Final Exam Fall 2018
Dec. 18, 2018

- **Answer all the following questions, full work must be shown.**
- **Calculators are not allowed.**
- **Duration of exam: 2.5 hours**

1. *[30 marks = 12 + 10 + 8]* Three (unrelated) questions on **Concavity/convexity and quasi-concavity/convexity**

   1a. *[4 marks each]*Let $f(x) = \sqrt{|x|}$ for $x \in R$

      i. Sketch the function.

      ii. Is $f(x)$ a concave or convex function? justify

      iii. Is $f(x)$ a quasi-concave or quasi-convex function? justify

   1b. Let $f(x) = x_1 x_2 \cdots x_n$ for $x \in R^n_{++}$, show that f is a quasi-concave function

   1c. Give an example of a function of R^3 which is strictly concave, justify

2. *[50 marks]* Three (unrelated) questions on **Optimization**

   2a. *[14 marks = 5 + 5 + 4]* Let $F(K,L) = K^{\frac{1}{3}} L^{\frac{1}{3}}$ be a firm's production function, where K and L denotes capital and labor respectively. If the price of output is p and the cost of capital and labor is r and w respectively. The firm's profit is given by

   $$\pi(K,L,p,r,w) = pK^{\frac{1}{3}} L^{\frac{1}{3}} - rK - wL$$

      i. Find the solution to the problem

      $$\pi^*(p,r,w) = \max_{K,L>0} \pi(K,L,p,r,w)$$

      satisfying first order conditions

      ii. Claim that the solution is in (i) is a global maximum

      iii. Find $\frac{\partial \pi^*}{\partial p}, \frac{\partial \pi^*}{\partial r}, \frac{\partial \pi^*}{\partial w}$

   2b. *[14 marks = 5 + 5 + 4]*Consider the following problem:

   $$\begin{cases} F(p_1,p_2,I) = \max_{x_1>0,x_2>0} \{\ln(x_1) + \ln(x_2)\} \\ \qquad\qquad s.t. \;\; p_1x_1 + p_2x_2 = I \end{cases}$$

      i. Solve the problem.

      ii. Claim that it is a local maximum by checking the properties of bordered Hessian matrix.

iii. claim that it is a global maximum.

2c. *[22 marks =12+5+5]* Solve the following problem:

$$\max_{x,y,z} \left\{x^2 + y^2 + z^2\right\}$$

$$\text{subject to } \begin{cases} 2x^2 + y^2 + z^2 \leq 10 \\ x + y + z = 0 \end{cases}$$

i. Find all solutions satisfying the first order conditions and Kuhn Tucker conditions.

ii. Find the global maximum by checking sufficient conditions

iii. Find the maximized function value of the problem. Suppose we change the two constraints to

$$\begin{cases} 2x^2 + y^2 + z^2 \leq 9.9 \\ x + y + z = 0.01 \end{cases}$$

Estimate maximized function value of the new problem by applying the envelope theorem.

3. *[20 marks]* Two (unrelated) questions on **Definiteness of matrices**

3a. *[5 marks]*Is the following matrix positive definite, negative definite, positive semidefinite, negative semidefinite, or indefinite? justify

$$A = \begin{pmatrix} 3 & 0 & 1 & 1 \\ 0 & 5 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

3b. *[15 marks]*Determine the value(s) of a for which the following matrix is positive definite, positive semidefinite, negative definite, negative semidefinite, or indefinite (There may be no values of a for which the matrix satisfies some of these conditions.)

$$A = \begin{pmatrix} a & 1 & b \\ 1 & -1 & 0 \\ b & 0 & -2 \end{pmatrix}$$