Homework 2 Solution

Choose the best answer

1. The opportunity cost of producing a bicycle refers to the
   a. out of pocket payments made to produce the bicycle.
   b. value of the goods that were given up to produce the bicycle.
   c. bicycle's retail price.
   d. marginal cost of the last bicycle produced.

2. Which of the following production functions exhibits a constant elasticity of substitution?
   a. $q = 3k + 2l$.
   b. $q = k^{0.5}l^{0.5}$.
   c. $q = 1 - \frac{1}{k} - \frac{1}{l}$.
   d. All of the above have a constant elasticity of substitution.

3. The shape of a firm's long run average cost curve is determined by
   a. the degree to which each input encounters diminishing marginal productivity.
   b. the underlying nature of the firm's production function when all inputs are able to be varied.
   c. how much the firm decides to produce.
   d. the way in which the firm's expansion path reacts to changes in the rental rate on capital.

4. The input demand functions that can be derived from cost functions are referred to as "conditional" demand functions because the functions:
   a. assume input costs are constant.
   b. express input demand as a function of output.
   c. depend on the assumption of profit maximization.
   d. assume constant returns to scale in production.

Analytical questions

1. A firm has production function

$$f(k,l) = (k^\rho + l^\rho)^{\frac{\gamma}{\rho}}, \quad \alpha \in (0,1),$$

Denote the price of output by p, capital price v, and labor price w.

a. Find the RTS between two inputs

$$RTS = \frac{f_l}{f_k} = \left(\frac{l}{k}\right)^{\rho-1}$$

b. Solve the cost minimization problem

$$\min_{k,l} vk + wl, \quad \text{s.t. } q = (k^\rho + l^\rho)^{\frac{\gamma}{\rho}}.$$

$$RTS = \frac{l^{\rho-1}}{k^{\rho-1}} = \frac{w}{v} \Rightarrow l^{\rho-1} = k^{\rho-1}\frac{w}{v}$$

$$l = w^{\frac{1}{\rho-1}} v^{-\frac{1}{\rho-1}} k$$

Plug in production function and solve for contingent input demands

$$q = (k^\rho + l^\rho)^{\frac{\gamma}{\rho}}$$

$$q^{\frac{1}{\gamma}} = \left(k^\rho + w^{\frac{\rho}{\rho-1}} v^{-\frac{\rho}{\rho-1}} k^\rho\right)^{\frac{1}{\rho}} = \left(1 + w^{\frac{\rho}{\rho-1}} v^{-\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}} k.$$

Therefore

$$k(w,v,q) = \frac{q^{\frac{1}{\gamma}}}{\left(w^{\frac{\rho}{\rho-1}} v^{-\frac{\rho}{\rho-1}} + 1\right)^{\frac{1}{\rho}}} = \frac{q^{\frac{1}{\gamma}}}{\left(w^{\frac{\rho}{\rho-1}} v^{-\frac{\rho}{\rho-1}} + v^{\frac{\rho}{\rho-1}} v^{-\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}}$$

$$= \frac{q^{\frac{1}{\gamma}}}{\left(w^{\frac{\rho}{\rho-1}} + v^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}} \left(v^{-\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}} = \frac{q^{\frac{1}{\gamma}} v^{\frac{1}{\rho-1}}}{\left(w^{\frac{\rho}{\rho-1}} + v^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}}.$$

By symmetry

$$l(w,v,q) = \frac{q^{\frac{1}{\gamma}} w^{\frac{1}{\rho-1}}}{\left(v^{\frac{\rho}{\rho-1}} + w^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}}.$$

c. Find the cost function $C(w,v,q)$.

$$C(w,v,q) = v\frac{q^{\frac{1}{\gamma}} v^{\frac{1}{\rho-1}}}{\left(w^{\frac{\rho}{\rho-1}} + v^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}} + w\frac{q^{\frac{1}{\gamma}} w^{\frac{1}{\rho-1}}}{\left(v^{\frac{\rho}{\rho-1}} + w^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}} = q^{\frac{1}{\gamma}} \left\{ \frac{v \cdot v^{\frac{1}{\rho-1}} + w \cdot w^{\frac{1}{\rho-1}}}{\left(v^{\frac{\rho}{\rho-1}} + w^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}} \right\}$$

$$= q^{\frac{1}{\gamma}} \left\{ \frac{v^{\frac{\rho}{\rho-1}} + w^{\frac{\rho}{\rho-1}}}{\left(v^{\frac{\rho}{\rho-1}} + w^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}} \right\} = q^{\frac{1}{\gamma}} \left(v^{\frac{\rho}{\rho-1}} + w^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}$$

$$= q^{\frac{1}{\gamma}} \left(v^{1-\sigma} + w^{1-\sigma}\right)^{\frac{1}{1-\sigma}}.$$

d. Verify Shepard's lemma (on labor).

$$\frac{\partial C(w,v,q)}{\partial w} = l(w,v,q)$$

$$\frac{\partial C}{\partial v} = \frac{\partial}{\partial v}\left\{q^{\frac{1}{\gamma}}\left(v^{\frac{\rho}{\rho-1}} + w^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}\right\} = q^{\frac{1}{\gamma}}\frac{\rho-1}{\rho}\left(v^{\frac{\rho}{\rho-1}} + w^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}-1}\frac{\rho}{\rho-1}v^{\frac{\rho}{\rho-1}-1}$$
$$= q^{\frac{1}{\gamma}}\left(v^{\frac{\rho}{\rho-1}} + w^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}} v^{\frac{1}{\rho-1}} = \frac{q^{\frac{1}{\gamma}}v^{\frac{1}{\rho-1}}}{\left(w^{\frac{\rho}{\rho-1}} + v^{\frac{\rho}{\rho-1}}\right)^{\frac{1}{\rho}}}$$

e.* Solve the cost minimization problem with a new production function:

$$\min_{k,l} vk + wl, \quad \text{s.t. } q = [(\alpha k)^\rho + (\beta l)^\rho]^{\frac{\gamma}{\rho}}.$$

[Hint: use change of variable to help you find the answer.]
We treat $k' = \alpha k$ and $l' = \beta l$, therefore $k = \frac{k'}{\alpha}$, $l = \frac{l'}{\beta}$ The cost minimization problem,

$$\min_{k,l} vk + wl, \quad \text{s.t. } q = [(\alpha k)^\rho + (\beta l)^\rho]^{\frac{\gamma}{\rho}},$$

can then be rewritten as

$$\min_{k',l'} \frac{v}{\alpha}k' + \frac{w}{\beta}l', \quad \text{s.t. } q = [(k')^\rho + (l')^\rho]^{\frac{\gamma}{\rho}}.$$

This is as if that the price of k' and l' become $v' = \frac{v}{\alpha}$ and $w' = \frac{w}{\beta}$. From the cost function above, we immediately have

$$C(v',w',q) = q^{\frac{1}{\gamma}}\left((v')^{1-\sigma} + (w')^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$$
$$\Rightarrow C(v,w,q) = q^{\frac{1}{\gamma}}\left((\frac{v}{\alpha})^{1-\sigma} + (\frac{w}{\beta})^{1-\sigma}\right)^{\frac{1}{1-\sigma}}.$$

2. A price-taking firm need three inputs, capital (k), labor (l), and raw material (m), for production of output (q). The production function is

$$q = f(k,l,m) = k^\alpha l^\beta m^\gamma.$$

The input price for k, l, and m are denoted by v, w, and r, respectively. α, β, γ are positive parameters.

a. Characterize conditions of α that make the production function exhibit diminishing marginal product of capital.

$$MP_k = \alpha k^{\alpha-1}l^\beta m^\gamma$$
$$\frac{\partial MP_k}{\partial k} = \alpha(\alpha-1)k^{\alpha-2}l^\beta m^\gamma$$

When $0 < \alpha < 1$, $\frac{\partial MP_k}{\partial k} < 0$, the production function exhibit diminishing marginal product of capital.

b. Characterize conditions of α, β, γ that make the production function exhibit increasing return to scale.

$$\begin{aligned} f(tk, tl, tm) &= (tk)^\alpha (tl)^\beta (tm)^\gamma = t^{\alpha+\beta+\gamma} k^\alpha l^\beta m^\gamma \\ &= t^{\alpha+\beta+\gamma} f(k, l, m) \end{aligned}$$

When $\alpha + \beta + \gamma > 1$, the production function exhibit increasing return to scale.

c. In the short run, suppose that only raw material can be adjusted. Capital and labor are fixed at k_1 and l_1, respectively. Compute the corresponding cost function $C^{SR}(q)$.

$$\min_m vk_1 + wl_1 + rm, \quad \text{s.t. } k_1^\alpha l_1^\beta m^\gamma = q$$

Solve m from the constraint

$$k_1^\alpha l_1^\beta m^\gamma = q \Rightarrow m^\gamma = q k_1^{-\alpha} l_1^{-\beta}$$

$$m^{SR}(w, v, m, k_1, l_1, q) = k_1^{-\frac{\alpha}{\gamma}} l_1^{-\frac{\beta}{\gamma}} q^{\frac{1}{\gamma}}$$

$$C^{SR}(q) = vk_1 + wl_1 + rm^{SR} = vk_1 + wl_1 + rk_1^{-\frac{\alpha}{\gamma}} l_1^{-\frac{\beta}{\gamma}} q^{\frac{1}{\gamma}}$$

d. Continue with part (c), let $\alpha = \beta = \gamma = 0.5$, $v = w = r = 2$, $k_1 = l_1 = 4$, and output price be p. Find the firm's supply function $S(p)$.

$$\begin{aligned} C^{SR}(q) &= vk_1 + wl_1 + rk_1^{-\frac{\alpha}{\gamma}} l_1^{-\frac{\beta}{\gamma}} q^{\frac{1}{\gamma}} \\ &= 2 * 4 + 2 * 4 + 2 * 4^{-1} 4^{-1} q^2 \\ &= 16 + \frac{1}{8} q^2 \end{aligned}$$

$$MC^{SR}(q) = \frac{1}{4} q = p$$

$$q = S(p) = 4p.$$

(Here the shut down point is at $p = 0$.)

e. In the "median" run, suppose that both labor and raw material can be adjusted. Capital is fixed at k_1. Compute the input demands $l^{MR}(w, v, r, q, k_1)$, $m^{MR}(w, v, r, q, k_1)$ and the corresponding cost function $C^{MR}(q)$.

$$\min_{l,m} vk_1 + wl + rm, \quad \text{s.t. } k_1^\alpha l^\beta m^\gamma = q$$

Treating k_1 as fixed, we can solve the problem using usual tangency condition. The derivation can be shown by usual Lagragian approach

$$\mathcal{L} = vk_1 + wl + rm + \lambda(q - k_1^\alpha l^\beta m^\gamma)$$

$$\frac{\partial \mathcal{L}}{\partial l} = w - \lambda k_1^\alpha b l^{\beta-1} m^\gamma = 0$$
$$\frac{\partial \mathcal{L}}{\partial m} = r - \lambda k_1^\alpha l^\beta c m^{\gamma-1} = 0$$
$$\Rightarrow RTS = \frac{MP_l}{MP_w} = \frac{\lambda k_1^\alpha \beta l^{\beta-1} m^\gamma}{\lambda k_1^\alpha l^\beta \gamma m^{\gamma-1}} = \frac{w}{r}$$
$$\frac{\beta m}{\gamma l} = \frac{w}{r} \Rightarrow m = \frac{w\gamma}{r\beta} l$$
$$k_1^\alpha l^\beta \left(\frac{w\gamma}{r\beta} l\right)^\gamma = q \Rightarrow l^{\beta+\gamma} = q k_1^{-\alpha} \frac{r^\gamma \beta^\gamma}{w^\gamma \gamma^\gamma}$$
$$l^{MR}(w,v,q,k_1) = \frac{r^{\frac{\gamma}{\beta+\gamma}} \beta^{\frac{\gamma}{\beta+\gamma}}}{w^{\frac{\gamma}{\beta+\gamma}} \gamma^{\frac{\gamma}{\beta+\gamma}}} k_1^{-\frac{\alpha}{\beta+\gamma}} q^{\frac{1}{\beta+\gamma}}$$

$$\begin{aligned} m^{MR}(w,v,q,k_1) &= \frac{w\gamma}{r\beta} l^{MR}(w,v,q,k_1) \\ &= \frac{w\gamma}{r\beta} \frac{r^{\frac{\gamma}{\beta+\gamma}} \beta^{\frac{\gamma}{\beta+\gamma}}}{w^{\frac{\gamma}{\beta+\gamma}} \gamma^{\frac{\gamma}{\beta+\gamma}}} k_1^{-\frac{\alpha}{\beta+\gamma}} q^{\frac{1}{\beta+\gamma}} \\ &= \frac{w^{\frac{\beta}{\beta+\gamma}} \gamma^{\frac{\beta}{\beta+\gamma}}}{r^{\frac{\beta}{\beta+\gamma}} \beta^{\frac{\beta}{\beta+\gamma}}} k_1^{-\frac{\alpha}{\beta+\gamma}} q^{\frac{1}{\beta+\gamma}} \end{aligned}$$

$$\begin{aligned} C^{MR}(q) &= vk_1 + wl^{MR} + rm^{SR} \\ &= vk_1 + w\frac{r^{\frac{\gamma}{\beta+\gamma}} \beta^{\frac{\gamma}{\beta+\gamma}}}{w^{\frac{\gamma}{\beta+\gamma}} \gamma^{\frac{\gamma}{\beta+\gamma}}} k_1^{-\frac{\alpha}{\beta+\gamma}} q^{\frac{1}{\beta+\gamma}} + r\frac{w^{\frac{\beta}{\beta+\gamma}} \gamma^{\frac{\beta}{\beta+\gamma}}}{r^{\frac{\beta}{\beta+\gamma}} \beta^{\frac{\beta}{\beta+\gamma}}} k_1^{-\frac{\alpha}{\beta+\gamma}} q^{\frac{1}{\beta+\gamma}} \\ &= vk_1 + \left(w\frac{r^{\frac{\gamma}{\beta+\gamma}} \beta^{\frac{\gamma}{\beta+\gamma}}}{w^{\frac{\gamma}{\beta+\gamma}} \gamma^{\frac{\gamma}{\beta+\gamma}}} + r\frac{w^{\frac{\beta}{\beta+\gamma}} \gamma^{\frac{\beta}{\beta+\gamma}}}{r^{\frac{\beta}{\beta+\gamma}} \beta^{\frac{\beta}{\beta+\gamma}}} \right) k_1^{-\frac{\alpha}{\beta+\gamma}} q^{\frac{1}{\beta+\gamma}} \end{aligned}$$

Textbook Exercise

10.1 a. By definition, total costs are lower when both q_1 and q_2 are produced by the same firm than when the same output levels are produced by different firms. $C(q_1, 0)$ simply means that a firm produces only q_1.

b. Let $q = q_1 + q_2$, where $q_1, q_2 > 0$. By assumption,

$$\frac{C(q_1, q_2)}{q} < \frac{C(q_1, 0)}{q_1},$$

implying

$$\frac{q_1 C(q_1, q_2)}{q} < C(q_1, 0).$$

Similarly,

$$\frac{q_2 C(q_1, q_2)}{q} < C(0, q_2).$$

Summing yields

$$C(q_1,q_2) < C(q_1,0) + C(0,q_2).$$

This proves economies of scope.

10.3 Given $q = \min(5k, 10l)$.

a. In the long run, no input should be wasted. Hence, $5k = 10l = q$, implying $k = 2l = q/5$. Thus,

$$\begin{aligned} C &= vk + wl \\ &= v(2l) + wl \\ &= v\left(\frac{q}{5}\right) + w\left(\frac{q}{10}\right) \\ &= \frac{q}{10}(2v + w). \end{aligned}$$

Therefore,

$$AC = \frac{C}{q} = \frac{2v+w}{10}$$

$$MC = \frac{\partial C}{\partial q} = \frac{2v+w}{10}.$$

b. $q = \min(50, 10l)$, when $k = 10$. There are two cases to consider. First, if $l < 5$, then $q = 10l$, implying $q < 50$. Hence,

$$STC = 10v + \left(\frac{q}{10}\right)w,$$

implying

$$SAC = \frac{STC}{q} = \frac{10v}{q} + \frac{w}{10},$$

$$SMC = \frac{\partial STC}{\partial q} = \frac{w}{10}.$$

If $l \geq 5$, then $q = 50$. It is impossible to produce more than 50 in the short run. Hence, $STC = SAC = SMC = \infty$ for $q > 50$.

Finally, right at $q = 50$, we have the same formula for total cost as above:

$$STC = 10v + \left(\frac{q}{10}\right)w.$$

SMC is technically not defined because the STC has different derivatives to the right and left of $q = 50$. However, SAC is well-defined, and is the same as the previous formula:

$$SAC = \frac{STC}{q} = \frac{10v}{q} + \frac{w}{10}.$$

c. Substituting $v = 1$ and $w = 3$ into the formulae from the previous parts, in the long run, $AC = MC = 1/2$. In the short run, for $q < 50$,

$$SAC = \frac{STC}{q} = \frac{10}{q} + \frac{3}{10},$$

$$SMC = \frac{\partial STC}{\partial q} = \frac{3}{10}.$$

10.4 Given $q = 2\sqrt{kl}$, $k = 100$.

a. Since $q = 2\sqrt{100\,l}$, $q = 20\sqrt{l}$. Rearranging,

$$\sqrt{l} = \frac{q}{20},$$

implying

$$l = \frac{q^2}{400}.$$

Hence,

$$SC = vk + wl = 1(100) + 4\left(\frac{q^2}{400}\right) = 100 + \frac{q^2}{100}.$$

$$SAC = \frac{STC}{q} = \frac{100}{q} + \frac{q}{100}.$$

b. We have

$$SMC = \frac{\partial SC}{\partial q} = \frac{q}{50}.$$

If $q = 25$,

$$SC = 100 + \left(\frac{25^2}{100}\right) = 106.25,$$

$$SAC = \frac{100}{25} + \frac{25}{100} = 4.25,$$

$$SMC = \frac{25}{50} = \frac{1}{2}.$$

If $q = 50$,

$$SC = 100 + \left(\frac{50^2}{100}\right) = 125,$$

$$SAC = \frac{100}{50} + \frac{50}{100} = 2.5,$$

$$SMC = \frac{50}{50} = 1.$$

If $q = 100$,

$$SC = 100 + \left(\frac{100^2}{100}\right) = 200,$$

$$SAC = \frac{100}{100} + \frac{100}{100} = 2,$$

$$SMC = \frac{100}{50} = 2.$$

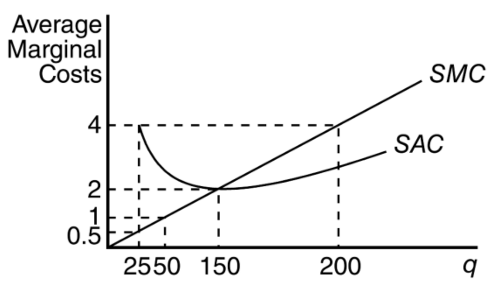
If $q = 200$,

$$SC = 100 + \left(\frac{200^2}{100}\right) = 500,$$

$$SAC = \frac{100}{200} + \frac{200}{100} = 2.5,$$

$$SMC = \frac{200}{50} = 4.$$

c.



d. As long as the marginal cost of producing one more unit is below the average-cost curve, average costs will be falling. Similarly, if the marginal cost of producing one more unit is higher than the average cost, then average costs will be rising. Therefore, the *SMC* curve must intersect the *SAC* curve at its lowest point.

e. Since $q = 2\sqrt{k_1 l}$, $q^2 = 4k_1 l$, implying

$$l = \frac{q^2}{4k_1}.$$

Substituting,

$$SC = vk_1 + wl = vk_1 + \frac{wq^2}{4k_1}.$$

f. Deriving the first-order condition from the previous expression,

$$\frac{\partial SC}{\partial k_1} = v - \frac{wq^2}{4k_1^2} = 0.$$

Rearranging,

$$k_1 = \frac{q}{2}\sqrt{\frac{w}{v}}.$$

g. Substituting first for l and then for k_1 into the cost function,

$$\begin{aligned} C &= vk_1 + wl(k_1) \\ &= vk_1 + w\frac{q^2}{4k_1} \\ &= v\left(\frac{q}{2}\sqrt{\frac{w}{v}}\right) + \frac{wq^2}{4}\left(\frac{2}{q}\sqrt{\frac{v}{w}}\right) \\ &= q\sqrt{vw}, \end{aligned}$$

(a special case of Example 10.2).

h. If $w = 4$ and $v = 1$, in the long run, $C = 2q$.

Let's examine the short run in different cases. Fixing $k_1 = 100$ in the short run,

$$SC\,(k_1 = 100) = 100 + \frac{q^2}{100}.$$

This is tangent to the long-run cost function for $q = 100$, as one can verify $SC = 200 = C$.

Fixing $k_1 = 200$ in the short run

$$SC\,(k_1 = 200) = 200 + \frac{q^2}{200}.$$

This is tangent to the long-run cost function for $q = 200$, as one can verify $SC = 400 = C$.

Finally, fixing $k_1 = 400$ in the short run,

$$SC\,(k_1 = 400) = 400 + \frac{q^2}{400}.$$

This is tangent to the long-run cost function for $q = 400$, as one can verify $SC = 800 = C$

10.7 a. As for many proofs involving duality, this one can be algebraically messy unless one sees the trick. Here the trick is to let

$$B = \left(v^{0.5} + w^{0.5}\right).$$

With this notation, $C = B^2 q.$ Using Shephard's lemma,

$$k = \frac{\partial C}{\partial v} = Bv^{-0.5}q,$$

$$l = \frac{\partial C}{\partial w} = Bw^{-0.5}q.$$

b. From part (a),

$$\frac{q}{k} = \frac{v^{0.5}}{B},$$

$$\frac{q}{l} = \frac{w^{0.5}}{B}.$$

Thus,

$$\frac{q}{k} + \frac{q}{l} = 1,$$

$$k^{-1} + l^{-1} = q^{-1}.$$

The production function then is $q = (k^{-1} + l^{-1})^{-1}$.

c. This is a CES production function with $r = -1.$ Hence,

$$s = \frac{1}{1-r} = 0.5.$$

Comparison to Example 8.2 shows the relationship between the parameters of the CES production function and its related cost function.