

# Kinematics and Dynamics of Phantom(TM) model 1.5 Haptic Interface

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## Abstract

In this report, derivation of the kinematic and dynamic equations of PHANToM(TM)\* model 1.5 manipulator are presented.

## 1 Introduction

In this document, we will follow the notation of [1] in representing rigid body transformations, and kinematics and dynamics calculations. The units are in MKS<sup>†</sup> system unless otherwise noted.

### Notation

The matrix

$$\mathcal{H} = \begin{bmatrix} & 0 \\ I_{3 \times 3} & 0 \\ & 0 \end{bmatrix} \quad (1)$$

is used to convert from homogeneous coordinates to Euclidean coordinates.

## 2 Kinematics

### 2.1 Forward Kinematics

Using the zero configuration and the naming convention shown in figure 1, the kinematic configuration of the manipulator is characterized by the following vectors and points:

$$w_1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T \quad (2)$$

$$w_2 = w_3 = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}^T \quad (3)$$

$$q_1 = \begin{bmatrix} 0 & 0 & -l_1 \end{bmatrix}^T \quad (4)$$

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\*Phantom is a trademark of SensAble Technologies, Inc. of Cambridge, MA.

<sup>†</sup>MKS: meter-kilogram-second.

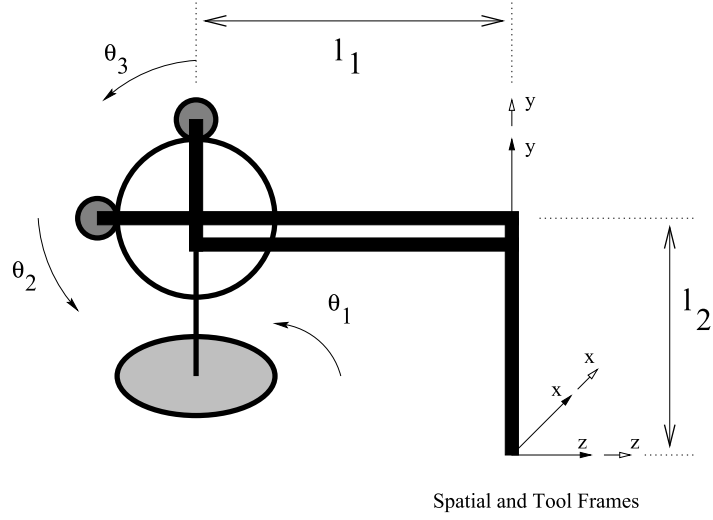


Figure 1: Zero configuration on the manipulator

$$q_2 = q_3 = \begin{bmatrix} 0 & l_2 & -l_1 \end{bmatrix}^T \quad (5)$$

$$\xi_i = \begin{bmatrix} -w_i \times q_i \\ w_i \end{bmatrix}, \quad i = 1, 2, 3. \quad (6)$$

As it can be easily seen from the side and top view illustrations in figure 2, the forward kinematic map is given as:

$$g_{st}(\theta) = \begin{bmatrix} R(\theta) & p(\theta) \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (7)$$

where

$$R(\theta) = e^{\hat{w}_1 \theta_1} e^{\hat{w}_3 \theta_3} I_{3 \times 3} \quad (8)$$

and

$$p(\theta) = \mathcal{H} \left( e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} \begin{bmatrix} I_{3 \times 3} & \begin{bmatrix} 0 \\ l_2 \\ 0 \\ 1 \end{bmatrix} \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} R(\theta) \begin{bmatrix} 0 \\ -l_2 \\ 0 \end{bmatrix} \\ 0 \end{bmatrix} \right). \quad (9)$$

In closed form,

$$g_{st}(\theta) = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \sin(\theta_3) & \cos(\theta_3) \sin(\theta_1) & \sin(\theta_1)(l_1 \cos(\theta_2) + l_2 \sin(\theta_3)) \\ 0 & \cos(\theta_3) & \sin(\theta_3) & l_2 - l_2 \cos(\theta_3) + l_1 \sin(\theta_2) \\ -\sin(\theta_1) & -\cos(\theta_1) \sin(\theta_3) & \cos(\theta_1) \cos(\theta_3) & -l_1 + \cos(\theta_1)(l_1 \cos(\theta_2) + l_2 \sin(\theta_3)) \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (10)$$

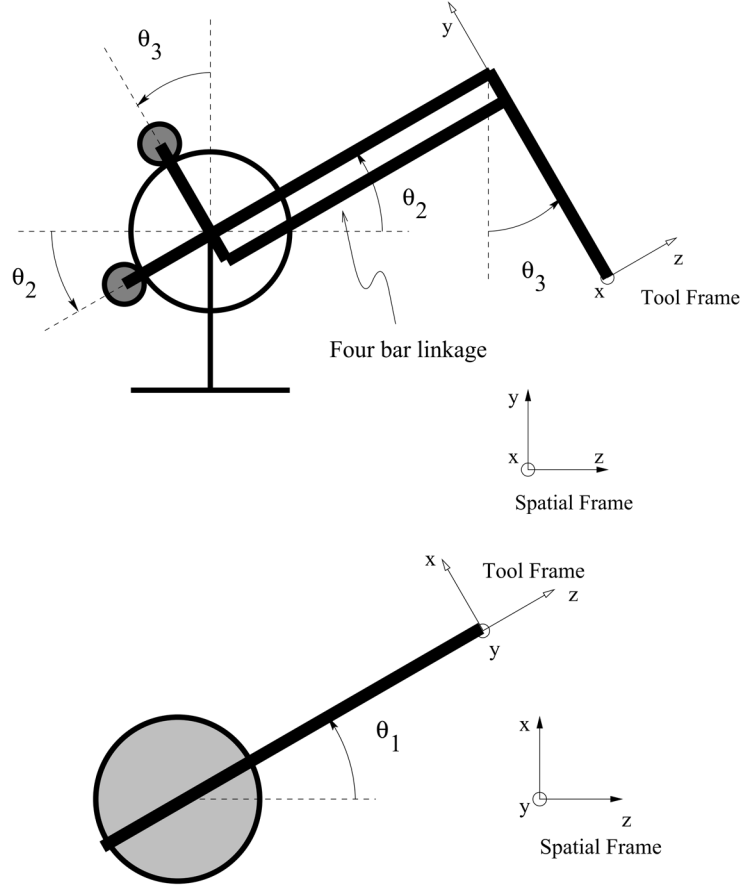


Figure 2: Side and top views

## 2.2 Inverse Kinematics

As this is a 3 DOF manipulator, the inverse kinematics problem is to find the set of  $(\theta_1, \theta_2, \theta_3)$  triples which move the manipulator to a desired end-effector position  $p_o = \begin{bmatrix} p_{ox} & p_{oy} & p_{oz} \end{bmatrix}^T$ .

$\theta_1$  can be calculated by inspections as

$$\theta_1 = \text{atan2}(p_{ox}, p_{oz} + l_1). \quad (11)$$

For,  $\theta_2$  and  $\theta_3$  consider the figure 3. In this figure

$$R = \sqrt{p_{ox}^2 + (p_{oz} + l_1)^2} \quad (12)$$

$$r = \sqrt{p_{ox}^2 + (p_{oy} - l_2)^2 + (p_{oz} + l_1)^2}. \quad (13)$$

By inspection

$$\beta = \text{atan2}(p_{oy} - l_2, R). \quad (14)$$

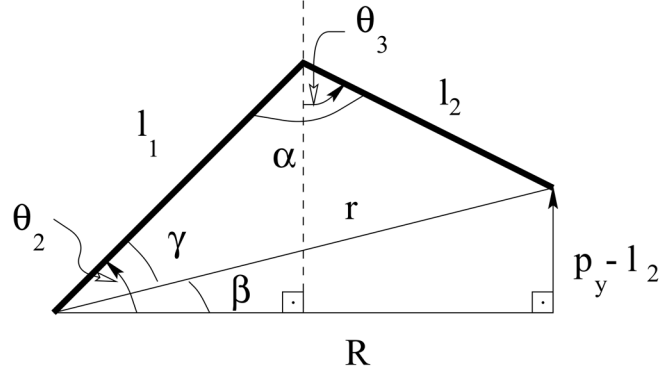


Figure 3: Side view for inverse kinematics calculations

If we write the law of cosines in the upper small triangle

$$l_1^2 + r^2 - 2l_1r \cos(\gamma) = l_2^2 \quad (15)$$

$$\gamma = \cos^{-1}\left(\frac{l_1^2 + r^2 - l_2^2}{2l_1r}\right). \quad (16)$$

In the physical workspace of the manipulator,  $\gamma > 0$ . Then,

$$\theta_2 = \gamma + \beta. \quad (17)$$

To calculate  $\theta_3$ , we write the law of cosines for the same triangle, but this time for the angle  $\alpha$ :

$$l_1^2 + l_2^2 - 2l_1l_2 \cos(\alpha) = r^2 \quad (18)$$

$$\alpha = \cos^{-1}\left(\frac{l_1^2 + l_2^2 - r^2}{2l_1l_2}\right) \quad (19)$$

This angle is also positive in the physical workspace of the manipulator. Then,

$$\theta_3 = \theta_2 + \alpha - \frac{\pi}{2}. \quad (20)$$

## 2.3 Manipulator Jacobian

Spatial Jacobian of the manipulator is given by

$$J^s(\theta) = \begin{bmatrix} \dots & (\frac{\partial g_{st}}{\partial \theta_i} g_{st}^{-1})^\vee & \dots \end{bmatrix} \quad (21)$$

$$= \begin{bmatrix} l_1 & -l_1 \sin(\theta_1) \sin(\theta_2) & \sin(\theta_1)(l_2 + l_1 \sin(\theta_2)) \\ 0 & l_1 \cos(\theta_2) & l_1(\cos(\theta_1) - \cos(\theta_2)) \\ 0 & -l_1 \cos(\theta_1) \sin(\theta_2) & \cos(\theta_1)(l_2 + l_1 \sin(\theta_2)) \\ 0 & 0 & -\cos(\theta_1) \\ 1 & 0 & 0 \\ 0 & 0 & \sin(\theta_1) \end{bmatrix}. \quad (22)$$

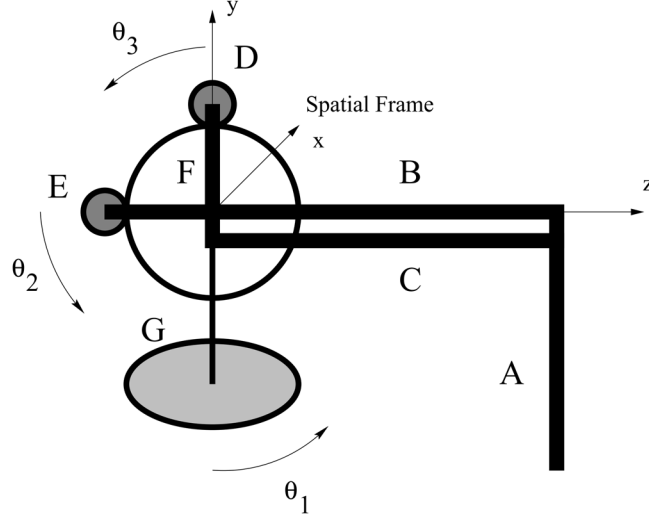


Figure 4: Segments used in dynamics analysis

Body Jacobian of the manipulator is calculated as

$$J^b(\theta) = \begin{bmatrix} \dots & (g_{st}^{-1} \frac{\partial g_{st}}{\partial \theta_i})^\vee & \dots \end{bmatrix} \quad (23)$$

$$= \begin{bmatrix} l_1 \cos(\theta_2) + l_2 \sin(\theta_3) & 0 & 0 \\ 0 & l_1 \cos(\theta_2 - \theta_3) & 0 \\ 0 & -l_1 \sin(\theta_2 - \theta_3) & l_2 \\ 0 & 0 & -1 \\ \cos(\theta_3) & 0 & 0 \\ \sin(\theta_3) & 0 & 0 \end{bmatrix}. \quad (24)$$

### 3 Dynamics

Lagrangian formulation will be used to derive the dynamic equations of the manipulator.

For ease of analysis, we have identified the segments A through G shown on figure 4. For each of the segments, or combinations of segments where appropriate, we will determine the rotation and position vectors of the rigid body transformation between the body and spatial frames, calculate  $\omega^b$  and  $v^s$ , derive the kinetic and potential energies of the segment(s) and finally calculate the inertial parameters of interest.

Note that the spatial frame used in dynamics calculations is centered at a different point than the one used in kinematic analysis. This is to simplify calculations, and does not effect the results.

Due to this change of coordinate frames, the vectors, points and the twists which characterize the configuration are changed as follows:

$$w_1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T \quad (25)$$

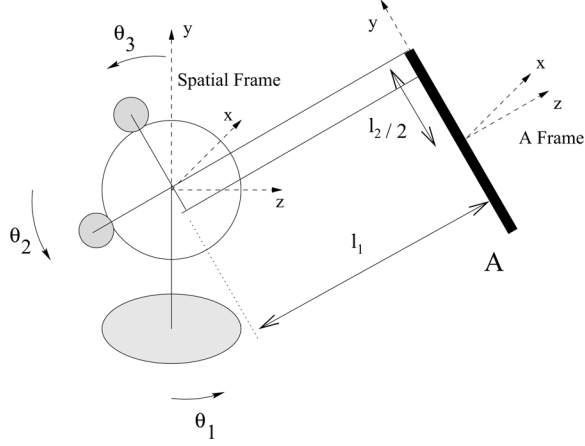


Figure 5: Segment A

$$w_2 = w_3 = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}^T \quad (26)$$

$$q_1 = q_2 = q_3 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T \quad (27)$$

$$\xi_i = \begin{bmatrix} -w_i \times q_i \\ w_i \end{bmatrix}, \quad i = 1, 2, 3 \quad (28)$$

Also, the following parameters of the manipulator are measured as:

$$l_1 = 0.215 \quad (29)$$

$$l_2 = 0.170 \quad (30)$$

$$l_3 = 0.0325 \quad (31)$$

The material used in the manipulator is assumed to be aluminum, except where noted, with a density of 1750kg/m<sup>3</sup>. Also, the motors used are estimated to be Maxon RE-025-055035.

### 3.1 Segment A

The rigid body rotation and the translation between the body frame of segment A and the spatial frame (Fig. 5) are given by

$$R_a(\theta) = e^{\hat{w}_1\theta_1} e^{\hat{w}_3\theta_3} I_{3 \times 3} \quad (32)$$

$$p_a(\theta) = \mathcal{H} \left( e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \begin{bmatrix} I_{3 \times 3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} R_a(\theta) \begin{bmatrix} 0 \\ -l_2/2 \\ 0 \end{bmatrix} \end{bmatrix} \right) \quad (33)$$

Then the potential and kinetic energy of segment A are

$$V_a(\theta) = m_a g p_{ay}(\theta) \quad (34)$$

$$T_a(\theta) = \frac{1}{2} (v_a^s{}^T M_a v_a^s + w_a^b{}^T I_a w_a^b), \quad (35)$$

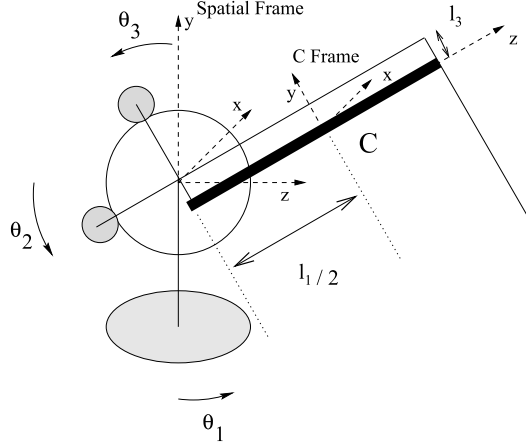


Figure 6: Segment C

where  $M_a = m_a I_{3 \times 3}$  and  $I_a$  are respectively the translational and rotational inertia matrices and

$$v_a^s = \dot{p}_a \quad (36)$$

$$w_a^b = (R_a^T \dot{R}_a)^\vee. \quad (37)$$

### 3.1.1 Inertial Parameters

Segment A is approximated as a hollow aluminum cylinder with internal diameter 3 mm, and outer diameter 8 mm. Then, the mass of segment A and its rotational inertia matrix are:

$$m_a = 0.0202 \quad (38)$$

$$I_a = \begin{bmatrix} I_{a_{xx}} & 0 & 0 \\ 0 & I_{a_{yy}} & 0 \\ 0 & 0 & I_{a_{zz}} \end{bmatrix} \quad (39)$$

$$I_{a_{xx}} = 0.4864 \times 10^{-4} \quad (40)$$

$$I_{a_{yy}} = 0.001843 \times 10^{-4} \quad (41)$$

$$I_{a_{zz}} = 0.4864 \times 10^{-4} \quad (42)$$

## 3.2 Segment C

The rigid body rotation and the translation between the body frame of segment C and the spatial frame (Fig. 6) are given by

$$R_c(\theta) = e^{\hat{w}_1 \theta_1} e^{\hat{w}_2 \theta_2} I_{3 \times 3} \quad (43)$$

$$p_c(\theta) = \mathcal{H} \left( e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_3 \theta_3} \begin{bmatrix} I_{3 \times 3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} R_c(\theta) \begin{bmatrix} 0 \\ 0 \\ -l_1/2 \end{bmatrix} \end{bmatrix} \right) \quad (44)$$

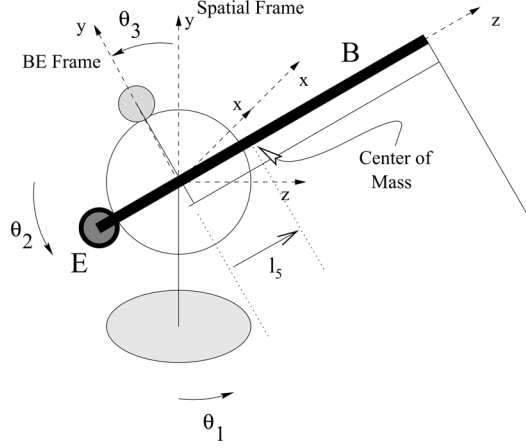


Figure 7: Segments B and E

Then, the potential and kinetic energy of segment C are

$$V_c(\theta) = m_c g p_{cy}(\theta) \quad (45)$$

$$T_c(\theta) = \frac{1}{2}(v_c^s{}^T M_c v_c^s + w_c^b{}^T I_c w_c^b), \quad (46)$$

where  $M_c = m_c I_{3 \times 3}$  and  $I_c$  are respectively the translational and rotational inertia matrices and

$$v_c^s = \dot{p}_c \quad (47)$$

$$w_c^b = (R_c^T \dot{R}_c)^\vee. \quad (48)$$

### 3.2.1 Inertial Parameters

Segment C is also approximated as a hollow aluminum cylinder with internal diameter 7.35 mm, and outer diameter 10.4 mm. Then, the mass of segment C and its rotational inertia matrix are:

$$m_c = 0.0249 \quad (49)$$

$$I_c = \begin{bmatrix} I_{cxx} & 0 & 0 \\ 0 & I_{cyy} & 0 \\ 0 & 0 & I_{czz} \end{bmatrix} \quad (50)$$

$$I_{cxx} = 0.959 \times 10^{-4} \quad (51)$$

$$I_{cyy} = 0.959 \times 10^{-4} \quad (52)$$

$$I_{czz} = 0.0051 \times 10^{-4} \quad (53)$$

### 3.3 Segments B and E

The rigid body rotation between the body frame of segment BE and the spatial frame (Fig. 7) is given by

$$R_{be}(\theta) = e^{\hat{w}_1 \theta_1} e^{\hat{w}_2 \theta_2} I_{3 \times 3}. \quad (54)$$



The translation, however, is equal to zero. Then, the potential and kinetic energy of segment BE are

$$V_{be}(\theta) = m_{be}g \sin(\theta_2)l_5 \quad (55)$$

$$T_{be}(\theta) = \frac{1}{2}(w_{be}^b)^T I_{be} w_{be}^b, \quad (56)$$

where  $m_{be}$  is the total mass of segment BE,  $I_{be}$  is its rotational inertia matrix and

$$w_{be}^b = (R_{be}^T \dot{R}_{be})^\vee. \quad (57)$$

### 3.3.1 Inertial Parameters

Segment BE is the combination of segment B, which is approximated as a hollow aluminum cylinder with internal diameter 7.35 mm, and outer diameter 10.5 mm, and an aluminum plate of dimensions 49×13×31 mm offset from the center of the coordinate frame by 20 mm in the  $-z$  direction, and segment E, which is the electric motor actuating axis 2. Then, the total mass of segment BE and its rotational inertia matrix are:

$$m_{be} = 0.2359 \quad (58)$$

$$I_{be} = \begin{bmatrix} I_{bexx} & 0 & 0 \\ 0 & I_{beyy} & 0 \\ 0 & 0 & I_{bezz} \end{bmatrix} \quad (59)$$

$$I_{bexx} = 11.09 \times 10^{-4}^\ddagger \quad (60)$$

$$I_{beyy} = 10.06 \times 10^{-4} \quad (61)$$

$$I_{bezz} = 0.591 \times 10^{-4} \quad (62)$$

The location of the center of mass of the segment BE is calculated as

$$l_5 = -36.8mm \quad (63)$$

### 3.4 Segments D and F

The rigid body rotation between the body frame of segment DF and the spatial frame (Fig. 8) is given by

$$R_{df}(\theta) = e^{\hat{w}_1\theta_1} e^{\hat{w}_3\theta_3} I_{3 \times 3}. \quad (64)$$

and the translation is equal to zero. Then, the potential and kinetic energy of segment DF are

$$V_{df}(\theta) = m_{df}g \cos(\theta_3)l_6 \quad (65)$$

$$T_{df}(\theta) = \frac{1}{2}(w_{df}^b)^T I_{df} w_{df}^b, \quad (66)$$

where  $m_{df}$  is the total mass of segment DF,  $I_{df}$  is its rotational inertia matrix and

$$w_{df}^b = (R_{df}^T \dot{R}_{df})^\vee. \quad (67)$$

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<sup>‡</sup>Motor inertia has been added to the inertia around the x-axis with a transmission ratio of 11.6:1.0 .

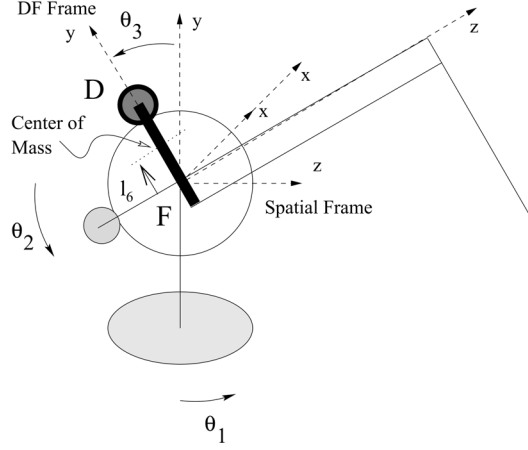


Figure 8: Segments D and F

### 3.4.1 Inertial Parameters

Segment DF is the combination of segment D, which is the electric motor actuating axis 2, and segment F, which is approximated as the combination of two stainless steel rods lying parallel to the x-axis of length 63 mm, diameter 5 mm and offset from the origin by 32.5 mm in the + and - y directions, and two rectangular tubes lying parallel to the y-axis of dimensions 7×39.5×9 mm, offset from the origin by ±28 mm along the x-axis. Then, the total mass of segment DF and its rotational inertia matrix are:

$$m_{df} = 0.1906 \quad (68)$$

$$I_{df} = \begin{bmatrix} I_{df_{xx}} & 0 & 0 \\ 0 & I_{df_{yy}} & 0 \\ 0 & 0 & I_{df_{zz}} \end{bmatrix} \quad (69)$$

$$I_{df_{xx}} = 7.11 \times 10^{-4} \text{§} \quad (70)$$

$$I_{df_{yy}} = 0.629 \times 10^{-4} \quad (71)$$

$$I_{df_{zz}} = 6.246 \times 10^{-4} \quad (72)$$

The location of the center of mass of the segment DF is calculated as

$$l_6 = 52.7 \text{ mm} \quad (73)$$

### 3.5 Segment G - the Base

The base can only rotate around the y-axis of the spatial frame. Then, its potential energy is zero, and its kinetic energy is given by

$$V_{base} = \frac{1}{2} \dot{\theta}_1 I_{base_{yy}} \dot{\theta}_1. \quad (74)$$

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§Motor inertia has been added to the inertia around the x-axis with a transmission ratio of 11.6:1.0 .

### 3.5.1 Inertial Parameters

The base is composed of a flat semicircular plate of radius 88 mm and thickness 2.1 mm, an aluminum ring of radius 88 mm and thickness 2.1 mm, a rectangular plate centered at the point of rotation that is 154×2×13.3 mm, a similar rectangular plate of dimensions 120×8×9 mm, a cylinder centered at the point of rotation of height 21.3 mm and radius 13 mm, two vertical shafts of dimensions 8×89×9 mm offset from the point of rotation by ± 55 mm in the x direction, a steel bar parallel to the x-axis of length 120 mm and diameter 6.5 mm, a circular plate lying in the y-z plane offset from the point of rotation by 51 mm and thickness 2.7 mm, a ring lying in the y-z plane offset from the point of rotation by 41 mm of radius 57 mm, thickness 2.2 mm and width (x dimension) of 18 mm, and the motor rotor inertia  $0.11 \times 10^{-4} \text{kg.m}^2$  through a transmission ratio of 13.3:1.0. Then, the only component of rotational inertia matrix of the base that is of interest is calculated as

$$I_{baseyy} = 11.87 \times 10^{-4} \quad (75)$$

### 3.6 Equations of Motion

Lagrangian of the manipulator is given by

$$L = T - V \quad (76)$$

$$= (T_a + T_c + T_{be} + T_{df} + T_{base}) - (V_a + V_c + V_{be} + V_{df} + V_{base}) \quad (77)$$

$$= \quad (78)$$

Then, the dynamic equations are calculated as

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = \tau_i, \quad i = 1, 2, 3. \quad (79)$$

$$\begin{bmatrix} M_{11} & 0 & 0 \\ 0 & M_{22} & M_{23} \\ 0 & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix} + \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & 0 & C_{23} \\ C_{31} & C_{32} & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} \quad (80)$$

where

$$\begin{aligned} M_{11} = & \left( \frac{1}{8} (4I_{a yy} + 4I_{a zz} + 8I_{base yy} + 4I_{be yy} + 4I_{be zz} + 4I_{c yy} + 4I_{c zz} + 4I_{df yy} + 4I_{df zz} + 4l_1^2 m_a + l_2^2 m_a + l_1^2 m_c + 4l_3^2 m_c) \right. \\ & + \frac{1}{8} (4I_{be yy} - 4I_{be zz} + 4I_{c yy} - 4I_{c zz} + l_1^2 (4m_a + m_c)) \cos(2\theta_2) \\ & \left. + \frac{1}{8} (4I_{a yy} - 4I_{a zz} + 4I_{df yy} - 4I_{df zz} - l_2^2 m_a - 4l_3^2 m_c) \cos(2\theta_3) + l_1(l_2 m_a + l_3 m_c) \cos(\theta_2) \sin(\theta_3) \right) \end{aligned} \quad (81)$$

$$M_{22} = \frac{1}{4} (4I_{be xx} + I_{c xx} + l_1^2 m_a) + l_1^2 m_c \quad (82)$$

$$M_{23} = -\frac{1}{2} l_1 (l_2 m_a + l_3 m_c) \sin(\theta_2 - \theta_3) \quad (83)$$

$$M_{32} = -\frac{1}{2} l_1 (l_2 m_a + l_3 m_c) \sin(\theta_2 - \theta_3) \quad (84)$$

$$M_{33} = \frac{1}{4} (4I_{a xx} + 4I_{df xx} + l_2^2 m_a + 4l_3^2 m_c) \quad (85)$$

$$\begin{aligned} C_{11} = & \frac{1}{8} \left( -2 \sin(\theta_2) \left( (4I_{be yy} - 4I_{be zz} + 4I_{c yy} - 4I_{c zz} + 4l_1^2 m_a + l_1^2 m_c) \cos(\theta_2) + 2l_1(l_2 m_a + l_3 m_c) \sin(\theta_3) \right) \dot{\theta}_2 \right. \\ & \left. + 2 \cos(\theta_3) \left( 2l_1(l_2 m_a + l_3 m_c) \cos(\theta_2) + (-4I_{a yy} + 4I_{a zz} - 4I_{df yy} + 4I_{df zz} + l_2^2 m_a + 4l_3^2 m_c) \sin(\theta_3) \right) \dot{\theta}_3 \right) \end{aligned} \quad (86)$$

$$C_{12} = -\frac{1}{8} \left( (4I_{be\,yy} - 4I_{be\,zz} + 4I_{c\,yy} - 4I_{c\,zz} + l_1^2(4m_a + m_c)) \sin(2\theta_2) + 4l_1(l_2m_a + l_3m_c) \sin(\theta_2) \sin(\theta_3) \right) \dot{\theta}_1 \quad (87)$$

$$C_{13} = -\frac{1}{8} (-4l_1(l_2m_a + l_3m_c) \cos(\theta_2) \cos(\theta_3) - (-4I_{a\,yy} + 4I_{a\,zz} - 4I_{df\,yy} + 4I_{df\,zz} + l_2^2m_a + 4l_3^2m_c) \sin(2\theta_3)) \dot{\theta}_1 \quad (88)$$

$$C_{21} = -C_{12} \quad (89)$$

$$C_{23} = \frac{1}{2} l_1(l_2m_a + l_3m_c) \cos(\theta_2 - \theta_3) \dot{\theta}_3 \quad (90)$$

$$C_{31} = -C_{13} \quad (91)$$

$$C_{32} = \frac{1}{2} l_1(l_2m_a + l_3m_c) \cos(\theta_2 - \theta_3) \dot{\theta}_2 \quad (92)$$

$$N_2 = \frac{1}{2} g(2l_1m_a + 2l_5m_{be} + l_1m_c) \cos(\theta_2) \quad (93)$$

$$N_3 = \frac{1}{2} g(l_2m_a + 2l_3m_c - 2l_6m_{df}) \sin(\theta_3). \quad (94)$$

Note that  $M(\theta)$  is positive definite and  $(\dot{M}(\theta) - 2C(\theta, \dot{\theta}))$  is skew symmetric.

## Appendix - Mathematica Code

## References

- [1] R. M. Murray and Z. Li and S. S. Sastry, *A Mathematical Introduction to Robotic Manipulation*, CRC Press, Inc., 1994.

## ■ Initialization

We need to load variational methods, and matrix manipulation packages.

```
<< Calculus`VariationalMethods`
<< LinearAlgebra`MatrixManipulation`;
```

We also need to load the screw calculus library of Murray, Li, Sastry (MLS) [1].

```
<< ~/surgery/phantom/PhantomDynamics/Screws.m
<< ~/surgery/phantom/PhantomDynamics/RobotLinks.m
```

And some definitions...

```
I33 = IdentityMatrix[3]; I44 = IdentityMatrix[4];
Origin = {{0}, {0}, {0}, {1}};
Vedge[what_] := {{-what[[2]][[3]]}, {what[[1]][[3]]}, {-what[[1]][[2]]}};
TopPart[a_] := a[{{1, 2, 3}, {1}}];
SE3inv[g_] := BlockMatrix[{{Transpose[SubMatrix[g, {1, 1}, {3, 3}]],
  -Transpose[SubMatrix[g, {1, 1}, {3, 3}] . SubMatrix[g, {1, 4}, {3, 1}]],
  {{0, 0, 0, 1}}}}];
```

Let's try the example on pp. 164–5 of MLS to test the EulerEquations[] function.

```
f[u_, v_, t_] := {{u'[t], v'[t]}.{{alpha + 2 beta Cos[v[t]], delta + beta Cos[v[t]]},
  {delta + beta Cos[v[t]], delta}}.{{u'[t]}, {v'[t]}}/
  2

EulerEquations[f[u, v, t], {u[t], v[t]}, t]

{{{1/2 (4 beta Sin[v[t]] u'[t] v'[t] + 2 beta Sin[v[t]] v'[t]^2 -
  2 (alpha + 2 beta Cos[v[t]]) u''[t] - 2 (delta + beta Cos[v[t]]) v''[t])}} ==
  0,
  {{-beta Sin[v[t]] u'[t]^2 - (delta + beta Cos[v[t]]) u''[t] - delta v''[t]}} == 0}

uv[u_, v_, t_] := {{u[t]}, {v[t]}};
ff[u_, v_, t_] :=
  D[Transpose[uv[u, v, t], t].{{alpha + 2 beta Cos[v[t]], delta + beta Cos[v[t]]},
    {delta + beta Cos[v[t]], delta}}.D[uv[u, v, t], t]/2
EulerEquations[ff[u, v, t], {u[t], v[t]}, t]

{{{1/2 (4 beta Sin[v[t]] u'[t] v'[t] + 2 beta Sin[v[t]] v'[t]^2 -
  2 (alpha + 2 beta Cos[v[t]]) u''[t] - 2 (delta + beta Cos[v[t]]) v''[t])}} ==
  0,
  {{-beta Sin[v[t]] u'[t]^2 - (delta + beta Cos[v[t]]) u''[t] - delta v''[t]}} == 0}
```

## ■ Phantom Kinematics

### ■ Common Definitions

First, some common definitions:

```
wfk1 = {0, 1, 0}; wfk1h = AxisToSkew[wfk1]; qfk1 = {0, 0, -l1};
wfk2 = {-1, 0, 0}; wfk2h = AxisToSkew[wfk2]; qfk2 = {0, l2, -l1};
wfk3 = {-1, 0, 0}; wfk3h = AxisToSkew[wfk3]; qfk3 = {0, l2, -l1};
```

## ■ Forward Kinematics

```
Rforkin[th1_, th2_, th3_] := SkewExp[wfk1h, th1] . SkewExp[wfk3h, th3] . I33;
pforkin[th1_, th2_, th3_] := Rforkin[th1, th2, th3] . {{0}, {-l2}, {0}} +
  TopPart[TwistExp[RevoluteTwist[qfk1, wfk1], th1] .
    TwistExp[RevoluteTwist[qfk2, wfk2], th2] . RPToHomogeneous[I33, {0, l2, 0}] . Origin];

gst[th1_, th2_, th3_] :=
  Simplify[RPToHomogeneous[Rforkin[th1, th2, th3], Flatten[pforkin[th1, th2, th3]]]];

gst[ee1, ee2, ee3]
{{Cos[ee1], -Sin[ee1] Sin[ee3], Cos[ee3] Sin[ee1], Sin[ee1] (l1 Cos[ee2] + l2 Sin[ee3])},
 {0, Cos[ee3], Sin[ee3], l2 - l2 Cos[ee3] + l1 Sin[ee2]}, {-Sin[ee1],
 -Cos[ee1] Sin[ee3], Cos[ee1] Cos[ee3], -l1 + Cos[ee1] (l1 Cos[ee2] + l2 Sin[ee3])},
 {0, 0, 0, 1}}
```

## ■ Jacobian

Spatial Jacobian is given by

```
Js1 = Transpose[{HomogeneousToTwist[
  Simplify[D[gst[th1, th2, th3], th1] . SE3inv[gst[th1, th2, th3]]]]];
Js2 = Transpose[{HomogeneousToTwist[
  Simplify[D[gst[th1, th2, th3], th2] . SE3inv[gst[th1, th2, th3]]]]];
Js3 = Transpose[{HomogeneousToTwist[
  Simplify[D[gst[th1, th2, th3], th3] . SE3inv[gst[th1, th2, th3]]]]];
Js = BlockMatrix[{{Js1, Js2, Js3}}]

{{l1, -l1 Sin[th1] Sin[th2], Sin[th1] (l2 + l1 Sin[th2])},
 {0, l1 Cos[th2], l1 (Cos[th1] - Cos[th2])},
 {0, -l1 Cos[th1] Sin[th2], Cos[th1] (l2 + l1 Sin[th2])}, {0, 0, -Cos[th1]}},
 {1, 0, 0}, {0, 0, Sin[th1]}}
```

MatrixForm[Js]

$$\begin{pmatrix} l1 & -l1 \sin[th1] \sin[th2] & \sin[th1] (l2 + l1 \sin[th2]) \\ 0 & l1 \cos[th2] & l1 (\cos[th1] - \cos[th2]) \\ 0 & -l1 \cos[th1] \sin[th2] & \cos[th1] (l2 + l1 \sin[th2]) \\ 0 & 0 & -\cos[th1] \\ 1 & 0 & 0 \\ 0 & 0 & \sin[th1] \end{pmatrix}$$

Body Jacobian is calculated as



```

Jb1 = Transpose[{HomogeneousToTwist[
  Simplify[SE3inv[gst[th1, th2, th3]].D[gst[th1, th2, th3], th1]]]];
Jb2 = Transpose[{HomogeneousToTwist[
  Simplify[SE3inv[gst[th1, th2, th3]].D[gst[th1, th2, th3], th2]]]];
Jb3 = Transpose[{HomogeneousToTwist[
  Simplify[SE3inv[gst[th1, th2, th3]].D[gst[th1, th2, th3], th3]]]];
Jb = BlockMatrix[{{Jb1, Jb2, Jb3}}]

{{11 Cos[th2] + 12 Sin[th3], 0, 0}, {0, 11 Cos[th2 - th3], 0}, {0, -11 Sin[th2 - th3], 12},
{0, 0, -1}, {Cos[th3], 0, 0}, {Sin[th3], 0, 0}}

MatrixForm[Jb]

```

$$\begin{pmatrix} 11 \cos[\text{th2}] + 12 \sin[\text{th3}] & 0 & 0 \\ 0 & 11 \cos[\text{th2} - \text{th3}] & 0 \\ 0 & -11 \sin[\text{th2} - \text{th3}] & 12 \\ 0 & 0 & -1 \\ \cos[\text{th3}] & 0 & 0 \\ \sin[\text{th3}] & 0 & 0 \end{pmatrix}$$

Let's define the adjoint of `gst`. We will use it to check if the spatial and body Jacobians of the manipulator are consistent.

```

Ad[th1_, th2_, th3_] := BlockMatrix[{{Rforkin[th1, th2, th3],
  AxisToSkew[Flatten[pforkin[th1, th2, th3]].Rforkin[th1, th2, th3]],
  {0 * I33, Rforkin[th1, th2, th3]}}];

Simplify[Js - Simplify[Ad[th1, th2, th3].Jb]]

{{0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

```

Yes, they check out.

## ■ Phantom Dynamics

Now, let's start calculating the equations governing the dynamics of the PHANToM.

### ■ Common Definitions

First, some common definitions:

```

w1 = {0, 1, 0}; w1h = AxisToSkew[w1];
w2 = {-1, 0, 0}; w2h = AxisToSkew[w2];
w3 = {-1, 0, 0}; w3h = AxisToSkew[w3];

```

## ■ Segment A

```

Ra[th1_, th2_, th3_, t_] := SkewExp[w1h, th1[t]] . SkewExp[w3h, th3[t]] . I33;
wab[th1_, th2_, th3_, t_] :=
  Vedge[Transpose[Ra[th1, th2, th3, t]] . D[Ra[th1, th2, th3, t], t]];
pa[th1_, th2_, th3_, t_] := Ra[th1, th2, th3, t] . {{0}, {-12/2}, {0}} +
  TopPart[TwistExp[RevoluteTwist[{0, 0, 0}, w1], th1[t]] .
    TwistExp[RevoluteTwist[{0, h, 0}, w2], th2[t]] .
    RPTToHomogeneous[I33, {0, h, 11}].Origin];
vas[th1_, th2_, th3_, t_] := D[pa[th1, th2, th3, t], t];
Ma = ma * I33; Ia = DiagonalMatrix[{Iaxx, Iayy, Iazz}];
Va[th1_, th2_, th3_, t_] := ma * g * pa[th1, th2, th3, t][[2]][[1]];
Ta[th1_, th2_, th3_, t_] :=
  Simplify[1/2 * (Transpose[vas[th1, th2, th3, t]] . Ma . vas[th1, th2, th3, t] +
    Transpose[wab[th1, th2, th3, t]] . Ia . wab[th1, th2, th3, t])];

Va[ee1, ee2, ee3, t]

```

$$g \, ma \left( h (1 - \cos[ee2[t]]) + h \cos[ee2[t]] - \frac{1}{2} 12 \cos[ee3[t]] + 11 \sin[ee2[t]] \right)$$

```
Ta[ee1, ee2, ee3, t]
```

$$\left\{ \left\{ \frac{1}{8} \left( \frac{1}{2} (4 I_{ayy} + 4 I_{azz} + 4 l_1^2 ma + l_2^2 ma + 4 l_1 l_2 ma \cos[2 ee2[t]] + \right. \right. \right. \\ \left. \left. \left. 4 I_{ayy} \cos[2 ee3[t]] - 4 I_{azz} \cos[2 ee3[t]] - l_2^2 ma \cos[2 ee3[t]] - \right. \right. \right. \\ \left. \left. \left. 4 l_1 l_2 ma \sin[ee2[t] - ee3[t]] + 4 l_1 l_2 ma \sin[ee2[t] + ee3[t]] \right) ee1'[t]^2 + \right. \right. \\ \left. \left. 4 l_1^2 ma ee2'[t]^2 - 4 l_1 l_2 ma \sin[ee2[t] - ee3[t]] ee2'[t] ee3'[t] + \right. \right. \\ \left. \left. (4 I_{axx} + l_2^2 ma) ee3'[t]^2 \right) \right\} \right\}$$

## ■ Segment C

```

Rc[th1_, th2_, th3_, t_] := SkewExp[w1h, th1[t]] . SkewExp[w2h, th2[t]] . I33;
wcb[th1_, th2_, th3_, t_] :=
  Vedge[Transpose[Rc[th1, th2, th3, t]] . D[Rc[th1, th2, th3, t], t]];
pc[th1_, th2_, th3_, t_] := Rc[th1, th2, th3, t] . {{0}, {0}, {11/2}} +
  TopPart[TwistExp[RevoluteTwist[{0, 0, 0}, w1], th1[t]] .
    TwistExp[RevoluteTwist[{0, h, 0}, w3], th3[t]] .
    RPTToHomogeneous[I33, {0, h - 13, 0}].Origin];
vcs[th1_, th2_, th3_, t_] := D[pc[th1, th2, th3, t], t];
Mc = mc * I33; Ic = DiagonalMatrix[{Icxx, Icy, Iczz}];
Vc[th1_, th2_, th3_, t_] := mc * g * pc[th1, th2, th3, t][[2]][[1]];
Tc[th1_, th2_, th3_, t_] :=
  Simplify[1/2 * (Transpose[vcs[th1, th2, th3, t]] . Mc . vcs[th1, th2, th3, t] +
    Transpose[wcb[th1, th2, th3, t]] . Ic . wcb[th1, th2, th3, t])];

Vc[ee1, ee2, ee3, t]

```

$$g \, mc \left( h (1 - \cos[ee3[t]]) + (h - 13) \cos[ee3[t]] + \frac{1}{2} 11 \sin[ee2[t]] \right)$$

**Tc[ee1, ee2, ee3, t]**

$$\left\{ \left\{ \frac{1}{8} \left( \frac{1}{2} (4 \text{Icyy} + 4 \text{Iczz} + 11^2 \text{mc} + 4 13^2 \text{mc} + (4 \text{Icyy} - 4 \text{Iczz} + 11^2 \text{mc}) \cos[2 \text{ee2}[t]] - 4 13^2 \text{mc} \cos[2 \text{ee3}[t]] - 4 11 13 \text{mc} \sin[\text{ee2}[t] - \text{ee3}[t]] + 4 11 13 \text{mc} \sin[\text{ee2}[t] + \text{ee3}[t]]) \text{ee1}'[t]^2 + (4 \text{Icxx} + 11^2 \text{mc}) \text{ee2}'[t]^2 - 4 11 13 \text{mc} \sin[\text{ee2}[t] - \text{ee3}[t]] \text{ee2}'[t] \text{ee3}'[t] + 4 13^2 \text{mc} \text{ee3}'[t]^2 \right) \right\} \right\}$$

## ■ Segments B and E

```
Rbe[th1_, th2_, th3_, t_] := SkewExp[w1h, th1[t]] . SkewExp[w2h, th2[t]] . I33;
wbeb[th1_, th2_, th3_, t_] :=
  Vedge[Transpose[Rbe[th1, th2, th3, t]] . D[Rbe[th1, th2, th3, t], t]];
vbes[th1_, th2_, th3_, t_] := {{0}, {0}, {0}};
Mbe = mbe * I33; Ibe = DiagonalMatrix[{Ibexx, Ibeyy, Ibezz}];
Vbe[th1_, th2_, th3_, t_] := mbe * g * Sin[th2[t]] * l5;
Tbe[th1_, th2_, th3_, t_] :=
  Simplify[1/2 * (Transpose[vbes[th1, th2, th3, t]] . Mbe . vbes[th1, th2, th3, t] +
    Transpose[wbeb[th1, th2, th3, t]] . Ibe . wbeb[th1, th2, th3, t])];
```

**Vbe[ee1, ee2, ee3, t]**

**g l5 mbe Sin[ee2[t]]**

**Tbe[ee1, ee2, ee3, t]**

$$\left\{ \left\{ \frac{1}{4} ((\text{Ibeyy} + \text{Ibezz} + (\text{Ibeyy} - \text{Ibezz}) \cos[2 \text{ee2}[t]]) \text{ee1}'[t]^2 + 2 \text{Ibexx} \text{ee2}'[t]^2) \right\} \right\}$$

## ■ Segments D and F

```
Rdf[th1_, th2_, th3_, t_] := SkewExp[w1h, th1[t]] . SkewExp[w3h, th3[t]] . I33;
wdfb[th1_, th2_, th3_, t_] :=
  Vedge[Transpose[Rdf[th1, th2, th3, t]] . D[Rdf[th1, th2, th3, t], t]];
vdfs[th1_, th2_, th3_, t_] := {{0}, {0}, {0}};
Mdf = mdf * I33; Idf = DiagonalMatrix[{Idfxx, Idfyy, Idfzz}];
Vdf[th1_, th2_, th3_, t_] := mdf * g * Cos[th3[t]] * l6;
Tdf[th1_, th2_, th3_, t_] :=
  Simplify[1/2 * (Transpose[vdfs[th1, th2, th3, t]] . Mdf . vdfs[th1, th2, th3, t] +
    Transpose[wdfb[th1, th2, th3, t]] . Idf . wdfb[th1, th2, th3, t])];
```

**Vdf[ee1, ee2, ee3, t]**

**g l6 mdf Cos[ee3[t]]**

**Tdf[ee1, ee2, ee3, t]**

$$\left\{ \left\{ \frac{1}{4} ((\text{Idfyy} + \text{Idfzz} + (\text{Idfyy} - \text{Idfzz}) \cos[2 \text{ee3}[t]]) \text{ee1}'[t]^2 + 2 \text{Idfxx} \text{ee3}'[t]^2) \right\} \right\}$$

## ■ Segment Base

```
Vbase[th1_, th2_, th3_, t_] := 0;
Tbase[th1_, th2_, th3_, t_] := Simplify[1/2 * D[th1[t], t] * Ibeyy * D[th1[t], t]];
```

Vbase[ee1, ee2, ee3, t]

0

Tbase[ee1, ee2, ee3, t]

$\frac{1}{2} \text{Ibaseyy ee1}'[t]^2$

## ■ Equations of Motion of the Whole Structure

The Lagrangian of the manipulator is

```
Simplify[(Ta[th1, th2, th3, t] + Tc[th1, th2, th3, t] +
  Tbe[th1, th2, th3, t] + Tdf[th1, th2, th3, t] + Tbase[th1, th2, th3, t]) -
  (Va[th1, th2, th3, t] + Vc[th1, th2, th3, t] + Vbe[th1, th2, th3, t] +
  Vdf[th1, th2, th3, t] + Vbase[th1, th2, th3, t])]

{ { 1/16 ( (4 Iayy + 4 Iazz + 8 Ibaseyy + 4 Ibeyy + 4 Ibezz +
  4 Icy y + 4 Iczz + 4 Idfy y + 4 Idfzz + 4 l1^2 ma + l2^2 ma + l1^2 mc + 4 l3^2 mc +
  (4 Ibeyy - 4 Ibezz + 4 Icy y - 4 Iczz + 4 l1^2 ma + l1^2 mc) Cos[2 th2[t]] +
  4 Iayy Cos[2 th3[t]] - 4 Iazz Cos[2 th3[t]] + 4 Idfy y Cos[2 th3[t]] -
  4 Idfzz Cos[2 th3[t]] - l2^2 ma Cos[2 th3[t]] - 4 l3^2 mc Cos[2 th3[t]] -
  4 l1 l2 ma Sin[th2[t] - th3[t]] - 4 l1 l3 mc Sin[th2[t] - th3[t]] +
  4 l1 l2 ma Sin[th2[t] + th3[t]] + 4 l1 l3 mc Sin[th2[t] + th3[t]]) th1'[t]^2 -
  2 (8 g h ma + 8 g h mc - 4 g l2 ma Cos[th3[t]] - 8 g l3 mc Cos[th3[t]] +
  8 g l6 mdf Cos[th3[t]] + 8 g l1 ma Sin[th2[t]] + 8 g l5 mbe Sin[th2[t]] +
  4 g l1 mc Sin[th2[t]] - (4 Ibexx + 4 Icx x + 4 l1^2 ma + l1^2 mc) th2'[t]^2 +
  4 l1 (l2 ma + l3 mc) Sin[th2[t] - th3[t]] th2'[t] th3'[t] - 4 Iaxx th3'[t]^2 -
  4 Idfxx th3'[t]^2 - l2^2 ma th3'[t]^2 - 4 l3^2 mc th3'[t]^2) ) } }
```

Let's write the Lagrangian as a function

```
L[th1_, th2_, th3_, t_] := Simplify[(Ta[th1, th2, th3, t] + Tc[th1, th2, th3, t] +
  Tbe[th1, th2, th3, t] + Tdf[th1, th2, th3, t] + Tbase[th1, th2, th3, t]) -
  (Va[th1, th2, th3, t] + Vc[th1, th2, th3, t] + Vbe[th1, th2, th3, t] +
  Vdf[th1, th2, th3, t] + Vbase[th1, th2, th3, t])][[1]][[1]];
```

This is the dynamics equaitons

**DyEq = FullSimplify[EulerEquations[L[th1, th2, th3, t], {th1[t], th2[t], th3[t]}, t]]**

$$\begin{aligned}
& \left\{ \frac{1}{8} (2 \text{th1}'[t] ((4 \text{Ibeyy} - 4 \text{Ibezz} + 4 \text{Icyy} - 4 \text{Iczz} + l_1^2 (4 \text{ma} + \text{mc})) \sin[2 \text{th2}[t]] + \right. \\
& \quad 4 l_1 (l_2 \text{ma} + l_3 \text{mc}) \sin[\text{th2}[t]] \sin[\text{th3}[t]] \text{th2}'[t] + \\
& \quad (-4 l_1 (l_2 \text{ma} + l_3 \text{mc}) \cos[\text{th2}[t]] \cos[\text{th3}[t]] - \\
& \quad (-4 \text{Iayy} + 4 \text{Iazz} - 4 \text{Idfyy} + 4 \text{Idfzz} + l_2^2 \text{ma} + 4 l_3^2 \text{mc}) \sin[2 \text{th3}[t]]) \text{th3}'[t]) - \\
& \quad (4 \text{Iayy} + 4 \text{Iazz} + 8 \text{Ibaseyy} + 4 \text{Ibeyy} + 4 \text{Ibezz} + 4 \text{Icyy} + \\
& \quad 4 \text{Iczz} + 4 \text{Idfyy} + 4 \text{Idfzz} + 4 l_1^2 \text{ma} + l_2^2 \text{ma} + l_1^2 \text{mc} + 4 l_3^2 \text{mc}) \text{th1}''[t] - \\
& \quad (4 \text{Ibeyy} - 4 \text{Ibezz} + 4 \text{Icyy} - 4 \text{Iczz} + l_1^2 (4 \text{ma} + \text{mc})) \cos[2 \text{th2}[t]] \text{th1}''[t] - \\
& \quad (4 \text{Iayy} - 4 \text{Iazz} + 4 \text{Idfyy} - 4 \text{Idfzz} - l_2^2 \text{ma} - 4 l_3^2 \text{mc}) \cos[2 \text{th3}[t]] \text{th1}''[t] - \\
& \quad 8 l_1 (l_2 \text{ma} + l_3 \text{mc}) \cos[\text{th2}[t]] \sin[\text{th3}[t]] \text{th1}''[t]) == 0, \\
& \frac{1}{8} (-4 g (2 l_1 \text{ma} + 2 l_5 \text{mbe} + l_1 \text{mc}) \cos[\text{th2}[t]] + \\
& \quad (- (4 \text{Ibeyy} - 4 \text{Ibezz} + 4 \text{Icyy} - 4 \text{Iczz} + l_1^2 (4 \text{ma} + \text{mc})) \sin[2 \text{th2}[t]] - \\
& \quad 4 l_1 (l_2 \text{ma} + l_3 \text{mc}) \sin[\text{th2}[t]] \sin[\text{th3}[t]]) \text{th1}'[t]^2 - \\
& \quad 4 l_1 (l_2 \text{ma} + l_3 \text{mc}) \cos[\text{th2}[t] - \text{th3}[t]] \text{th3}'[t]^2 - \\
& \quad 2 (4 (\text{Ibexx} + \text{Icxx} + l_1^2 \text{ma}) + l_1^2 \text{mc}) \text{th2}''[t] + \\
& \quad 4 l_1 (l_2 \text{ma} + l_3 \text{mc}) \sin[\text{th2}[t] - \text{th3}[t]] \text{th3}''[t]) == 0, \\
& \frac{1}{8} \\
& \quad (-4 g (l_2 \text{ma} + 2 l_3 \text{mc} - 2 l_6 \text{mdf}) \sin[\text{th3}[t]] + (4 l_1 (l_2 \text{ma} + l_3 \text{mc}) \cos[\text{th2}[t]] \cos[\text{th3}[t]] + \\
& \quad (-4 \text{Iayy} + 4 \text{Iazz} - 4 \text{Idfyy} + 4 \text{Idfzz} + l_2^2 \text{ma} + 4 l_3^2 \text{mc}) \sin[2 \text{th3}[t]]) \text{th1}'[t]^2 + \\
& \quad 4 l_1 (l_2 \text{ma} + l_3 \text{mc}) (\cos[\text{th2}[t] - \text{th3}[t]] \text{th2}'[t]^2 + \sin[\text{th2}[t] - \text{th3}[t]] \text{th2}''[t]) - \\
& \quad 2 (4 \text{Iaxx} + 4 \text{Idfxx} + l_2^2 \text{ma} + 4 l_3^2 \text{mc}) \text{th3}''[t]) == \\
& \quad 0 \}
\end{aligned}$$

With signs corrected to follow the convention in MLS, the motor torques at each axis are equal to:

**Tau1 = Collect[-DyEq[[1]][[1]], {th1''[t], th2''[t], th3''[t], th1'[t], th2'[t], th3'[t]}]**

$$\begin{aligned}
& \text{th1}'[t] \left( \frac{1}{4} (- (4 \text{Ibeyy} - 4 \text{Ibezz} + 4 \text{Icyy} - 4 \text{Iczz} + l_1^2 (4 \text{ma} + \text{mc})) \sin[2 \text{th2}[t]] - \right. \\
& \quad 4 l_1 (l_2 \text{ma} + l_3 \text{mc}) \sin[\text{th2}[t]] \sin[\text{th3}[t]]) \text{th2}'[t] + \\
& \quad \frac{1}{4} (4 l_1 (l_2 \text{ma} + l_3 \text{mc}) \cos[\text{th2}[t]] \cos[\text{th3}[t]] + \\
& \quad (-4 \text{Iayy} + 4 \text{Iazz} - 4 \text{Idfyy} + 4 \text{Idfzz} + l_2^2 \text{ma} + 4 l_3^2 \text{mc}) \sin[2 \text{th3}[t]]) \text{th3}'[t] \Big) + \\
& \left( \frac{1}{8} (4 \text{Iayy} + 4 \text{Iazz} + 8 \text{Ibaseyy} + 4 \text{Ibeyy} + 4 \text{Ibezz} + \right. \\
& \quad 4 \text{Icyy} + 4 \text{Iczz} + 4 \text{Idfyy} + 4 \text{Idfzz} + 4 l_1^2 \text{ma} + l_2^2 \text{ma} + l_1^2 \text{mc} + 4 l_3^2 \text{mc}) + \\
& \quad \frac{1}{8} (4 \text{Ibeyy} - 4 \text{Ibezz} + 4 \text{Icyy} - 4 \text{Iczz} + l_1^2 (4 \text{ma} + \text{mc})) \cos[2 \text{th2}[t]] + \\
& \quad \frac{1}{8} (4 \text{Iayy} - 4 \text{Iazz} + 4 \text{Idfyy} - 4 \text{Idfzz} - l_2^2 \text{ma} - 4 l_3^2 \text{mc}) \cos[2 \text{th3}[t]] + \\
& \quad \left. l_1 (l_2 \text{ma} + l_3 \text{mc}) \cos[\text{th2}[t]] \sin[\text{th3}[t]] \right) \\
& \text{th1}''[t]
\end{aligned}$$

```

Tau2 =
Collect[-DyEq[[2]][[1]], {th1''[t], th2''[t], th3''[t], th1'[t], th2'[t], th3'[t]}]


$$\begin{aligned} & \frac{1}{2} g (2 l1 ma + 2 l5 mbe + l1 mc) \cos[th2[t]] + \\ & \frac{1}{8} ((4 Ibeyy - 4 Ibezz + 4 Icy - 4 Icz + l1^2 (4 ma + mc)) \sin[2 th2[t]] + \\ & \quad 4 l1 (l2 ma + l3 mc) \sin[th2[t]] \sin[th3[t]]) th1'[t]^2 + \\ & \frac{1}{2} l1 (l2 ma + l3 mc) \cos[th2[t] - th3[t]] th3'[t]^2 + \\ & \frac{1}{4} (4 (Ibexx + Icxx + l1^2 ma) + l1^2 mc) th2''[t] - \\ & \frac{1}{2} l1 (l2 ma + l3 mc) \sin[th2[t] - th3[t]] th3''[t] \end{aligned}$$


Tau3 =
Collect[-DyEq[[3]][[1]], {th1''[t], th2''[t], th3''[t], th1'[t], th2'[t], th3'[t]}]


$$\begin{aligned} & \frac{1}{2} g (l2 ma + 2 l3 mc - 2 l6 mdf) \sin[th3[t]] + \frac{1}{8} (-4 l1 (l2 ma + l3 mc) \cos[th2[t]] \cos[th3[t]] - \\ & \quad (-4 Iayy + 4 Iazz - 4 Idfy + 4 Idfz + l2^2 ma + 4 l3^2 mc) \sin[2 th3[t]]) th1'[t]^2 - \\ & \frac{1}{2} l1 (l2 ma + l3 mc) \cos[th2[t] - th3[t]] th2'[t]^2 - \\ & \frac{1}{2} l1 (l2 ma + l3 mc) \sin[th2[t] - th3[t]] th2''[t] + \\ & \frac{1}{4} (4 Iaxx + 4 Idfxx + l2^2 ma + 4 l3^2 mc) th3''[t] \end{aligned}$$


```

Then these are the elements of the M, C and N matrices

$$\begin{aligned}
M11 = & \left( \frac{1}{8} (4 I_{ayy} + 4 I_{azz} + 8 I_{baseyy} + 4 I_{beyy} + 4 I_{bezz} + \right. \\
& 4 I_{cyy} + 4 I_{czz} + 4 I_{dfyy} + 4 I_{dfzz} + 4 l_1^2 m_a + 4 l_2^2 m_a + 4 l_1^2 m_c + 4 l_3^2 m_c) + \\
& \frac{1}{8} (4 I_{beyy} - 4 I_{bezz} + 4 I_{cyy} - 4 I_{czz} + l_1^2 (4 m_a + m_c)) \cos[2 \theta_2[t]] + \\
& \frac{1}{8} (4 I_{ayy} - 4 I_{azz} + 4 I_{dfyy} - 4 I_{dfzz} - l_2^2 m_a - 4 l_3^2 m_c) \cos[2 \theta_3[t]] + \\
& \left. l_1 (l_2 m_a + l_3 m_c) \cos[\theta_2[t]] \sin[\theta_3[t]] \right); \\
M12 = & 0; \\
M13 = & 0; \\
M21 = & 0; \\
M22 = & \frac{1}{4} (4 (I_{bexx} + I_{cxx} + l_1^2 m_a) + l_1^2 m_c) \theta_2''[t]; \\
M23 = & -\frac{1}{2} l_1 (l_2 m_a + l_3 m_c) \sin[\theta_2[t] - \theta_3[t]]; \\
M31 = & 0; \\
M32 = & -\frac{1}{2} l_1 (l_2 m_a + l_3 m_c) \sin[\theta_2[t] - \theta_3[t]] \theta_2''[t]; \\
M33 = & \frac{1}{4} (4 I_{axx} + 4 I_{dxx} + l_2^2 m_a + 4 l_3^2 m_c); \\
C21 = & \frac{1}{8} ((4 I_{beyy} - 4 I_{bezz} + 4 I_{cyy} - 4 I_{czz} + l_1^2 (4 m_a + m_c)) \sin[2 \theta_2[t]] + \\
& 4 l_1 (l_2 m_a + l_3 m_c) \sin[\theta_2[t]] \sin[\theta_3[t]]) \theta_1'[t]; \\
C22 = & 0; \\
C23 = & \frac{1}{2} l_1 (l_2 m_a + l_3 m_c) \cos[\theta_2[t] - \theta_3[t]] \theta_3'[t]; \\
C31 = & \frac{1}{8} (-4 l_1 (l_2 m_a + l_3 m_c) \cos[\theta_2[t]] \cos[\theta_3[t]] - \\
& (-4 I_{ayy} + 4 I_{azz} - 4 I_{dfyy} + 4 I_{dfzz} + l_2^2 m_a + 4 l_3^2 m_c) \sin[2 \theta_3[t]]) \theta_1'[t]; \\
C32 = & \frac{1}{2} l_1 (l_2 m_a + l_3 m_c) \cos[\theta_2[t] - \theta_3[t]] \theta_2'[t]; \\
C33 = & 0; \\
N1 = & 0; \\
N2 = & \frac{1}{2} g (2 l_1 m_a + 2 l_5 m_{be} + l_1 m_c) \cos[\theta_2[t]]; \\
N3 = & \frac{1}{2} g (l_2 m_a + 2 l_3 m_c - 2 l_6 m_{df}) \sin[\theta_3[t]];
\end{aligned}$$

We will choose the first row of C so that  $(\dot{M} - 2C)$  is skew symmetric.

$$\begin{aligned}
C12 = & -C21; \\
C13 = & -C31;
\end{aligned}$$

```

C11x = Collect[
  Simplify[ ( th1'[t] ( (1/4) (- (4 Ibeyy - 4 Ibezz + 4 Icy - 4 Iczz + l1^2 (4 ma + mc)) Sin[2 th2[t]] -
    4 l1 (l2 ma + l3 mc) Sin[th2[t]] Sin[th3[t]]) th2'[t] +
    (1/4) (4 l1 (l2 ma + l3 mc) Cos[th2[t]] Cos[th3[t]] +
    (-4 Iayy + 4 Iazz - 4 Idfy + 4 Idfz + l2^2 ma + 4 l3^2 mc) Sin[2 th3[t]]) th3'[t] ) ) -
    C12 th2'[t] - C13 th3'[t] ],
  th1'[t] ]

1/8 th1'[t] (-2 Sin[th2[t]] ((4 Ibeyy - 4 Ibezz + 4 Icy - 4 Iczz + 4 l1^2 ma + l1^2 mc) Cos[th2[t]] +
  2 l1 (l2 ma + l3 mc) Sin[th3[t]]) th2'[t] +
  2 Cos[th3[t]] (2 l1 (l2 ma + l3 mc) Cos[th2[t]] +
  (-4 Iayy + 4 Iazz - 4 Idfy + 4 Idfz + l2^2 ma + 4 l3^2 mc) Sin[th3[t]])
  th3'[t] )

```

Then

```

C11 = 1/8 (-2 Sin[th2[t]] ((4 Ibeyy - 4 Ibezz + 4 Icy - 4 Iczz + 4 l1^2 ma + l1^2 mc) Cos[th2[t]] +
  2 l1 (l2 ma + l3 mc) Sin[th3[t]]) th2'[t] +
  2 Cos[th3[t]] (2 l1 (l2 ma + l3 mc) Cos[th2[t]] +
  (-4 Iayy + 4 Iazz - 4 Idfy + 4 Idfz + l2^2 ma + 4 l3^2 mc) Sin[th3[t]])
  th3'[t] );

```

Let's check if  $(\dot{M} - 2C)$  is skew symmetric :

```

Simplify[D[M11, t] - 2 * C11]

0

```

Yes, it holds!

## ■ Numerical Values

```

h = 0;
l1 = 0.215; l2 = 0.170; l3 = 0.0325; l5 = -0.0368; l6 = 0.0527;
ma = 0.0202;
Iaxx = 0.4864 * 10^-4; Iayy = 0.0018 * 10^-4; Iazz = 0.4864 * 10^-4;
mc = 0.0249;
Icxx = 0.959 * 10^-4; Icy = 0.959 * 10^-4; Iczz = 0.0051 * 10^-4;
mbe = 0.2359;
Ibexx = 11.09 * 10^-4; Ibeyy = 10.06 * 10^-4; Ibezz = 0.591 * 10^-4;
mdf = 0.1906;
Idfxx = 7.11 * 10^-4; Idfy = 0.629 * 10^-4; Idfz = 6.246 * 10^-4;
Ibaseyy = 11.87 * 10^-4;
g = 9.81;

```



**Tau1**

$$\begin{aligned} & \text{th1}'[t] \left( \frac{1}{4} (-0.00905514 \sin[2 \text{th2}[t]] - 0.00364919 \sin[\text{th2}[t]] \sin[\text{th3}[t]]) \text{th2}'[t] + \right. \\ & \quad \left. \frac{1}{4} (0.00364919 \cos[\text{th2}[t]] \cos[\text{th3}[t]] + 0.00312962 \sin[2 \text{th3}[t]]) \text{th3}'[t] \right) + \\ & (0.00283279 + 0.00113189 \cos[2 \text{th2}[t]] - \\ & \quad 0.000391203 \cos[2 \text{th3}[t]] + 0.000912299 \cos[\text{th2}[t]] \sin[\text{th3}[t]]) \\ & \text{th1}''[t] \end{aligned}$$

**Tau2**

$$\begin{aligned} & -0.016298 \cos[\text{th2}[t]] + \\ & \frac{1}{8} (0.00905514 \sin[2 \text{th2}[t]] + 0.00364919 \sin[\text{th2}[t]] \sin[\text{th3}[t]]) \text{th1}'[t]^2 + \\ & 0.000456149 \cos[\text{th2}[t] - \text{th3}[t]] \text{th3}'[t]^2 + 0.0024264 \text{th2}''[t] - \\ & 0.000456149 \sin[\text{th2}[t] - \text{th3}[t]] \text{th3}''[t] \end{aligned}$$

**Tau3**

$$\begin{aligned} & -0.0737552 \sin[\text{th3}[t]] + \\ & \frac{1}{8} (-0.00364919 \cos[\text{th2}[t]] \cos[\text{th3}[t]] - 0.00312962 \sin[2 \text{th3}[t]]) \text{th1}'[t]^2 - \\ & 0.000456149 \cos[\text{th2}[t] - \text{th3}[t]] \text{th2}'[t]^2 - 0.000456149 \sin[\text{th2}[t] - \text{th3}[t]] \text{th2}''[t] + \\ & 0.000931886 \text{th3}''[t] \end{aligned}$$