DIFFRACTION

P47 – Optics: Unit 7

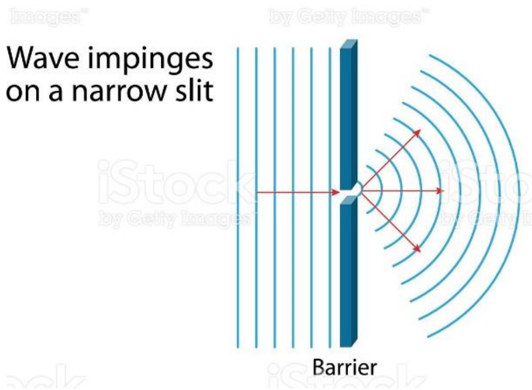
Course Outline

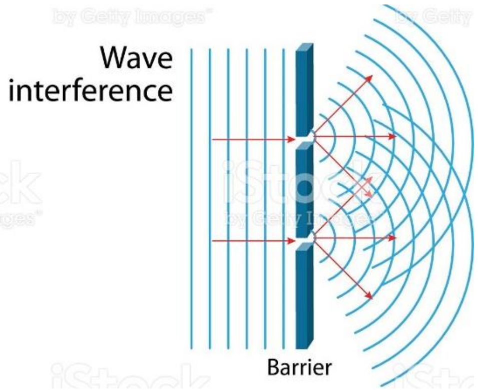
~~Unit 1: Electromagnetic Waves~~
~~Unit 2: Interaction with Matter~~
~~Unit 3: Geometric Optics~~

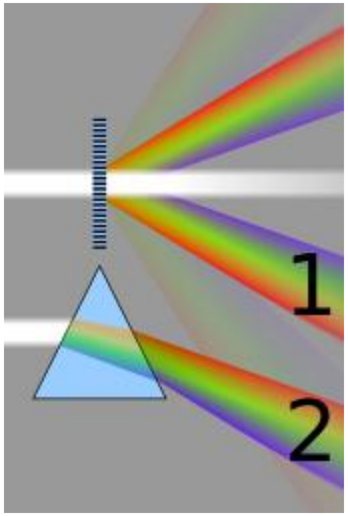
~~Unit 4: Superposition of Waves~~
~~Unit 5: Polarization~~
~~Unit 6: Interference~~

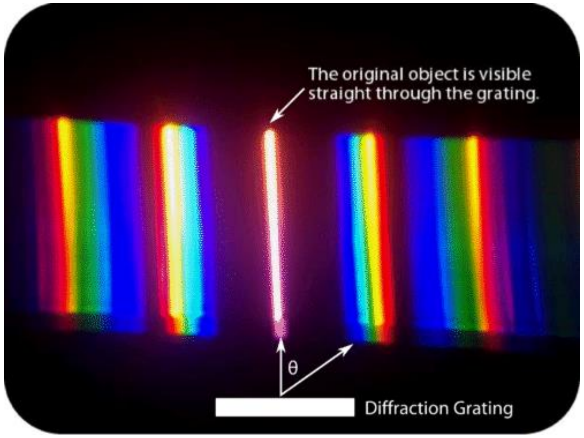
Unit 7: Diffraction
Unit 8: Fourier Optics
Unit 9: Modern Optics

Diffraction









Single Slit Diffraction – The Huygens-Fresnel Principle

Light propagating through an aperture is a sum of "spherical wavelets"

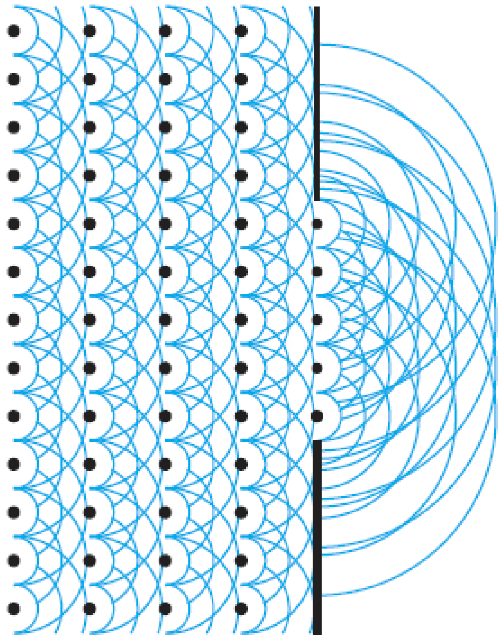
$$\boldsymbol{E}(\boldsymbol{r}) = C \iint \boldsymbol{E}(x', y', 0)\, \boldsymbol{u}(k, \boldsymbol{R})\, dx' dy'$$

Incident light (usually plane wave)

Diffracted "wavelet" (downstream propagation)

This isn't nearly as simple as it may seem.

Most of the challenge of diffraction theory involves doing this in a way you can actually solve!



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Scalar Diffraction Theory

- We'll treat the field as a scalar wave, ignoring polarization.

Vector Helmholtz Equation

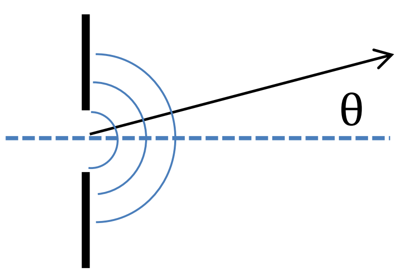
$$\nabla^2 \mathbf{E}(\mathbf{r}) + k^2 \mathbf{E}(\mathbf{r}) = 0$$

$$\boldsymbol{E}(\boldsymbol{r}, k) = \hat{\boldsymbol{E}}(\boldsymbol{r}, k) u(\boldsymbol{r}, k)^*$$

Scalar Helmholtz Equation

$$\nabla^2 E(\mathbf{r}) + k^2 E(\mathbf{r}) = 0$$

$$E(\boldsymbol{r}, k) = E_0 \frac{e^{ikr}}{r} u(\theta, k)$$



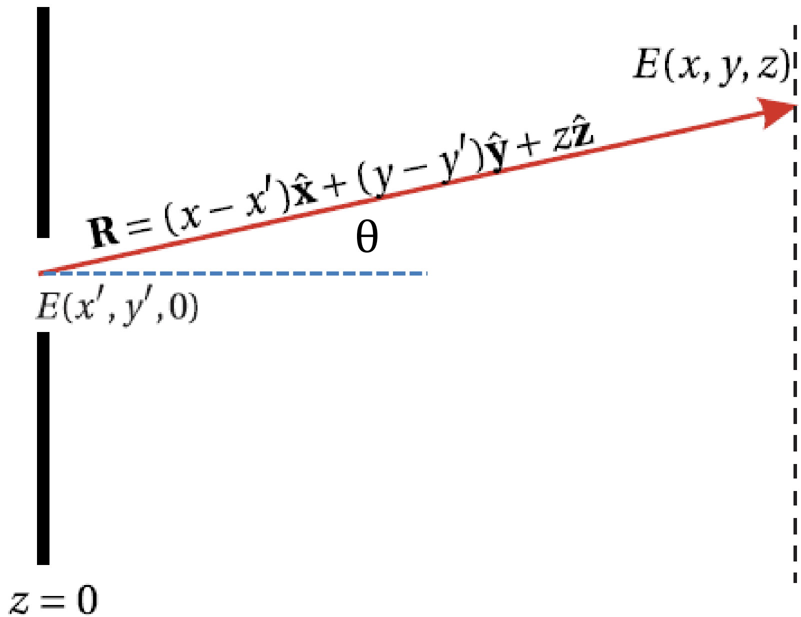
This works fine as long as $\theta \le \frac{\lambda}{d} \ll 1$

Equivalent to requiring that $\hat{\boldsymbol{E}}$ can be approximated as constant in space

*Vector diffraction theory is graduate-level electrodynamics, and well beyond the scope of this course. See Jackson *Electrodynamics* Ch. 10.

Coordinates in Diffraction Problems

The wavelet function $\mathbf{u}(k, \boldsymbol{R})$ depends on position in a non-trivial way:



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$$R = \sqrt{(x - x')^2 + (y - y')^2 + z^2}$$

$$R = \sqrt{(\rho - \rho')^2 + z^2}$$

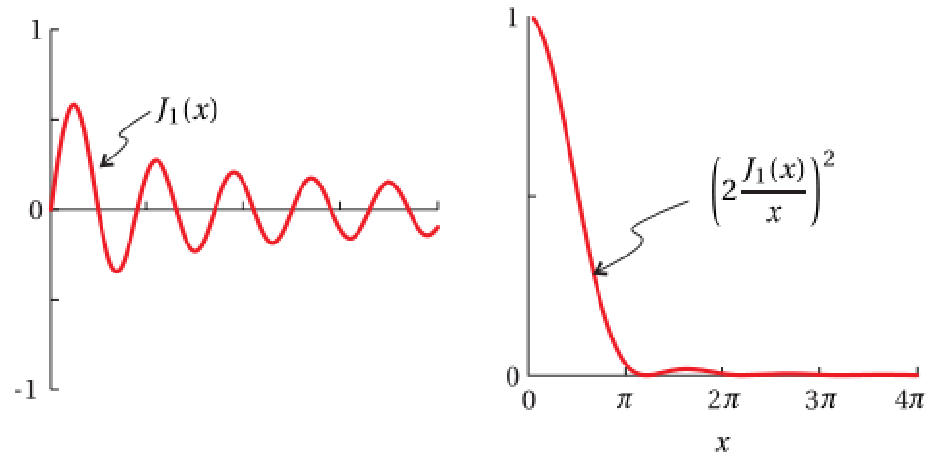
Choose *Cartesian* or *cylindrical* coordinates depending on the symmetry of the aperture (if any)

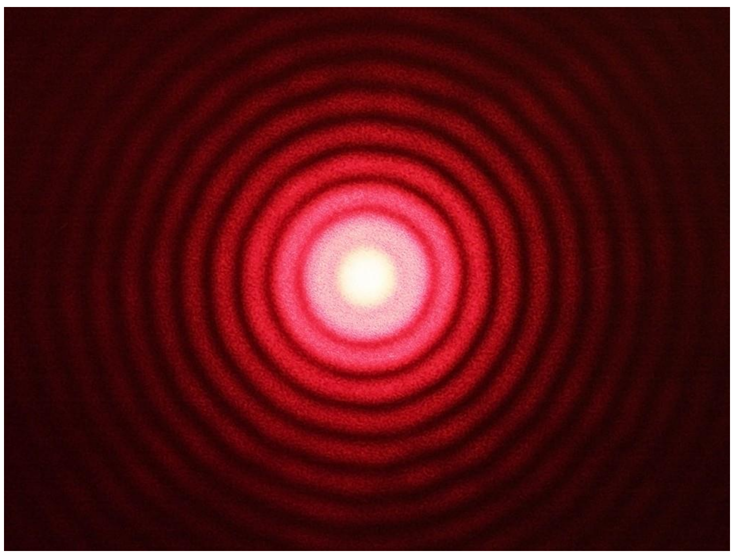
Airy Disk and Resolution Limit

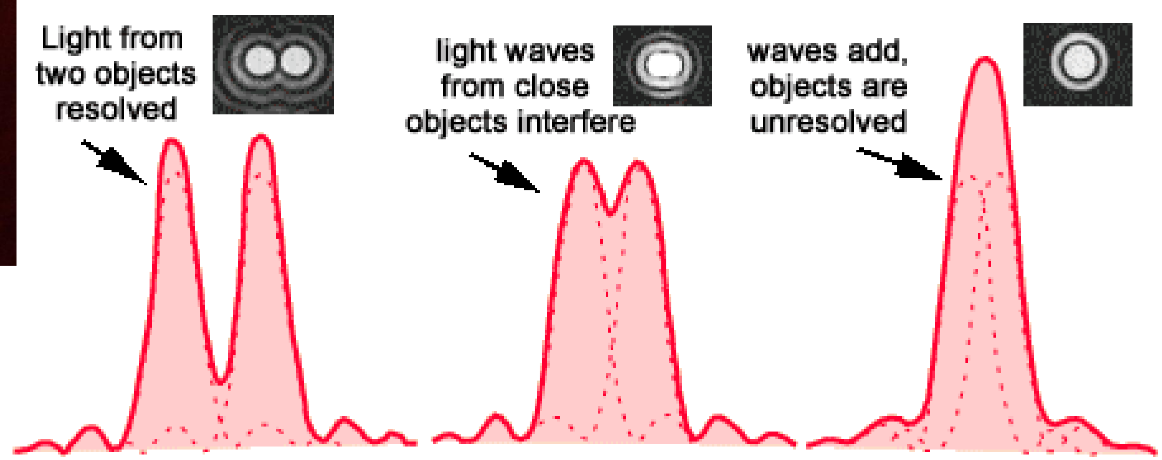
Irradiance from circular aperture:

$$I(\rho, z) = I_0 \left(\frac{\pi a^2}{\lambda z}\right)^2 \left[2\,\frac{J_1(ka\rho/z)}{ka\rho/z}\right]^2$$

First-order Bessel function:



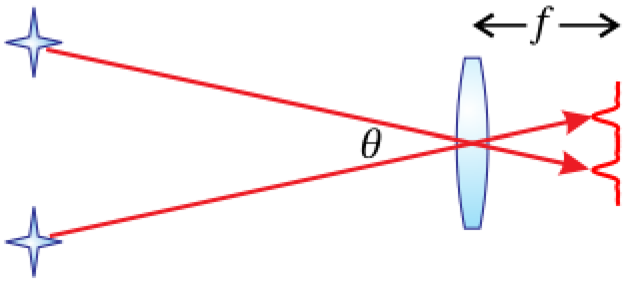




Airy Disk and Resolution Limit

Irradiance from lens aperture, diameter, D:

$$I(\rho, f) = I_0 \left(\frac{\pi D^2}{4\lambda f}\right)^2 \left[2 \frac{J_1(kD\rho/2f)}{kD\rho/2f}\right]^2$$



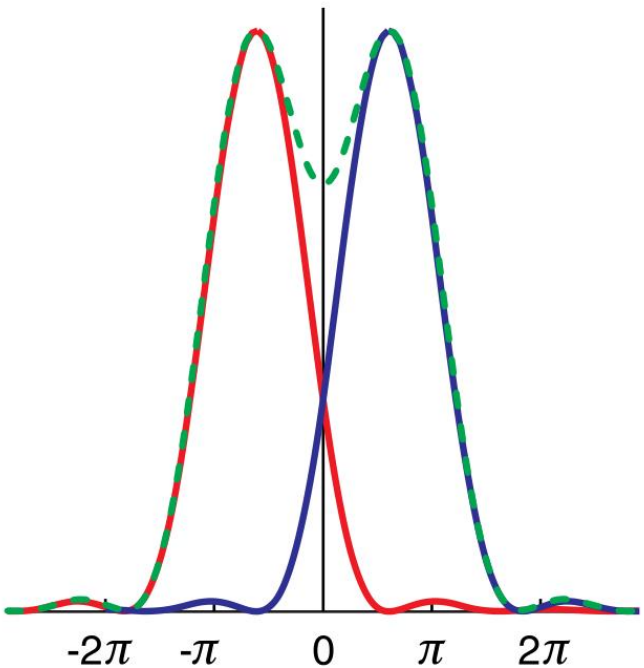
Resolution Limit:
When the first zero of one peak is at the same position as the second primary peak

Meet this condition when: $\frac{kD\rho}{2f} = 1.22\pi$

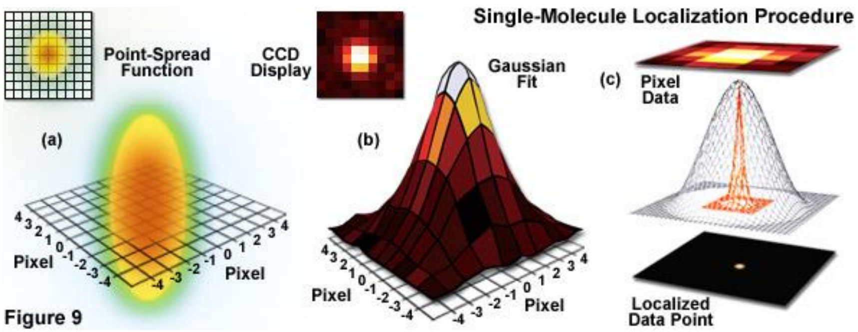
Have minimum resolvable angle between two objects as:

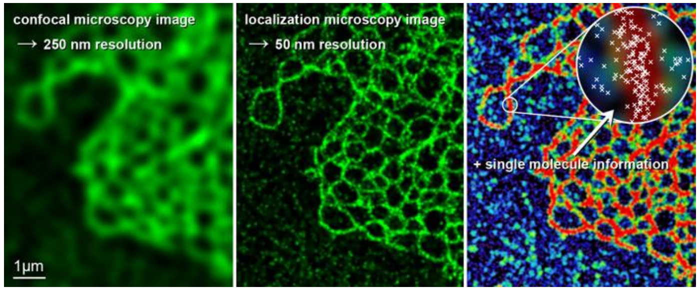
$$\theta_{min} \simeq \frac{\rho}{f} = \frac{1.22\lambda}{D}$$

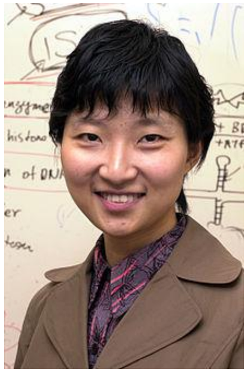
The Diffraction Limit!

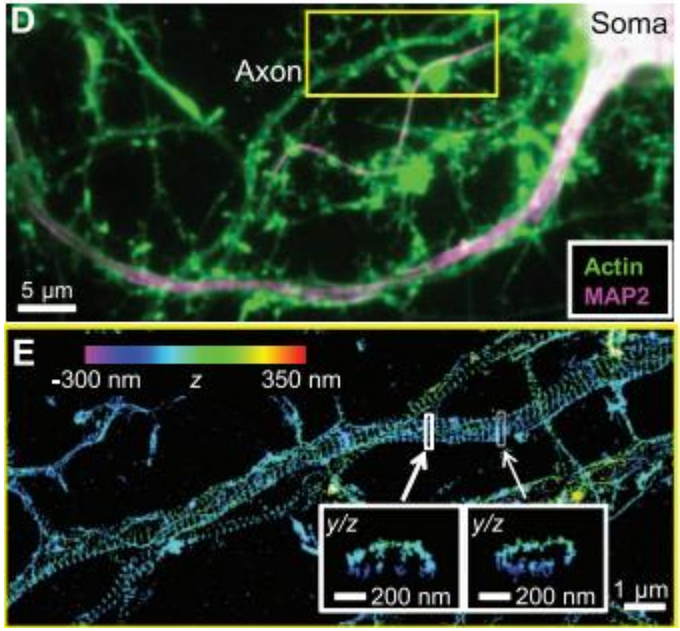


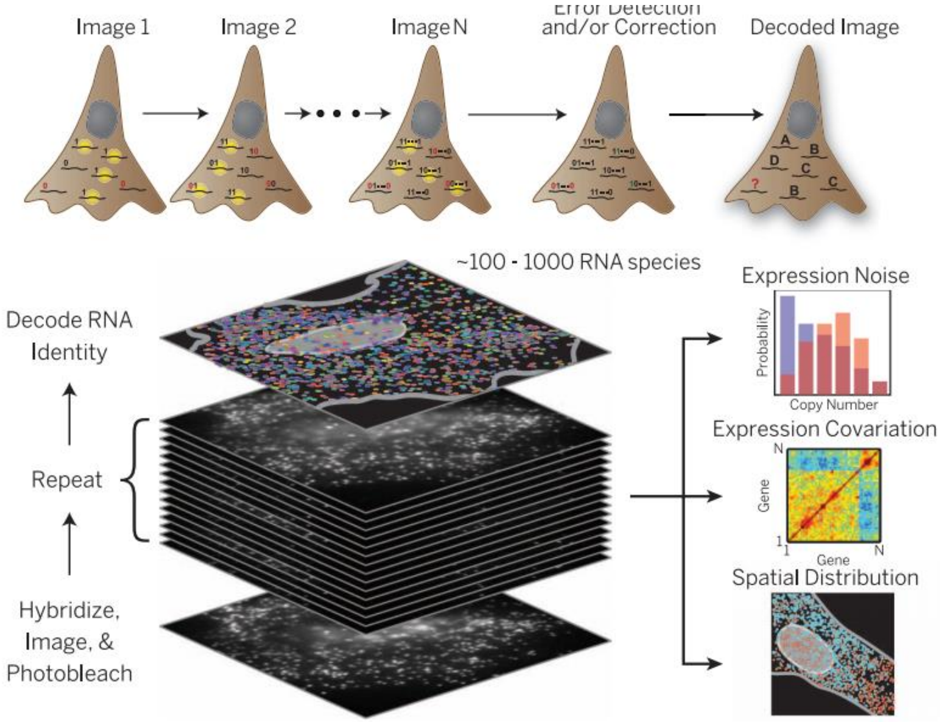
Illuminating biology at the nanoscale with single-molecule and super-resolution fluorescence microscopy



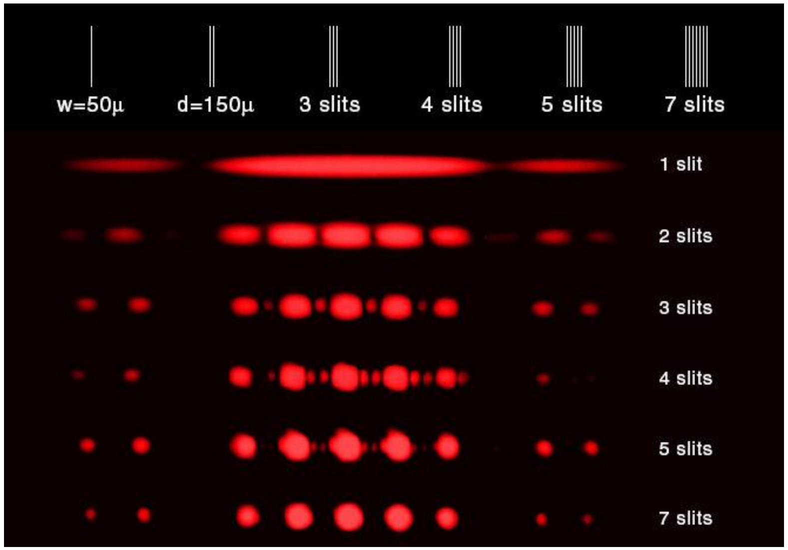


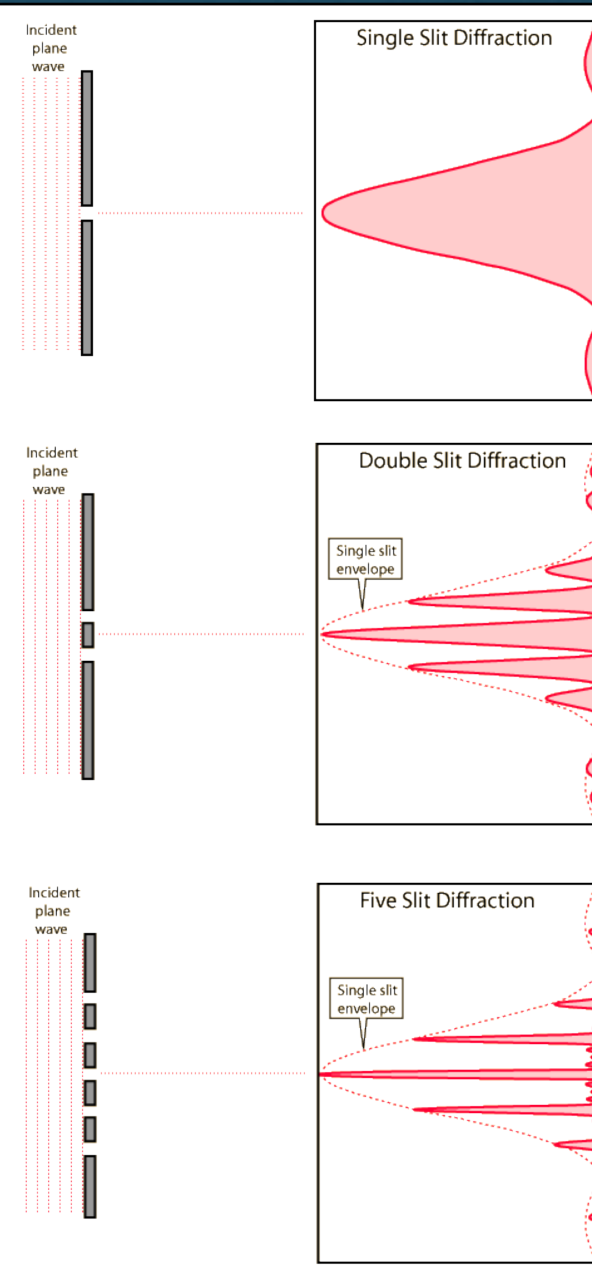


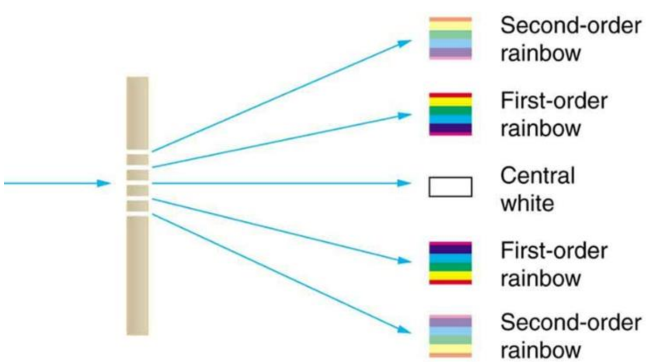


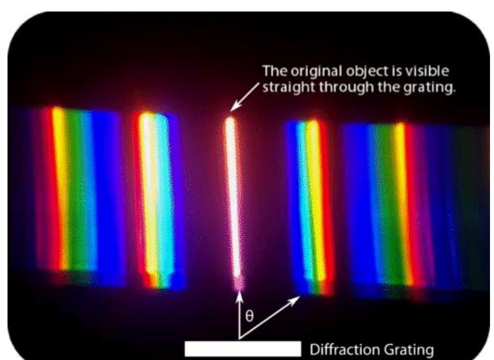


Multiple Slit Diffraction









Equivalent Diffraction Gratings and Spectrometers

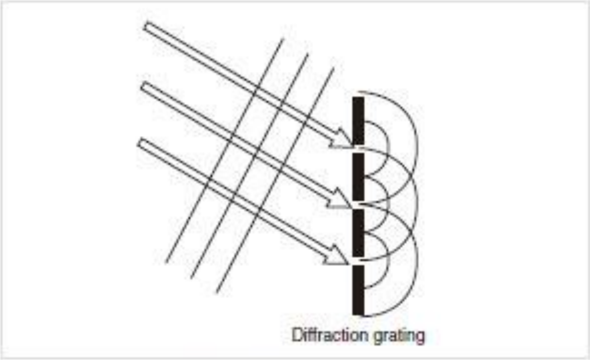


Fig.4 Diffraction Grating with Row of Slits

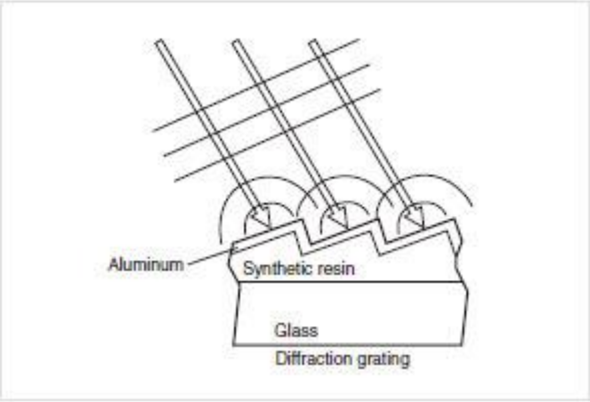
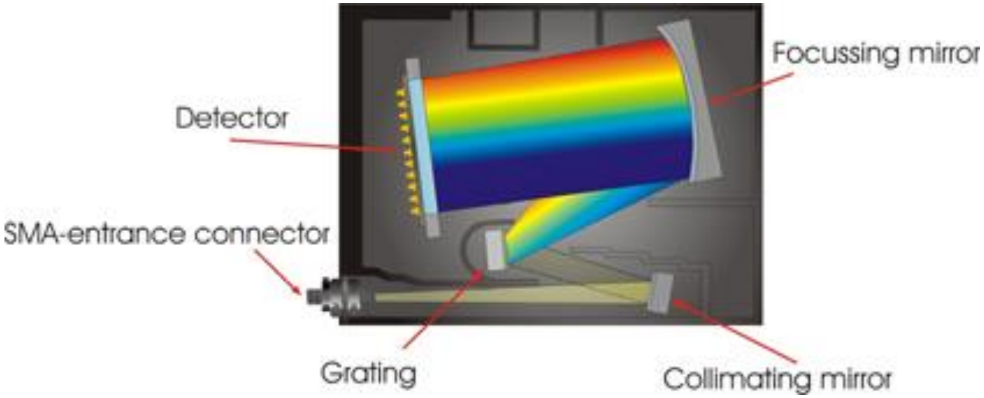
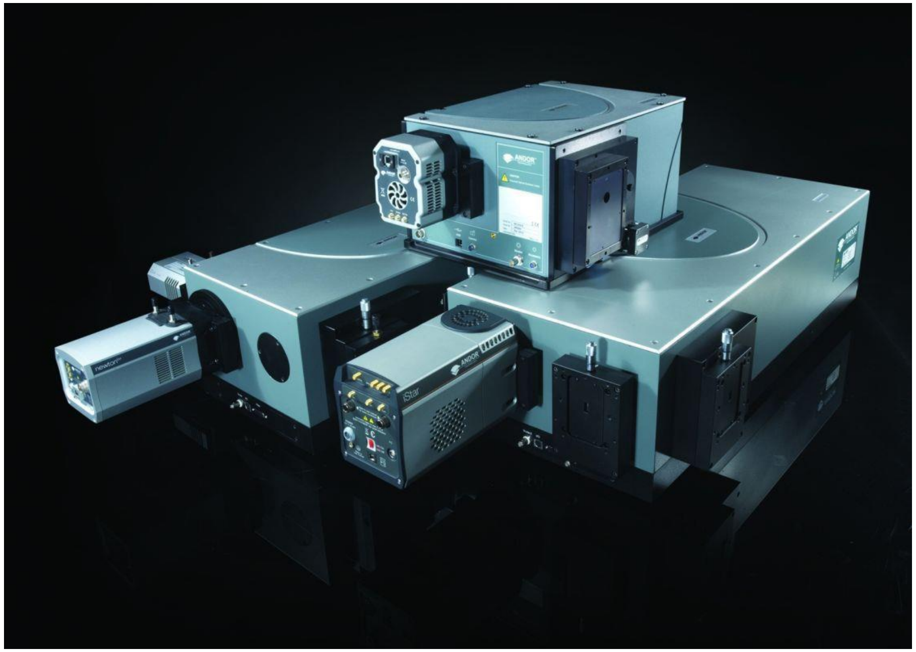
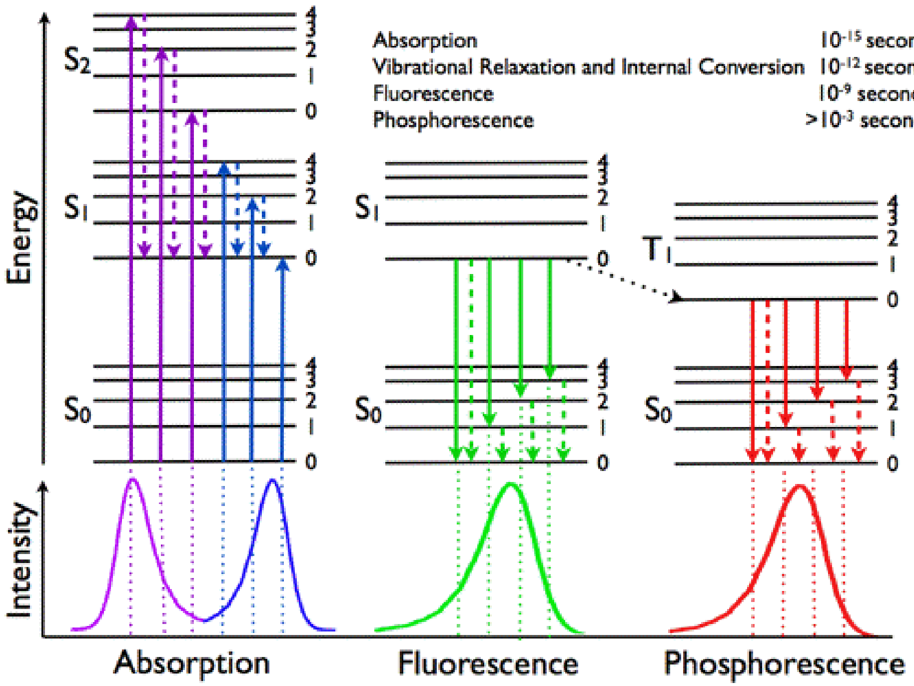


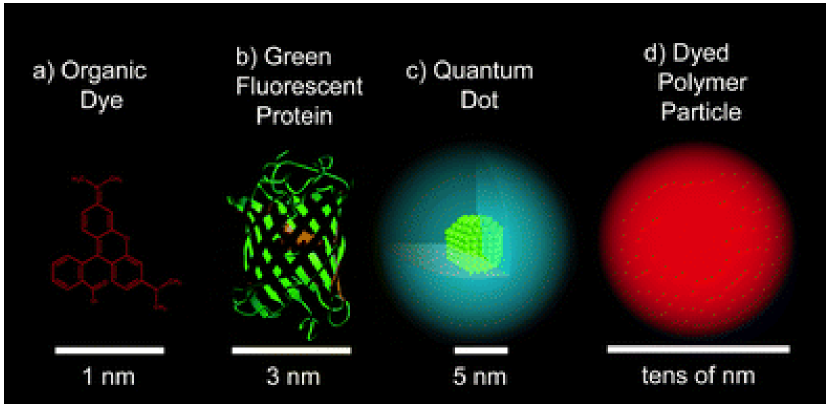
Fig.5 Reflective Blazed Diffraction Grating

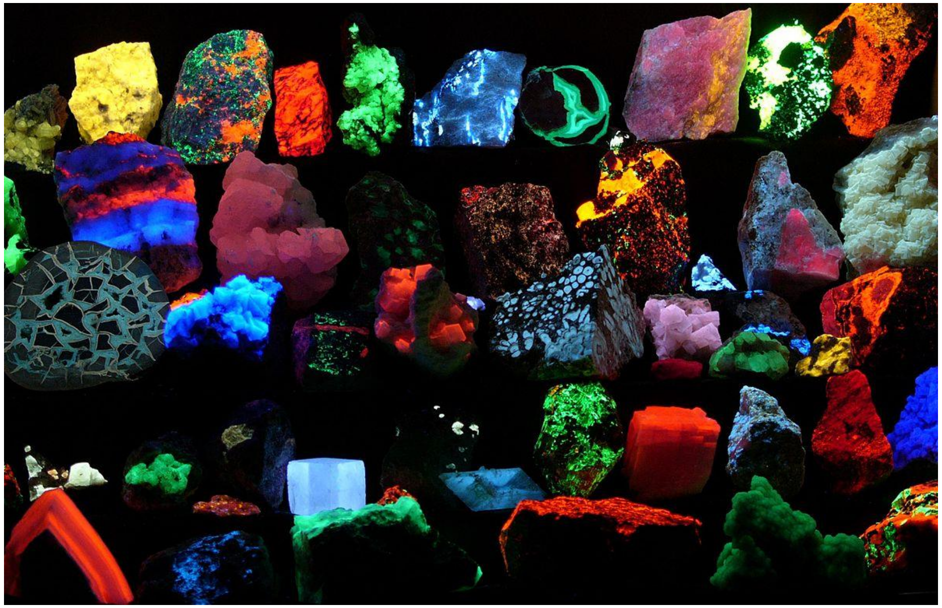




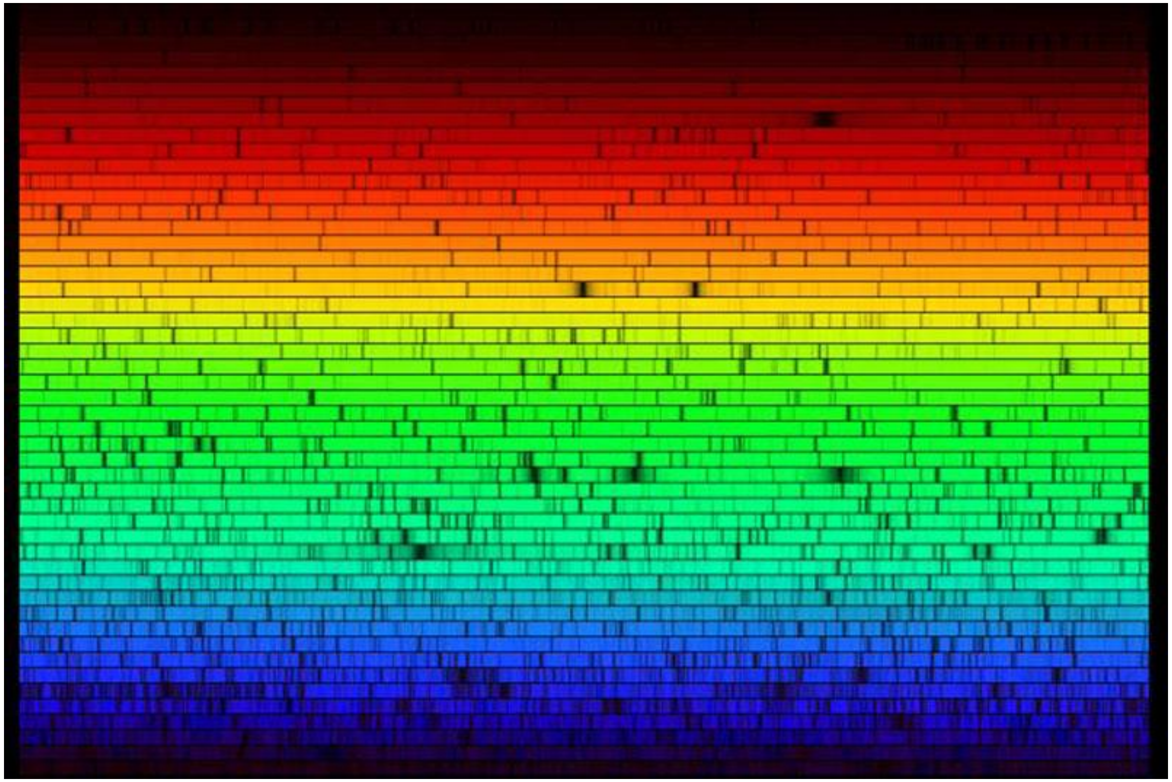
Luminescence Spectroscopy



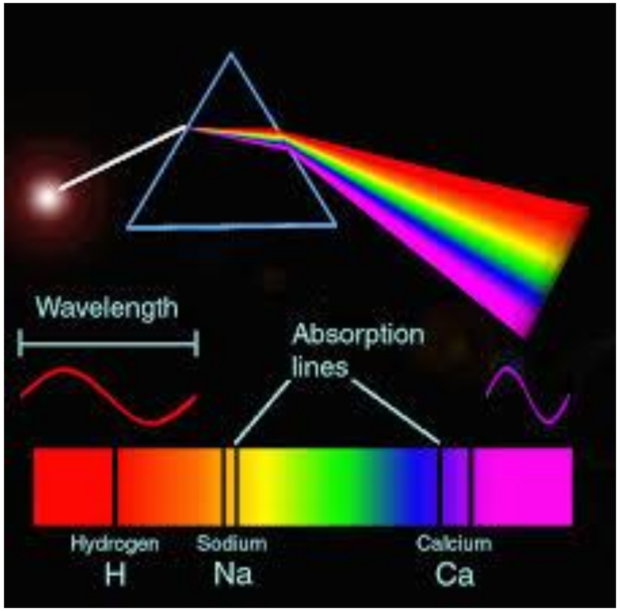


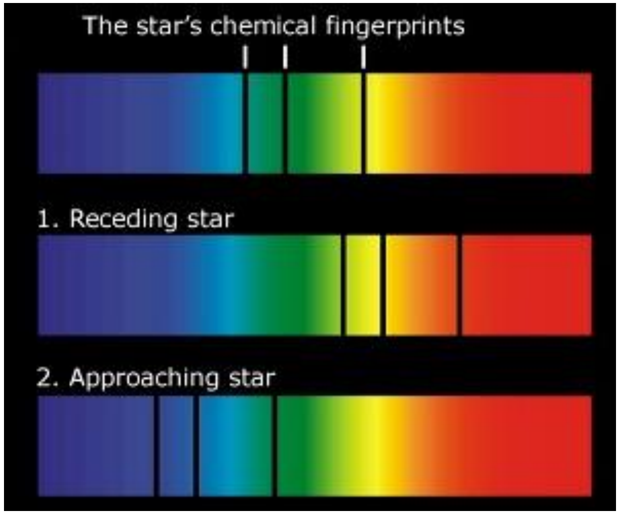


Stellar Spectroscopy

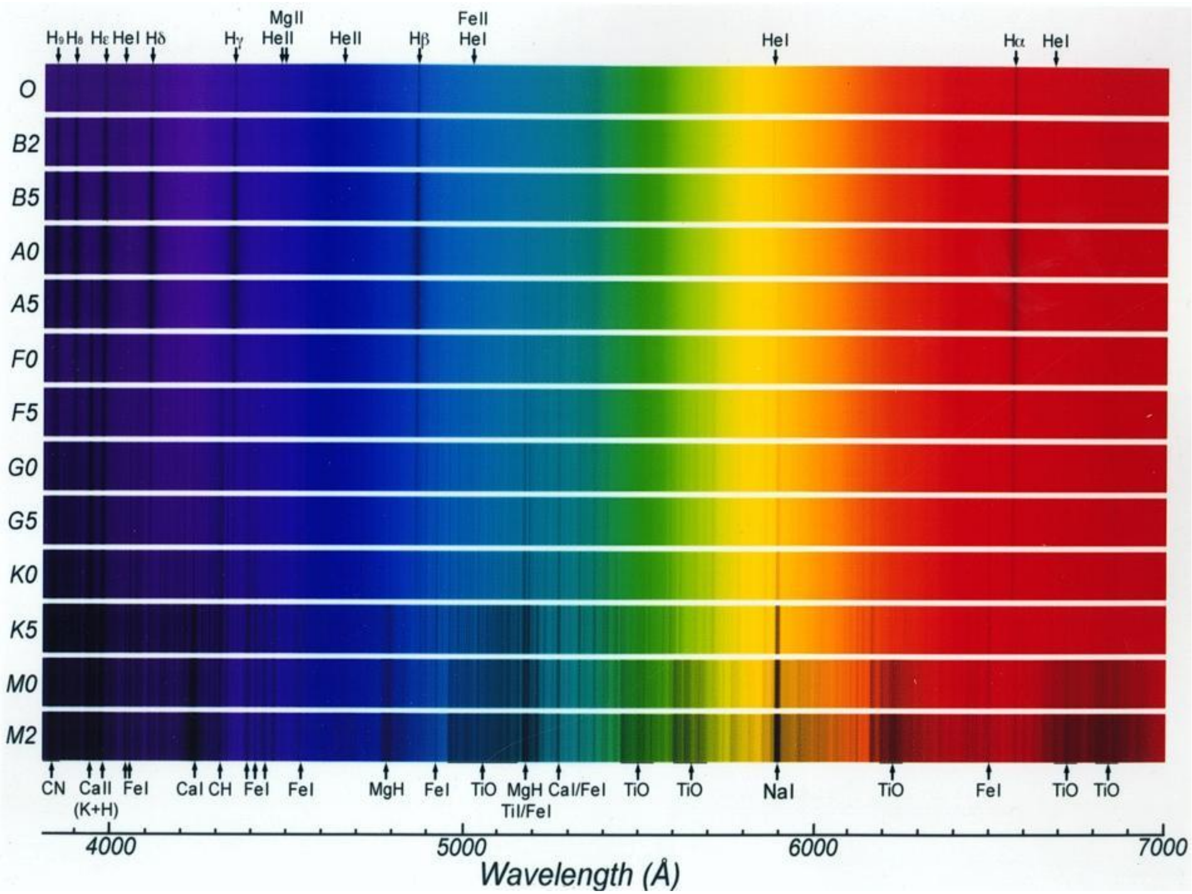


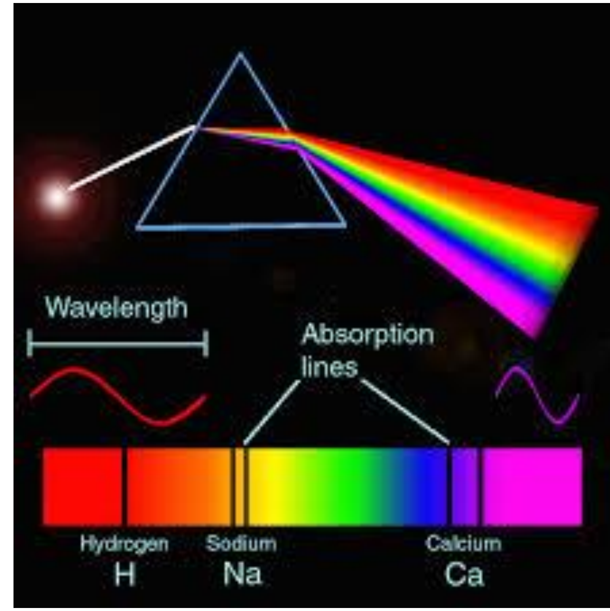
our sun

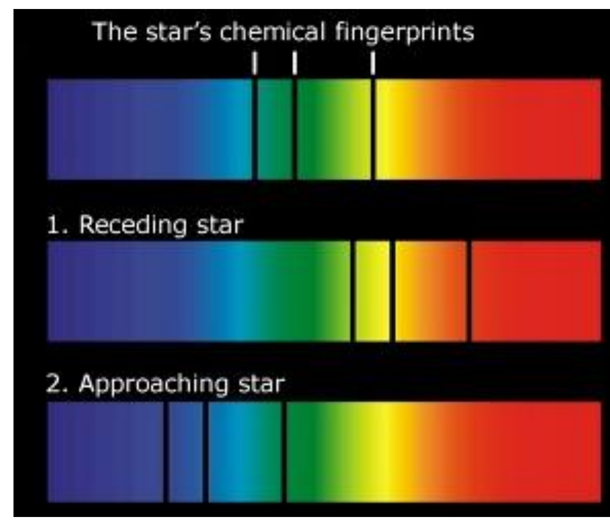




Stellar Spectroscopy

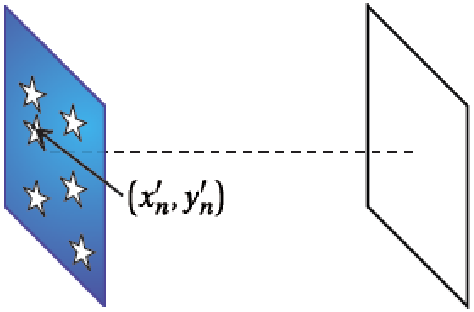






other suns

Array Theorem



Calculate the diffraction pattern caused by N identical apertures with $E_{\text{aperture}}(x', y', 0)$

Each aperture has a location (x'_n, y'_n), so that we use $E_{\text{aperture}}(x' - x'_n, y' - y'_n, 0)$

$$E(x,y,z) = -i\frac{e^{ikz}e^{i\frac{k}{2z}(x^2+y^2)}}{\lambda z}\sum_{n=1}^{N}\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dy' E_{\text{aperture}}(x'-x'_n, y'-y'_n, 0)\, e^{-i\frac{k}{z}(xx'+yy')}$$

Change of variables: $x'' \equiv x' - x'_n$ and $y'' \equiv y' - y'_n$

$$E(x,y,z) = -i\frac{e^{ikz}e^{i\frac{k}{2z}(x^2+y^2)}}{\lambda z}\sum_{n=1}^{N}\int_{-\infty}^{\infty}dx''\int_{-\infty}^{\infty}dy'' E_{\text{aperture}}(x'', y'', 0)\; e^{-i\frac{k}{z}[x(x''+x'_n)+y(y''+y'_n)]}$$

$$E(x,y,z) = \left[\sum_{n=1}^{N} e^{-i\frac{k}{z}(xx'_n+yy'_n)}\right]\left[-i\frac{e^{ikz}e^{i\frac{k}{2z}(x^2+y^2)}}{\lambda z}\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dy' E_{\text{aperture}}(x', y', 0)\, e^{-i\frac{k}{z}(xx'+yy')}\right]$$

Array Theorem in the Real World:

Discrete Fourier Transform of 2D Images

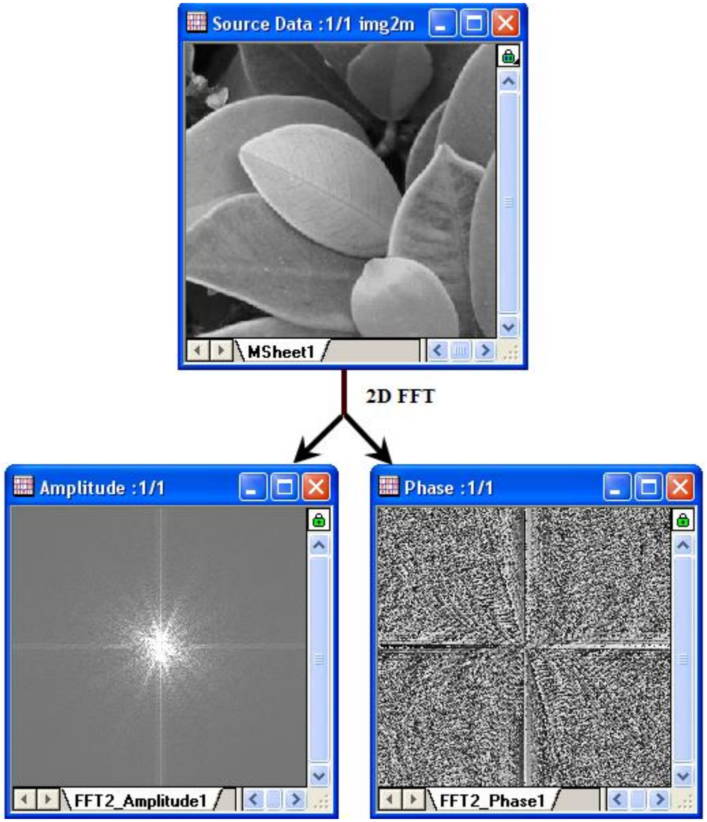
For a list of N sample data points (1D)

$$F(u) = \frac{1}{N}\sum_{n=0}^{N-1} f(n)\, e^{-i(2\pi n u)/N}$$

For an MxN array of sample data points (2D)

$$F(u,v) = \frac{1}{NM}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1} f(n,m)\, e^{-2\pi i\left(\frac{nu}{N}+\frac{mv}{M}\right)}$$

A lot of work has been put into algorithms to calculate this as efficiently as possible. (FFTW)



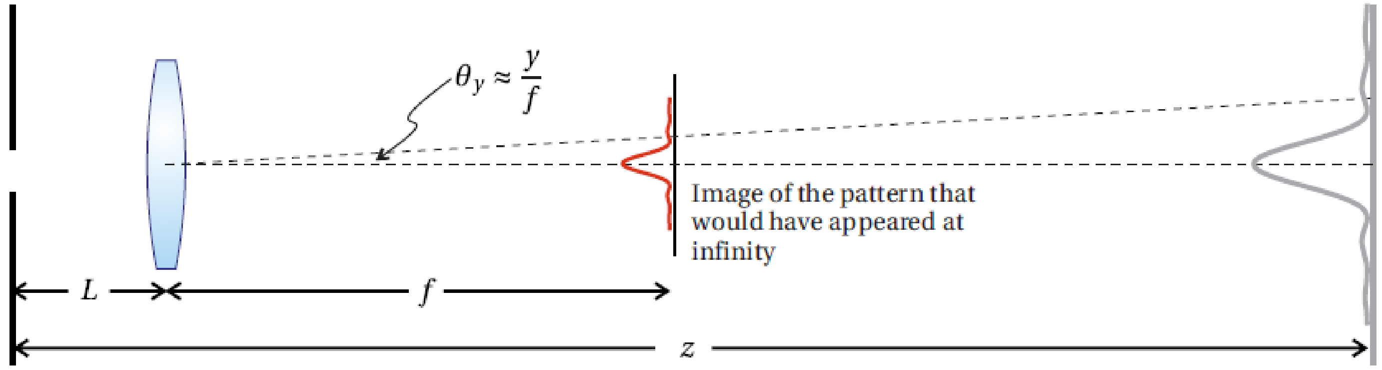
www.originlab.com

Fourier Optics: The Basic Idea

$$E(x,y,z) \cong -\frac{i e^{ikz} e^{i\frac{k}{2z}(x^2+y^2)}}{\lambda z} \iint\limits_{\text{aperture}} E(x',y',0)\, e^{-i\frac{k}{z}(xx'+yy')} dx' dy'$$

1. The intensity distribution in the far-field (Fraunhofer limit) is the 2D Fourier transform of the intensity distribution at the source.

2. A (perfect) lens causes the image at z=∞ to form at its focal distance.



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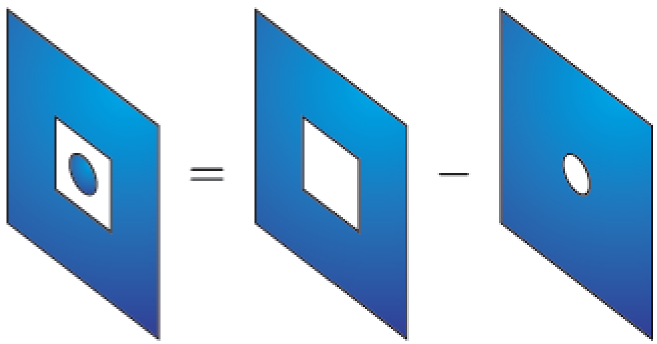
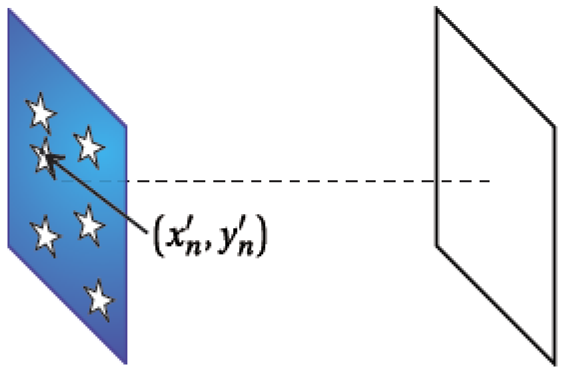
Babinet's Principle



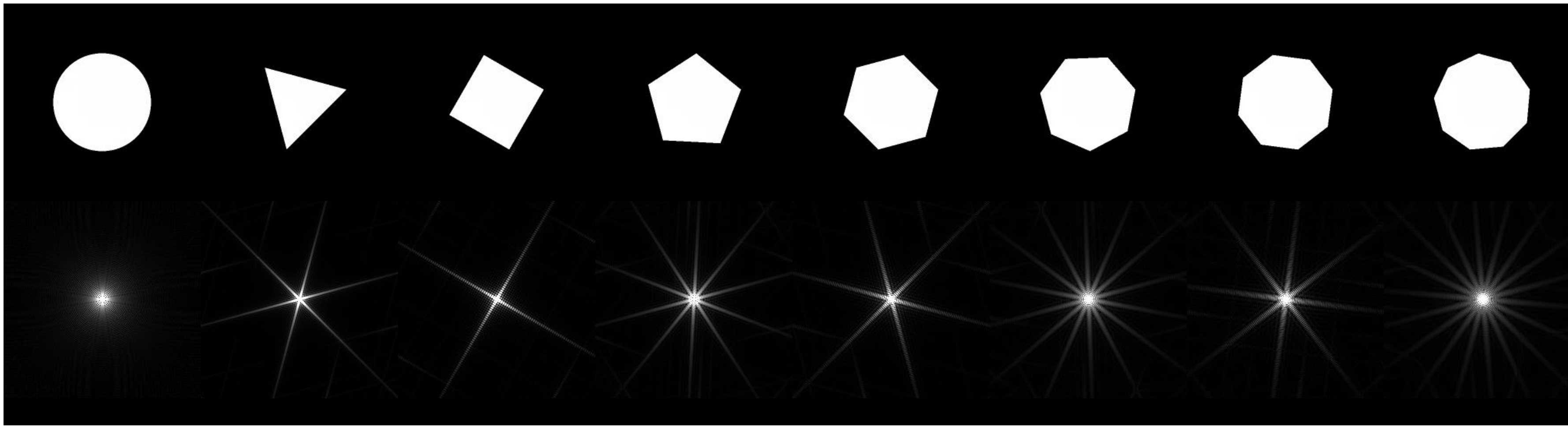
Array Theorem

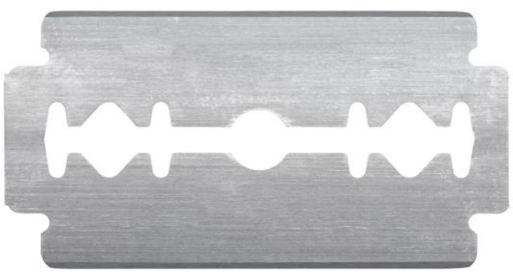


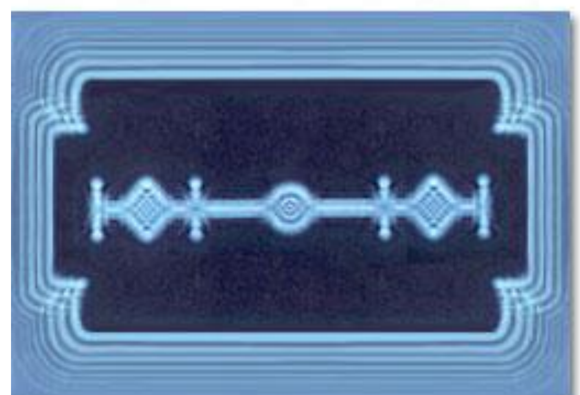
Diffraction from a complicated aperture is just
the *coherent* sum of diffraction by its component parts

Reminder: add the electric field amplitudes, not the intensities!

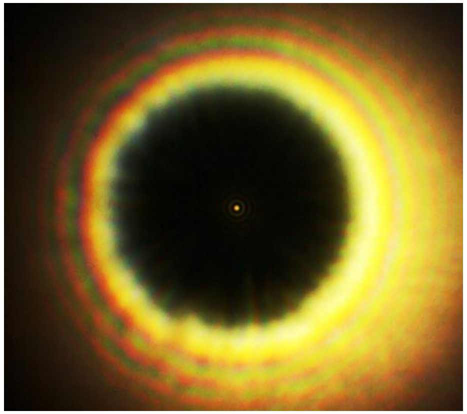
Diffraction





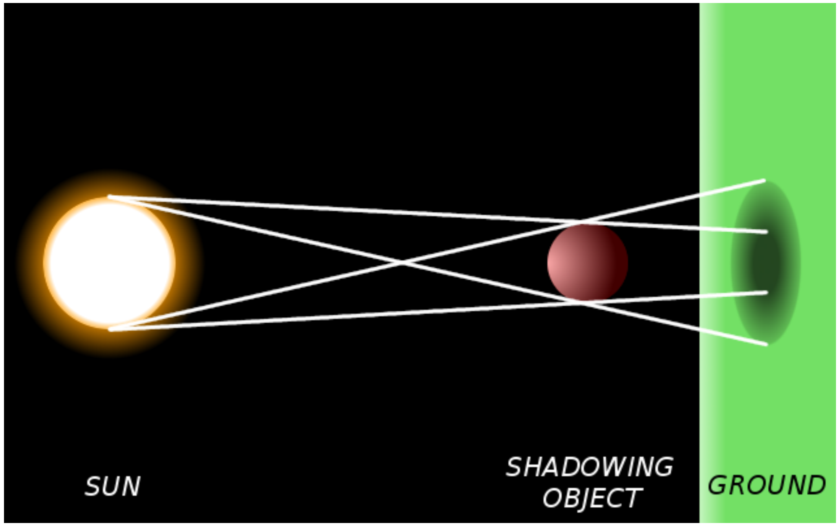


The Arago Spot

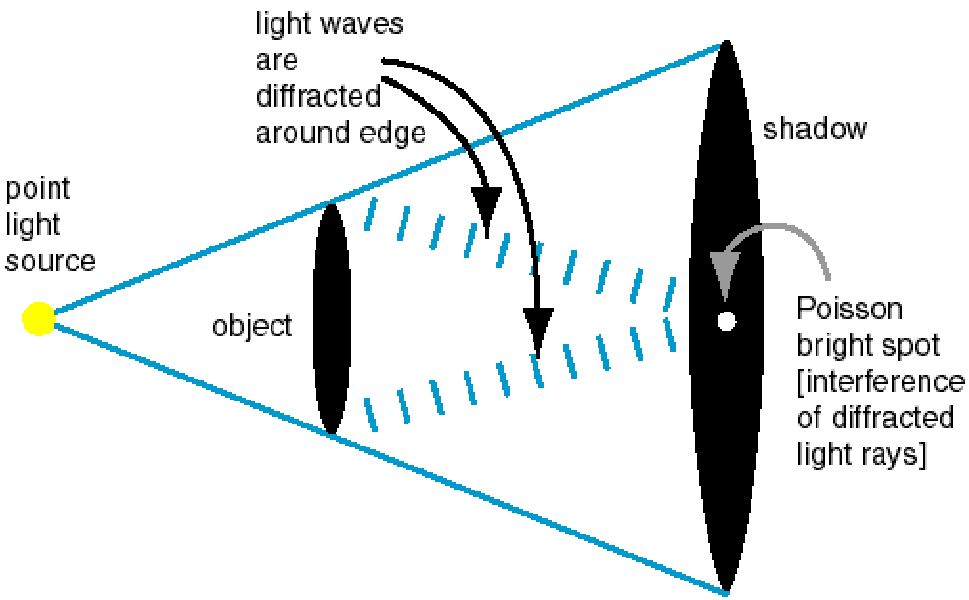


What is the “correct” nature of light?

Corpuscular Theory of Light

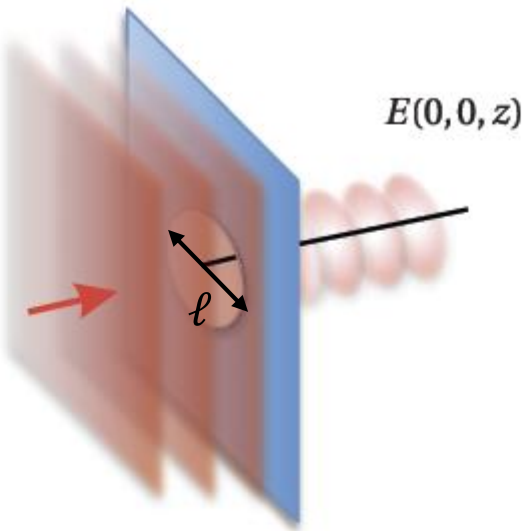


Wave Theory of Light



Fresnel-Kirchoff – Circular Aperture

Example: What is the on-axis electric field due of a plane wave diffracting through a circular aperture?



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$$E(0,0,z) = -\frac{i}{\lambda}\iint\limits_{\text{aperture}} E(x',y',0)\frac{e^{ik\sqrt{x'^2+y'^2+z^2}}}{\sqrt{x'^2+y'^2+z^2}}\,dx'dy'$$

Change to cylindrical coordinates!

$$= -\frac{iE_0}{\lambda}\int_0^{2\pi} d\phi' \int_0^{\ell/2} \frac{e^{ik\sqrt{\rho'^2+z^2}}}{\sqrt{\rho'^2+z^2}}\rho'\,d\rho'$$

Integration over ϕ is trivial, integral over ρ isn't too bad…

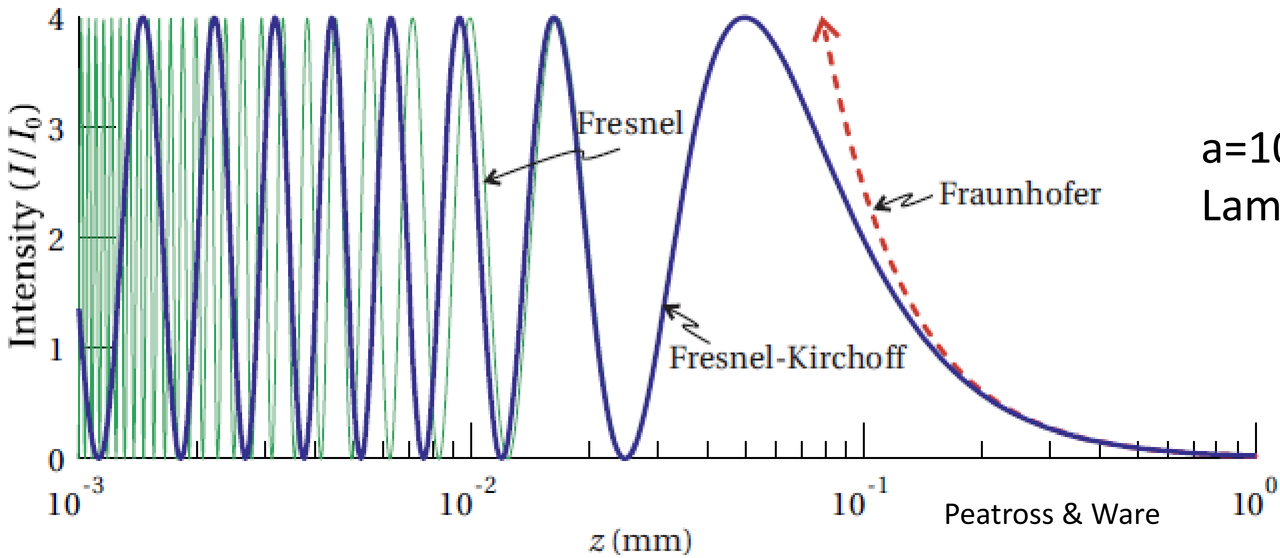
$$= -\frac{iE_0}{\lambda}2\pi\left.\frac{e^{ik\sqrt{\rho'^2+z^2}}}{ik}\right|_0^{\ell/2} = -E_0\left(e^{ik\sqrt{(\ell/2)^2+z^2}} - e^{ikz}\right)$$

On-Axis Intensity Behind Circular Aperture

$$I(0,0,z) \propto E(0,0,z)\,E^*(0,0,z)$$

$$\propto |E_0|^2 \left(e^{ik\sqrt{(\ell/2)^2+z^2}} - e^{ikz}\right)\left(e^{-ik\sqrt{(\ell/2)^2+z^2}} - e^{-ikz}\right)$$

$$\propto 2|E_0|^2 \left[1-\cos\left(k\sqrt{(\ell/2)^2+z^2} - kz\right)\right]$$



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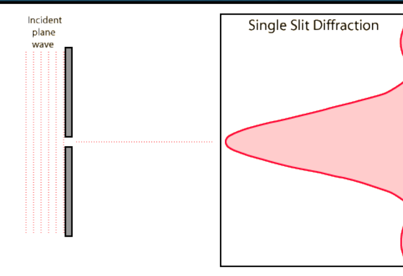
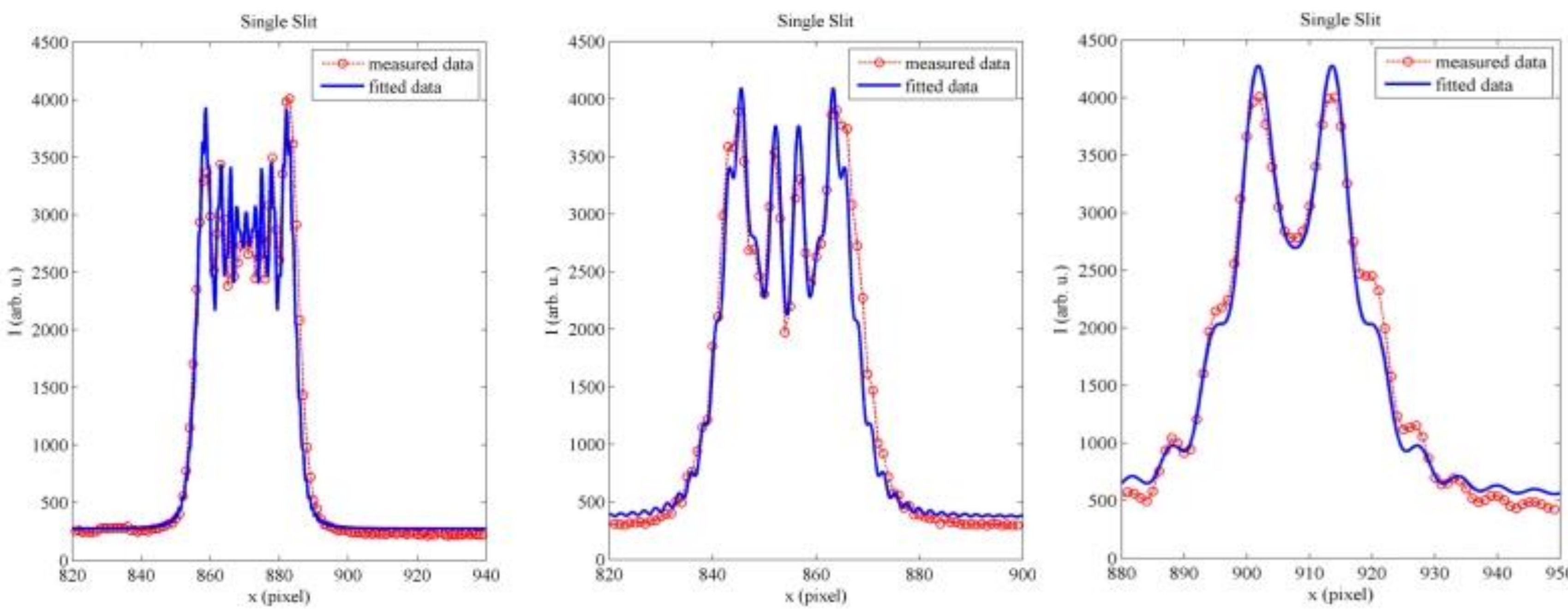
a=10 microns
Lambda = 500 nm

Fresnel Diffraction – Single Slit



$$E(x,y,z) \cong -\frac{i e^{ikz} e^{i\frac{k}{2z}(x^2+y^2)}}{\lambda z} \iint\limits_{\text{aperture}} E(x',y',0)\, \boxed{e^{i\frac{k}{2z}(x'^2+y'^2)}}\, e^{-i\frac{k}{z}(xx'+yy')}\, dx'\, dy'$$



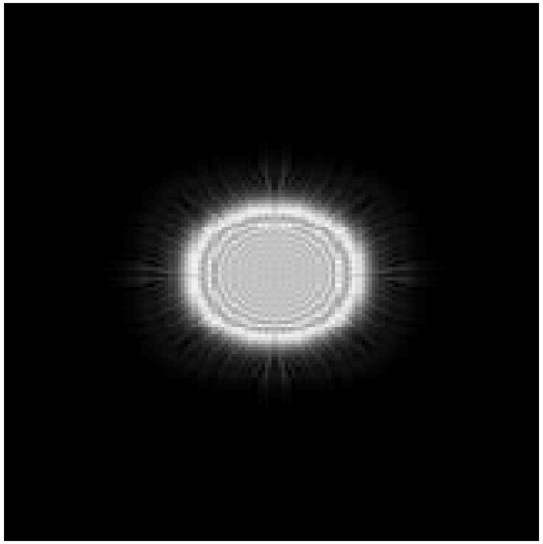
Fresnel Diffraction – Circular Aperture

- Intensity pattern at varying distances behind a circular aperture:

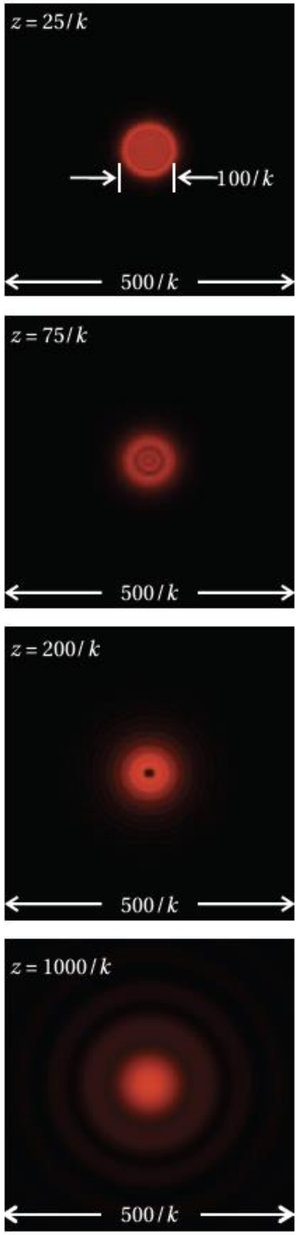
$$E(\rho,z) = -iE_0 \frac{2\pi e^{ikz} e^{i\frac{k\rho^2}{2z}}}{\lambda z} \int_0^{\ell/2} \rho' \, d\rho' \, e^{i\frac{k\rho'^2}{2z}} J_0\left(\frac{k\rho\rho'}{z}\right)$$

Bessel Function

- Has to be computed numerically



Credit: MuthuKutty at en.wikipedia



Limits of Diffraction

Fraunhofer:

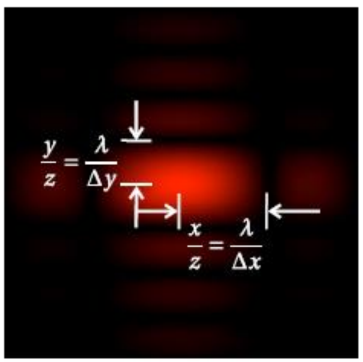
$$\lambda > \frac{d^2}{z}$$

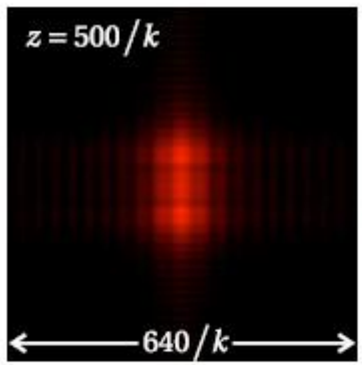
Fresnel:

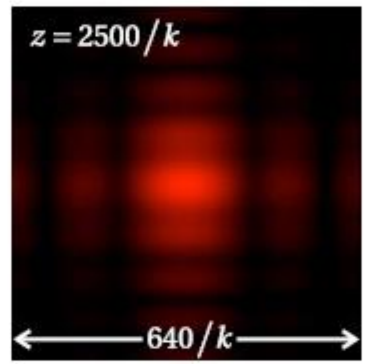
$$\lambda \cong \frac{d^2}{z}$$

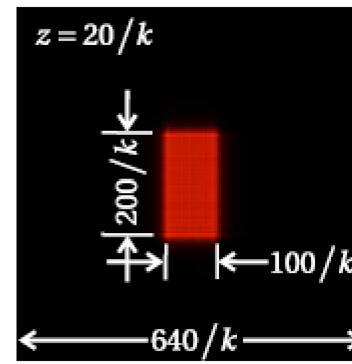
Geometric optics:

$$\lambda \ll d, z$$
$$(\lambda \to 0)$$









Imagine illuminating the same slit with different wavelengths
Keeping everything else the same:

← Microwave

X-ray →

Diffraction