Chapter 2

Mathematical Modeling

In this chapter, we give a complete description of the crane model, a derivation of the equations of motion, and the corresponding state-space model. To derive a set of equations of motion that model the system dynamics, we use the Lagrangian approach.

2.1 Model Description

A rotary crane consists of a trolley that moves radially along a rotating jib. The jib rotates in a horizontal plane. The combined movements of the jib and the trolley enable positioning of the trolley and consequently the load over any point in the work space. The variation in the length of the hoisting cable is important for picking up the load, putting it down, and moving it away from obstacles. It also can be used as a part of the control strategy.

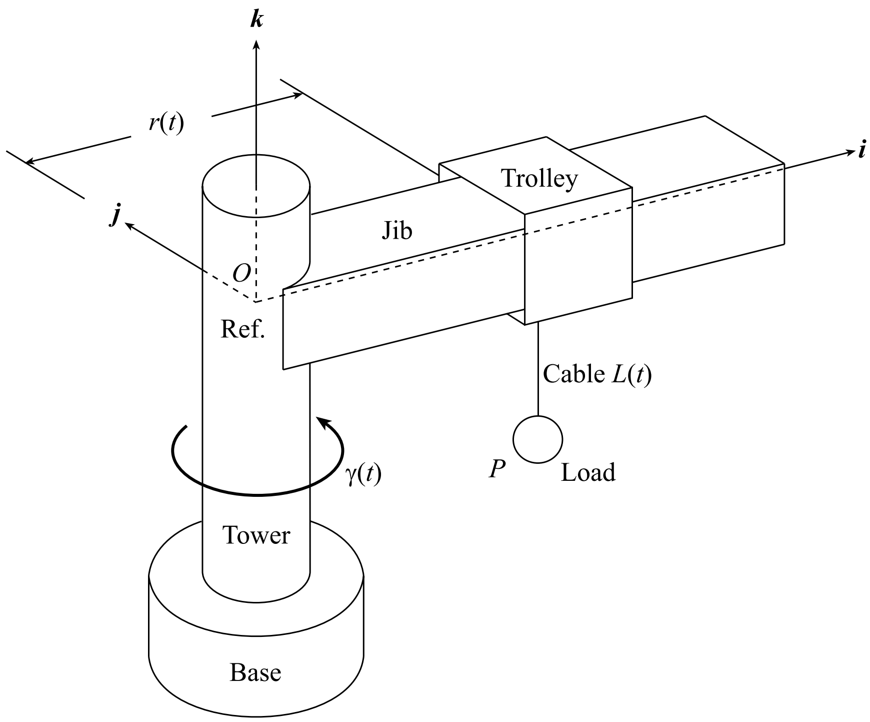

Figure 2.1: A 3D model of a rotary crane.

As shown in Figure 2.1, the structure of the crane consists of

(a) A tower that holds the jib of the crane; it is responsible for the rotational motion of the crane.

(b) A base that is usually fixed to the ground to prevent any oscillations.

(c) A jib that is mounted to the tower.

(d) A trolley that slides over the jib in a transverse direction.

(e) A suspension system of cables and pulleys. In the very general case, the length of the cable can be changed during load transport or at least at the pickup and end points. The process of changing the cable length is called hoisting.

2.2 System Parameters

To derive the equations of motion, one needs to define clearly the system parameters. As shown in Figure 2.1, a right-handed Cartesian coordinate system (xyz) is centered at a reference point that lies in the plane of the jib at the center of the crane tower, with its positive z-axis being along the tower upward axis. The $x-$ and $y-$axes are in the plane of the jib, with the x-axis being along the jib. The xyz coordinate system is attached to the moving jib. The jib rotates and traces an angle $\gamma(t)$. The trolley moves on the jib with its position $r(t)$ being the distance measured from the reference point of the xyz coordinate system to the suspension point of the payload cable on the trolley. The angle $\gamma(t)$ and the radial distance $r(t)$ are the inputs to the system. They are used to control the system behavior. We model the load as a point mass. The interaction between the load dynamics and the crane dynamics is neglected due to the assumption that the mass of the crane being very large compared to that of the load. We start by defining the velocity of the trolley in the jib-fixed coordinate system as

$$v_x = \dot{r}\mathbf{i} \tag{2.1}$$

and its acceleration

$$a_x = \ddot{r}\mathbf{i} \tag{2.2}$$

The angular velocity of the jib is

$$\omega = \dot{\gamma}\mathbf{k} \tag{2.3}$$

and its angular acceleration is

$$\alpha = \ddot{\gamma}\mathbf{k} \tag{2.4}$$

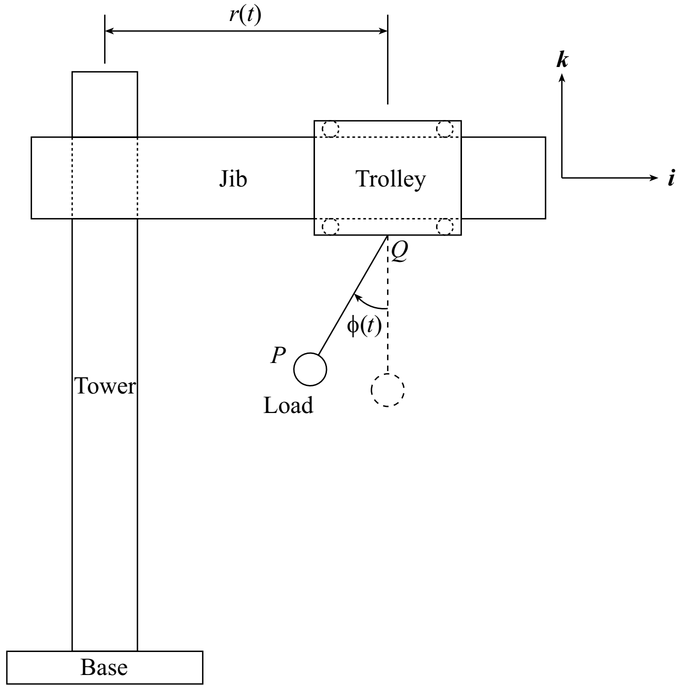

Figure 2.2: Side view of the crane showing the in-plane angle ϕ.

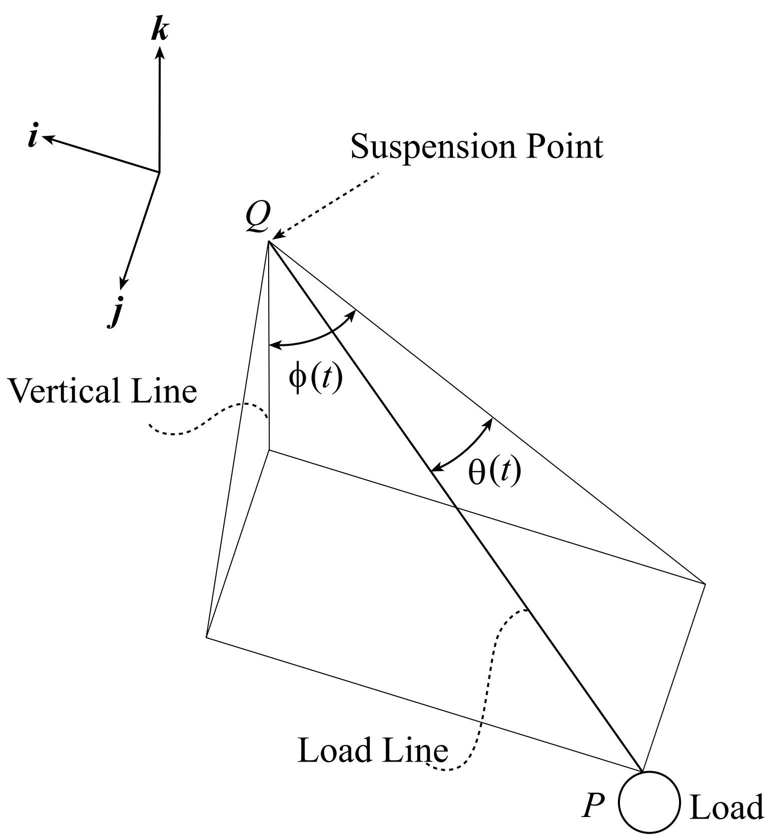

Figure 2.3: Oscillation angles $\phi(t)$ and $\theta(t)$ of the load.

The load pendulations are characterized by two angles, ϕ and θ. The angle ϕ is the angle which the cable makes with the z-axis in the xz-plane, a shown in Figure 2.2. The out-of-plane angle θ is the angle which the cable makes with the xz-plane. So it is clear now that the objective of the controller is to move the payload while keeping ϕ and θ as small as possible, Figure 2.3.

2.3 Derivation of the Equations of Motion

The first step in deriving the equations of motion using the Lagrangian approach is to find the position P(t) of the load with respect to the reference point O. In the jib-fixed coordinate system, the load position is

$$P(t) = [r(t) - L(t)\cos\theta(t)\sin\phi(t)]\mathbf{i} + [L(t)\sin\theta(t)]\mathbf{j} - [L(t)\cos\theta(t)\cos\phi(t)]\mathbf{k} \tag{2.5}$$

To determine the kinetic energy of the load, we need to determine the velocity $\dot{P}$(t) of the load. Since the jib is moving,

$$\dot{P}(t) = \frac{\partial P(t)}{\partial t} + \omega(t) \times P(t) \tag{2.6}$$

where

$$\omega(t) = \dot{\gamma}(t)\mathbf{k} \tag{2.7}$$

Hence, the absolute velocity of the payload is

$$\begin{aligned}\dot{P}(t) =& [\dot{r}(t) - \dot{L}(t)\sin\phi(t)\cos\theta(t) - L(t)(\dot{\gamma}(t)\sin\theta(t)\\ &- \dot{\theta}(t)\sin\theta(t)\sin\phi(t) + \dot{\phi}(t)\cos\theta(t)\cos\phi(t))]\mathbf{i}\\ &+ [\dot{L}(t)\sin\theta(t) + r(t)\dot{\gamma}(t) + L(t)\cos\theta(t)(\dot{\theta}(t) - \dot{\gamma}(t)\sin\phi(t))]\mathbf{j}\\ &+ [-\dot{L}(t)\cos\theta(t)\cos\phi(t) + L(t)(\dot{\theta}\sin\theta(t)\cos\phi(t) + \dot{\phi}(t)\sin\phi(t)\cos\theta(t))]\mathbf{k}\end{aligned} \tag{2.8}$$

The kinetic energy of the load is

$$K_E = \frac{1}{2} m_{Load}[\dot{P}(t).\dot{P}(t)] \tag{2.9}$$

or

$$\begin{aligned}K_E =& \frac{1}{2} m_{Load}\{[\dot{L}(t)\sin\theta(t) + r(t)\dot{\gamma}(t) + L(t)\cos\theta(t)(\dot{\theta}(t) - \dot{\gamma}(t)\sin\theta(t))]^2 \\ &+[\dot{r}(t) - \dot{L}(t)\sin\phi(t)\cos\theta(t) - L(t)(\dot{\gamma}(t)\sin\theta(t) - \dot{\theta}(t)\sin\theta(t)\sin\phi(t) \\ &+\dot{\phi}(t)\cos\theta(t)\cos\phi(t))]^2 + [-\dot{L}(t)\cos\phi(t)\cos\theta(t) + L(t)(\dot{\theta}(t)\sin\theta(t)\cos\phi(t) \\ &+\dot{\phi}(t)\cos\theta(t)\sin\phi(t))]^2\}\end{aligned} \tag{2.10}$$

The potential energy of the payload is given by

$$P_E = -m_{Load} g L(t)\cos\theta(t)\cos\phi(t) \tag{2.11}$$

Finally, the Lagrangian $\mathcal{L}$ is given by

$$\mathcal{L} = K_E - P_E \tag{2.12}$$

or

$$\begin{aligned}\mathcal{L} =& \frac{1}{2} m_{Load}\{2gL(t)\cos\theta(t)\cos\phi(t) + [\dot{L}(t)\sin\theta(t) + r(t)\dot{\gamma}(t) \\ &+ L(t)\cos\theta(t)(\dot{\theta}(t) - \dot{\gamma}(t)\sin\theta(t))]^2 + [\dot{r}(t) - \dot{L}(t)\sin\phi(t)\cos\theta(t) \\ &- L(t)(\dot{\gamma}(t)\sin\theta(t) - \dot{\theta}(t)\sin\theta(t)\sin\phi(t) + \dot{\phi}(t)\cos\theta(t)\cos\phi(t))]^2 \\ &+ [-\dot{L}(t)\cos\phi(t)\cos\theta(t) + L(t)(\dot{\theta}(t)\sin\theta(t)\cos\phi(t) \\ &+ \dot{\phi}(t)\cos\theta(t)\sin\phi(t))]^2\}\end{aligned} \tag{2.13}$$

The Euler Lagrange equations corresponding to $\mathcal{L}$ are

$$\frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial x_i}) - \frac{\partial \mathcal{L}}{\partial x_i} = 0 \tag{2.14}$$

where $x_1 = \theta$ and $x_2 = \phi$. This will yield the following two nonlinear equations of motion:

$$\begin{aligned}
&L(t)\ddot{\theta}(t) + 2\dot{L}(t)\dot{\theta}(t) - 2L(t)\dot{\gamma}(t)\cos\phi(t)\cos^2\theta(t)\dot{\phi}(t) + \frac{1}{2}L(t)\sin 2\theta(t)\dot{\phi}^2(t) \\
&- \frac{1}{2}L(t)\dot{\gamma}^2(t)\sin 2\theta(t)\cos^2\phi(t) + g\sin\theta(t)\cos\phi(t) + 2\dot{r}(t)\dot{\gamma}(t)\cos\theta(t) \\
&- r(t)\dot{\gamma}^2(t)\sin\phi(t)\sin\theta(t) + \ddot{r}(t)\sin\theta(t)\sin\phi(t) - 2\dot{L}(t)\dot{\gamma}(t)\sin\phi(t) \\
&+ r(t)\ddot{\gamma}(t)\cos\theta(t) - L(t)\ddot{\gamma}(t)\sin\phi(t) = 0
\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}
&L(t)\cos\theta(t)\ddot{\phi}(t) + 2\dot{L}(t)\cos\theta(t)\dot{\phi}(t) + 2L(t)\dot{\gamma}(t)\cos\theta(t)\cos\phi(t)\dot{\theta}(t) \\
&- 2L(t)\sin\theta(t)\dot{\theta}(t)\dot{\phi}(t) + g\sin\phi(t) + 2\dot{L}(t)\dot{\gamma}(t)\cos\phi(t)\sin\theta(t) \\
&+ \cos\phi(t)\dot{\gamma}^2(t)[r(t) - L(t)\sin\phi(t)\cos\theta(t)] + L(t)\ddot{\gamma}(t)\sin\theta(t)\cos\phi(t) \\
&- \ddot{r}(t)\cos\phi(t) = 0
\end{aligned} \tag{2.16}$$

For our case, the cable length L(t) is set equal to a constant value, then

$$\frac{dL}{dt} = 0 \tag{2.17}$$

Substituting equation (2.17) into equations (2.15) and (2.16) yields

$$\begin{aligned}
&\ddot{\theta}(t) - 2\dot{\gamma}(t)\cos\phi(t)\cos^2\theta(t)\dot{\phi}(t) + \frac{1}{2}\sin 2\theta(t)\dot{\phi}^2(t) \\
&- \frac{1}{2}\dot{\gamma}^2(t)\sin 2\theta(t)\cos^2\phi(t) + \frac{g}{L}\sin\theta(t)\cos\phi(t) + \frac{2}{L}\dot{r}(t)\dot{\gamma}(t)\cos\theta(t) \\
&- \frac{1}{L}r(t)\dot{\gamma}^2(t)\sin\phi(t)\sin\theta(t) + \frac{1}{L}\ddot{r}(t)\sin\theta(t)\sin\phi(t) + \frac{1}{L}r(t)\ddot{\gamma}(t)\cos\theta(t) \\
&- \ddot{\gamma}(t)\sin\phi(t) = 0
\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}
&\cos\theta(t)\ddot{\phi}(t) + 2\dot{\gamma}(t)\cos\theta(t)\cos\phi(t)\dot{\theta}(t) - 2\sin\theta(t)\dot{\theta}(t)\dot{\phi}(t) \\
&+ \frac{g}{L}\sin\phi(t) + \cos\phi(t)\dot{\gamma}^2(t)[\frac{r(t)}{L} - \sin\phi(t)\cos\theta(t)] \\
&+ \ddot{\gamma}(t)\sin\theta(t)\cos\phi(t) - \frac{1}{L}\ddot{r}(t)\cos\phi(t) = 0
\end{aligned} \tag{2.19}$$

2.4 State-Space Model of the Crane

For easier manipulation of the crane parameters, we reformulate the equations of motion in state-space form. The following equations are used later to simulate the system dynamics. To this end, we let

$$x_1 = \theta(t) \tag{2.20}$$

$$x_2 = \phi(t) \tag{2.21}$$

$$x_3 = r(t) \tag{2.22}$$

$$x_4 = \gamma(t) \tag{2.23}$$

$$x_5 = \dot{\theta}(t) \tag{2.24}$$

$$x_6 = \dot{\phi}(t) \tag{2.25}$$

$$x_7 = \dot{r}(t) \tag{2.26}$$

$$x_8 = \dot{\gamma}(t) \tag{2.27}$$

$$U_1 = \ddot{r}(t) \tag{2.28}$$

$$U_2 = \ddot{\gamma}(t) \tag{2.29}$$

Hence,

$$\dot{x}_1 = x_5 \tag{2.30}$$

$$\dot{x}_2 = x_6 \tag{2.31}$$

$$\dot{x}_3 = x_7 \tag{2.32}$$

$$\dot{x}_4 = x_8 \tag{2.33}$$

$$\dot{x}_7 = U_1 = \ddot{r}(t) \tag{2.34}$$

$$\dot{x}_8 = U_2 = \ddot{\gamma}(t) \tag{2.35}$$

Then, it follows from equations (2.18) and (2.19) that

$$\begin{aligned}\dot{x}_5 = &-\frac{1}{2L}(2g\cos x_2 \sin x_1 + 4x_7x_8\cos x_1 - Lx_8{}^2 \sin 2x_1 \cos^2 x_2 \\ &- 2x_3x_8{}^2 \sin x_1 \sin x_2 - 4Lx_6x_8\cos x_2\cos^2 x_1 + Lx_6{}^2 \sin 2x_1 + 2\sin x_1 \sin x_2 U_1 \\ &+ 2x_3\cos x_1 U_2 - 2L\sin x_2 U_2)\end{aligned} \tag{2.36}$$

$$\begin{aligned}\dot{x_6} = &-\frac{1}{L\cos x_1}(g\sin x_2 + x_3x_8{}^2\cos x_2 - Lx_8{}^2 \sin x_2 \cos x_1 \cos x_2 \\ &+ 2Lx_5x_8\cos x_1\cos x_2 - 2Lx_5x_6\sin x_1 - \cos x_2 U_1 + L\sin x_1\cos x_2 U_2)\end{aligned} \tag{2.37}$$