Exercise 11 (Assignment 4) Solutions

1. $L(x,\lambda^*,\mu^*) = x_2x_3 + x_1x_3 + \lambda^*\left(x_2^2+x_3^2-1\right) - (x_1x_3-3)$
$= \lambda^*\left(x_2^2+x_3^3\right) + x_2x_3 + 3 - \lambda^*$
For solutions 1 and 2: $\lambda^* = -\frac{1}{2}$, thus, $L(x,\lambda^*,\mu^*) = -\frac{1}{2}\left(x_2^2+x_3^2\right) + x_2x_3 + \frac{7}{2}$
is concave function of x
For solutions 3 and 4: $\lambda^* = \frac{1}{2}$, thus $L(x,\lambda^*,\mu^*) = \frac{1}{2}\left(x_2^2+x_3^2\right) + x_2x_3 + \frac{5}{2}$
is convex function of x
It follows from Sufficient condition #1 (for constrained optimization) that Solutions 1 and 2 are global maximizer and Solutions 3 and 4 are global minimizer.

2. The problem is the same as

$$\max_{x,y}\left\{-x^2-y^2-2x+2y\right\} \quad \text{subject to } x^2+y^2 \le 4$$

The Lagrange function is

$$\mathcal{L}(x,y,\lambda) = -x^2-y^2-2x+2y+\lambda\left(4-x^2-y^2\right)$$

Solution to FOC and KTC with $\lambda^* \ge 0$: $(x^*,y^*,\lambda^*) = (-1,1,0)$
$(-1,1,0)$ is a global solution since $\mathcal{L}(x,y,\lambda^*)$ is concave

3. Lagrange function

$$\mathcal{L}(x,y,\lambda,\mu) = xy + \lambda(100-x-y) + \mu(40-x)$$

$$\text{FOC} \quad : \quad \begin{cases} y-\lambda-\mu = 0 \\ x-\lambda = 0 \end{cases}$$

$$\text{KTC} \quad : \quad \begin{cases} \lambda(100-x-y) = 0 \\ \mu(40-x) = 0 \end{cases}$$

Case 1: $\lambda = \mu = 0 \implies x = y = 0$
Case 2: $\lambda = 0, \mu \neq 0 \implies$ no solution
Case 3: $\lambda \neq 0, \mu = 0 \implies x = y = \lambda = 50$ rejected since $x >$40
Case 4: $\lambda \neq 0, \mu \neq 0 \implies x = 40, y = 60, \lambda = 40, \mu = 20$
(a): one solution $(x^*,y^*,\lambda^*,\mu^*) = (40,60,40,20)$
(b): since the functions $xy, \lambda^*(100-x-y), \mu^*(40-x)$ are all quasi-concave and $f'(x^*,y^*) = (y^*,x^*) \neq 0$. (x^*,y^*) is a global maximum (sufficient condition #2)

4. The Lagrange function to the problem is

$$\mathcal{L}(x_1,x_2,\lambda) = x_1^2x_2 + \lambda\left(3-2x_1^2-x_2^2\right)$$

(a): FOC:

$$\begin{aligned}
\mathcal{L}_{x_1} &= 2x_1x_2 - 4\lambda x_1 = 0 && (1)\\
\mathcal{L}_{x_2} &= x_1^2 - 2\lambda x_2 = 0 && (2)\\
\mathcal{L}_{\lambda} &= 3 - 2x_1^2 - x_2^2 = 0 && (3)
\end{aligned}$$

if $x_1 = 0$, then $x_2 = \sqrt{3}$ or $-\sqrt{3}$, $\lambda = 0$
if $x_1 \neq 0$, then from (1) and (2), $x_2 = 2\lambda$ and $x_1^2 = 4\lambda^2$, substitute them into (3) gives $\lambda^2 = 1/4, \lambda = \pm.5$.
Thus solutions are $(x_1^*, x_2^*, \lambda^*) = v_i$, where:

$$\begin{aligned}
v_1 &= \left(0, \sqrt{3}, 0\right), v_2 = \left(0, -\sqrt{3}, 0\right),\\
v_3 &= (1, 1, 0.5), v_4 = (-1, 1, 0.5)\\
v_5 &= (1, -1, -0.5), v_6 = (-1, -1, -0.5)
\end{aligned}$$

(b): The bordered Hessian matrix is

$$B = \begin{pmatrix} 0 & 4x_1 & 2x_2 \\ 4x_1 & 2x_2 - 4\lambda & 2x_1 \\ 2x_2 & 2x_1 & -2\lambda \end{pmatrix}$$

thus the bordered Hessian matrix corresponding to v_i's are

$$\begin{aligned}
B_1 &= \begin{pmatrix} 0 & 0 & 2\sqrt{3} \\ 0 & 2\sqrt{3} & 0 \\ 2\sqrt{3} & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & -2\sqrt{3} \\ 0 & -2\sqrt{3} & 0 \\ -2\sqrt{3} & 0 & 0 \end{pmatrix}\\
B_3 &= \begin{pmatrix} 0 & 4 & 2 \\ 4 & 0 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \qquad B_4 = \begin{pmatrix} 0 & -4 & 2 \\ -4 & 0 & -2 \\ 2 & -2 & -1 \end{pmatrix}\\
B_5 &= \begin{pmatrix} 0 & 4 & -2 \\ 4 & 0 & 2 \\ -2 & 2 & 1 \end{pmatrix}, \qquad B_6 = \begin{pmatrix} 0 & -4 & -2 \\ -4 & 0 & -2 \\ -2 & -2 & 1 \end{pmatrix}
\end{aligned}$$

with

$$\begin{aligned}
\det(B_1) &= -\left(2\sqrt{3}\right)^3 < 0, \quad \det(B_2) = \left(2\sqrt{3}\right)^3 > 0\\
\det(B_3) &= \det(B_4) = 48 > 0\\
\det(B_5) &= \det(B_6) = -48 < 0
\end{aligned}$$

thus, v_2, v_3, v_4 are local maximum, and v_1, v_5, v_6 are local minimum
(c) The constraint set is obviously a compact set, thus from Sufficient condition #3

a global maximum exists, which must be one of v_2, v_3, v_4, since

$$f\left(0, -\sqrt{3}\right) = 0, \quad f(1,1) = f(-1,1) = 1$$

the maximum is reached at $(1,1)$ or $(-1,1)$, with corresponding $\lambda^* = 0.5$. Note that the first two sufficient

conditions for maximization is satisfied.

(d) maximized value $f\left(x_1^*, x_2^*\right) = 1$

(e) Consider the following problem:

$$f^*(a) = \max_{x_1, x_2} \left\{x_1^2 x_2\right\} \text{ subject to } 2x_1^2 + x_2^2 = a$$

The Lagrange function:

$$\mathcal{L}(x_1, x_2, \lambda, a) = x_1^2 x_2 + \lambda\left(a - 2x_1^2 - x_2^2\right)$$

By the Envelop Theorem,

$$\left.\frac{d}{da} f^*(a)\right|_{a=3} = \lambda^* = 0.5$$

thus

$$f^*(2.9) \approx f^*(3) + \left.\frac{d}{da} f^*(a)\right|_{a=3} \times (2.9 - 3) = 1 - 0.5 \times 0.1 = 0.95$$