Quantum Error Correction and Stabilizer Codes

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Overview

1. Quantum Computing

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2. Quantum Errors

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3. Quantum Error Correction

Quantum Computing

	Data	Operations
Classical	Bits	Gates
Quantum	Vectors	(Unitary) Operators

Quantum States

Qubits:

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

where

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

unit vectors in $\mathbb{C}^2$, i.e. $\alpha^2 + \beta^2 = 1$.

Quantum Operations

Quantum Gates (Unitary Operators):

$$U(|\psi\rangle) = \alpha U |0\rangle + \beta U |1\rangle$$

such that

$$U^* U = UU^* = I$$

in $M_2(\mathbb{C})$.

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$$(|\phi\rangle + |\psi\rangle) \otimes |\gamma\rangle = |\phi\rangle \otimes |\gamma\rangle + |\psi\rangle \otimes |\gamma\rangle$$

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symmetric: $|\phi\rangle \otimes |\psi\rangle = |\psi\rangle \otimes |\phi\rangle$

Tensor Example

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$$\begin{aligned} |\psi\rangle \otimes |\phi\rangle &= (\alpha|0\rangle + \beta|1\rangle) \otimes (a|0\rangle + b|1\rangle) \\ &= \alpha|0\rangle \otimes a|0\rangle + \alpha|0\rangle \otimes b|1\rangle + \beta|1\rangle \otimes a|0\rangle + \beta|1\rangle \otimes b|1\rangle \\ &= a\alpha|00\rangle + b\alpha|01\rangle + a\beta|10\rangle + b\beta|11\rangle \end{aligned}$$

Multi-Qubit Gates

- Tensors of single-qubit gates $X \otimes X, I \otimes X$

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- Other: CNOT, SWAP

Common Gates

$$
\begin{aligned}
X|0\rangle &= |1\rangle & \qquad X|1\rangle &= |0\rangle \\
Z|0\rangle &= |0\rangle & \qquad Z|1\rangle &= -|1\rangle \\
H|0\rangle &= |+\rangle & \qquad H|1\rangle &= |-\rangle \\
\mathrm{CNOT}|00\rangle &= |00\rangle & \qquad \mathrm{CNOT}|01\rangle &= |01\rangle \\
\mathrm{CNOT}|10\rangle &= |11\rangle & \qquad \mathrm{CNOT}|11\rangle &= |10\rangle
\end{aligned}
$$

where

$$
|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)
$$

Quantum Errors

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Quantum Errors

- X errors: $X\left|0\right\rangle = \left|1\right\rangle, X\left|1\right\rangle = \left|0\right\rangle$
- Z errors: $Z\left|0\right\rangle = \left|0\right\rangle, Z\left|1\right\rangle = -\left|1\right\rangle$

No-Cloning

There does not exist an operator U such that

$$U |\psi\rangle = |\psi\rangle \otimes |\psi\rangle$$

Bit Flip Code

Encoding: With input state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, encode via
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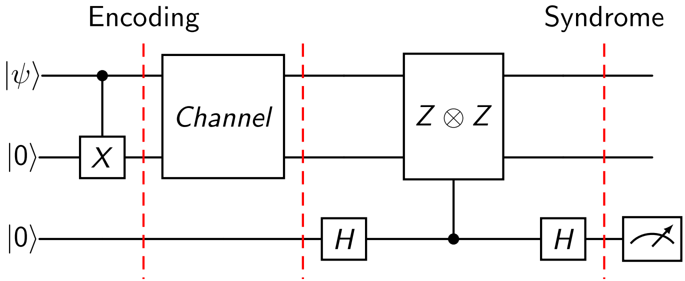
Notice that this code has (in classical terms) distance 1!

Bit Flip Circuit



Example

X-Error detection example.

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X-Error correction example.

Conclusion

A similar method can be used to correct Z-errors (attached file).

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We can extend both of these to correct arbitrary numbers of errors, at the cost of much larger codes.

Thank You!

[1] Jason Crann. *MATH 5821 Lecture Notes*. 2024.

[2] Craig Gidney. *Quirk*. `https://github.com/Strilanc/Quirk`.

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