ECON 3123: Macroeconomic Theory 1

Problem Set #4

Due Date: May 19, 2019

1. The Solow growth model with a Cobb-Douglas production function

In this question, you will study the Solow growth model with a Cobb-Douglas production function. Consider the following model of economic growth:

$$K_{t+1} = (1-\delta)K_t + I_t \tag{1}$$

$$Y_t = \mathcal{A}_t K_t^{\alpha} N_t^{1-\alpha} \tag{2}$$

$$S_t = sY_t \tag{3}$$

$$I_t = S_t \tag{4}$$

$$\mathcal{A}_t = \mathcal{A} \quad \text{for all } t \tag{5}$$

$$N_t = N \quad \text{for all } t \tag{6}$$

(a) Is the production function (2) characterized by constant returns to scale? Explain.

Yes. Note that $\mathcal{A}_t(xK_t)^{\alpha}(xN_t)^{1-\alpha} = (x^{\alpha}x^{1-\alpha})\mathcal{A}_t K_t^{\alpha} N_t^{1-\alpha} = x\mathcal{A}_t K_t^{\alpha} N_t^{1-\alpha}$ for any x.

(b) Transform the production function (2) into a relation between output per worker, y_t, and capital per worker, k_t.

Due to Equations (5) and (6), $Y_t = \mathcal{A}K_t^{\alpha}N^{1-\alpha}$. By divide this expression by N, we have

$$y_t = \frac{Y_t}{N} = \mathcal{A}K_t^{\alpha}\frac{N^{1-\alpha}}{N} = \mathcal{A}K_t^{\alpha}N^{-\alpha} = \mathcal{A}\left(\frac{K_t}{N}\right)^{\alpha} = \mathcal{A}k_t^{\alpha}.$$

(c) Show that $k_{t+1} = (1-\delta)k_t + s\mathcal{A}k_t^{\alpha}$.

Dividing Equation (1) by N yields the following: $k_{t+1} = (1-\delta)k_t + \frac{I_t}{N}$. Note that $I_t = S_t = sY_t$. Therefore, $\frac{I_t}{N} = sy_t = s\mathcal{A}k_t^{\alpha}$ given the result in (b).

(d) Show that the steady-state level of capital per worker, k^*, is given by $\left(\frac{s\mathcal{A}}{\delta}\right)^{1/(1-\alpha)}$.

The result in (c) implies that k^* is characterized by $k^* = (1-\delta)k^* + s\mathcal{A}(k^*)^\alpha$. This reduces to the condition that $\delta k^* = s\mathcal{A}(k^*)^\alpha$. It is equivalent to $(k^*)^{1-\alpha} = \frac{s\mathcal{A}}{\delta}$, which leads to the desired result.

(e) Give an expression for output per worker in the steady state, y^*, and consumption per worker in the steady state, c^*. For c^*, assume that $T = 0$.

$$y^* = \mathcal{A}(k^*)^\alpha = \mathcal{A}\left(\frac{s\mathcal{A}}{\delta}\right)^{\alpha/(1-\alpha)}.$$
$$c^* = (1-s)y^* = (1-s)\mathcal{A}\left(\frac{s\mathcal{A}}{\delta}\right)^{\alpha/(1-\alpha)}.$$

(f) Explain what happens to the steady state output, Y^*, and output per worker, y^*, when N doubles.

Output per worker, y^*, does not depend on N as illustrated above. Thus, y^* does not change. Output, Y^*, is equal to y^*N. As N doubles, Y^* also doubles.

2. Numerical analysis of the Solow model

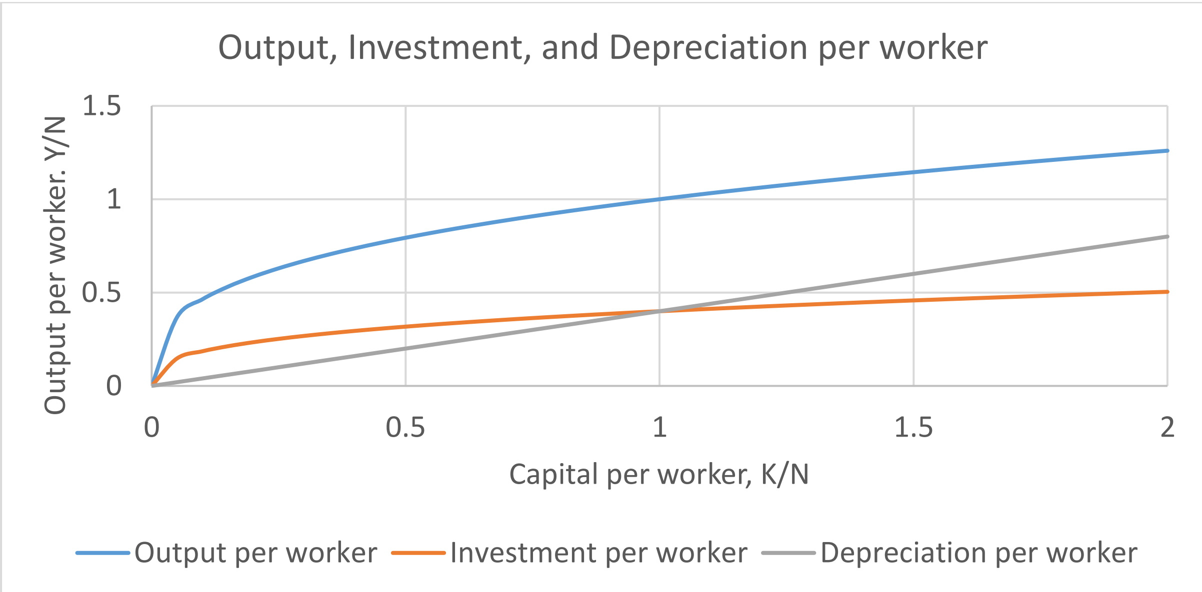
Consider the model in question 1. Here, we will draw some figures using Excel or your favorite software to understand the working of the Solow model.

(a) Let $s = 0.4$, $\delta = 0.4$, $\mathcal{A} = 1, \alpha = 1/3$, and $N = 1$. Using the results in 1(d), compute the value of k^* and y^*.

$$k^* = y^* = 1.$$

(b) Draw a graph similar to Figure 11-2 (i.e., plot y_t, sy_t, and δk_t against k_t). You may want to begin with constructing a grid for the value of k_t between 0 and 2 spaced by 0.05 (HINT: look at my spreadsheet below).

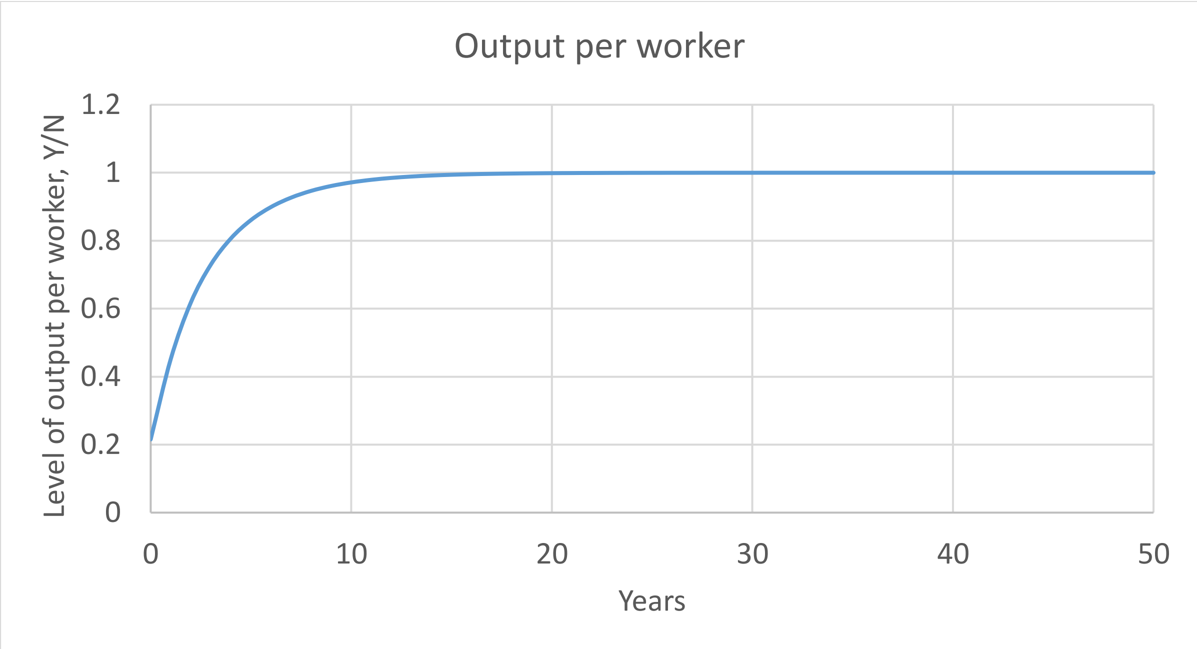
	A	B	C	D	E	F	G
1	capital per worker	output per worker	investment per work	depreciation per wor		Parameters	
2	0	=$G$4*A2^$G$5	=$G$2*B2	=$G$3*A2		s	0.4
3	=A2+0.05	=$G$4*A3^$G$5	=$G$2*B3	=$G$3*A3		delta	0.4
4	=A3+0.05	=$G$4*A4^$G$5	=$G$2*B4	=$G$3*A4		A	=1
5	=A4+0.05	=$G$4*A5^$G$5	=$G$2*B5	=$G$3*A5		alpha	=1/3
6	=A5+0.05	=$G$4*A6^$G$5	=$G$2*B6	=$G$3*A6		N	=1

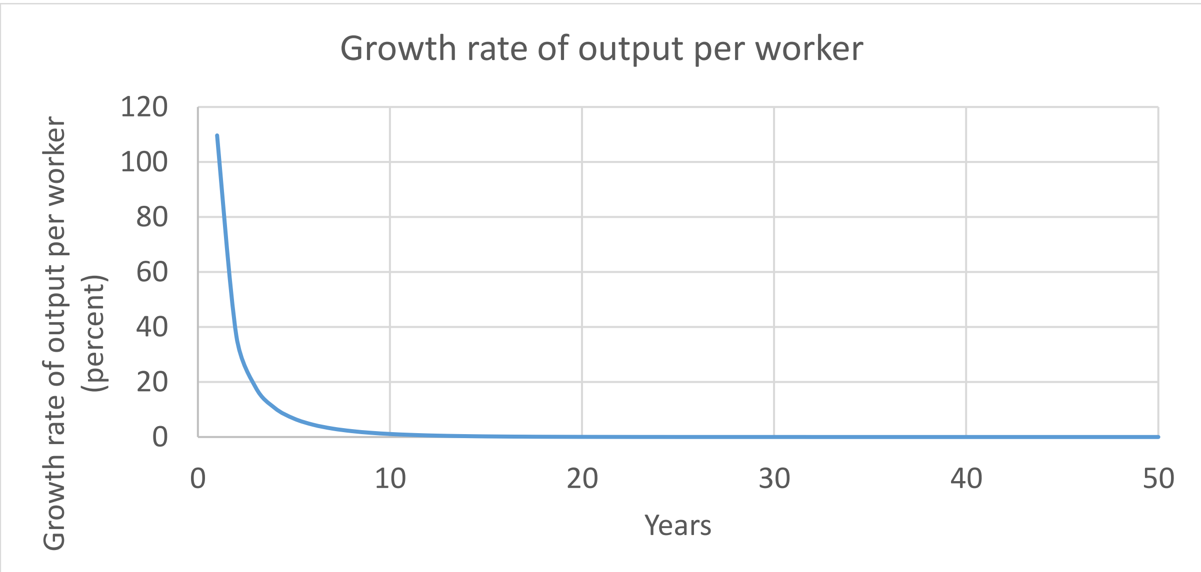


(c) Suppose that this economy in year 0 is very poor. Specifically, $k_0 = 0.01$. Show how this economy grows between year 0 and year 50 using figures similar to Figure 11-7. You need to plot y_t and the growth rate of y_t against t (HINT: look at my spreadsheet below).

	A	B	C	D	E	F	G
1	Years	Capital per worker	Output per worker	Growth rate of outpu		Parameters	
2	0	0.01	=\$G\$4*B2^\$G\$5			s	0.4
3	=A2+1	=(1-\$G\$3)*B2+\$G\$2*C2	=\$G\$4*B3^\$G\$5	=(C3-C2)/C2*100		delta	0.4
4	=A3+1	=(1-\$G\$3)*B3+\$G\$2*C3	=\$G\$4*B4^\$G\$5	=(C4-C3)/C3*100		A	=1
5	=A4+1	=(1-\$G\$3)*B4+\$G\$2*C4	=\$G\$4*B5^\$G\$5	=(C5-C4)/C4*100		alpha	=1/3
6	=A5+1	=(1-\$G\$3)*B5+\$G\$2*C5	=\$G\$4*B6^\$G\$5	=(C6-C5)/C5*100		N	=1
7	=A6+1	=(1-\$G\$3)*B6+\$G\$2*C6	=\$G\$4*B7^\$G\$5	=(C7-C6)/C6*100			
8	=A7+1	=(1-\$G\$3)*B7+\$G\$2*C7	=\$G\$4*B8^\$G\$5	=(C8-C7)/C7*100			

The economy grows as capital accumulates.





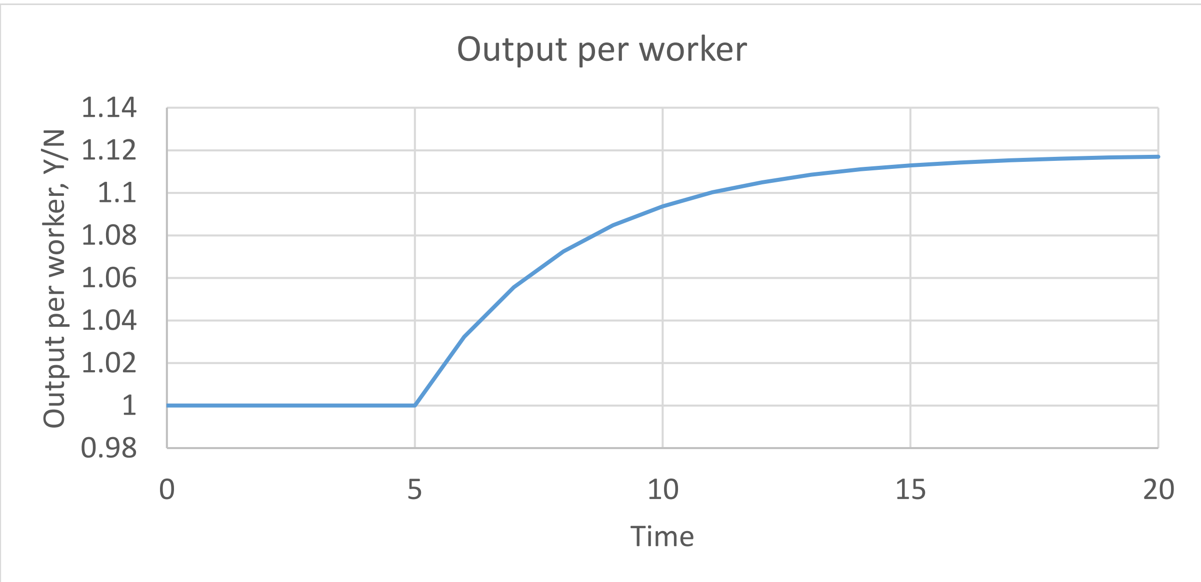
(d) How long does it take for this economy to grow half-way close to its steady state? That is, after how many years does k_t becomes greater than $k^*/2$?

$k_3 = 0.39 < \frac{k^*}{2} = 0.5 < k_4 = 0.53$. Thus, it takes only four years for this economy grow half-way close to the steady state.

(e) Suppose that the economy stays in a steady state associated with $s = 0.4$ for $t \leq 5$. Then, from year 6, the saving rate increases from 0.4 to 0.5. Draw a graph showing how y_t evolves between year 0 and year 20, which should be similar to Figure 11-4 (HINT: look at my spreadsheet below).

	A	B	C	D	E	F
1	Years	Capital per worker	Output per worker		Parameters	
2	0	=1	=$F$4*B2^$F$5		s0	0.4
3	=A2+1	=(1-$F$3)*B2+$F$2*C2	=$F$4*B3^$F$5		delta	0.4
4	=A3+1	=(1-$F$3)*B3+$F$2*C3	=$F$4*B4^$F$5		A	=1
5	=A4+1	=(1-$F$3)*B4+$F$2*C4	=$F$4*B5^$F$5		alpha	=1/3
6	=A5+1	=(1-$F$3)*B5+$F$2*C5	=$F$4*B6^$F$5		N	=1
7	=A6+1	=(1-$F$3)*B6+$F$2*C6	=$F$4*B7^$F$5			
8	=A7+1	=(1-$F$3)*B7+$F$8*C7	=$F$4*B8^$F$5		s1	0.5
9	=A8+1	=(1-$F$3)*B8+$F$8*C8	=$F$4*B9^$F$5			
10	=A9+1	=(1-$F$3)*B9+$F$8*C9	=$F$4*B10^$F$5			

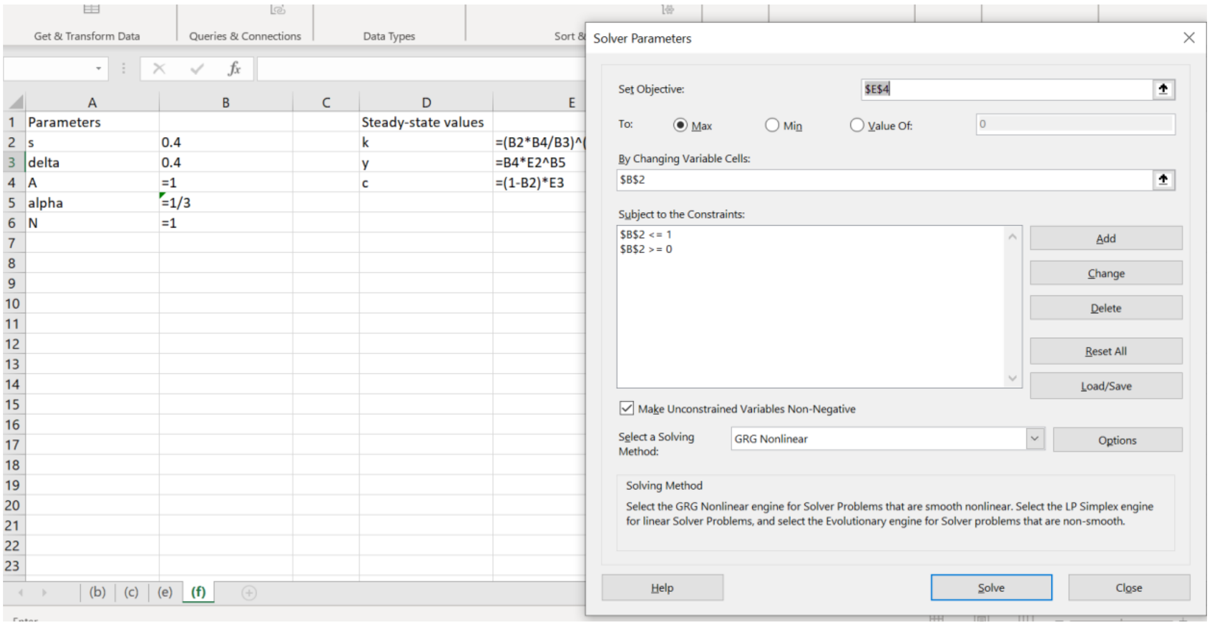
As the saving rate increases, this economy invests more and accumulates more capital. As a result, output per worker increases over time to the new, higher steady state level.



(f) Let’s find the golden rule saving rate, which maximizes the steady-state consumption per worker in this economy. Rather than analytically maximizing c^* with respect to $s \in [0,1]$, we will use a numerical optimization toolbox in Excel. We first need to add “Solver Add-in.” Click File, Options, and Add-ins. Select Excel Add-ins, click Go, select “Solver Add-in,” and click Ok. Then, write a spreadsheet which calculates the value of c^* as a function of s (HINT: look at my spreadsheet below).

	A	B	C	D	E
1	Parameters			Steady-state values	
2	s	0.4		k	=(B2*B4/B3)^(1/(1-B5))
3	delta	0.4		y	=B4*E2^B5
4	A	=1		c	=(1-B2)*E3
5	alpha	=1/3			
6	N	=1			

Then, open “Solver” under Data tab. Note that we want to maximize c^* by changing s subject to the conditions that $s \geq 0$ and $s \leq 1$. What is the golden rule saving rate s_G (HINT: look at my spreadsheet below)?



$$s_G = 0.33.$$