Quantum Error Correction and Stabilizer Codes

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Overview

1. Quantum Computing

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2. Quantum Errors

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3. Quantum Error Correction

Quantum Computing

	Data	Operations
Classical	Bits	Gates
Quantum	Vectors	(Unitary) Operators

Quantum States

Qubits:

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

where

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad\qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

unit vectors in $\mathbb{C}^2$, i.e. $\alpha^2 + \beta^2 = 1$.

Quantum Operations

Quantum Gates (Unitary Operators):

$$U(|\psi\rangle) = \alpha U|0\rangle + \beta U|1\rangle$$

such that

$$U^* U = UU^* = I$$

in $M_2(\mathbb{C})$.

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$$(|\phi\rangle + |\psi\rangle) \otimes |\gamma\rangle = |\phi\rangle \otimes |\gamma\rangle + |\psi\rangle \otimes |\gamma\rangle$$

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symmetric: $|\phi\rangle \otimes |\psi\rangle = |\psi\rangle \otimes |\phi\rangle$

Tensor Example

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$$

$$
\begin{aligned}
|\psi\rangle \otimes |\phi\rangle &= (\alpha\,|0\rangle + \beta\,|1\rangle) \otimes (a\,|0\rangle + b\,|1\rangle) \\
&= \alpha\,|0\rangle \otimes a\,|0\rangle + \alpha\,|0\rangle \otimes b\,|1\rangle + \beta\,|1\rangle \otimes a\,|0\rangle + \beta\,|1\rangle \otimes b\,|1\rangle \\
&= a\alpha\,|00\rangle + b\alpha\,|01\rangle + a\beta\,|10\rangle + b\beta\,|11\rangle
\end{aligned}
$$

Multi-Qubit Gates

- Tensors of single-qubit gates $X \otimes X, I \otimes X$

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- Other: CNOT, SWAP

Common Gates

$$
\begin{aligned}
X|0\rangle &= |1\rangle & X|1\rangle &= |0\rangle \\
Z|0\rangle &= |0\rangle & Z|1\rangle &= -|1\rangle \\
H|0\rangle &= |+\rangle & H|1\rangle &= |-\rangle \\
\text{CNOT}|00\rangle &= |00\rangle & \text{CNOT}|01\rangle &= |01\rangle \\
\text{CNOT}|10\rangle &= |11\rangle & \text{CNOT}|11\rangle &= |10\rangle
\end{aligned}
$$

where

$$
|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)
$$

Quantum Errors

- X errors: $X|0\rangle = |1\rangle, X|1\rangle = |0\rangle$

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- X errors: $X|0\rangle = |1\rangle, X|1\rangle = |0\rangle$
- Z errors: $Z|0\rangle = |0\rangle, Z|1\rangle = -|1\rangle$

No-Cloning

There does not exist an operator U such that

$$U|\psi\rangle = |\psi\rangle \otimes |\psi\rangle$$

Bit Flip Code

Encoding: With input state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, encode via $\mathrm{CNOT}(|\psi\rangle \otimes |0\rangle) \otimes |0\rangle = (\alpha|00\rangle + \beta|11\rangle) \otimes |0\rangle$

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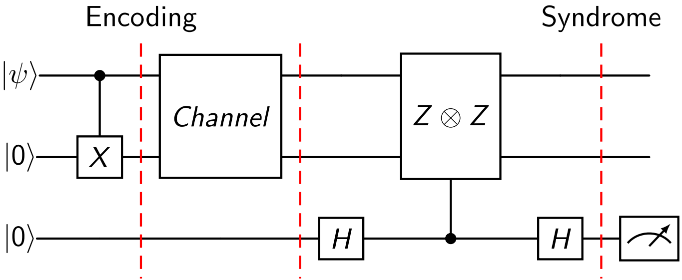
Notice that this code has (in classical terms) distance 2!
We can detect a single error but correct none since we cannot determine the position of the error.

Bit Flip Detection Circuit



Example

Bit-Flip Detection Code

Example

Bit-Flip Detection Code
Bit-Flip Correction Code

Conclusion

A similar method can be used to correct Z-errors.

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These codes can be combined to correct an arbitrary error, and extended to correct more qubits.

Thank You! I

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Thank You! II

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