Derivative Securities (FINA 3203)
Solutions to Problem Set 6

In the following three questions, the assumptions of Merton's model hold. Let's assume a firm has assets of \$100 with $\sigma = 40\%$, the continuously-compounded expected return of assets is $\mu = 15\%$, the dividend yield is $q = 0$, and the continuous-compounded risk-free rate is $r = 8\%$.

Question 1: Credit Spread (3/10)

Solution:

- The value of the firm satisfies:

$$V_T = V_0 \times \exp\left[(\mu - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\epsilon\right] \tag{1}$$

  where $\epsilon \sim N(0,1)$. The firm defaults if:

$$V_T < F = 120$$

$$\mathbb{P}_0\left(V_T < F\right) = \mathbb{P}(\epsilon < -d_{2,\mu}) = \mathcal{N}(-d_{2,\mu})$$

  where

$$d_{2,\mu} = \frac{\ln\left(\frac{V_0}{F}\right) + \left(\mu - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

  We plug in numbers and get the default probability $\mathbb{P}(V_T < F) = 0.4258$. Attention: in the expression for $d_{2,\mu}$ we use μ instead of r, because we want to figure out the "actual" probability of default, instead of the risk-neutral probability of default.

- The value of equity can be calculated using risk-neutral pricing formula:

$$\begin{aligned} E_0 &= e^{-rT}\mathbb{E}^*\left[(V_T - F)\mathbf{1}(V_T \geq F)\right] \\ &= \text{Call}(V_0, F, r, \sigma, T) \\ &= V_0\mathcal{N}(d_1) - Fe^{-rT}\mathcal{N}(d_2) \end{aligned} \tag{2}$$

where:

$$d_1 = \frac{\ln\left(\frac{S_t}{F}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}},$$
$$d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\ln\left(\frac{S_t}{F}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

The value of debt can also be calculated using risk-neutral pricing formula:

$$\begin{aligned} D_0 &= e^{-rT}\mathbb{E}_0^*\left[V_T\mathbf{1}(V_T < F)\right] + e^{-rT}\mathbb{E}_0^*\left[F\mathbf{1}(V_T \geq F)\right] \\ &= Fe^{-rT} - e^{-rT}\mathbb{E}_0^*\left[(F - V_T)\mathbf{1}(V_T < F)\right] \\ &= Fe^{-rT} - \text{Put}(V_0, F, r, \sigma, T), \end{aligned} \tag{3}$$

where the Black-Scholes-Merton formula for the European put is

$$\text{Put}(V_0, F, r, \sigma, T) = Fe^{-rT}\mathcal{N}(-d_2) - V_0\mathcal{N}(-d_1).$$

This is the formula we derived in our lecture notes. If we plug the numbers into (2) and (3), we can obtain the result $E_0 = 41.78$ and $D_0 = 58.22$.

The credit spread $y - r$ is defined as follows:

$$\begin{aligned} y - r &= \frac{1}{T}\ln\frac{F}{D_0} - r \\ &= -r - \frac{1}{T}\ln\frac{D_0}{F} \\ &= -r - \frac{1}{T}\ln\frac{Fe^{-rT} - \text{Put}(V_0, F, r, \sigma, T)}{F} \end{aligned} \tag{4}$$

Note that

$$\frac{Fe^{-rT} - \text{Put}(V_0, F, r, \sigma, T)}{F} = e^{-rT} - \text{Put}(\frac{V_0}{F}, 1, r, \sigma, T) \tag{5}$$

Combining (5) and (4), it leads to

$$y - r = -\frac{1}{T}\ln\left[1 - e^{rT}\text{Put}(\frac{V_0}{F}, 1, r, \sigma, T)\right] \tag{6}$$

Plugging the numbers, we can get the result for the credit spread $y - r = 6.47\%$.

Questions 2: Expected Loss Given Default (4/10)

Solution:

1. Let's compute yield, default probability, and expected loss (rate) given default.

– The value of the firm satisfies:

$$V_T = V_0 \times \exp\left[(\mu - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\epsilon\right], \quad \text{with} \;\; \epsilon \sim N(0,1).$$

The probability of default is

$$\mathbb{P}_0(V_T < F) = \mathbb{P}_0(\epsilon < -d_{2,\mu}) = \mathcal{N}(-d_{2,\mu}) \tag{7}$$

where

$$d_{2,\mu} = \frac{\ln\left(\frac{S_t}{F}\right) + \left(\mu - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

Plugging in numbers, the physical default probability $\mathbb{P}_0(V_T < F)$ can be computed:

* $T = 1$: $\mathbb{P}_0(V_T < F) = 0.4305$.
* $T = 3$: $\mathbb{P}_0(V_T < F) = 0.3809$.
* $T = 10$: $\mathbb{P}_0(V_T < F) = 0.2900$.

From (6), it holds that

$$y - r = -\frac{1}{T}\ln\left[1 - e^{rT}\mathrm{Put}(\frac{V_0}{F}, 1, r, \sigma, T)\right]. \tag{8}$$

Plugging in numbers, we get the credit spreads:

* $T = 1$: $y - r = 13.54\%$.
* $T = 3$: $y - r = 7.01\%$.
* $T = 10$: $y - r = 3.16\%$.

2. The expected loss given default is

$$\mathbb{E}_0\left[F - V_T | V_T < F\right] = \frac{\mathbb{E}_0\left[(F - V_T)\mathbf{1}(V_T < F)\right]}{\mathbb{E}_0\left[\mathbf{1}(V_T < F)\right]} \tag{9}$$

The numerator is

$$\begin{aligned}\mathbb{E}_0\left[(F - V_T)\mathbf{1}(V_T < F)\right] &= F\mathbb{E}_0\left[\mathbf{1}(V_T < F)\right] - \mathbb{E}_0\left[V_T\mathbf{1}(V_T < F)\right] \\ &= F\mathbb{P}_0(V_T < F) - \mathbb{E}_0\left[V_T\mathbf{1}(V_T < F)\right] \\ &= F\mathcal{N}(-d_{2,\mu}) - V_0 e^{\mu T}\mathcal{N}(-d_{1,\mu}) \end{aligned} \tag{10}$$

where

$$d_{1,\mu} = \frac{\ln\left(\frac{S_t}{F}\right) + \left(\mu + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

The denominator is simply the default probability:

$$\mathbb{E}_0\left[\mathbf{1}(V_T < F)\right] = \mathcal{N}(-d_{2,\mu}). \tag{11}$$

Plugging (10) and (11) into (9), we can get

$$\mathbb{E}_0\left[F - V_T | V_T < F\right] = F - V_0 e^{\mu T} \frac{\mathcal{N}(-d_{1,\mu})}{\mathcal{N}(-d_{2,\mu})}. \tag{12}$$

The expected loss rate given default is

$$\mathbb{E}_0\left[\frac{F - V_T}{F}\middle| V_T < F\right] = \frac{F - V_0 e^{\mu T}\frac{\mathcal{N}(-d_{1,\mu})}{\mathcal{N}(-d_{2,\mu})}}{F} = 1 - \frac{V_0}{F} e^{\mu T} \frac{\mathcal{N}(-d_{1,\mu})}{\mathcal{N}(-d_{2,\mu})}$$

Plugging numbers, we can get

* $T = 1$: $\mathbb{E}_0\left[F - V_T | V_T < F\right] = 23.72$, $\mathbb{E}_0\left[\frac{F - V_T}{F}\middle| V_T < F\right] = 0.2372$
* $T = 3$: $\mathbb{E}_0\left[F - V_T | V_T < F\right] = 34.27$, $\mathbb{E}_0\left[\frac{F - V_T}{F}\middle| V_T < F\right] = 0.3427$
* $T = 10$: $\mathbb{E}_0\left[F - V_T | V_T < F\right] = 46.67$, $\mathbb{E}_0\left[\frac{F - V_T}{F}\middle| V_T < F\right] = 0.4667$

3. Consider the Taylor expansion $\ln(1-x) \approx -x$ with $x \approx 0$. Applying the approximation to equation (6), we have

$$y - r \approx \frac{1}{T} \times e^{rT} \times \text{Put}(\frac{V_0}{F}, 1, r, \sigma, T). \tag{13}$$

By risk-neutral pricing formula, it holds that

$$e^{rT} \times \text{Put}(\frac{V_0}{F}, 1, r, \sigma, T) = \mathbb{P}_0^*(V_T < F) \times \mathbb{E}_0^*\left[1 - \frac{V_T}{F}\middle| V_T < F\right]. \tag{14}$$

Thus, the approximation of credit spread is

$$y - r \approx \frac{1}{T} \times \underbrace{\mathbb{P}_0^*(V_T < F)}_{\text{Risk-Neutral Prob Default}} \times \underbrace{\mathbb{E}_0^*\left[1 - \frac{V_T}{F}\middle| V_T < F\right]}_{\text{Risk-Neutral Expected Loss Rate Given Default}} \tag{15}$$

Plugging in numbers, we can get the approximated credit spreads:

- $T = 1$: $\widehat{y - r} = \frac{0.5 \times 25.34}{100} = 12.67\%$
- $T = 3$: $\widehat{y - r} = \frac{0.5 \times 37.91}{3 \times 100} = 6.31\%$
- $T = 10$: $\widehat{y - r} = \frac{0.5 \times 54.18}{10 \times 100} = 2.71\%$

The approximation is accurate.
If you approximate $y - r$ using physical default probability $\mathbb{P}_0(V_T < F)$ and physical expected loss rate given default $\mathbb{E}_0\left[1 - \frac{V_T}{F}\middle| V_T < F\right]$. The following are the approximations of credit spreads based on physical probability measures:

- $T = 1$: $\widehat{y - r} = \frac{0.4305 \times 23.72}{100} = 10.21\%$
- $T = 3$: $\widehat{y - r} = \frac{0.3809 \times 34.27}{3 \times 100} = 4.35\%$
- $T = 10$: $\widehat{y - r} = \frac{0.2900 \times 46.67}{10 \times 100} = 1.35\%$

The approximation is less accurate.

Question 3: Expected Recovery (3/10)

SOLUTION:

1. The value of the firm satisfies:

$$V_T = V_0 \times \exp\left[(\mu - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\epsilon\right], \quad \text{with} \;\; \epsilon \sim N(0,1).$$

   The formula is the same as above. The default probability is

$$\mathbb{P}_0(V_T < F) = \mathbb{P}_0(\epsilon < -d_{2,\mu}) = \mathcal{N}(-d_{2,\mu}) \tag{16}$$

   where

$$d_{2,\mu} = \frac{\ln\left(\frac{S_t}{F}\right) + \left(\mu - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

   Plugging in numbers, the default probability $\mathbb{P}_0(V_T < F)$ can be computed: $\mathbb{P}_0(V_T < F) = 0.4343$.

   The bond price is given by

$$D_0 = Fe^{-rT} - \text{Put}(V_0, F, r, \sigma, T). \tag{17}$$

   Plugging in numbers, we can get the price of bond $D_0 = 67.46$.

   The yield to maturity is

$$y = \frac{1}{T}\ln\left(\frac{F}{D_0}\right). \tag{18}$$

   Plugging in numbers, we can get the result of yield to maturity $y = 16.3\%$.

2. When the firm defaults, $V_T < F$. The final payment is $D_T = V_T$. Thus, the expected recovery is

$$\mathbb{E}_0\left[V_T | V_T < F\right] = \frac{\mathbb{E}_0\left[V_T \mathbf{1}(V_T < F)\right]}{\mathbb{E}_0\left[\mathbf{1}(V_T < F)\right]}. \tag{19}$$

   Following the derivation in (10), we have

$$\mathbb{E}_0\left[V_T | V_T < F\right] = V_0 e^{\mu T}\frac{\mathcal{N}(-d_{1,\mu})}{\mathcal{N}(-d_{2,\mu})}. \tag{20}$$

   Plugging in numbers, we can get the expected recovery $\mathbb{E}_0\left[V_T | V_T < F\right] = 70.55$.