TWO-SIDED WEIGHTED BOUNDS ON FUNDAMENTAL SOLUTION TO FRACTIONAL SCHRÖDINGER OPERATOR

D. KINZEBULATOV AND YU. A. SEMËNOV

ABSTRACT. We establish sharp two-sided weighted bounds on the fundamental solution to the fractional Schrödinger operator using the method of desingularizing weights.

In [MS0], Milman and Semenov developed an approach to study of the integral kernels of semigroups which are not necessarily ultracontractive by transferring them to appropriately chosen weighted spaces where they become ultracontractive [MS1, MS2]. In the special case of the Schrödinger semigroup generated by $-\Delta - V$, with potential $V(x) = \delta \frac{(d-2)^2}{4}|x|^{-2}$, $0 < \delta \le 1$, $d \ge 3$, having a critical-order singularity at x = 0 (which makes invalid the standard two-sided Gaussian bounds on its integral kernel) this method yields sharp two-sided weighted bounds on the integral kernel; the transition to the weighted space effectively removes the singularity, hence the name the method of desingularizing weights.

In [KSSz], we employed the method of desingularizing weights to establish sharp two-sided weighted bounds on the fundamental solution to the non-local operator

$$(-\Delta)^{\frac{\alpha}{2}} + b \cdot \nabla, \quad b(x) = \delta(d - \alpha)^{-2} 2c_{\alpha}^{-2} |x|^{-\alpha} x, \quad 0 < \delta < 1, \quad 1 < \alpha < 2, \quad d \ge 3,$$

$$c(\alpha, p, d) = \frac{\gamma(\frac{d}{p} - \alpha)}{\gamma(\frac{d}{p})}, \quad \gamma(\alpha) = \frac{2^{\alpha} \pi^{\frac{d}{2}} \Gamma(\frac{\alpha}{2})}{\Gamma(\frac{d}{2} - \frac{\alpha}{2})}, \quad 1$$

In this paper, we specify our arguments in [KSSz] to the operator

$$(-\Delta)^{\frac{\alpha}{2}} - V, \quad V(x) = \delta c_{\alpha}^{-2} |x|^{-\alpha}, \quad 0 < \delta < 1, \quad 0 < \alpha < 2, \quad c_{\alpha} := c\left(\frac{\alpha}{2}, 2, d\right),$$

and obtain sharp two-sided weighted bounds on its fundamental solution. These bounds are known for $0 < \delta \le 1$, see [BGJP], where the authors use a different technique. Concerning $(-\Delta)^{\frac{\alpha}{2}} + c|x|^{-\alpha}$, c > 0, see [CKSV] and [JW].

1. The method of desingularizing weights relies on two assumptions: the Sobolev embedding property, and a "desingularizing" (L^1, L^1) bound on the weighted semigroup. Let X be a locally compact topological space and μ a σ -finite Borel measure on X. Let Λ be a non-negative selfadjoint operator in the (complex) Hilbert space $L^2 = L^2(X,\mu)$ with the inner product $\langle f,g \rangle = \int_X f \bar{g} d\mu$. We assume that Λ possesses the Sobolev embedding property:

There are constants j > 1 and $c_S > 0$ such that, for all $f \in D(\Lambda^{\frac{1}{2}})$,

$$c_S \|f\|_{2i}^2 \le \|\Lambda^{\frac{1}{2}}f\|_2^2 \tag{M_1}$$

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but $e^{-t\Lambda} \upharpoonright L^1 \cap L^2$, t > 0, cannot be extended by continuity to a bounded map on L^1 and the ultracontractive estimate

$$||e^{-t\Lambda}f||_{\infty} \le c_t ||f||_1, \ f \in L^1 \cap L^{\infty}, \ t > 0$$

is not valid.

In this case we will be assuming that there exists a family of real valued weights $\varphi = \{\varphi_s\}_{s>0}$ on X such that, for all s>0,

$$\varphi_s, \ 1/\varphi_s \in L^2_{loc}(X,\mu)$$
 (M₂)

and there exists a constant c_1 independent of s such that, for all $0 < t \le s$,

$$\|\varphi_s e^{-t\Lambda} \varphi_s^{-1} f\|_1 \le c_1 \|f\|_1, \quad f \in \mathcal{D} := \varphi_s L_{com}^{\infty}(X, \mu).$$
 (M₃)

Theorem A ([MS2]). In addition to $(M_1) - (M_3)$ assume that

$$\inf_{s>0, x\in X} |\varphi_s(x)| \ge c_0 > 0. \tag{M_4}$$

Then e^{-tA} , t > 0 are integral operators, and there is a constant $C = C(j, c_s, c_1, c_0)$ such that, for all t > 0 and μ a.e. $x, y \in X$,

$$|e^{-t\Lambda}(x,y)| \le Ct^{-j'}|\varphi_t(x)\varphi_t(y)|, \quad j' = j/(j-1).$$
 (NIE_w)

For the sake of completeness, we recall the proof of Theorem A in Appendix A.

In applications of Theorem A to concrete operators the main difficulty consists in verification of the (L^1, L^1) bound (M_3) . In [MS2], (M_3) is proved for the Schrödinger operator by means of the theory of m-sectorial operators and the Stampacchia criterion in L^2 . However, attempts to apply that argument to $(-\Delta)^{\frac{\alpha}{2}}$, $\alpha < 2$, are quite problematic since $(-\Delta)^{\frac{\alpha}{2}}$ lacks the local properties of $-\Delta$. In [KSSz], we developed a new approach to the proof of (M_3) by means of the Lumer-Phillips Theorem applied to specially constructed C_0 semigroups in L^1 which approximate $\varphi_s e^{-t\Lambda} \varphi_s^{-1}$. Thus, in contrast to [MS2], where the (L^1, L^1) bound is proved using the L^2 theory, here we stay within the L^1 theory. For $\alpha = 2$, the approximation semigroups are constructed replacing |x| by $|x|_{\varepsilon} = \sqrt{|x|^2 + \varepsilon}$, $\varepsilon > 0$, both in the potential and in the weights, see below. For $\alpha < 2$, the construction of the approximation semigroups is more subtle, and is a key observation.

2. We now state our result on $(-\Delta)^{\frac{\alpha}{2}} - V$ in detail. According to the Hardy-Rellich inequality $\|(-\Delta)^{\frac{\alpha}{4}}f\|_2^2 \geq c^{-2}(\frac{\alpha}{2},2,d)\||x|^{-\frac{\alpha}{2}}f\|_2^2$ (see [KPS, Lemma 2.7]) the form difference $\Lambda = (-\Delta)^{\frac{\alpha}{2}} \dot{-} V$ is well defined [Ka, Ch.VI, sect 2.5].

Define β by $\delta c_{\alpha}^{-2} = \frac{\gamma(\beta)}{\gamma(\alpha+\beta)}$, and let $\varphi(x) \equiv \varphi_s(x) = \eta(s^{-\frac{1}{\alpha}}|x|)$, where

$$\eta(r) = \begin{cases} r^{-d+\beta}, & 0 < r < 1, \\ \frac{1}{2}, & r \ge 2. \end{cases}$$

Theorem 1. Under constraints $0 < \delta < 1$ and $0 < \alpha < 2$, $e^{-t\Lambda}$ is an integral operator for each t > 0. The weighted Nash initial estimate

$$e^{-t\Lambda}(x,y) \le ct^{-\frac{d}{\alpha}}\varphi_t(x)\varphi_t(y), \quad c = c_{d,\delta,\alpha},$$

is valid for all t > 0, $x, y \in \mathbb{R}^d - \{0\}$.

Proof of Theorem 1. We verify the assumptions of Theorem A:

- (M_1) follows from the Hardy-Rellich inequality and the uniform Sobolev inequality $\|(-\Delta)^{\frac{\alpha}{4}}f\|_2^2 \ge c_S \|f\|_{2j}^2$, $j = \frac{d}{d-\alpha}$.
 - $(M_2),\,(M_4)$ are immediate from the definition of φ_s .
 - (M_3) Our goal is to prove the following (L^1, L^1) bound:

$$\|\varphi e^{-t\Lambda}\varphi^{-1}h\|_1 \le e^{c\frac{t}{s}}\|h\|_1, \quad h \in L^1 \cap L^2, \ t > 0.$$
 (•)

Proof of (\bullet) . In L^1 define operator $\Lambda^{\varepsilon} = (-\Delta)^{\frac{\alpha}{2}} - V_{\varepsilon}$, $V_{\varepsilon}(x) = \delta c_{\alpha}^{-2} |x|_{\varepsilon}^{-\alpha}$, $\varepsilon > 0$, $D(\Lambda^{\varepsilon}) = D((-\Delta)^{\frac{\alpha}{2}})$,

$$Q = \phi_n \Lambda^{\varepsilon} \phi_n^{-1}, \quad D(Q) = \phi_n D(\Lambda^{\varepsilon}), \quad F_{\varepsilon,n}^t = \phi_n e^{-t\Lambda^{\varepsilon}} \phi_n^{-1}, \quad \phi_n(x) = e^{-\frac{\Lambda^{\varepsilon}}{n}} \varphi(x), \quad n = 1, 2, \dots$$

Here $\phi_n D(\Lambda^{\varepsilon}) := \{\phi_n u \mid u \in D(\Lambda^{\varepsilon})\}$. We also note that $e^{-t(-\Delta)^{\frac{\alpha}{2}}}$, $e^{-t\Lambda^{\varepsilon}} : \mathcal{M} \to \mathcal{M}$ where $\mathcal{M} = C_u$ or $\mathcal{M} = L^1$; and $\varphi = \varphi^{(1)} + \varphi^{(u)}$, $\varphi^{(1)} \in D((-\Delta)^{\frac{\alpha}{2}}_{L^1})$, $\varphi^{(u)} \in D((-\Delta)^{\frac{\alpha}{2}}_{C_u})$. $C_u \equiv C_u(\mathbb{R}^d)$ stands for the Banach space of uniformly continuous functions endowed with the supremum norm.

Since $\phi_n, \phi_n^{-1} \in L^{\infty}$, the operators $Q, F_{\varepsilon,n}^t$ are well defined.

1. Clearly, $F_{\varepsilon,n}^t$ is a quasi bounded C_0 semigroup in L^1 , say e^{-tG} . Set

$$M := \phi_n (1 + (-\Delta)^{\frac{\alpha}{2}})^{-1} [L^1 \cap C_u] = \phi_n (\lambda_{\varepsilon} + \Lambda^{\varepsilon})^{-1} [L^1 \cap C_u], \quad 0 < \lambda_{\varepsilon} \in \rho(-\Lambda^{\varepsilon}).$$

Clearly, M is a dense subspace of L^1 , $M \subset D(Q)$ and $M \subset D(G)$. Moreover, $Q \upharpoonright M \subset G$. Indeed, for $f = \phi_n u \in M$,

$$Gf = s - L^{1} - \lim_{t \downarrow 0} t^{-1} (1 - e^{-tG}) f = \phi_{n} s - L^{1} - \lim_{t \downarrow 0} t^{-1} (1 - e^{-t\Lambda^{\varepsilon}}) u = \phi_{n} \Lambda^{\varepsilon} u = Qf.$$

Thus $Q \upharpoonright M$ is closable and $\tilde{Q} := (Q \upharpoonright M)^{clos} \subset G$.

Next, let us show that $R(\lambda_{\varepsilon} + \tilde{Q})$ is dense in L^1 . If $\langle (\lambda_{\varepsilon} + \tilde{Q})h, v \rangle = 0$ for all $h \in D(\tilde{Q})$ and some $v \in L^{\infty}$, $||v||_{\infty} = 1$, then taking $h \in M$ we would have $\langle (\lambda_{\varepsilon} + Q)\phi_n(\lambda_{\varepsilon} + \Lambda^{\varepsilon})^{-1}g, v \rangle = 0$, $g \in L^1 \cap C_u$, or $\langle \phi_n g, v \rangle = 0$. Choosing $g = e^{\frac{\Delta}{k}}(\chi_m v)$, where $\chi_m \in C_c^{\infty}$ with $\chi_m(x) = 1$ when $x \in B(0, m)$, we would have $\lim_{k \uparrow \infty} \langle \phi_n g, v \rangle = \langle \phi_n \chi_m, |v|^2 \rangle = 0$, and so $v \equiv 0$. Thus, $R(\lambda_{\varepsilon} + \tilde{Q})$ is dense in L^1 .

2. The main step:

Proposition 1. There is a constant $\hat{c} = \hat{c}(d, \alpha, \delta)$ such that

$$\lambda + \tilde{Q}$$
 is accretive whenever $\lambda \geq \hat{c}s^{-1}$.

Taking Proposition 1 for granted we immediately establish the bound

$$\|e^{-tG}\|_{1\to 1} \equiv \|\phi_n e^{-t\Lambda^{\varepsilon}} \phi_n^{-1}\|_{1\to 1} \le e^{\omega t}, \quad \omega = \hat{c}s^{-1}.$$
 (*)

Indeed, the facts: \tilde{Q} is closed and $R(\lambda_{\varepsilon} + \tilde{Q})$ is dense in L^1 together with Proposition 1 imply $R(\lambda_{\varepsilon} + \tilde{Q}) = L^1$. But then, by the Lumer-Phillips Theorem, $\lambda + \tilde{Q}$ is the (minus) generator of a contraction C_0 semigroup, and $\tilde{Q} = G$ due to $\tilde{Q} \subset G$. Incidentally, M is a core of G.

In turn, (\star) easily yields

$$\|\varphi e^{-t\Lambda^{\varepsilon}}\varphi^{-1}h\|_1 \le e^{\omega t}\|h\|_1, \quad h \in L^1 \cap L^2. \tag{**}$$

Indeed, (\star) implies that $\lim_{n\uparrow\infty} \|\phi_n e^{-t\Lambda^{\varepsilon}} v\|_1 \le e^{\omega t} \lim_{n\uparrow\infty} \|\phi_n v\|_1$ for all $v \in L^1 \cap L^2$. But

$$\lim_{n\uparrow\infty}\|\phi_nv\|_1=\lim_{n\uparrow\infty}\langle\varphi,e^{-\frac{\Lambda^\varepsilon}{n}}|v|\rangle=\langle\varphi,|v|\rangle<\infty,$$

$$\lim_{n \uparrow \infty} \|\phi_n e^{-t\Lambda^{\varepsilon}} v\|_1 = \lim_{n \uparrow \infty} \langle \varphi, e^{-\frac{\Lambda^{\varepsilon}}{n}} | e^{-t\Lambda^{\varepsilon}} v | \rangle = \langle \varphi, | e^{-t\Lambda^{\varepsilon}} v | \rangle < \infty.$$

Therefore, taking $v = \varphi^{-1}h$ we arrive at $(\star\star)$. Finally, it is seen that $\varphi e^{-t\Lambda^{\varepsilon}}\varphi^{-1}$ preserves positivity, so (\bullet) follows from $(\star\star)$ by noticing that $e^{-t\Lambda^{\varepsilon}}|g|\uparrow e^{-t\Lambda}|g|\mathcal{L}^d$ a.e.

Let us write down a simple consequence of $(\star\star)$:

Corollary 1. For all t > 0, $x \in \mathbb{R}^d - \{0\}$ and all small $\varepsilon > 0$, there is a constant \hat{c} , such that $e^{-t\Lambda^{\varepsilon}}\varphi_t \leq e^{\hat{c}}\varphi_t$ and $\langle e^{-t\Lambda^{\varepsilon}}(x,\cdot)\rangle \leq 2e^{\hat{c}}\varphi_t(x)$.

Proof of Proposition 1. First we note that, for $f = \phi_n u \in M$,

$$\langle Qf, \frac{f}{|f|} \rangle = \langle \phi_n \Lambda^{\varepsilon} u, \frac{f}{|f|} \rangle = \lim_{t \downarrow 0} t^{-1} \langle \phi_n (1 - e^{-t\Lambda^{\varepsilon}}) u, \frac{f}{|f|} \rangle,$$

$$\operatorname{Re} \langle Qf, \frac{f}{|f|} \rangle \geq \lim_{t \downarrow 0} t^{-1} \langle (1 - e^{-t\Lambda^{\varepsilon}}) | u |, \phi_n \rangle$$

$$= \langle \Lambda^{\varepsilon} e^{-\frac{\Lambda^{\varepsilon}}{n}} | u |, \varphi \rangle.$$

Let us emphasize that $e^{-t\Lambda^{\varepsilon}}$ is a holomorphic semigroup due to the Hille Perturbation Theorem (see e.g. [Ka, Ch. IX, sect. 2.2]).

We are going to estimate $J := \langle \Lambda^{\varepsilon} e^{-\frac{\Lambda^{\varepsilon}}{n}} | u |, \varphi \rangle$ from below using the equality

$$(-\Delta)^{\frac{\alpha}{2}}\varphi = -I_{2-\alpha}\Delta\varphi,$$

where $I_{\nu} \equiv (-\Delta)^{-\frac{\nu}{2}}$.

Using the equality $(-\Delta)^{\frac{\alpha}{2}}\tilde{\varphi}_1 = V\tilde{\varphi}_1$, where $\tilde{\varphi}_1(x) = |x|^{-d+\beta}$ (see e.g. [KPS]), we have

$$(-\Delta)^{\frac{\alpha}{2}}\varphi_1 = V\tilde{\varphi}_1 - I_{2-\alpha}\Delta(\varphi_1 - \tilde{\varphi}_1) = V\tilde{\varphi}_1 - I_{2-\alpha}\mathbf{1}_{B^c(0,1)}\Delta(\varphi_1 - \tilde{\varphi}_1). \quad B^c(0,1) := \mathbb{R}^d - B(0,1).$$

Routine calculation shows that $-I_{2-\alpha}(\mathbf{1}_{B^c(0,1)}\Delta(\varphi_1-\tilde{\varphi}_1)\geq -C_1$ for a constant C_1 .

Since $\Lambda^{\varepsilon}\varphi_1 = (-\Delta)^{\frac{\alpha}{2}}\varphi_1 - V_{\varepsilon}\varphi_1$ and $V\tilde{\varphi}_1 - V_{\varepsilon}\varphi_1 \ge -V_{\varepsilon}(\varphi_1 - \tilde{\varphi}_1) \ge -\delta c_{\alpha}^{-2}$, we obtain by scaling the bound

$$J \ge -(\delta c_{\alpha}^{-2} + C_1)s^{-1} \|e^{-\frac{\Lambda^{\varepsilon}}{n}}\|_{1 \to 1} \|\phi_n^{-1} f\|_{1},$$

or due to $\phi_n \ge \frac{1}{2}$,

$$J \ge -2Cs^{-1} \|e^{-\frac{\Lambda^{\varepsilon}}{n}}\|_{1\to 1} \|f\|_{1}, \quad C = C_{1} + \delta c_{\alpha}^{-2}.$$

Noticing that $\|e^{-\frac{\Lambda^{\varepsilon}}{n}}\|_{1\to 1} \le e^{\delta c_{\alpha}^{-2} \varepsilon^{-2} n^{-1}} = 1 + o(n)$ and taking $\lambda = 3Cs^{-1}$ we arrive at

$$\operatorname{Re}\langle (\lambda + Q)f, \frac{f}{|f|} \rangle \ge 0 \qquad f \in M.$$

Clearly, the latter holds for all $f \in D(\tilde{Q})$.

The proof of (\bullet) is completed. We have verified all the assumptions of (M_1) - (M_4) of Theorem A. The latter now yields the assertion of Theorem 1.

Having at hand Theorem 1 and Corollary 1, it is a simple matter to obtain the upper and lower bounds of the form

$$e^{-t\Lambda}(x,y) \approx e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y)\varphi_t(x)\varphi_t(y).$$

Here $e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y) \approx t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}}$. $(a(z) \approx b(z) \text{ means that } c^{-1}b(z) \leq a(z) \leq cb(z)$ for some constant c>1 and all admissible z).

Proof of the upper bound. $e^{-t\Lambda}(x,y) \leq Ce^{-tA}(x,y)\varphi_t(x)\varphi_t(y)$ $(t>0,x,y\neq 0)$. (For brevity here and below $(-\Delta)^{\frac{\alpha}{2}}=:A$.)

By scaling, it suffices consider t = 1. Since $e^{-A}(x, y) \approx 1 \wedge |x - y|^{-d - \alpha}$ $(x \neq y)$, Theorem 1 yields, for $|x|, |y| \leq 2R$,

$$e^{-\Lambda^{\varepsilon}}(x,y) \le C_R e^{-A}(x,y)\varphi(x)\varphi(y), \quad (\varphi \equiv \varphi_1)$$

By symmetry, it remains to prove this estimate for $|x| \le |y|$, |y| > 2R. First we note that for $|x| \le |y|$, |y| > 2R, $|z| \le R$ and $0 \le \tau < 1$,

$$e^{-(1-\tau)A}(z,y) \le e^{-A}(x,y).$$

Thus, by the Duhamel formula $e^{-\Lambda^{\varepsilon}} = e^{-A} + \int_0^1 e^{-\tau \Lambda^{\varepsilon}} V_{\varepsilon} e^{-(1-\tau)A} d\tau$,

$$e^{-\Lambda^{\varepsilon}}(x,y) \leq e^{-A}(x,y) \left(1 + \int_{0}^{1} e^{-\tau\Lambda^{\varepsilon}} V_{\varepsilon}(x) d\tau\right) + \int_{0}^{1} \langle e^{-\tau\Lambda^{\varepsilon}}(x,z) V_{\varepsilon}(z) \mathbf{1}_{B^{c}(0,R)}(z) e^{-(1-\tau)A}(z,y) \rangle_{z} d\tau$$

$$\leq e^{-A}(x,y) \left(1 + \int_{0}^{1} e^{-\tau\Lambda^{\varepsilon}} V_{\varepsilon}(x) d\tau\right) + V(R) \int_{0}^{1} \langle e^{-\tau\Lambda^{\varepsilon}}(x,z) e^{-(1-\tau)A}(z,y) \rangle_{z} d\tau.$$

Now fix R by $\delta c_{\alpha}^{-2} R^{-\alpha} = \frac{1}{2}$. Then

$$V(R) \int_0^1 \langle e^{-\tau \Lambda^{\varepsilon}}(x,z) e^{-(1-\tau)A}(z,y) \rangle_z d\tau \leq \frac{1}{2} \int_0^1 \langle e^{-\tau \Lambda^{\varepsilon}}(x,z) e^{-(1-\tau)\Lambda^{\varepsilon}}(z,y) \rangle_z d\tau = \frac{1}{2} e^{-\Lambda^{\varepsilon}}(x,y),$$

and so

$$\frac{1}{2}e^{-\Lambda^{\varepsilon}}(x,y) \le e^{-A}(x,y) \left(1 + \int_{0}^{1} e^{-\tau\Lambda^{\varepsilon}} V_{\varepsilon}(x) d\tau\right).$$

Next, by the Duhamel formula and Corollary 1,

$$1 + \int_0^1 e^{-\tau \Lambda^{\varepsilon}} V_{\varepsilon}(x) d\tau = \langle e^{-\Lambda^{\varepsilon}}(x, \cdot) \rangle \le 2e^{\hat{c}} \varphi(x),$$

and hence $e^{-\Lambda^{\varepsilon}}(x,y) \leq 4e^{\hat{c}}e^{-A}(x,y)\varphi(x) \leq 8e^{\hat{c}}e^{-A}(x,y)\varphi(x)\varphi(y)$.

Finally, setting $C = C_R \vee (8e^{\hat{c}})$ and using $e^{-\Lambda^{\varepsilon}}|f| \uparrow e^{-\Lambda}|f|$ we end the proof of the upper bound.

Proof of the lower bound. $e^{-t\Lambda}(x,y) \ge Ce^{-tA}(x,y)\varphi_t(x)\varphi_t(y)$ $(C>0, x,y\neq 0).$

Proposition 2. Define $g = \varphi h$, $\varphi \equiv \varphi_s$, $0 \le h \in S$ -the L.Schwartz space of test functions. There is a constant $\hat{\mu} > 0$ such that, for all $0 < t \le s$,

$$e^{-\frac{\hat{\mu}}{s}t}\langle g\rangle \le \langle \varphi e^{-t\Lambda}\varphi^{-1}g\rangle$$

Proof of Proposition 2. Set $g_n = \phi_n h$, $\phi_n(x) = e^{-\frac{\Lambda^{\varepsilon}}{n}} \varphi(x)$, $\varphi \equiv \varphi_s$. Let $\mu > 0$ be a constant. Then $(\mu = \frac{\hat{\mu}}{s})$

$$\langle g_n \rangle - \langle \phi_n e^{-t(\Lambda^{\varepsilon} - \mu)} h \rangle = -\mu \int_0^t \langle \varphi, e^{-\tau(\Lambda^{\varepsilon} - \mu)} e^{-\frac{\Lambda^{\varepsilon}}{n}} h \rangle d\tau + \int_0^t \langle \varphi, \Lambda^{\varepsilon} e^{-\tau(\Lambda^{\varepsilon} - \mu)} e^{-\frac{\Lambda^{\varepsilon}}{n}} h \rangle d\tau.$$

Note that $\Lambda^{\varepsilon}\varphi = \Lambda^{\varepsilon}\tilde{\varphi} + \Lambda^{\varepsilon}(\varphi - \tilde{\varphi}) = \mathbf{1}_{B(0,1)}(V - V_{\varepsilon})\varphi + v_{\varepsilon}$, where $\tilde{\varphi}(x) = (s^{-\frac{1}{\alpha}}|x|)^{-d+\beta}$. Routine calculation shows that $||v_{\varepsilon}||_{\infty} \leq \frac{\mu_1}{s}$, $\mu_1 \neq \mu_1(\varepsilon)$. Thus

$$\int_0^t \langle v_\varepsilon, e^{-\tau(\Lambda^\varepsilon - \mu)} e^{-\frac{\Lambda^\varepsilon}{n}} h \rangle d\tau \leq \frac{\mu_1}{s} \int_0^t \langle e^{-\tau(\Lambda^\varepsilon - \mu)} e^{-\frac{\Lambda^\varepsilon}{n}} h \rangle d\tau \leq \frac{2\mu_1}{s} \int_0^t \langle \varphi, e^{-\tau(\Lambda^\varepsilon - \mu)} e^{-\frac{\Lambda^\varepsilon}{n}} h \rangle d\tau.$$

Taking $\hat{\mu} = 2\mu_1$, we have

$$\langle g_n \rangle - \langle \phi_n e^{-t(\Lambda^{\varepsilon} - \mu)} h \rangle \le \int_0^t \langle \mathbf{1}_{B(0,1)} (V - V_{\varepsilon}) \varphi, e^{-(\tau + \frac{1}{n})\Lambda^{\varepsilon}} h \rangle e^{\mu \tau} d\tau$$
, or sending $n \to \infty$,

$$\langle g \rangle - e^{\frac{\hat{\mu}}{s}t} \langle \varphi e^{-t\Lambda^{\varepsilon}} h \rangle \le e^{\hat{\mu}} \int_0^t \langle \mathbf{1}_{B(0,1)} (V - V_{\varepsilon}) \varphi, e^{-\tau \Lambda^{\varepsilon}} h \rangle d\tau.$$

Set $W_{\varepsilon} = 1_{B(0,1)}(V - V_{\varepsilon})\varphi^2$ and $F_{\varepsilon}^{\tau} = \varphi e^{-\tau \Lambda^{\varepsilon}}\varphi^{-1}$. Note that $W_{\varepsilon} \in L^1$ due to $2(d - \beta) + \alpha < d$, and $\|F_{\varepsilon}^{\tau}f\|_1 \le e^{\frac{\hat{c}}{s}\tau}\|f\|_1$, $f \in L^1$ due to Proposition 1. Therefore,

$$\int_0^t \langle \mathbf{1}_{B(0,1)}(V - V_{\varepsilon})\varphi, e^{-\tau \Lambda^{\varepsilon}} h \rangle d\tau = \int_0^t \langle F_{\varepsilon}^{\tau} W_{\varepsilon}, \varphi^{-1} h \rangle \leq 2e^{\hat{c}} s \|W_{\varepsilon}\|_1 \|h\|_{\infty} \to 0 \text{ as } \varepsilon \downarrow 0.$$

We also need the following consequence of the upper bound and Proposition 2.

Corollary 2. Fix t > 0. Set $g := \varphi h$, $\varphi = \varphi_t$, $0 \le h \in \mathcal{S}$ with sprt $h \in B(0, R_0)$ for some $R_0 < \infty$. Then there are $0 < r_t < R_0 \lor t^{\frac{\alpha}{2}} < R_{t,R_0}$ such that, for all $r \in [0, r_t]$ and $R \in [2R_{t,R_0}, \infty[$,

$$e^{-\hat{\mu}-1}\langle g\rangle \leq \langle \mathbf{1}_{R,r}\varphi e^{-t\Lambda}\varphi^{-1}g\rangle, \qquad \mathbf{1}_{R,r}:=\mathbf{1}_{B(0,R)}-\mathbf{1}_{B(0,r)}, \ \mathbf{1}_{R,0}:=\mathbf{1}_{B(0,R)}.$$

In particular, $e^{-\hat{\mu}-1}\varphi_t(x) \leq e^{-t\Lambda}\varphi_t \mathbf{1}_{R,r}(x)$ for every $x \neq 0$.

Proof of Corollary 2. By the upper bound,

$$\langle \mathbf{1}_{B(0,r)} \varphi e^{-t\Lambda} \varphi^{-1} g \rangle \leq C \langle \mathbf{1}_{B(0,r)} \varphi^{2}, e^{-tA} g \rangle$$

$$\leq C C_{1} t^{-\frac{d}{\alpha}} \| \mathbf{1}_{B(0,r)} \varphi^{2} \|_{1} \| g \|_{1}$$

$$= o(r_{t}) \| g \|_{1}, \quad o(r_{t}) \to 0 \text{ as } r_{t} \downarrow 0;$$

$$\langle \mathbf{1}_{B^{c}(0,R)} \varphi e^{-t\Lambda} \varphi^{-1} g \rangle \leq C \langle \mathbf{1}_{B^{c}(0,R)} \varphi^{2}, e^{-tA} g \rangle$$

$$\leq C \langle e^{-tA} \mathbf{1}_{B^{c}(0,R)}, g \mathbf{1}_{B(0,R_{0})} \rangle, \text{ where } R \geq 2R_{t,R_{0}} \geq 2(R_{0} \vee t^{\frac{\alpha}{2}})$$

$$\leq C \sup_{x \in B(0,R_{0})} e^{-tA} \mathbf{1}_{B^{c}(0,R)}(x) \| g \|_{1}$$

$$\leq C \tilde{C} C_{d} R_{t,R_{0}}^{-\frac{\alpha}{2}} \| g \|_{1}$$

$$= o(R_{t,R_{0}}) \| g \|_{1}, \quad o(R_{t,R_{0}}) \to 0 \text{ as } R_{t,R_{0}} \uparrow \infty$$

due to $e^{-tA}(x,y) \leq \tilde{C}(t|x-y|^{-d-\alpha} \wedge t^{-\frac{d}{\alpha}}) \leq \tilde{C}2^{d+\frac{\alpha}{2}}|y|^{-d-\frac{\alpha}{2}}$ if $|x| \leq R_0$ and $|y| \geq R$. We are left to apply Proposition 2.

Now we are in position to apply the so-called 3q argument. Set $q_t(x,\cdot) = e^{-t\Lambda}(x,\cdot)\varphi_t^{-1}(x)\varphi_t^{-1}(\cdot)$. (a) Let $x,y \in B^c(0,1), x \neq y$. Clearly,

$$q_3(x,y) \ge \varphi_3^{-1}(x)\varphi_3^{-1}(y)e^{-3\Lambda}(x,y) \ge e^{-3\Lambda}(x,y) \ge e^{-3\Lambda}(x,y).$$

(b) Let $x, y \in B(0,1)$, $0 < |x| \le |y|$. By the reproduction property, since $e^{-t\Lambda}$ is positivity preserving,

$$\begin{split} q_3(x,y) &\geq \varphi_3^{-1}(x)\varphi_3^{-1}(y)\langle e^{-\Lambda}(x,\cdot)e^{-2\Lambda}(\cdot,y)\mathbf{1}_{R,r}(\cdot)\rangle \\ &= \varphi_3^{-1}(x)\varphi_3^{-1}(y)\langle e^{-\Lambda}(x,\cdot)\varphi_1(\cdot)\varphi_1^{-1}(\cdot)e^{-2\Lambda}(\cdot,y)\mathbf{1}_{R,r}(\cdot)\rangle \\ &\geq \varphi_3^{-1}(x)\varphi_3^{-1}(y)\langle e^{-\Lambda}(x,\cdot)\varphi_1(\cdot)\mathbf{1}_{R,r}(\cdot)\rangle \inf_{r\leq |z|\leq R} \varphi_1^{-1}(z)e^{-2\Lambda}(z,y) \\ & (\text{here we are using Corollary 2}) \\ &\geq e^{-\hat{\mu}-1}\varphi_3^{-1}(x)\varphi_1(x)\varphi_1^{-1}(r)\varphi_3^{-1}(y)\inf_{r\leq |z|\leq R} e^{-2\Lambda}(z,y) \\ &= C_{r,R}\varphi_3^{-1}(y)\inf_{r\leq |z|\leq R} e^{-2\Lambda}(y,z); \\ e^{-2\Lambda}(y,z) &\geq \langle e^{-\Lambda}(y,\cdot)\varphi_1(\cdot)\varphi_1^{-1}(\cdot)e^{-\Lambda}(\cdot,z)\mathbf{1}_{R,r}(\cdot)\rangle \\ & (\text{again we are using Corollary 2}) \\ &\geq e^{-\hat{\mu}-1}\varphi_1(y)\varphi_1^{-1}(r)\inf_{r\leq |z|,|\cdot|\leq R} e^{-\Lambda}(\cdot,z). \end{split}$$

Therefore

$$q_{3}(x,y) \geq C'_{r,R} \inf_{r \leq |z|,|\cdot| \leq R} e^{-A}(\cdot,z) \geq C'''_{r,R} e^{-3A}(x,y).$$
(c) Let $x \in B(0,1), x \neq 0, y \in B^{c}(0,1)$. Then
$$q_{3}(x,y) \geq \varphi_{3}^{-1}(x)\varphi_{3}^{-1}(y)\langle e^{-\Lambda}(x,\cdot)\varphi_{1}(\cdot)\varphi_{1}^{-1}(\cdot)e^{-2A}(\cdot,y)\mathbf{1}_{R,r}(\cdot)\rangle$$

$$\geq \varphi_{1}^{-1}(x)\langle e^{-\Lambda}(x,\cdot)\varphi_{1}(\cdot)\varphi_{1}^{-1}(\cdot)e^{-2A}(\cdot,y)\mathbf{1}_{R,r}(\cdot)\rangle$$

$$\geq e^{-\hat{\mu}-1} \inf_{r < |z| < R} \varphi_{1}^{-1}(z)e^{-2A}(z,y) \geq e^{-\hat{\mu}-1}\varphi_{1}^{-1}(r) \inf_{r < |z| < R} e^{-2A}(z,y)$$

$$\geq C_{R,r}e^{-3A}(x,y).$$

Finally, by (a),(b),(c), $q_3(x,y) \ge Ce^{-3A}(x,y)$ or $e^{-3\Lambda}(x,y) \ge Ce^{-3A}(x,y)\varphi_3(x)\varphi_3(y)$. The scaling argument ends the proof of the lower bound.

APPENDIX A. PROOF OF THEOREM A

Set $L_{\varphi}^2 = L^2(X, \varphi^2 d\mu)$, and define a unitary map $\Phi: L_{\varphi}^2 \to L^2$ by $\Phi f = \varphi f$. Then the operator $\Lambda_{\varphi} = \Phi^{-1} \Lambda \Phi$ of domain $D(\Lambda_{\varphi}) = \Phi^{-1} D(\Lambda)$ is selfadjont on L_{φ}^2 and $\|e^{-t\Lambda_{\varphi}}\|_{2\to 2, \varphi} = \|e^{-t\Lambda}\|_{2\to 2} \le 1$ for all $t \ge 0$. Here and below the subscript φ indicates that the corresponding quantities are related to the measure $\varphi^2 d\mu$.

Let $f = \varphi^{-1}h$, $h \in L_{com}^{\infty}$, and so $f \in L_{\varphi}^{2} \cap L_{\varphi}^{1}$ by (M_{2}) . Let $u_{t} = e^{-t\Lambda_{\varphi}}f$. Then $\varphi u_{t} = e^{-t\Lambda}\varphi f$ and

$$\langle \Lambda_{\varphi} u_t, u_t \rangle_{\varphi} = \|\Lambda^{\frac{1}{2}} \varphi u_t\|_2^2 \ge c_S \|\varphi u_t\|_{2j}^2$$

$$\ge c_S \|\varphi u_t\|_2^{2 + \frac{2}{j'}} \|\varphi u_t\|_1^{-\frac{2}{j'}}$$

$$= c_S \langle u_t, u_t \rangle_{\varphi}^{1 + \frac{1}{j'}} \|\varphi^{-1} \varphi e^{-t\Lambda} \varphi^{-1} \varphi^2 f\|_1^{-\frac{2}{j'}},$$

where (M_1) and Hölder's inequality have been used.

Clearly, $-\frac{1}{2}\frac{d}{dt}\langle u_t, u_t\rangle_{\varphi} = \langle \Lambda_{\varphi}u_t, u_t\rangle_{\varphi}$. Setting $w := \langle u_t, u_t\rangle_{\varphi}$ and using (M_4) we have

$$\frac{d}{dt}w^{-\frac{1}{j'}} \ge \frac{2}{j'}c_S(c_0^{-1}\|\varphi e^{-t\Lambda}\varphi^{-1}\varphi^2 f\|_1)^{-\frac{2}{j'}}.$$

By our choice of $f, \varphi^2 f = \varphi h \in \mathcal{D}$. Therefore we can apply (M_3) and obtain

$$\frac{d}{dt}w^{-\frac{1}{j'}} \ge \frac{2}{j'}c_S(c_1c_0^{-1}||f||_{1,\varphi})^{-\frac{2}{j'}}, \ t \le s.$$

Integrating this inequality over [0, t] gives

$$||e^{-t\Lambda_{\varphi_s}}f||_{2,\varphi_s} \le ct^{-\frac{j'}{2}}||f||_{1,\varphi_s}, \ t \le s.$$

Since $f \in \varphi^{-1}L_{com}^{\infty}$ and $\varphi^{-1}L_{com}^{\infty}$ is a dense subspace of L_{φ}^{1} , the last inequality yields

$$||e^{-t\Lambda_{\varphi_s}}||_{1\to 2, \varphi_s} \le ct^{-\frac{j'}{2}}, \ t \le s$$

and (NIE_w) follows.

References

[BGJP] K. Bogdan, T. Grzywny, T. Jakubowski and D. Pilarczyk, Fractional Laplacian with Hardy potential, Comm. Partial Differential Equations, 44 (2019), p. 20-50.

[CKSV] S. Cho, P. Kim, R. Song, Z. Vondraček, Factorization and estimates of Dirichlet heat kernels for non-local operators with critical killings, arXiv:1809.01782 (2018), 43 p.

[JW] T. Jakubowski and J. Wang. Heat kernel estimates for fractional Schrödinger operators with negative Hardy potential, arXiv:1809.02425 (2018), 26 p.

[Ka] T. Kato. Perturbation Theory for Linear Operators. Springer-Verlag Berlin Heidelberg, 1995.

[KSSz] D. Kinzebulatov, Yu. A. Semënov and K. Szczypkowski. Heat kernel of fractional Laplacian with Hardy drift via desingularizing weights. Preprint, arXiv:1904.07363 (2019), 19 p.

[KPS] V.F. Kovalenko, M. A. Perelmuter and Yu. A. Semënov, Schrödinger operators with $L_W^{1/2}(\mathbb{R}^l)$ -potentials, J. Math. Phys., **22** (1981), p. 1033-1044.

[MS0] P.D. Milman and Yu. A. Semënov. Desingularizing weights and heat kernel bounds, Preprint (1998).

[MS1] P. D. Milman and Yu. A. Semënov. Heat kernel bounds and desingularizing weights, J. Funct. Anal., 202 (2003), p. 1-24.

[MS2] P. D. Milman and Yu. A. Semënov. Global heat kernel bounds via desingularizing weights, J. Funct. Anal., 212 (2004), p. 373-398.

Université Laval, Département de mathématiques et de statistique, 1045 av. de la Médecine, Québec, QC, G1V 0A6, Canada

E-mail address: damir.kinzebulatov@mat.ulaval.ca

University of Toronto, Department of Mathematics, 40 St. George Str, Toronto, ON, M5S 2E4, Canada

E-mail address: semenov.yu.a@gmail.com