Homework 3 Solution

Choose the best answer

1. In order to maximize profits, a firm should produce at the output level for which
   a. average cost is minimized.
   b. marginal revenue equals marginal cost.
   c. marginal cost is minimized.
   d. price minus average cost is as large as possible.

2. For the cost function $C = q^{0.8}v^{0.4}w^{0.6}$, which of the following statements are true:
I. The function exhibits decreasing average cost.
II. The function is homogeneous of degree 1 in v and w.
III. The elasticity of marginal cost with respect to v exceeds the elasticity with respect to w.
   a. None is true.
   b. I, II, and III.
   c. I only.
   d. I and II.

3. A firm's demand for labor is known as a "derived demand" because
   a. the firm gains utility from hiring more labor.
   b. the amount of labor hired depends upon how much output the firm can sell.
   c. the wage rate paid to workers is derived from the market for labor.
   d. it is derived from the demand for capital.

4. If a firm is a price taker in both the input and output markets, its marginal revenue product of labor is given by
   a. the price of its output times labor's marginal physical productivity.
   b. the marginal value product of labor.
   c. the marginal revenue product of capital times the ratio of the wage rate to the rental rate on capital.
   d. all of the above.

Analytical questions

1. Consider the following production function

$$q = f(k, l) = \sqrt{k} + \sqrt{l}$$

   a. Does this production function has increasing, decreasing, or constant return to scale?

For $t > 1$,

$$f(tk, tl) = \sqrt{tk} + \sqrt{tl} = \sqrt{t}(\sqrt{k} + \sqrt{l}) = \sqrt{t}q < tq$$

Therefore, decreasing return to scale.

b. Solve the profit max problem when the output price is p and factor prices are v and w respectively. Find the output supply function and input demand functions.

$$\pi = \max_{k,l} \{pf(k,l) - vk - wl\}$$

$$k(p,v,w) = \frac{p^2}{4v^2}, \; l(p,v,w) = \frac{p^2}{4w^2}$$
$$q(p,v,w) = \frac{p}{2}\left(\frac{1}{v} + \frac{1}{w}\right)$$

c. Find the profit function and verify the Evnelop results: $\frac{\partial \pi}{\partial p} = q(p,v,w)$, $\frac{\partial \pi}{\partial v} = -k(p,v,w)$, $\frac{\partial \pi}{\partial w} = -l(p,v,w)$.

$$\pi(p,v,w) = pq(p,v,w) - vk(p,v,w) - wl(p,v,w) = \frac{p^2}{4}\left(\frac{1}{v} + \frac{1}{w}\right)$$

Verification are straight forward.

d. Solve the cost minimization problem and find the conditional input demand functions and the cost function. Is the cost function concave or convex in output?

$$\min_{k,l}\{vk + wl\}, \quad \text{s.t. } f(k,l) = q$$

$$k(q,v,w) = \frac{q^2w^2}{(v+w)^2}, \; l(q,v,w) = \frac{q^2v^2}{(v+w)^2}$$
$$c(q,v,w) = \frac{vwq^2}{v+w}$$
$$\frac{\partial^2 c}{\partial q^2} = \frac{2vw}{v+w} > 0, \text{ convex}$$

e. Take the cost function from part (d), solve the profit max problem with respect to output and confirm the profit function is the same as the profit function obtained in (c).

$$\max_q\{pq - c(q,v,w)\}$$

$$MC(q) = p \Rightarrow \frac{2vwq}{v+w} = p \Rightarrow q(p,v,w) = \frac{p}{2}\left(\frac{1}{v} + \frac{1}{w}\right)$$

$$\pi(p,v,w) = pq(p,v,w) - c(q(p,v,w)v,w) = \frac{p^2}{2}\left(\frac{1}{v}+\frac{1}{w}\right) - \frac{vw}{v+w}\left[\frac{p}{2}\left(\frac{1}{v}+\frac{1}{w}\right)\right]^2 = \frac{p^2}{4}\left(\frac{1}{v}+\frac{1}{w}\right)$$

f. In the short-run, suppose k is fixed at k_1, show that the short-run cost function is no less than long-run cost function from (d), and the two costs are the same when conditional demand for k is equal to k_1.

$$SC(q,v,w,k_1) = w(q-\sqrt{k_1})^2 + vk_1$$

Define the difference of short-run and long-run cost function as

$$\begin{aligned} g(k_1) &= SC(q,v,w,k_1) - c(q,v,w) \\ &= w(q-\sqrt{k_1})^2 + vk_1 - \frac{vwq^2}{v+w} \end{aligned}$$

Take derivative w.r.t. k_1

$$g'(k_1) = w(q-\sqrt{k_1}) \times (-k_1^{-\frac{1}{2}}) + v = v + w - wqk_1^{-\frac{1}{2}}$$

Second order condition

$$g''(k_1) = \frac{1}{2}wqk_1^{-\frac{3}{2}} > 0, \text{ convex.}$$

Therefore, there is a unique minimum at $g'(k_1) = 0$, compute this minimum

$$k_1^{\frac{1}{2}} = \frac{wq}{v+w} \Rightarrow k_1^* = \frac{q^2w^2}{(v+w)^2} = k(q,v,w),$$

which is the conditional input demand of the long run. Plug in, we can verify that $g(k_1^*) = 0$.

2. Firm uses capital k, and labor l, to produce output, q. The firm is a price taker in the output market and in both input markets, denoted by (p,v,w) respectively. The firm's supply function is

$$q(p,v,w) = mp^a v^{-1} w^{-2},$$

its demand function for capital is

$$k(p,v,w) = 3p^4 v^b w^{-2},$$

and its demand function for labor is

$$l(p,v,w) = np^4 v^c w^{-3}.$$

What are the values of the constants a, b, c, m, and n? Explain your reasoning in each case. [Hint: these functions are homogeneous of degree zero in prices, and the Hotelling lemma]

Using the homogeneity property, we know that scaling up both inputs and output prices shall not change the function. Therefore

$$q(tp, tv, tw) = m(tp)^a(tv)^{-1}(tw)^{-2} = mp^a v^{-1} w^{-2} t^{a-1-2} = q(p, v, w),$$

which implies $a = 3$. Similarly, we have

$$4 + b - 2 = 0 \Rightarrow b = -2$$

$$4 + c - 3 = 0 \Rightarrow c = -1.$$

By Hotelling's lemma

$$\begin{aligned}
\frac{\partial \pi}{\partial v} &= -k(p, v, w) = -3p^4 v^{-2} w^{-2}, \\
\frac{\partial \pi}{\partial w} &= -l(p, v, w) = -np^4 v^{-1} w^{-3}, \\
\frac{\partial \pi}{\partial p} &= q(p, v, w) = mp^3 v^{-1} w^{-2}.
\end{aligned}$$

By Young's theorem, $\pi_{vw} = \pi_{wv}$, $\pi_{vp} = \pi_{pv}$, therefore, we have

$$\begin{aligned}
\frac{\partial k(p, v, w)}{\partial w} &= \frac{\partial}{\partial w}(-\frac{\partial \pi}{\partial v}) = \frac{\partial}{\partial v}(-\frac{\partial \pi}{\partial w}) = \frac{\partial l(p, v, w)}{\partial v} \\
&\Rightarrow -6p^4 v^{-2} w^{-3} = -np^4 v^{-2} w^{-3} \Rightarrow n = 6. \\
\frac{\partial k(p, v, w)}{\partial p} &= \frac{\partial}{\partial p}(-\frac{\partial \pi}{\partial v}) = \frac{\partial}{\partial v}(-\frac{\partial \pi}{\partial p}) = -\frac{\partial q(p, v, w)}{\partial v} \\
&\Rightarrow 12p^3 v^{-2} w^{-2} = mp^3 v^{-2} w^{-2} \Rightarrow m = 12.
\end{aligned}$$

3. A price-taking firm has production function

$$q = f(k, l) = k^{\frac{1}{3}} l^{\frac{1}{3}}.$$

The capital rental price is $v = 2$ and labor price is $w = 3$.

a. In the short run, capital is fixed at the level k_1. Set up the cost minimization problem and find the short-run cost function.

$$\min_l vk_1 + wl, \quad \text{s.t. } q = k_1^{\frac{1}{3}} l^{\frac{1}{3}}$$

From the constraint, we have

$$l^{\frac{1}{3}} = qk_1^{-\frac{1}{3}} \Rightarrow l = q^3 k_1^{-1}.$$

Short-run cost function is therefore

$$\begin{aligned}
SC(v, w, q, k_1) &= vk_1 + wk_1^{-1} q^3 \\
&= 2k_1 + 3k_1^{-1} q^3
\end{aligned}$$

b. Find the short-run supply function given price P. What is the shut-down price?

$$SMC(v, w, q, k_1) = 9k_1^{-1}q^2 = P$$

Short-run supply function is

$$q = \left(\frac{k_1 P}{9}\right)^{\frac{1}{2}}.$$

$$SAVC = \frac{3k_1^{-1}q^3}{q} = 3k_1^{-1}q^2$$

Because the short-run average variable cost has its minimum at $q = 0$, the shut-down price is $P = 0$.

c. Find the long-run supply function.

$$\min_{k,l} vk + wl, \quad \text{s.t. } q = k^{\frac{1}{3}} l^{\frac{1}{3}}$$

Tangent condition

$$\frac{f_l}{f_k} = \frac{k}{l} = \frac{w}{v} = \frac{3}{2} \Rightarrow k = \frac{3}{2}l$$

$$q = \left(\frac{3}{2}l\right)^{\frac{1}{3}} l^{\frac{1}{3}} = 3^{\frac{1}{3}} 2^{-\frac{1}{3}} l^{\frac{2}{3}}$$

$$l^{\frac{2}{3}} = q3^{-\frac{1}{3}}2^{\frac{1}{3}} \Rightarrow l(v, w, q) = q^{\frac{3}{2}} 3^{-\frac{1}{2}} 2^{\frac{1}{2}}$$

$$k(v, w, q) = \frac{3}{2} \times q^{\frac{3}{2}} 3^{-\frac{1}{2}} 2^{\frac{1}{2}} = q^{\frac{3}{2}} 3^{\frac{1}{2}} 2^{-\frac{1}{2}}$$

$$\begin{aligned} C(v, w, q) &= vk(v, w, q) + wl(v, w, q) \\ &= 2 \times q^{\frac{3}{2}} 3^{\frac{1}{2}} 2^{-\frac{1}{2}} + 3 \times q^{\frac{3}{2}} 3^{-\frac{1}{2}} 2^{\frac{1}{2}} = 2^{\frac{3}{2}} 3^{\frac{1}{2}} q^{\frac{3}{2}} \end{aligned}$$

$$MC = \frac{3}{2} 2^{\frac{3}{2}} 3^{\frac{1}{2}} q^{\frac{1}{2}} = 2^{\frac{1}{2}} 3^{\frac{3}{2}} q^{\frac{1}{2}} = P$$

$$\Rightarrow q^{\frac{1}{2}} = 2^{-\frac{1}{2}} 3^{-\frac{3}{2}} P$$

The long-run supply function is

$$q = 2^{-1} 3^{-3} P^2 = \frac{P^2}{54}$$

* (Students do not need to answer this part) For the long-run zero-profit condition, because

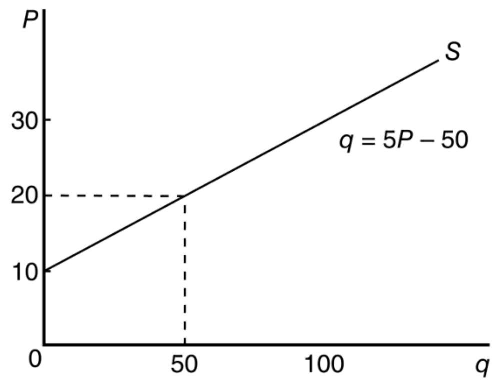
$$AC(q) = 2^{\frac{3}{2}} 3^{\frac{1}{2}} q^{\frac{1}{2}},$$

has a minimum at $q = 0$. So firm start supplying as long as $P > 0$. Here, we do not assume a fixed cost other than capital k in the long run.

11.1 a. $MC = \partial C / \partial q = 0.2q + 10.$ Setting $MC = P = 20,$ yields $q^* = 50.$

b. $\pi = Pq - C = 1000 - 800 = 200.$

c.



11.3 a. Since $q = 2\sqrt{l}$, $q^2 = 4l$.

$$C = wl = \frac{wq^2}{4}.$$

b. $\pi = Pq - TC = \frac{2P^2}{w} - \frac{P^2}{w} = \frac{P^2}{w}.$

This is homogeneous of degree 1 in P and w.

c. Profit maximization requires

$$P = MC = \frac{2wq}{4}.$$

Solving for q yields $q = 2P/w$.

The result can also be derived from Shephard's lemma:

$$q = \frac{\partial \pi}{\partial P} = \frac{2P}{w}.$$

The shutdown price is 0.

d. From the production function, $l = q^2/4$. Replacing q from the supply function, we get $l = P^2/w^2$. Shephard's lemma gives the same result:

$$l = -\frac{\partial \pi}{\partial w} = \frac{P^2}{w^2}.$$

e. Intuitive properties include the following:

- Total cost increases with wages and output.
- Profits increase with output price and decrease with wages. The function is homogeneous of degree 1 in the prices.
- Supply increases with output price and decreases with the wage. The function is homogeneous of degree 0 in input and output prices.

- Labor demand increases with output price and decreases with the wage. Input demand is homogeneous of degree 0 with respect in output and input prices.

11.7 a. If $q = a + bP$,

$$MR = P + q\frac{dP}{dq} = \frac{q-a}{b} + q\left(\frac{1}{b}\right) = \frac{2q-a}{b}.$$

Hence,

$$q = \frac{a + bMR}{2}.$$

Because the distance between the vertical axis and the demand curve is $q = a + bP$, the marginal revenue curve must bisect this distance for any line parallel to the horizontal axis.

b. If $q = a + bP$, $b < 0$, and $P = (q-a)/b$, then

$$MR = \frac{2q-a}{b},$$

$$P - MR = -\frac{1}{b}q.$$

c. The constant elasticity demand curve is $q = aP^{b}$, where b is the price elasticity of demand.

$$MR = P + q\frac{\partial P}{\partial q} = \left(\frac{q}{a}\right)^{1/b} + \left[\frac{(q/a)^{1/b}}{b}\right].$$

Thus, vertical distance is

$$P - MR = \frac{-(q/a)^{\frac{1}{b}}}{b} = \frac{-P}{b}.$$

This is positive because $b < 0$.

d. If $e_{q,P} < 0$ (downward sloping demand curve), then marginal revenue will be less than price. Hence, vertical distance will be given by $P - MR$.

Since $MR = P + q\frac{dP}{dq}$, the vertical distance is

$$-q\frac{dP}{dq}.$$

Since $dq/dP = b$ is the slope of the tangent linear demand curve, the distance becomes $-q/b$ as in part (b).

e.