EC2174 Midterm fall 2018

Oct. 18, 2018, 1:30 - 2:40pm

- *Answer all questions, full work must be shown*
- *Calculators are not allowed*

1. *[22 (5 +5+5+7) marks]* Let $A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 0 & 2 & 3 \end{pmatrix}$

   (a) Compute the determinant of A using cofactor expansion.

   (b) State whether A is invertible. Briefly justify your answer.

   (c) What is the rank of A?

   (d) Use elementary row operations to solve $Ax = b$, where $b = (2, 1, 0)$

2. *[18 (6 each) marks]* For what values of α and β the following system of linear equations

$$\begin{cases} x_1 - x_3 = -1 \\ x_2 - x_3 = \beta \\ -x_1 - x_2 + \alpha x_3 = 1 \end{cases}$$

   (a) has exactly one solution (you don't need to get the solution)?

   (b) has no solution?

   (c) have infinitely many solutions?

3. *[30 (5+2+6+6+6+5) marks]* Let $x = (x_1, x_2)'$, $A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{pmatrix}$, and $f(x) = (x_1, x_2, 1)\, A \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}$

   (a) Find $f(x)$.

   (b) Is f a homogeneous function? Briefly justify your answer.

   (c) Find the gradient and Hessian matrix of f defined in (a).

   (d) Sketch the level curve $f(x) = 0$

   (e) Verify that $(2, -2)$ is a point on the level curve in (d). Find the gradient vector of the function at the point $(2, -2)$ and draw it in the same graph in part (d) starting at $(2, -2)$

   (f) Let $A = \{x \in R^2 : f(x) \geq 0\}$, sketch the set

4. *[20 (4+8+8) marks]*

Consider the equations

$$\begin{cases} x + y^2 + z^3 + e^z w^2 = 2 \\ e^{2x} - y + xz^2 + w\ln(w) = 0 \end{cases} \tag{1}$$

(a) Verify that $(x, y, z, w) = (0, 1, 0, 1)$ is a point satisfying Equations (1).

(b) Argue that Equations (1) implicitly define (z, w) as differentiable function of (x, y) for (x, y, z, w) close to $(0, 1, 0, 1)$.

(c) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial w}{\partial x}$ at the point $(0, 1, 0, 1)$

5. *[10 (5 each) marks]* Let $f \in C^2$ be a function of two variables (with partial derivatives $f_1', f_2', f_{11}'', f_{12}'', f_{22}''$) and $g(x) = f(2x, e^x)$ for $x \in R$, find $g'(x)$ and $g''(x)$

Theorems

(might be needed for some of the questions)

- **Implicit function theorem to multi-variable, multi-function**: If the functions $F^1(x_1, ..., x_m, y_1, ...y_n), ...,$ are continuously differentiable. Suppose further that

$$\begin{aligned} F^1\left(x_1^0, ..., x_m^0, y_1^0, ...y_n^0\right) &= 0 \\ &\vdots \\ F^n\left(x_1^0, ..., x_m^0, y_1^0, ...y_n^0\right) &= 0 \end{aligned}$$

and

$$|J| = \begin{vmatrix} \frac{\partial F^1}{\partial y_1} & \frac{\partial F^1}{\partial y_2} & \cdots & \frac{\partial F^1}{\partial y_n} \\ \frac{\partial F^2}{\partial y_1} & \frac{\partial F^2}{\partial y_2} & \cdots & \frac{\partial F^2}{\partial y_n} \\ & & & \\ \frac{\partial F^n}{\partial y_1} & \frac{\partial F^n}{\partial y_2} & \cdots & \frac{\partial F^n}{\partial y_n} \end{vmatrix} \neq 0$$

at $(x_1^0, ..., x_m^0, y_1^0, ...y_n^0)$, then equation $F^1(x_1, ..., x_m, y_1, ...y_n) = 0, ..., F^n(x_1, ..., x_m, y_1, ..., y_n) = 0$ defines $y_1, ..., y_n$ as a continuously differentiable functions of $x_1, x_2, ..., x_m$:

$$\begin{aligned} y_1 &= f_1(x_1, x_2, ..., x_m) \\ &\vdots \\ y_n &= f_n(x_1, x_2, ..., x_m) \end{aligned}$$

for $(x_1, ..., x_m, y_1, ..., y_n)$ close to $(x_1^0, ..., x_m^0, y_1^0, ...y_n^0)$.