HEAT KERNEL OF FRACTIONAL LAPLACIAN WITH HARDY DRIFT VIA DESINGULARIZING WEIGHTS

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Abstract. We establish sharp two-sided bounds on the heat kernel of the fractional Laplacian, perturbed by a drift having critical-order singularity, using the method of desingularizing weights.

1. In 1998, Milman and Semënov [MS0] introduced the method of desingularizing weights to establish two-sided weighted bounds on the heat kernel of the Schrödinger operator $-\Delta - V$, $V(x) = \delta\big(\frac{d-2}{2}\big)^2|x|^{-2}$, $0<\delta\leq 1$ in $L^2(\mathbb{R}^d, dx)$, $d \geq 3$ [MS1, MS2]. The corresponding C_0 semigroup is not ultra-contractive, but becomes one after transferring it to an appropriate weighted space.

In this paper we use the desingularization method to obtain sharp two-sided weighted bounds on the heat kernel of the operator

$$(-\Delta)^{\frac{\alpha}{2}} + b\cdot\nabla, \quad b(x) = c|x|^{-\alpha}x, \quad c>0, \quad 1<\alpha<2.$$

The vector field b has a model critical-order singularity at $x=0$. The standard upper bound in terms of the heat kernel of $(-\Delta)^{\frac{\alpha}{2}}$ does not hold.

The desingularization method rests on two assumptions: the Sobolev embedding property, and a "desingularizing" (L^1, L^1) bound on the weighted semigroup. Namely, let X be a locally compact space and μ a σ-finite Borel measure on X. Set

$$\langle u, v\rangle = \langle u\bar{v}\rangle := \int_X u\bar{v}d\mu.$$

Let $-\Lambda$ be the generator of a C_0 contraction semigroup $e^{-t\Lambda}$, $t>0$, in the (complex) Banach space $L^p = L^p(X,\mu)$ for any $p\in[2,\infty[$. Assume that Λ, Λ^* possess the Sobolev-type embedding property: There are constants $j>1$ and $c_S>0$ such that

$$\operatorname{Re}\langle \Lambda f, f\rangle \geq c_S\|f\|_{2j}^2, \quad f\in D(\Lambda), \tag{N_1}$$

$$\operatorname{Re}\langle \Lambda^* g, g\rangle \geq c_S\|g\|_{2j}^2, \quad g\in D(\Lambda^*), \tag{N_1^*}$$

where $\|\cdot\|_p = \|\cdot\|_{L^p}$, but $e^{-t\Lambda}\restriction L^1\cap L^p$ cannot be extended by continuity to a bounded map on L^1 and the ultra-contraction estimate

$$\|e^{-t\Lambda}f\|_\infty \leq c(t)\|f\|_1, \;\; f\in L^1\cap L^\infty, \; t>0$$

is not valid.

In this case we will be assuming that there exists a family of real valued weights $\varphi = \{\varphi_s\}_{s>0}$ on X such that, for all $s>0$,

$$\varphi_s, \; 1/\varphi_s \in L^2_{\rm loc}(X,\mu) \tag{N_2}$$

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and there exists constant c_1, independent of s such that, for all $0 < t \le s$

$$\|\varphi_s e^{-t\Lambda}\varphi_s^{-1} f\|_1 \le c_1 \|f\|_1, \quad f \in \mathcal{D} := \varphi_s L^\infty_{\rm com}(X,\mu). \tag{N_3}$$

The following general theorem is the point of departure for the desingularization method in the non-selfadjoint setting:

Theorem A. *In addition to* (N_1)-(N_3) *assume that*

$$\inf_{s>0,\, x \in X} |\varphi_s(x)| \ge c_0 > 0. \tag{N_4}$$

Then, for each $t > 0$, $e^{-t\Lambda}$ *is integral operator, and there is a constant* $C = C(j, c_s, c_1, c_0)$ *such that the weighted Nash initial estimate*

$$|e^{-t\Lambda}(x,y)| \le C t^{-j'} |\varphi_t(y)|, \quad j' = j/(j-1). \tag{NIE_w}$$

is valid for μ *a.e.* $x, y \in X$.

Proof. 1. There exists a constant c_2 such that the inequality

$$\|e^{-t\Lambda}\varphi f\|_2 \le c_2 t^{-\frac{j'}{2}} \|\varphi^2 f\|_1 \quad (\varphi \equiv \varphi_s) \tag{$*$}$$

is valid for all $f \in \varphi^{-1} L^\infty_{com}$ and $0 < t \le s$.

Indeed, set $L^2_\varphi = L^2(X, \varphi^2 d\mu)$, define a unitary map $\Phi : L^2_\varphi \to L^2$ by $\Phi f = \varphi f$. Set $\Lambda_\varphi = \Phi^{-1}\Lambda\Phi$ of domain $D(\Lambda_\varphi) = \Phi^{-1} D(\Lambda)$. Then $\|e^{-t\Lambda_\varphi}\|_{2,\varphi \to 2,\varphi} = \|e^{-t\Lambda}\|_{2\to 2} \le 1$ for all $t \ge 0$. Here and below $\|\cdot\|_{p\to q} = \|\cdot\|_{L^p \to L^q}$, and the subscript φ indicates that the corresponding quantities are related to the measure $\varphi^2 d\mu$.

Let $f = \varphi^{-1} h$, $h \in L^\infty_{com}$, and so $f \in L^2_\varphi \cap L^1_\varphi$ by (N_2). Let $u_t = e^{-t\Lambda_\varphi} f$. Then $\varphi u_t = e^{-t\Lambda}\varphi f$ and

$$\begin{aligned} \mathrm{Re}\langle \Lambda_\varphi u_t, u_t\rangle_\varphi &\ge c_S \|\varphi u_t\|_{2j}^2 \\ &\ge c_S \|\varphi u_t\|_2^{2+\frac{2}{j'}} \|\varphi u_t\|_1^{-\frac{2}{j'}} \\ &= c_S \langle u_t, u_t\rangle_\varphi^{1+\frac{1}{j'}} \|\varphi^{-1}\varphi e^{-t\Lambda}\varphi^{-1}\varphi^2 f\|_1^{-\frac{2}{j'}}, \end{aligned}$$

where (N_1) and Hölder's inequality have been used.

Clearly, $-\frac{1}{2}\frac{d}{dt}\langle u_t, u_t\rangle_\varphi = \mathrm{Re}\langle \Lambda_\varphi u_t, u_t\rangle_\varphi$. Setting $w := \langle u_t, u_t\rangle_\varphi$ and using (N_4) we have

$$\frac{d}{dt} w^{-\frac{1}{j'}} \ge \frac{2}{j'} c_S (c_0^{-1} \|\varphi e^{-t\Lambda}\varphi^{-1}\varphi^2 f\|_1)^{-\frac{2}{j'}}.$$

By our choice of f, $\varphi^2 f = \varphi h \in \mathcal{D}$. Therefore we can apply (N_3) and obtain

$$\frac{d}{dt} w^{-\frac{1}{j'}} \ge \frac{2}{j'} c_S (c_1 c_0^{-1} \|f\|_{1,\varphi})^{-\frac{2}{j'}}, \quad t \le s.$$

Integrating this inequality over $[0,t]$ gives

$$\|e^{-t\Lambda_\varphi} f\|_{2,\varphi} \le c_2 t^{-\frac{j'}{2}} \|f\|_{1,\varphi}, \quad t \le s,$$

or

$$\|e^{-t\Lambda}\varphi f\|_2 \le c_2 t^{-\frac{j'}{2}} \|f\|_{1,\varphi},$$

i.e. $(*)$.

2. Next, we claim that there is a constant $c_3 > 0$ such that

$$\|e^{-t\Lambda}\|_{2\to\infty} \le c_3 t^{-\frac{j'}{2}}. \tag{$**$}$$

Indeed, since Λ is accretive, Λ^* is accretive as well. Since $e^{-t\Lambda}$ is a contraction on all L^p, $2 \le p < \infty$, we have

$$\|e^{-t\Lambda^*}g\|_1 \le \|g\|_1, \quad g \in L^2 \cap L^1.$$

Thus, arguing as above (with $\varphi \equiv 1$) and using (N_1^*), we have $\|e^{-t\Lambda^*}\|_{1\to 2} \le c_3 t^{-\frac{j'}{2}}$, and so via duality $(**)$.

3. Combining $(*)$ and $(**)$, we obtain, for all $f \in \varphi^{-1}L^\infty_{\rm com}$,

$$\|e^{-2t\Lambda}\varphi f\|_\infty \le c_3 t^{-\frac{j'}{2}}\|e^{-t\Lambda}\varphi f\|_2 \le c_3 c_2 t^{-j'}\|\varphi^2 f\|_1.$$

The latter yields (after redefinition on a null set) (NIE_w). The proof of Theorem A is completed. $\square$

Remark. (N_1^*) provides the bound $\|e^{-t\Lambda}\|_{2\to\infty} \le ct^{-\frac{j'}{2}}$, needed to prove (NIE_w). There are other means to obtain the (L^2, L^∞) bound, e.g. replacing (N_1^*) by $\mathrm{Re}\langle\Lambda f, |f|^{p-1}\mathrm{sgn}\, f\rangle \ge c_S\|f\|_{pj}^p$, $f \in D(\Lambda)$, for all $p \ge 2$, and then arguing as in [KiS1, proof of Theorem 4.3].

In applications of Theorem A to concrete operators the main difficulty consists in verification of the (L^1, L^1) bound (N_3). In this paper we develop a new approach to the proof of (N_3) for $\Lambda \supset (-\Delta)^{\frac{\alpha}{2}} + c|x|^{-\alpha}x\cdot\nabla$, $c > 0$, by verifying the hypotheses of the Lumer-Phillips Theorem for specially constructed C_0 semigroups approximating $\varphi_s e^{-t\Lambda}\varphi_s^{-1}$ in L^1. This construction of the approximating semigroups is a key observation.

2. We now state our main result concerning $(-\Delta)^{\frac{\alpha}{2}} + c|x|^{-\alpha}x\cdot\nabla$, $1 < \alpha < 2$, $c > 0$, in detail. Let $d \ge 3$. Set

$$c(\alpha, p, d) := \frac{\gamma(\frac{d}{p} - \alpha)}{\gamma(\frac{d}{p})}, \quad \gamma(\alpha) := \frac{2^\alpha \pi^{\frac{d}{2}}\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{d}{2} - \frac{\alpha}{2})}, \quad 1 < p < \frac{d}{\alpha}.$$

Set

$$b(x) := \kappa|x|^{-\alpha}x, \qquad \kappa := \delta(d-\alpha)^{-1}2c^{-2}\left(\frac{\alpha}{2}, 2, d\right), \quad 0 < \delta < 1.$$

Proposition 1. *$\Lambda := (-\Delta)^{\frac{\alpha}{2}} + b\cdot\nabla$, $D(\Lambda) = D((-\Delta)^{\frac{\alpha}{2}}) = \mathcal{W}^{\alpha,2}$, is the (minus) generator of a holomorphic semigroup in L^2.*

We prove Proposition 1 below by showing that $b\cdot\nabla$ is Rellich's perturbation of $(-\Delta)^{\frac{\alpha}{2}}$.

Define β by $\frac{\gamma(\beta)}{(\beta-\alpha)\gamma(\beta-\alpha)} = \kappa$. This choice of β entails that $|x|^{-d+\beta}$ is a Lyapunov function to the formal operator $\Lambda^* = (-\Delta)^{\frac{\alpha}{2}} - \nabla\cdot b$, i.e. $\Lambda^*|x|^{-d+\beta} = 0$, cf. Appendix A.

Let η be a $C^2(]0,\infty[)$ function such that

$$\eta(r) = \begin{cases} r^{-d+\beta}, & 0 < r < 1, \\ \frac{1}{2}, & r \ge 2. \end{cases}$$

Theorem 1. *$e^{-t\Lambda}$ is an integral operator for each $t > 0$; there exists a constant C such that the weighted Nash initial estimate*

$$e^{-t\Lambda}(x,y) \le Ct^{-j'}\varphi_t(y), \quad j' = \frac{d}{\alpha}, \quad \varphi_t(y) = \eta(t^{-\frac{1}{\alpha}}|y|)$$

is valid for all $x, y \in \mathbb{R}^d$, $y \neq 0$ *and* $t > 0$.

Having at hand Theorem 1, we obtain below the following.

Theorem 2. $e^{-t\Lambda}(x,y) \approx e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y)\varphi_t(y), \quad x,y \in \mathbb{R}^d, y \neq 0, \quad t > 0.$

Here $e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y) \approx t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}}$. ($a(z) \approx b(z)$ means that $c^{-1}b(z) \leq a(z) \leq cb(z)$ for some constant $c > 1$ and all admissible z).

Sharp two-sided weighted bounds for the heat kernel of $(-\Delta)^{\frac{\alpha}{2}} - \delta c_\alpha^{-2}|x|^{-\alpha}$, $0 < \alpha < 2$, $0 < \delta \leq 1$ is the subject of [BGJP]. Our method gives a short and transparent operator-theoretic proof of these bounds for $0 < \delta < 1$ [KiS2]. Concerning $(-\Delta)^{\frac{\alpha}{2}} + c|x|^{-\alpha}$, $c > 0$, see [CKSV] and [JW].

1. Proof of Proposition 1

For brevity, write $\|\cdot\| \equiv \|\cdot\|_{2\to 2}$ and $A \equiv (-\Delta)^{\frac{\alpha}{2}}$ in L^2.

Define $T = b \cdot \nabla(\zeta + A)^{-1}$, $\mathrm{Re}\zeta > 0$, and note that

$$
\begin{aligned}
\|T\| &\leq \||b|(\zeta + A)^{-1+\frac{1}{\alpha}}\|\|\nabla(\zeta + A)^{-\frac{1}{\alpha}}\| \\
&\quad \text{(we are using } \|\nabla g\|_2 = \|(-\Delta)^{\frac{1}{2}}g\|_2) \\
&\leq \||b|(\mathrm{Re}\zeta + A)^{-1+\frac{1}{\alpha}}\|\|A^{\frac{1}{\alpha}}(\zeta + A)^{-\frac{1}{\alpha}}\| \\
&\quad \text{(by the Spectral Theorem, } \|A^{\frac{1}{\alpha}}(\zeta + A)^{-\frac{1}{\alpha}}\| \leq 1) \\
&\leq \||b|(-\Delta)^{-\frac{\alpha-1}{2}}\| \\
&\quad \text{(we are using [KPS, Lemma 2.7])} \\
&= \kappa c(\alpha - 1, 2, d) < \delta \ (< 1)
\end{aligned}
$$

because $c(\alpha - 1, 2, d) < (d - \alpha)2^{-1}c^2(\frac{\alpha}{2}, 2, d)$ or, equivalently,

$$
F(\alpha) \equiv (d-\alpha)\Gamma\big(\frac{d-2+2\alpha}{4}\big)\big[\Gamma\big(\frac{d-\alpha}{4}\big)\big]^2 - 4\Gamma\big(\frac{d+2-2\alpha}{4}\big)\big[\Gamma\big(\frac{d+\alpha}{4}\big)\big]^2 > 0
$$

(the latter is due to $\frac{d^2}{dt^2}\log\Gamma(t) \geq 0$ and $F(2) = 0$ $\big((d-2)\Gamma(\frac{d-2}{4}) = 4\Gamma(\frac{d+2}{4})\big)$).

Thus, the Neumann series for $(\zeta + \Lambda)^{-1} = (\zeta + A)^{-1}(1 + T)^{-1}$ converges, and

$$
\|(\zeta + \Lambda)^{-1}\| \leq (1 - \delta)^{-1}|\zeta|^{-1}, \quad \mathrm{Re}\zeta > 0,
$$

i.e. $-\Lambda$ is the generator of a holomorphic semigroup. □

2. Proof of Theorem 1

First, we are going to verify the assumptions of Theorem A for the operators

$$
P^\varepsilon := (-\Delta)^{\frac{\alpha}{2}} + b_\varepsilon \cdot \nabla + U_\varepsilon \quad \text{in } L^2, \qquad D(P^\varepsilon) = D((-\Delta)^{\frac{\alpha}{2}}_2) \equiv \mathcal{W}^{\alpha,2} \ \text{(Bessel potential space)},
$$

where $\varepsilon > 0$,

$$
b_\varepsilon(x) = \kappa|x|_\varepsilon^{-\alpha}x, \quad |x|_\varepsilon := \sqrt{|x|^2 + \varepsilon}, \qquad U_\varepsilon(x) := \alpha\kappa\varepsilon|x|_\varepsilon^{-\alpha-2} \ (> 0),
$$

and for the weights φ_s defined in Theorem 1.

P^ε, $\varepsilon > 0$, is the generator of a C_0 semigroup in L^2 (for example, by the Hille Perturbation Theorem [Ka, Ch. IX, sect. 2.2]). Similarly, $\Lambda^\varepsilon := (-\Delta)^{\frac{\alpha}{2}} + b_\varepsilon \cdot \nabla$ generates a C_0 semigroup in

L^2. Moreover, it is well known that $e^{-t\Lambda^\varepsilon}L^2_+ \subset L^2_+$ and $\|e^{-tP^\varepsilon}f\|_\infty \leq \|e^{-t\Lambda^\varepsilon}|f|\|_\infty \leq \|f\|_\infty$, $f \in L^2 \cap L^\infty$. It follows from (N_1) (see below) that e^{-tP^ε} is a contraction in L^2. In particular, e^{-tP^ε} is a C_0 contraction semigroup in all L^p, $2 \leq p < \infty$.

(N_1): *There is a constant $c > 0$ such that, for all $f \in D(P^\varepsilon)$ and $\varepsilon > 0$,*

$$\mathrm{Re}\langle P^\varepsilon f, f\rangle \geq c\|f\|_{2j}^2, \quad j = \frac{d}{d-\alpha}.$$

Proof. Indeed, $\mathrm{Re}\langle P^\varepsilon f, f\rangle = \|(-\Delta)^{\frac{\alpha}{4}}f\|_2^2 + \kappa \mathrm{Re}\langle |x|_\varepsilon^{-\alpha}x\cdot\nabla f, f\rangle + \langle U_\varepsilon f, f\rangle$ and

$$\kappa\mathrm{Re}\langle |x|_\varepsilon^{-\alpha}x\cdot\nabla f, f\rangle = -\kappa\frac{d-\alpha}{2}\||x|^{-\frac{\alpha}{2}}f\|_2^2 - \frac{1}{2}\langle U_\varepsilon f, f\rangle.$$

Now applying the Hardy-Rellich inequality $\|(-\Delta)^{\frac{\alpha}{4}}f\|_2^2 \geq c^{-2}(\frac{\alpha}{2},2,d)\||x|^{-\frac{\alpha}{2}}f\|_2^2$ (see [KPS, Lemma 2.7]) and the uniform Sobolev inequality $\|(-\Delta)^{\frac{\alpha}{4}}f\|_2^2 \geq c_S\|f\|_{2j}^2$, we obtain (N_1) with $c = (1-\delta)c_S$. □

(N_1^*): *There is a constant $c > 0$ such that, for all $g \in D((P^\varepsilon)^*)$ and $\varepsilon > 0$,*

$$\mathrm{Re}\langle (P^\varepsilon)^* g, g\rangle \geq c\|g\|_{2j}^2.$$

Proof. Since $D((P^\varepsilon)^*) = D((-\Delta)_2^{\frac{\alpha}{2}}) \equiv D(P^\varepsilon)$, (N_1^*) is a consequence of (N_1). □

$(N_2), (N_4)$: $\varphi^{\pm 1} \in L^2_{\mathrm{loc}}$ and $\inf_{s>0,\, x\in\mathbb{R}^d}\varphi_s(x) \geq \frac{1}{2}$. By the construction of φ, (N_2),(N_4) are valid.

(N_3): *There exists a constant $\omega > 0$ such that, for all $0 < t \leq s$*

$$\|\varphi_s e^{-tP^\varepsilon}\varphi_s^{-1}h\|_1 \leq e^{\omega\frac{t}{s}}\|h\|_1, \quad h \in L^1 \cap L^2, \quad \omega \neq \omega(\varepsilon).$$

See the proof of (N_3) below.

Thus, Theorem A applies and yields

$$\|e^{-tP^\varepsilon}\varphi_t f\|_\infty \leq Ct^{-j'}\|\varphi_t^2 f\|_1, \quad C \neq C(\varepsilon), \quad f \in L^1_\varphi. \tag{$\star$}$$

It remains to take $\varepsilon \downarrow 0$ in $(\star)$. In Remark 1 we prove that $e^{-tP^\varepsilon} \to e^{-t\Lambda}$ strongly in L^2. The latter and $(\star)$ clearly yield $\|e^{-t\Lambda}\varphi_t f\|_\infty \leq Ct^{-j'}\|\varphi_t^2 f\|_1$ and hence Theorem 1.

Proof of (N_3). In L^1 define operators

$$P^\varepsilon := (-\Delta)^{\frac{\alpha}{2}} + b_\varepsilon\cdot\nabla + U_\varepsilon, \qquad D(P^\varepsilon) = D((-\Delta)_1^{\frac{\alpha}{2}}) \equiv \mathcal{W}^{\alpha,1},$$

$$(P^\varepsilon)^* := (-\Delta)^{\frac{\alpha}{2}} - \nabla\cdot b_\varepsilon + U_\varepsilon = (-\Delta)^{\frac{\alpha}{2}} - b_\varepsilon\cdot\nabla - W_\varepsilon, \qquad D((P^\varepsilon)^*) = D((-\Delta)_1^{\frac{\alpha}{2}}),$$

where $W_\varepsilon(x) = (d-\alpha)\kappa|x|_\varepsilon^{-\alpha}$. Note that for each $\varepsilon > 0$ e^{-tP^ε}, $e^{-t(P^\varepsilon)^*}$ can be viewed as C_0 semigroups in L^1 and $C_u = \{f \in C(\mathbb{R}^d) \mid f$ are uniformly continuous and bounded$\}$ with the sup-norm (e.g. by the Hille Perturbation Theorem).

Set

$$\phi_n(x) = \big(e^{-\frac{(P^\varepsilon)^*}{n}}\varphi\big)(x), \qquad \varphi \equiv \varphi_s, \qquad n = 1, 2, \dots$$

Since $\varphi = \varphi_{(1)} + \varphi_{(u)}$, $\varphi_{(1)} \in D((-\Delta)_1^{\frac{\alpha}{2}})$, $\varphi_{(u)} \in D((-\Delta)_{C_u}^{\frac{\alpha}{2}})$, the weights ϕ_n are well defined.

Remark. We emphasize that this choice of ϕ_n, the regularization of φ, is the key observation that allows to carry out the method in the case $\alpha < 2$.

Put

$$Q=\phi_n P^\varepsilon \phi_n^{-1}, \quad D(Q)=\phi_n D(P^\varepsilon)=\phi_n D((-\Delta)^{\frac{\alpha}{2}}), \quad F^t_{\varepsilon,n}=\phi_n e^{-tP^\varepsilon}\phi_n^{-1}.$$

Here $\phi_n D(P^\varepsilon):=\{\phi_n u \mid u\in D(P^\varepsilon)\}$. Since $\phi_n\geq \frac{1}{2}$ and $\phi_n,\phi_n^{-1}\in L^\infty$, these operators are well defined. In particular, $F^t_{\varepsilon,n}$ is a quasi bounded C_0 semigroup in L^1, say e^{-tG}. Set

$$\begin{aligned} M :=& \phi_n(1+(-\Delta)^{\frac{\alpha}{2}})^{-1}[L^1\cap C_u] \\ =& \phi_n(\lambda_\varepsilon+P^\varepsilon)^{-1}[L^1\cap C_u], \quad 0<\lambda_\varepsilon\in\rho(-P^\varepsilon). \end{aligned}$$

Clearly, M is a dense subspace of L^1, $M\subset D(Q)$ and $M\subset D(G)$. Moreover, $Q\upharpoonright M\subset G$. Indeed, for $f=\phi_n u\in M$,

$$Gf=s\text{-}L^1\text{-}\lim_{t\downarrow 0}t^{-1}(1-e^{-tG})f=\phi_n s\text{-}L^1\text{-}\lim_{t\downarrow 0}t^{-1}(1-e^{-tP^\varepsilon})u=\phi_n P^\varepsilon u=Qf.$$

Thus $Q\upharpoonright M$ is closable and $\tilde{Q}:=(Q\upharpoonright M)^{\rm clos}\subset G$.

Next, let us show that $R(\lambda_\varepsilon+\tilde{Q})$ is dense in L^1. If $\langle(\lambda_\varepsilon+\tilde{Q})h,v\rangle=0$ for all $h\in D(\tilde{Q})$ and some $v\in L^\infty$, $\|v\|_\infty=1$, then taking $h\in M$ we would have $\langle(\lambda_\varepsilon+Q)\phi_n(\lambda_\varepsilon+P^\varepsilon)^{-1}g,v\rangle=0$, $g\in L^1\cap C_u$, or $\langle\phi_n g,v\rangle=0$. Choosing $g=e^{\frac{\Delta}{k}}(\chi_m v)$, where $\chi_m\in C_c^\infty$ with $\chi_m(x)=1$ when $x\in B(0,m)$, we would have $\lim_{k\uparrow\infty}\langle\phi_n g,v\rangle=\langle\phi_n\chi_m,|v|^2\rangle=0$, and so $v\equiv 0$. Thus, $R(\lambda_\varepsilon+\tilde{Q})$ is dense in L^1.

Proposition 2 (The main step)**.** *There is a constant $\hat{c}=\hat{c}(d,\alpha,\delta)$ such that*

$$\lambda+\tilde{Q} \textit{ is accretive whenever } \lambda\geq\hat{c}s^{-1}.$$

Taking Proposition 2 for granted, we immediately establish the bound

$$\|e^{-tG}\|_{1\to 1}\equiv\|\phi_n e^{-tP^\varepsilon}\phi_n^{-1}\|_{1\to 1}\leq e^{\omega t}, \quad \omega=\hat{c}s^{-1}. \tag{$\star\star$}$$

Indeed, the facts: $\tilde{Q}$ is closed and $R(\lambda_\varepsilon+\tilde{Q})$ is dense in L^1 together with Proposition 2 imply $R(\lambda_\varepsilon+\tilde{Q})=L^1$ (Appendix B). But then, by the Lumer-Phillips Theorem, $\lambda+\tilde{Q}$ is the (minus) generator of a contraction semigroup, and $\tilde{Q}=G$ due to $\tilde{Q}\subset G$.

In turn, $(\star\star)$ easily yields (N_3). Indeed, $(\star\star)$ implies that $\lim_{n\uparrow\infty}\|\phi_n e^{-tP^\varepsilon}v\|_1\leq e^{\omega t}\lim_{n\uparrow\infty}\|\phi_n v\|_1$ for all $v\in L^1\cap L^2$. But

$$\lim_{n\uparrow\infty}\|\phi_n v\|_1=\lim_{n\uparrow\infty}\langle\varphi,e^{-\frac{P^\varepsilon}{n}}|v|\rangle=\langle\varphi,|v|\rangle<\infty,$$

$$\lim_{n\uparrow\infty}\|\phi_n e^{-tP^\varepsilon}v\|_1=\lim_{n\uparrow\infty}\langle\varphi,e^{-\frac{P^\varepsilon}{n}}|e^{-tP^\varepsilon}v|\rangle=\langle\varphi,|e^{-tP^\varepsilon}v|\rangle<\infty.$$

Therefore, taking $v=\varphi^{-1}h$ we arrive at (N_3).

Proof of Proposition 2. First we note that, for $f=\phi_n u\in M$,

$$\begin{aligned} \langle Qf,\frac{f}{|f|}\rangle =& \langle\phi_n P^\varepsilon u,\frac{f}{|f|}\rangle=\lim_{t\downarrow 0}t^{-1}\langle\phi_n(1-e^{-tP^\varepsilon})u,\frac{f}{|f|}\rangle, \\ {\rm Re}\langle Qf,\frac{f}{|f|}\rangle \geq& \lim_{t\downarrow 0}t^{-1}\langle(1-e^{-tP^\varepsilon})|u|,\phi_n\rangle \\ =& \langle P^\varepsilon e^{-\frac{P^\varepsilon}{n}}|u|,\varphi\rangle. \end{aligned}$$

We emphasize that e^{-tP^ε} is holomorphic due to Hille's Perturbation Theorem.

We are going to estimate $J := \langle P^\varepsilon e^{-\frac{P^\varepsilon}{n}}|u|, \varphi\rangle$ from below using the representation

$$(-\Delta)^{\frac{\alpha}{2}}\varphi = -I_{2-\alpha}\Delta\varphi,$$

where $I_\nu \equiv (-\Delta)^{-\frac{\nu}{2}}$.

Since $e^{-t(P^\varepsilon)^*}$ is a C_0 semigroup in L^1 and C_u, and $\varphi = \varphi_{(1)} + \varphi_{(u)}$, $\varphi_{(1)} \in D((-\Delta)_1^{\frac{\alpha}{2}})$, $\varphi_{(u)} \in D((-\Delta)_{C_u}^{\frac{\alpha}{2}})$, $(P^\varepsilon)^*\varphi$ is well defined and belongs to $L^1 + C_u = \{w + v \mid w \in L^1, v \in C_u\}$.

Define $\tilde{\varphi}_1(x) = |x|^{-d+\beta}$, $V(x) := (\beta-\alpha)\kappa|x|^{-\alpha}$ $(= \frac{\gamma(\beta)}{\gamma(\beta-\alpha)}|x|^{-\alpha}$ by the choice of $\beta)$. Using the identity $(-\Delta)^{\frac{\alpha}{2}}\tilde{\varphi}_1 = V\tilde{\varphi}_1$ (see Appendix A), we obtain

$$\begin{aligned}(-\Delta)^{\frac{\alpha}{2}}\varphi_1 &= -I_{2-\alpha}\mathbf{1}_{B(0,1)}\Delta\tilde{\varphi}_1 - I_{2-\alpha}\mathbf{1}_{B^c(0,1)}\Delta\varphi_1 \qquad (B^c(0,1) := \mathbb{R}^d - B(0,1)) \\ &= V\tilde{\varphi}_1 - I_{2-\alpha}\mathbf{1}_{B^c(0,1)}\Delta(\varphi_1 - \tilde{\varphi}_1).\end{aligned}$$

Routine calculation shows that $-I_{2-\alpha}\mathbf{1}_{B^c(0,1)}\Delta(\varphi_1 - \tilde{\varphi}_1) \geq -c_0$ for a constant c_0.

Also, by straightforward calculation, $-\big(b_\varepsilon \cdot \nabla + W_\varepsilon\big)\varphi_1 \geq -V\tilde{\varphi}_1 - c_1$ for a constant c_1.

Therefore,

$$(P^\varepsilon)^*\varphi_1 = (-\Delta)^{\frac{\alpha}{2}}\varphi_1 - \big(b_\varepsilon \cdot \nabla + W_\varepsilon\big)\varphi_1 \geq -C, \qquad C := c_0 + c_1,$$

so, by scaling,

$$J = \langle e^{-\frac{P^\varepsilon}{n}}|u|, (P^\varepsilon)^*\varphi\rangle \geq -Cs^{-1}\|e^{-\frac{P^\varepsilon}{n}}|u|\|_1 \geq -Cs^{-1}\|e^{-\frac{P^\varepsilon}{n}}\|_{1\to1}\|\phi_n^{-1}f\|_1,$$

or due to $\phi_n \geq \frac{1}{2}$,

$$J \geq -2Cs^{-1}\|e^{-\frac{P^\varepsilon}{n}}\|_{1\to1}\|f\|_1.$$

Noticing that $\|W_\varepsilon\|_\infty \leq c\varepsilon^{-\frac{\alpha}{2}}$, $c := \kappa(d-\alpha)$, we have $\|e^{-\frac{P^\varepsilon}{n}}\|_{1\to1} \leq e^{c\varepsilon^{-\frac{\alpha}{2}}n^{-1}} = 1 + o(n)$. Taking $\lambda = 3Cs^{-1}$ we obtain that

$$\mathrm{Re}\langle(\lambda + Q)f, \frac{f}{|f|}\rangle \geq 0 \qquad f \in M.$$

The latter holds for all $f \in D(\tilde{Q})$. The proof of Proposition 2 is completed. □

The proof of (N_3) is completed. The proof of Theorem 1 is completed. □

Remark 1 (Proof of $e^{-tP^\varepsilon} \overset{s}{\to} e^{-t\Lambda}$)**.** It suffices to show that $(\mu + P^\varepsilon)^{-1} \overset{s}{\to} (\mu + \Lambda)^{-1}$ for a $\mu > 0$.

First, we show that $(\mu + \Lambda^\varepsilon)^{-1} \overset{s}{\to} (\mu + \Lambda)^{-1}$. We will use notation introduced in the proof of Proposition 1 above. Recall: $(\mu + \Lambda)^{-1} = (\mu + A)^{-1}(1 + T)^{-1}$, $\|(\mu + \Lambda)^{-1}\| \leq (1-\delta)^{-1}\mu^{-1}$. Since $\|(T - T_\varepsilon)f\|_2 \leq \||b - b_\varepsilon|(\mu + A)^{-1}|\nabla f|\|_2 \to 0$ for every $f \in C_c^\infty$ by the Dominated Convergence Theorem, we have $T_\varepsilon \overset{s}{\to} T$. Therefore, $(\mu + \Lambda^\varepsilon)^{-1} \overset{s}{\to} (\mu + \Lambda)^{-1}$.

We show that $(\mu + P^\varepsilon)^{-1} - (\mu + \Lambda^\varepsilon)^{-1} \overset{s}{\to} 0$. Set $S = (\mu + A)^{-1+\frac{1}{\alpha}}b \cdot \nabla(\mu + A)^{-\frac{1}{\alpha}}$ and $S_\varepsilon = (\mu + A)^{-1+\frac{1}{\alpha}}b_\varepsilon \cdot \nabla(\mu + A)^{-\frac{1}{\alpha}}$. Then $\sup_\varepsilon \|S_\varepsilon\|, \|S\| < 1$ and

$$(\mu + \Lambda^\varepsilon)^{-1} = (\mu + A)^{-\frac{1}{\alpha}}(1 + S_\varepsilon)^{-1}(\mu + A)^{-1+\frac{1}{\alpha}}, \quad \mu > 0.$$

Now, let $h \in L^2 \cap L^\infty$. Then

$$\|(\mu + P^\varepsilon)^{-1}h - (\mu + \Lambda^\varepsilon)^{-1}h\|_2 = \|(\mu + \Lambda^\varepsilon)^{-1}U_\varepsilon(\mu + P^\varepsilon)^{-1}h\|_2 \leq K_1 + K_2,$$

$$\begin{aligned}K_1 &= \|(\mu+\Lambda^\varepsilon)^{-1}U_\varepsilon \mathbf{1}_{B(0,1)}(\mu+P^\varepsilon)^{-1}h\|_2\\ &\leq \|(\mu+\Lambda^\varepsilon)^{-1}|x|^{-\alpha+1}\|\||x|^{\alpha-1}U_\varepsilon\mathbf{1}_{B(0,1)}\|_2\mu^{-1}\|h\|_\infty\\ &\leq C\mu^{-1}\|h\|_\infty\|\varepsilon|x|_\varepsilon^{-1}\mathbf{1}_{B(0,1)}\|_2 \to 0,\end{aligned}$$

$$K_2 = \|(\mu+\Lambda^\varepsilon)^{-1}U_\varepsilon\mathbf{1}_{B^c(0,1)}(\mu+P^\varepsilon)^{-1}h\|_2 \leq \kappa\alpha\varepsilon(1-\delta)^{-1}\mu^{-2}\|h\|_2 \to 0.$$

The convergence $e^{-tP^\varepsilon} \overset{s}{\to} e^{-t\Lambda}$ is established.

Similar arguments show that $e^{-t(P^\varepsilon)^*} \overset{s}{\to} e^{-t\Lambda^*}$.

Remark 2. In the assumptions of Theorem 1, $e^{-t\Lambda}$ is contraction in L^2. Indeed, $(e^{-tP^\varepsilon})_{\varepsilon>0}$ are contractions (due to (N_1), see the proof of Theorem 1), so the result follows from Remark 1.

Remark 3. Above we could have constructed an operator realization Λ of $(-\Delta)^{\frac{\alpha}{2}}+b\cdot\nabla$ on L^2 for $b(x) := \delta_2 c^{-2}(\frac{\alpha-1}{2},2,d)|x|^{-\alpha}x$, $0<\delta_2<1$, by following the arguments in [KiS1, Section 4]. Note that

$$c^{-1}(\alpha-1,2,d) < c^{-2}(\frac{\alpha-1}{2},2,d)$$

(indeed, $\Gamma(\frac{d+2-2\alpha}{4})[\Gamma(\frac{d-1+\alpha}{4})]^2-\Gamma(\frac{d-2+2\alpha}{4})[\Gamma(\frac{d+1-\alpha}{4})]^2>0$), i.e. these assumptions are less restrictive than the ones needed in the proof of Proposition 1.

Then, in particular,

$$\|e^{-t\Lambda}f\|_q \leq c_r t^{-j'\left(\frac{1}{r}-\frac{1}{q}\right)}\|f\|_r, \quad f\in L^r\cap L^q, \quad 2\leq r<q\leq\infty$$

(arguing as in the proof of [KiS1, Theorem 4.3]).

The following inequalities, which will be needed in the proof of Theorem 2 below, are simple consequences of (N_3) and $(\star)$:

Corollary 1.

$$e^{-t(P^\varepsilon)^*}\varphi(x) \leq c\varphi(x), \quad \langle e^{-t(P^\varepsilon)^*}(x,\cdot)\rangle \leq 2c\varphi(x) \quad x\neq 0,\ s\geq t>0.$$

3. Proof of Theorem 2: The upper bound $e^{-t\Lambda}(x,y) \leq Ce^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y)\varphi_t(y)$ $(y\neq 0)$.

For brevity, everywhere below $(-\Delta)^{\frac{\alpha}{2}} =: A$.

By scaling, it suffices to consider $t=1$. It suffices to prove the bound $(\varepsilon>0)$

$$e^{-(P^\varepsilon)^*}(x,y) \leq Ce^{-A}(x,y)\varphi(x), \quad C\neq C(\varepsilon), \quad \varphi\equiv\varphi_1.$$

Let $R>1$ to be chosen later.

The case $|x|,|y|\leq 2R$.

Since $e^{-A}(x,y)\approx 1\wedge|x-y|^{-d-\alpha}$ $(x\neq y)$, the Nash initial estimate $e^{-t(P^\varepsilon)^*}(x,y)\leq Ct^{-j'}\varphi(x)$ (Theorem 1) yields

$$e^{-(P^\varepsilon)^*}(x,y) \leq C_R e^{-A}(x,y)\varphi(x), \quad C_R\neq C_R(\varepsilon).$$

To consider the other cases we will be using the Duhamel formula,

$$\begin{aligned}e^{-(P^\varepsilon)^*} &= e^{-A}+\int_0^1 e^{-\tau(P^\varepsilon)^*}(B_{\varepsilon,R}+B^c_{\varepsilon,R})e^{-(1-\tau)A}d\tau\\ &=: e^{-A}+K_R+K_R^c,\end{aligned}$$

where $B_{\varepsilon,R} := \mathbf{1}_{B(0,R)}B_\varepsilon$, $B^c_{\varepsilon,R} := \mathbf{1}_{B^c(0,R)}B_\varepsilon$ and $B_\varepsilon := b_\varepsilon\cdot\nabla + W_\varepsilon$ (recall, $W_\varepsilon(x) = \kappa(d-\alpha)|x|_\varepsilon^{-\alpha}$, $b_\varepsilon(x) = \kappa|x|_\varepsilon^{-\alpha}x$).

Below we prove that $K_R(x,y)$, $K_R^c(x,y) \leq C_R' e^{-A}(x,y)\varphi(x)$, which would yield the upper bound. We will need the following.

Lemma 1. *Set* $E^t(x,y) = t\big(|x-y|^{-d-\alpha-1}\wedge t^{-\frac{d+\alpha+1}{\alpha}}\big)$, $E^tf(x) := \langle E^t(x,\cdot)f(\cdot)\rangle$.

Let $0<t\leq 1$. *Then*

(i) $|\nabla_x e^{-tA}(x,y)| \leq c_0E^t(x,y)$;

(ii) $\int_0^t\langle e^{-(t-\tau)A}(x,\cdot)E^\tau(\cdot,y)\rangle d\tau \leq c_1e^{-tA}(x,y)$;

(iii) $\int_0^t\langle E^{t-\tau}(x,\cdot)E^\tau(\cdot,y)\rangle d\tau \leq c_2E^t(x,y)$.

Proof. For the proof of (i), (ii) see e.g. [BJ]. Essentially the same argument yields (iii). For the sake of completeness, we provide the details:

$$\begin{aligned}
E^t(x,z)\wedge E^\tau(z,y) &= (t|x-z|^{-d-\alpha-1}\wedge t^{-\frac{d+1}{\alpha}})\wedge(\tau|z-y|^{-d-\alpha-1}\wedge\tau^{-\frac{d+1}{\alpha}})\\
&\leq C_0\left(\frac{t+\tau}{2}\right)^{-\frac{d+1}{\alpha}}\wedge\left[(t+\tau)\left(\frac{|x-z|+|z-y|}{2}\right)^{-d-\alpha-1}\right] \qquad (C_0>1)\\
&\leq C(t+\tau)^{-\frac{d+1}{\alpha}}\wedge\big[(t+\tau)(|x-y|)^{-d-\alpha-1}\big] = CE^{t+\tau}(x,y),
\end{aligned}$$

so (iii) follows from the inequality $ac = (a\wedge c)(a\vee c) \leq (a\wedge c)(a+c)$ $(a,c\geq 0)$:

$$\int_0^t\langle E^{t-\tau}(x,\cdot)E^\tau(\cdot,y)\rangle d\tau \leq E^{t+\tau}(x,y)\int_0^t\langle E^{t-\tau}(x,\cdot)+E^\tau(\cdot,y)\rangle d\tau,$$

where, routine calculation shows, $\int_0^t\langle E^{t-\tau}(x,\cdot)+E^\tau(\cdot,y)\rangle d\tau \leq c_2<\infty$ (we use that $t\leq 1$). $\square$

The case $|y|>2R$, $0<|x|\leq|y|$.

Claim 1. *If* $|y|>2R$, $0<|x|\leq|y|$, *then*

$$K_R(x,y) \equiv \int_0^1\big\langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)B_{\varepsilon,R}(\cdot)e^{-(1-\tau)A}(\cdot,y)\big\rangle d\tau \leq \hat{C}e^{-A}(x,y)\varphi(x), \quad \hat{C}\neq\hat{C}(\varepsilon).$$

Proof. Claim 1 clearly follows from

(j) $\int_0^t\langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)\mathbf{1}_{B(0,R)}(\cdot)W_\varepsilon(\cdot)e^{-(t-\tau)A}(\cdot,y)\rangle d\tau \leq c_4e^{-tA}(x,y)\varphi(x)$,

and, in view of Lemma 1(i), from

(jj) $\int_0^t\langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)\mathbf{1}_{B(0,R)}(\cdot)Z_\varepsilon(\cdot)E^{t-\tau}(\cdot,y)\rangle d\tau \leq c_3e^{-tA}(x,y)\varphi(x)$, where $Z_\varepsilon(x) := |x|_\varepsilon^{-\alpha}|x|$.

Let us prove (jj):

$$\int_0^t \langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)\mathbf{1}_{B(0,R)}(\cdot)Z_\varepsilon(\cdot)E^{t-\tau}(\cdot,y)\rangle d\tau$$

(we are using $E^{t-\tau}(\cdot,y) \leq Ce^{-(t-\tau)A}(\cdot,y)|\cdot-y|^{-1}$)

$$\leq C\int_0^t \langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)\mathbf{1}_{B(0,R)}(\cdot)Z_\varepsilon(\cdot)e^{-(t-\tau)A}(\cdot,y)|\cdot-y|^{-1}\rangle d\tau$$

(we are using $\mathbf{1}_{B(0,R)}(\cdot)|\cdot-y|^{-1} \leq |\cdot|^{-1}$)

$$\leq C'\int_0^t \langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)\mathbf{1}_{B(0,R)}(\cdot)W_\varepsilon(\cdot)e^{-(t-\tau)A}(\cdot,y)\rangle d\tau$$

(we are using $\mathbf{1}_{B(0,R)}(\cdot)e^{-(t-\tau)A}(\cdot,y) \leq e^{-tA}(x,y)$)

$$\leq C''e^{-tA}(x,y)\int_0^t \langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)\mathbf{1}_{B(0,R)}(\cdot)W_\varepsilon(\cdot)\rangle d\tau.$$

According to the Duhamel formula $e^{-t(P^\varepsilon)^*} = e^{-tA} + \int_0^t e^{-\tau(P^\varepsilon)^*}(b_\varepsilon\cdot\nabla + W_\varepsilon)e^{-(t-\tau)A}d\tau$,

$$1 + \int_0^t \langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)W_\varepsilon(\cdot)\rangle d\tau = \langle e^{-t(P^\varepsilon)^*}(x,\cdot)\rangle.$$

Using the inequality $\langle e^{-t(P^\varepsilon)^*}(x,\cdot)\rangle \leq 2c\varphi(x)$ from Corollary 1, it is seen that

$$\int_0^t \langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)W_\varepsilon(\cdot)\rangle d\tau \leq 2c\varphi(x).$$

The latter and the previous estimate yield (jj). Incidentally, we have also proved (j). □

Claim 2. *If* $|y| > 2R$, $|x| \leq |y|$, *then*

$$K_R^c(x,y) \equiv \int_0^1 \langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)B^c_{\varepsilon,R}(\cdot)e^{-(1-\tau)A}(\cdot,y)\rangle d\tau \leq Ce^{-A}(x,y)\varphi(x).$$

Proof. Lemma 1(i) yields

$$|B^c_{\varepsilon,R}(\cdot)e^{-(\tau-\tau')A}(\cdot,y)| \leq C_0\big(R^{-\alpha}e^{-(\tau-\tau')A}(\cdot,y) + R^{-\alpha+1}E^{\tau-\tau'}(\cdot,y)\big), \tag{$*$}$$

$$\begin{aligned} K_R^c(x,y) &\equiv \int_0^1 \langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)B^c_{\varepsilon,R}(\cdot)e^{-(1-\tau)A}(\cdot,y)\rangle d\tau \\ &\leq C_0R^{-\alpha}\int_0^1 \big\langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)e^{-(1-\tau)A}(\cdot,y)\big\rangle d\tau + C_0R^{-\alpha+1}\int_0^1 \big\langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)E^{1-\tau}(\cdot,y)\big\rangle d\tau. \end{aligned} \tag{$**$}$$

1. Let us estimate the first term in the RHS of $(**)$. By the Duhamel formula,

$$\begin{aligned} &\int_0^1 e^{-\tau(P^\varepsilon)^*}e^{-(1-\tau)A}d\tau \\ &= \int_0^1 e^{-\tau A}e^{-(1-\tau)A}d\tau + \int_0^1\int_0^\tau e^{-\tau'(P^\varepsilon)^*}(B_{\varepsilon,R} + B^c_{\varepsilon,R})e^{-(\tau-\tau')A}d\tau' e^{-(1-\tau)A}d\tau \\ &\equiv e^{-A} + I_R + I_R^c. \end{aligned}$$

We have $I_R = \int_0^1 I_R^\tau e^{-(1-\tau)A} d\tau$, where $I_R^\tau := \int_0^\tau e^{-\tau'(P^\varepsilon)^*} B_{\varepsilon,R} e^{-(\tau-\tau')A} d\tau'$. By Claim 1,

$$|I_R^\tau(x,y)| \leq \hat{C} e^{-\tau A}(x,y)\varphi(x) \qquad \text{and so } |I_R(x,y)| \leq \hat{C} e^{-A}(x,y)\varphi(x).$$

In turn, $I_R^c = \int_0^1 (I_R^c)^\tau e^{-(1-\tau)A} d\tau$, where $(I_R^c)^\tau := \int_0^\tau e^{-\tau'(P^\varepsilon)^*} B_{\varepsilon,R}^c e^{-(\tau-\tau')A} d\tau'$, so

$$\begin{aligned} |(I_R^c)^\tau(x,y)| \leq C_0 R^{-\alpha} &\int_0^\tau \big\langle e^{-\tau'(P^\varepsilon)^*}(x,\cdot) e^{-(\tau-\tau')A}(\cdot,y)\big\rangle d\tau' \\ &+ C_0 R^{-\alpha+1} \int_0^\tau \big\langle e^{-\tau'(P^\varepsilon)^*}(x,\cdot) E^{\tau-\tau'}(\cdot,y)\big\rangle d\tau'. \end{aligned}$$

Then

$$\begin{aligned} |I_R^c(x,y)| \leq C_0 R^{-\alpha} &\int_0^1 \int_0^\tau \big(e^{-\tau'(P^\varepsilon)^*} e^{-(\tau-\tau')A} e^{-(1-\tau)A}\big)(x,y) d\tau' d\tau \\ &+ C_0 R^{-\alpha+1} \int_0^1 \int_0^\tau \big(e^{-\tau'(P^\varepsilon)^*} E^{\tau-\tau'} e^{-(1-\tau)A}\big)(x,y) d\tau' d\tau, \end{aligned}$$

where we estimate the first and second integrals as follows.

$$\begin{aligned} &\int_0^1 \int_0^\tau \big(e^{-\tau'(P^\varepsilon)^*} e^{-(1-\tau')A}\big)(x,y) d\tau' d\tau \\ &\leq \int_0^1 \int_0^1 \big(e^{-\tau'(P^\varepsilon)^*} e^{-(1-\tau')A}\big)(x,y) d\tau' d\tau = \int_0^1 \big\langle e^{-\tau'(P^\varepsilon)^*}(x,\cdot) e^{-(1-\tau')A}(\cdot,y)\big\rangle d\tau', \end{aligned}$$

$$\begin{aligned} &\int_0^1 \int_0^\tau \big(e^{-\tau'(P^\varepsilon)^*} E^{\tau-\tau'} e^{-(1-\tau)A}\big)(x,y) d\tau' d\tau \\ &\text{(we are changing the order of integration in } \tau \text{ and } \tau') \\ &= \int_0^1 \int_{\tau'}^1 \big(e^{-\tau'(P^\varepsilon)^*} E^{\tau-\tau'} e^{-(1-\tau)A}\big)(x,y) d\tau d\tau' \\ &\text{(by Lemma 1}(ii)\text{, } \int_{\tau'}^1 (E^{\tau-\tau'} e^{-(1-\tau)A})(\cdot,y) d\tau \leq c_1 e^{-(1-\tau')A}(\cdot,y)) \\ &\leq c_1 \int_0^1 \big\langle e^{-\tau'(P^\varepsilon)^*}(x,\cdot) e^{-(1-\tau')A}(\cdot,y)\big\rangle d\tau'. \end{aligned}$$

Thus,

$$|I_R^c(x,y)| \leq C_0(R^{-\alpha} + c_1 R^{-\alpha+1}) \int_0^1 \big\langle e^{-\tau(P^\varepsilon)^*}(x,\cdot) e^{-(1-\tau)A}(\cdot,y)\big\rangle d\tau.$$

Therefore, for $R > 1$ such that $C_0(R^{-\alpha} + c_1 R^{-\alpha+1}) \leq \frac{1}{2}$,

$$\begin{aligned} &\int_0^1 \langle e^{-\tau(P^\varepsilon)^*}(x,\cdot) e^{-(1-\tau)A}(\cdot,y)\rangle d\tau \\ &\leq e^{-A}(x,y) + \frac{1}{2} \int_0^1 \langle e^{-\tau(P^\varepsilon)^*}(x,\cdot) e^{-(1-\tau)A}(\cdot,y)\rangle d\tau + \hat{C} e^{-A}(x,y)\varphi(x), \end{aligned}$$

i.e. $\int_0^1 \langle e^{-\tau(P^\varepsilon)^*}(x,\cdot) e^{-(1-\tau)A}(\cdot,y)\rangle d\tau \leq 2(2+\hat{C}) e^{-A}(x,y)\varphi(x)$.

2. Let us estimate the second term in the RHS of $(**)$. By the Duhamel formula

$$\begin{aligned}&\int_0^1 e^{-\tau(P^\varepsilon)^*}E^{1-\tau}d\tau\\&=\int_0^1 e^{-\tau A}E^{1-\tau}d\tau+\int_0^1\int_0^\tau e^{-\tau'(P^\varepsilon)^*}(B_{\varepsilon,R}+B^c_{\varepsilon,R})e^{-(\tau-\tau')A}d\tau' E^{1-\tau}d\tau\\&\equiv\int_0^1 e^{-\tau A}E^{1-\tau}d\tau+J_R+J_R^c,\end{aligned}$$

where, by Lemma 1(ii), $\int_0^1\langle e^{-\tau A}(x,\cdot)E^{1-\tau}(\cdot,y)\rangle ds\leq c_1e^{-A}(x,y)$. Let us estimate J_R and J_R^c.

We have $J_R=\int_0^1 J_R^\tau E^{1-\tau}d\tau$, where $J_R^\tau:=\int_0^\tau e^{-\tau'(P^\varepsilon)^*}B_{\varepsilon,R}e^{-(\tau-\tau')A}d\tau'$. By Claim 1,

$$|J_R^\tau(x,y)|\leq \hat{C}e^{-\tau A}(x,y)\varphi(x),\qquad \text{and so by Lemma 1}(ii),$$

$$|J_R(x,y)|\leq C_1e^{-A}(x,y)\varphi(x).$$

In turn, $J_R^c=\int_0^1(J_R^c)^\tau E^{1-\tau}d\tau$, where $(J_R^c)^\tau:=\int_0^\tau e^{-\tau'(P^\varepsilon)^*}B^c_{\varepsilon,R}e^{-(\tau-\tau')A}d\tau'$. By $(*)$ and Lemma 1(ii), $|(J_R^c)^\tau(x,y)|\leq C_0R^{-\alpha}\int_0^\tau\big(e^{-\tau'(P^\varepsilon)^*}e^{-(\tau-\tau')A}\big)(x,y)d\tau'+C_0R^{-\alpha+1}\int_0^\tau\big(e^{-\tau'(P^\varepsilon)^*}E^{\tau-\tau'}\big)(x,y)d\tau'$. Due to Lemma 1$(ii)$,$(iii)$,

$$\begin{aligned}|J_R^c(x,y)|\leq C_0c_1R^{-\alpha}&\int_0^1\langle e^{-\tau'(P^\varepsilon)^*}(x,\cdot)e^{-(1-\tau')A}(\cdot,y)\rangle d\tau'\\&+C_0c_2R^{-\alpha+1}\int_0^1\langle e^{-\tau'(P^\varepsilon)^*}(x,\cdot)E^{1-\tau'}(\cdot,y)\rangle d\tau'.\end{aligned}$$

Thus, for $R>1$ such that $C_0c_1R^{-\alpha},C_0c_2R^{-\alpha+1}\leq\frac12$,

$$\begin{aligned}&\int_0^1\langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)E^{1-\tau}(\cdot,y)\rangle d\tau\leq c_1e^{-A}(x,y)+\frac12\int_0^1\langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)e^{-(1-\tau)A}(\cdot,y)\rangle d\tau\\&+\frac12\int_0^1\langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)E^{1-\tau}(\cdot,y)\rangle d\tau+C_1e^{-A}(x,y)\varphi(x).\end{aligned}$$

Using **1** we arrive at $\int_0^1\langle e^{-\tau(P^\varepsilon)^*}(x,\cdot)E^{1-\tau}(\cdot,y)\rangle d\tau\leq 2(2c_1+2+\hat{C}+C_1)e^{-A}(x,y)\varphi(x)$.

Now **1** and **2** applied in $(**)$ yield Claim 2. $\square$

The case $|x|>2R$, $|y|\leq|x|$ is treated similarly, so we omit the details.

The proof of the upper bound is completed.

4. Proof of Theorem 2: The lower bound $e^{-t\Lambda}(x,y)\geq Ce^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y)\varphi_t(y)$ $(C>0,x,y\neq 0)$.

Proposition 3. *Define $g=\varphi h$, $\varphi\equiv\varphi_s$, $0\leq h\in\mathcal{S}$-the L. Schwartz space of test functions. There is a constant $0<\hat{\mu}$ such that, for all $0<t\leq s$,*

$$e^{-\frac{\hat{\mu}}{s}t}\langle g\rangle\leq\langle\varphi e^{-t\Lambda}\varphi^{-1}g\rangle.$$

Proof. Set $g_n=\phi_nh$, $\phi_n(x)=(e^{-\frac{(P^\varepsilon)^*}{n}}\varphi)(x)$. Then

$$\langle g_n\rangle-\langle\phi_ne^{-t(P^\varepsilon-\mu)}h\rangle=-\mu\int_0^t\langle\varphi,e^{-\tau(P^\varepsilon-\mu)}e^{-\frac{P^\varepsilon}{n}}h\rangle d\tau+\int_0^t\langle\varphi,P^\varepsilon e^{-\tau(P^\varepsilon-\mu)}e^{-\frac{P^\varepsilon}{n}}h\rangle d\tau,$$

where $\mu = \frac{\hat{\mu}}{s} > 0$ is to be chosen. Let $\tilde{\varphi}(x) = (s^{-\frac{1}{\alpha}}|x|)^{-d+\beta}$. Write $(P^\varepsilon)^*\varphi = (P^\varepsilon)^*\tilde{\varphi} + (P^\varepsilon)^*(\varphi - \tilde{\varphi}) = \mathbf{1}_{B(0,1)}(V - V_\varepsilon)\varphi + v_\varepsilon$, $V(x) \equiv V(|x|) = \kappa(\beta-\alpha)|x|^{-\alpha}$, $V_\varepsilon(x) \equiv V_\varepsilon(|x|) := V(|x|_\varepsilon)$. Routine calculation shows that $\|v_\varepsilon\|_\infty \leq \frac{\mu_1}{s}$ for a $\mu_1 \neq \mu_1(\varepsilon)$ (cf. the proof of Proposition 2). Thus

$$\int_0^t \langle v_\varepsilon, e^{-\tau(P^\varepsilon - \mu)} e^{-\frac{P^\varepsilon}{n}} h\rangle d\tau \leq \frac{\mu_1}{s}\int_0^t \langle e^{-\tau(P^\varepsilon - \mu)} e^{-\frac{P^\varepsilon}{n}} h\rangle d\tau \leq \frac{2\mu_1}{s}\int_0^t \langle \varphi, e^{-\tau(P^\varepsilon - \mu)} e^{-\frac{P^\varepsilon}{n}} h\rangle d\tau.$$

Taking $\hat{\mu} = 2\mu_1$, we have

$$\langle g_n\rangle - \langle \phi_n e^{-t(P^\varepsilon - \mu)} h\rangle \leq \int_0^t \langle \mathbf{1}_{B(0,1)}(V - V_\varepsilon)\varphi, e^{-(\tau + \frac{1}{n})P^\varepsilon} h\rangle e^{\mu\tau} d\tau,$$

or, sending $n \to \infty$,

$$\langle g\rangle - e^{\frac{\hat{\mu}}{s}t}\langle \varphi e^{-tP^\varepsilon} h\rangle \leq e^{\hat{\mu}} \int_0^t \langle \mathbf{1}_{B(0,1)}(V - V_\varepsilon)\varphi, e^{-\tau P^\varepsilon} h\rangle d\tau. \tag{$\diamond$}$$

It remains to take $\varepsilon \downarrow 0$ in ($\diamond$). Since $\|e^{-\tau P^\varepsilon} h\|_\infty \leq \|h\|_\infty$ and

$$\mathbf{1}_{B(0,1)}|V - V_\varepsilon|\varphi \leq 2\varphi \mathbf{1}_{B(0,1)} V \leq C\mathbf{1}_{B(0,1)}|x|^{-d+\beta-\alpha}, \qquad d - \beta + \alpha < d,$$

the RHS of ($\diamond$) tends to 0 as $\varepsilon \downarrow 0$ due to the Dominated Convergence Theorem. The latter, $e^{-tP^\varepsilon} h \to e^{-t\Lambda} h$ strongly in L^2 (see Remark 1) and (N_3) yield Proposition 3. □

We also need the following consequence of the upper bound and Proposition 3.

Proposition 4. *Fix $t > 0$. Set $g := \varphi h$, $\varphi = \varphi_t$, $0 \leq h \in \mathcal{S}$ with $\operatorname{sprt} h \subset B(0, R_0)$ for some $R_0 \geq 1$. Then there are $0 < r_t < R_0 \vee t^{\frac{2}{\alpha}} < R_{t,R_0}$ such that, for all $r \in [0, r_t]$ and $R \in [2R_{t,R_0}, \infty[$,*

$$e^{-\hat{\mu}-1}\langle g\rangle \leq \langle \mathbf{1}_{R,r}\varphi e^{-t\Lambda}\varphi^{-1} g\rangle, \qquad \mathbf{1}_{R,r} := \mathbf{1}_{B(0,R)} - \mathbf{1}_{B(0,r)}.$$

In particular,

$$e^{-\hat{\mu}-1}\varphi(x) \leq e^{-t\Lambda^*}\varphi \mathbf{1}_{R,r}(x) \quad \textit{for all } x \in B(0, R_0).$$

Proof. By the upper bound,

$$\begin{aligned}
\langle \mathbf{1}_{B(0,r)}\varphi_t e^{-t\Lambda}\varphi_t^{-1} g\rangle &\leq C\langle \mathbf{1}_{B(0,r)}\varphi_t, e^{-tA} g\rangle \\
&\leq CC_1 t^{-\frac{d}{\alpha}}\|\mathbf{1}_{B(0,r)}\varphi_t\|_1 \|g\|_1 \\
&= o(r_t)\|g\|_1, \quad o(r_t) \to 0 \text{ as } r_t \downarrow 0; \\
\langle \mathbf{1}_{B^c(0,R)}\varphi_t e^{-t\Lambda}\varphi_t^{-1} g\rangle &\leq C\langle \mathbf{1}_{B^c(0,R)}\varphi_t, e^{-tA} g\rangle \\
&\leq C\langle e^{-tA}\mathbf{1}_{B^c(0,R)}, g\mathbf{1}_{B(0,R_0)}\rangle, \text{ where } R \geq 2R_{t,R_0} \geq 2(R_0 \vee t^{\frac{2}{\alpha}}) \\
&\leq C \sup_{x \in B(0,R_0)} e^{-tA}\mathbf{1}_{B^c(0,R)}(x)\|g\|_1 \\
&\leq C\tilde{C}C_d R_{t,R_0}^{-\frac{\alpha}{2}}\|g\|_1 \\
&= o(R_{t,R_0})\|g\|_1, \quad o(R_{t,R_0}) \to 0 \text{ as } R_{t,R_0} \uparrow \infty
\end{aligned}$$

due to $e^{-tA}(x,y) \leq \tilde{C}(t|x-y|^{-d-\alpha} \wedge t^{-\frac{d}{\alpha}}) \leq \tilde{C}2^{d+\frac{\alpha}{2}}|y|^{-d-\frac{\alpha}{2}}$ if $|x| \leq R_0$ and $|y| \geq R$.

It remains to apply Proposition 3. □

Proposition 5. $\langle h\rangle = \langle e^{-t\Lambda^*} h\rangle$ *for every $h \in L^1$, $t > 0$.*

Proof. We have, for $h \in \mathcal{S}$,

$$\begin{aligned}\langle h\rangle - \langle e^{-t(P^\varepsilon)^*}h\rangle &= \int_0^t \langle 1, (P^\varepsilon)^* e^{-\tau (P^\varepsilon)^*}h\rangle d\tau = \int_0^t \langle U_\varepsilon e^{-\tau(P^\varepsilon)^*}h\rangle d\tau \\ &= \int_0^t \langle \mathbf{1}_{B^c(0,1)} U_\varepsilon e^{-\tau(P^\varepsilon)^*}h\rangle d\tau + \int_0^t \langle \mathbf{1}_{B(0,1)} U_\varepsilon e^{-\tau(P^\varepsilon)^*}h\rangle d\tau.\end{aligned}$$

It is clear that $\langle \mathbf{1}_{B^c(0,1)} U_\varepsilon e^{-\tau(P^\varepsilon)^*}h\rangle \leq \|\mathbf{1}_{B^c(0,1)}U_\varepsilon\|_\infty \|h\|_1 \to 0$ as $\varepsilon \downarrow 0$, and so the first integral converges to 0. Let us estimate the second integral:

$$\int_0^t \langle \mathbf{1}_{B(0,1)} U_\varepsilon e^{-\tau(P^\varepsilon)^*}h\rangle d\tau = \int_0^t \langle e^{-\tau P^\varepsilon}\mathbf{1}_{B(0,1)}U_\varepsilon, h\rangle d\tau$$

(we are using the upper bound $e^{-tP^\varepsilon}(x,y) \leq Ce^{-tA}(x,y)\varphi_t(y)$)

$$\leq C\int_0^t \langle e^{-\tau A}\varphi \mathbf{1}_{B(0,1)}U_\varepsilon, |h|\rangle d\tau$$

$$\leq Ct\|h\|_\infty \|\varphi \mathbf{1}_{B(0,1)}U_\varepsilon\|_1 \to 0 \text{ as } \varepsilon \downarrow 0 \text{ due to } d-\beta+\alpha<d.$$

Thus, $\langle h\rangle = \lim_\varepsilon \langle e^{-t(P^\varepsilon)^*}h\rangle$. Next, since $e^{-t(P^\varepsilon)^*}h \to e^{-t\Lambda^*}h$ strongly in L^2 (see Remark 1), we may suppose that $e^{-t(P^\varepsilon)^*}h \to e^{-t\Lambda^*}h$ a.e. The upper bound $e^{-t(P^\varepsilon)^*}(x,y) \leq Ce^{-tA}(x,y)\varphi_t(x)$, yields $|e^{-t(P^\varepsilon)^*}h| \leq C\varphi_t e^{-tA}|h| \in L^1$, and so $\lim_\varepsilon \langle e^{-t(P^\varepsilon)^*}h\rangle = \langle e^{-t\Lambda^*}h\rangle$ by the Dominated Convergence Theorem. Thus, equality $\langle h\rangle = \langle e^{-t\Lambda^*}h\rangle$ holds for every $h \in \mathcal{S}$ and hence for every $h \in L^1$. $\square$

Proposition 6. *Fix $t>0$. Let $0 \leq h \in \mathcal{S}$ with $\operatorname{sprt} h \subset B(0,R_0)$ for some $R_0 \geq 1$. Then there are $0 < r_t < R_0 \vee t^{\frac{2}{\alpha}} < R_{t,R_0}$ such that, for all $r \in [0,r_t]$ and $R \in [2R_{t,R_0},\infty[$,*

$$\frac{1}{2}\langle h\rangle \leq \langle \mathbf{1}_{R,r} e^{-t\Lambda^*}h\rangle.$$

In particular,

$$\frac{1}{2} \leq e^{-t\Lambda}\mathbf{1}_{R,r}(x) \quad \text{for all } x \in B(0,R_0).$$

Proof. We follow the argument in the proof of Proposition 4. By the upper bound,

$$\begin{aligned}\langle \mathbf{1}_{B(0,r)} e^{-t\Lambda^*}h\rangle &\leq C\langle \mathbf{1}_{B(0,r)}\varphi_t, e^{-tA}h\rangle \\ &\leq CC_1 t^{-\frac{d}{\alpha}}\|\mathbf{1}_{B(0,r)}\varphi_t\|_1\|h\|_1 \\ &= o(r_t)\|h\|_1, \quad o(r_t) \to 0 \text{ as } r_t \downarrow 0;\end{aligned}$$

$$\begin{aligned}\langle \mathbf{1}_{B^c(0,R)} e^{-t\Lambda^*}h\rangle &\leq C\langle \mathbf{1}_{B^c(0,R)}\varphi_t, e^{-tA}h\rangle \\ &\leq C\langle e^{-tA}\mathbf{1}_{B^c(0,R)}, h\mathbf{1}_{B(0,R_0)}\rangle, \text{ where } R \geq 2R_{t,R_0} \geq 2(R_0 \vee t^{\frac{2}{\alpha}}) \\ &\leq C\sup_{x \in B(0,R_0)} e^{-tA}\mathbf{1}_{B^c(0,R)}(x)\|h\|_1 \\ &\leq C\tilde{C}C_d R_{t,R_0}^{-\frac{\alpha}{2}}\|h\|_1 \\ &= o(R_{t,R_0})\|h\|_1, \quad o(R_{t,R_0}) \to 0 \text{ as } R_{t,R_0} \uparrow \infty\end{aligned}$$

due to $e^{-tA}(x,y) \leq \tilde{C}(t|x-y|^{-d-\alpha} \wedge t^{-\frac{d}{\alpha}}) \leq \tilde{C}2^{d+\frac{\alpha}{2}}|y|^{-d-\frac{\alpha}{2}}$ if $|x| \leq R_0$ and $|y| \geq R$.

The last two estimates and Proposition 5 yield $\frac{1}{2}\langle h\rangle \leq \langle \mathbf{1}_{R,r}e^{-t\Lambda^*}h\rangle$. $\square$

Claim 3. *For every $r>0$ there exist a constant $t(r)>0$ such that*

$$e^{-t\Lambda^*}(x,y) \geq \frac{1}{2}e^{-tA}(x,y) \quad \text{for all } |x|\geq r,\ |y|\geq r, \quad 0<t\leq t(r).$$

Proof. By the Duhamel formula,

$$e^{-t(P^\varepsilon)^*}(x,y) \geq e^{-tA}(x,y)+M_t(x,y), \qquad M_t(x,y) \equiv \int_0^t \langle e^{-(t-\tau)(P^\varepsilon)^*}(x,\cdot)b_\varepsilon(\cdot)\cdot\nabla . e^{-\tau A}(\cdot,y)\rangle d\tau.$$

By Lemma 1(i),

$$\begin{aligned}|M_t(x,y)| &\leq c_1 \int_0^t \big\langle e^{-(t-\tau)(P^\varepsilon)^*}(x,\cdot)|\cdot|^{1-\alpha}E^\tau(\cdot,y)\big\rangle d\tau \\ &\text{(we apply the upper bound)} \\ &\leq c_1 C \int_0^t \varphi_{t-\tau}(x)\big\langle e^{-(t-\tau)A}(x,\cdot)|\cdot|^{1-\alpha}E^\tau(\cdot,y)\big\rangle d\tau \\ &\text{(since } |x|\geq r, \text{ we may select } t=t(r)>0 \text{ sufficiently small so that } \varphi_{t-\tau}(x)=\tfrac{1}{2}) \\ &\leq \frac{c_1 C}{2}\int_0^t \big\langle e^{-(t-\tau)A}(x,\cdot)|\cdot|^{1-\alpha}E^\tau(\cdot,y)\big\rangle d\tau =: J(|\cdot|^{1-\alpha}).\end{aligned}$$

Next, select $\gamma>0$ sufficiently small ($\gamma \ll r$) so that, for all $0<\tau<t$, $|x|$, $|y|\geq r$,

$$\begin{aligned}\mathbf{1}_{B(0,\gamma)}(\cdot)e^{-(t-\tau)A}(x,\cdot) &\leq C_5 e^{-tA}(x,0),\\ \mathbf{1}_{B(0,\gamma)}(\cdot)e^{-\tau A}(\cdot,y) &\leq C_6 e^{-tA}(0,y),\\ \mathbf{1}_{B(0,\gamma)}(\cdot)E^\tau(\cdot,y) &\leq C_7 e^{-tA}(0,y).\end{aligned}$$

Using the inequality

$$e^{-tA}(x,z)e^{-\tau A}(z,y) \leq K e^{-(t+\tau)A}(x,y)\big(e^{-tA}(x,z)+e^{-\tau A}(z,y)\big), \tag{$*$}$$

which holds for a constant $K=K(d,\alpha)$, all $x,z,y\in\mathbb{R}^d$ and $t,\tau>0$ (see e.g. [BJ]), we have

$$\begin{aligned}J(\mathbf{1}_{B(0,\gamma)}|\cdot|^{1-\alpha}) &\leq c\int_0^t \langle \mathbf{1}_{B(0,\gamma)}(\cdot)|\cdot|^{1-\alpha}\rangle d\tau (e^{-tA}(x,0)+e^{-tA}(0,y))e^{-2tA}(x,y)\\ &\leq cC(r)\gamma^{d-\alpha+1}te^{-tA}(x,y).\end{aligned} \tag{$**$}$$

In turn,

$$J(\mathbf{1}_{B^c(0,\gamma)}|\cdot|^{1-\alpha}) \leq \frac{c_1 C}{2}C_0\gamma^{1-\alpha}t^{1-\frac{1}{\alpha}}e^{-tA}(x,y), \tag{$***$}$$

follows immediately from

$$\int_0^t \langle e^{-(t-\tau)A}(x,\cdot)E^\tau(\cdot,y)\rangle d\tau \leq C_0 t^{1-\frac{1}{\alpha}}e^{-tA}(x,y)$$

proved in Appendix C.

Thus, putting $t=\gamma^{2\alpha}$ and selecting $\gamma>0$ sufficiently small in ($**$) and ($***$), we have

$$|M_t(x,y)| \leq \frac{1}{2}e^{-tA}(x,y).$$

Thus,

$$e^{-t(P^\varepsilon)^*}(x,y) \geq \frac{1}{2}e^{-tA}(x,y), \quad |x|\geq r,\ |y|\geq r,\ 0<t\leq t(r).$$

Finally, using L^2-strong convergence $e^{-t(P^\varepsilon)^*} \to e^{-t\Lambda^*}$ (see Remark 1), we complete the proof of the Claim. $\square$

Claim 4. *For every $r>0$ there exists a constant $c(r)>0$ such that*

$$e^{-\Lambda^*}(x,y) \geq c(r) e^{-A}(x,y) \quad \text{for all } |x| \geq r,\ |y| \geq r, \quad x \neq y.$$

Proof. By the reproduction property,

$$\begin{aligned} e^{-2t_0\Lambda^*}(x,y) &\geq \langle e^{-t_0\Lambda^*}(x,\cdot)\mathbf{1}_{B^c(0,r)}(\cdot)e^{-t_0\Lambda^*}(\cdot,y)\rangle \\ &\quad \text{(we are applying Claim 3)} \\ &\geq c_1^2 \langle e^{-t_0 A}(x,\cdot)\mathbf{1}_{B^c(0,r)}(\cdot)e^{-t_0 A}(\cdot,y)\rangle, \qquad c_1 := \frac{1}{2},\ t_0 = t(r). \end{aligned}$$

Consider the following cases:

1) If $(r \leq)\ |x|,|y| \leq r_m$, where $r_m\ (> r)$ is to be chosen, then the above inequality yields $e^{-2t_0\Lambda^*}(x,y) \geq C_{r_m} > 0$, and so

$$e^{-2t_0\Lambda^*}(x,y) \geq C_{1,r_m} e^{-2t_0 A}(x,y), \quad C_{1,r_m} > 0.$$

2) If $|x|,|y| > r_m$, then

$$\begin{aligned} e^{-2t_0\Lambda^*}(x,y) &\geq c_1^2\big(e^{-2t_0 A}(x,y) - \langle e^{-t_0 A}(x,\cdot)\mathbf{1}_{B(0,r)}(\cdot)e^{-t_0 A}(\cdot,y)\rangle\big) \\ &\quad \text{(we are applying $(*)$)} \\ &\geq c_1^2 e^{-2t_0 A}(x,y)\big(1 - K\langle \mathbf{1}_{B(0,r)}(\cdot)(e^{-t_0 A}(x,\cdot) + e^{-t_0 A}(\cdot,y))\rangle\big) \\ &\geq c_1^2 e^{-2t_0 A}(x,y)\big(1 - K_1\langle \mathbf{1}_{B(0,r)}\rangle (r_m - r)^{-d-\alpha}\big) \\ &\quad \text{(we select r_m sufficiently large)} \\ &\geq C_{2,r_m} e^{-2t_0 A}(x,y) \quad C_{2,r_m} > 0. \end{aligned}$$

3) If $r \leq |x| \leq r_m$, $|y| > r_m$, then

$$\begin{aligned} e^{-2t_0\Lambda^*}(x,y) &\geq c_1^2 \langle e^{-t_0 A}(x,\cdot)\mathbf{1}_{B^c(0,r)}(\cdot)e^{-t_0 A}(\cdot,y)\rangle \\ &\geq C_{3,r_m}\langle e^{-t_0 A}(x,\cdot)\mathbf{1}_{B^c(0,r)}(\cdot)\rangle (r + |y|)^{-d-\alpha} \\ &\geq C_{4,r_m} e^{-2t_0 A}(0,y) \geq C_{5,r_m} e^{-2t_0 A}(x,y), \quad C_{i,r_m} > 0\ (i = 3,4,5). \end{aligned}$$

4) If $r \leq |y| \leq r_m$, $|x| > r_m$, then, by the symmetry of $e^{-t_0 A}$, $e^{-2t_0\Lambda^*}(x,y) \geq C_{5,r_m} e^{-2t_0 A}(x,y)$.

Thus, we have proved that $e^{-2t_0\Lambda^*}(x,y) \geq c_2 e^{-2t_0 A}(x,y)$, $c_2 > 0$, for all $|x|,|y| \geq r$. Continuing this process, we obtain the assertion of the claim. $\square$

We are in position to complete the proof of the lower bound using the so-called $3q$ argument.
Set $q_t(x,y) := \varphi^{-1}(x) e^{-t\Lambda^*}(x,y)$ $(\varphi \equiv \varphi_1)$.
(a) Let $x,y \in B^c(0,1)$, $x \neq y$. Then by Claim 3

$$q_3(x,y) \geq \varphi^{-1}(x) e^{-3\Lambda^*}(x,y) \geq e^{-3\Lambda^*}(x,y) \geq c e^{-3A}(x,y).$$

Now, fix $R_0 = 1$.

(b) Let $x \in B(0,1)$, $|y| \geq r$, $x \neq y$. By the reproduction property,

$$\begin{aligned} q_2(x,y) &\geq \varphi^{-1}(x)\langle e^{-\Lambda^*}(x,\cdot)\varphi^{-1}(\cdot)\varphi(\cdot)e^{-\Lambda^*}(\cdot,y)\mathbf{1}_{R,r}(\cdot)\rangle \\ &\geq \varphi^{-1}(x)\varphi^{-1}(y)\langle e^{-\Lambda^*}(x,\cdot)\varphi(\cdot)e^{-\Lambda^*}(\cdot,y)\mathbf{1}_{R,r}(\cdot)\rangle \\ &\text{(we are applying Proposition 4)} \\ &\geq e^{-\hat{\mu}-1}\varphi^{-1}(y)\inf_{r\leq|z|\leq R} e^{-\Lambda^*}(z,y) \\ &\text{(we are applying Claim 4)} \\ &\geq e^{-\hat{\mu}-1}\varphi^{-1}(y)c(r)e^{-A}(x,y) \\ &\geq C_1(r)e^{-A}(x,y). \end{aligned}$$

(b') Let $x \in B(0,1), |y| \geq 1$ $(> r)$, $x \neq y$. Arguing as in (b), we obtain

$$q_3(x,y) \geq C_2 e^{-3A}(x,y).$$

(c) Let $|x| \geq r$, $y \in B(0,1)$, $x \neq y$. We have

$$\begin{aligned} q_2(x,y) &\geq \varphi^{-1}(x)\langle e^{-\Lambda^*}(x,\cdot)e^{-\Lambda^*}(\cdot,y)\mathbf{1}_{R,r}(\cdot)\rangle \\ &= \varphi^{-1}(x)\langle e^{-\Lambda^*}(x,\cdot)e^{-\Lambda}(y,\cdot)\mathbf{1}_{R,r}(\cdot)\rangle \\ &\text{(we are applying Claim 4)} \\ &\geq \varphi^{-1}(x)c(r)\langle e^{-A}(x,\cdot)e^{-\Lambda}(y,\cdot)\mathbf{1}_{R,r}(\cdot)\rangle \\ &\geq C_3(r)(R+|x|)^{-d-\alpha}\langle e^{-\Lambda}(y,\cdot)\mathbf{1}_{R,r}(\cdot)\rangle \\ &\text{(we are applying Proposition 6)} \\ &\geq C_3(r)2^{-1}(R+|x|)^{-d-\alpha} \geq C_4(r)e^{-2A}(x,y). \end{aligned}$$

(c') Let $|x| \geq 1(> r)$, $y \in B(0,1)$, $x \neq y$. Arguing as in (c), we obtain

$$q_3(x,y) \geq C_5(r)e^{-3A}(x,y).$$

(d) Let $x, y \in B(0,1)$, $x \neq y$. By the reproduction property,

$$\begin{aligned} q_3(x,y) &\geq \varphi^{-1}(x)\langle e^{-\Lambda^*}(x,\cdot)e^{-2\Lambda^*}(\cdot,y)\mathbf{1}_{R,r}(\cdot)\rangle \\ &\text{(we are using (c))} \\ &\geq \varphi^{-1}(x)C_4(r)\langle e^{-\Lambda^*}(x,\cdot)\varphi(\cdot)e^{-2A}(\cdot,y)\mathbf{1}_{R,r}(\cdot)\rangle \\ &\text{(we are using } e^{-2A}(z,y) \geq c_{r,R} > 0 \text{ for } r \leq |z| \leq R,\ |y| \leq 1) \\ &\geq C_4 c_{r,R}\varphi^{-1}(x)\langle e^{-\Lambda^*}(x,\cdot)\mathbf{1}_{R,r}(\cdot)\varphi(\cdot)\rangle \\ &\text{(we are applying Proposition 4)} \\ &\geq C_4 c_{r,R} e^{-\hat{\mu}-1} \geq C_5(r,R)e^{-3A}(x,y). \end{aligned}$$

By (a), (b'), (c'), (d), $q^3(x,y) \geq Ce^{-3A}(x,y)$ for all $x, y \in \mathbb{R}^d$, $x \neq y$, and so $e^{-3\Lambda^*}(x,y) \geq Ce^{-3A}(x,y)\varphi(x)$. Now the scaling argument yields the lower bound.

Appendix A.

Set $I_\alpha = (-\Delta)^{-\frac{\alpha}{2}}$, the Riesz potential defined by the formula

$$I_\alpha f(x) := \frac{1}{\gamma(\alpha)}\langle |x-\cdot|^{-d+\alpha} f(\cdot)\rangle, \quad \gamma(\alpha) := \frac{2^\alpha \pi^{\frac{d}{2}}\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{d}{2}-\frac{\alpha}{2})}.$$

The identity

$$\frac{\gamma(\beta-\alpha)}{\gamma(\beta)}|x|^{-d+\beta} = I_\alpha |x|^{-d+\beta-\alpha}, \quad 0<\alpha<\beta<d, \tag{$\star$}$$

follows e.g. from $I_\beta = I_\alpha I_{\beta-\alpha}$.

In the proof of Theorem 1 we use a consequence of ($\star$):

$$(-\Delta)^{\frac{\alpha}{2}}|x|^{-d+\beta} = V(x)|x|^{-d+\beta}, \quad V(x) = \frac{\gamma(\beta)}{\gamma(\beta-\alpha)}|x|^{-\alpha},$$

(i.e. $\tilde{\varphi}_1(x) = |x|^{-d+\beta}$ is a Lyapunov's function to the formal operator $(-\Delta)^{\frac{\alpha}{2}} - V$).

Appendix B.

Let P be a closed operator on L^1 such that $\mathrm{Re}\langle(\lambda+P)f, \frac{f}{|f|}\rangle \geq 0$ for all $f\in D(P)$, and $R(\mu+P)$ is dense in L^1 for a $\mu>\lambda$.

Then $R(\mu+P)=L^1$.

Indeed, let $y_n \in R(\mu+P)$, $n=1,2,\dots$, be a Cauchy sequence in L^1; $y_n = (\mu+P)x_n$, $x_n\in D(P)$. Write $[f,g] := \langle f, \frac{g}{|g|}\rangle$. Then

$$\begin{aligned}(\mu-\lambda)\|x_n-x_m\|_1 &= (\mu-\lambda)[x_n-x_m, x_n-x_m] \\ &\leq (\mu-\lambda)[x_n-x_m,x_n-x_m] + [(\lambda+P)(x_n-x_m), x_n-x_m] \\ &= [(\mu+P)(x_n-x_m), x_n-x_m] \leq \|y_n-y_m\|_1.\end{aligned}$$

Thus, $\{x_n\}$ is itself a Cauchy sequence in L^1. Since P is closed, the result follows.

Appendix C.

Let us show that

$$\int_0^t \langle e^{-(t-\tau)A}(x,\cdot)E^\tau(\cdot,y)\rangle d\tau \lesssim t^{1-\frac{1}{\alpha}} e^{-tA}(x,y) \quad \text{ for all } x,y\in\mathbb{R}^d, \quad t>0.$$

Indeed,

$$\begin{aligned}&e^{-(t-\tau)A}(x,z)E^\tau(z,y) \approx e^{-(t-\tau)A}(x,z)e^{-\tau A}(z,y)(|z-y|^{-1}\wedge \tau^{-\frac{1}{\alpha}}) \\ &\text{(we are applying } (*)) \\ &\lesssim e^{-tA}(x,y)\big(e^{-(t-\tau)A}(x,z) + e^{-\tau A}(z,y)\big)\big(|z-y|^{-1}\wedge\tau^{-\frac{1}{\alpha}}\big).\end{aligned}$$

Therefore, using $e^{-tA}(x,z) \lesssim (t|x-z|^{-d-\alpha})\wedge t^{-\frac{d}{\alpha}} \lesssim |x-z|^{-d}\wedge t^{-\frac{d}{\alpha}}$, we obtain

$$\begin{aligned}e^{-(t-\tau)A}(x,z)E^\tau(z,y) &\lesssim e^{-tA}(x,y)\big[(|x-z|^{-d}\wedge(t-\tau)^{-\frac{d}{\alpha}}) + (|z-y|^{-d}\wedge\tau^{-\frac{d}{\alpha}})\big](|z-y|^{-1}\wedge\tau^{-\frac{1}{\alpha}}) \\ &=: e^{-tA}(x,y)\, I,\end{aligned}$$

where, it is easily seen using Young's inequality,

$$I \lesssim |x-z|^{-d-1} \wedge (t-\tau)^{-\frac{d+1}{\alpha}} + |z-y|^{-d-1} \wedge \tau^{-\frac{d+1}{\alpha}}, \quad \text{and so } \int_0^t \langle I \rangle_z d\tau \lesssim t^{1-\frac{1}{\alpha}}.$$

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