FELLER EVOLUTION FAMILIES AND PARABOLIC EQUATIONS WITH FORM-BOUNDED VECTOR FIELDS

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ABSTRACT. We show that the weak solutions of parabolic equation $\partial_t u - \Delta u + b(t,x) \cdot \nabla u = 0$, $(t,x) \in (0,\infty) \times \mathbb{R}^d$, $d \geqslant 3$, for $b(t,x)$ in a wide class of time-dependent vector fields capturing critical order singularities, constitute a Feller evolution family and, thus, determine a Feller process. Our proof uses an a priori estimate on the L^p-norm of the gradient of solution in terms of the L^q-norm of the gradient of initial function, and an iterative procedure that moves the problem of convergence in L^∞ to L^p.

1. Introduction and results

1.1. Consider Cauchy problem

$$(\partial_t - \Delta + b(t,x) \cdot \nabla) u = 0, \qquad (t,x) \in (0,\infty) \times \mathbb{R}^d, \tag{1}$$

$$u(+0,x) = f(x), \tag{2}$$

where $d \geqslant 3$, $b \in L^1_{\rm loc}([0,\infty) \times \mathbb{R}^d, \mathbb{R}^d)$, $f \in L^2_{\rm loc}(\mathbb{R}^d)$.

We prove that for b in a wide class of time-dependent vector fields capturing critical order singularities the unique weak solution of (1), (2) for the initial function f in space $C_\infty(\mathbb{R}^d) := \{f \in C(\mathbb{R}^d) : \lim_{x \to \infty} f(x) = 0\}$ (endowed with sup-norm $\|\cdot\|_\infty$) is given by a Feller evolution family, i.e. a family of bounded linear operators $(U(t,s))_{0 \leqslant s \leqslant t < \infty} \subset \mathcal{L}\big(C_\infty(\mathbb{R}^d)\big)$ such that:

(**E1**) $U(s,s) = \mathrm{Id}$, $U(t,s) = U(t,r)U(r,s)$ for all $0 \leqslant s \leqslant r \leqslant t$,

(**E2**) mapping $(t,s) \mapsto U(t,s)$ is strongly continuous in $C_\infty(\mathbb{R}^d)$,

(**E3**) operators $U(t,s)$ are positivity-preserving and L^∞-contractive:

$$U(t,s)f \geqslant 0 \quad \text{if} \quad f \geqslant 0, \quad \text{and} \quad \|U(t,s)f\|_\infty \leqslant \|f\|_\infty, \quad 0 \leqslant s \leqslant t,$$

(**E4**) function $u(t) := U(t,s)f$ $(t > s)$ is a weak solution of equation (1).

It is well known that the operators $(U(t,s))_{0 \leqslant s \leqslant t < \infty}$ determine the (sub-Markov) transition probability function of a Feller process X_t (in particular, a Hunt process), see e.g. [1, Theorem 2.22]. X_t is related to the differential operator in (1) via (**E4**). The problem of constructing a Brownian motion perturbed by a locally unbounded drift b has been thoroughly studied in the literature, motivated by applications as well as by the search for the maximal general class of drifts b such that the associated diffusion exists (see [5] and references therein).

In the present paper, we consider the following class of drifts:

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DEFINITION 1. The parabolic class of form-bounded vector fields $\mathbf{F}_{\beta,\mathcal{P}} = \mathbf{F}_{\beta,\mathcal{P}}(-\Delta)$ consists of vector fields $b \in L^2_{\rm loc}\big([0,\infty) \times \mathbb{R}^d, \mathbb{R}^d\big)$ such that

$$\int_0^\infty \|b(t,\cdot)\varphi(t,\cdot)\|_2^2 dt \leqslant \beta \int_0^\infty \|\nabla\varphi(t,\cdot)\|_2^2 dt + \int_0^\infty g(t)\|\varphi(t,\cdot)\|_2^2 dt \qquad (\mathbf{BC})$$

for some $\beta < \infty$ and $g = g_\beta \in L^1_{\rm loc}([0,\infty))$, $g \geqslant 0$, for all $\varphi \in C_c^\infty([0,\infty) \times \mathbb{R}^d)$.

$\|\cdot\|_2$ is the norm in $L^2(\mathbb{R}^d)$.

It is clear that $b \in \mathbf{F}_{\beta,\mathcal{P}} \Leftrightarrow cb \in \mathbf{F}_{c^2\beta,\mathcal{P}}$, $c \neq 0$.

Example 1. 1. If $b : \mathbb{R}^d \to \mathbb{R}^d$, $b = b_1 + b_2$, $|b_1| \in L^{d,\infty}(\mathbb{R}^d)$ (weak L^d space), $|b_2| \in L^\infty(\mathbb{R}^d)$, then $b \in \mathbf{F}_{\beta,\mathcal{P}}$ with

$$\sqrt{\beta} = \|b_1\|_{d,\infty}\Omega_d^{-\frac{1}{d}}\frac{2}{d-2}, \qquad \Omega_d := \pi^{\frac{d}{2}}\Gamma\left(\frac{d}{2}+1\right)$$

(using Strichartz inequality with sharp constants [3, Prop 2.5, 2.6, Cor. 2.9]). In particular, $b(x) = x|x|^{-2}$ belongs to $\mathbf{F}_{\beta,\mathcal{P}}$ with $\beta = (2/(d-2))^2$ (and $g \equiv 0$) (Hardy inequality). More generally, any vector field $b(t,x)$ such that for some c_1, $c_2 > 0$

$$|b(t,x)|^2 \leqslant c_1|x-x_0|^{-2} + c_2|t-t_0|^{-1}\big(\log(e+|t-t_0|^{-1})\big)^{-1-\varepsilon}, \quad \varepsilon > 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}^d,$$

belongs to the class $\mathbf{F}_{\beta,\mathcal{P}}$ with $\beta = c_1\,(2/(d-2))^2$. The above examples show that the Gaussian bounds on the fundamental solution of $\partial_t - \Delta + b(t,x)\cdot\nabla$, $b \in \mathbf{F}_{\beta,\mathcal{P}}$, are, in general, not valid.

2. If $h \in L^2(\mathbb{R})$, $T : \mathbb{R}^d \to \mathbb{R}$ is a linear map, then the vector field $b(x) = h(Tx)a$, where $a \in \mathbb{R}^d$, is in $\mathbf{F}_{\beta,\mathcal{P}}$ with appropriate β, but $|b|$ may not be in $L^{d,\infty}_{\rm loc}(\mathbb{R}^d)$.

3. Let $b : \mathbb{R}^d \to \mathbb{R}^d$. If b^2 is in the Campanato-Morrey class

$$M_p := \left\{ v \in L^p : \|v\|_{M_p} := \sup_{x \in \mathbb{R}^d, r>0} r^{2-\frac{d}{p}}\|\mathbf{1}_{B(x,r)}v\|_p < \infty \right\}$$

for some $p > 1$, then $b \in \mathbf{F}_{\beta,\mathcal{P}}$ with $\beta = \beta(\|b^2\|_{M_p})$. Here $\mathbf{1}_{B(x,r)}$ is the characteristic function of the open ball of radius r centered at x.

4. Set $L^qL^p := L^q\big([0,\infty), L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)\big)$. We have:

$$|b| \in L^qL^p \text{ with } \frac{d}{p} + \frac{2}{q} \leqslant 1 \qquad \Rightarrow \qquad b \in \mathbf{F}_{0,\mathcal{P}} := \bigcap_{\beta>0} \mathbf{F}_{\beta,\mathcal{P}}$$

(using the Hölder inequality and the Sobolev embedding theorem).

The class $\mathbf{F}_{\beta,\mathcal{P}}$ contains vector fields having critical order singularities: replacing a $b \in \mathbf{F}_{\beta,\mathcal{P}}$ in (1) with cb, $c > 1$, in general destroys e.g. the uniqueness of weak solution of Cauchy problem (1), (2) (see [4, Example 5]). The class $\mathbf{F}_{0,\mathcal{P}}$ doesn't contain vector fields having critical order singularities.

The explicit dependence on the value of the relative bound β is a crucial feature of our results.

We consider only real Banach spaces. Throughout this paper we use the following notation:

$$\langle g \rangle = \langle g(\cdot) \rangle := \int_{\mathbb{R}^d} g(x)dx.$$

Let $\langle g,h \rangle$ denote the $(L^p, L^{p'})$ pairing, so that

$$\langle g\,,\,h \rangle := \int_{\mathbb{R}^d} g(x)h(x)dx \qquad \big(g \in L^p(\mathbb{R}^d), h \in L^{p'}(\mathbb{R}^d)\big).$$

Before formulating the main result, let us remind the reader the definition of a weak solution to Cauchy problem (1), (2).

DEFINITION 2. A real-valued function $u \in L^\infty_{\rm loc}((0,\infty), L^2_{\rm loc}(\mathbb{R}^d))$ is said to be a weak solution of equation (1) if ∇u (understood in the sense of distributions) is in $L^1_{\rm loc}((0,\infty)\times\mathbb{R}^d,\mathbb{R}^d)$, $b\cdot\nabla u \in L^1_{\rm loc}((0,\infty)\times\mathbb{R}^d)$, and

$$\int_0^\infty \langle u, \partial_t \psi\rangle dt - \int_0^\infty \langle u, \Delta\psi\rangle dt + \int_0^\infty \langle b\cdot\nabla u, \psi\rangle dt = 0 \tag{3}$$

for all $\psi \in C_c^\infty((0,\infty)\times\mathbb{R})$.

DEFINITION 3. A weak solution of (1) is said to be a weak solution to Cauchy problem (1), (2) if $\lim_{t\to+0}\langle u(t),\xi\rangle = \langle f,\xi\rangle$ for all $\xi \in L^2(\mathbb{R}^d)$ having compact support.

Theorem 1 (Main result)**.** *Let $d \geqslant 3$. Suppose a vector field $b(\cdot,\cdot)$ belongs to the class $\mathbf{F}_{\beta,\mathcal{P}}$. If $\beta < d^{-2}$, then there exists a Feller evolution family $(U(t,s))_{0\leqslant s\leqslant t} \subset \mathcal{L}\big(C_\infty(\mathbb{R}^d)\big)$ that produces the weak solution to Cauchy problem* (1), (2)*, i.e.* (**E1**)–(**E4**) *hold true.*

Theorem 1 in the stationary case $b:\mathbb{R}^d\to\mathbb{R}^d$ and under the extra assumption $|b| \in L^2(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ is due to [4]. The extra assumption is used there in the verification that the constructed limit of approximating semigroups is strongly continuous in $C_\infty(\mathbb{R}^d)$ (i.e. in the verification of the assumptions of the Trotter approximation theorem in $C_\infty(\mathbb{R}^d)$). We run their iterative procedure differently, so that it automatically yields strong continuity. (Generally speaking, unless b is sufficiently regular in t, the non-stationary case presents the next level of difficulty compared to the stationary case. It is the inherent flexibility of the method of [4] (which, we believe, goes beyond $\partial_t - \Delta + b(t,x)\cdot\nabla$) that allows us to carry out the construction of the process for a non-stationary $b(\cdot,\cdot) \in \mathbf{F}_{\beta,\mathcal{P}}$.)

Let us also note that, in the assumptions of Theorem 1, given $p > (1-\sqrt{\beta/4})^{-1}$, the formula

$$U_p(t,s) := \Big(U(t,s)|_{L^p(\mathbb{R}^d)\cap C_\infty(\mathbb{R}^d)}\Big)^{\rm clos}_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)},$$

determines a (strongly continuous) evolution family in $\mathcal{L}\big(L^p(\mathbb{R}^d)\big)$, cf. [6]. The proof is obtained from Theorem 1, estimate (8) below and the Dominated Convergence Theorem.

We now briefly comment on the relationship between this work and the existing results.

1. First, for $|b| \in L^qL^p$ (cf. Example 1.3), $\frac{d}{p}+\frac{2}{q}<1$, the associated diffusion has been constructed in [5] as the strong solution of the SDE $dX_t = b(t,X_t)dt + \frac{1}{2}dW_t$, $X_0 = x_0 \in \mathbb{R}^d$.

2. Recall the definition of the parabolic Kato class $\mathbf{K}^{d+1}_{\beta,\mathcal{P}}$:

$$\mathbf{K}^{d+1}_{\beta,\mathcal{P}} := \left\{ b \in L^1_{\rm loc}([0,\infty)\times\mathbb{R}^d,\mathbb{R}^d) : \inf_{r>0} k^{1,1}(b,r) \leqslant \beta,\ \inf_{r>0} k^\infty(b,r) \leqslant \beta \right\},$$

where

$$k^{1,1}(b,r) := \sup_{u\geqslant 0,\, x\in\mathbb{R}^d} \int_u^{u+r}\int_{\mathbb{R}^d} \Gamma_{t-u}(x-y)\frac{|b(t,y)|}{\sqrt{t-u}}dydt,$$

$$k^\infty(b,r) := \sup_{u\geqslant r,\, x\in\mathbb{R}^d} \int_u^{u+r}\int_{\mathbb{R}^d} \Gamma_{u+r-t}(x-y)\frac{|b(t-r,y)|}{\sqrt{u+r-t}}dydt,$$

and $\Gamma_t(z) := (4\pi t)^{-\frac{d}{2}} e^{-\frac{|z|^2}{4t}}$. If $b \in \mathbf{K}^{d+1}_{\beta,\mathcal{P}}$ with $\beta > 0$ sufficiently small, then the fundamental solution of (1) admits local in time Gaussian upper and lower bounds, see [7], which, in turn, yield the corresponding Feller evolution family (in $C_b(\mathbb{R}^d) := \{f \in C(\mathbb{R}^d) : \sup_x |f(x)| < \infty\}$ endowed with the sup-norm). Note that $\mathbf{K}^{d+1}_{0,\mathcal{P}} - \mathbf{F}_{\beta,\mathcal{P}} \neq \varnothing$, where $\mathbf{K}^{d+1}_{0,\mathcal{P}} := \cap_{\beta>0}\mathbf{K}^{d+1}_{\beta,\mathcal{P}}$ (on the other hand, $L^d(\mathbb{R}^d,\mathbb{R}^d) - \mathbf{K}^{d+1}_{\beta,\mathcal{P}} \cap \{\mathsf{f} : \mathbb{R}^d \to \mathbb{R}^d\} \neq \varnothing$).

3. In the stationary case $b : \mathbb{R}^d \to \mathbb{R}^d$, it has been shown in [2] that the associated Feller process exists for vector fields b in the class

$$\mathbf{F}^{\frac{1}{2}}_\beta := \left\{ b \in L^1_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R}^d) : \big\||b|^{\frac{1}{2}}(\lambda-\Delta)^{-\frac{1}{4}}\big\|^2_{L_2\to L_2} \leqslant \sqrt{\beta} \ \text{ for some } \lambda = \lambda_\beta > 0 \right\}.$$

In particular, the class $\mathbf{F}^{\frac{1}{2}}_\beta$ contains vector fields of the form $b := b_1 + b_2$, where $b_1 \in \mathbf{F}_\beta := \mathbf{F}_{\beta,\mathcal{P}} \cap \{\mathsf{f} : \mathbb{R}^d \to \mathbb{R}^d\}$, $b_2 \in \mathbf{K}^{d+1}_\beta := \mathbf{K}^{d+1}_{\beta,\mathcal{P}} \cap \{\mathsf{f} : \mathbb{R}^d \to \mathbb{R}^d\}$.

Remark 1. We leave out the L^p-theory of $\partial_t - \Delta + b(t,x)\cdot\nabla$ with $b \in \mathbf{F}_{\beta,\mathcal{P}}$, $1 < \beta < 4$, or with b in a parabolic analogue of the class $\mathbf{F}^{\frac{1}{2}}_\beta$.

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2. Proof of Theorem 1

We will need a regular approximation of b: vector fields $\{b_m\}_{m=1}^\infty \subset C_c^\infty([0,\infty)\times\mathbb{R}^d,\mathbb{R}^d)$ that satisfy $b_m \to b$ in $L^2_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^d,\mathbb{R}^d)$, and

$$\int_0^\infty \|b_m(t,\cdot)\varphi(t,\cdot)\|_2^2 dt \leqslant \left(\beta + \frac{1}{m}\right)\int_0^\infty \|\nabla\varphi(t,\cdot)\|_2^2 dt + \int_0^\infty g(t)\|\varphi(t,\cdot)\|_2^2 dt \qquad (\mathbf{BC}_m)$$

for all $\varphi \in C_c^\infty([0,\infty)\times\mathbb{R}^d)$. (Such b_m's can be constructed by the formula $b_m := \eta_m * \mathbf{1}_m b$, where $\mathbf{1}_m$ is the characteristic function of set $\{(t,x) \in \mathbb{R}\times\mathbb{R}^d : |b(t,x)| \leqslant m, |x| \leqslant m, 0 \leqslant |t| \leqslant m\}$, $*$ is the convolution on $\mathbb{R}\times\mathbb{R}^d$, and $\{\eta_m\} \subset C_c^\infty(\mathbb{R}\times\mathbb{R}^d)$ is an appropriate family of mollifiers.)

Due to the strict inequality $\beta < d^{-2}$, we may assume without loss of generality that b_m's satisfy $(\mathbf{BC}_m)$ with β in place of $\beta + \frac{1}{m}$.

The construction of the Feller evolution family goes as follows. Fix some $T > 0$. Denote

$$D_T := \{(s,t) \in \mathbb{R}^2 : 0 \leqslant s \leqslant t \leqslant T\}.$$

Let $(U_m(t,s))_{0\leqslant s\leqslant t} \subset \mathcal{L}(C_\infty(\mathbb{R}^d))$ be the Feller evolution family for the equation

$$(\partial_t - \Delta + b_m(t,x)\cdot\nabla)u = 0. \tag{4}$$

Given a $f \in C_c^\infty(\mathbb{R}^d)$, we define

$$Uf := \lim_{m\to\infty} U_m f \quad \text{in} \quad L^\infty\big(D_T, C_\infty(\mathbb{R}^d)\big) \tag{5}$$

Assuming that the convergence in (5) has been established, we note that U_m is L^∞-contractive and $C_c^\infty(\mathbb{R}^d)$ is dense in $C_\infty(\mathbb{R}^d)$, so $U = (U(t,s))_{0\leqslant s\leqslant t}$ extends to a strongly continuous family of bounded linear operators in $\mathcal{L}\big(C_\infty(\mathbb{R}^d)\big)$, which we denote again by $(U(t,s))_{0\leqslant s\leqslant t}$.

Proposition 1. *In the assumptions of Theorem 1 $(U(t,s))_{0\leqslant s\leqslant t}$ defined by* (5) *satisfies* (**E1**)-(**E4**).

The main difficulty is in establishing the convergence in (5). The proof of the convergence uses a parabolic variant of the iterative procedure of [4].

2.1. **Proof of the convergence in (5): a parabolic variant of the iterative procedure of Kovalenko-Semenov.** Fix $f \in C_c^\infty(\mathbb{R}^d)$. Set

$$u_m(t) = U_m(t,s)f, \quad t \geqslant s.$$

Lemma 1 (a priori estimate)**.** *Let $d \geqslant 3$. Suppose b is in $\mathbf{F}_{\beta,\mathcal{P}}$ with $\beta < d^{-2}$, $q \in \left(d, \beta^{-\frac{1}{2}}\right)$. Then*

$$\|\nabla u_m\|_{L^\infty([s,\tau],L^q(\mathbb{R}^d))} + C_1\|\nabla u_m\|_{L^q([s,\tau],L^{\frac{qd}{d-2}}(\mathbb{R}^d))} \leqslant C\|\nabla f\|_q, \quad s \leqslant \tau \leqslant T,$$

where constants $C_1 = C_1(q,\beta) > 0$, $C = C(q,T) < \infty$, do not depend on m or (s,τ).

Remark 2. The a priori estimate of Lemma 1 is one of the main results of the paper. It is the basis for the approach as a whole (for the corresponding result in the elliptic case see [4, Lemma 5]).

We subtract the approximating equations (4) for b_m, b_n, and integrate to obtain:

Lemma 2. *Suppose $b \in \mathbf{F}_{\beta,\mathcal{P}}$ with $\beta < 4$. Let $0 < \alpha < 1$. There exist $h > 0$, $k = k(\beta) > 1$ and a m_0 such that for all $m, n \geqslant m_0$, for all $p \geqslant p_0 > \frac{2}{2-\sqrt{\beta}}$ we have*

$$\|u_m - u_n\|_{L^{\frac{p}{1-\alpha}}([s,s+h],L^{\frac{pd}{d-2+2\alpha}}(\mathbb{R}^d))} \leqslant \left(C_0\beta\|\nabla u_m\|^2_{L^{2\lambda'}([s,s+h],L^{2\sigma'}(\mathbb{R}^d))}\right)^{\frac{1}{p}} (p^{2k})^{\frac{1}{p}} \|u_m - u_n\|^{1-\frac{2}{p}}_{L^{(p-2)\lambda}([s,s+h],L^{(p-2)\sigma}(\mathbb{R}^d))}, \quad (6)$$

for any σ such that $1 < \sigma < \frac{d}{d-2+2\alpha}$, $\frac{1}{\sigma} + \frac{1}{\sigma'} = 1$, and $\frac{1/(1-\alpha)}{\lambda} = \frac{d/(d-2+2\alpha)}{\sigma}$, $\frac{1}{\lambda} + \frac{1}{\lambda'} = 1$, for a constant $C_0 = C_0(h) < \infty$ that doesn't depend on m or $s \leqslant T$.

The a priori estimate of Lemma 1 allows to iterate the inequality (6) (with a proper choice of α, λ and σ) in order to obtain an L^∞-norm in the left-hand side, and an L^p-norm ($p < \infty$) (of $u_m - u_n$) in the right-hand side. Set

$$D_{T,h} := D_T \cap \{(s,t) : 0 \leqslant t - s \leqslant h\}, \quad h < T.$$

Lemma 3. *In the assumptions of Theorem 1, for any $p_0 > \frac{2}{2-\sqrt{\beta}}$ there exist $h > 0$, constants $B < \infty$ and $\gamma := \left(1 - \frac{\sigma d}{d+2}\right)\left(1 - \frac{\sigma d}{d+2} + \frac{2\sigma}{p_0}\right)^{-1} > 0$ ($1 < \sigma < \frac{d+2}{d}$) independent of m, n such that*

$$\|U_m f - U_n f\|_{L^\infty(D_{T,h}\times\mathbb{R}^d)} \leqslant B \sup_{0\leqslant s\leqslant T-h} \|U_m f - U_n f\|^\gamma_{L^{p_0}([s,s+h],L^{p_0}(\mathbb{R}^d))} \quad \text{for all } n, m. \quad (7)$$

Remark 3. Lemma 3 is the key result. It moves the problem of convergence of $\{U_m f\}$ in L^∞ to a space having much weaker topology (locally).

That $\{U_m f\}$ does indeed converge in the weaker topology of the right-hand side of (7) will follow from the following

Lemma 4. *Suppose $b \in \mathbf{F}_{\beta,\mathcal{P}}$ with $\beta < 1$. The sequence $\{U_m f\}$ from Lemma 3 is fundamental in $L^\infty(D_T, L^r(\mathbb{R}^d))$, $2 \leqslant r < \infty$.*

Let us prove the convergence in (5). Fix $f \in C_c^\infty(\mathbb{R}^d)$, and choose $r = 2$ in Lemma 4. Then $r > \frac{2}{2-\sqrt{\beta}}$ since β is less than 1, and we can take $p_0 := r$ in Lemma 3. Now, Lemma 3 and Lemma 4 imply that there exists $h > 0$ such that the sequence $\{U_m f\}$ is fundamental in $L^\infty(D_{T,h}, C_\infty(\mathbb{R}^d))$. By the reproduction property, $\{U_m f\}$ is fundamental in $L^\infty(D_T, C_\infty(\mathbb{R}^d))$. The convergence in (5) follows.

The proof of Theorem 1 is completed.

Remark 4. Note that the constraint on β in Theorem 1 (in addition to $\beta < 1$) comes solely from Lemma 1.

3. Proofs of Lemmas 1 – 4 and Proposition 1

Preliminaries. 1. We will use the following well known fact (which we use below for u_m). Suppose that b belongs to $\mathbf{F}_{\beta,\mathcal{P}}$ with $\beta < 1$. If $p > (1-\sqrt{\beta/4})^{-1}$, $f \in L^p(\mathbb{R}^d)$, then the (unique) weak solution u of the equation (1) such that

$$\lim_{t\to+0}\langle u(t),\xi\rangle = \langle f,\xi\rangle$$

for all $\xi \in L^{p'}(\mathbb{R}^d)$ having compact support, $\frac{1}{p}+\frac{1}{p'}=1$, satisfies

$$\sup_{t\in[0,\tau]}\|u(t)\|_p^p + C_1\int_0^\tau \langle (\nabla(u|u|^{\frac{p}{2}-1}))^2\rangle dt \leqslant C_2\|f\|_p^p, \tag{8}$$

where $0 < C_i = C_i(\beta,g,p) < \infty$, $i=1,2$ (see Appendix A for the proof for u_m which, in turn, is sufficient to conclude (8) for u as above).

2. Let g be the function from the condition (**BC**). Set

$$G(h) := \sup_{0\leqslant s\leqslant T-h}\int_s^{s+h} g(t)dt.$$

Clearly, $G(h) = o(h)$ (i.e. $G(h)\to 0$ as $h\to 0$).

Proof of Lemma 1. It suffices to prove Lemma 1 for $s\leqslant\tau\leqslant s+h$, for a small h, uniformly in s.

We consider smooth approximating vector fields $b_m := \eta_m * \mathbf{1}_m b$, not just truncations $\mathbf{1}_m b$ of b (cf. the beginning of Section 2), because the intermediate calculations below involve third order derivatives of u.

In what follows, we omit index m where possible: $u(t) := u_m(t)$ $(= U_m(t,s)f, t\geqslant s)$. Denote $w=\nabla u$, $w_r = \frac{\partial}{\partial x_r}u$, $1\leqslant r\leqslant d$. Define

$$\varphi_r := -\frac{\partial}{\partial x_r}\left(w_r|w|^{q-2}\right), \quad 1\leqslant r\leqslant d,$$

$$I_q = \int_s^\tau \left\langle |w|^{q-2}\sum_{r=1}^d |\nabla w_r|^2\right\rangle dt \geqslant 0, \quad J_q = \int_s^\tau \langle |w|^{q-2}|\nabla|w||^2\rangle dt \geqslant 0.$$

Now, we are going to 'differentiate the equation without differentiating its coefficients'. That is, we multiply the equation in (1) by the 'test function' φ_r, integrate in t and x, and then sum over r to get

$$S := \sum_{r=1}^d\int_s^\tau\left\langle \varphi_r,\frac{\partial u}{\partial t}\right\rangle dt = \sum_{r=1}^d\int_s^\tau\langle\varphi_r,\Delta u\rangle dt - \sum_{r=1}^d\int_s^\tau\langle\varphi_r, b_m\cdot w\rangle dt =: S_1+S_2.$$

We can re-write

$$S = \frac{1}{q}\int_s^\tau\frac{\partial}{\partial t}\langle|w|^q\rangle\,dt = \frac{1}{q}\langle|w(\tau)|^q\rangle - \frac{1}{q}\langle|\nabla f|^q\rangle$$

(the fact that $w(s)=\nabla f$ follows by differentiating in x_i, for each $1\le i\le d$, the equation in (1) and the initial function f, solving the resulting Cauchy problem, and then integrating its solution in x_i to see that it is indeed the derivative of v in x_i). Further,

$$S_1=-\sum_{r=1}^d\int_s^\tau\left\langle\frac{\partial}{\partial x_r}\left(w_r|w|^{q-2}\right),\Delta u\right\rangle dt=-\sum_{r=1}^d\int_s^\tau\left\langle\nabla\left(w_r|w|^{q-2}\right),\nabla w_r\right\rangle dt$$
$$=-\int_s^\tau\left\langle|w|^{q-2}\sum_{r=1}^d|\nabla w_r|^2\right\rangle dt-\frac{1}{2}\int_s^\tau\langle\nabla|w|^{q-2},\nabla|w|^2\rangle dt=-I_q-(q-2)J_q.$$

Next,

$$S_2=\int_s^\tau\langle|w|^{q-2}\Delta u,b_m\cdot w\rangle dt+\int_s^\tau\left\langle w\cdot\nabla|w|^{q-2},b_m\cdot w\right\rangle dt=:W_1+W_2.$$

Let us estimate W_1 and W_2 as follows. By the inequality $ac\le\frac{\gamma}{4}a^2+\frac{1}{\gamma}c^2$ $(\gamma>0)$, we have

$$\begin{aligned}|W_1|&\leqslant\int_s^\tau\langle|w|^{\frac{q-2}{2}}|\Delta u||w|^{\frac{q-2}{2}}|b_m||w|\rangle dt\\&\leqslant\frac{\gamma}{4}\int_s^\tau\langle|w|^{q-2}|\Delta u|^2\rangle dt+\frac{1}{\gamma}\int_s^\tau\left\langle\left(|b_m||w|^{\frac{q}{2}}\right)^2\right\rangle dt\\&\quad\text{(we use }(\mathbf{BC}_m)\text{, where we omit }1/m\text{ in }\beta+1/m)\\&\leqslant\frac{\gamma}{4}\int_s^\tau\langle|w|^{q-2}|\Delta u|^2\rangle dt+\frac{1}{\gamma}\left[\beta\frac{q^2}{4}J_q+\int_s^\tau g(t)\langle|w|^q\rangle\right]\end{aligned}$$

In turn, representing $|\Delta u|^2=(\nabla\cdot w)^2$ and integrating by parts twice we obtain:

$$\int_s^\tau\langle|w|^{q-2}|\Delta u|^2\rangle dt=-\int_s^\tau\langle w\cdot\nabla|w|^{q-2},\Delta u\rangle dt+\sum_{r=1}^d\int_s^\tau\left\langle w\cdot\nabla w_r,\nabla_r|w|^{q-2}\right\rangle dt+I_q$$
$$=:-F+H+I_q,$$

where we estimate, using quadratic estimates of the form $ac\le\varkappa a^2+\frac{1}{4\varkappa}c^2$ $(\varkappa>0)$,

$$|F|\leqslant(q-2)\left(\frac{1}{4\varkappa}\int_s^\tau\langle|w|^{q-2}|\Delta u|^2\rangle dt+\varkappa J_q\right),\quad|H|\leqslant(q-2)\left(\frac{1}{2}I_q+\frac{1}{2}J_q\right).$$

Thus, we obtain

$$\left(1-\frac{q-2}{4\varkappa}\right)\int_s^\tau\langle|w|^{q-2}|\Delta u|^2\rangle dt\leqslant I_q+(q-2)\left(\varkappa J_q+\frac{1}{2}I_q+\frac{1}{2}J_q\right),\quad\varkappa>\frac{q-2}{4},$$

so

$$|W_1|\leqslant\frac{\gamma}{4}\frac{4\varkappa}{4\varkappa-q+2}\left(I_q+(q-2)\left(\varkappa J_q+\frac{1}{2}I_q+\frac{1}{2}J_q\right)\right)+\frac{1}{\gamma}\left[\beta\frac{q^2}{4}J_q+\int_s^\tau g(t)\langle|w|^q\rangle\right].$$

Next, using $ac \le \nu a^2 + \frac{1}{4\nu}c^2$ ($\nu > 0$), we obtain

$$
\begin{aligned}
|W_2| &\leqslant (q-2)\int_s^\tau \langle |w|^{q-2}|\nabla|w|||b_m||w|\rangle dt = (q-2)\int_s^\tau \langle |w|^{\frac{q-2}{2}}|\nabla|w|||b_m||w|^{\frac{q}{2}}\rangle dt \\
&\leqslant (q-2)\left[\nu\int_s^\tau \langle |w|^{q-2}|\nabla|w||^2\rangle dt + \frac{1}{4\nu}\int_s^\tau \left\langle \left(|b_m||w|^{\frac{q}{2}}\right)^2\right\rangle dt\right] \\
&\text{(we use } (\mathbf{BC}_m)) \\
&\leqslant (q-2)\left[\nu J_q + \frac{\beta}{4\nu}\frac{q^2}{4}J_q + \frac{1}{4\nu}\int_s^\tau g(t)\langle |w|^q\rangle dt\right].
\end{aligned}
$$

Thus, identity $S = S_1 + S_2$ transforms into

$$
\frac{1}{q}\langle |w(\tau)|^q\rangle - \frac{1}{q}\langle |\nabla f|^q\rangle + I_q + (q-2)J_q = W_1 + W_2,
$$

and, in view of the above estimates on $|W_1|$, $|W_2|$, implies

$$
\frac{1}{q}\langle |w(\tau)|^q\rangle + N\, I_q + M\, J_q \leqslant \frac{1}{q}\langle |\nabla f|^q\rangle + \left(\frac{q-2}{4\nu} + \frac{1}{\gamma}\right)\int_s^\tau g(t)\langle |w|^q\rangle dt, \tag{9}
$$

where

$$
N := 1 - \frac{\gamma\varkappa}{4\varkappa - q + 2}\big(1 + \frac{1}{2}(q-2)\big),
$$

$$
M := q - 2 - (q-2)\left(\nu + \frac{\beta}{16\nu}q^2\right) - \frac{\beta}{\gamma}\frac{q^2}{4} - \frac{\gamma\varkappa}{4\varkappa - q + 2}(q-2)\left(\varkappa + \frac{1}{2}\right).
$$

We fix

$$
\nu := q\sqrt{\beta}/4, \quad \varkappa := \frac{q-1}{2}, \quad \gamma := \frac{q\sqrt{\beta}}{q-1}.
$$

Since $\sqrt{\beta} < q^{-1}$, we have $N > 0$. Then, in view of the inequality $I_q \geqslant J_q$, we have

$$
N\, I_q + M\, J_q \geqslant \left(q - 1 - \frac{q\sqrt{\beta}}{2}(2q-3)\right)J_q, \qquad \text{where, clearly, } q - 1 - \frac{q\sqrt{\beta}}{2}(2q-3) > \frac{1}{2}.
$$

Then, applying the Sobolev embedding theorem to $\frac{q^2}{4}J_q$ $(= \int_s^\tau \langle |\nabla |w|^{\frac{q}{2}}|^2\rangle dt)$, and recalling that $w = \nabla u$, we obtain from (9):

$$
\frac{1}{q}\langle |\nabla u(\tau)|^q\rangle + \frac{2C_d}{q^2}\|\nabla u\|^q_{L^q([s,\tau],L^{\frac{qd}{d-2}}(\mathbb{R}^d))} \leqslant \frac{1}{q}\langle |\nabla f|^q\rangle + \left(\frac{q-2}{4\nu} + \frac{1}{\gamma}\right)\int_s^\tau g(t)\langle |\nabla u(t)|^q\rangle dt,
$$

where $C_d > 0$ is the constant in the Sobolev embedding theorem.

Estimating $\int_s^\tau g(t)\langle |\nabla u|^q\rangle dt \leqslant G(h)\sup_{t\in[s,\tau]}\langle |\nabla u(t)|^q\rangle$, and selecting h $(\geqslant \tau - s)$ sufficiently small, so that $\left(\frac{q-2}{4\nu} + \frac{1}{\gamma}\right)G(h) < \frac{1}{2q}$ (recall that $G(h) = o(h)$, cf. the beginning of Section 3), we obtain

$$
\frac{1}{2}\sup_{t\in[s,\tau]}\langle |\nabla u(t)|^q\rangle + \frac{2C_d}{q}\|\nabla u\|^q_{L^q([s,\tau],L^{\frac{qd}{d-2}}(\mathbb{R}^d))} \leqslant \langle |\nabla f|^q\rangle,
$$

which completes the proof.

Proof of Lemma 2. Set $r = r_{m,n} := u_m - u_n$. Then r satisfies

$$\partial_t r = \Delta r - b_m(t,x) \cdot \nabla r - \big(b_m(t,x) - b_n(t,x)\big) \cdot \nabla u_n. \tag{10}$$

Set $\eta := r|r|^{\frac{p-2}{2}}$. We multiply equation (10) by $r|r|^{p-2}$ and integrate to obtain the identity

$$\frac{1}{p}\|\eta(\tau)\|_2^2 + \frac{4(p-1)}{p^2}\int_s^\tau \|\nabla\eta\|_2^2 dt = -\frac{2}{p}\int_s^\tau \langle \nabla\eta, b_m\eta\rangle dt - \int_s^\tau \langle \eta|\eta|^{1-\frac{2}{p}}, (b_m - b_n)\cdot\nabla u_n\rangle dt \tag{11}$$

(note that by definition $\eta(s) \equiv 0$). We estimate the right-hand side of (11). Using $ac \leqslant \varepsilon a^2 + \frac{1}{4\varepsilon}c^2$ ($\varepsilon > 0$) and ($\mathbf{BC}_m$), we obtain:

$$\begin{aligned}\left|\int_s^\tau \langle\nabla\eta, b_m\eta\rangle dt\right| &\leqslant \varepsilon\int_s^\tau \langle (b_m\eta)^2\rangle dt + \frac{1}{4\varepsilon}\int_s^\tau\langle|\nabla\eta|^2\rangle dt \\ &\leqslant \varepsilon\beta\int_s^\tau\langle|\nabla\eta|^2\rangle dt + \varepsilon\int_s^\tau g(t)\langle\eta^2\rangle dt + \frac{1}{4\varepsilon}\int_s^\tau\langle|\nabla\eta|^2\rangle dt.\end{aligned}$$

Next, using $|b_m - b_n| \leqslant |b_m| + |b_n|$, $ac \leqslant \delta a^2 + \frac{1}{4\delta}c^2$ ($\delta > 0$), and ($\mathbf{BC}_m$), we find

$$\begin{aligned}\left|\int_s^\tau \langle \eta|\eta|^{1-\frac{2}{p}}, (b_m - b_n)\cdot\nabla u_n\rangle dt\right| &\leqslant \int_s^\tau \langle |b_m - b_n||\eta|, |\eta|^{1-\frac{2}{p}}|\nabla u_n|\rangle dt \\ &\leqslant \delta\int_s^\tau\langle (b_m\eta)^2\rangle dt + \delta\int_s^\tau \langle (b_n\eta)^2\rangle dt + 2\frac{1}{4\delta}\int_s^\tau \langle |\eta|^{2-\frac{4}{p}}|\nabla u_n|^2\rangle dt \\ &\leqslant 2\delta\left(\beta\int_s^\tau\langle|\nabla\eta|^2\rangle dt + \int_s^\tau g(t)\langle\eta^2\rangle dt\right) + 2\frac{1}{4\delta}\int_s^\tau\langle|\eta|^{2-\frac{4}{p}}|\nabla u_n|^2\rangle dt.\end{aligned}$$

Thus, applying the last two estimates in the right-hand side of (11), we obtain:

$$\begin{aligned}&\frac{1}{p}\|\eta(\tau)\|_2^2 + \left(\frac{4(p-1)}{p^2} - \frac{2}{p}\left(\varepsilon\beta + \frac{1}{4\varepsilon}\right) - 2\beta\delta\right)\int_s^\tau\langle|\nabla\eta|^2\rangle dt \\ &\qquad\leqslant \frac{1}{2\delta}\int_s^\tau \langle|\eta|^{2-\frac{4}{p}}|\nabla u_n|^2\rangle dt + \left(\frac{2}{p}\varepsilon + 2\delta\right)\int_s^\tau g(t)\langle\eta^2\rangle dt.\end{aligned}$$

Set

$$P := \frac{4(p-1)}{p^2} - \frac{2}{p}\left(\varepsilon\beta + \frac{1}{4\varepsilon}\right) - 2\beta\delta \qquad \text{with } \varepsilon := \frac{1}{2\sqrt{\beta}}.$$

Estimating $\int_s^\tau g(t)\langle\eta^2\rangle dt \leqslant G(h)\sup_{t\in[s,\tau]}\|\eta(t)\|_2^2$, we have:

$$\left(\frac{1}{p} - \left(\frac{1}{p\sqrt{\beta}} + 2\delta\right)G(h)\right)\sup_{t\in[s,\tau]}\|\eta(t)\|_2^2 + P\int_s^\tau\langle|\nabla\eta|^2\rangle dt \leqslant \frac{1}{2\delta}\int_s^\tau\langle|\eta|^{2-\frac{4}{p}}|\nabla u_n|^2\rangle dt. \tag{12}$$

Since $p_0 > \frac{2}{2-\sqrt{\beta}}$, we can fix k so that $\frac{4(p_0-1)}{p_0^2} - \frac{2}{p_0}\sqrt{\beta} \geqslant \frac{2}{p_0^k}$. The last inequality remains valid if we replace p_0 with any $p > p_0$. Fix δ by

$$\delta := \frac{1}{2\beta}\left(\frac{4(p-1)}{p^2} - \frac{2}{p}\sqrt{\beta} - \frac{1}{p^k}\right) \geqslant \frac{1}{2\beta p^k}.$$

Then

$$P = \frac{4(p-1)}{p^2} - \frac{2}{p}\sqrt{\beta} - 2\beta\delta = \frac{1}{p^k}.$$

In the next Steps 1 and 2 we estimate the left-hand side and the right-hand side of (12).

Step 1. Given $0<\alpha<1$, we can choose $k>1$ so that for all $n \geqslant m_0$,

$$\frac{c_0}{p^k}\|r\|^p_{L^{\frac{p}{1-\alpha}}([s,\tau],L^{\frac{pd}{d-2+2\alpha}}(\mathbb{R}^d))} \leqslant \text{the LHS of (12).} \tag{13}$$

for some constant $c_0<\infty$.

Indeed, applying the Sobolev embedding theorem in the spatial variables, we obtain from (12):

$$\left(\frac{1}{p}-\left(\frac{1}{p\sqrt{\beta}}+2\delta\right)G(h)\right)\sup_{t\in[s,\tau]}\|r(t)\|_p^p+\frac{C_d}{p^k}\|r\|^p_{L^p([s,\tau],L^{\frac{pd}{d-2}}(\mathbb{R}^d))} \leqslant \text{the LHS of (12).}$$

Since $\delta \leqslant \frac{c}{p}$, $c:=\frac{1}{\beta}(2-\sqrt{\beta})$, we can select h sufficiently small (we use that $G(h)=o(h)$), so that for all $p \geqslant p_0$

$$\begin{aligned}
&\frac{1}{p}-\left(\frac{1}{p\sqrt{\beta}}+2\delta\right)G(h) \geqslant \\
&\frac{1}{p}\left(1-\left(\frac{1}{\sqrt{\beta}}+2c\right)G(h)\right) \geqslant \frac{1}{2p} \\
&\qquad\qquad\text{(we use that } k>1) \\
&\qquad\qquad \geqslant \frac{1}{2p^k}.
\end{aligned}$$

Thus, we have

$$\frac{1}{2p^k}\sup_{t\in[s,\tau]}\|r(t)\|_p^p+\frac{C_d}{p^k}\|r\|^p_{L^p([s,\tau],L^{\frac{pd}{d-2}}(\mathbb{R}^d))} \leqslant \text{the LHS of (12).}$$

Using first the Hölder inequality, and then the Young inequality we obtain:

$$\begin{aligned}
\|r\|^p_{L^{\frac{p}{1-\alpha}}([s,\tau],L^{\frac{pd}{d-2+2\alpha}}(\mathbb{R}^d))} &\leqslant \|r\|^{\alpha p}_{L^\infty([s,\tau],L^p(\mathbb{R}^d))}\|r\|^{(1-\alpha)p}_{L^p([s,\tau],L^{\frac{pd}{d-2}}(\mathbb{R}^d))} \\
&\leqslant \alpha\|r\|^p_{L^\infty([s,\tau],L^p(\mathbb{R}^d))}+(1-\alpha)\|r\|^p_{L^p([s,\tau],L^{\frac{pd}{d-2}}(\mathbb{R}^d))},
\end{aligned}$$

which yields (13).

Step 2: With σ, σ' and λ, λ' as in the formulation of the lemma, we have

$$\text{the RHS of (12)} \leqslant \beta p^k\|\nabla u_n\|^2_{L^{2\lambda'}([s,\tau],L^{2\sigma'}(\mathbb{R}^d))}\|r\|^{p-2}_{L^{(p-2)\lambda}([s,\tau],L^{(p-2)\sigma}(\mathbb{R}^d))} \tag{14}$$

Indeed, since $\delta \geqslant \frac{1}{2\beta p^k}$, the RHS of (12) $=\frac{1}{2\delta}\int_s^\tau\langle|\eta|^{2-\frac{4}{p}}|\nabla u_n|^2\rangle dt \leqslant \beta p^k\int_s^\tau\langle|\eta|^{2-\frac{4}{p}}|\nabla u_n|^2\rangle dt$. In turn,

$$\begin{aligned}
\int_s^\tau\langle|\eta|^{2-\frac{4}{p}}|\nabla u_n|^2\rangle dt &\leqslant \int_s^\tau\langle|\nabla u_n|^{2\sigma'}\rangle^{\frac{1}{\sigma'}}\langle|\eta|^{\left(2-\frac{4}{p}\right)\sigma}\rangle^{\frac{1}{\sigma}}dt \\
&= \int_s^\tau\|\nabla u_n\|^2_{L^{2\sigma'}(\mathbb{R}^d)}\|r\|^{p-2}_{L^{(p-2)\sigma}(\mathbb{R}^d)}dt \\
&\leqslant \left(\int_s^\tau\|\nabla u_n\|^{2\lambda'}_{L^{2\sigma'}(\mathbb{R}^d)}dt\right)^{\frac{1}{\lambda'}}\left(\int_s^\tau\|r\|^{(p-2)\lambda}_{L^{(p-2)\sigma}(\mathbb{R}^d)}dt\right)^{\frac{1}{\lambda}} \\
&= \|\nabla u_n\|^2_{L^{2\lambda'}([s,\tau],L^{2\sigma'}(\mathbb{R}^d))}\|r\|^{p-2}_{L^{(p-2)\lambda}([s,\tau],L^{(p-2)\sigma}(\mathbb{R}^d))},
\end{aligned}$$

which yields (14).

Substituting the estimates (13) and (14) into (12), and taking $\tau:=s+h$, we arrive at the required estimate (6).

Proof of Lemma 3. The proof of Lemma 3 follows closely the proof of [4, Lemma 7]. Consider the inequality of Lemma 2:

$$
\begin{aligned}
&\|u_m-u_n\|_{L^{\frac{p}{1-\alpha}}([s,s+h],L^{\frac{pd}{d-2+2\alpha}}(\mathbb{R}^d))} \\
&\qquad\leqslant \Big(C_0\beta\|\nabla u_m\|^2_{L^{2\lambda'}([s,s+h],L^{2\sigma'}(\mathbb{R}^d))}\Big)^{\frac{1}{p}}(p^{2k})^{\frac{1}{p}}\|u_m-u_n\|^{1-\frac{2}{p}}_{L^{(p-2)\lambda}([s,s+h],L^{(p-2)\sigma}(\mathbb{R}^d))}, \qquad (15)
\end{aligned}
$$

where λ is defined by $\frac{1/(1-\alpha)}{\lambda}=\frac{d/(d-2+2\alpha)}{\sigma}$, and $\frac{1}{\lambda}+\frac{1}{\lambda'}=1$ (it is easy to see that $\lambda'=\frac{\sigma'(d-2+2\alpha)}{d(1-\alpha)}$). We fix $\alpha:=\frac{2}{d+2}$ (we keep α to make the calculations easier to follow) and $1<\sigma<\frac{d}{d-2+2\alpha}$ so that $\sigma'>\frac{d}{2(1-\alpha)}$, determined from $\frac{1}{\sigma}+\frac{1}{\sigma'}=1$, is sufficiently close to $\frac{d}{2(1-\alpha)}$. We apply the a priori estimate of Lemma 1:

$$
\begin{aligned}
&\|\nabla u_m\|^2_{L^{2\lambda'}([s,s+h],L^{2\sigma'}(\mathbb{R}^d))}\\
&\text{(we use the Hölder inequality)}\\
&\leqslant \|\nabla u_m\|^{\alpha}_{L^\infty([s,s+h],L^q(\mathbb{R}^d))}\|\nabla u_m\|^{1-\alpha}_{L^q([s,s+h],L^{\frac{qd}{d-2}}(\mathbb{R}^d))}\\
&\text{(we use Young's inequality)}\\
&\leqslant \alpha\|\nabla u_m\|_{L^\infty([s,s+h],L^q(\mathbb{R}^d))}+(1-\alpha)\|\nabla u_m\|_{L^q([s,s+h],L^{\frac{qd}{d-2}}(\mathbb{R}^d))}\\
&\text{(we use Lemma 1)}\\
&\leqslant C\|\nabla f\|_q =: D<\infty,
\end{aligned}
$$

where q is determined from $\sigma'=\frac{1}{2}\frac{qd}{d-2+2\alpha}$ (such q ($\in(d,1/\sqrt{\beta})$) in Lemma 1 is admissible, in view of the assumptions on β in Theorem 1). Then (15) yields

$$
\|u_m-u_n\|_{L^{\frac{p}{1-\alpha}}([s,s+h],L^{\frac{pd}{d-2+2\alpha}}(\mathbb{R}^d))}\leqslant D^{\frac{1}{p}}(p^{2k})^{\frac{1}{p}}\|u_m-u_n\|^{1-\frac{2}{p}}_{L^{(p-2)\lambda}([s,s+h],L^{(p-2)\sigma}(\mathbb{R}^d))}. \qquad (16)
$$

In order to iterate the inequality (16), choose any $p_0>\frac{2}{2-\sqrt{\beta}}$ and construct a sequence $\{p_l\}_{l\geqslant 0}$ by successively assuming $\sigma(p_1-2)=p_0$, $\sigma(p_2-2)=\frac{p_1 d}{d-2+2\alpha}$, $\sigma(p_3-2)=\frac{p_2 d}{d-2+2\alpha}$ etc, so that

$$
p_l=(a-1)^{-1}\left(a^l\left(\frac{p_0}{\sigma}+2\right)-a^{l-1}\frac{p_0}{\sigma}-2\right),\quad a:=\frac{1}{\sigma}\frac{d}{d-2+2\alpha}>1. \qquad (17)
$$

Clearly,

$$
c_1a^l\leqslant p_l\leqslant c_2a^l,\qquad \text{where } c_1:=p_1a^{-1},\quad c_2:=c_1(a-1)^{-1}, \qquad (18)
$$

and so $p_l\to\infty$ as $l\to\infty$.

Now, we iterate inequality (16), starting with $p=p_0$, to obtain

$$
\|u_m-u_n\|_{L^{\frac{p_l}{1-\alpha}}([s,s+h],L^{\frac{p_l d}{d-2+2\alpha}}(\mathbb{R}^d))}\leqslant D^{\alpha_l}\Gamma_l\|u_m-u_n\|^{\gamma_l}_{L^{p_0\lambda}([s,s+h],L^{p_0\sigma}(\mathbb{R}^d))}, \qquad (19)
$$

where

$$
\gamma_l:=\left(1-\frac{2}{p_1}\right)\dots\left(1-\frac{2}{p_l}\right),
$$

$$\alpha_l := \frac{1}{p_1}\Big(1-\frac{2}{p_2}\Big)\Big(1-\frac{2}{p_3}\Big)\dots\Big(1-\frac{2}{p_l}\Big)+ \\ \frac{1}{p_2}\Big(1-\frac{2}{p_3}\Big)\Big(1-\frac{2}{p_4}\Big)\dots\Big(1-\frac{2}{p_l}\Big)+\dots+\frac{1}{p_{l-1}}\Big(1-\frac{2}{p_l}\Big)+\frac{1}{p_l},$$

$$\Gamma_l := \left(p_l^{p_l^{-1}}\,p_{l-1}^{p_{l-1}^{-1}(1-2p_l^{-1})}\,p_{l-2}^{p_{l-2}^{-1}(1-2p_{l-1}^{-1})(1-2p_l^{-1})}\dots p_1^{p_1^{-1}(1-2p_2^{-1})\dots(1-2p_l^{-1})}\right)^{2k}.$$

We wish to take $l\to\infty$ in (19): since $p_l\to\infty$ as $l\to\infty$, this would yield the required inequality (7) provided that sequences $\{\alpha_l\}$, $\{\Gamma_l\}$ are bounded from above, and $\{\gamma_l\}$ is bounded from below by a positive constant. Note that $\alpha_l = a^l - \frac{1}{p_l(a-1)}$, $\gamma_l = p_0\frac{a^{l-1}}{\sigma p_l}$. In view of (17),

$$\sup_l \alpha_l \leqslant \left(\frac{p_0}{\sigma}+2-\frac{p_0(d-2+2\alpha)}{d}\right)^{-1}<\infty, \quad \sup_l \gamma_l<\infty, \tag{20}$$

$$\inf_l \gamma_l > \left(1-\frac{\sigma(d-2+2\alpha)}{d}\right)\left(1-\frac{\sigma(d-2+2\alpha)}{d}+\frac{2\sigma}{p_0}\right)^{-1}>0. \tag{21}$$

Further, noticing that (cf. (17)) $\Gamma_l^{1/2k} = p_l^{p_l^{-1}}\,p_{l-1}^{ap_l^{-1}}\,p_{l-2}^{a^2p_l^{-1}}\dots p_1^{a^{l-1}p_l^{-1}}$, we have by (18)

$$\Gamma_l^{1/2k} \leqslant (c_1a^l)^{(c_2a^l)^{-1}}(c_1a^{l-1})^{(c_2a^{l-1})^{-1}}\dots(c_1a)^{(c_2a)^{-1}} = \\ \left(c_1^{(a^l-1)/(a^l(a-1))}a^{\sum_{j=1}^l ja^{-j}}\right)^{c_2^{-1}} \leqslant \left(c_1^{(a-1)^{-1}}c_2^{a(a-1)^{-1}}\right)^{c_2^{-1}}<\infty. \tag{22}$$

Now, estimates (20), (21) and (22) imply that we can take $l\to\infty$ in (16):

$$\|u_m-u_n\|_{L^\infty([s,s+h],L^\infty(\mathbb{R}^d))}\leqslant B\|u_m-u_n\|^\gamma_{L^{p_0}([s,s+h],L^{p_0}(\mathbb{R}^d))}.$$

Taking sup in $0\leqslant s\leqslant T-h$ in both sides of the inequality, we obtain (7) in Lemma 3.

Remark 5. The main concern of the iterative procedure has been to keep $\inf_l\gamma_l>0$: if $\gamma_l\downarrow 0$, then the result of the iterations ($\|U_mf-U_nf\|_{L^\infty(D_T\times\mathbb{R}^d)}\leqslant C$) would be useless for the purpose of proving Theorem 1.

Proof of Lemma 4. By the reproduction property, and in view of (8), it suffices to show that $\{U_mf\}$ is fundamental in $L^\infty(D_{T,h},L^2(\mathbb{R}^d))$ for some $h>0$. We show this in three steps:

Step 1. Define

$$\rho_\delta(x) := (1+\delta|x|^2)^{-\frac{1}{2}}, \quad \delta>0, \quad x\in\mathbb{R}^d.$$

In Step 1, we are going to show that there is an $h=h(g)>0$ (g is from the condition ($\mathbf{BC}_m$)) such that for any $\varepsilon>0$ there is a $0<\delta<1$ such that

$$\|(1-\rho_\delta)^{\frac{1}{2}}\,U_mf\|_{L^\infty(D_{T,h},L^2(\mathbb{R}^d))}<\varepsilon \quad \text{for all } m. \tag{23}$$

Indeed, set $u_m(t)=U_m(t,s)f$ $(t\geqslant s)$. Set

$$J := \int_s^\tau \langle(1-\rho_\delta)(\nabla u_m)^2\rangle dt.$$

We multiply the equation in (1) by $(1-\rho_\delta)u_m$ and integrate by parts to get

$$\langle (1-\rho_\delta)u_m^2(\tau)\rangle - \langle (1-\rho_\delta)f^2\rangle + 2J = \int_s^\tau \langle u_m^2, (-\Delta\rho_\delta)\rangle dt - 2\int_s^\tau \langle (1-\rho_\delta)u_m b_m, \nabla u_m\rangle dt. \tag{24}$$

Estimating the last term by applying the inequality $2ac \leqslant \gamma a^2 + \frac{1}{\gamma}c^2$ ($\gamma > 0$) and the condition ($\mathbf{BC}_m$), we get:

$$\begin{aligned}
&-2\int_s^\tau \langle (1-\rho_\delta)u_m b_m, \nabla u_m\rangle dt \\
&\leqslant \gamma J + \frac{1}{\gamma}\int_s^\tau \langle (1-\rho_\delta)b_m^2 u_m^2\rangle dt \\
&\leqslant \gamma J + \frac{\beta}{\gamma}\int_s^\tau \langle (\nabla(u_m\sqrt{1-\rho_\delta}))^2\rangle dt + \frac{1}{\gamma}\int_s^\tau \langle g(t)(1-\rho_\delta)u_m^2\rangle dt.
\end{aligned}$$

We compute:

$$\begin{aligned}
&\int_s^\tau \langle (\nabla(u_m\sqrt{1-\rho_\delta}))^2\rangle dt \\
&= J + \int_s^\tau \langle u^2(\nabla\sqrt{1-\rho_\delta})^2\rangle dt + \frac{1}{2}\int_s^\tau \langle u^2, (-\Delta\rho_\delta)\rangle dt \\
&= J + \int_s^\tau \left\langle u^2, \frac{\delta^2 x^2\rho^6}{4(1-\rho)}\right\rangle dt + \int_s^\tau \left\langle u^2, \frac{\rho^3\delta}{2}(d-3\rho^2\delta x^2)\right\rangle dt.
\end{aligned}$$

Thus, estimating $\int_s^\tau \langle g(t)(1-\rho_\delta)u_m^2\rangle dt \leqslant G(h)\sup_{t\in[s,\tau]}\langle (1-\rho_\delta)u_m^2(t)\rangle$, we obtain from (24):

$$\begin{aligned}
&\left(1-\frac{G(h)}{\gamma}\right)\sup_{t\in[s,\tau]}\langle (1-\rho_\delta)u_m^2(t)\rangle + \left(2-\gamma-\frac{\beta}{\gamma}\right)J \\
&\leqslant \langle (1-\rho_\delta)f^2\rangle + \frac{\beta}{\gamma}\int_s^\tau \left\langle u^2, \frac{\delta^2x^2\rho^6}{4(1-\rho)}\right\rangle dt + \left(1-\frac{\beta}{\gamma}\right)\int_s^\tau \left\langle u^2, \frac{\rho^3\delta}{2}(d-3\rho^2\delta x^2)\right\rangle dt.
\end{aligned}$$

Now, fix $\gamma > 0$ by the condition $2-\gamma-\frac{\beta}{\gamma} > 0$, and then fix h by the condition $1-\frac{1}{\gamma}G(h) > 0$ (recall that $G(h) = o(h)$). Noting that $\frac{\delta^2x^2\rho^6(x)}{4(1-\rho(x))} \leqslant \frac{\delta}{2}\rho(x)$, $\frac{\rho^3(x)\delta}{2}(d-3\rho^2(x)\delta x^2) \leqslant \delta\frac{d-3}{2}\rho(x)$, $\int_s^\tau \langle \rho_\delta u^2\rangle dt \leqslant hC\|f\|_2^2$ (by (8) with $p=2$), we obtain:

$$\left(1-\frac{G(h)}{\gamma}\right)\sup_{t\in[s,\tau]}\langle (1-\rho_\delta)u_m^2(t)\rangle \leqslant \langle (1-\rho_\delta)f^2\rangle + \delta hC\left(\frac{\beta}{2\gamma} + \left(1-\frac{\beta}{\gamma}\right)\frac{d-3}{2}\right)\|f\|_2^2.$$

Since $\rho_\delta \to 1$ uniformly on the support of $f \in C_c^\infty(\mathbb{R}^d)$ as $\delta \to 0$, the right-hand side of the inequality can be made arbitrarily small by taking sufficiently small δ, i.e. we have proved (23).

Step 2. In Step 2, we are going to show that there is an $h = h(g) > 0$ such that for a given $\varepsilon > 0$ and $\delta := \delta(\varepsilon)$ from Step 1 there is a $n_0 = n_0(\varepsilon)$ such that

$$\|\rho_\delta^{\frac{1}{2}}(U_mf - U_nf)\|_{L^\infty(D_{T,h}, L^2(\mathbb{R}^d))} < \varepsilon \quad \text{for all } m, n \geqslant n_0. \tag{25}$$

Indeed, by the equation for $r(t) := u_m(t) - u_n(t)$ ($= U_m(t,s)f - U_n(t,s)f$),

$$\int_s^\tau \left\langle \rho_\delta r\frac{\partial r}{\partial t}\right\rangle dt + \int_s^\tau \langle \rho_\delta r(-\Delta r)\rangle dt = -\int_s^\tau \langle \rho_\delta r, b_m\cdot\nabla r\rangle dt - \int_s^\tau \langle \rho_\delta r, (b_m-b_n)\cdot\nabla u_n\rangle dt.$$

Integrating by parts in the second term in the left-hand side, and applying the inequality $ac \leqslant \frac{1}{2}a^2 + \frac{1}{2}c^2$ to the first term in the right-hand side, we obtain:

$$\langle \rho_\delta r^2(\tau)\rangle + \int_s^\tau \langle \rho_\delta (\nabla r)^2\rangle dt + 2\int_s^\tau \langle r\nabla\rho_\delta, \nabla r\rangle dt \leqslant \int_s^\tau \langle \rho_\delta b_m^2 r^2\rangle dt - 2\int_s^\tau \langle \rho_\delta r, (b_m - b_n)\cdot \nabla u_n\rangle dt$$

or

$$\langle \rho_\delta r^2(\tau)\rangle + \int_s^\tau \langle \rho_\delta (\nabla r)^2\rangle dt + K \leqslant L + Z.$$

We have

$$K = \int_s^\tau \langle \nabla \rho_\delta, \nabla r^2\rangle dt = \int_s^\tau \langle (-\Delta\rho_\delta) r^2\rangle dt = \int_s^\tau \langle \left(\delta d \rho_\delta^3 - 3\delta^2|x|^2\rho_\delta^5\right) r^2\rangle dt \geqslant 0.$$

Next, using $(\mathbf{BC}_m)$ we obtain

$$\begin{aligned}
L &= \int_s^\tau \langle \rho_\delta b_m^2 r^2\rangle dt \\
&\leqslant \beta \int_s^\tau \langle (\nabla(\sqrt{\rho_\delta} r))^2\rangle dt + \int_s^\tau g(t)\langle \rho_\delta r^2\rangle dt \\
&\left(\text{here we use } \frac{(\nabla\rho_\delta(x))^2}{\rho_\delta(x)} = \delta^2|x|^2\rho^5\right) \\
&= \frac{\beta}{4}\int_s^\tau \langle \delta^2|x|^2\rho_\delta^5 r^2\rangle dt + \frac{\beta}{2}K + \beta\int_s^\tau \langle \rho_\delta(\nabla r)^2\rangle dt + \int_s^\tau g(t)\langle\rho_\delta r^2\rangle dt.
\end{aligned}$$

Now we combine the above bound on L and the estimates

$$\int_s^\tau g(t)\langle \rho_\delta r^2\rangle dt \leqslant G(h) \sup_{t\in[s,\tau]} \langle \rho_\delta r^2(t)\rangle, \quad \int_s^\tau \langle \delta^2|x|^2\rho_\delta^5 r^2\rangle dt \leqslant h\,\delta \sup_{t\in[s,\tau]}\langle\rho_\delta r^2(t)\rangle,$$

obtaining:

$$\left(1 - G(h) - \frac{\beta\delta h}{4}\right)\sup_{t\in[s,\tau]}\langle \rho_\delta r^2(t)\rangle + (1-\beta)\int_s^\tau \langle \rho_\delta(\nabla r)^2\rangle dt + \left(1 - \frac{\beta}{2}\right)K \leqslant Z. \tag{26}$$

Fix $h > 0$ by the condition $1 - G(h) - \frac{\beta\delta h}{4} \geqslant \frac{1}{2}$ (recall that $G(h) = o(h)$, β, $\delta < 1$).

Finally, we estimate the term Z as follows:

$$\begin{aligned}
Z &= -2\int_s^\tau \langle \rho_\delta r(b_m - b_n), \nabla u_n\rangle dt \\
&\leqslant \varepsilon \int_s^\tau (\nabla u_n)^2 dt + \frac{1}{\varepsilon}\int_s^\tau \langle \rho_\delta^2 r^2 (b_m - b_n)^2\rangle dt \\
&\text{(here we use } \int_s^\tau (\nabla u_n)^2 dt \leqslant C\|f\|_2^2\text{, see Appendix A with } p = 2) \\
&\leqslant \varepsilon C\|f\|_2^2 + \frac{1}{\varepsilon}\int_s^\tau \langle \rho_\delta^2 r^2(b_m - b_n)^2\rangle dt, \\
&\leqslant \varepsilon C\|f\|_2^2 + \frac{1}{\varepsilon}\int_s^\tau \langle (1 - \mathbf{1}_{B(0,R)})\rho_\delta^2 r^2 (b_m - b_n)^2\rangle dt + \frac{1}{\varepsilon}\int_s^\tau \langle \mathbf{1}_{B(0,R)}\rho_\delta^2 r^2(b_m - b_n)^2\rangle dt \\
&=: \varepsilon C\|f\|_2^2 + \frac{1}{\varepsilon}Z_1 + \frac{1}{\varepsilon}Z_2.
\end{aligned}$$

In turn,

$$Z_1 \leqslant 2(1 + \delta R^2)^{-\frac{1}{2}}\left(\int_s^\tau \langle \rho_\delta b_m^2 r^2\rangle dt + \int_s^\tau \langle \rho_\delta b_n^2 r^2\rangle dt\right).$$

Estimating the terms in the brackets in the last inequality in the same way as L, and substituting the resulting estimate on Z into (26), we obtain:

$$\left(1 - G(h) - \frac{\beta\delta h}{4} - \frac{1}{\varepsilon}(1+\delta R^2)^{-\frac{1}{2}}C_1\right) \sup_{t\in[s,\tau]} \langle \rho_\delta r^2(t)\rangle$$
$$+ \left(1 - \beta - \frac{4\beta}{\varepsilon}(1+\delta R^2)^{-\frac{1}{2}}\right) \int_s^\tau \langle \rho_\delta (\nabla r)^2\rangle dt + \left(1 - \frac{\beta}{2} - \frac{2\beta}{\varepsilon}(1+\delta R^2)^{-\frac{1}{2}}\right) K \leqslant \varepsilon C\|f\|_2^2 + \frac{1}{\varepsilon}Z_2,$$

where $C_1 := 4\left(G(h) + \frac{\beta\delta h}{4}\right)$.

Choose $R = R(\varepsilon,\delta) > 0$ sufficiently large to ensure that the coefficients of $\int_s^\tau \langle \rho_\delta(\nabla r)^2\rangle dt$, K remain positive and, moreover, the coefficient of $\sup_{t\in[s,\tau]}\langle \rho_\delta r^2(t)\rangle$ is greater or equal to $\frac{1}{4}$ (since $1 - G(h) - \frac{\beta\delta h}{4} \geqslant \frac{1}{2}$). Then the previous inequality yields

$$\frac{1}{4}\sup_{t\in[s,\tau]} \langle \rho_\delta r^2(t)\rangle \leqslant \varepsilon C\|f\|_2^2 + \frac{1}{\varepsilon}Z_2. \tag{27}$$

Since U_m is L^∞-contractive, $\|r(\tau)\|_\infty \leqslant 2\|f\|_\infty$ and so there is a $n_0 = n_0(R,\varepsilon)$ such that

$$\begin{aligned} Z_2 &= \int_s^\tau \langle \mathbf{1}_{B(0,R)}\rho_\delta^2 r^2 (b_m - b_n)^2\rangle dt \\ &\leqslant 4\|f\|_\infty^2 \int_s^\tau \langle \mathbf{1}_{B(0,R)}(b_m - b_n)^2\rangle dt < \varepsilon^2 \end{aligned}$$

for all $(s,\tau) \in D_{T,h}$ for all $m,n \geqslant n_0$ since $b_m \to b$ in $L^2_{\rm loc}([s,s+h]\times\mathbb{R}^d,\mathbb{R}^d)$.

Thus, in view of (27)

$$\sup_{t\in[s,\tau]} \langle \rho_\delta r^2(t)\rangle < 4(C\|f\|_2^2 + 1)\,\varepsilon.$$

Therefore, we have proved (25).

Step 3. Set $\|\cdot\| := \|\cdot\|_{L^\infty(D_{T,h},L^2(\mathbb{R}^d))}$. The results of Step 1 and Step 2 yield: for any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) < 1$, and an $n_0 = n_0(\varepsilon)$ such that

$$\begin{aligned} \|U_m f - U_n f\|^2 &= \|(1-\rho_\delta)^{\frac{1}{2}}(U_m f - U_n f)\|^2 + \|\rho_\delta^{\frac{1}{2}}(U_m f - U_n f)\|^2 \\ &\leqslant 2\|(1-\rho_\delta)^{\frac{1}{2}}U_m f\|^2 + 2\|(1-\rho_\delta)^{\frac{1}{2}}U_n f\|^2 + \|\rho_\delta^{\frac{1}{2}}(U_m f - U_n f)\|^2 < 5\varepsilon \end{aligned}$$

for all $m,n \geqslant n_0$.

The latter implies that $\{U_m f\}$ is fundamental in $L^\infty(D_{T,h},L^2(\mathbb{R}^d))$, as required.

Proof of Proposition 1. In Section 2.1 we proved the existence of $Uf := {}_{L^\infty(D_T\times\mathbb{R}^d)}\text{-}\lim_{m\to\infty} U_m f$, $f \in C_c^\infty(\mathbb{R}^d)$. Since $C_c^\infty(\mathbb{R}^d)$ in dense in $C_\infty(\mathbb{R}^d)$, and U_m is L^∞-contractive, U extends by continuity to $C_\infty(\mathbb{R}^d)$. Thus, the property (**E2**) is established.

The properties (**E1**) and (**E3**) follow from (5) and the analogous properties of U_m.

We are left to prove (**E4**). Set $u(t) = U(t,0)f$ $(t \geqslant 0)$, $f \in C_\infty(\mathbb{R}^d)$. In order to verify that u is a weak solution of (1), we have to show that $b\cdot\nabla u \in L^1_{\rm loc}((0,\infty)\times\mathbb{R}^d)$. Since $b \in L^2_{\rm loc}([0,\infty)\times\mathbb{R}^d,\mathbb{R}^d)$, it suffices to show that $\nabla u \in L^2_{\rm loc}((0,\infty)\times\mathbb{R}^d,\mathbb{R}^d)$. Fix $k > \frac{d}{2}$. Set

$$\theta_\delta(x) := (1+\delta|x|^2)^{-k}, \quad \delta > 0, \quad x \in \mathbb{R}^d.$$

It is easy to see that $\theta_\delta \in L^1(\mathbb{R}^d)$.

Set $u_m(t) = U_m(t,0)f$ $(t \geqslant 0)$.

Claim 1. *There exist an $h>0$ and a $\delta>0$ such that for all m*

$$\int_0^h \langle \theta_\delta (\nabla u_m)^2\rangle dt \leqslant c_1 \langle \theta_\delta f^2\rangle + c_2\sqrt{\delta}\|f\|_\infty^2, \quad f \in C_\infty(\mathbb{R}^d), \tag{28}$$

where constants $c_1, c_2<\infty$ don't depend on m.

Proof of Claim 1. For all m,

$$C_0\int_0^h \langle \theta_\delta (\nabla u_m)^2\rangle dt \leqslant \langle \theta_\delta f^2\rangle + C_1 k\sqrt{\delta}\bigg(\int_0^h \langle \theta_\delta u_m^2\rangle dt + \int_0^h \langle \theta_\delta (\nabla u_m)^2\rangle dt\bigg), \tag{29}$$

where $0<C_0, C_1<\infty$ do not depend on m or δ. The proof is similar to the proof of Lemma 4 (Step 1) but with $1-\rho_\delta$ replaced by θ_δ. By (29),

$$(C_0 - C_1 k\sqrt{\delta})\int_0^h \langle \theta_\delta (\nabla u_m)^2\rangle dt \leqslant \langle \theta_\delta f^2\rangle + C_1 k\sqrt{\delta}\int_0^h \langle \theta_\delta u_m^2\rangle dt \quad \text{for all } m.$$

We choose $\delta>0$ by the condition $C_0 - C_1 k\sqrt{\delta}>0$. Recalling that U_m is L^∞-contractive and $\theta_\delta \in L^1$, we obtain $\int_0^h \langle \theta_\delta u_m^2\rangle dt \leqslant C_3\|f\|_\infty^2$. This yields (28). ☐

We fix h and δ from Claim 1. By (28), the sequence $\{\nabla u_m|_{[0,h]\times \bar{B}(0,R)}\}$ is weakly relatively compact in $L^2([0,h]\times \bar{B}(0,R),\mathbb{R}^d)$, where $\bar{B}(0,R)$ is the closed ball of radius $R>0$ arbitrarily fixed. Hence, $\nabla u|_{(0,h)\times B(0,R)}$ (understood in the sense of distributions) is in $L^2([0,h]\times \bar{B}(0,R),\mathbb{R}^d)$. It follows that $\nabla u \in L^2_{\rm loc}((0,\infty)\times\mathbb{R}^d,\mathbb{R}^d)$.

(Note that if $f\in C_\infty(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$, then $\nabla u \in L^2_{\rm loc}((0,\infty)\times\mathbb{R}^d,\mathbb{R}^d)$ also follows from (8) with $p=2$.)

It remains to show that u satisfies the integral identity (3). Clearly,

$$\int_0^\infty \langle u_m, \partial_t \psi\rangle dt - \int_0^\infty \langle u_m, \Delta\psi\rangle dt + \int_0^\infty \langle (b_m - b)\cdot\nabla u_m, \psi\rangle dt + \int_0^\infty \langle b\cdot \nabla u_m, \psi\rangle dt = 0. \tag{30}$$

Without loss of generality, we consider only the test functions ψ with spt $\psi \subset (0,h)\times B(0,R)$, for some $R>0$. Since $u_m \to u$ in $C([0,h], C_\infty(\mathbb{R}^d))$ by (5), we can pass to the limit $m\to\infty$ in the first two terms in the left-hand side of (30). By the Hölder inequality,

$$\left|\int_0^\infty \langle (b_m-b)\cdot\nabla u_m, \psi\rangle dt\right| \leqslant S^{\frac{1}{2}}\left(\int_0^\infty \langle (b_m-b)^2|\psi|\rangle dt\right)^{\frac{1}{2}},$$

where $S:=\sup_m \int_s^T \langle |\nabla u_m|^2|\psi|\rangle dt<\infty$ by (28). Therefore, since $b_m\to b$ in $L^2_{\rm loc}([0,\infty)\times\mathbb{R}^d,\mathbb{R}^d)$ and spt ψ is compact, the third term the left-hand side of (30) tends to 0 as $m\to\infty$. Finally, we can pass to the limit $m\to\infty$ in the fourth term in (30) because $\{\nabla u_m|_{[0,h]\times\bar{B}(0,R)}\}$ is weakly relatively compact in $L^2([0,h]\times\bar{B}(0,R))$, see (28), and $|b\psi|\in L^2([0,h]\times\bar{B}(0,R))$.

Appendix A.

Proof of (8). We omit index m: $u=u_m$. Without loss of generality, we may assume that $\tau\leqslant h$ for a small h, and that $f\geqslant 0$, so $u\geqslant 0$. Multiply the equation (1) by u^{p-1} and integrate to get

$$R := \int_0^\tau \langle u^{p-1}, \partial_t u\rangle dt = \int_0^\tau \langle u^{p-1},\Delta u\rangle dt - \int_0^\tau \langle u^{p-1}, b_m\cdot\nabla u\rangle dt =: R_1 + R_2.$$

We have

$$R = \frac{1}{p}\langle u^p(\tau)\rangle - \frac{1}{p}\langle f^p\rangle, \quad R_1 = -(p-1)\frac{4}{p^2}\int_0^\tau \langle (\nabla u^{\frac{p}{2}})^2\rangle dt.$$

Using the inequality $ac \leqslant \nu a^2 + \frac{1}{4\nu}c^2$ $(\nu > 0)$ and the condition $(\mathbf{BC}_m)$, we obtain:

$$R_2 = -\frac{2}{p}\int_0^\tau \langle u^{\frac{p}{2}}, b_m \cdot \nabla u^{\frac{p}{2}}\rangle dt \leqslant \frac{2}{p}\nu\int_0^\tau \langle (\nabla u^{\frac{p}{2}})^2\rangle dt + \frac{1}{2p\nu}\biggl(\beta\int_0^\tau \langle (\nabla u^{\frac{p}{2}})^2\rangle dt + \int_0^\tau \langle g(t)u^p\rangle dt\biggr).$$

Therefore,

$$\frac{1}{p}\langle u^p(\tau)\rangle + \left(\frac{4(p-1)}{p^2} - \frac{2}{p}\nu - \frac{\beta}{2p\nu}\right)\int_0^\tau \langle (\nabla u^{\frac{p}{2}})^2\rangle dt \leqslant \frac{1}{p}\langle f^p\rangle + \frac{\beta}{2p\nu}\int_0^\tau g(t)\langle u^p\rangle dt$$

The maximum of $\nu \mapsto \frac{4(p-1)}{p^2} - \frac{2}{p}\nu - \frac{\beta}{2p\nu}$, attained at $\sqrt{\beta/4}$, is positive if and only if $p > (1-\sqrt{\beta/4})^{-1}$. Set $\nu := \sqrt{\beta/4}$. Estimating $\int_0^\tau g(t)\langle u^p\rangle dt \leqslant G(h)\sup_{t\in[0,\tau]}\langle u^p(t)\rangle$, and selecting h sufficiently small, so that $1 - \frac{\beta}{2\nu}G(h) > 0$ (recall that $G(h) = o(h)$), we obtain

$$\frac{1}{p}\left(1 - \frac{\beta}{2\nu}G(h)\right)\sup_{t\in[0,\tau]}\langle u^p(t)\rangle + \left(\frac{4(p-1)}{p^2} - \frac{2}{p}\nu - \frac{\beta}{2p\nu}\right)\int_0^\tau \langle (\nabla u^{\frac{p}{2}})^2\rangle dt \leqslant \frac{1}{p}\langle f^p\rangle.$$

which yields (8). □

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