FOURIER OPTICS

P47 – Optics: Unit 8

Course Outline

~~Unit 1: Electromagnetic Waves~~
~~Unit 2: Interaction with Matter~~
~~Unit 3: Geometric Optics~~

~~Unit 4: Superposition of Waves~~
~~Unit 5: Polarization~~
~~Unit 6: Interference~~

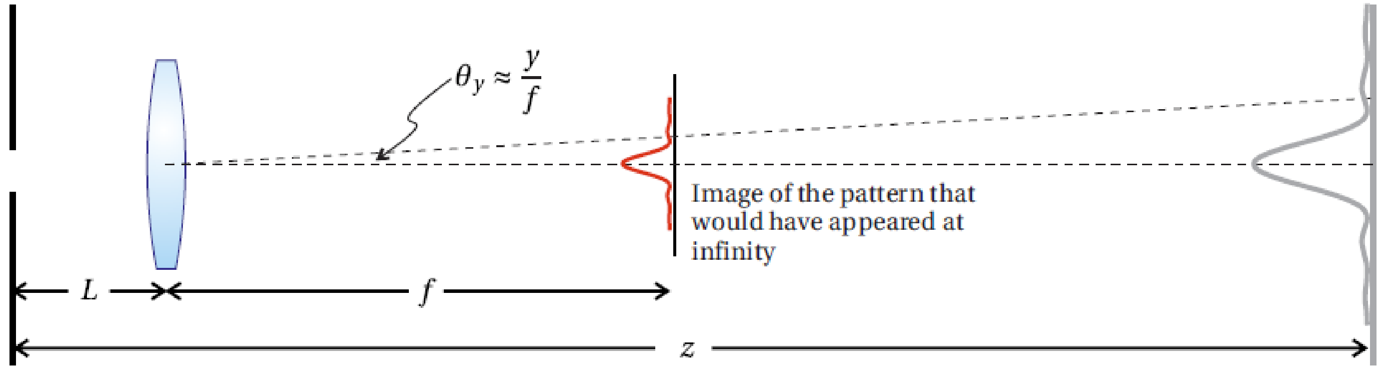
~~Unit 7: Diffraction~~
Unit 8: Fourier Optics
Unit 9: Modern Optics

Fourier Optics: The Basic Idea

$$E(x,y,z) \cong -\frac{i e^{ikz} e^{i\frac{k}{2z}(x^2+y^2)}}{\lambda z} \iint_{\text{aperture}} E(x',y',0)\, e^{-i\frac{k}{z}(xx'+yy')}\, dx'\, dy'$$

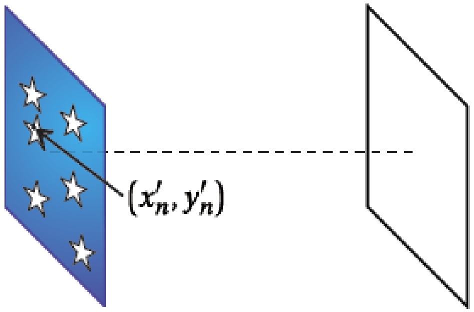
1. The intensity distribution in the far-field (Fraunhofer limit) is the 2D Fourier transform of the intensity distribution at the source.

2. A (perfect) lens causes the image at z=∞ to form at its focal distance.



Peatross & Ware

Array Theorem



Calculate the diffraction pattern caused by N identical apertures with $E_{\text{aperture}}(x', y', 0)$

Each aperture has a location (x_n', y_n'), so that we use $E_{\text{aperture}}(x' - x_n', y' - y_n', 0)$

$$E(x,y,z) = -i\frac{e^{ikz}e^{i\frac{k}{2z}(x^2+y^2)}}{\lambda z}\sum_{n=1}^{N}\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dy' E_{\text{aperture}}(x'-x_n', y'-y_n', 0)\, e^{-i\frac{k}{z}(xx'+yy')}$$

Change of variables: $x'' \equiv x' - x_n'$ and $y'' \equiv y' - y_n'$

$$E(x,y,z) = -i\frac{e^{ikz}e^{i\frac{k}{2z}(x^2+y^2)}}{\lambda z}\sum_{n=1}^{N}\int_{-\infty}^{\infty}dx''\int_{-\infty}^{\infty}dy'' E_{\text{aperture}}(x'', y'', 0)\; e^{-i\frac{k}{z}[x(x''+x_n')+y(y''+y_n')]}$$

$$E(x,y,z) = \left[\sum_{n=1}^{N} e^{-i\frac{k}{z}(xx_n'+yy_n')}\right]\left[-i\frac{e^{ikz}e^{i\frac{k}{2z}(x^2+y^2)}}{\lambda z}\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dy' E_{\text{aperture}}(x', y', 0)\, e^{-i\frac{k}{z}(xx'+yy')}\right]$$

Array Theorem in the Real World:

Discrete Fourier Transform of 2D Images

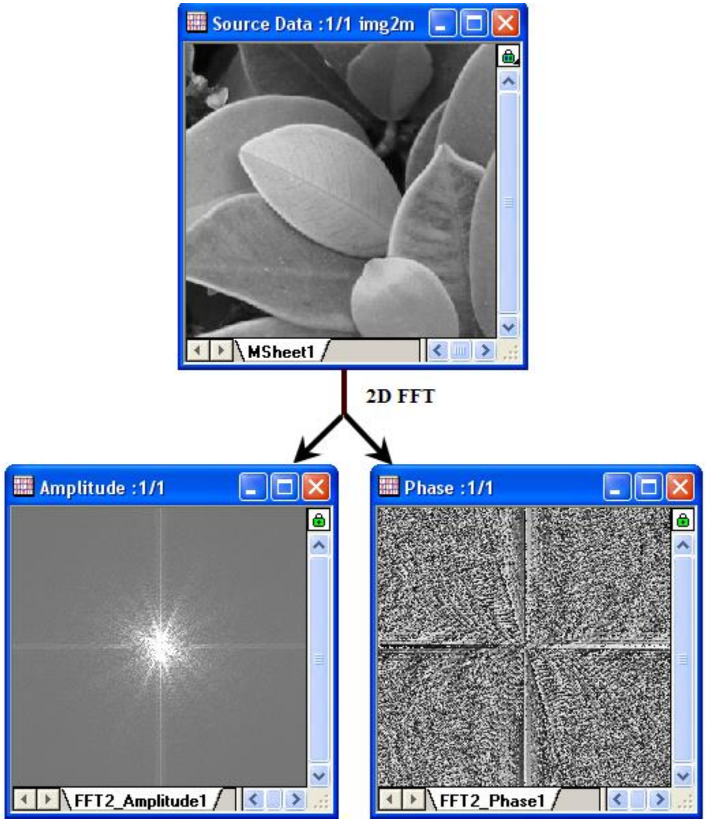
For a list of N sample data points (1D)

$$F(u) = \frac{1}{N}\sum_{n=0}^{N-1} f(n)\, e^{-i(2\pi n u)/N}$$

For an MxN array of sample data points (2D)

$$F(u,v) = \frac{1}{NM}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1} f(n,m)\, e^{-2\pi i\left(\frac{nu}{N}+\frac{mv}{M}\right)}$$

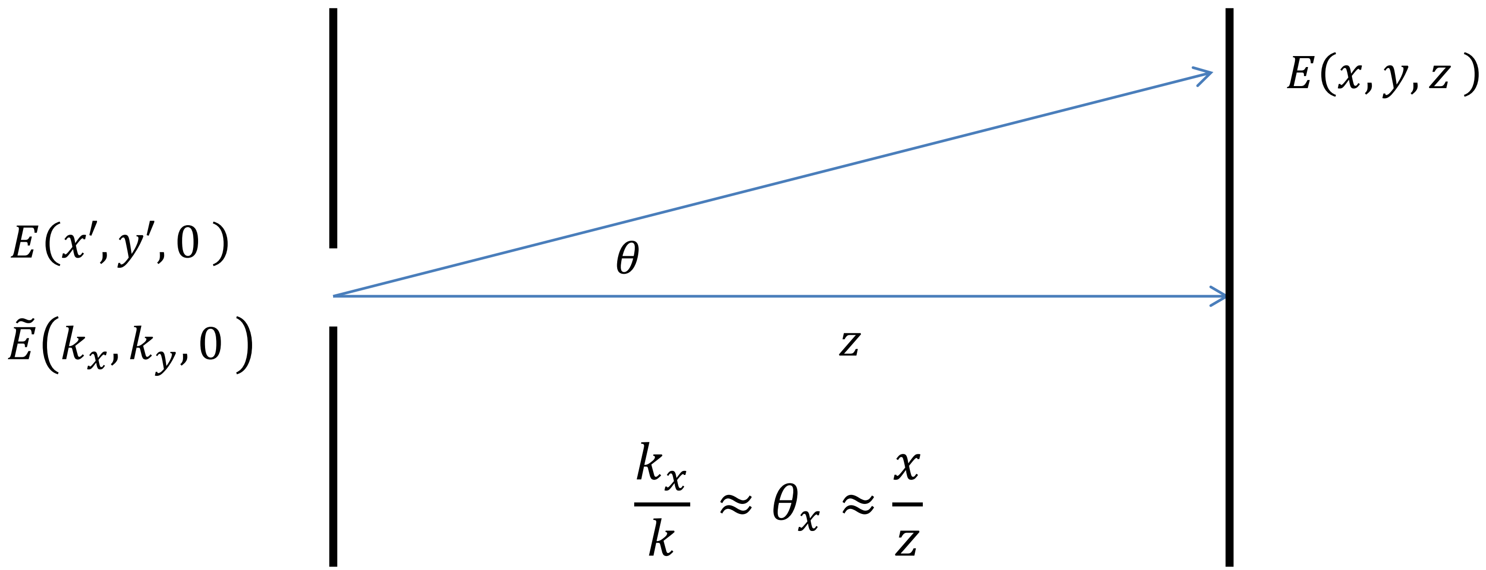
A lot of work has been put into algorithms to calculate this as efficiently as possible. (FFTW)



www.originlab.com

Fourier Optics

Connection between the spatial frequency at a source, and the intensity in the far-field

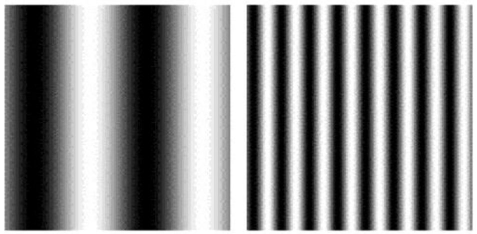


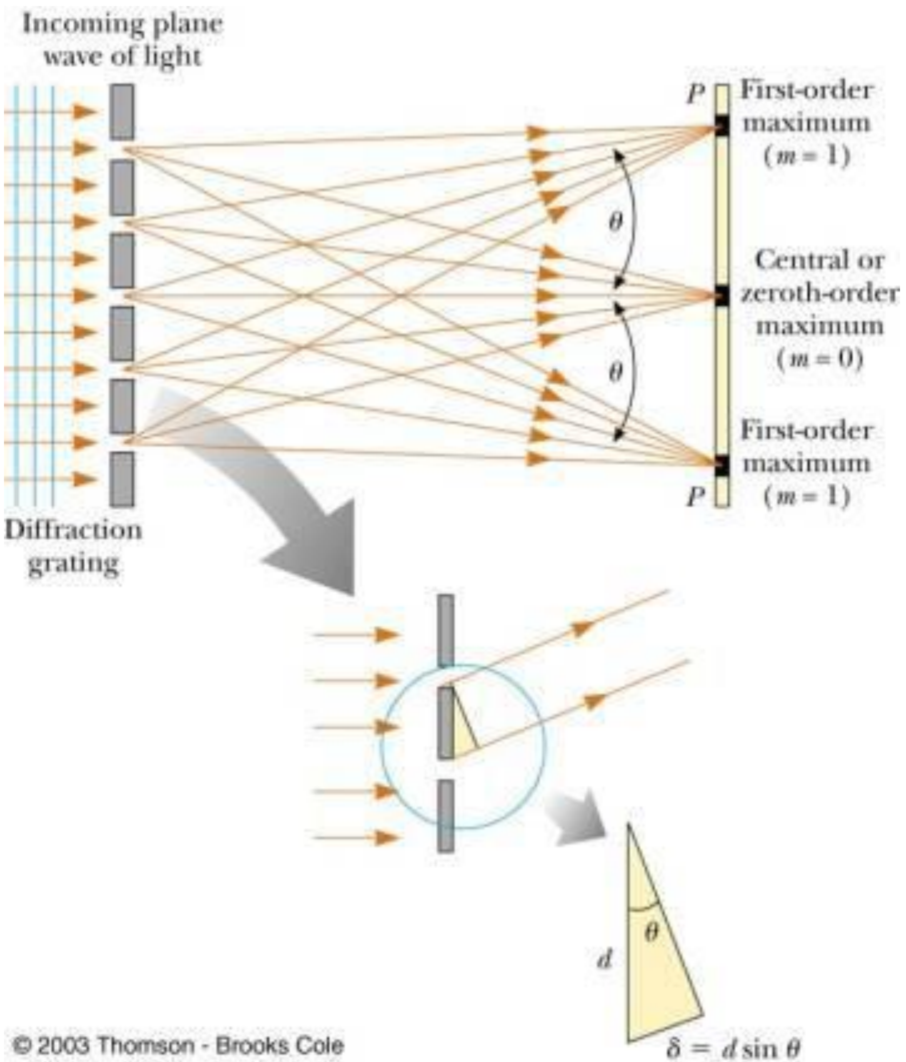
$$\frac{k_x}{k} \approx \theta_x \approx \frac{x}{z}$$

$$\frac{k_y}{k} \approx \theta_y \approx \frac{y}{z}$$

Spatial Frequency

- Spatial information in images can be broken down into Fourier components, at high and low frequency
- units of 1/length
(e.g. cycles/mm)





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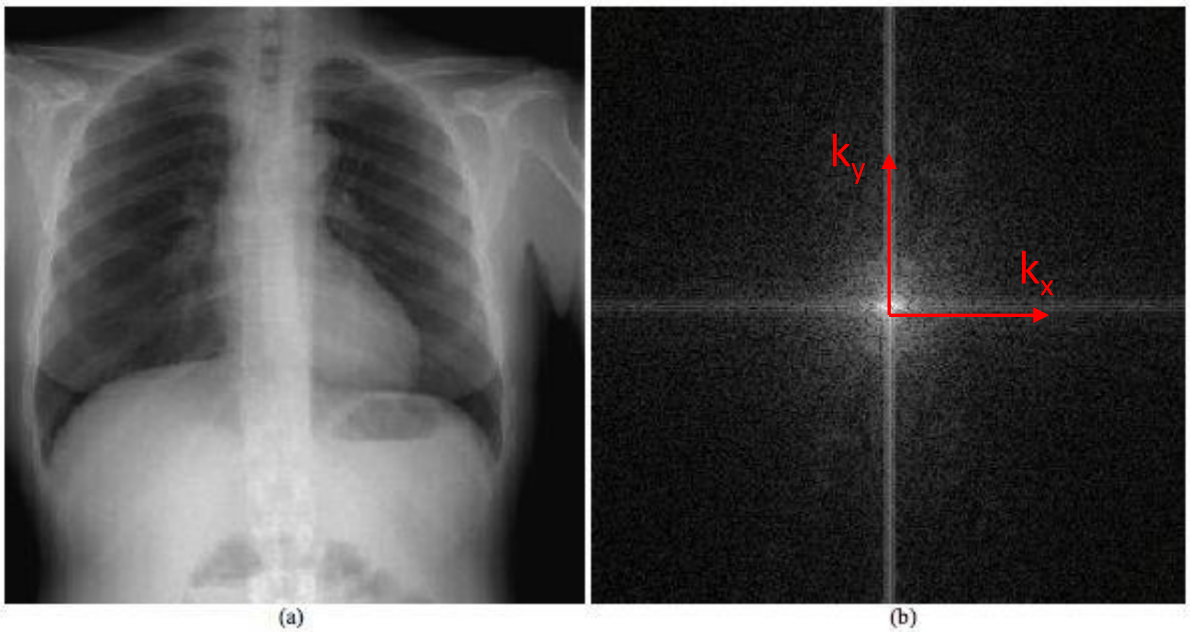
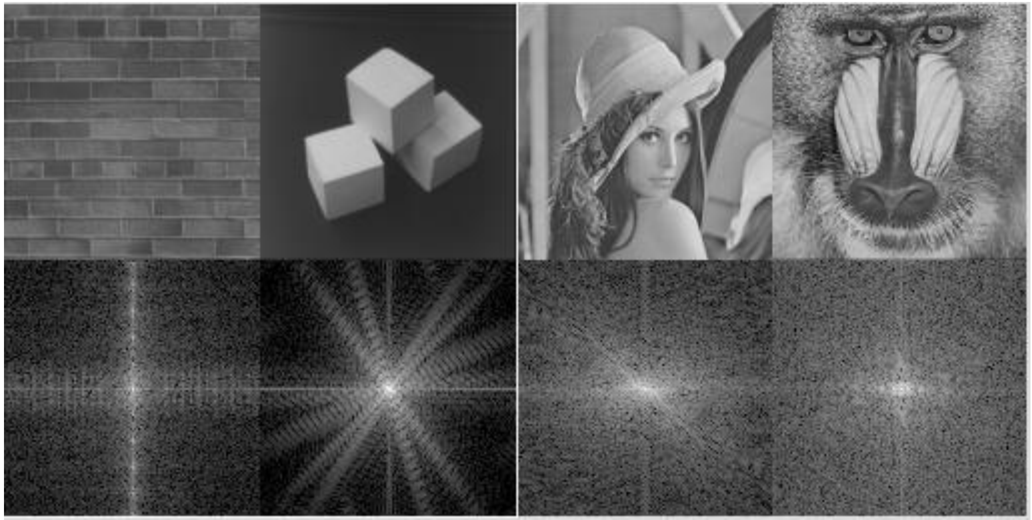
Spatial Frequency



- Low spatial frequency information is at the center of the Fourier plane
- high spatial frequency information is at the edges

Fourier Filter Image Processing



Original image

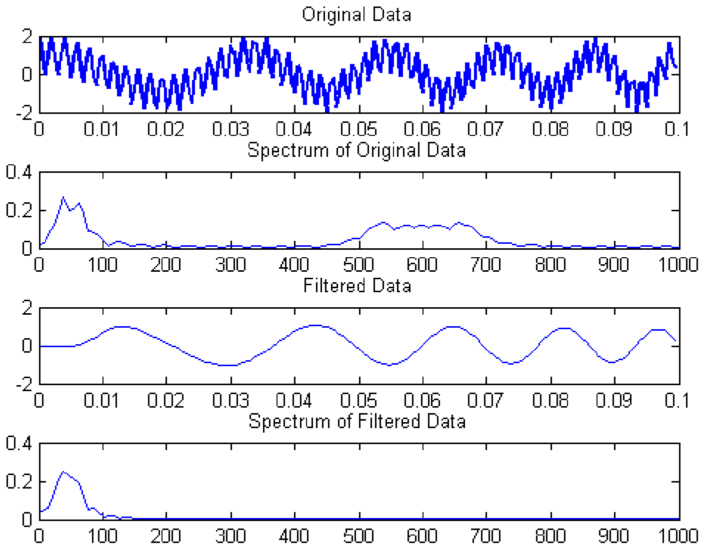
Power spectrum with mask that filters low frequencies

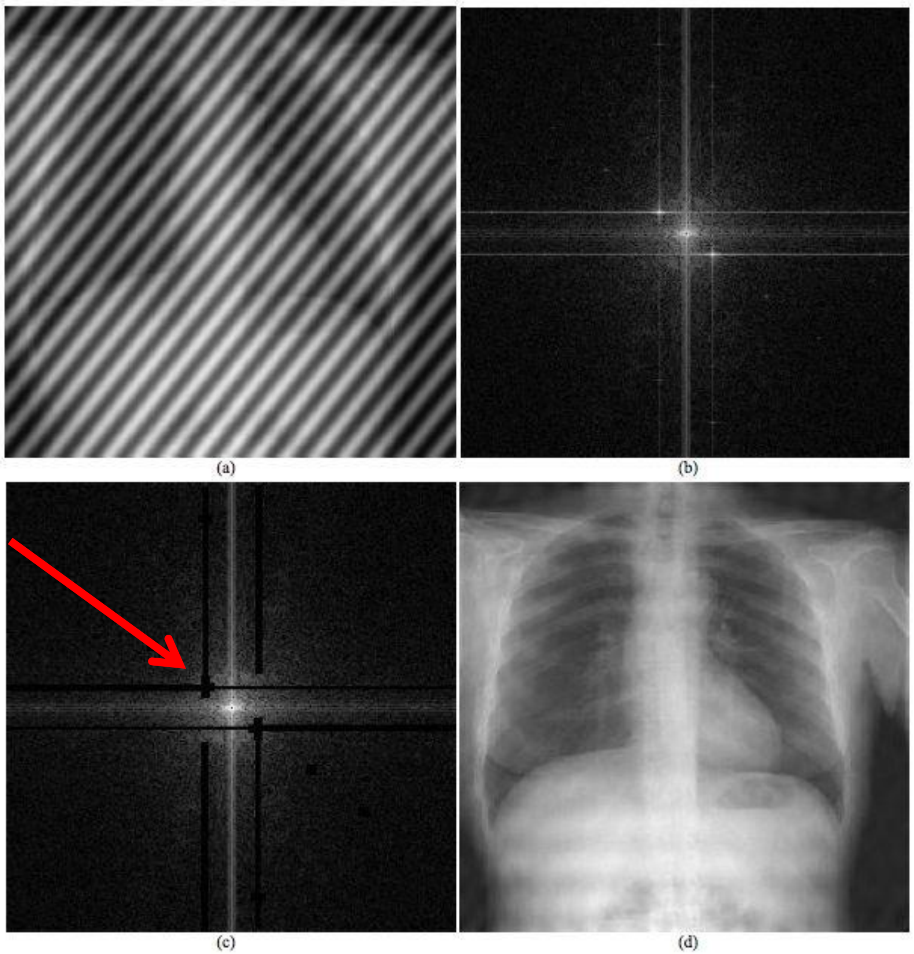
Result of inverse transform

Power spectrum with mask that passes low frequencies

Result of inverse transform

Fourier Filter Image Processing



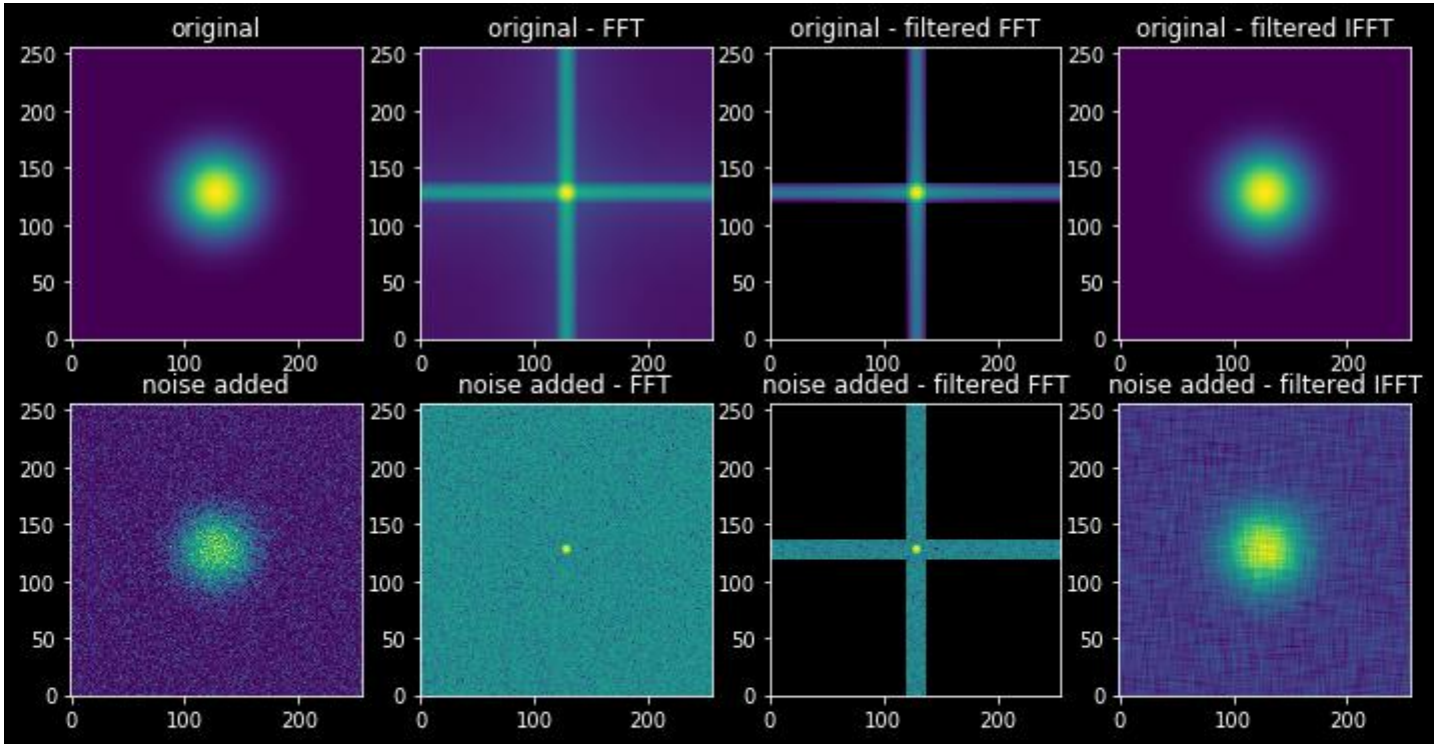


can filter out unwanted noise if it's sufficiently distinct in spatial frequency from the signal of interest

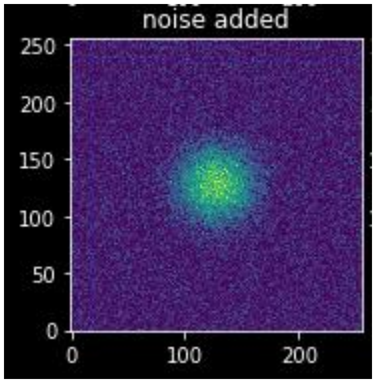
Block out spatial frequencies associated with the diagonal lines to recover most of the information in the original image

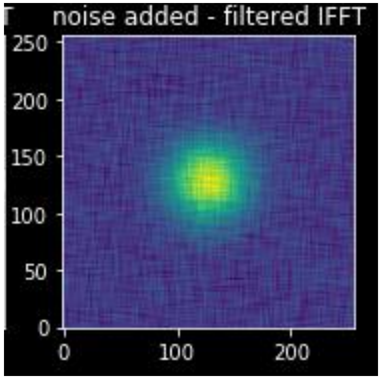
These techniques are pretty important in medical imaging of all kinds

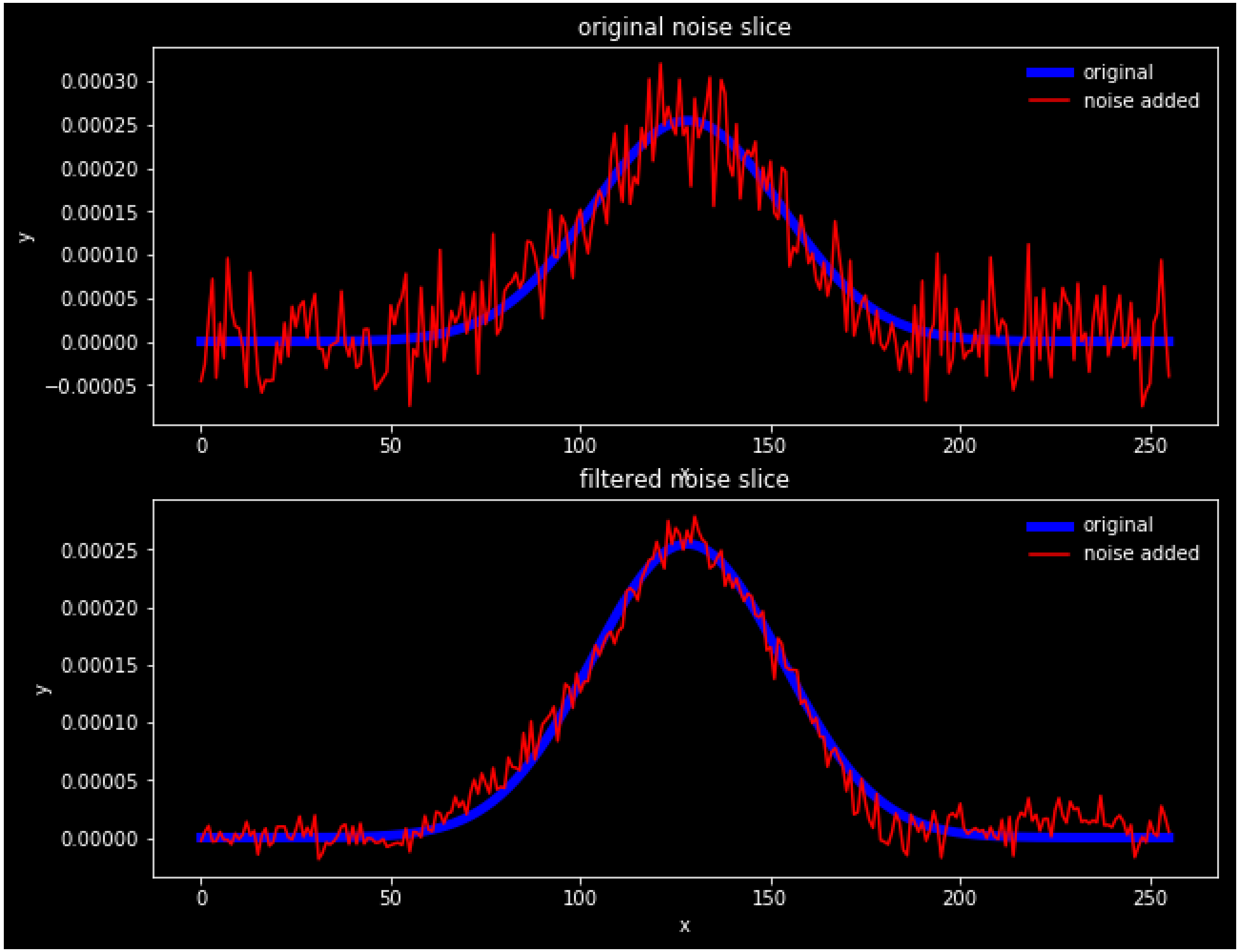
Fourier Filter Image Processing



Fourier Filter Image Processing





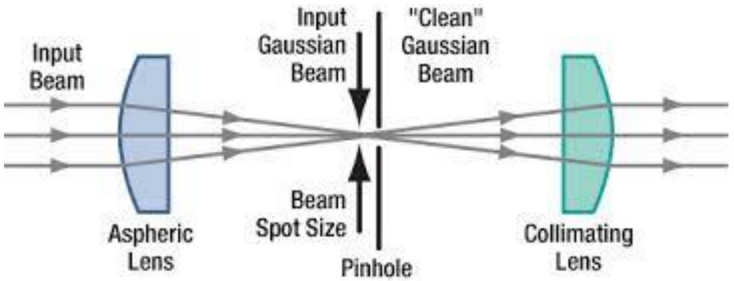


Common Fourier Transforms

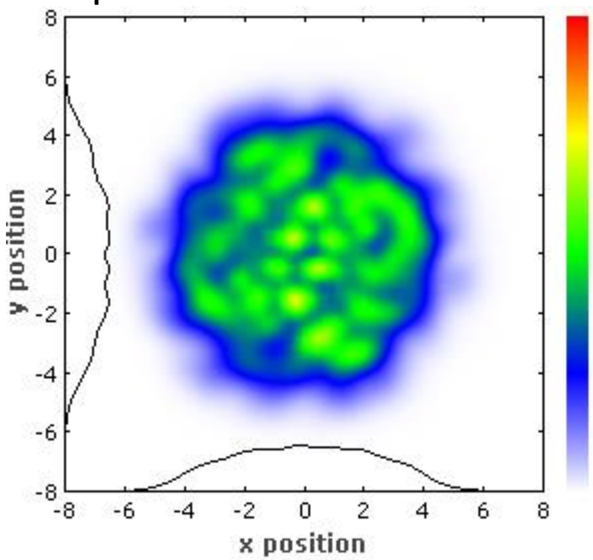
- Gaussian $\overset{\mathcal{F}}{\leftrightarrow}$ Gaussian $\mathcal{F}\left[e^{-t^2/2}\right] = e^{-\omega^2/2}$
- Exponential $\overset{\mathcal{F}}{\leftrightarrow}$ Lorentzian $\mathcal{F}\left[e^{-|t|}\right] = \dfrac{2}{1+\omega^2}$
- Square $\overset{\mathcal{F}}{\leftrightarrow}$ Sinc $\mathcal{F}[\mathrm{rect}(t)] = \mathrm{sinc}\,(\omega/2)$
- Constant $\overset{\mathcal{F}}{\leftrightarrow}$ δ Function $\mathcal{F}[1/2\pi] = \delta(\omega)$

Spatial Filtering of a Laser

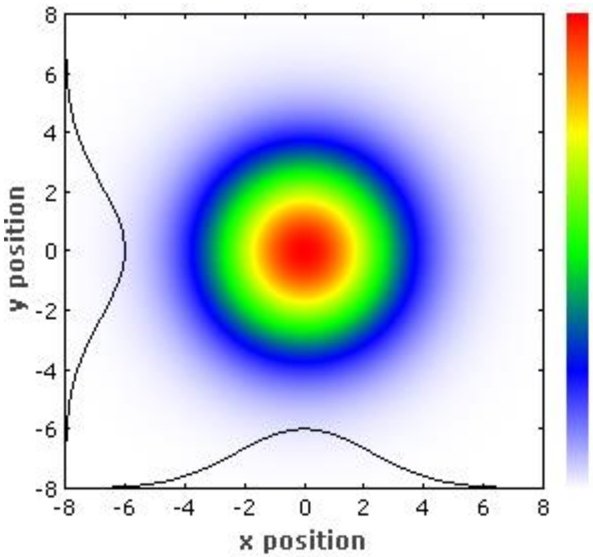
Removing high spatial frequency components from the Fourier/focal plane results in a smoother laser beam, without speckle and distortion



Input beam



Output beam



Linear Systems

Given some input signal: $f(x, y)$

We get an output signal: $F(X, Y) = \mathcal{L}[f(x, y)]$

Linear Systems

Given some input signal: $\alpha f_1(x,y) + \beta f_2(x,y)$

We get an output signal:

$$\mathcal{L}[\alpha f_1(x,y) + \beta f_2(x,y)] = \alpha F_1(X,Y) + \beta F_2(X,Y)$$

Linear Systems

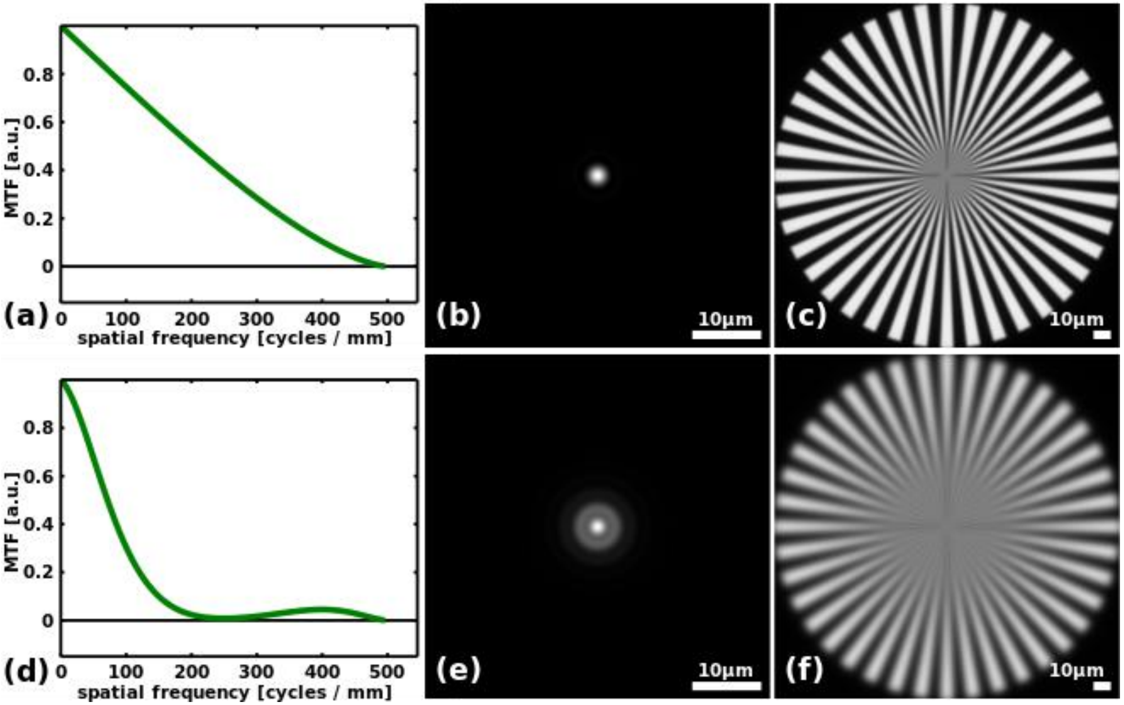
In optics the linear operator $\mathcal{L}$ is the Fourier transform, which we'll write as $\mathcal{F}$. We can simplify our treatment of optical systems by defining a few new general classes of functions:

- Aperture Function
- Point Spread Function
- Transfer Function

These general ideas apply to a much wider variety of physical systems: e.g. linear electronic circuits

What's the Point?

We need better tools than the **Rayleigh resolution criterion** for determining how faithfully our optical systems transmit information from source to image:



Modulation Transfer Function

Aperture (pupil) Function

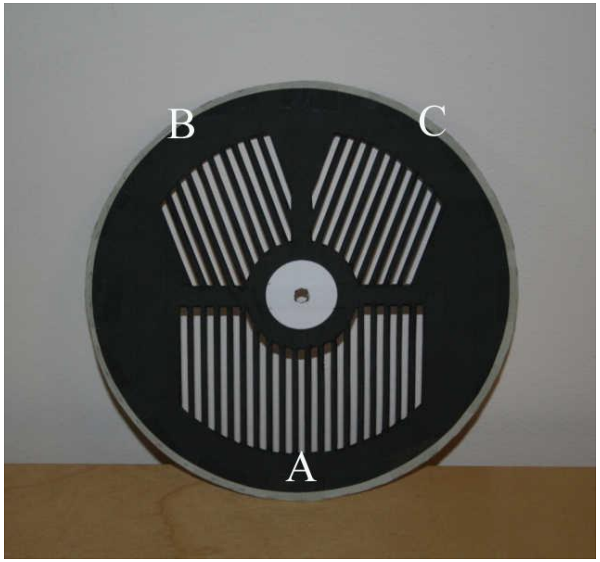
$$\mathcal{A}(x,y) = \mathcal{A}_0(x,y)e^{i\,\phi(x,y)}$$

- Describes the effect of an optical system on light that passes through it.

- For a plane wave $E_i = E_0 e^{i(kz-\omega t)}$ incident on an amplitude mask, the field after the mask is simply

$$E_{\mathcal{A}}(x,y) = E_0\mathcal{A}_0(x,y)e^{i\,\phi(x,y)} = E_0\mathcal{A}_0(x,y)$$

Where $\mathcal{A}_0(x,y)$ is a function describing the transmission through the mask



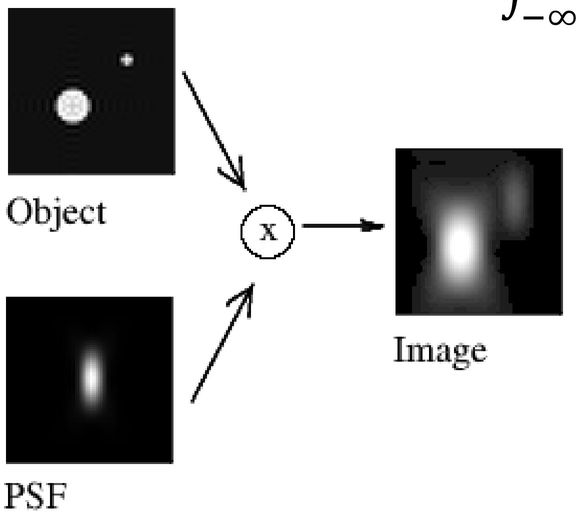
Bahtinov mask (geoastro.com)

Point Spread Function

imaging in the "real world"

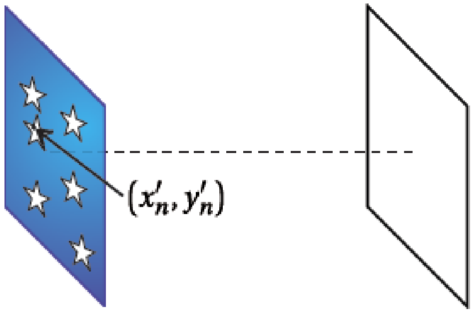
- What if we don't have a point object?
- The image formed by a *real, finite* imaging system is the *convolution* of the ideal image with the point spread function.

$$I(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_0(x',y') S(x-x',y-y')\, dx'\, dy'$$



Practically speaking, the ideal image gets blurred out in a very specific way determined by the point spread function

Array Theorem



Calculate the diffraction pattern caused by N identical apertures with $E_{\text{aperture}}(x', y', 0)$

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Change of variables: $x'' \equiv x' - x_n'$ and $y'' \equiv y' - y_n'$

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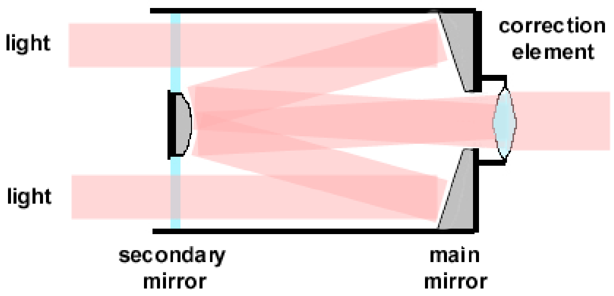
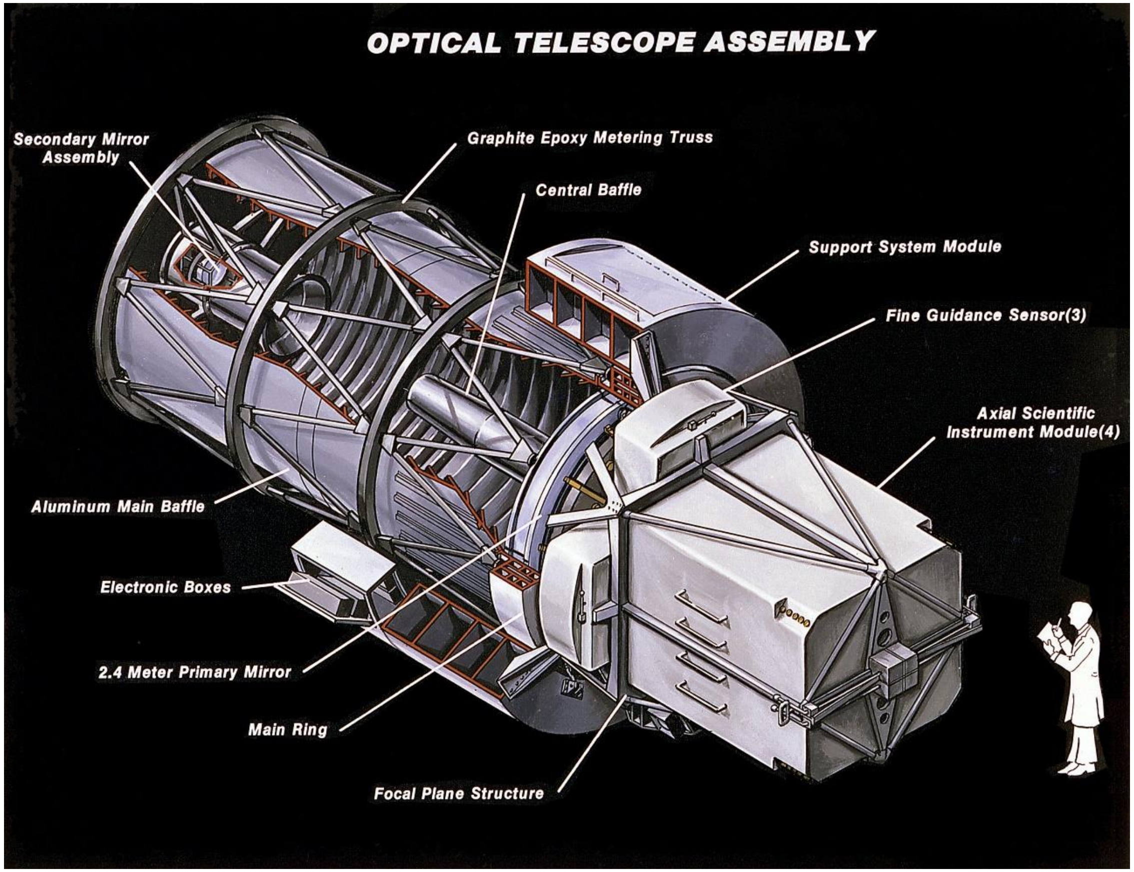
Example: Annular Aperture



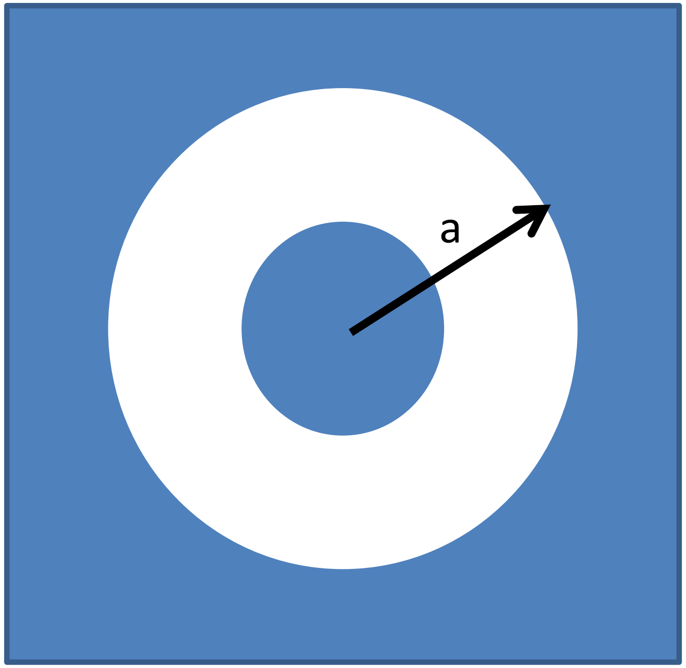


It hurts the resolution a bit, but how much?

Autocorrelation of an annular aperture: HW!

Example: Annular Aperture

$$\mathcal{A}(\rho) = \begin{cases} 1 & \epsilon a < \rho < a; \\ 0 & elsewhere \end{cases}$$

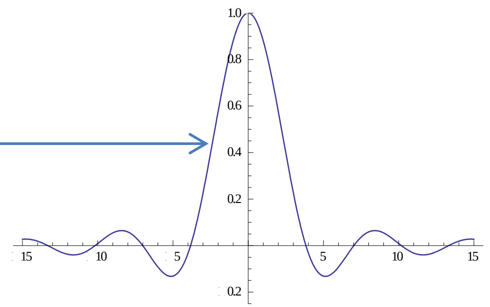


Example: PSF of Annular Aperture $\quad \mathcal{P} = \mathcal{F}[\mathcal{A}_0]^2$

What is the Fourier transform of this function? $\mathcal{A}_\epsilon(\rho) = \begin{cases} 1 & \epsilon a < \rho < a; \\ 0 & elsewhere \end{cases}$

Start with the (known) Fourier transform of a circular aperture of radius a $(\epsilon = 0)$

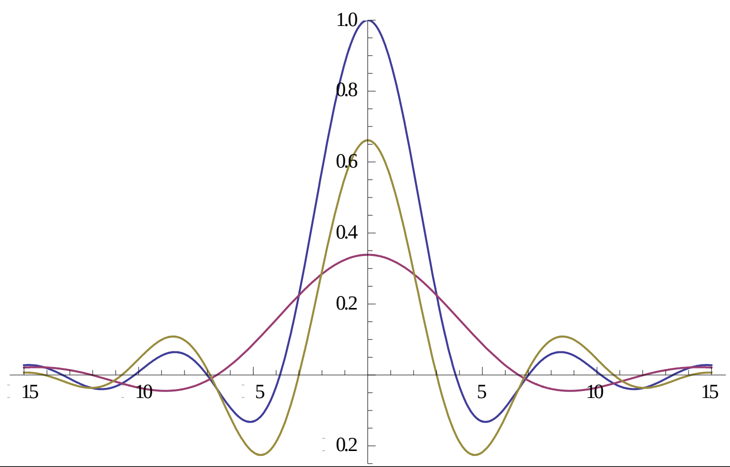
$$\mathcal{F}[\mathcal{A}_{\epsilon=0}] = 2\pi a^2 \frac{\mathrm{J}_1(a\rho)}{a\rho}$$



Subtract off the Fourier transform of circle of radius ϵa (Babinet's principle!)

$$\mathcal{F}[\mathcal{A}_\epsilon] = 2\pi a^2 \left(\frac{\mathrm{J}_1(a\rho)}{a\rho}\right) - 2\pi\epsilon^2 a^2 \left(\frac{\mathrm{J}_1(\epsilon a\rho)}{\epsilon a\rho}\right)$$

$$= 2\pi a^2 \left[\left(\frac{\mathrm{J}_1(ka)}{ka}\right) - \epsilon^2 \left(\frac{\mathrm{J}_1(\epsilon a\rho)}{\epsilon a\rho}\right)\right]$$



Example: PSF of Annular Aperture

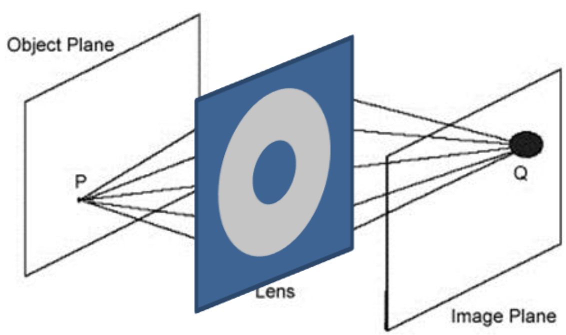
To get the point spread function:

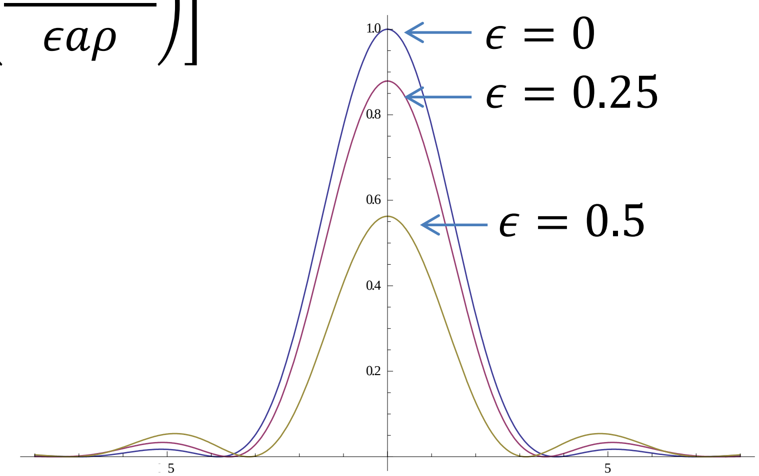
$$\mathcal{P} = \mathcal{F}[\mathcal{A}_\epsilon]^2 = (2\pi a^2)^2 \left[\left(\frac{J_1(a\rho)}{a\rho}\right) - \epsilon^2 \left(\frac{J_1(\epsilon a\rho)}{\epsilon a\rho}\right)\right]^2$$

What's the point of this?

This is the intensity pattern you'll see from a point source:

$$I(\rho) = I_{\epsilon=0}(0)\,(2\pi a^2)^2 \left[\left(\frac{J_1(a\rho)}{a\rho}\right) - \epsilon^2 \left(\frac{J_1(\epsilon a\rho)}{\epsilon a\rho}\right)\right]^2$$

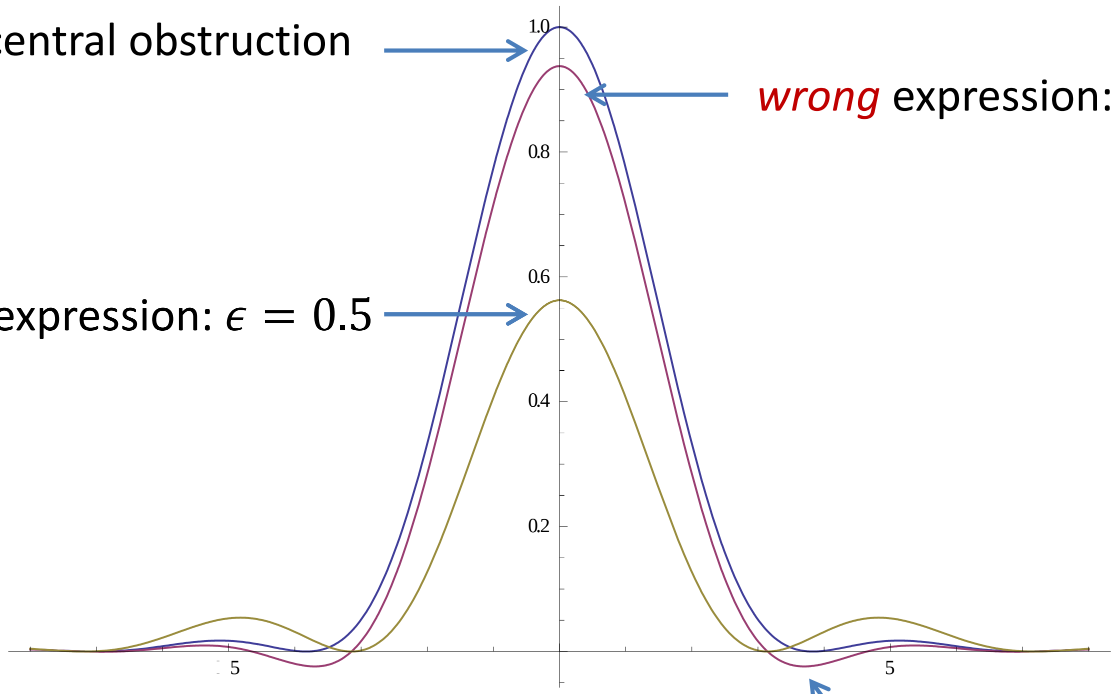




Example: PSF of Annular Aperture

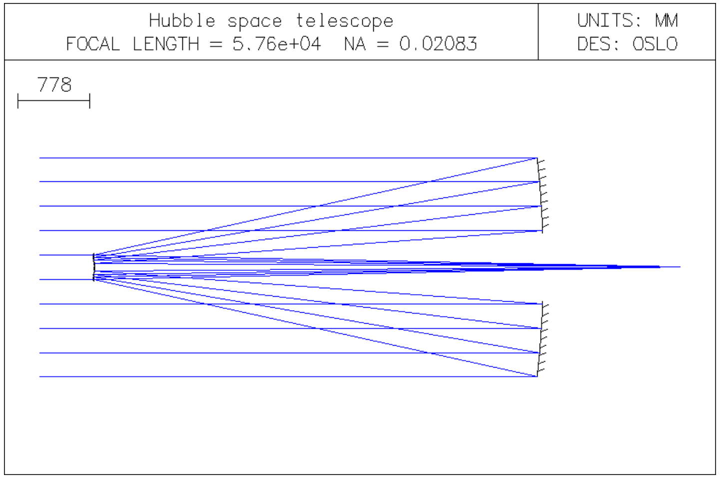
Note: Don't leave out the cross terms!

$$\mathcal{P} = \left[\left(\frac{J_1(ka)}{ka}\right) - \epsilon^2\left(\frac{J_1(k\epsilon a)}{k\epsilon a}\right)\right]^2 \quad \neq \quad \left[\left(\frac{J_1(ka)}{ka}\right)^2 - \epsilon^2\left(\frac{J_1(k\epsilon a)}{k\epsilon a}\right)^2\right]$$

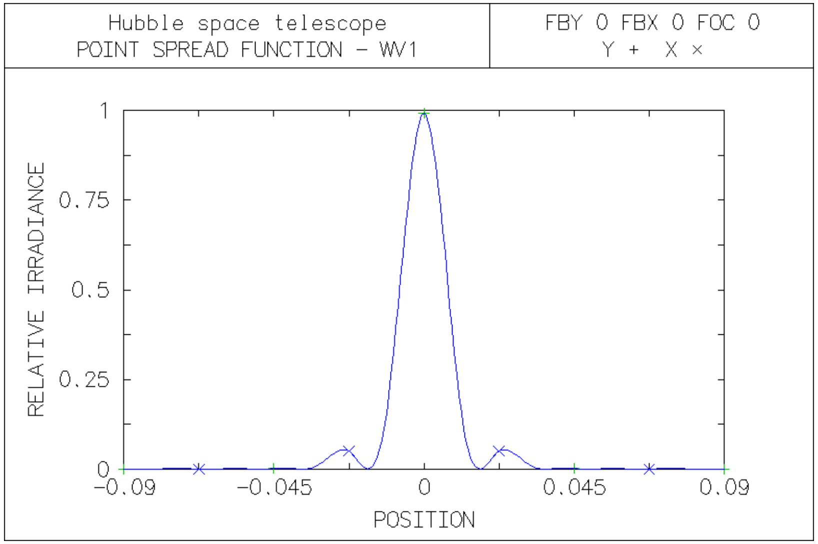


Example: The Hubble PSF Calculation

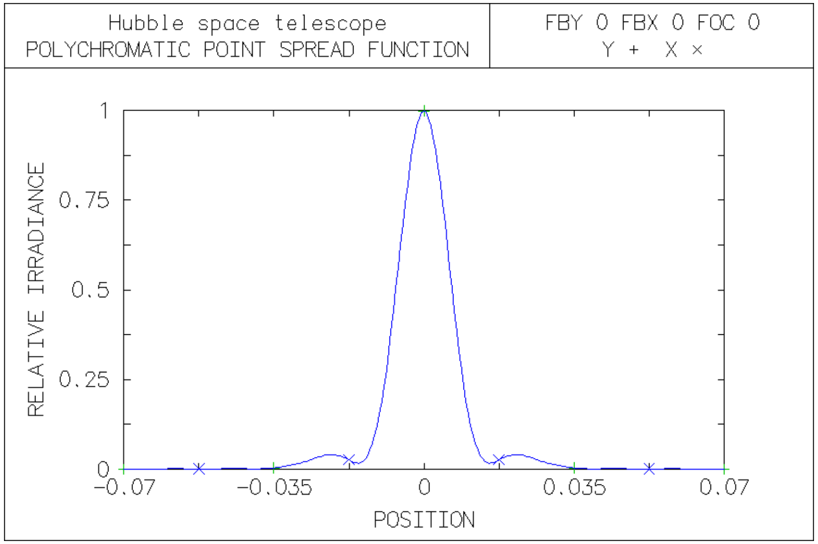
$$r_p = 1.104\ m$$
$$r_s = 0.135\ m$$
$$(\epsilon = 0.122\ )$$



Single Wavelength $\lambda = 632\ nm$



$\lambda = 500 - 700\ nm$



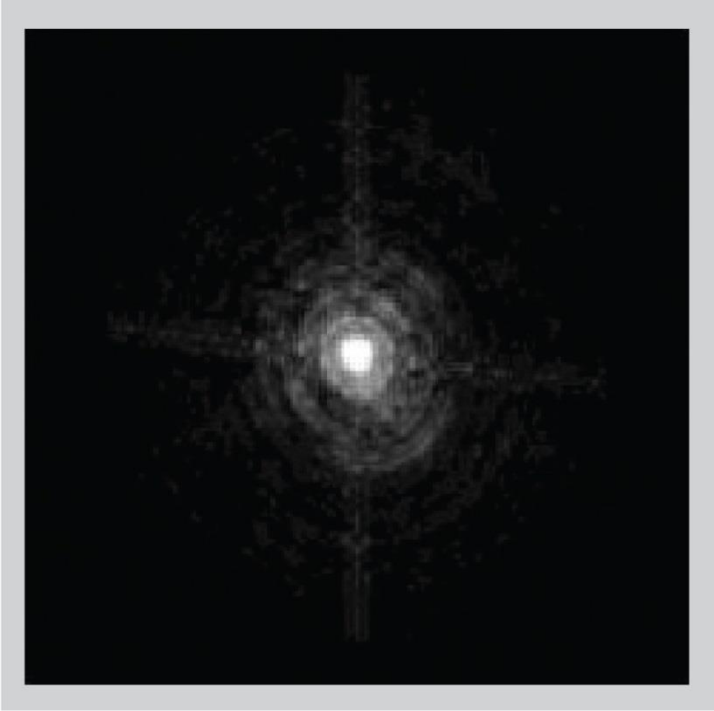
Example: The Hubble PSF Calculation

Point-source diffraction pattern

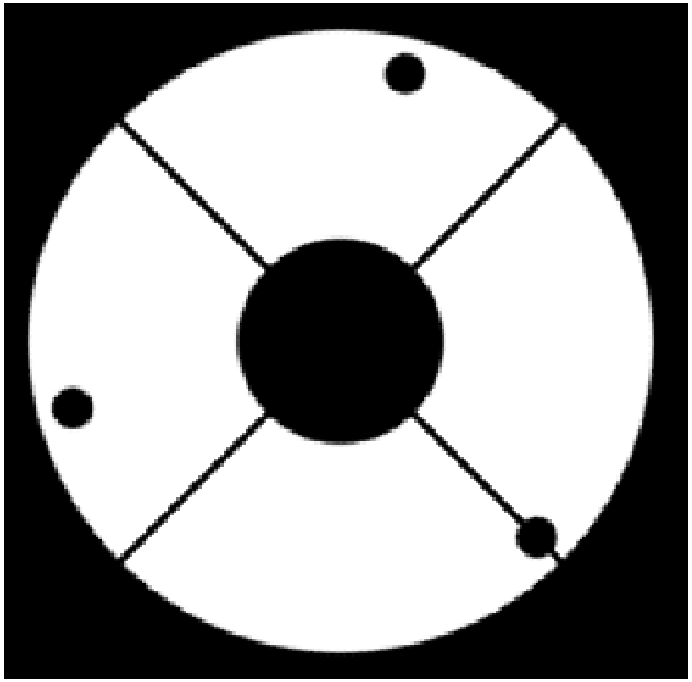
There's spatial frequency information hiding in plain sight here

Hubble aperture model:

- Secondary mirror obstruction
- Radial supports for secondary
- Three pads holding primary mirror



$$\mathcal{F} \leftrightarrow$$



Convolution, Correlation, and Autocorrelation

The convolution of two one-dimensional functions is:

$$f \circledast g = \int_{-\infty}^{\infty} f(x')g(x-x')\,dx'$$

The (cross) correlation of two functions is

$$f \odot g = \int_{-\infty}^{\infty} f^*(x')g(x+x')\,dx'$$

Note that if ***either*** function is ***even***

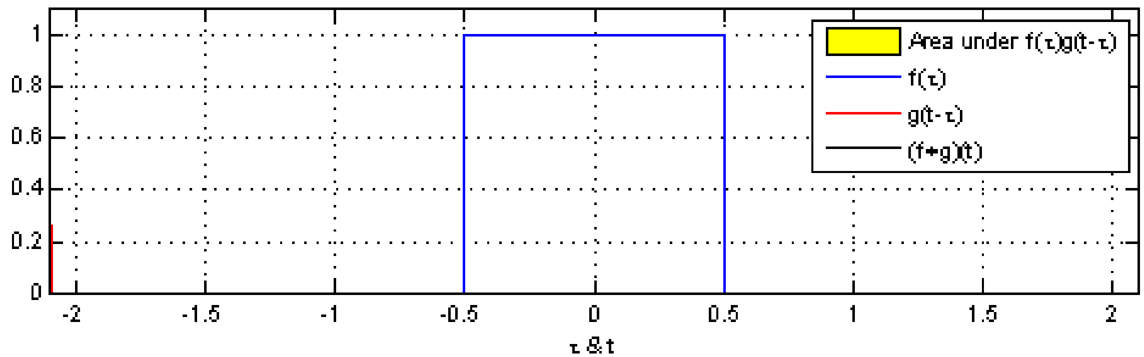
$$f \circledast g = f \odot g$$

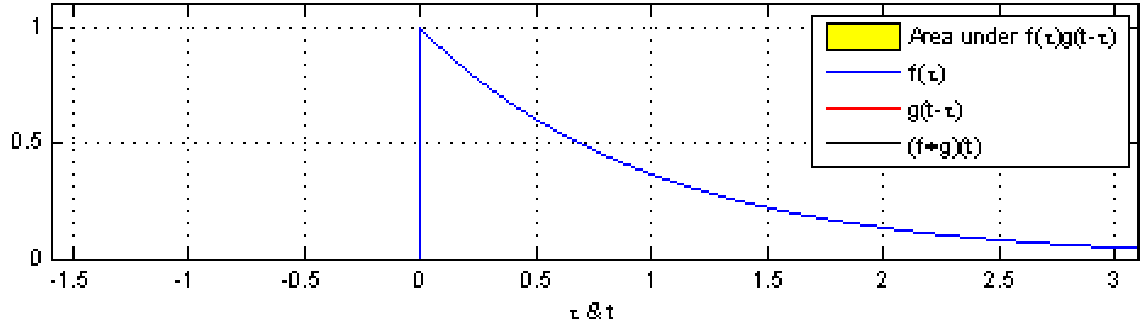
Correlation of a function with itself is called an *autocorrelation*

$$f \odot f = \int_{-\infty}^{\infty} f(x')f(x+x')\,dx'$$

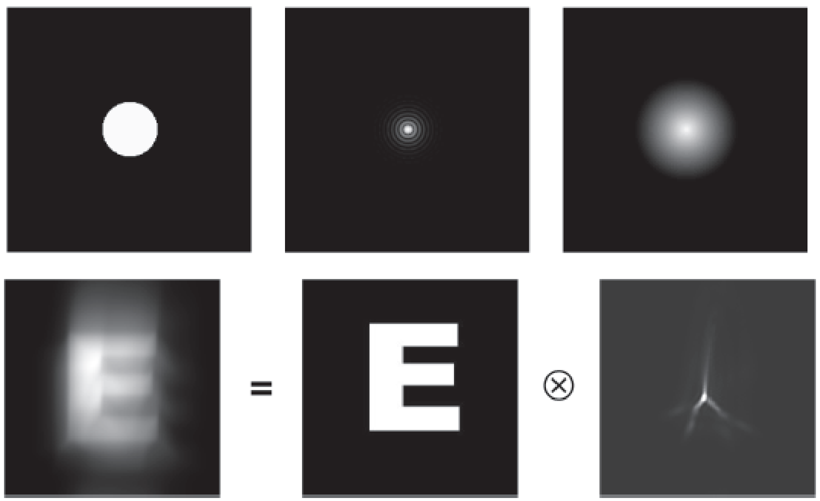
Convolution

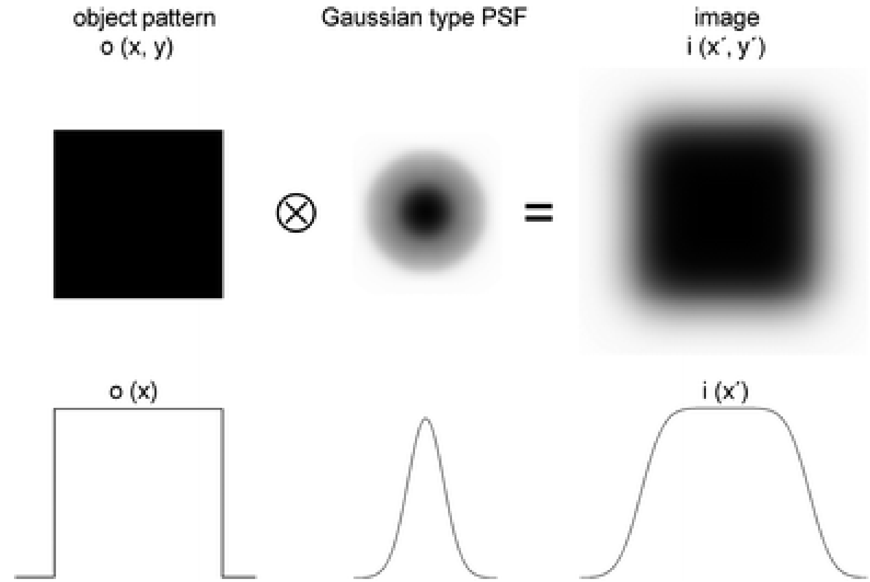
$$f \circledast g = \int_{-\infty}^{\infty} f(x')g(x-x')\,dx'$$

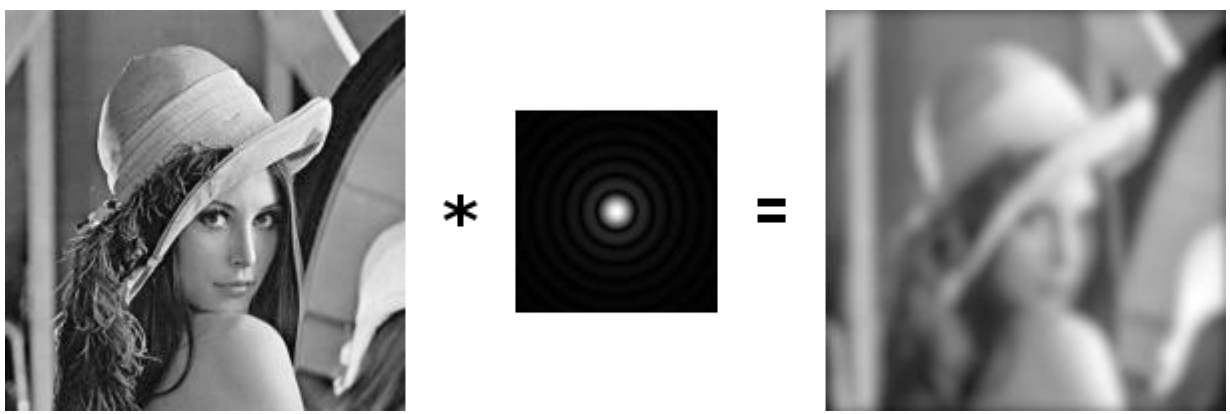




Convolution Examples







Convolution Theorem

$$\mathcal{F}[f * g] = \mathcal{F}[f]\,\mathcal{F}[g]$$

(proof in notes)

Why is this useful?

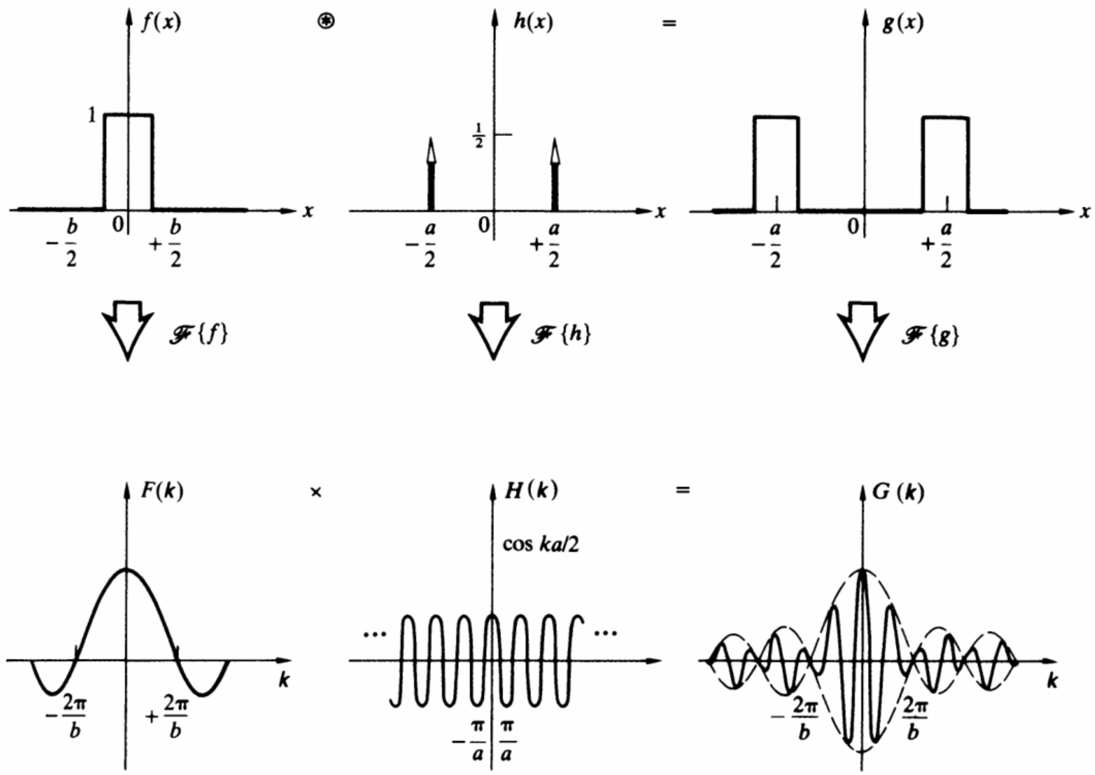
To convolve two functions, Fourier transform them, multiply them together and then apply the inverse Fourier transform on their product!

$$f * g = \mathcal{F}^{-1}\,[\,\mathcal{F}[f]\,\mathcal{F}[g]\,]$$

If you already know the Fourier (and inverse) transforms it makes calculating the convolution much easier!

Convolution Theorem

$$f \circledast h = f \circledast h$$



$$\mathcal{F}[f] \times \mathcal{F}[h] = \mathcal{F}[f * h]$$

Example: Convolution of two Gaussians

$$f(x) = Ae^{-x^2/\alpha^2} \qquad g(x) = Be^{-x^2/\beta^2}$$

$$f * g = ?$$

$$\mathcal{F}[f] = A\alpha\sqrt{\pi}e^{-\alpha^2 k^2/4} \qquad \mathcal{F}[g] = B\beta\sqrt{\pi}e^{-\beta^2 k^2/4}$$

$$\mathcal{F}[f]\,\mathcal{F}[g] = AB\alpha\beta\pi e^{-(\alpha^2+\beta^2)k^2/4}$$

$$f * g = \mathcal{F}^{-1}[\mathcal{F}[f]\,\mathcal{F}[g]] \qquad = AB\alpha\beta\sqrt{\pi}e^{-\frac{x^2}{(\alpha^2+\beta^2)}}$$

The convolution is still a Gaussian and the width is that of the original Gaussians summed *in quadrature!*

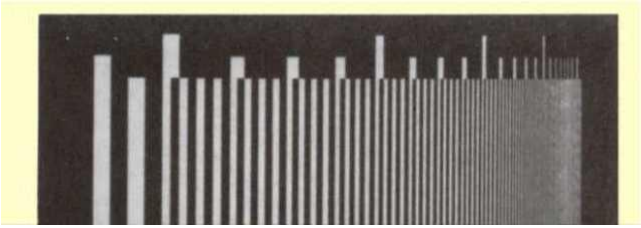
Evaluating Fidelity of Optical Systems

Remember this is a linear system: $g(X,Y) = \mathcal{L}[f(x,y)]$

If we put in a signal at a specific spatial frequency, we can ask how well that information is reproduced at the output of the system (the final image)

The Modulation Transfer Function (MTF) measures loss of contrast for sharp lines at different spatial frequencies

$$\text{Modulation} = \frac{I_{max}-I_{min}}{I_{max}+I_{min}}$$



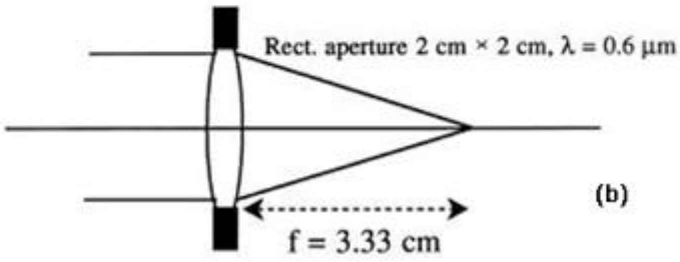
Full contrast

No contrast

Calculating the Modulation Transfer Function

MTF is simply the *autocorrelation* of the aperture function of the system!

$$\mathcal{M} = \mathcal{A} \odot \mathcal{A}$$

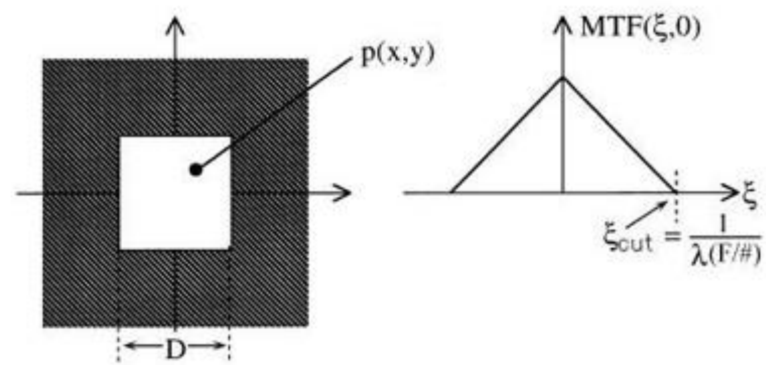


Aperture function:

$$\mathcal{A}_0 = \mathrm{rect}(x/D)$$

Rescale to spatial-frequency units:

$$k = \frac{x}{\lambda f}$$



The autocorrelation of a rectangle is easy: it's just a triangle…

$$\mathrm{MTF}(k) = 1 - \frac{k}{k_c}$$

Note: The cutoff frequency above which no spatial information makes it through the system (diffraction limit!)

$$k_c = \frac{\lambda D}{f}$$

Modulation Transfer Function

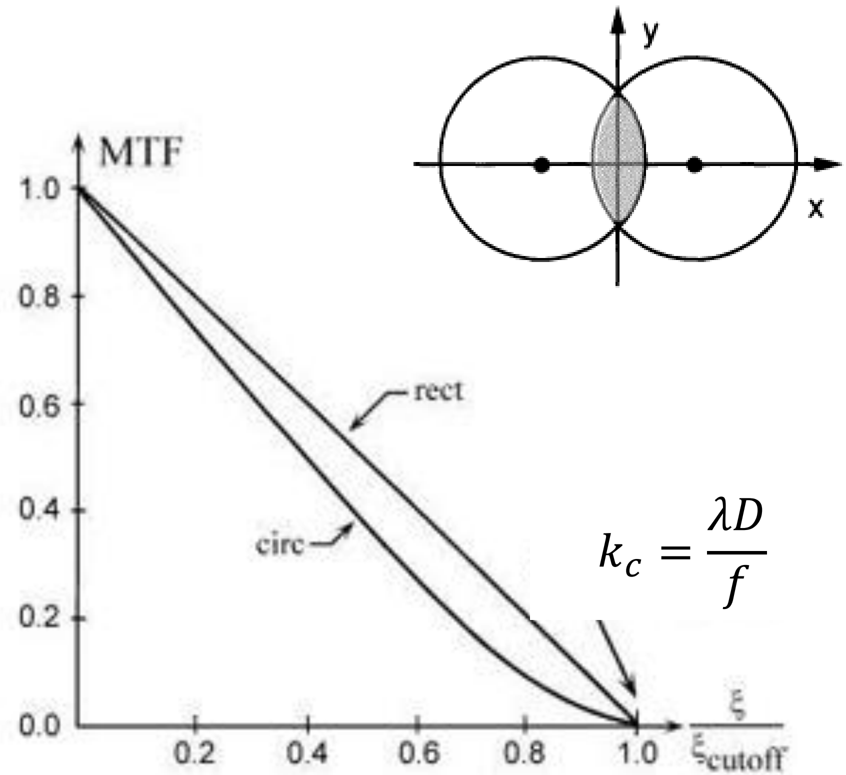
(Square vs Circular Aperture)

- The cutoff frequency is the same for square and circular apertures
- BUT the frequency/wavelength dependence is different
- (less light = less resolving power!)

Low spatial frequencies make it through fine

Higher spatial frequencies have less contrast, falling to zero at the cutoff

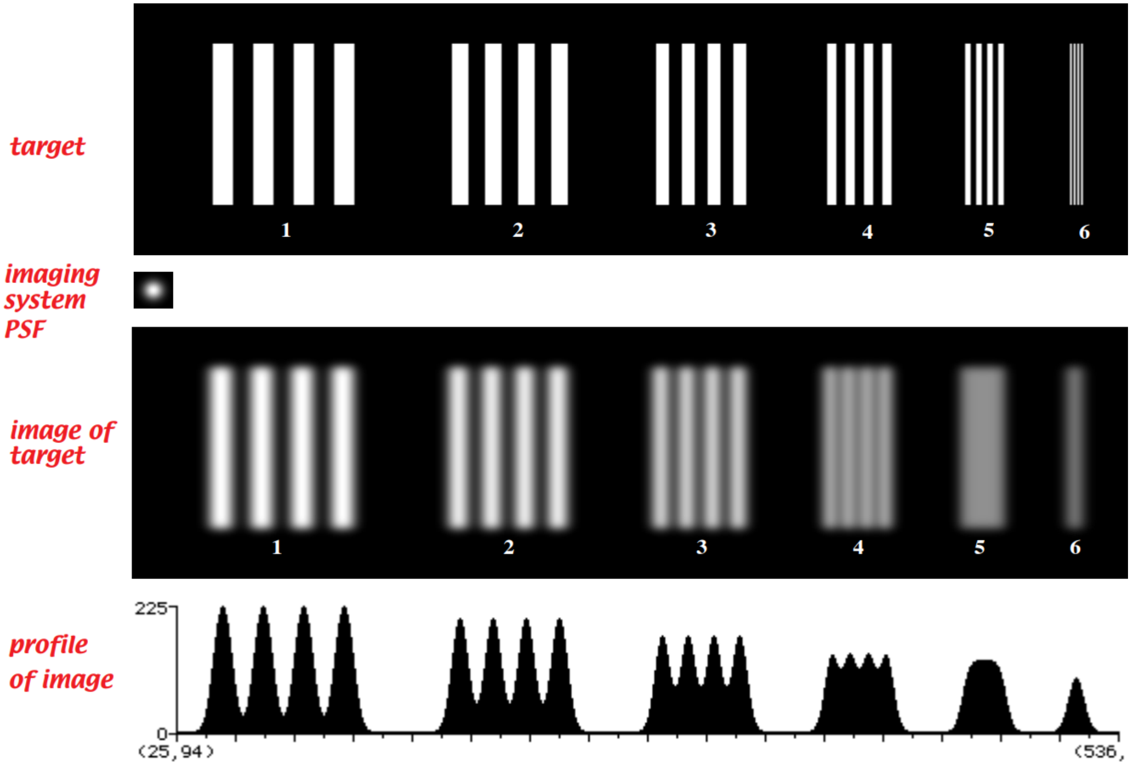
The larger the aperture, the higher the cutoff frequency (more resolution)

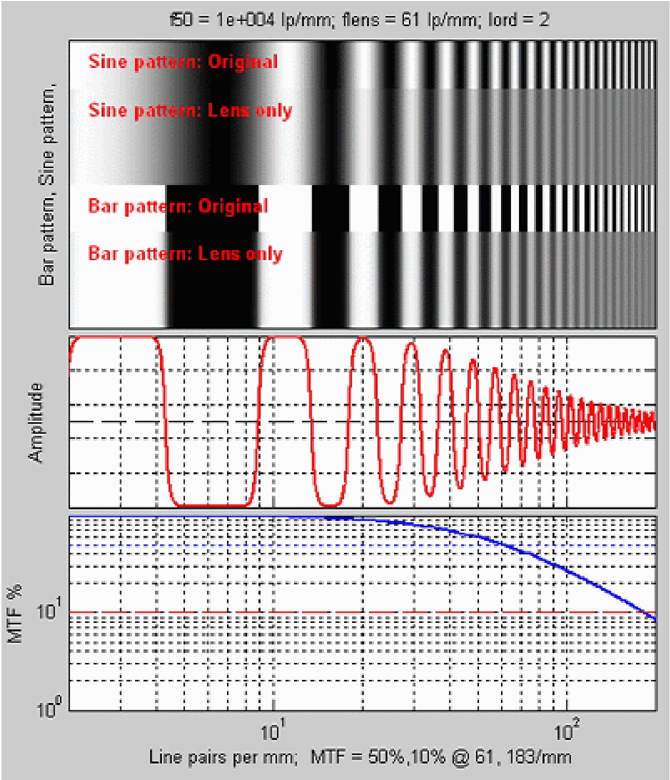


$$k_c = \frac{\lambda D}{f}$$

Measuring Modulation Transfer Function

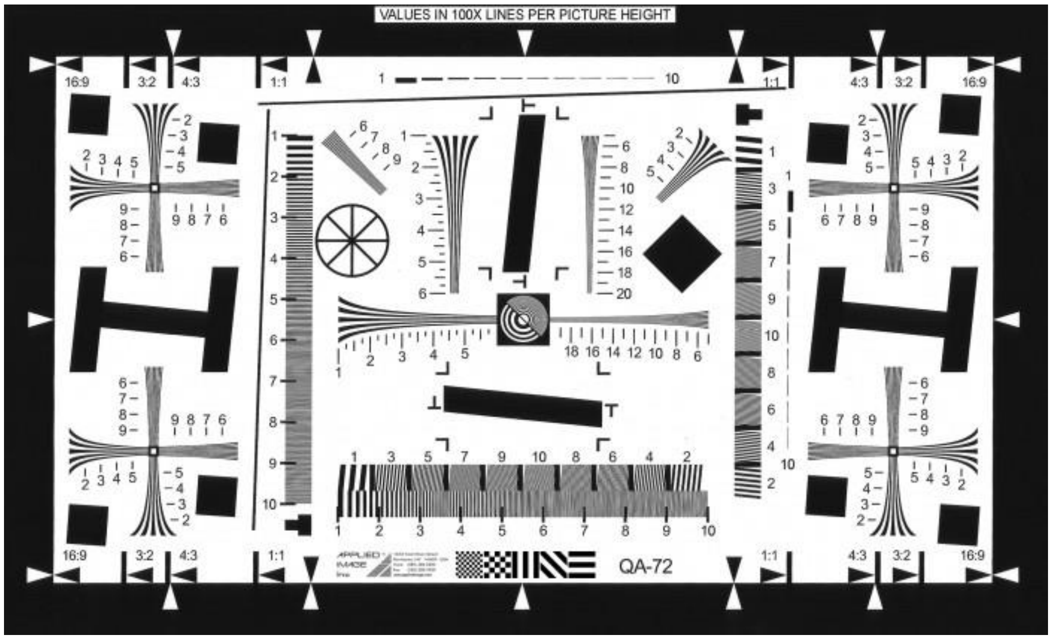
- MTF for real optical elements is normally *measured*
- most common to simply image a target object of various spatial frequencies
- takes aberrations into account

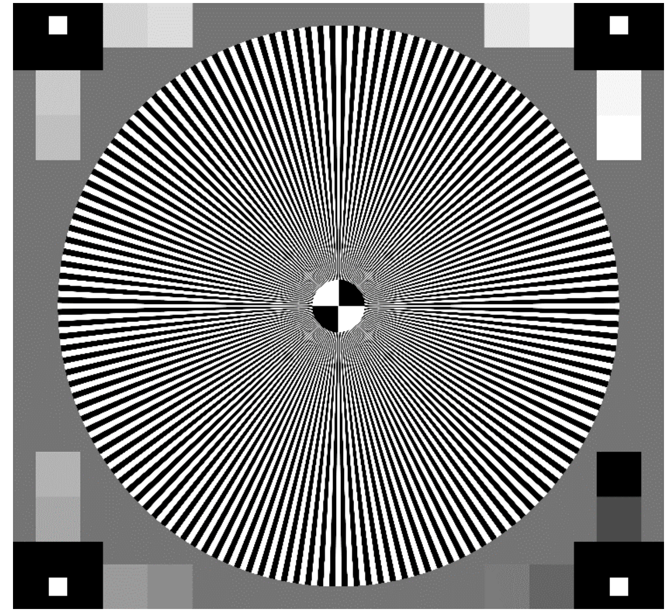


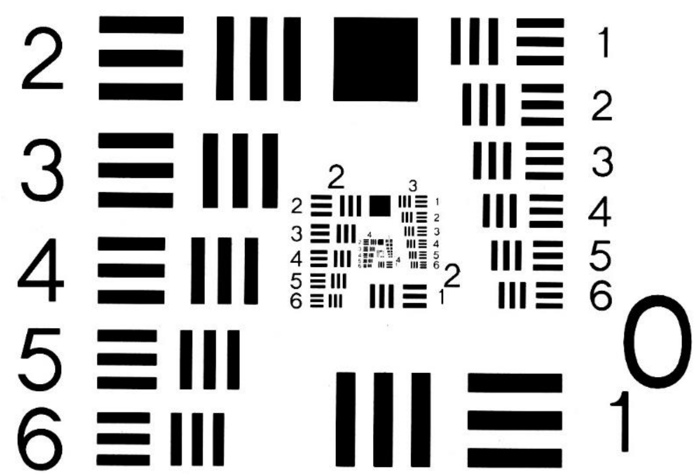


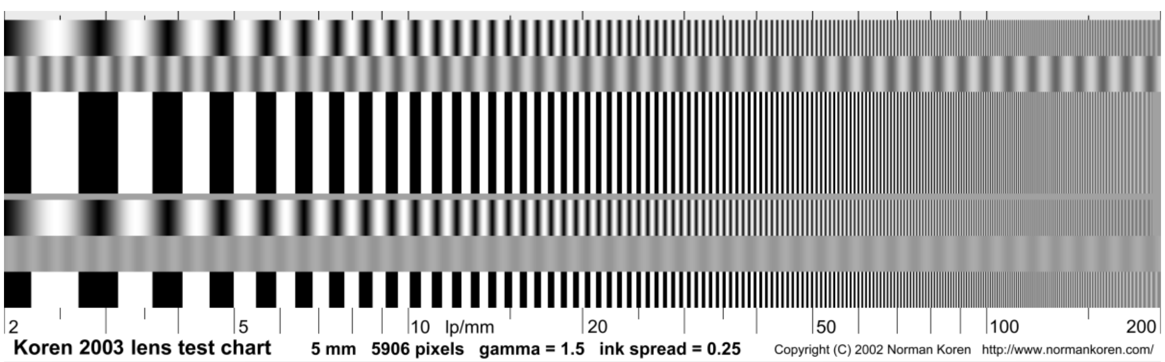
Evaluating MTF and Resolution

Are *many* different test patterns…









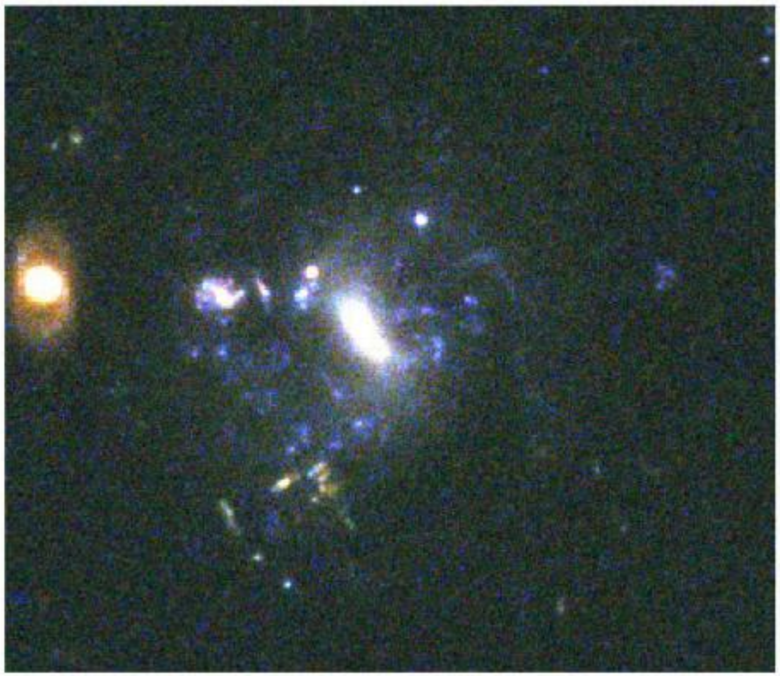
Real-World Diffraction Limit

$$k_c = \frac{\lambda D}{f}$$

Reality: Aperture usually isn't the limiting factor,
it's aberrations (esp. atmospheric conditions)

Subaru Telescope (D = 8 m)

Hubble (D = 2.4 m)



Fourier Optics