ECON3133
Microeconomic Theory II

Tutorial #9: Game Theory (cont.)

Today’s tutorial:

- Mixed strategies in 2 player games
- Tragedy of the Commons & games with externalities
- Sequential Games

Mixed Strategies

		2	
		L	H
1	L	40,20	10,10
	H	10,10	20,40

- In general, a static game may have Pure Strategy Nash equilibria and Mixed Strategy equilibria
- In this game, we have:
  - Pure Strategy NE:

(L, L) ; (H, H)

· 2 of them.

- Does either player have a dominant strategy?

1: L , H
2: L , H

No dominant strategies.

Mixed Strategies

			2	
			L	H
			q	$1-q$
1	L	p	40,20	10,10
	H	$1-p$	10,10	20,40

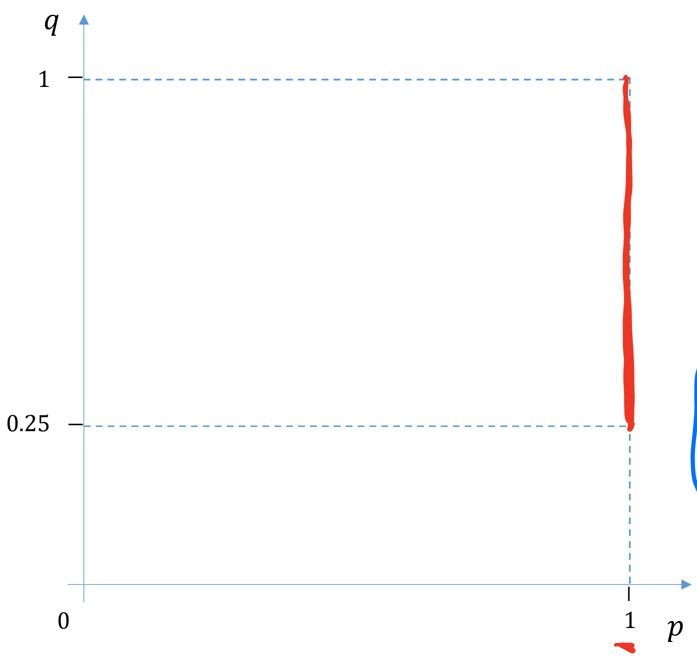
- Both players choose their strategies according to a probability distribution:
  - Player 1 plays L with probability p
  - Player 2 plays L with probability q
- Player 1 chooses best response taking account of q
- Player 2 chooses best response taking account of p
- Players maximise expected utility (assumed to be their pay-offs) subject to the other player's probability distribution

Mixed Strategies

			2	
			L	H
			q	$1-q$
1	L	p	40,20	10,10
	H	$1-p$	10,10	20,40

- Player 1:
- Expected (utility of) pay-off if player 1 always chooses L is:
- $E[u_1(L)] = 40\text{q} + 10(1-\text{q})$
  $= 30q + 10$
- Expected (utility of) pay-off if player 1 always chooses H is:
- $E[u_1(H)] = 10\text{q} + 20(1-\text{q})$
  $= 20 - 10q$

Mixed Strategies



- Player 1:
- If $E[u_1(L)] > E[u_1(H)]$ => player 1 always plays L
  - We have:

$$E[u_1(L)] > E[u_1(H)]$$

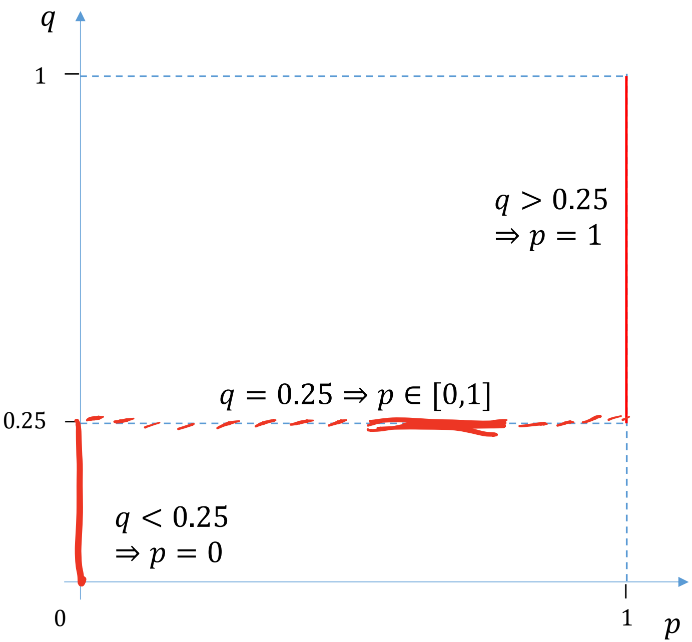
$$\Rightarrow \quad 30q + 10 > 20 - 10q$$

$$40q > 10$$

$$\Rightarrow q > 0.25$$

- So for $q > 0.25$
  - Player 1 always plays L
  - $p^* = 1$

Mixed Strategies



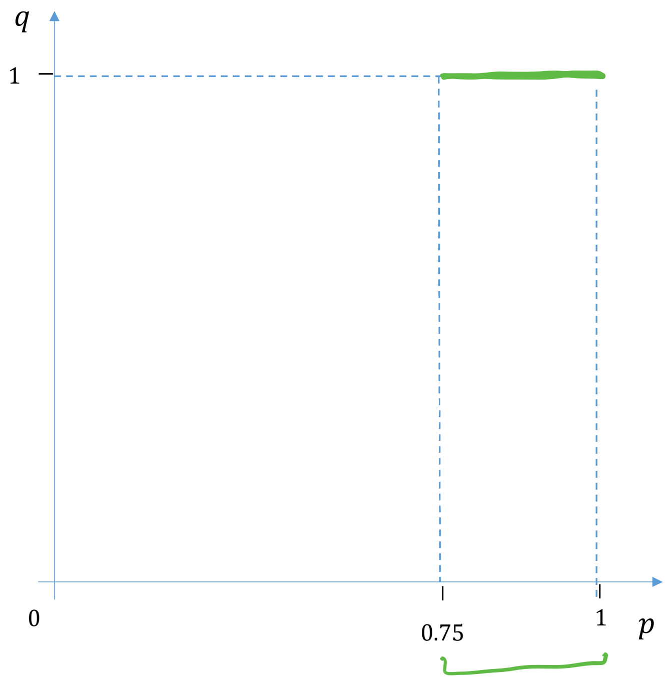
- Player 1:
- If $E[u_1(L)] < E[u_1(H)]$ => player 1 always plays H
  - ie for $q < 0.25$
  - Player 1 always plays H
  - $p^* = 0$
- If $E[u_1(L)] = E[u_1(H)]$ => player 1 indifferent between L and H
  - ie for $q = 0.25$, $p^* \in [0,1]$
    - (can be anywhere in $[0,1]$)
- Gives Player 1's best response curve in terms of p and q

Mixed Strategies

			2	
			L	H
			q	$1-q$
1	L	p	40,20	10,10
	H	$1-p$	10,10	20,40

- Player 2:
- Expected (utility of) pay-off if player 2 always chooses L is:
- $E[u_2(L)] = 20\text{p} + 10(1 - \text{p})$
  $= 10p + 10$
- Expected (utility of) pay-off if player 2 always chooses H is:
- $E[u_2(H)] = 10\text{p} + 40(1 - \text{p})$
  $= 40 - 30p$

Mixed Strategies



- Player 2:
- If $E[u_2(L)] > E[u_2(H)]$ => player 2 always plays L
  - We have:

$$E[u_2(L)] > E[u_2(H)]$$

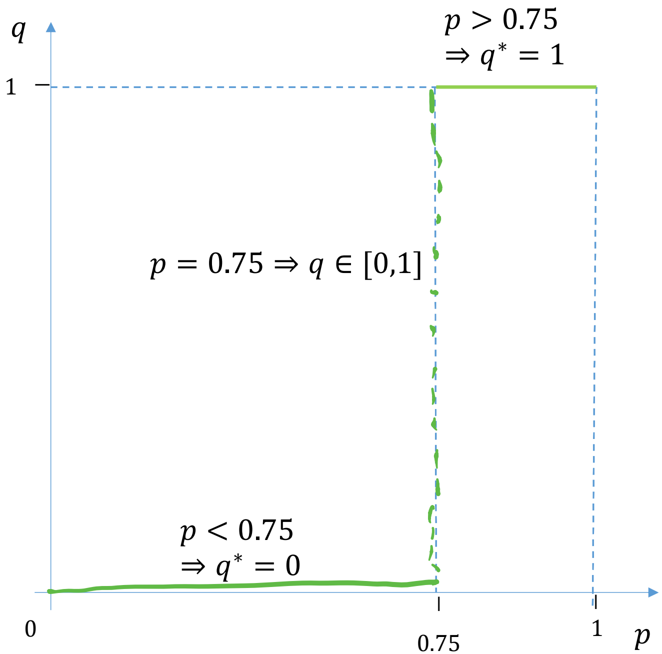
$$\Rightarrow \quad 10p + 10 > 40 - 30p.$$

$$40p > 30$$

$$p^* > 0.75$$

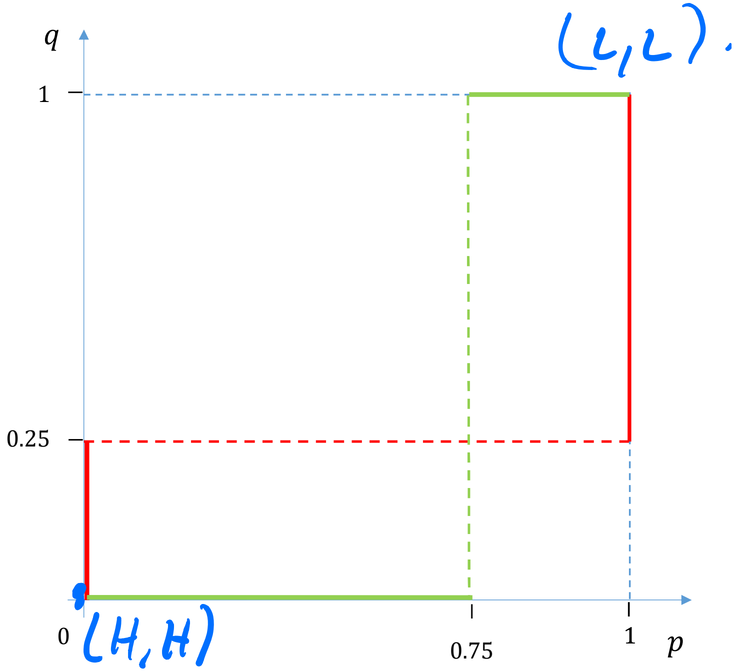
- So for $p > 0.75$
  - Player 2 always plays L
  - $q^* = 1$

Mixed Strategies



- Player 2:
- If $E[u_2(L)] < E[u_2(H)]$ => player 2 always plays H
  - ie for $\mathrm{p} < 0.75$
  - Player 2 always plays H
  - $q^* = 0$
- If $E[u_2(L)] = E[u_2(H)]$ => player 2 indifferent between L and H
  - ie for $\mathrm{p} = 0.75$, $q^* \in [0,1]$
    - (can be anywhere in $[0,1]$)
- Gives Player 2's best response curve in terms of p and q

Mixed Strategies



- Best responses:

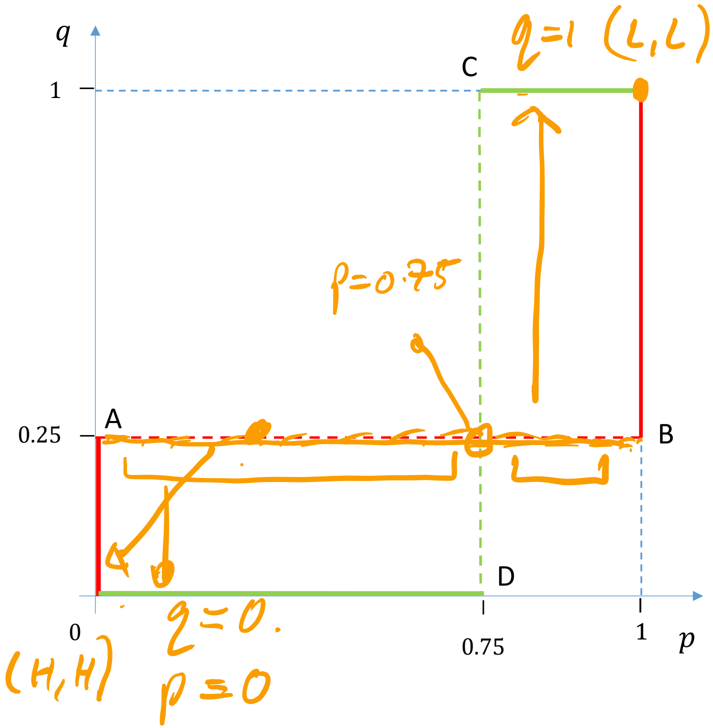
1) If player 2 has $q < 0.25$, player 1's best response is $p^* = 0$
   - Then player 2's best response is $q^* = 0$
   - And player 1's best response is still $p^* = 0$ H,H

- Result: a pure strategy Nash equilibrium at ~~(L, L)~~

2) If player 2 has $q > 0.25$, player 1's best response is $p^* = 1$
   - Then player 2's best response is $q^* = 1$
   - And player 1's best response is still $p^* = 1$

- Result: a pure strategy Nash equilibrium at ~~(H, H)~~ (L,L).

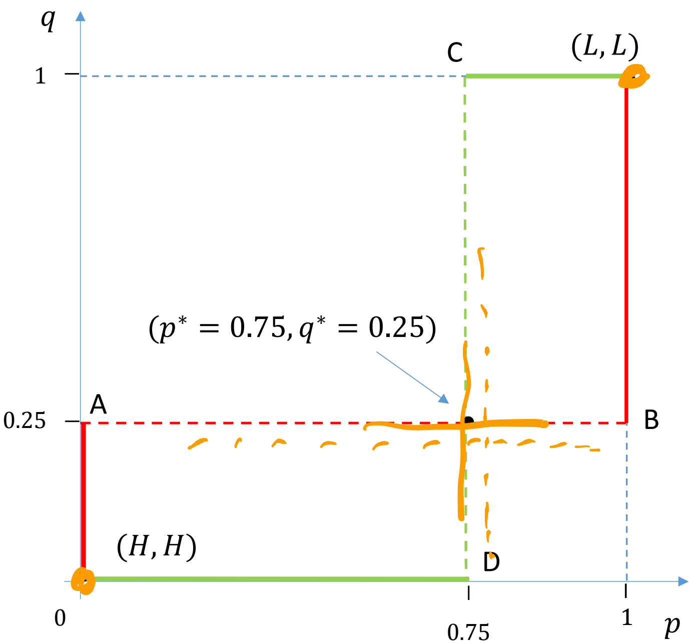
Mixed Strategies



- Best responses (cont.):
- What if player 2 plays $q = 0.25$?
  - Player 1 is indifferent between any $p \in [0,1]$ so can play anywhere on line AB
  - If player 1 plays $p < 0.75$, then player 2 plays $q = 0$; then player 1 plays $p = 0$ and we are back to a certain Nash equilibrium
  - If player 1 plays $p > 0.75$, then player 2 plays $q = 1$; then player 1 plays $p = 1$ and we are back to a certain Nash equilibrium
  - If player 1 plays $p = 0.75$, then player 2 is indifferent between any $q \in [0,1]$ so can play anywhere on line CD
    - But no reason to move from $q = 0.25$
- Gives a mixed strategy Nash equilibrium

$$(p^* = 0.75, q^* = 0.25)$$

Mixed Strategies



- Summary:
- In this game we have 3 NE:
  - 2 pure strategy: $(L, L), (H, H)$
  - 1 mixed strategy: $(p^* = 0.75, q^* = 0.25)$
- Notice that we have an odd number of NE

$p^* = 0.75$

$q^* = 0.25$

Mixed Strategies

		2	
		L	H
1	L	10,10	0,20
	H	20,0	20,20

- Consider this game:
- What are the pure strategy NE?

  (H, H).

- Are there any dominant strategies?

  1: H 2: H (H, H).

- What about mixed strategies?

Mixed Strategies

		2	
		L q	H
1	L p	10,10	0,20
	H	20,0	20,20

Note that in this case both players had a dominant strategy of playing H

- Player 1
- $E[u_1(L)] = 10\text{q}$ ✓
- $E[u_1(H)] = 20\text{q} + 20(1 - \text{q})$

  $= 20$ ✓
- $E[u_1(L)] < E[u_1(H)] \Rightarrow 10q < 20$ $\Rightarrow q < 2$.
- $q < 2$, which is always the case, and so player 1 always plays H

  (H, H)
- Player 2
- $E[u_2(L)] = 10\text{p}$
- $E[u_2(H)] = 20\text{p} + 20(1 - p)$

  $= 20$
- $E[u_2(L)] < E[u_1(H)] \Rightarrow 10p < 20$
- p$<$ 2, which is always the case, and so player 2 always plays H
- So we have 1 pure strategy NE and no mixed strategy NE

(H, H)

Tragedy of the commons

- Suppose there are two hotels (1 and 2) whose guests (q_1, q_2) share access to a beach and the ocean
- The value of each guest at the hotels (v_1, v_2) depends on the cleanliness of the beach and of the sea, which in turn depends (negatively) on how many people use them:
  - $v_1 = 180 - (q_1 + q_2)$
  - $v_2 = 180 - (q_1 + q_2)$
- Profits of each hotel (π_1, π_2) are given by:
  - $\pi_1 = v_1(q_1, q_2)q_1 = 180q_1 - q_1^2 - q_1 q_2$
  - $\pi_2 = v_2(q_1, q_2)q_2 = 180q_2 - q_2^2 - q_1 q_2$
- Questions:
- 1) Is there a Nash equilibrium in terms of the number of guests, q_1^*, q_2^* at each hotel?
- 2) What are profits at the Nash equilbrium and how do they compare to the case where the hotels collude?

Tragedy of the commons

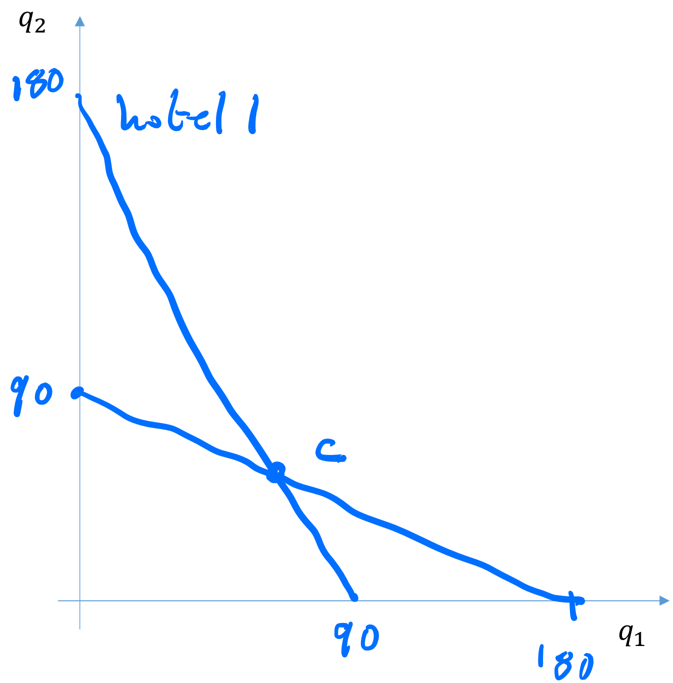
- Solution:
- The hotels have the following maximisation problem:
  - $\max_{q_1} \pi_1 = 180q_1 - q_1^2 - q_1 q_2$
  - $\max_{q_2} \pi_2 = 180q_2 - q_2^2 - q_1 q_2$
- The FOCs give the best response functions:

1: $180 - 2q_1 - q_2 = 0$

1: $\Rightarrow q_2 = 180 - 2q_1$

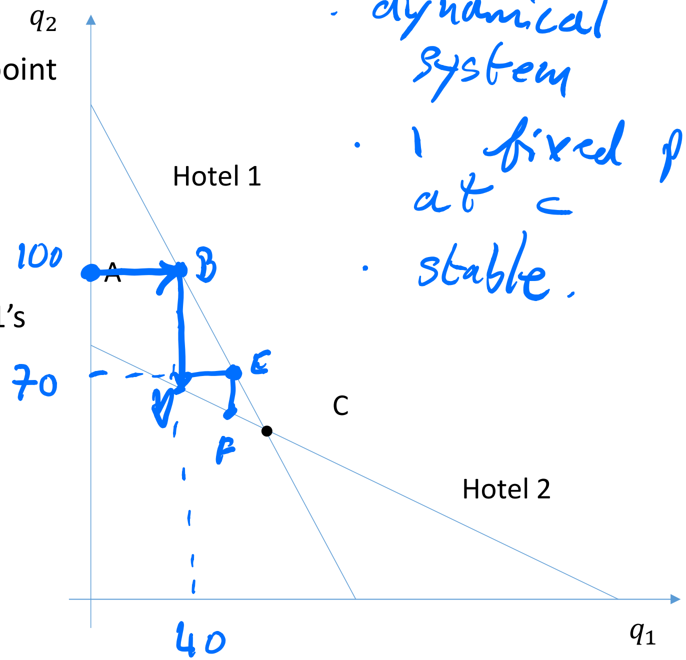
2: $180 - 2q_2 - q_1 = 0$

2: $\Rightarrow q_2 = 90 - \frac{q_1}{2}$



Tragedy of the commons

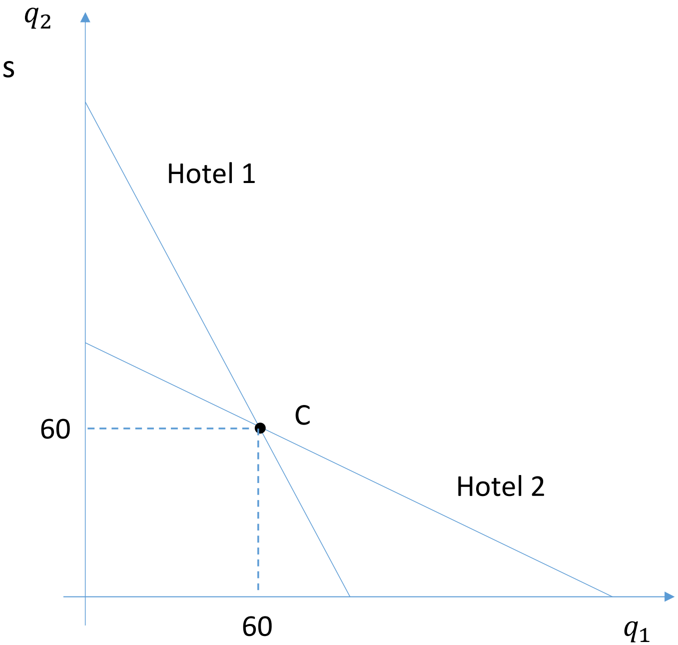
- Solution:
- Suppose that hotel 2 accepted 100 guests at point A; what is hotel 1's best response?
  - $q_2 = 180 - 2q_1$
  - $q_{1,BR} = 40$
- Then what is hotel 2's best response to hotel 1's response?
  - $q_2 = 90 - \frac{q_1}{2}$ 40
  - $q_{2,BR} = 70$
- Where does this process stop?
  - Point C



- dynamical system
- 1 fixed point at C
- stable.

Tragedy of the commons

- Solution:
- At point C we solve the best response functions simultaneously to give:
  - $q_2 = 180 - 2q_1 = 90 - \frac{q_1}{2}$
  - ie $q_1^* = 60$
  - $q_2^* = 60$
- Total profits here are:



$\pi_1 = V_1(q_1^*, q_2^*) q_1^*$
$= [180 - (60+60)] 60$
$= 60^2$
$= 3600$
$\pi_2 = 3600$

$\max_Q (180 - Q) Q$

FOC: $180 - 2Q = 0$
$\Rightarrow \underline{\underline{Q^* = 90}}$

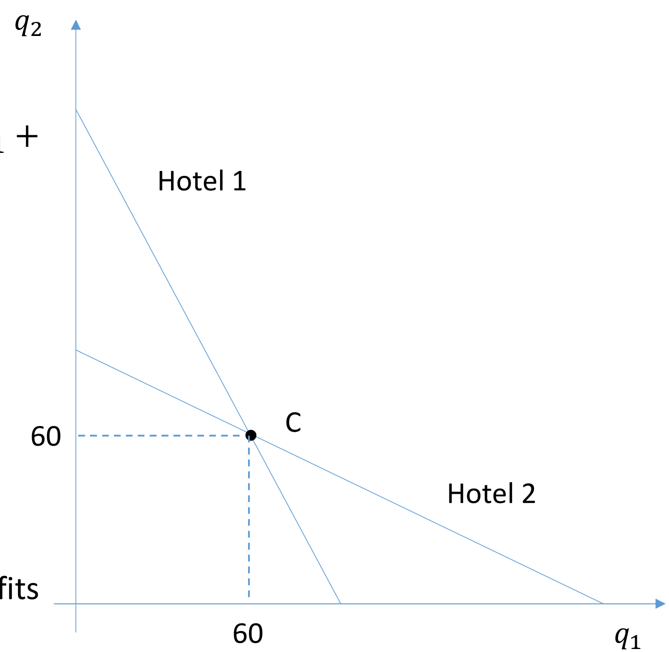
$\pi = \pi_1 + \pi_2 = 7200$

- Competitive: 120 ; collusion 90.

$\Pi_{collusion} = (180 - 90)90 = 90^2 = 8100.$

Tragedy of the commons

- Solution:
- With collusion, the hotels act as a monopoly to maximise total profits Π with respect to $Q = q_1 + q_2$:
- $\max_Q \Pi = (180 - Q)Q$
- FOC:
  - $180 - 2Q = 0$
  - $\Rightarrow\ Q^* = 90$
- And total profits are $(180 - 90) \times 90 = 8100$
- So with collusion, total output is lower, but profits are higher



Tragedy of the commons

- Now suppose that the value of each guest at the hotels still depends on the number of guests, but is now given by:
  - $v_1 = 90 - \left(\frac{q_1}{4} + q_2\right)$
  - $v_2 = 90 - \left(\frac{q_2}{4} + q_1\right)$

  The value to a hotel of its own guests is higher than the value to it of the other hotel's guests
- Profits of each hotel (π_1, π_2) are given by:
  - $\pi_1 = \underline{v_1(q_1, q_2)q_1} = \underline{90q_1 - \frac{q_1^2}{4} - q_1 q_2}$
  - $\pi_2 = v_2(q_1, q_2)q_2 = \underline{90q_2 - \frac{q_2^2}{4} - q_1 q_2}$
- Questions:
- 1) Is there a Nash equilibrium in terms of the number of guests, q_1^*, q_2^* at each hotel?

Tragedy of the commons

- Solution:
- In this case we have the best response functions as follows:

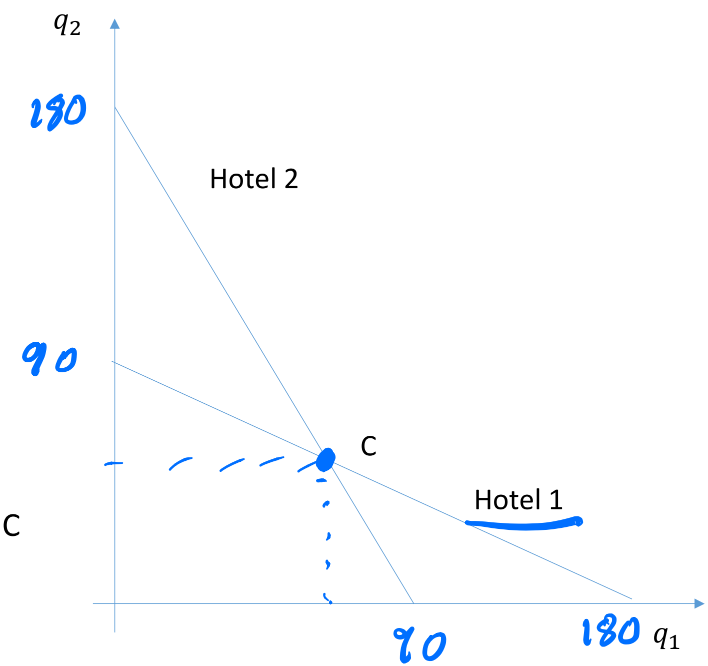
FOC 1: $90 - \frac{q_1}{2} - q_2 = 0$

1: $\Rightarrow q_2 = 90 - \frac{q_1}{2}$

2: $90 - \frac{q_2}{2} - q_1 = 0$

$\Rightarrow q_2 = 180 - 2q_1$

- Note that the simultaneous equilibrium is at point C with $q_1^* = q_2^* = 60$
- But do we ever get there?



Tragedy of the commons

- Solution:
- Consider the point B with $q_1 = 30, q_2 = 80$
- What is hotel 1's best response?

$$q_1^* = (90 - q_2) \times 2 = 20$$

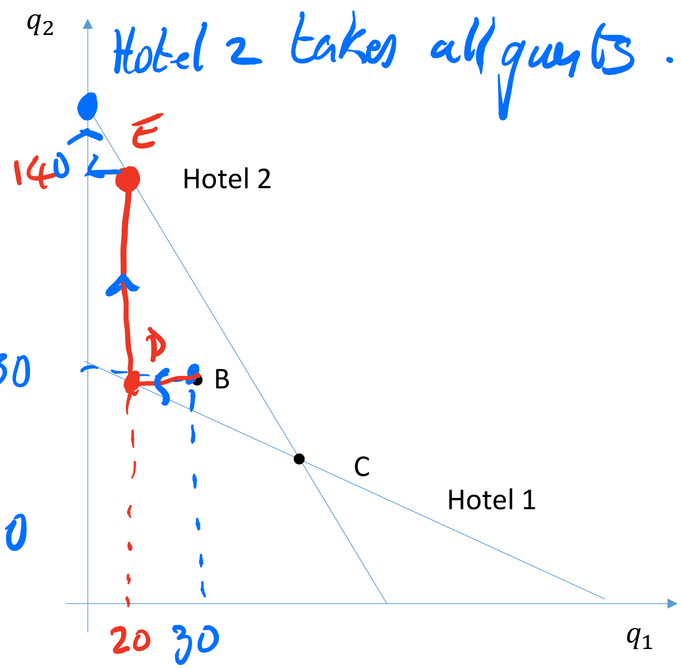
- And then hotel 2's best response

$$q_2^* = 180 - 2q_1^* = 180 - 40 = 140.$$

- Then hotel 1's best response

$$q_1^* = (90 - q_2) \times 2.$$
$$= (90 - 140) \times 2 = -100 < 0$$

- What has happened here?



Tragedy of the commons

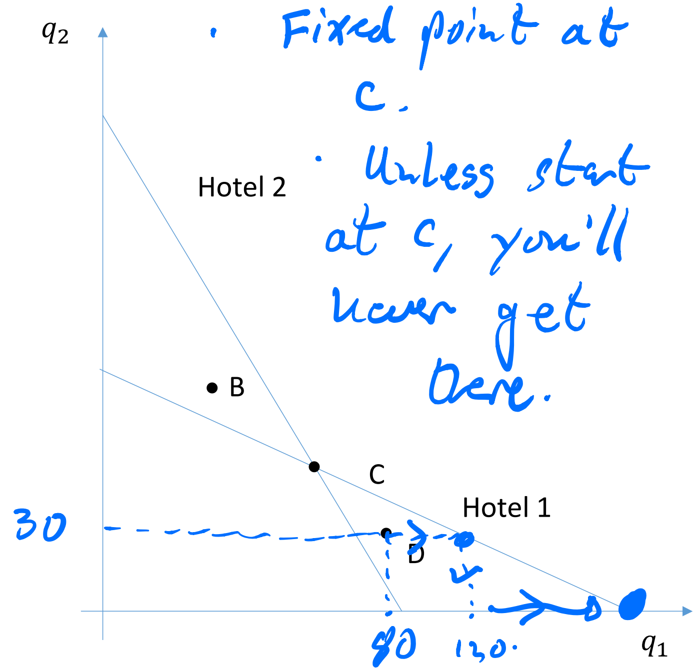
- Solution:
- Consider the point D with $q_1 = 80, q_2 = 30$
- What is hotel 1's best response?

  $(90-30) \times 2 = 120$

- And then hotel 2's best response

  $$q_2^* = 180 - 2q_1^* = 180 - 240 = -60$$

- So we have a simultaneous equilibrium (a fixed point of the system) but we will never move there

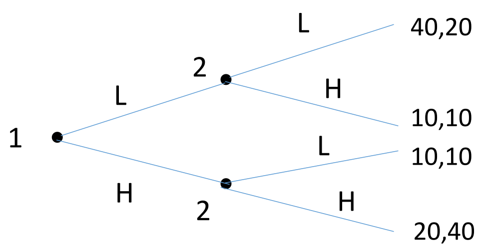


Sequential games

<u>Normal form of the static game</u>

		2	
		L	H
1	L	40,20	10,10
	H	10,10	20,40

<u>Extensive form of the sequential game</u>



- Consider the game that we had previously, but now suppose that player 1 plays first, followed by player 2

Sequential games

Normal form of the static game

		2	
		L	H
1	L	40,20	10,10
	H	10,10	20,40

Normal form of the sequential game

		HH	HL	LH	LL
1	L	10,10		40,20	
	H		10,10		10,10

- We may write the normal form for the sequential game as a 'product of strategies' $\{L, H\} \times \{L, H\}$
- Written in this way, we may identify the NE

  1: (L, LH)

  2: (L, LL)

  3: (H, HH)

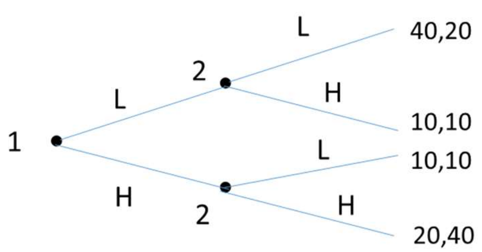
- What can we say about these equilibria?

Sequential games

<u>Normal form of the sequential game</u>

		HH	HL	LH	LL
1	L	10,10	10,10	40,20	40,20
	H	20,40	10,10	20,40	10,10

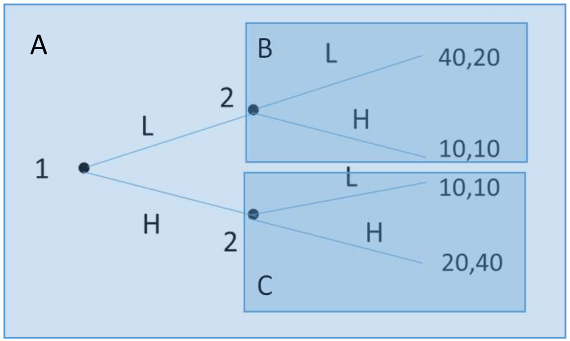
<u>Extensive form of the sequential game</u>



- Consider the NE (H, HH):
- This may be considered a way to ensure the highest pay-off for player 2 (40):
    - Player 2 threatens always to play H
    - If Player 1 plays L then the pay-off to Player 1 is 10 rather than 20
    - So Player 1 prefers to play H and so Player 2 maximises their pay-off in the game
- But is this threat credible?
    - If Player 1 plays L instead of H, then Player 2 will play L rather than H
    - A non-credible threat

Sequential games

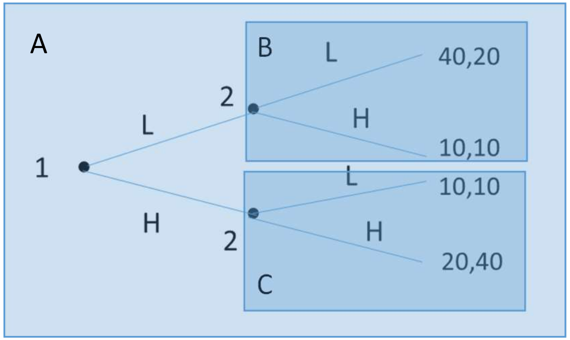
<u>Extensive form of the sequential game</u>



- A sub-game perfect equilibrium excludes non-credible threats
- A sub-game perfect equilibrium requires that there is a NE at each sub-game
- Sub-game A:
  - 3 NE: $(L, LH), (L, LL), (H, HH)$
- Sub-game B: Player 2 only plays
  - NE: L; pay-off $(40{,}20)$
  - Rules out NE (H, HH)
- Sub-game C: Player 2 only plays
  - NE: H; pay-off $(20{,}40)$
  - Rules out NE (L, LL)

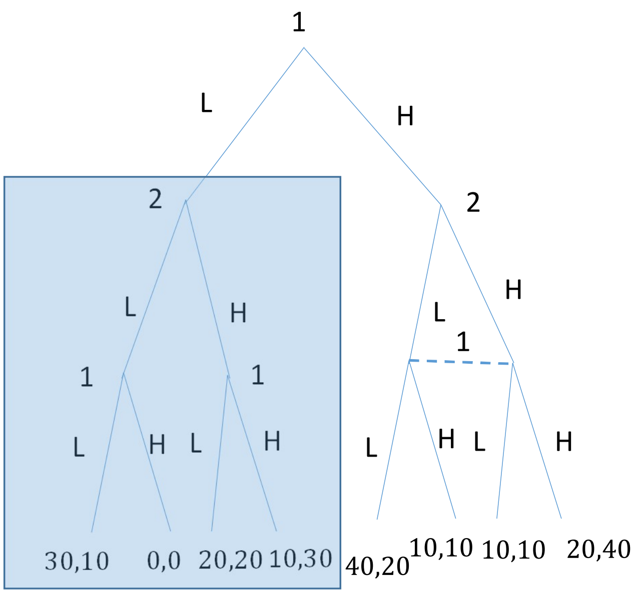
Sequential games

Extensive form of the sequential game



- We may use backward induction to find the sub-game perfect equilibrium and the equilibrium path
- Sub-game B: Player 2 only plays
  - NE: L; pay-off $(40{,}20)$
- Sub-game C: Player 2 only plays
  - NE: H; pay-off $(20{,}40)$
- Then player 1 faces pay-offs $(40{,}20)$ if L is chosen and $(20{,}40)$ if H is chosen
- Player 1 prefers to play L and so the sub-game perfect equilibrium and equilibrium path is given by $(L, (L, H))$

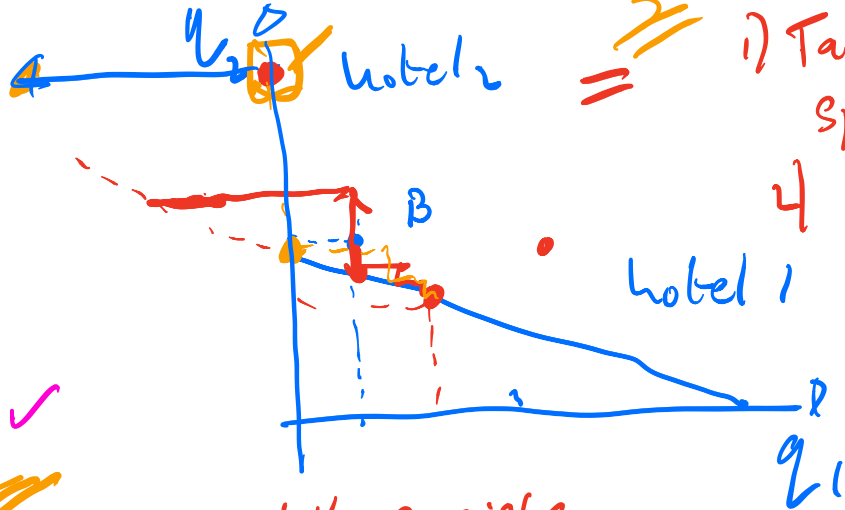
Sequential games



- Backward induction may be used to find the SPE and equilibrium path of more complicated games
- The right hand sub-game has 3 NE found earlier:
  - (L, L), (H, H), $(p^* = 0.75, q^* = 0.25)$
  - With pay-offs $(40{,}20)$, $(20{,}40)$ and $\mathrm{E}[u_1] = 17.5$
- Backward induction of the left hand sub-game gives SPE (L, H) and pay-off $(20{,}20)$
- We then have:

RHS sub-game NE	Outcome
(L, L)	$\{H; (L, L)\}$
(H, H)	$\{L; (L, H)\}$ or $\{H; (H, H)\}$
$(p^* = 0.75, q^* = 0.25)$	$\{L; (L, H)\}$

to be dynamics .



= 3 1) Take points in (q_1, q_2) space.

2) Decide who goes first.

3) See where you get to.

2 NE still exists.

Find the simultaneous solution (q_1^*, q_2^*) : NE

4 "Do we need to check dynamics in exams"

NO