ECON 3113 Microeconomic Theory I
Lecture 4: Theory of Utility Maximization

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Mar 2020

Introduction

- Given a preference relation (over possible consumption bundles) encapsulated in the utility function, and given a set of feasible/affordable consumption bundles, what bundle would the consumer go for?
- This is a problem of **constrained maximization**.
- Mathematical approach and graphical approach are complementary.

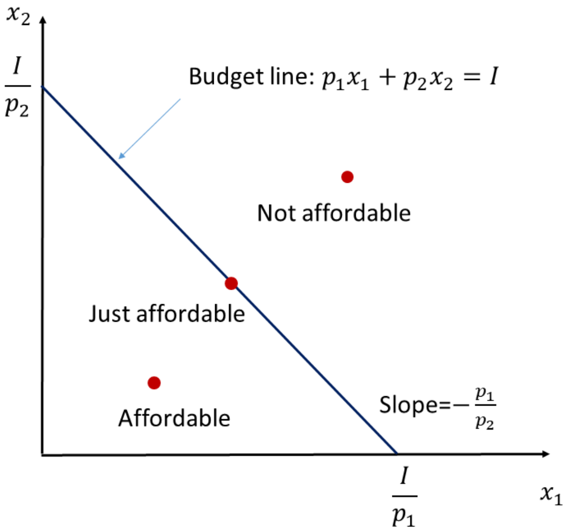
Affordable Bundles

- Let's begin with the constraints.
- Suppose there are n goods available and the respective prices are p_1, p_2, ..., p_n.
- Suppose the consumer has an income of I dollars to spend.
- The set of all feasible/affordable bundles is

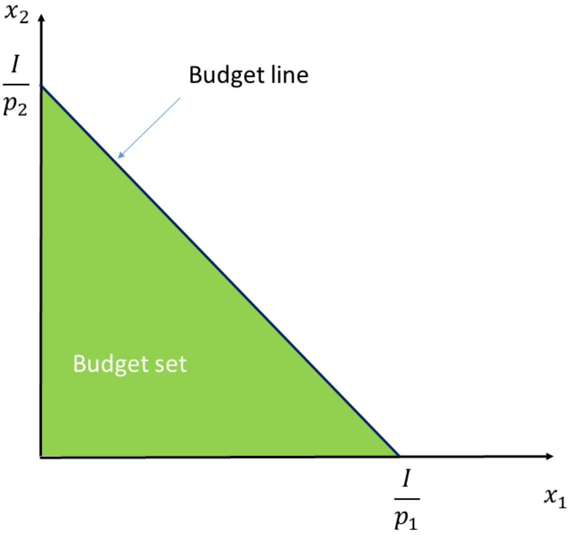
$$\left\{(x_1, x_2, ..., x_n) \geq 0 : p_1 x_1 + p_2 x_2 + ... + p_n x_n \leq I\right\}.$$

- The inequality $p_1 x_1 + p_2 x_2 + ... + p_n x_n \leq I$ is the consumer's **budget constraint**.
- The budget constraint is scale-invariant: multiplying all prices and income by a common factor will leave the budget constraint unchanged.

Affordable Bundles: Diagram

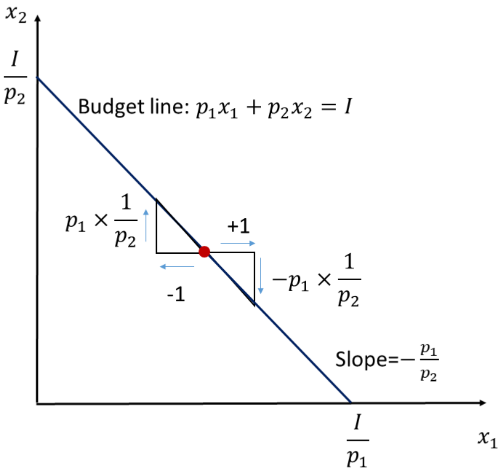


Budget Set



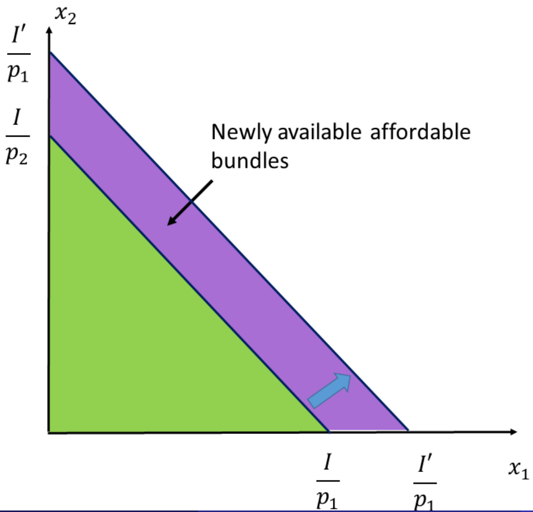
Moving along the Budget Line

- Slope of budget line is $-p_1/p_2$: giving up 1 unit of good 1 allows the consumer to buy p_1/p_2 more units of good 2.



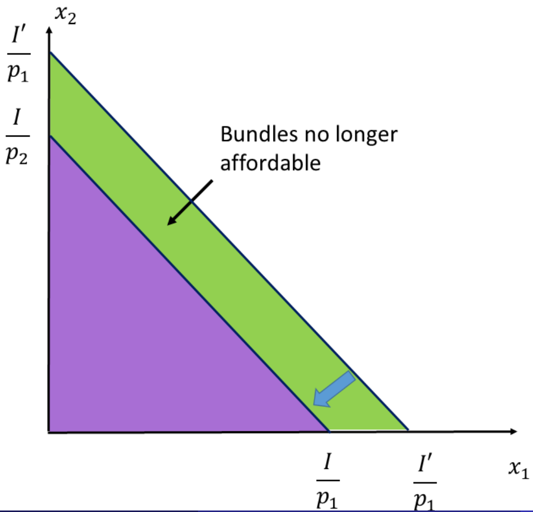
Change in the Budget Constraint: Increase in Income

- An increase in consumer's income from I to I' gives rise to a parallel outward shift of the budget line.



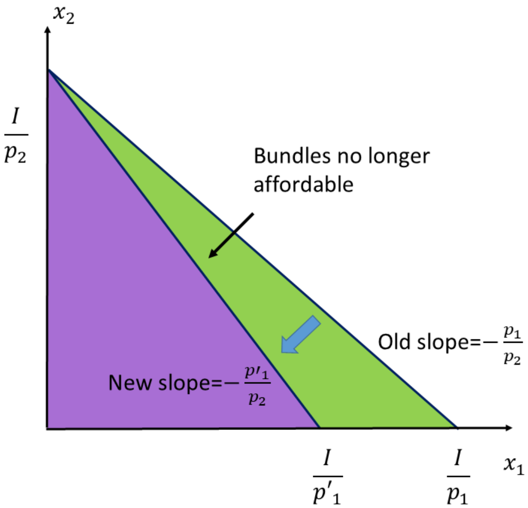
Change in the Budget Constraint: Decrease in Income

- An decrease in consumer's income from I' to I gives rise to a parallel inward shift of the budget line.



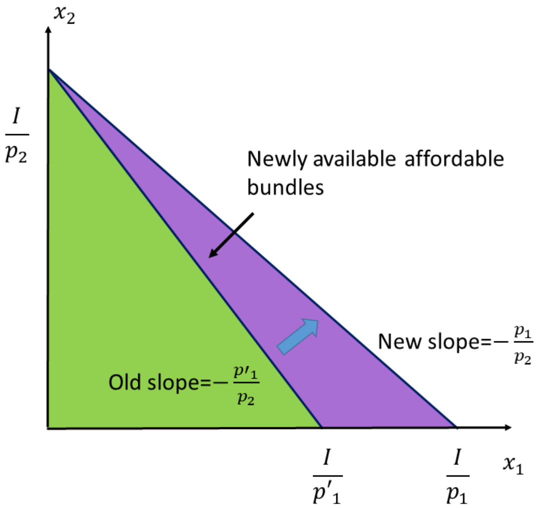
Change in Budget Constraint: Increase in Good 1's Price

- An increase in the price of goods 1 from p_1 to p_1' gives rise to an inward rotation of the budget line around the vertical intercept.



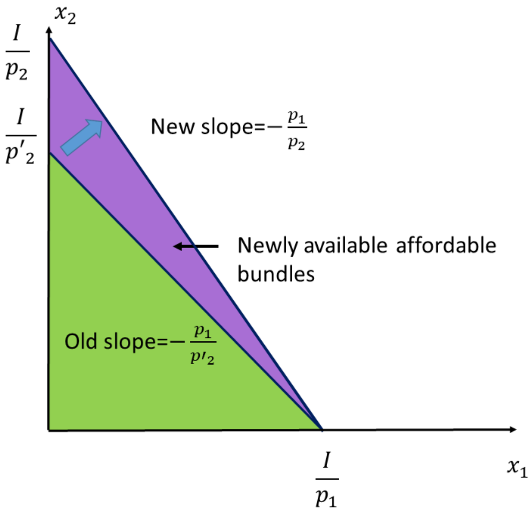
Change in Budget Constraint: Decrease in Good 1's Price

- A decrease in the price of goods 1 from p_1' to p_1 gives rise to an outward rotation of the budget line around the vertical intercept.



Change in Budget Constraint: Decrease in Good 2's Price

- A decrease in the price of goods 2 from p_2' to p_2 gives rise to a inward rotation of the budget line around the horizontal intercept.



Consumer's Problem

- We will assume throughout that the consumer has a complete and transitive preference relation over all consumption bundles, enabling a utility function u representing it.
- The consumer's problem is to choose a feasible bundle to maximize her utility. Formally, she solves

$$\max_{(x_1, x_2, \ldots, x_n) \geq 0} u\left(x_1, x_2, \ldots, x_n\right)$$

subject to the budget constraint

$$p_1 x_1 + p_2 x_2 + \ldots + p_n x_n \leq I.$$

- Prices and income are **exogenous**.
- The consumption bundle chosen is **endogenous**: it depends on prices and income.

Consumer's Problem: The Two-Goods Case

- With only two goods, good 1 and good 2, the problem is simply

$$\max_{(x_1,x_2)\geq 0} u\left(x_1, x_2\right)$$

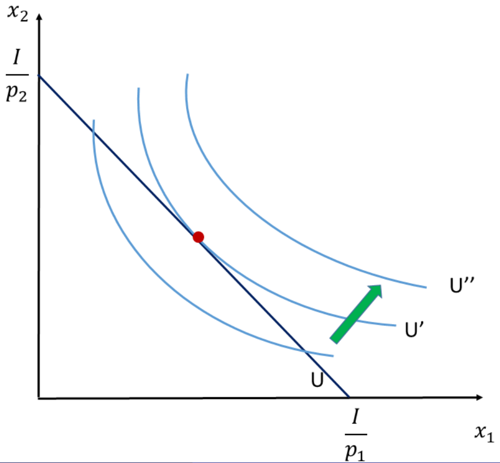
subject to the budget constraint

$$p_1 x_1 + p_2 x_2 \leq I.$$

- The solution(s) to the problem may be
  - unique or non-unique
  - interior or boundary

Solving the Consumer's Problem Graphically

- Superposing the budget line with the indifference curves.
- Look for the bundle/point in the budget set that lies on the indifference curve with the highest utility.

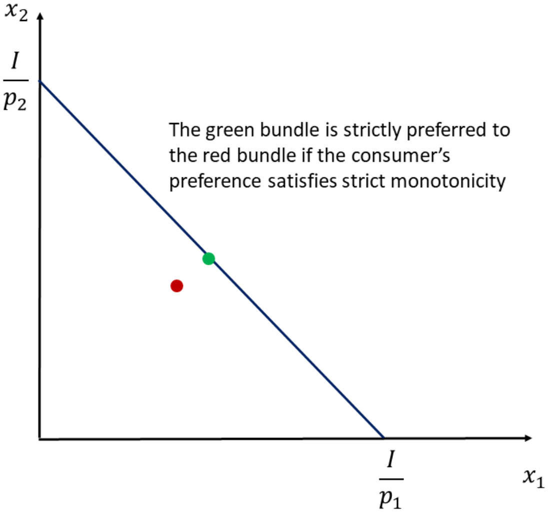


Walras' Law: Spend All You Have

- Monotonicity of preferences implies that an optimal bundle lies on the budget line.
- Strict monotonicity of preferences implies all optimal bundles lie on the budget line.
- With a strictly monotone preference, the optimal bundle $(x_1^*, x_2^*, ..., x_n^*)$ necessarily satisfies

$$p_1 x_1^* + p_2 x_2^* + ... + p_n x_n^* = I.$$

Walras' Law: Spend All You Have

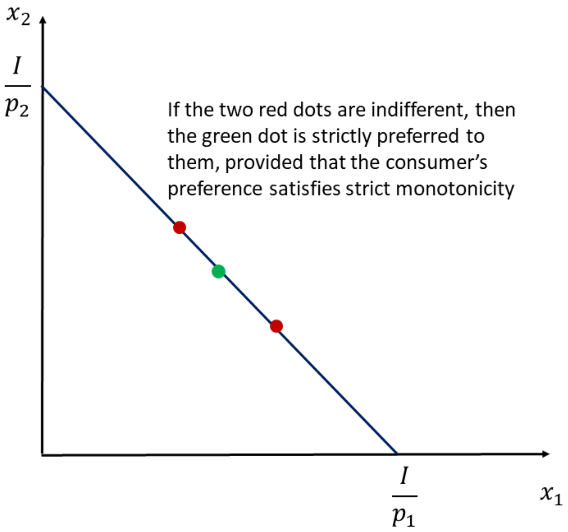


Existence and Uniqueness of Consumer's Problem

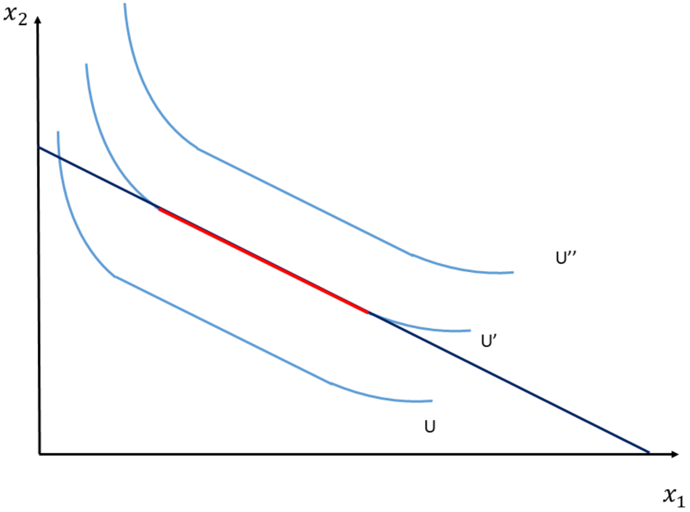
Theorem

If the consumer's preference is continuous, then an optimal consumption bundle exists.
If, in addition, it is strictly convex (equivalently, the utility function u is strictly quasi-concave), then the optimal consumption bundle is unique.

Existence and Uniqueness of Consumer's Problem



Uniqueness and Non-uniqueness of Optimal Bundles



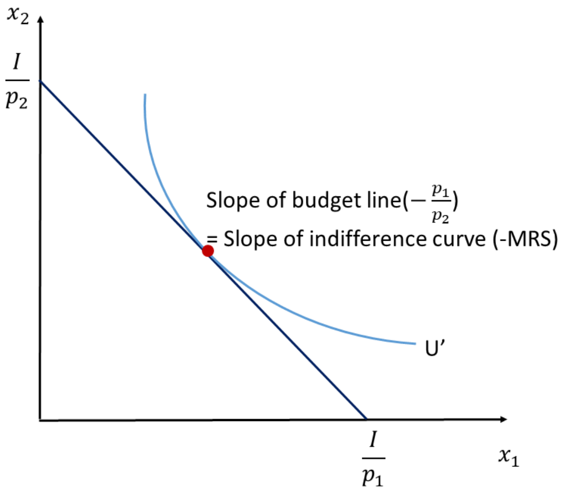
Solving the Consumer's Problem Graphically: Strictly Convex Preference

- Recall strict convexity of preference is equivalent to Diminishing Marginal Rate of Substitution (DMRS).
- If the optimal bundle (x_1^*, x_2^*) is interior, it is uniquely determined by aligning the slope of the budget line with the slope of the indifference curve:

$$\underbrace{MRS}_{\text{(negative of) slope of indifference curve}} = \underbrace{\frac{p_1}{p_2}}_{\text{(negative of) slope of budget line}} .$$

(Tangency)

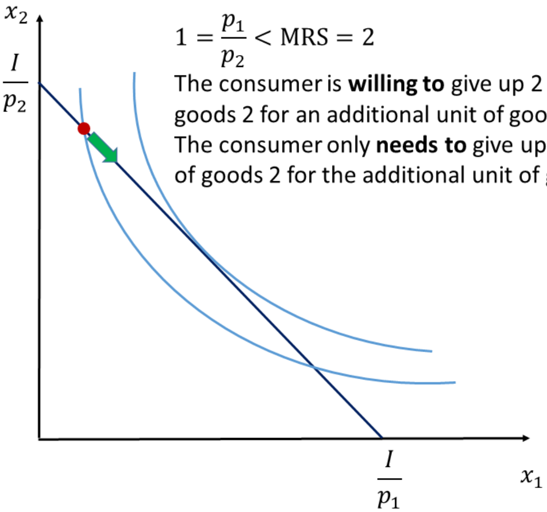
Solving the Consumer's Problem Graphically: Strictly Convex Preference



Economic Interpretation of the Tangency Condition

- What bundle would make the consumer willing to stay put?
- Start with any bundle $(x_1, x_2) > (0, 0)$. If she wants to increase his consumption of goods 1 by one unit,
  - the amount of goods 2 she is *willing to* give up is MRS;
  - the amount of goods 2 she *has to* give up is $p_1 \times \frac{1}{p_2}$.
- She wants to consume more of goods 1 if $\frac{p_1}{p_2} < MRS$.

Economic Interpretation of the Tangency Condition



$$1 = \frac{p_1}{p_2} < \text{MRS} = 2$$

The consumer is **willing to** give up 2 units of goods 2 for an additional unit of goods 1.
The consumer only **needs to** give up 1 unit of goods 2 for the additional unit of goods 1.

Economic Interpretation of the Tangency Condition

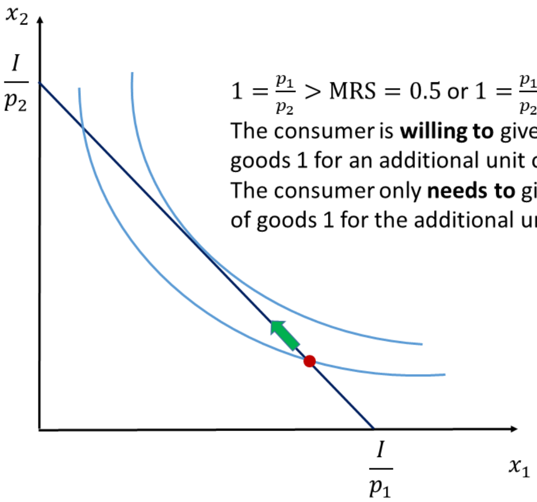
- If she wants to increase his consumption of goods 2 by one unit,
  - the amount of goods 1 she is *willing to* give up is $1/MRS$;
  - the amount of goods 1 she *has to* give up is $p_2 \times \frac{1}{p_1}$.
- She wants to consume more of goods 2 if $\frac{p_1}{p_2} > MRS$.

Economic Interpretation of the Tangency Condition

$1 = \frac{p_1}{p_2} > \text{MRS} = 0.5$ or $1 = \frac{p_1}{p_2} < \frac{1}{MRS} = 2$

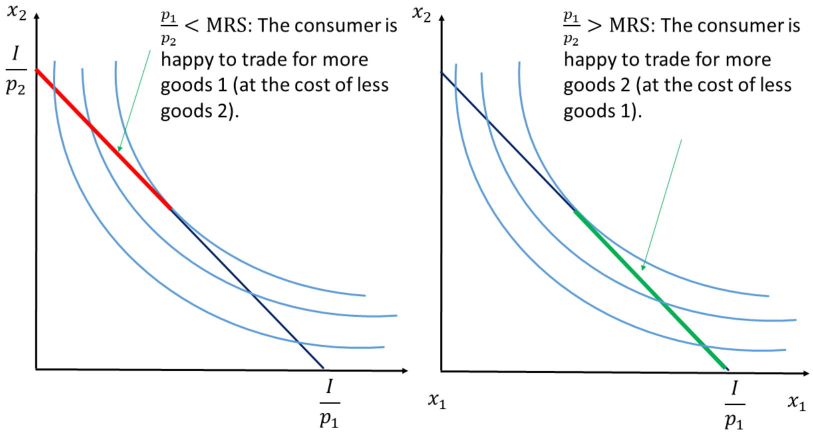
The consumer is **willing to** give up 2 units of goods 1 for an additional unit of goods 2.

The consumer only **needs to** give up 1 unit of goods 1 for the additional unit of goods 2.



Economic Interpretation of the Tangency Condition

- Only if an interior bundle has $MRS = p_1 / p_2$, the consumer does not want to move away from it.

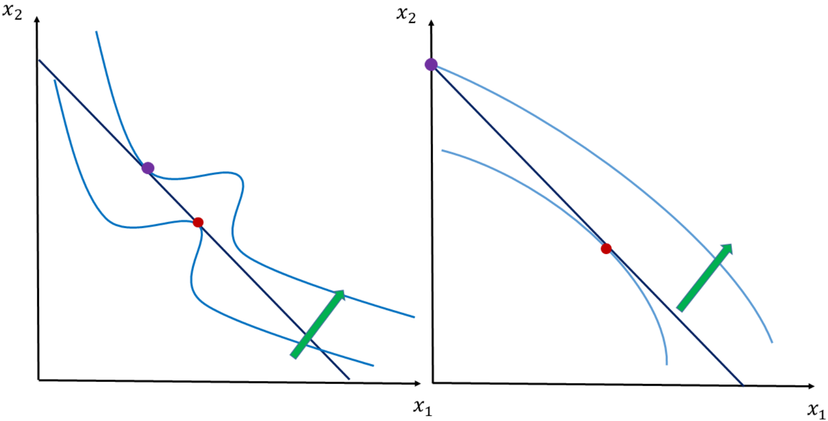


Caveats

- A bundle that satisfies the tangency condition may not be optimal.
- A optimal bundle may not satisfy the tangency condition.

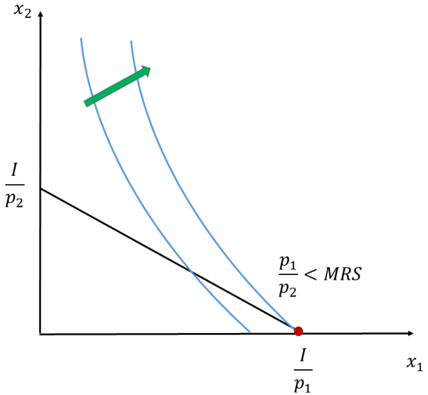
When does tangency condition fails to guarantee optimum?

- If preference is not convex.



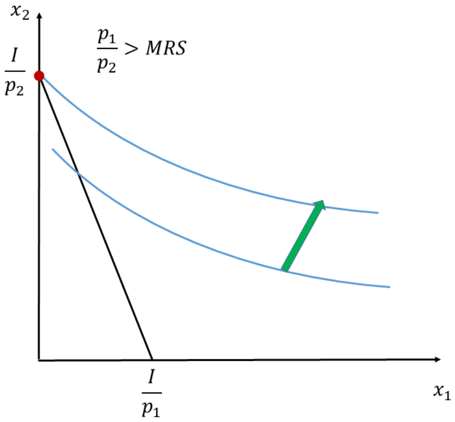
When does the optimal bundle fails tangency condition?

- If the optimal bundle lies on the boundary – a **corner solution**.
- Start with a bundle $(x_1, x_2) = (I/p_1, 0)$ and suppose at this bundle $MRS > p_1/p_2$.
    - If possible, the consumer would love to trade for more goods 1, but she has already spent all the income on goods 1.



When does the optimal bundle fails tangency condition?

- Start with a bundle $(x_1, x_2) = (0, I/p_2)$ and suppose at this bundle $MRS < p_1/p_2$.
  - If possible, the consumer would love to trade for more goods 2, but she has already spent all the income on goods 2.



Generalizing the Tangency Condition: First-order Condition

- The optimal bundle necessarily (x_1^*, x_2^*) satisfies either one of the three conditions below.

1. $$MRS\left(x_1^*, x_2^*\right) = \frac{p_1}{p_2}, \text{ or}$$

2. $$MRS\left(x_1^*, x_2^*\right) > \frac{p_1}{p_2} \text{ and } x_2^* = 0, \text{ or}$$

3. $$MRS\left(x_1^*, x_2^*\right) < \frac{p_1}{p_2} \text{ and } x_1^* = 0.$$

Solving the Consumer's Problem Mathematically

- If the utility function $u(x_1, x_2)$ is differentiable, strictly increasing, and satisfying DMRS, the first-order condition (FOC) is sufficient.
  - That is, if bundle (x_1^*, x_2^*) satisfies FOC, it must be optimal.
- Suppose further that the solution is interior, it can be identified by solving the system

$$\begin{aligned} p_1 x_1 + p_2 x_2 &= I\text{; and} && \text{(Walras' law)} \\ \frac{\partial u / \partial x_1}{\partial u / \partial x_2} &= \frac{p_1}{p_2} && \text{(Tangency)} \end{aligned}$$

Example: Cobb-Douglas

- Suppose the consumer has a Cobb-Douglas utility function:

$$u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha},$$

  for some $\alpha \in [0, 1]$.

- As $u(x_1, 0) = u(0, x_2) = 0 < u(x_1, x_2)$ for any positive x_1, x_2, the optimal bundle is interior for certain.
  - Recall that we have checked that DMRS is satisfied for Cobb-Douglas utility.

- As

$$MRS = \frac{\partial u / \partial x_1}{\partial u / \partial x_2} = \frac{\alpha x_1^{\alpha-1} x_2^{1-\alpha}}{(1-\alpha) x_1^{\alpha} x_2^{-\alpha}} = \frac{\alpha}{1-\alpha} \frac{x_2}{x_1},$$

  the tangency condition reads

$$\frac{\alpha}{1-\alpha} \frac{x_2}{x_1} = \frac{p_1}{p_2} \Leftrightarrow x_2 = \frac{1-\alpha}{\alpha} \frac{p_1}{p_2} x_1.$$

Example: Cobb-Douglas

- Substitute this into the budget line:

$$p_1 x_1 + p_2 \underbrace{\left(\frac{1-\alpha}{\alpha} \frac{p_1}{p_2} x_1 \right)}_{x_2} = I \Rightarrow x_1^* = \frac{\alpha I}{p_1}.$$

- Substitute this back to the tangency condition gives:

$$x_2^* = (1-\alpha) \frac{I}{p_2}.$$

- The consumer spends a constant proportion of her income of each goods:

$$\frac{p_1 x_1^*}{I} = \alpha \text{ and } \frac{p_2 x_2^*}{I} = 1 - \alpha.$$

Example: Quasi-Linear

- Suppose the consumer has a quasi-linear utility function:

$$u(x_1, x_2) = v(x_1) + x_2,$$

  for some strictly increasing and strictly concave function v.

- The MRS is given by

$$MRS = \frac{\partial u / \partial x_1}{\partial u / \partial x_2} = v'(x_1),$$

  so it is strictly decreasing in x_1 but independent of x_2.

  - Strict concavity of v implies DMRS.

Example: Quasi-Linear

- If the solution is interior, then it is necessary that

$$v'\left(x_1^*\right) = \frac{p_1}{p_2} \text{ and } x_1^* \in \left(0, \frac{I}{p_1}\right),$$

  and x_2^* can be backed out from the budget line: $x_2^* = I - p_1 x_1^*$.
- If there is no solution to the equation above, then we will have to look for corner solution.
  - Either

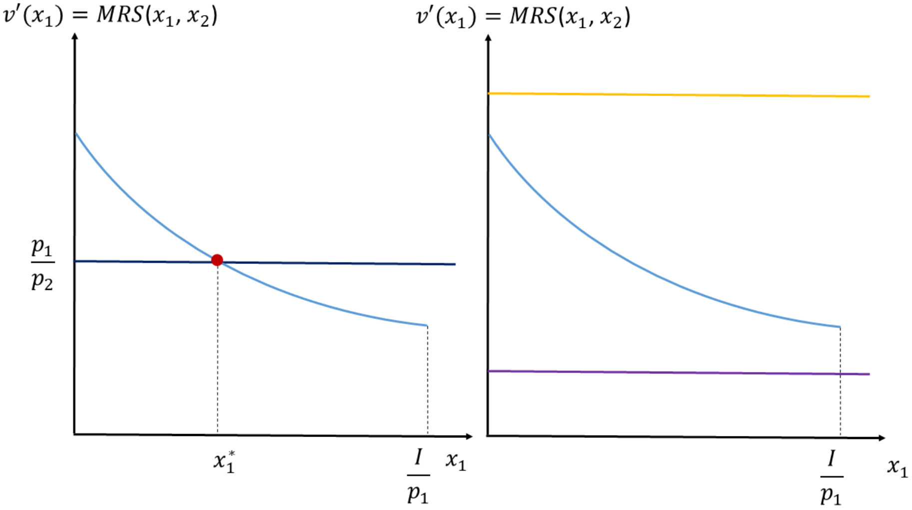
$$v'\left(\frac{I}{p_1}\right) > \frac{p_1}{p_2},$$

    in which case $x_1^* = \frac{I}{p_1}$ and $x_2^* = 0$.
  - Or

$$v'\left(0\right) < \frac{p_1}{p_2},$$

    in which case $x_1^* = 0$ and $x_2^* = \frac{I}{p_2}$.

Example: Quasi-Linear



Example: Quasi-Linear

- In sum,

$$x_1^* = \begin{cases} 0 & \text{if } \frac{p_1}{p_2} > v'(0) \\ (v')^{-1}\left(\frac{p_1}{p_2}\right) & \text{if } \frac{p_1}{p_2} \in \left[v'\left(\frac{I}{p_1}\right), v'(0)\right] \\ \frac{I}{p_1} & \text{if } \frac{p_1}{p_2} < v'\left(\frac{I}{p_1}\right) \end{cases}.$$

  and

$$x_2^* = I - p_1 x_1^*.$$

- Note that the consumption of goods 1 is independent of income I, except when the consumer is spending all her income on it.
- We say there is no income effect for goods 1 (provided that the optimal consumption is interior).

Example: Quasi-Linear

- Quasi-linear utility is often used to capture the notion of willingness to pay.
- Treat goods 2 as money, so $p_2 = 1$. Also we can without loss set $v(0) = 0$.
- What is the consumer's willingness to pay for the 1st unit of goods 1?

$$v(0) + I = v(1) + I - WTP_1 \Leftrightarrow WTP_1 = v(1).$$

- What is her willingness to pay for the 2nd unit?

$$\begin{aligned} & v(1) + I - WTP_1 = v(2) + I - WTP_1 - WTP_2 \\ \Leftrightarrow \quad & WTP_2 = v(2) - v(1). \end{aligned}$$

- $v'(x)$ is the consumer's willingness to pay for the x-th unit (at the margin).

Example: Perfect Substitutes

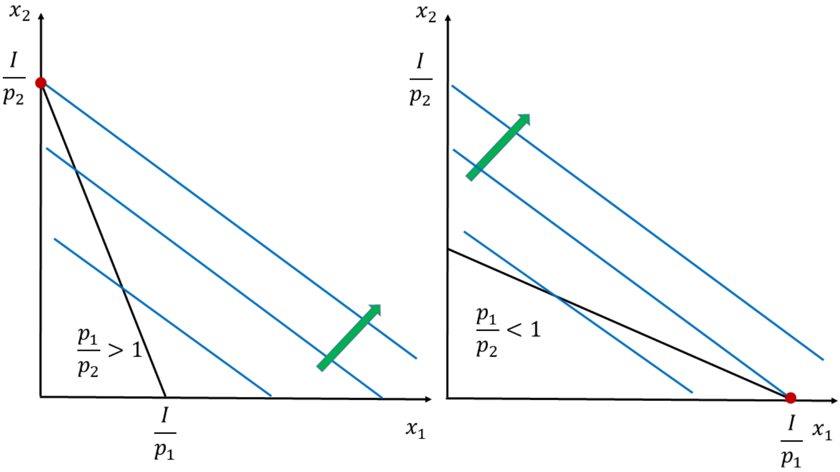
- Suppose the utility function is linear.

$$u(x_1, x_2) = x_1 + x_2.$$

  The two goods are **perfect substitutes**.

- If $p_1 > p_2$, then the consumer finds it optimal to spend all the income on goods 2, so $(x_1^*, x_2^*) = (0, I/p_2)$.
- If $p_1 < p_2$, then the consumer finds it optimal to spend all the income on goods 1, so $(x_1^*, x_2^*) = (I/p_1, 0)$.
- If $p_1 = p_2$, then any way of spending up the income is optimal.

Example: Perfect Substitutes



Example: Perfect Complements

- Suppose the utility function is

$$u(x_1, x_2) = \min\{x_1, x_2\}.$$

  The two goods are **perfect complements**.
- If $x_1 > x_2$, the consumer can improve her utility by buying less of goods 1 and more of goods 2.
- If $x_1 < x_2$, the consumer can improve his utility by buying less of goods 2 and more of goods 1.
- At the optimum, we must have $x_1 = x_2$.
- Substitute this into the budget line gives:
  $p_1 x_1 + p_2 x_1 = I \Rightarrow x_1^* = x_2^* = I/(p_1 + p_2)$.

Example: Perfect Complements