery respectively.
- By the independence axiom, $L_n = \alpha L_n + (1-\alpha) L_n \succ \alpha L_n + (1-\alpha) L_1$.
- By the independence axiom again (and that the individual is consequentialist), for any $\beta > \alpha$,

$$\begin{aligned}
& \beta L_n + (1-\beta) L_1 \\
= \; & \frac{\beta-\alpha}{1-\alpha} L_n + \frac{1-\beta}{1-\alpha} \left[\alpha L_n + (1-\alpha) L_1\right] \\
\succ \; & \frac{\beta-\alpha}{1-\alpha} \left[\alpha L_n + (1-\alpha) L_1\right] + \frac{1-\beta}{1-\alpha} \left[\alpha L_n + (1-\alpha) L_1\right] \\
= \; & \alpha L_n + (1-\alpha) L_1 .
\end{aligned}$$

- Therefore, for compound lottery of the form $\alpha L_n + (1-\alpha) L_1$, the larger the probability of L_n, the better it is.

Proof (Optional)

- Continuity axiom: for any degenerate lottery L_i, there is some $\alpha \in [0, 1]$ such that $L_i \sim (1 - \alpha) L_1 + \alpha L_n$.
  - It follows from above that this value α is unique for each L_i.
- Let's call this α by $u(x_i)$, so by definition:

$$L_i \sim (1 - u(x_i)) L_1 + u(x_i) L_n.$$

Proof (Optional)

- Any lottery $L = (p_1, p_2, ..., p_n)$ can be written as $L = p_1 L_1 + p_2 L_2 + ... + p_n L_n$ (consequentialist).
- Applying the independence axiom to iteratively replace L_i with $(1 - u(x_i)) L_1 + u(x_i) L_n$ preserves indifference:

$$
\begin{aligned}
L &\sim p_1 \left[(1 - u(x_1)) L_1 + u(x_1) L_n \right] + p_2 L_2 + ... + p_n L_n \\
&\sim p_1 \left[(1 - u(x_1)) L_1 + u(x_1) L_n \right] \\
&\quad + p_2 \left[(1 - u(x_2)) L_1 + u(x_2) L_n \right] + ... + p_n L_n \\
&\sim ... \sim \\
&\sim p_1 \left[(1 - u(x_1)) L_1 + u(x_1) L_n \right] \\
&\quad + p_2 \left[(1 - u(x_2)) L_1 + u(x_2) L_n \right] + ... + \\
&\quad + p_n \left[(1 - u(x_n)) L_1 + u(x_n) L_n \right].
\end{aligned}
$$

Proof (Optional)

- Therefore, lottery L is indifferent to the lottery

$$\left(1 - \underbrace{\left[p_1 u\left(x_1\right) + ... + p_n u\left(x_n\right)\right]}_{E_L[u(x)]}\right) L_1 + \underbrace{\left[p_1 u\left(x_1\right) + ... + p_n u\left(x_n\right)\right]}_{E_L[u(x)]} L_n$$

- Lottery $L' = \left(p'_1, p'_2, ..., p'_n\right)$ is indifferent to the lottery

$$\left(1 - \underbrace{\left[p'_1 u\left(x_1\right) + ... + p'_n u\left(x_n\right)\right]}_{E_{L'}[u(x)]}\right) L_1 + \underbrace{\left[p'_1 u\left(x_1\right) + ... + p'_n u\left(x_n\right)\right]}_{E_{L'}[u(x)]} L_n.$$

- By transitivity, $L \succsim L'$ if and only if

$$\begin{aligned} & \left(1 - E_L\left[u\left(x\right)\right]\right) L_1 + E_L\left[u\left(x\right)\right] L_n \\ \succsim \quad & \left(1 - E_{L'}\left[u\left(x\right)\right]\right) L_1 + E_{L'}\left[u\left(x\right)\right] L_n \end{aligned}$$

- By the result 3 pages before, this happens if and only if

$$E_L\left[u\left(x\right)\right] \geq E_{L'}\left[u\left(x\right)\right].$$

Expected Utility

- In expected-utility representation, the vN-M utility is **not purely ordinal**.
- Two vN-M utility functions represent the same preference for lotteries if and only if one is a **positive linear transformation** of the other.
  - A positive linear transformation takes the form: $f(x) = A + Bx$ for some numbers $B > 0$ and A.
- If $u(x)$ represents $\succsim$, so is $A + Bu(x)$:

$$
\begin{aligned}
& E_L[u(x)] \geq E_{L'}[u(x)] \\
\Leftrightarrow \quad & A \times E_L[u(x)] + B \geq A \times E_{L'}[u(x)] + B \\
\Leftrightarrow \quad & E_L[Au(x) + B] \geq E_{L'}[Au(x) + B]
\end{aligned}
$$

- We will see below that other transformation may not preserve the preference.

Risk Attitude

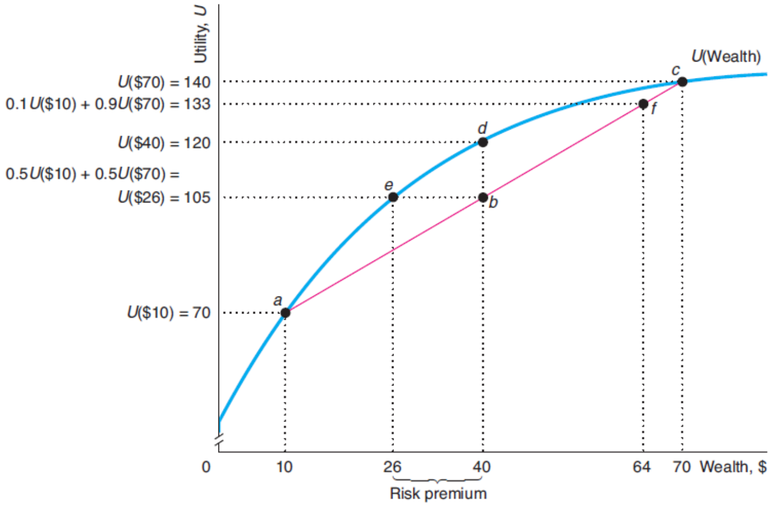
- For the rest of this lecture, we focus on the lotteries over wealth.
- The vN-M utility function is therefore defined over wealth.
- The **shape of an individual's vN-M utility function** determines his/her risk attitude.
- She is $\begin{cases} \textbf{risk-averse} \\ \textbf{risk-loving} \\ \textbf{risk-neutral} \end{cases}$ if her vN-M utility function is $\begin{cases} \textbf{concave} \\ \textbf{convex} \\ \textbf{linear} \end{cases}$.

Example

- Lottery 1: Gives wealth \$40 for certain.
- Lottery 2: Gives wealth \$70 with probability 50%, and gives wealth \$10 with probability 50%.
- Two lotteries have identical expected values:

$$E_{L1}[x] = 1 \times 40 = 40 = \frac{1}{2} \times 70 + \frac{1}{2} \times 10 = E_{L2}[x].$$

Risk Aversion: Concave vN-M Utility Function



- The marginal utility for money is diminishing as wealth increases.

Risk Aversion

- Her expected utility of lottery 1:

$$E_{L1}\left[U\left(x\right)\right] = 1 \times U\left(40\right) = 120.$$

- Her expected utility of lottery 2:

$$E_{L2}\left[U\left(x\right)\right] = \frac{1}{2} \times U\left(70\right) + \frac{1}{2} \times U\left(10\right) = \frac{1}{2}\left(140\right) + \frac{1}{2}\left(70\right) = 105.$$

- She strictly prefers lottery 1, and so would any risk-averse person.

Risk Premium and Certainty Equivalent

- For a person with vN-M utility $U(\cdot)$, the **certainty equivalent** ψ of lottery L is the guaranteed amount of money that she would view as equally desirable as lottery L. That is,

$$U(\psi) = E_L[U(x)].$$

- The **risk premium** of lottery L is the difference between its expected value and and certainty equivalent.

$$\text{Risk premium} = E_L[x] - \psi.$$

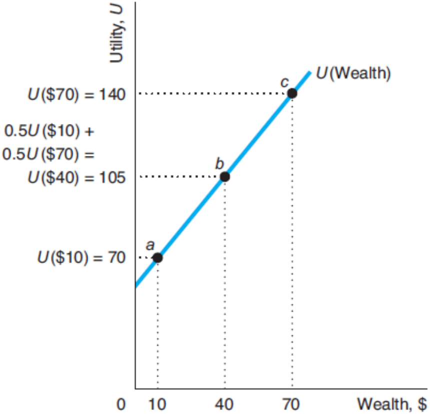
- In other words, the risk premium is the amount (of expected value) that the person is willing to give up to avoid the risk involved in lottery L altogether.

Risk Aversion: Example

- The certainty equivalent of lottery 2 is the sure amount that gives her an expected utility equal to $E_{L2}\left[U\left(x\right)\right] = 105$. Inspecting her vN-M utility function, it is equal to $\psi = 26$.
- The risk premium of lottery 2 is thus $E_{L2}\left[x\right] - \psi = 40 - 26 = 14$.

Risk Neutrality

- An individual is **risk-neutral** if her vN-M utility function is **linear**.
    - Her marginal utility for money is constant as wealth increases.
- A risk-neutral person evaluates lotteries by their **expected values**.
- She has a **zero risk premium** for all lotteries.



Risk Neutrality

- Her expected utility of lottery 1:

$$E_{L1}\left[U\left(x\right)\right] = 1 \times U\left(40\right) = 105.$$

- Her expected utility of lottery 2:

$$E_{L2}\left[U\left(x\right)\right] = \frac{1}{2} \times U\left(70\right) + \frac{1}{2} \times U\left(10\right) = \frac{1}{2}\left(140\right) + \frac{1}{2}\left(70\right) = 105.$$

- A risk-neutral person is indifferent between lotteries with equal expected value.

vN-M utility function is preserved only with positive linear transformation

- In this example, $U(x) = \frac{175}{3} + \frac{7}{6}x$.
- If we apply transformation $\sqrt{u}$ (which is strictly increasing) to it, it becomes $\tilde{U}(x) = \sqrt{\frac{175}{3} + \frac{7}{6}x}$.
- "Expected utility" of lottery 1 becomes:

$$E_{L1}\left[\tilde{U}(x)\right] = 1 \times \tilde{U}(40) = \sqrt{105} \approx 10.25.$$

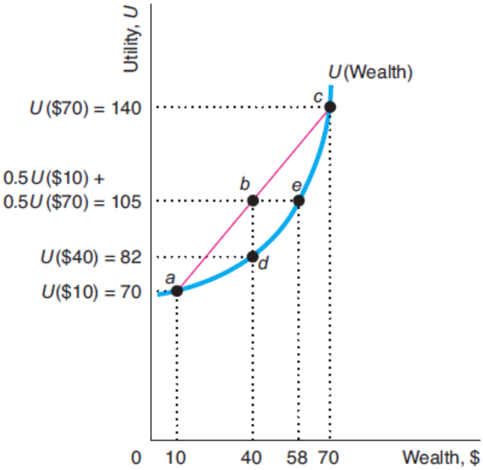
- "Expected utility" of lottery 2 becomes:

$$E_{L2}[U(x)] = \frac{1}{2} \times \tilde{U}(70) + \frac{1}{2} \times \tilde{U}(10) = \frac{1}{2}\sqrt{140} + \frac{1}{2}\sqrt{70} \approx 10.10.$$

- Therefore, lottery 1 has a higher "expected utility".

Risk Loving

- An individual is **risk-loving** if her vN-M utility function is **convex**.
    - Her marginal utility for money is increasing as wealth increases.
- A risk-loving person has a **negative risk premium**.



Risk Loving

- Her expected utility of lottery 1:

$$E_{L1}\left[U\left(x\right)\right] = 1 \times U\left(40\right) = 82.$$

- Her expected utility of lottery 2:

$$E_{L2}\left[U\left(x\right)\right] = \frac{1}{2} \times U\left(70\right) + \frac{1}{2} \times U\left(10\right) = \frac{1}{2}\left(140\right) + \frac{1}{2}\left(70\right) = 105.$$

- A risk-loving person would strictly prefer lottery 2.

State-preference Approach to Choice under Uncertainty

- Suppose you are endowed with wealth $\$w$ and a car worth $\$L$ to you.
- There is a probability p that the car will be stolen.
- Two states: good state (car not stolen) and bad state (car stolen)
- Your wealth in the good state is $W_g = w$ and your wealth in the bad state is $W_b = w - L$.
- Your expected utility is

$$(1-p)\,U\left(W_g\right) + pU\left(W_b\right).$$

- Suppose you are risk averse, so your U is concave.

Insurance

- Suppose an insurance company offers you a deal: you can get a coverage q for the loss of car at a premium of πq.
  - π is the premium rate/ premium per dollar of coverage.
- Insurance purchase allows you to transfer your wealth across the two states (at some exchange rate):

$$W_g = w - \pi q \text{ and } W_b = w - \pi q - L + q.$$

- By varying choice of $q \in [0, L]$, you can attain any combination of (W_g, W_b) that satisfies

$$\begin{aligned} &(1-\pi) W_g + \pi W_b = w - \pi L, \\ &W_g \leq w,\ W_b \leq w - \pi L \end{aligned}$$

Insurance

- The insurance purchase problem can be formulated as

$$\max_{W_g \le w, W_b \le w - \pi L} (1-p)\,U(W_g) + pU(W_b)$$

subject to the "budget constraint"

$$\underbrace{(1-\pi)}_{\text{like price of good } W_g} \times W_g + \underbrace{\pi}_{\text{like price of good } W_b} \times W_b = \underbrace{w - \pi L}_{\text{like income}}.$$

- The marginal rate of substitution is

$$MRS = \frac{(1-p)\,U'(W_g)}{pU'(W_b)}.$$

- The price ratio is

$$\frac{\text{Price of } W_g}{\text{Price of } W_b} = \frac{1-\pi}{\pi}.$$

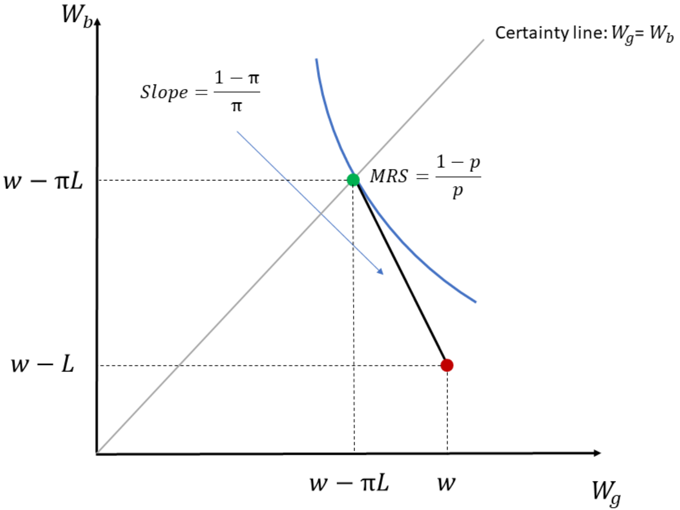
Insurance

- At an interior solution, *MRS* equals price ratio, so

$$\frac{1-p}{p} \times \frac{U'\left(W_g\right)}{U'\left(W_b\right)} = \frac{1-\pi}{\pi}.$$

- The expected profit of the insurance company is $\pi q - pq$.
- If the insurance company is risk-neutral and faces extremely intense competition, it will offer **actuarially fair rate**: $\pi = p$.
- In this case, the FOC gives $W_g = W_b$, so you will opt for full coverage: $q = L$.
    - This conclusion does not depend on the particular vN-M utility form.

Actuarially Fair Insurance



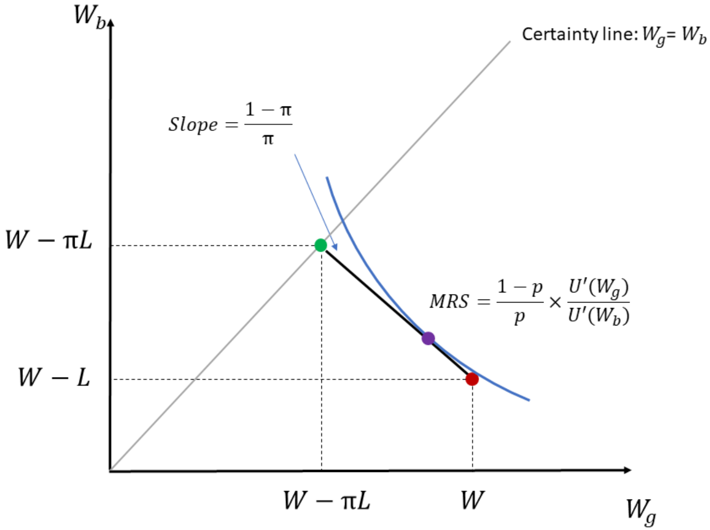
Insurance

- If the insurance company can make positive expected profit, then $\pi > p$.
- In this case,

$$\begin{aligned} \frac{U'(W_g)}{U'(W_b)} &= \frac{p}{1-p}\frac{1-\pi}{\pi} < 1 \\ \Rightarrow \quad U'(W_g) &< U'(W_b) \end{aligned}$$

- Risk-aversion implies concavity of U, so $W_g > W_b$, equivalently, $q < L$.
- You will go for partial coverage only.

Actuarially Unfair Insurance



Insurance: Another formulation

- Alternatively, we can formulate the insurance problem as one of choosing coverage directly:

$$\max_{q\in[0,L]} (1-p)\,U\left(w-\pi q\right)+pU\left(w-\pi q-L+q\right)$$

- FOC:

$$\begin{aligned} & (1-p)\,(-\pi)\,U'\left(w-\pi q\right)+p\,(1-\pi)\,U'\left(w-\pi q-L+q\right)=0 \\ \Leftrightarrow \quad & \frac{1-p}{p}\times\frac{U'\left(w-\pi q\right)}{U'\left(w-\pi q-L+q\right)}=\frac{1-\pi}{\pi} \end{aligned}$$

- If $\pi=p$, then $w-\pi q=w-\pi q-L+q \Leftrightarrow q=L$ (full coverage).
- If $\pi<p$, then $w-\pi q>w-\pi q-L+q \Leftrightarrow q<L$ (partial coverage).

Asset Investment

- Suppose you are endowed with wealth $\$w$ and you can invest in an asset that costs π per unit.
- Each unit of the asset pays $\$R > \pi$ in the good state and pays nothing in the bad state.
- The probability of good state is $1 - p$.
- If you buy x units of the asset, your wealth in the two states are respectively

$$W_g = w - \pi x + Rx \text{ and } W_b = w - \pi x.$$

- By varying choice of x, you can attain any combination of (W_g, W_b) that satisfies

$$\pi W_g + (R - \pi) W_b = wR.$$

Asset Investment

- This investment problem can therefore be formulated as

$$\max_{W_g, W_b} (1-p)\, U(W_g) + pU(W_b)$$

subject to budget constraint

$$\pi W_g + (R-\pi)\, W_b = wR.$$

- The marginal rate of substitution is

$$MRS = \frac{(1-p)\, U'(W_g)}{pU'(W_b)}.$$

- The price ratio is

$$\frac{\text{Price of } W_g}{\text{Price of } W_b} = \frac{\pi}{R-\pi}.$$

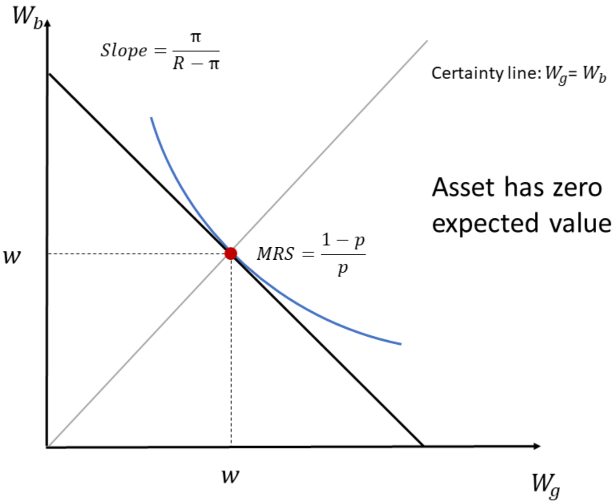
Asset Investment

- At an interior solution, *MRS* equals price ratio, so

$$\frac{1-p}{p} \times \frac{U'\left(W_g\right)}{U'\left(W_b\right)} = \frac{\pi}{R-\pi}.$$

- If the asset is actuarially fair, $\pi = (1-p)\,R$.
- In this case, the FOC gives $W_g = W_b$, so you will not invest in the asset: $x = 0$.
  - This conclusion does not depend on the particular vN-M utility form.

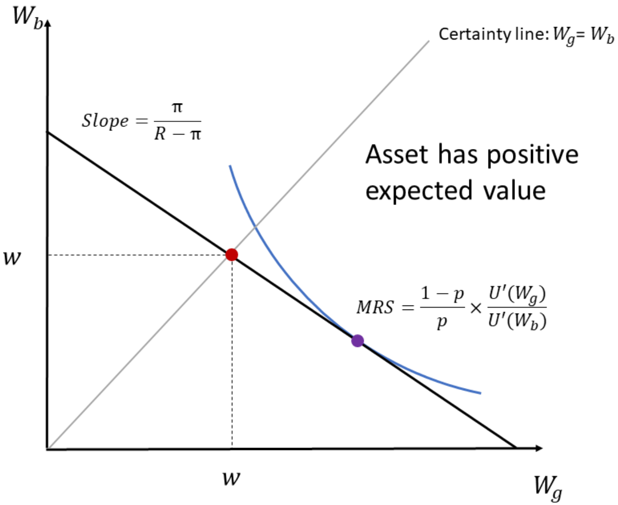
Asset Investment with Zero Expected Value



Asset Investment with Positive Expected Value

- If the asset has a positive expected value, $\pi < (1-p)R$, then FOC gives
$$\frac{U'(W_g)}{U'(W_b)} = \frac{\pi}{R-\pi} \times \frac{p}{1-p} < 1.$$
- Therefore, $W_g > W_b$ and you are a buyer of the asset.

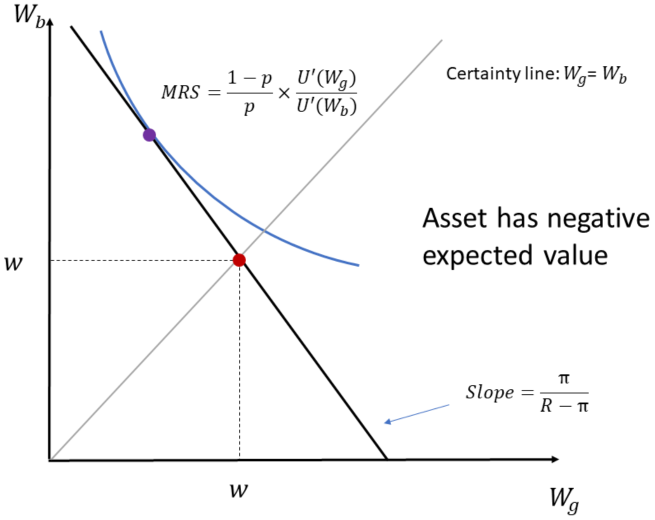
Asset with Positive Expected Value



Asset Investment with Negative Expected Value

- If the asset has a negative expected value, $\pi > (1-p)\,R$, then FOC gives
$$\frac{U'(W_g)}{U'(W_b)} = \frac{\pi}{R-\pi} \times \frac{p}{1-p} > 1.$$
- Therefore, $W_g < W_b$ and you are a short-seller of the asset (if it is feasible).

Asset with Negative Expected Value



Asset Investment: Another formulation

- Alternatively, we can formulate the investment problem as one of choosing x directly:

$$\max_{x} (1-p)\, U(w - \pi x + Rx) + pU(w - \pi x)$$

- FOC:

$$\begin{aligned} & (1-p)(R-\pi)\, U'(w - \pi x + Rx) + p(-\pi)\, U'(w - \pi x) = 0 \\ \Leftrightarrow \quad & \frac{U'(w - \pi x + Rx)}{U'(w - \pi x)} = \frac{\pi}{R-\pi} \times \frac{p}{1-p}. \end{aligned}$$

- If $\pi = (1-p)R$, then $w - \pi x + Rx = w - \pi x \Leftrightarrow x = 0$.
- If $\pi < (1-p)R$, then $w - \pi x + Rx > w - \pi x \Leftrightarrow x > 0$.
- If $\pi > (1-p)R$, then $w - \pi x + Rx < w - \pi x \Leftrightarrow x < 0$.

Summary

- **Expected utility** If a complete transitive preference over lottery satisfies the independence axiom and the continuity axiom, then it can be represented by expected utility.
- **Risk preference** An individual is risk averse if her vN-M utility function is concave. She is risk neutral if her vN-M utility function is linear.
- **Insurance** A risk-averse individual fully (partially) insure against her potential loss if charged actuarially fair (unfair) rate.
- **Investment** A risk-averse individual invests in an asset if and only if it has a positive expected value.