MATC46 Assignment 1

River McCubbin

January 28, 2021

Question 1: (a) Solve the system $\frac{d\bar{x}}{dt} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \bar{x}$.

First, determine eigenvalues:

$$\begin{aligned} 0 &= \det(A - rI) \\ &= \begin{vmatrix} 2-r & -1 \\ 1 & -r \end{vmatrix} \\ &= (2-r)(-r) \\ &= r^2 - 2r + 1 \\ r &= \frac{2 \pm \sqrt{4-4}}{2} \\ &= 1 \end{aligned}$$

Therefore $r_1 = r_2 = 1$.
Find eigenvectors:

$$\begin{aligned} (A - rI)\xi &= 0 \\ \begin{bmatrix} 2-1 & -1 \\ 1 & -1 \end{bmatrix} \xi &= 0 \\ \xi_1 - \xi_2 &= 0 && \text{since the second equation is redundant} \\ \xi &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

Find a generalized eigenvector:

$$\begin{aligned} (A - rI)\eta &= \xi \\ \begin{bmatrix} 2-1 & -1 \\ 1 & -1 \end{bmatrix} \eta &= \xi \\ \eta_1 - \eta_2 &= 1 \\ \eta &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

Therefore the solution to the system is

$$x = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^t \left[t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right]$$

(b) Classify the type of the trajectories around the origin.
This is a nodal source, since $r_1 = r_2 > 0$.

(c) Determine stability or instability.
$r_1 = r_2 > 0$, therefore this is an unstable improper node.

(d) Sketch the chart of the trajectories in vicinity of the origin in original Cartesian coordinates.
See attached image.

Question 2: (a) Solve the system $\frac{d\bar{x}}{dt} = \begin{bmatrix} 1 & 3 \\ -6 & -5 \end{bmatrix} \bar{x}$.
First, determine the eigenvalues:

$$\begin{aligned} 0 &= \det(A - rI) \\ &= \begin{vmatrix} 1-r & 3 \\ -6 & -5-r \end{vmatrix} \\ &= (1-r)(-5-r) - (3)(-6) \\ &= r^2 + 4r + 13 \\ r &= \frac{4 \pm \sqrt{16 - 4(13)}}{2} \\ &= -2 \pm 3i \end{aligned}$$

take $r = -2 + 3i$ as an eigenvalue.
Find eigenvectors:

$$\begin{aligned} (A - rI)\xi &= 0 \\ \begin{bmatrix} 1-(-2+3i) & 3 \\ -6 & -5-(-2+3i) \end{bmatrix} \xi &= 0 \\ (3-3i)\xi_1 + 3\xi_2 &= 0 \\ \xi &= \begin{bmatrix} 1 \\ -1+i \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

convert to terms of sine and cosine

$$\begin{aligned} e^{rt} &= e^{(-2+3i)t} \\ &= e^{-2t}(\cos(3t) + i\sin(3t)) \end{aligned}$$

Therefore the solution to the system is:

$$\begin{aligned} x &= e^{rt}\xi \\ &= e^{-2t}(\cos(3t) + i\sin(3t)) \begin{bmatrix} 1 \\ -1+i \end{bmatrix} \\ &= e^{-2t}\left(c_1 \begin{bmatrix} \cos(3t) \\ -\cos(3t) - \sin(3t) \end{bmatrix} + c_2 \begin{bmatrix} \sin(3t) \\ \cos(3t) - \sin(3t) \end{bmatrix}\right) \end{aligned}$$

(b) Classify the type of the trajectories around the origin.
$\lambda < 0$ therefore this is a spiral point.

(c) Determine stability or instability.
$\lambda < 0$ therefore this is an asymptotically stable spiral point.

(d) Sketch the chart of the trajectories in the vicinity of the origin in original Cartesian coordinates.
See attached image.