ECON3133
Microeconomic Theory II

Tutorial #2: The (Total) Cost function

Today’s tutorial: the (Total) Cost function

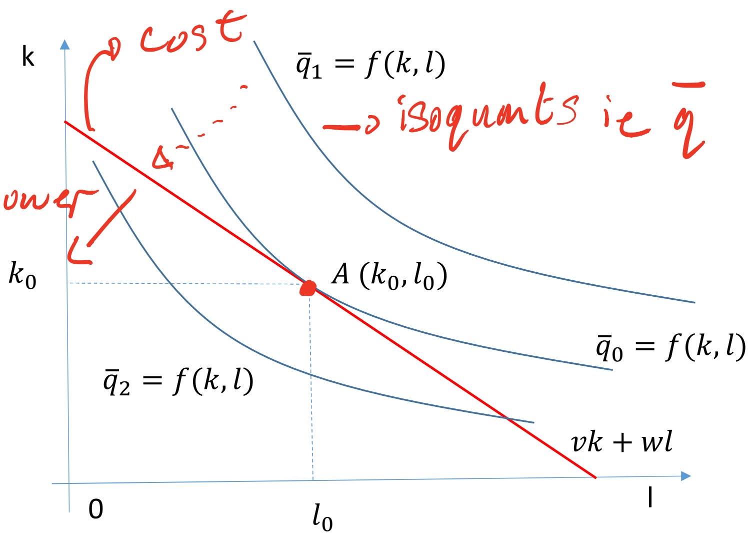
- The economic problem that we are solving: minimize total cost subject to producing a given level of output
- The solution to the problem 1: graphical
- What happens when input costs change? Elasticity of substitution and the cost expansion path
- The solution to the problem 2: the Lagrangian approach
- Comparative statics, the Envelope Theorem and Shephard’s Lemma

The Cost Minimisation problem

$c = vk + wl$

- The problem that we are solving is:
  - $\min_{k,l} vk + wl \text{ s.t. } \bar{q} = f(k, l)$
- Or in words:
  - Given costs v and w of k and l respectively, and production function $f(k, \bar{l})$, choose amounts of k and l to produce a given amount, $\bar{q}$, at minimum cost
- The solution depends on:
  - The amount to be produced
  - The shape of the isoquants
  - The relative magnitudes of v and w
- solution k^*, l^*

- Example: $\bar{q} = k^{\alpha} l^{\beta}$

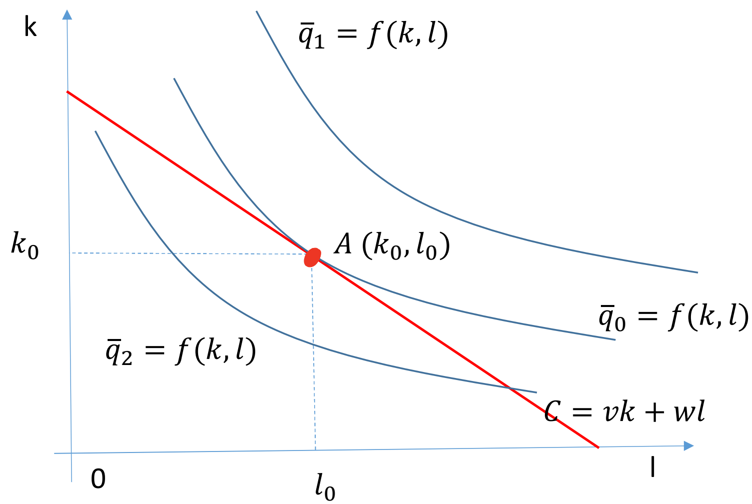


The Cost Minimisation problem: graphical solution

- Cost is minimized at the point where the isoquant is tangent to the cost line:
  - $\left.\frac{dk}{dl}\right|_{C=vk+wl} = \left.\frac{dk}{dl}\right|_{q=\bar{q}}$
- The slope of $C = vk + wl$ is:
- $\left.\frac{dk}{dl}\right|_{C=vk+} = -\frac{w}{v}$
- The slope of $\bar{q} = k^{\alpha} l^{\beta}$ is:
- $\left.\frac{dk}{dl}\right|_{q=\bar{q}} = -\frac{\partial f}{\partial l} \Big/ \frac{\partial f}{\partial k}$
- And so the equilibrium condition is:

$$\boxed{\frac{w}{v} = \frac{\partial f}{\partial l} \Big/ \frac{\partial f}{\partial k}}$$

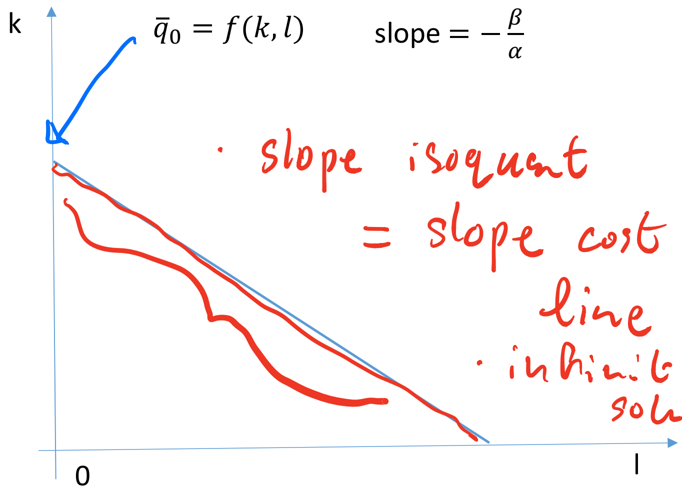
- Example: $\bar{q} = k^{\alpha} l^{\beta}$



Note that we have a unique solution here because of the shape of the isoquants

Equilibrium depends on the shape of the isoquants

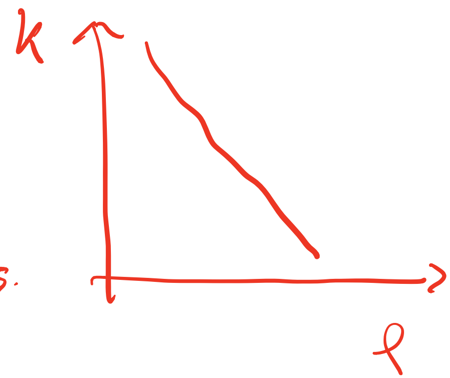
- Example 1: Linear production function $q = \alpha k + \beta l$, α, β constants



· slope isoquant = slope cost line

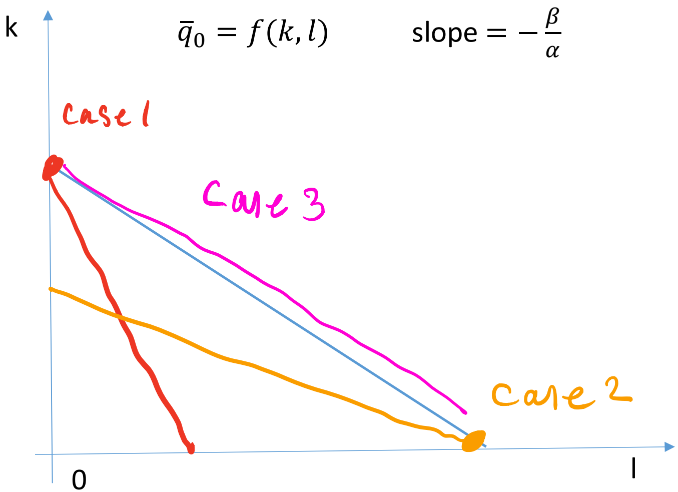
· infinite solutions.

- Given v and w in $C = vk + wl$, and slope $-\frac{w}{v}$, where is the equilibrium?



Equilibrium depends on the shape of the isoquants

- Example 1: Linear production function $q = \alpha k + \beta l$, α, β constants

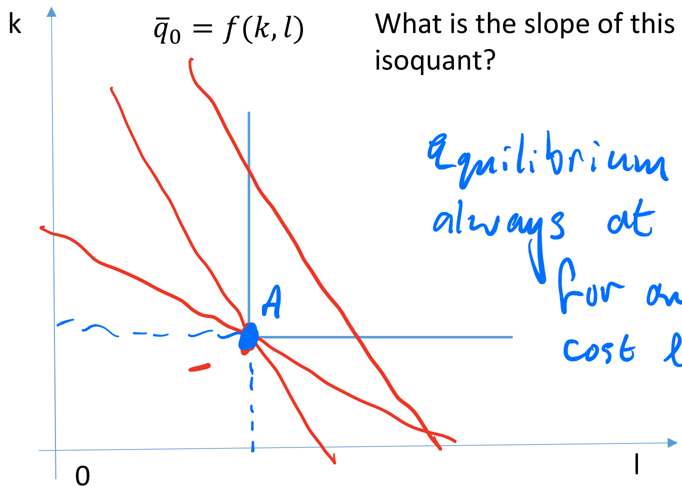


- Given v and w in $C = vk + wl$, and slope $-\frac{w}{v}$, where is the equilibrium?
- Case 1): $\frac{w}{v} > \frac{\beta}{\alpha}$ $\quad k^* = \frac{\bar{q}}{\alpha}$, $l^* = 0$.
- Case 2): $\frac{w}{v} < \frac{\beta}{\alpha}$ $\quad k^* = 0$, $l^* = \frac{\bar{q}}{\beta}$
- Case 3): $\frac{w}{v} = \frac{\beta}{\alpha}$

infinite solutions

Equilibrium depends on the shape of the isoquants

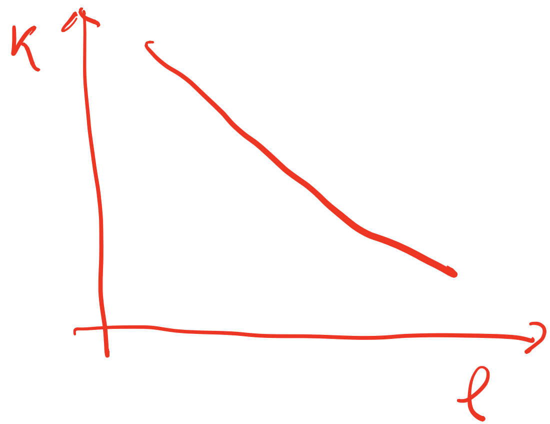
- Example 2: Fixed Proportions production function $q = \min[\alpha k, \beta l]$, $\alpha, \beta > 0$



What is the slope of this isoquant?

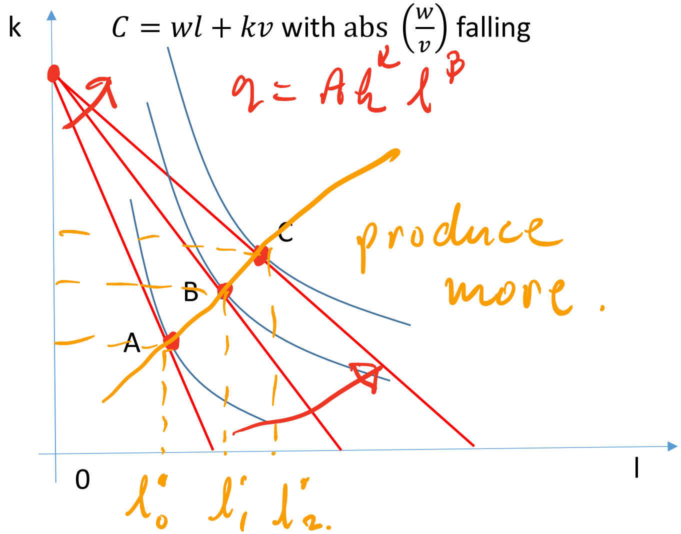
equilibrium always at A for any cost line

- Given v and w in $C = vk + wl$, and slope $-\frac{w}{v}$, where is the equilibrium?



Comparative Statics I: what happens when $\frac{w}{v}$ changes?

- What happens to optimal k and l (and therefore $\frac{k}{l}$) when $\frac{w}{v}$ changes?

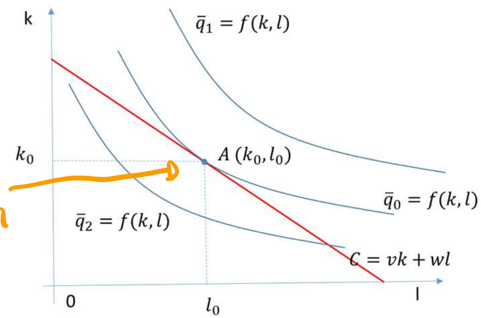


- Example: Suppose that wages fall relative to the cost of capital (ie abs $\left(\frac{w}{v}\right)$ falls). What happens to optimal k and l?
- Falling abs $\left(\frac{w}{v}\right)$ causes the cost line to become less steep
- In the diagram, it looks like k^* and l^* both increase when w falls relative to v
- Can we be more precise?
- Consider the elasticity of substitution between k and l
- optimal ratio $\frac{k^*}{l^*}$?

Comparative Statics: what happens when $\frac{w}{v}$ changes?

- Example: what happens to equilibrium $\frac{k}{l}$ when wages fall by 20% relative to the cost of capital?
- Re-call the definition of the elasticity of substitution (σ):
  - $\sigma = \dfrac{\%\ \text{change in}\ \frac{k}{l}}{\%\ \text{change in}\ RTS}$ → slope of isoquant
- Remember that cost is minimised where $RTS = \frac{w}{v}$ → tangency condition
- Therefore at minimum cost $\sigma = \dfrac{\%\ \text{change in}\ \frac{k}{l}}{\%\ \text{change in}\ \frac{w}{v}}$
- Therefore, if we know the value of σ, we know the equilibrium

  % change in $\frac{k}{l}$ given a % change in $\frac{w}{v}$



% ch. in $\frac{k}{l}$ in equilibrium $= \sigma \times$ % change in $\frac{w}{v}$

Comparative Statics I: what happens when $\frac{w}{v}$ changes?

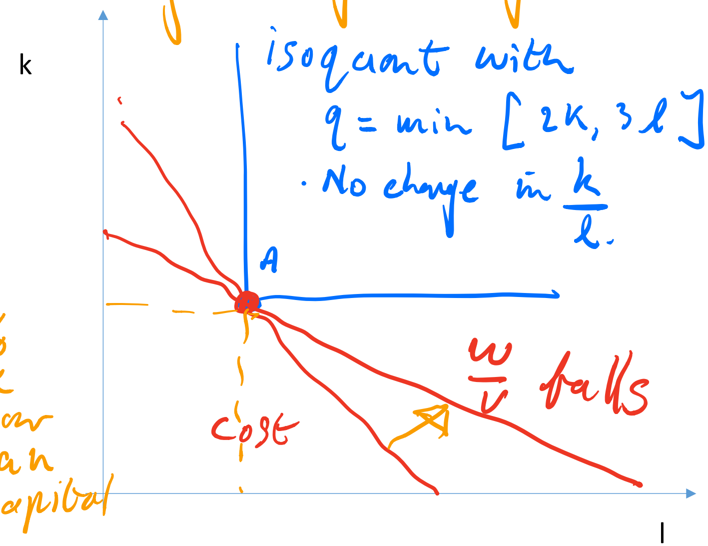
- Example: Assuming the production functions below, what happens to equilibrium $\frac{k}{l}$ when wages fall by 20% relative to the cost of capital?

-20%

$\frac{w}{v}$ falling = change is negative.

Production function	σ	% change in $\frac{k}{l}$
$q = 2k + 3l$	∞	∞
$q = \min(2k, 3l)$	0	0
CD $q = 10k^{1/2}l^{1/2}$	1	$1 \times -0.20 = -20\%$
CES $q = \left(k^{1/4} + l^{1/4}\right)^4$	$\rho = \frac{1}{4}$, $\frac{1}{1-\frac{1}{4}} = \frac{4}{3}$	$\frac{4}{3} \times -0.20 = 26.7\%$

20% more labor than capital



$\frac{k}{l}$ falls (l relatively cheaper)

Reminder: The Lagrangian Multiplier approach to constrained optimisation

constrained problem

- The problem: minimize cost, $C = wl + vk$, with respect to k, l subject to producing $\bar{q}$ with production function $q = f(k, l)$
    - In this case, assume a production function $q = k^{1/2}l^{1/2}$ $\alpha = \beta = \frac{1}{2}$.
- We may use a Lagrangian to solve this problem. The steps are as follows:

1) Form the Lagrangian function which has the form $\mathcal{L} = wl + vk + \lambda[\bar{q} - k^{1/2}l^{1/2}]$, where λ is another variable

now an unconstrained problem

2) Find the 3 partial derivatives with respect to k, l and λ and set them to zero:

FOCs $= 0$

$$\frac{\partial \mathcal{L}}{\partial k} = v - \lambda \frac{1}{2} k^{-1/2} l^{1/2} = 0 \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial l} = w - \lambda \frac{1}{2} k^{1/2} l^{-1/2} = 0 \qquad (2)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \bar{q} - k^{1/2} l^{1/2} = 0 \qquad (3)$$

result is minimum cost under the constraint.

Therefore $C^* = \mathcal{L}^*$

The Lagrangian Multiplier approach

3) Solve the three equations in three unknowns: k, l, λ

- Divide equation 1) by equation2) and re-arrange to give l in terms of w and v

$$\frac{v}{w} = \frac{l}{k} \longrightarrow l = k\frac{v}{w} \quad 4)$$

eq 4)

- Substitute into equation 3) to give k in terms of w, v and $\bar{q}$

$$\bar{q} - k^{\frac{1}{2}}\left(k\frac{v}{w}\right)^{\frac{1}{2}} = 0 \longrightarrow k^* = \bar{q}\left(\frac{w}{v}\right)^{\frac{1}{2}}$$

- Find the corresponding expression for l in terms of w, v and $\bar{q}$

$$l^* = \bar{q}\left(\frac{v}{w}\right)^{\frac{1}{2}}.$$

$$\lambda^* = 2(vw)^{\frac{1}{2}}$$

Contingent/conditional input demand functions

- What have we done here?
- We have solved the problem: $\min_{k,l} vk + wl \text{ s.t. } \bar{q} = f(k,l)$
- The solution gives three equations in terms of parameters only, of w, v and $\bar{q}$:
  - $k^*(w,v,\bar{q}) = \bar{q}\left(\frac{w}{v}\right)^{1/2}$
  - $l^*(w,v,\bar{q}) = \bar{q}\left(\frac{v}{w}\right)^{1/2}$
  - $\lambda^*(w,v,\bar{q}) = 2(vw)^{1/2}$

k^*, l^*, λ^* only dependent on w, v and $\bar{q}$

- The equations for k^* and l^* are called the contingent/conditional demand functions for k^* and l^*
- We may write them as:
  - $k^c(w,v,\bar{q}) = \bar{q}\left(\frac{w}{v}\right)^{1/2}, \; l^c(w,v,\bar{q}) = \bar{q}\left(\frac{v}{w}\right)^{1/2}$

Note that the contingent demand functions depend only on exogenous variables and parameters

Contingent/conditional input demand functions

- And so now we have:
  - $\min_{k,l} vk + wl$ s.t. $\bar{q} = f(k,l)$ gives contingent/conditional demand functions $k^*(w,v,\bar{q})$ and $l^*(w,v,\bar{q})$
- The optimal total cost of producing $\bar{q}$ is then:

$C = vk + wl$

  - $C^*_{q=\bar{q}} = vk^c(w,v,\bar{q}) + wl^c(w,v,\bar{q})$
- And we may write:
  - $C^*_{q=\bar{q}} = C^*_{q=\bar{q}}(\mathrm{w},\mathrm{v},\bar{q})$

Notice that as a result of the optimisation, the minimum cost $C^*_{q=\bar{q}}$ depends only on w, v and $\bar{q}$

- When written in this way (ie in terms of exogenous variables and parameters only – in this case w, v and $\bar{q}$), $C^*_{q=\bar{q}}(\mathrm{w},\mathrm{v},\bar{q})$ is known as the total cost function
- Any optimised function written in terms of exogenous variables and parameters only is known as a ‘Value function’
  - Another example:
    - The utility function in terms of prices of the goods, income and parameters of the utility function

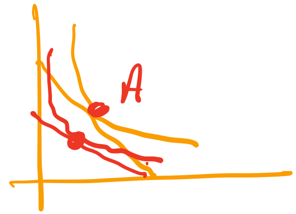
$$\underbrace{u(P_x, P_y, I)}_{\text{value function.}} \qquad x^a y^a$$

Comparative statics and the envelope theorem

$C^* | q = \bar{q}$

- How does minimum cost change when one of the exogenous variables or parameters changes?
- For example, how does minimum cost change when wages increase?
  - That is, what is $\frac{\partial C^*_{q=\bar{q}}}{\partial w}$?
- A prediction: the answer is $\frac{\partial C^*_{q=\bar{q}}}{\partial w} = \bar{q}\left(\frac{v}{w}\right)^{1/2}$
- How can we derive the expression that we need?
- We have the total cost function $C^*_{q=\bar{q}} = vk^c(w,v,\bar{q}) + wl^c(w,v,\bar{q}) = v\bar{q}\left(\frac{w}{v}\right)^{1/2} + w\bar{q}\left(\frac{v}{w}\right)^{1/2}$
- Then $\frac{\partial C^*_{q=\bar{q}}}{\partial w} = \frac{1}{2}v\bar{q}\left(\frac{w}{v}\right)^{-1/2}\frac{1}{v} + \bar{q}\left(\frac{v}{w}\right)^{1/2} - \frac{1}{2}\mathrm{w}\,\bar{q}v^{1/2}w^{-3/2}$

where is point? B



Comparative statics and the Envelope Theorem

- We have:
- Then $\frac{\partial C^*_{q=\bar{q}}}{\partial w} = \frac{1}{2}\, v\bar{q}\left(\frac{w}{v}\right)^{-1/2}\frac{1}{v} + \bar{q}\left(\frac{v}{w}\right)^{1/2} - \frac{1}{2}\mathrm{w}\;\bar{q}v^{1/2}w^{-3/2}$ simplify.

$$= \frac{1}{2}\,\bar{q}\left(\frac{w}{v}\right)^{-\frac{1}{2}} + \bar{q}\left(\frac{v}{w}\right)^{\frac{1}{2}} - \frac{1}{2}\,\bar{q}\,v^{\frac{1}{2}}\,w^{-\frac{1}{2}}.$$

$$= \bar{q}\left(\frac{v}{w}\right)^{\frac{1}{2}},$$

Remember the prediction:

the answer is $\frac{\partial C^*_{q=\bar{q}}}{\partial w} = \bar{q}\left(\frac{v}{w}\right)^{1/2}$

Comparative statics and the Envelope Theorem

- How could we predict the answer without having to do the calculation?
- The approach:
- We write the Lagrangian function including the optimised values for the unknowns (k^*, l^*, λ^*) :
  - $\mathcal{L}^* = vk^* + wl^* + \lambda^*\left[\bar{q} - k^{*\,1/2} l^{*\,1/2}\right]$
- Then $\frac{\partial C^*_{q=\bar{q}}}{\partial w} = \frac{\partial \mathcal{L}^*}{\partial w} = l^*$

$$= \bar{q}\left(\frac{v}{w}\right)^{1/2}$$

- So we differentiate the Lagrangian with respect to w and then use the solution for l^* that we found from the optimisation

$\mathcal{L} = vk + wl + \lambda\left[\bar{q} - k^{\frac{1}{2}} l^{\frac{1}{2}}\right]$

- substitute k^*, l^*, λ^*

Remember the prediction:

the answer is $\frac{\partial C^*_{q=\bar{q}}}{\partial w} = \bar{q}\left(\frac{v}{w}\right)^{1/2}$

1) Start with $\mathcal{L} = vk + wl + \lambda\left[\bar{q} - k^{\frac{1}{2}} l^{\frac{1}{2}}\right]$
2) find k^*, l^*, λ^*
3) substitute into $\mathcal{L}$:

$vk^* + wl^* + \lambda^*\left[\bar{q} - k^{\frac{1}{2}} l^{\frac{1}{2}}\right]$

Comparative statics and the Envelope Theorem

cost fixed output

- Exercise: In the cost minimisation problem $\min_{k,l} \underline{vk + wl}$ s.t. $\bar{q} = k^{1/2} l^{1/2}$ with contingent input demand functions $\underline{k^*(w, v, \bar{q}) = \bar{q}\left(\frac{w}{v}\right)}^{1/2}$ and $\underline{l^*(w, v, \bar{q}) = \bar{q}\left(\frac{v}{w}\right)}^{1/2}$ and $\lambda^*(w, v, \bar{q}) = 2v\left(\frac{\cancel{v}\,w}{\cancel{w}\,v}\right)^{1/2} = 2w\left(\frac{v}{w}\right)^{1/2}$, $= 2(vw)^{1/2}$ find the following:

1) $\frac{\partial C^*_{q=\bar{q}}}{\partial v}$ ie how do minimum costs change if the cost of capital changes?

2) $\frac{\partial C^*}{\partial \bar{q}}$ ie how do minimum costs change if desired output changes?

There are two ways to do this:

Comparative statics and the Envelope Theorem

Method 1: Using the Cost function $C^*(w, v, \bar{q})$

1) $\frac{\partial C^*_{q=\bar{q}}}{\partial v}$

$$C^*_{q=\bar{q}} = wl^c + vk^c = w\bar{q}\left(\frac{v}{w}\right)^{1/2} + v\,\bar{q}\left(\frac{w}{v}\right)^{1/2}$$

$$= 2\bar{q}(vw)^{1/2}$$

$$\frac{\partial C^*_{q=\bar{q}}}{\partial v} = \bar{q}v^{-1/2}w^{1/2}$$

$$= \bar{q}\left(\frac{w}{v}\right)^{1/2}$$

$$= k^c$$

Method 2: Using the Lagrangian function and the Envelope theorem

- $\mathcal{L}^* = vk^* + wl^* + \lambda^*\left[\bar{q} - k^{*\,1/2}l^{*\,1/2}\right]$

- $\frac{\partial C^*|_{q=\bar{q}}}{\partial v} = \frac{\partial \mathcal{L}^*}{\partial v}$

$$= k^*$$

$$= \bar{q}\left(\frac{w}{v}\right)^{\frac{1}{2}}$$

Comparative statics and the Envelope Theorem

Method 1: Using the Cost function $C^*(w, v, \bar{q})$

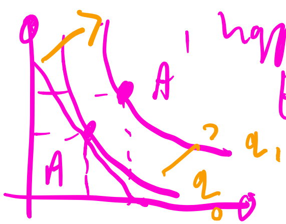
2) $\frac{\partial C^*}{\partial \bar{q}}$

$$C^*_{q=\bar{q}} = wl^c + vk^c = w\bar{q}\left(\frac{v}{w}\right)^{1/2} + v\,\bar{q}\left(\frac{w}{v}\right)^{1/2}$$

$$= 2\bar{q}(vw)^{1/2}$$

$$\frac{\partial C^*}{\partial \bar{q}} = \overset{2}{\cancel{\bar{q}}}(vw)^{1/2} = 2(vw)^{1/2}$$

what happens to cost?



Method 2: Using the Lagrangian function and the Envelope theorem

- $\mathcal{L}^* = vk^* + wl^* + \lambda^*\left[\bar{q} - k^{*\,1/2} l^{*\,1/2}\right]$

$$\frac{\partial \mathcal{L}^*}{\partial \bar{q}} = \lambda^*$$

$$= 2(vw)^{\frac{1}{2}}$$

Why does the Envelope Theorem work?

- Consider the optimisation process using the Lagrangian:

$$\min_{k,l} vk + wl \text{ s.t. } \bar{q} = f(k,l)$$

- Step 1: Form and solve the Lagrangian $\mathcal{L} = wl + vk + \lambda[\bar{q} - f(k,l)]$
  - This gives contingent/conditional demand functions $k^*(w,v,\bar{q})$, $l^*(w,v,\bar{q})$, and a solution for λ^* that depend only on $w, v, \bar{q}$

- Step 2: Substitute the optimal values, $k^*(w,v,\bar{q})$, $l^*(w,v,\bar{q})$, and $\lambda^*(w,v,\bar{q})$ back into the Lagrangian:

$$\mathcal{L}^* = vk^*(w,v,\bar{q}) + wl^*(w,v,\bar{q}) + \lambda^*[\bar{q} - f(k^*(w,v,\bar{q}), l^*(w,v,\bar{q}))]$$

  - The optimised Lagrangian therefore depends on $k^*, l^*, \lambda^*, w, v, \bar{q}$

$$\mathcal{L}^*(k^*, l^*, \lambda^*, w, v, \bar{q}) = \mathcal{L}(k^*(w,v,\bar{q}), l^*(w,v,\bar{q}), \lambda^*(w,v,\bar{q}), w, v, \bar{q})$$

• everything depends only on $w, v, \bar{q}$
• either directly, or indirectly in k^*, l^*, λ^*

Why does the Envelope Theorem work?

- Step 3: Differentiate both sides with respect to one of the exogenous variables or parameters eg w:
  - $\mathcal{L}^*(k^*, l^*, \lambda^*, w, v, \bar{q}) = \mathcal{L}(k^*(w, v, \bar{q}), l^*(w, v, \bar{q}), \lambda^*(w, v, \bar{q}), w, v, \bar{q})$

$\frac{\partial \mathcal{L}^*}{\partial w}$

$$\frac{\partial \mathcal{L}^*(v,w,\bar{q})}{\partial w} = \underbrace{\frac{\partial \mathcal{L}}{\partial k^*}}_{=0}\frac{\partial k^*}{\partial w} + \underbrace{\frac{\partial \mathcal{L}}{\partial l^*}}_{=0}\frac{\partial l^*}{\partial w} + \underbrace{\frac{\partial \mathcal{L}}{\partial \lambda^*}}_{=0}\frac{\partial \lambda^*}{\partial w} + \underbrace{\frac{\partial \mathcal{L}}{\partial w}}_{\neq 0}$$

use chain rule.

- Remember the first order conditions in the optimisation. What are the values of $\frac{\partial \mathcal{L}}{\partial k^*}, \frac{\partial \mathcal{L}}{\partial l^*}, \frac{\partial \mathcal{L}}{\partial \lambda^*}$?
- Remember that k^*, l^* and λ^* were chosen so that $\mathcal{L}$ is at a ~~maximum~~. minimum
- Therefore, $\frac{\partial \mathcal{L}}{\partial k^*} = \frac{\partial \mathcal{L}}{\partial l^*} = \frac{\partial \mathcal{L}}{\partial \lambda^*} = 0$
- And therefore:

$$\frac{\partial C^*}{\partial w} = \frac{\partial \mathcal{L}^*(v,w,\bar{q})}{\partial w} = \frac{\partial \mathcal{L}}{\partial w}$$

direct differentiation wrt w

$\frac{\partial \mathcal{L}}{\partial k} = 0$ at optimum.

$\frac{\partial \mathcal{L}}{\partial l} = 0$ at optimum

$\frac{\partial \mathcal{L}}{\partial \lambda} = 0$ " "

Why does the Envelope Theorem work?

- And therefore, in the cost minimisation problem with $\mathcal{L}^* = vk^* + wl^* + \lambda^*[\bar{q} - f(k^*, l^*)]$
- We have the total cost function $C^*(w, v, \bar{q})$
- We also have:

$$\frac{\partial C^*_{q=\bar{q}}}{\partial v} = \frac{\partial \mathcal{L}^*(v,w,\bar{q})}{\partial v} = k^* \qquad \frac{\partial C^*_{q=\bar{q}}}{\partial w} = \frac{\partial \mathcal{L}^*(v,w,\bar{q})}{\partial w} = l^* \qquad \frac{\partial C^*}{\partial \bar{q}} = \frac{\partial \mathcal{L}^*(v,w,\bar{q})}{\partial \bar{q}} = \lambda^*$$

- These relationships are known as 'Shephard's Lemma'

· contingent demands are $k^* = k^c$

$l^* = l^c$

Appendix: How does the Lagrange Multiplier approach work?

100% optional.

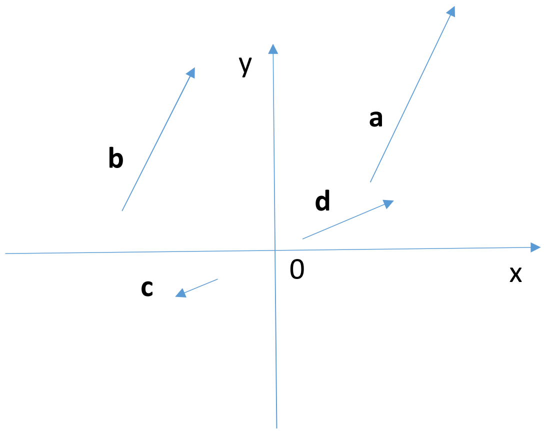
- The Lagrange Multiplier approach uses some ideas from linear algebra and multivariable calculus.
- The main ideas that we need are:

Linear algebra	
1	Two vectors are equal if their components are equal, and they are parallel if they are scalar multiples of each other
2	Any point on a curve has a tangent vector to the curve at that point
3	At any point on a curve, the gradient vector is 'normal' ie perpendicular to the tangent vector
The $U(x, y)$ optimization	
4	In the constrained optimization problem, $U(x, y)$ is maximized at the (unique) point where $U(x, y)$ curve is tangent to the budget constraint
5	At this point, the gradient of the $U(x, y)$ curve points in the same direction as the gradient of the budget constraint ie they are parallel
6	Since the gradient of $U(x, y)$ is parallel to the gradient of the budget constraint at this point, this gives first order conditions for an unconstrained optimisation
7	The function that is the subject of the unconstrained optimisation is the Lagrangian

Appendix: How does the Lagrange Multiplier approach work?

1. Vectors are equal if their components are equal, and parallel if they are scalar multiples of each other. Here, a scalar just means a number.

For example, the vectors $\mathbf{a} = 6i + 12j$ and $\mathbf{b} = 6i + 12j$ are equal, and $\mathbf{c} = -i - 3j$ and $\mathbf{d} = 2i + 6j$ are parallel (since $\mathbf{d} = -2\mathbf{c}$)



- Note that we can position vectors anywhere on the graph
- We can write any vector in terms of the standard unit vectors i and j:

$$\mathbf{u} = u_1 i + u_2 j, \mathbf{v} = v_1 i + v_2 j$$

- Equality of $\mathbf{u}$, $\mathbf{v}$ requires that

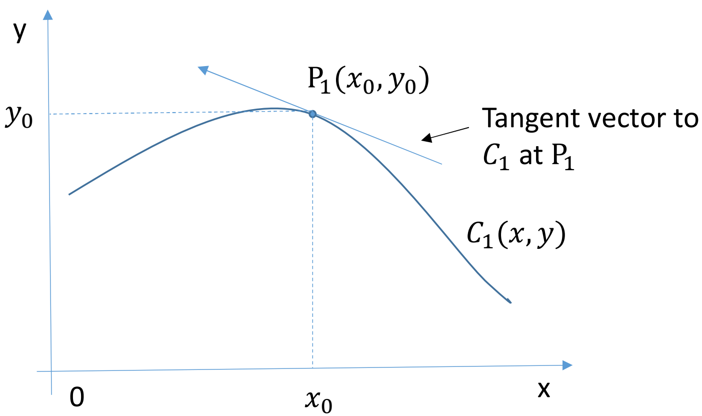
  $u_1 = v_1,\, u_2 = v_2$

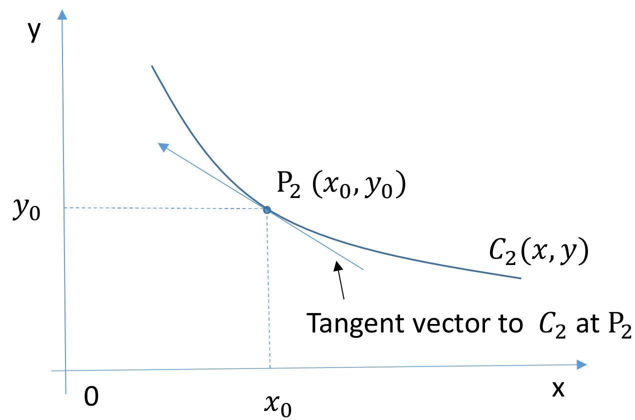
- $\mathbf{u}$, $\mathbf{v}$ parallel requires that $\mathbf{u} = \lambda \mathbf{v}$ for some scalar (ie number) λ

Appendix: How does the Lagrange Multiplier approach work?

2. At any point on a curve, there is a tangent vector to the curve.

Examples:

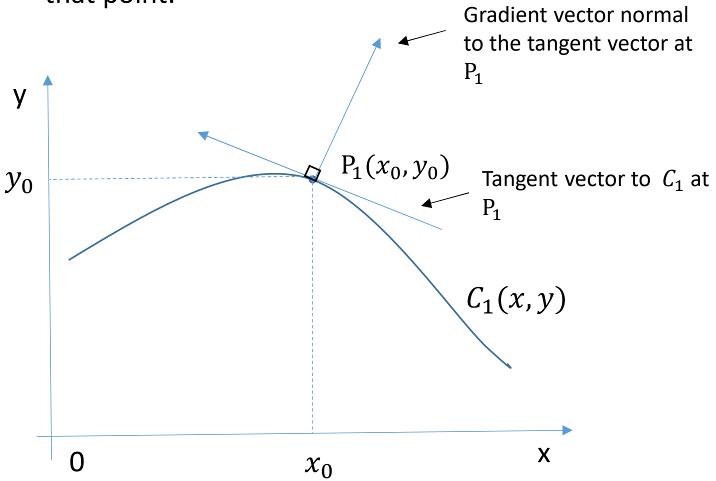


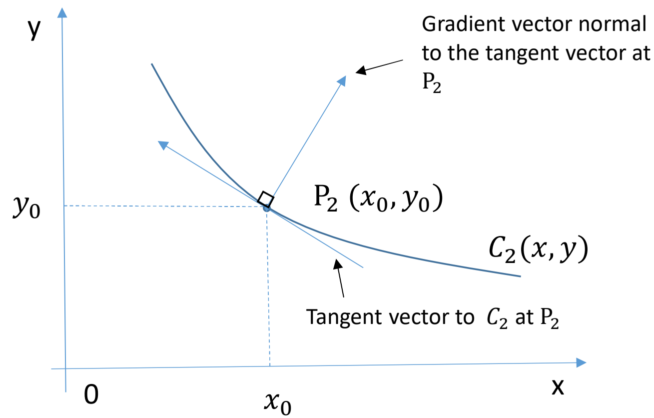


- In terms of economics, what kind of curve does curve C_2 look like?
  - An indifference curve $U(x_0, y_0)$

Appendix: How does the Lagrange Multiplier approach work?

3. At any point on a curve, the gradient vector at that point is 'normal' (ie perpendicular) to the tangent vector at that point.

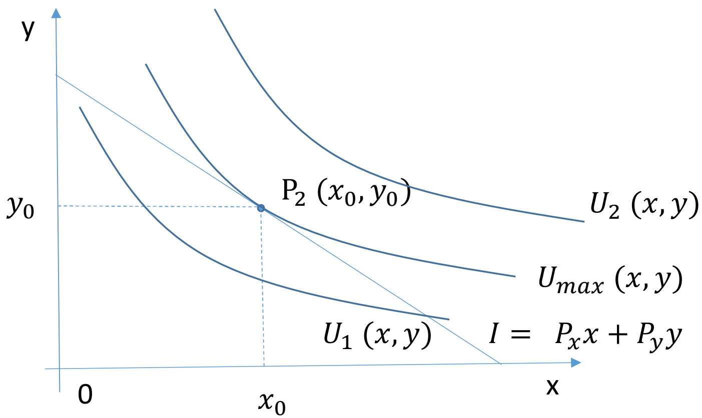




- For a given function $f(x, y)$, the gradient vector is defined as $\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial y}{\partial x} \mathrm{j}$

Appendix: How does the Lagrange Multiplier approach work?

4. In the constrained optimization problem, $U(x, y)$ is maximized at the (unique) point where $U(x, y)$ curve is tangent to the budget constraint

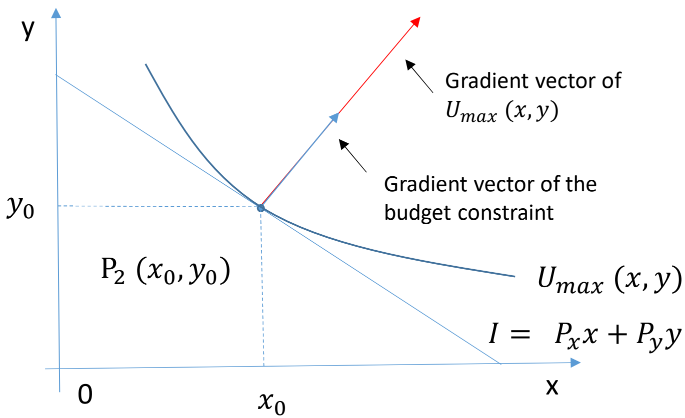


- Given income I, utility is maximized at point $P_2\ (x_0, y_0)$ and is $U_{max}\ (x_0, y_0)$
  - Income is too low to achieve U_2, and we have enough income to do better than U_1
- The shape of the $U(x, y)$ curve means that in this case the tangency point is unique
  - Depends on the form of the utility function

So the $U_{max}(x, y)$ curve and the budget constraint have tangent vectors at $P_2(x_0, y_0)$ that are parallel

Appendix: How does the Lagrange Multiplier approach work?

5. At P_2 (x_0, y_0) (ie the U_{max} point), the <u>gradient</u> of the $U_{max}(x_0, y_0)$ curve points in the same direction as the <u>gradient</u> of the budget constraint ie they are parallel



- The gradient vector of $U_{max}(x_0, y_0)$ is parallel to the gradient vector of the budget constraint
- Note that their lengths may well be different, but they are parallel
- In symbols:

  $\nabla U_{max}(x_0, y_0) = \lambda \nabla I$, with λ a scalar (ie a number)

- What does this mean?

Appendix: How does the Lagrange Multiplier approach work?

6. The $\nabla U_{max}(x_0, y_0) = \lambda \nabla I$ statement is equivalent to the first order conditions of an unconstrained optimization

To see this, consider the following:

- $\nabla U_{max}(x_0, y_0) = \lambda \nabla I$ may be written $\frac{\partial U_{max}(x_0,y_0)}{\partial x} i + \frac{\partial U_{max}(x_0,y_0)}{\partial y} j = \lambda[\frac{\partial I}{\partial x} i + \frac{\partial I}{\partial y} j]$
- Since $\frac{\partial I}{\partial x} = P_x$, and $\frac{\partial I}{\partial y} = P_y$, this is equivalent to $\frac{\partial U_{max}(x_0,y_0)}{\partial x} i + \frac{\partial U_{max}(x_0,y_0)}{\partial y} j = P_x i + \lambda P_y j$
- Both sides of the equation are vectors, and because they are equal, their i and j components are equal
  - ie $\frac{\partial U_{max}(x_0,y_0)}{\partial x}$ = λP_x and $\frac{\partial U_{max}(x_0,y_0)}{\partial y}$ = λP_y
- Re-arranging each equation gives:
  - $\frac{\partial U_{max}(x_0,y_0)}{\partial x}$ - $\lambda P_x = 0$
  - $\frac{\partial U_{max}(x_0,y_0)}{\partial y}$ - $\lambda P_y = 0$
- These are nothing more than the first order conditions of an optimization with respect to x and y of the function $\mathcal{L}$:

$$\max_{x,y} \mathcal{L} = U(x, y) + \lambda[\mathrm{I} - P_x x + P_y y]$$

Appendix: How does the Lagrange Multiplier approach work?

7. The $\nabla U_{max}(x_0, y_0) = \lambda \nabla I$ statement is equivalent to the Lagrangian, $\mathcal{L}$, that we started with

<u>Lagrangian</u>

$$\max_{x,y} \mathcal{L} = U(x,y) + \lambda[\mathrm{I} - P_x x + P_y y]$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial U(x,y)}{\partial x} - \lambda P_x = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{\partial U(x,y)}{\partial y} - \lambda P_y = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \mathrm{I} - P_x x + P_y y = 0$$

- Solve for x_0, y_0

<u>Gradient utility function = gradient budget constraint</u>

$$\nabla U_{max}(x_0, y_0) = \lambda \nabla I$$

$$\frac{\partial U_{max}(x_0, y_0)}{\partial x} - \lambda P_x = 0$$

$$\frac{\partial U_{max}(x_0, y_0)}{\partial y} - \lambda P_y = 0$$

$$\mathrm{I} - P_x x + P_y y = 0$$

- x_0, y_0 solve these equations

- So that's why we use a Lagrangian for constrained optimization!