Interactions with Matter

Radiation of Accelerating Charges:

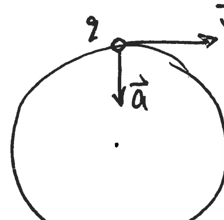
I) The Point Charge

stationary: $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$

accelerating: $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{a}{c^2}\sin\theta\,\frac{q}{r}\,\hat{\theta}$

- radiation is always emitted perpendicular to $\vec{a}$ ($\vec{k} \perp \vec{a}$)

Ex: Synchrotron radiation
- the strong forward emission from circulating charges



radiated power is

$$P = \frac{1}{4\pi\epsilon_0}\cdot\frac{2}{3}\cdot\frac{q^2a^2}{c^3}\gamma^4$$

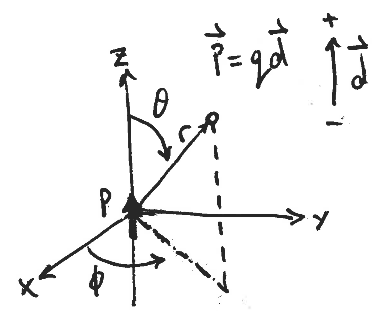
(where $\gamma = 1/\sqrt{1 - v^2/c^2}$)

II) The Electric Dipole

$r \gg d$ for both cases

Stationary: $\vec{E} = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$

Oscillating: $\vec{E} = \frac{p_0 k^2}{4\pi\epsilon_0}\sin\theta\,\frac{e^{i(kr-\omega t)}}{r}\,\hat{\theta}$



when $p = p_0 e^{i(kr-\omega t)}$, no radiation emitted in $\vec{d}$ direction

Note: for both the point charge and dipole, acceleration/oscillation leads to radiated field with $E \propto 1/r$ in direction $k \perp \vec{a}, \vec{d}$

Atomic Absorption & Emission of Radiation

- atoms are built of charges in a bound state
- lowest energy state = "ground state", but many higher energy "excited states"

Quantum Picture: $|0\rangle \uparrow |1\rangle$, $\omega = \frac{E_1 - E_0}{\hbar}$

Lorentz Oscillator Model: (charged masses on a spring)

(works surprisingly well in many cases)

The Driven Oscillator:

$$\vec{F}_{spring} = -m\omega_0^2 \vec{x} \quad \& \quad \vec{F}_{E\text{-field}} = q\vec{E}(t) = q\vec{E}_0 e^{i\omega t}$$

then the (undamped) equation of motion is:

$$m\frac{d^2\vec{x}}{dt^2} + m\omega_0^2\vec{x} = q\vec{E}_0 e^{i\omega t} \qquad [\text{remember: } \mathrm{Re}(e^{i\omega t}) = \cos\omega t]$$

has the solution $x(t) = x_0 e^{i\omega t}$

$$\Rightarrow x(t) = \frac{q/m}{(\omega_0^2 - \omega^2)} E_0 e^{i\omega t} = \frac{q/m}{(\omega_0^2 - \omega^2)} E(t)$$

so the oscillator just follows the field

now add a damping term: $\vec{F}_{damp} = -\gamma \frac{d\vec{x}}{dt}$

$$\Rightarrow \frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{qE_0}{m} e^{i\omega t}$$

$$\Rightarrow -\omega^2 x(t) + i\gamma x(t) + \omega_0^2 x(t) = \frac{q}{m} E_0 e^{i\omega t}$$

$$\Rightarrow x(t) = \frac{q/m}{(\omega_0^2 - \omega^2 + i\gamma\omega)} E(t)$$

in the context of bound charges of an atom, this relates to Polarization

$$P(t) = qx(t) = \frac{q^2/m}{(\omega_0^2 - \omega^2 + i\gamma\omega)} E_0 e^{i\omega t}$$

[same as Hecht eq 3.68, but with damping – see Hecht eq 3.72]

and for N atoms

$$P(t) = qNx(t) = \frac{q^2N/m}{(\omega_0^2 - \omega^2 + i\gamma\omega)} E_0 e^{i\omega t}$$

Note: this is complex valued, so get a phase shift from complex part

$$|z| = \sqrt{\tilde{z}\tilde{z}^*} = a^2 + b^2 \quad \& \quad \phi = \tan^{-1}\left(\frac{\tilde{z} - \tilde{z}^*}{\tilde{z} + \tilde{z}^*}\right) = \tan^{-1}(b/a)$$

When light interacts with matter it

1) causes charges in the matter to oscillate at the same frequency, but with a magnitude and phase that depends on $\omega - \omega_0$

2) the oscillating charges re-radiate (scattering)

3) the incident and scattered waves both continue to propagate

Light in Bulk Matter

What is the primary difference to our previous vacuum?

electric permittivity & magnetic permeability constants

$$\epsilon_0, \mu_0 \xrightarrow[\text{in matter}]{} \underbrace{\epsilon = K_E \epsilon_0}_{\text{relative permitivity}}, \quad \underbrace{\mu = K_m \mu_0}_{\text{relative permeability}}, \quad v = 1/\sqrt{\epsilon\mu}$$

index of refraction: $n \equiv c/v = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}}$ (this root is negative for only very special cases)

unless dealing with magnetic materials, $\frac{\mu}{\mu_0} \sim 1$

$$\Rightarrow n \simeq \sqrt{\epsilon/\epsilon_0} = \sqrt{K_E}$$

Now, given the formula for the polarization of a single dipole, let's connect to the macroscopic properties of a material (ie: ϵ and n)

$$\vec{P}(t) = \frac{q^2 N/m}{(\omega_0^2 - \omega^2 + i\gamma\omega)} \vec{E}(t)$$

- have N oscillators of charge q per unit volume
- we're pretending there's only 1 resonance freq and no interactions

Assume our bulk matter is a <u>linear</u> dielectric

$$\vec{P} = \epsilon_0 \chi_E \vec{E} \qquad \chi_E : \text{electric susceptibility}$$

$$= \epsilon_0 (K_E - 1)\vec{E} \qquad K_E : \text{relative permittivity (or dielectric constant)}$$

$$= \epsilon_0 \left(\frac{\epsilon}{\epsilon_0} - 1\right)\vec{E}$$

$$= (\epsilon - \epsilon_0)\vec{E}$$

then $\epsilon = \epsilon_0 + \frac{\vec{P}}{\vec{E}} = \frac{q^2 N/m}{(\omega_0^2 - \omega^2 + i\gamma\omega)} + \epsilon_0$

[Caveat: changes to a sum for multiple resonances and modified if the oscillators interact] (see Hecht eqn 3.73)

now remembering the connection between ϵ and the speed of the wave

$n = \frac{c}{v} = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} = \sqrt{K_E K_m}$, but $K_m \approx 1$ for <u>almost</u> all transparent materials

$$n^2 = K_E = \epsilon/\epsilon_0 = 1 + \frac{Nq^2}{\epsilon_0 m}\left(\frac{1}{\omega_0^2 - \omega^2 + i\gamma\omega}\right)$$

It bears repeating: this is overly simple $\longrightarrow$ usually have multiple resonances

$$\Longrightarrow \quad n^2 = 1 + \frac{Nq^2}{\epsilon_0 m}\sum_n \frac{f_n}{\omega_n^2 - \omega^2 + i\gamma\omega} \qquad \text{(Hecht eqn 3.72)}$$

Also, there may be interactions between oscillators in a dense medium, so that they're not independent (Hecht eqn 3.73)

Limiting cases are worth considering

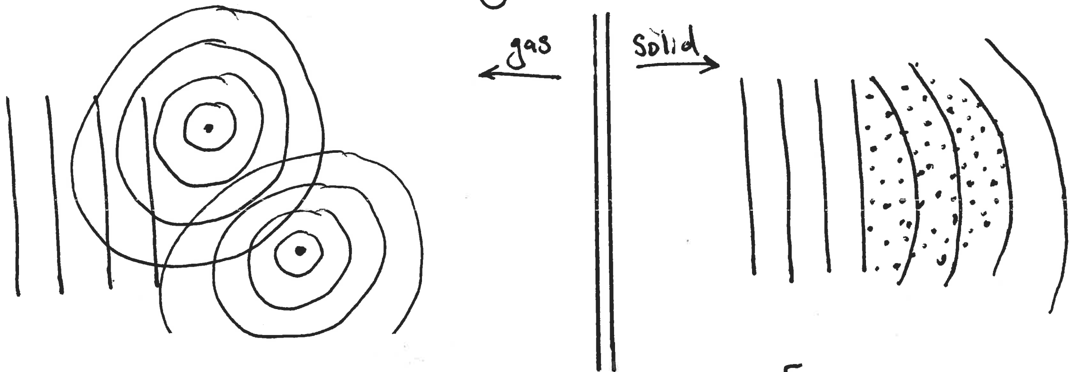
A) $\omega_0^2 - \omega^2 \gg \gamma\omega$ [negligible absorption]

B) $\epsilon/\epsilon_0 \ll 1$ [very small index, dilute gas]

C) $\omega \simeq \omega_0$ [near resonance]

Microscopic - Macroscopic Connection

What effect does the N charged oscillators have on the EM wave when the density of N changes?



[dilute medium]

- essentially independent scattered waves
- lots of lateral scattering

[dense medium]

- constructive interference going forward
- destructive interference on sides (very little lateral scattering)

The scattered waves experience a phase lag $\longrightarrow$ interferes with incident wave
~ this is why light travels slower when $n > 1$

Velocity of light in a medium: $v = c/n$

Velocity of light "between" atoms: $v = c$

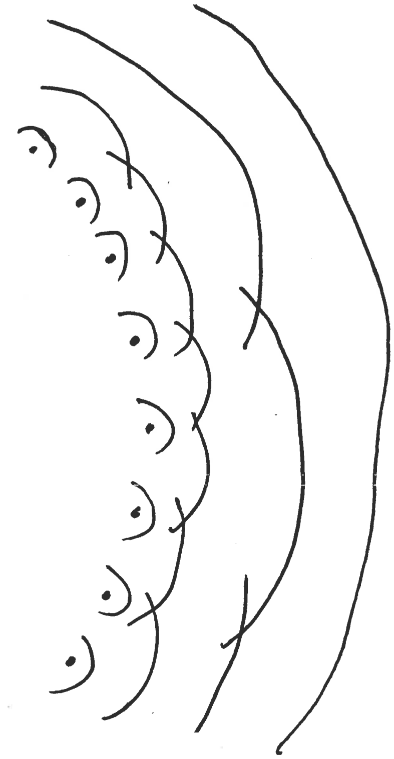
$n > 1 \longrightarrow$ scattered phase lags (velocity slower than c)

$n < 1 \longrightarrow$ scattered phase leads (velocity <u>faster</u> than c?!)

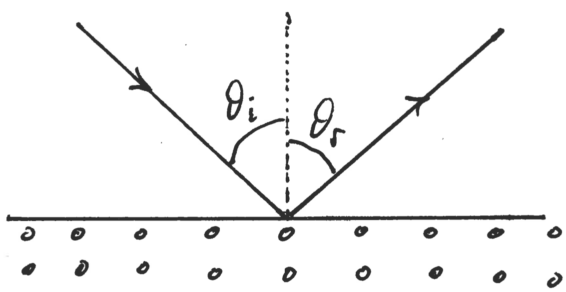
<u>Remember</u>: these are <u>phase</u> velocities, which motivates other definitions of velocity

Huygens' Principle

propagation modeled as emission of a bunch of "wavelets" from each point on a surface of equal phase

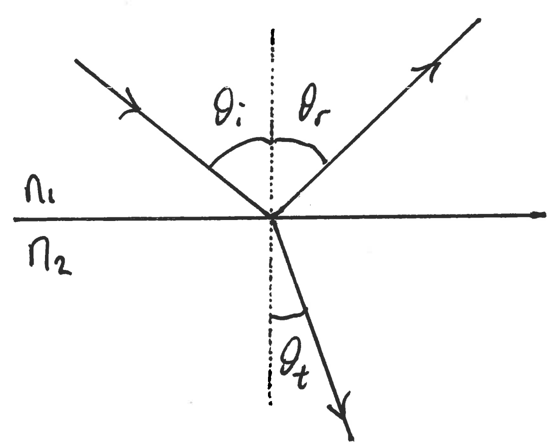


Law of Reflection



$$\theta_i = \theta_r$$

Law of Refraction



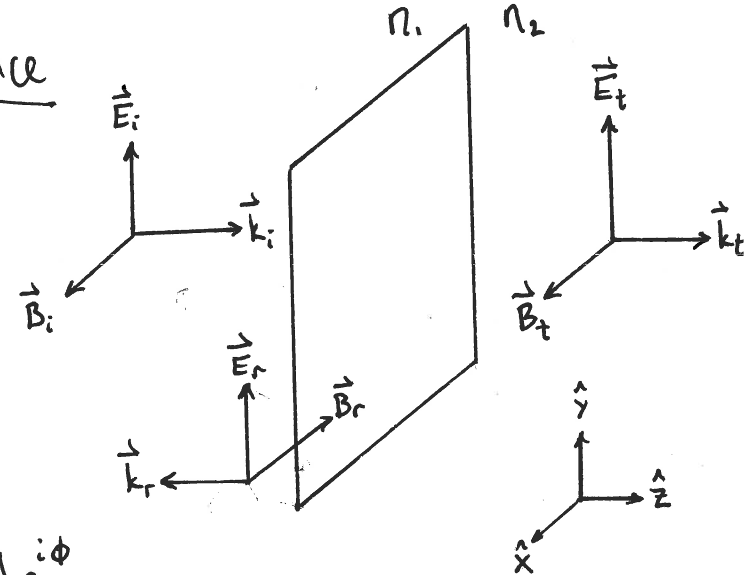
Snell's Law

$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

Reflection / Refraction at an Interface

What happens when a plane wave passes from one medium to another

First, consider case of <u>normal incidence</u>



In most general case, have complex vector wave equations

$$\tilde{\mathbf{E}}(\vec{r},t) = \tilde{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}\,\hat{u}$$

$$\tilde{\mathbf{B}}(\vec{r},t) = \tilde{B}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}\,(\hat{k}\times\hat{u})$$

where the phase has been absorbed: $\tilde{A}_0 = A_0 e^{i\phi}$

and remembering that $|E| = c|B|$ in vacuum $\longrightarrow$ $|E_0| = v|B_0|$ in matter

$$\tilde{\mathbf{B}}(\vec{r},t) = \frac{n}{c}\tilde{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}\,(\hat{k}\times\hat{u})$$

[Note: drop the ~ notation to simplify]

<u>incident</u>	<u>reflected</u>	<u>transmitted</u>
$\vec{E}_i(z,t) = E_{0i}\, e^{i(k_i z - \omega t)}\,\hat{y}$	$E_{0r}\, e^{i(-k_r z - \omega t)}\,\hat{y}$	$E_{0t}\, e^{i(k_t z - \omega t)}\,\hat{y}$
$\vec{B}_i(z,t) = B_{0i}\, e^{i(k_i z - \omega t)}\,\hat{x}$	$-\frac{n_1}{c} E_{0r}\, e^{i(-k_r z - \omega t)}\,\hat{x}$	$\frac{n_t}{c} E_{0t}\, e^{i(k_t z - \omega t)}\,\hat{x}$

$\left(B_{0i} = \frac{n_1}{c} E_{0i}\right)$

(note the minus signs)

for continuity at the material interface, enforce EM boundary conditions

<u>fields // to interface</u>

i) $E_{\parallel}^{(1)} = E_{\parallel}^{(2)}$

ii) $\frac{1}{\mu_1} B_{\parallel}^{(1)} = \frac{1}{\mu_2} B_{\parallel}^{(2)}$

$\left(B_{\parallel}^{(1)} \simeq B_{\parallel}^{(2)}\right)$

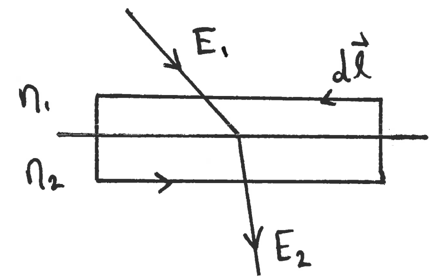
<u>fields ⊥ to interface</u>

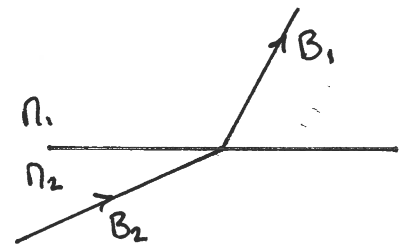
iii) $\epsilon_1 E_{\perp}^{(1)} = \epsilon_2 E_{\perp}^{(2)}$ (assuming bound charges, no surface charge density σ_f)

iv) $B_{\perp}^{(1)} = B_{\perp}^{(2)}$

Where do these boundary conditions come from?

[We won't rigorously derive these, but it's easy to explain since we only care about the case for linear media with no free surface charge or current (see Griffiths 7.3.6)]





Faraday's Law (integral form)

$$\oint_C \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_A \vec{B}\cdot d\vec{A}$$

take the area (flux) to zero

$$\oint_C \vec{E}\cdot d\vec{l} = 0$$

$$E_{\parallel}^{(1)} - E_{\parallel}^{(2)} = 0$$

Ampere's Law

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 \cancelto{0}{I_{enc}} + \frac{d}{dt}\int_A \epsilon\vec{E}\cdot d\vec{A}$$

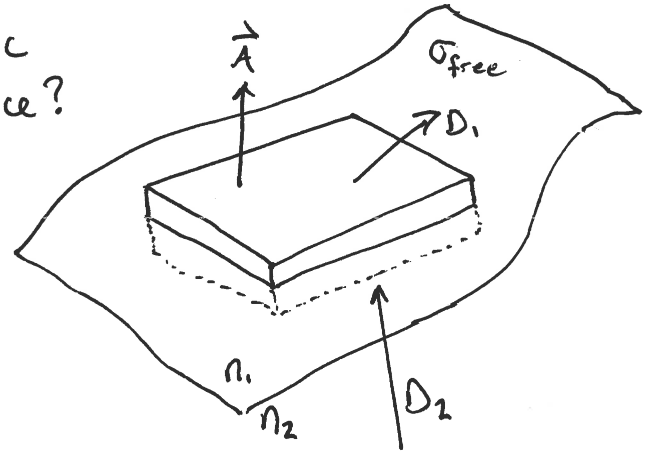
(same trick: $A\to 0$)

$$\oint_C \vec{B}\cdot d\vec{l} = 0$$

$$B_{\parallel}^{(1)} - B_{\parallel}^{(2)} = 0$$

So the fields // to the interface must be equal to be continuous

How do the $\perp$ fields change at dielectric interface?



Gauss' Law

$$\oint_A \vec{D}\cdot d\vec{A} = Q_{free}^{enc} \qquad (\vec{D}_1 = \epsilon_1\vec{E}_1)$$

$$\vec{D}_1\cdot\vec{A} - \vec{D}_2\cdot\vec{A} = \cancelto{0}{\sigma_{free}} A$$

$$\epsilon_1 E_{\perp}^{(1)} = \epsilon_2 E_{\perp}^{(2)}$$

(and similar for $\oint_A \vec{B}\cdot d\vec{A} = 0$: $B_{\perp}^{(1)} = B_{\perp}^{(2)}$)

Now, given these BC's and the field definitions, have

$$E_{\parallel}^{(1)} = E_{\parallel}^{(2)} \qquad B_{\parallel}^{(1)} = B_{\parallel}^{(2)}$$

$$E_{oi} + E_{or} = E_{ot} \qquad n_1 (E_{oi} - E_{or}) = n_2 E_{ot} \qquad (E = \tfrac{c}{n} B)$$

and can solve for E_{ot} & E_{or} in terms of E_{oi}

<u>Amplitude Reflection Coefficient</u>: $r = E_{or}/E_{oi} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)$

<u>Amplitude Transmission Coefficient</u>: $t = E_{ot}/E_{oi} = \left(\frac{2n_1}{n_1 + n_2}\right)$

What about powers involved? ($I \propto E^2$)

<u>Reflectance</u>: $R = I_r/I_i = r^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$

<u>Transmittance</u>: $T = I_t/I_i = \frac{n_2}{n_1} t^2 = \frac{n_2}{n_1}\left(\frac{2n_1}{n_1 + n_2}\right)^2$

[needed since velocity changes between media → changes energy delevered per unit time and area]

Conservation requires:

$$R + T = 1$$

A few comments:

- the above amplitude and power ratios are for <u>normal</u> incidence; we'll expand these to the general Fresnel Eqn's for arb. angle incedence
- the sign of r changes from $n_1 > n_2$ to $n_1 < n_2$ (180° phase change)
- r gets larger when n_1/n_2 is very different from 1

  t gets smaller when " " " " "

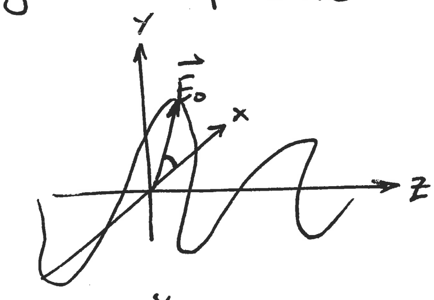
A digression to discuss polarization

imagine a plane wave $\vec{E} = \vec{E}_0 \cos(kz - \omega t)$

assume $\vec{E}_0$ is constant in time : "linear" polarization

Could even write $\vec{E}_0$ as a superposition of two orthogonal components

$$\vec{E}_0 = E_{0x}\hat{i} + E_{0y}\hat{j}$$

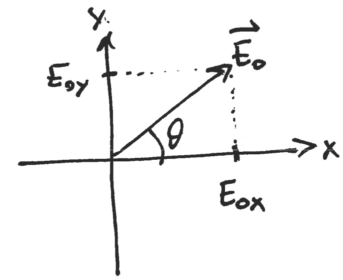


Where it's easy to see that

$$E_0^2 = E_{0x}^2 + E_{0y}^2$$

$$E_{0x} = E_0 \cos\theta$$

$$E_{0y} = E_0 \sin\theta$$



and can simply write $\vec{E}(z,t) = \left(E_{0x}\hat{i} + E_{0y}\hat{j}\right)\cos(kz - \omega t)$

Linear polarization means the E-field is oscillating along a line (or plane), and the angle is variable

Note: the B-field is still there, orthogonal to E all the time, we're just not concerned with it at the moment

☆ In general, the phase of the x & y components don't have to be equal

Ex: $\vec{E}(z,t) = \hat{i} E_{0x} \cos(kz - \omega t + \phi_x) + \hat{j} E_{0y} \cos(kz - \omega t + \phi_y)$

- magnitude and phase of E become time-dependent
- for now, it's sufficient to know we can decompose any polarization into a sum of orthogonal linear polarizations

EM Waves at Interface for Arbitrary Angle

Remember, in general have wave functions of the form

$$\vec{E}_i(\vec{r},t) = \vec{E}_{0i}\, e^{i(\vec{k}_i\cdot\vec{r} - \omega t)} \quad \& \quad \vec{B}_i(\vec{r},t) = \frac{n_i}{c}(\vec{k}_i \times \vec{E}_i)$$

(and similarly for $\vec{E}_r$ & $\vec{E}_t$, just with different polarization and $\vec{k}$ - see above)

Before, the fields were always // to the interface; now we have to consider components $\perp$ to interface, too

How to define polarization with respect to the interface?

<u>S-polarized/Transverse Electric</u>: $\vec{E} \perp$ plane of incidence

<u>P-polarized/Transverse Magnetic</u>: $\vec{E}$ // plane of incidence

Each polarization state (S and P) have separate calculations for reflection & transmission coefficients

Can add the polarization components back up at the end to get general result for total transmitted and reflected waves

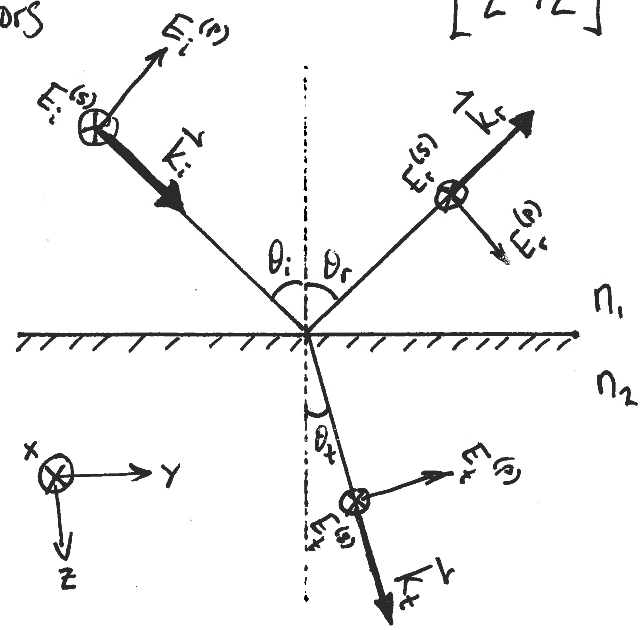
Same prescription as for normal incidence case, just more to keep track of

1) develope expressions for i, r, t fields of form: $\vec{E}(r,t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

2) apply EM boundary conditions to get expressions of the form:

$$(\quad)e^{i(\vec{k}_i\cdot\vec{r} - \omega_i t)} + (\quad)e^{i(\vec{k}_r\cdot\vec{r} - \omega_r t)} = (\quad)e^{i(\vec{k}_t\cdot\vec{r} - \omega_t t)}$$

First, let's look at constraints on propagation vectors



<u>incident</u>: $\vec{k}_i \cdot \vec{r} = k_i (y \sin\theta_i + z\cos\theta_i)$

<u>reflected</u>: $\vec{k}_r \cdot \vec{r} = k_r (y \sin\theta_r - z\cos\theta_r)$

<u>transmitted</u>: $\vec{k}_t \cdot \vec{r} = k_t (y \sin\theta_t + z\cos\theta_t)$

All of the BC's, defined at $z=0$, have the form:

$$(\ )e^{i(k_i y \sin\theta_i - \omega_i t)} + (\ )e^{i(k_r y \sin\theta_r - \omega_r t)}$$

$$= (\ )e^{i(k_t y \sin\theta_t - \omega_t t)}$$

To be satisfied at all possible values of y & t, must have

I) $\omega_i = \omega_r = \omega_t = \omega$

II) $k_i \sin\theta_i = k_r \sin\theta_r = k_t \sin\theta_t$ (remember: $k = \frac{\omega}{v} = \frac{n\omega}{c}$)

$$\Rightarrow \underbrace{\underbrace{n_i \sin\theta_i = n_r \sin\theta_r}_{A} = n_t \sin\theta_t}_{B}$$

A) Since $n_i = n_r$, then $\theta_i = \theta_r$ $\longrightarrow$ <u>Law of Reflection</u>

B) $n_i \sin\theta_i = n_t \sin\theta_t$ $\longrightarrow$ <u>Snell's Law</u>

So with the exponential parts of the BC's being equal, we can solve for the field amplitudes

Break the fields down into vector components

$$\underline{\underline{\vec{E} = \vec{E}^{(s)} + \vec{E}^{(p)}}}$$ (orthogonal linear polarization states)

$$\vec{E}_i = E_i^{(P)}(\hat{y}\cos\theta_i - \hat{z}\sin\theta_i) + \hat{x}E_i^{(S)}$$
$$\vec{E}_r = E_r^{(P)}(\hat{y}\cos\theta_r + \hat{z}\sin\theta_r) + \hat{x}E_r^{(S)}$$
$$\vec{E}_t = E_t^{(P)}(\hat{y}\cos\theta_t + \hat{z}\sin\theta_r) + \hat{x}E_t^{(S)}$$

$$\underline{\vec{B} = \vec{B}^{(S)} + \vec{B}^{(P)}}$$

$$\vec{B}_i = \frac{n_i}{c}\left[-\hat{x}E_i^{(P)} + E_i^{(S)}(-\hat{z}\sin\theta_i + \hat{y}\cos\theta_i)\right]$$
$$\vec{B}_r = \frac{n_r}{c}\left[\hat{x}E_r^{(P)} + E_i^{(S)}(-\hat{z}\sin\theta_r - \hat{y}\cos\theta_r)\right]$$
$$B_t = \frac{n_t}{c}\left[-\hat{x}E_t^{(P)} + E_t^{(S)}(-\hat{z}\sin\theta_t + \hat{y}\cos\theta_t)\right]$$

Now we can solve the transmission/reflection problem separately for the two orthogonal linear polarization states $\underbrace{(s \text{ and } p, \text{ or } \perp \text{ and } /\!/)}$

notations may be used interchangably along with TE and TM

Simplify the problem: <u>S-polarized component only</u>

$$\vec{E}_i = E_i^{(S)}\hat{x} \qquad \vec{B}_i = \frac{n_i}{c}E_i^{(S)}(-\hat{z}\sin\theta_i + \hat{y}\cos\theta_i)$$
$$\vec{E}_r = \vec{E}_r^{(S)}\hat{x} \qquad \vec{B}_r = \frac{n_r}{c}E_r^{(S)}(-\hat{z}\sin\theta_r - \hat{y}\cos\theta_r)$$
$$\vec{E}_t = \vec{E}_t^{(S)}\hat{x} \qquad \vec{B}_t = \frac{n_t}{c}E_t^{(S)}(-\hat{z}\sin\theta_t + \hat{y}\cos\theta_t)$$

Apply boundary conditions:

i) $E_{/\!/}^{(1)} = E_{/\!/}^{(2)} \implies \boxed{\vec{E}_i^{(S)} + \vec{E}_r^{(S)} = \vec{E}_t^{(S)}}$ (just $\hat{x}$-components... easy!)

ii) $B_{/\!/}^{(1)} = B_{/\!/}^{(2)} \implies \vec{B}_i^{(S)} + \vec{B}_r^{(S)} = \vec{B}_t^{(S)}$ (only $\hat{y}$-components // to surface, $\hat{z}$ gives redundant set of constraints)

$$\implies \frac{n_i}{c}E_i^{(S)}\cos\theta_i - \frac{n_r}{c}E_r^{(S)}\cos\theta_r = \frac{n_t}{c}E_t^{(S)}\cos\theta_t$$

$$\implies n_i\cos\theta_i(E_i^{(S)} - E_r^{(S)}) = n_t E_t^{(S)}\cos\theta_t \implies \boxed{E_i^{(S)} - E_r^{(S)} = \frac{n_t\cos\theta_t}{n_i\cos\theta_i}E_t^{(S)}} \quad \begin{pmatrix}\theta_i = \theta_r \\ n_i = n_r\end{pmatrix}$$

So have 2 equations and 3 unknowns

Can eliminate 1 of the 3 fields and calculate the ratio for the remaining two (same as we did for normal coincidence)

$$r_s = E_r^{(s)} / E_i^{(s)} \quad \& \quad t_s = E_t^{(s)} / E_i^{(s)}$$

Usually have to do some work to express these relations in a form that's convenient for a given problem

Simple substitution gives $r_s = \dfrac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t}$ ($n_i, n_t, \theta_i, \theta_t$ are not independent → Snell's Law)

1) want to eliminate n_i & n_t?

$$\Rightarrow r_s = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$$

(by using Snell's Law and product-to-sum trig. identities)

2) eliminate θ_t instead?

rewrite Snell's Law: $\theta_t = \sin^{-1}\left(\frac{n_i}{n_t} \sin\theta_i\right)$

then $\cos\theta_t = \sqrt{1 - \left(\frac{n_i}{n_t} \sin\theta_i\right)^2}$

$$\Rightarrow r_s = \frac{\frac{n_i}{n_t}\cos\theta_i - \sqrt{1 - \left(n_i/n_t \sin\theta_i\right)^2}}{\frac{n_i}{n_t}\cos\theta_i + \sqrt{1 - \left(n_i/n_t \sin\theta_i\right)^2}}$$

Similar (simple) substitution gives amplitude transmission coefficient

$$t_s = \frac{2 n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t}$$

The same algorithm can be applied to get r_p & t_p

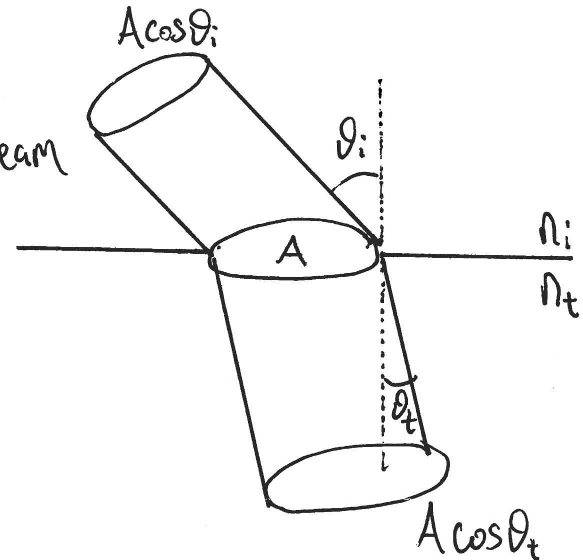
- consider only P-polarized components of i, r, and t EM fields
- apply boundary conditions: iii) $\epsilon_1 E_\perp^{(1)} = \epsilon_2 E_\perp^{(2)}$, iv) $B_\perp^{(1)} = B_\perp^{(2)}$
- obtain 2 equations of 3 unknowns (field amplitudes)
- eliminate 1 of these 3 fields and calculate ratio of remaining two
- apply Snell's Law & Law of Reflection throughout, as needed

$$r_p = \frac{n_t \cos\theta_i - n_i \cos\theta_t}{n_i \cos\theta_t + n_t \cos\theta_i} \qquad t_p = \frac{2 n_i \cos\theta_i}{n_i \cos\theta_t + n_t \cos\theta_i}$$

Reflectance/Transmittance for Arbitrary Angle of Incidence

We still have $R = |r|^2$, but there are two effects to account for in the transmitted beam

1) change in velocity of propagation
2) change in cross-sectional area



$$R = \frac{I_r A\cos\theta_r}{I_i A\cos\theta_i} = \frac{I_r}{I_i} = |r|^2$$

(can be $r_{\parallel}$ or $r_\perp$)

$$T = \frac{I_t A\cos\theta_t}{I_i A\cos\theta_i} = \frac{I_t \cos\theta_t}{I_i \cos\theta_i} = \frac{n_t \cos\theta_t}{n_i \cos\theta_i}\left(\frac{E_{ot}}{E_{oi}}\right)^2 = \frac{n_t\cos\theta_t}{n_i\cos\theta_i}\, t^2$$

($t_{\parallel}$ or $t_\perp$)

$$R + T = 1, \quad R_{\parallel} + T_{\parallel} = 1, \quad R_\perp + T_\perp = 1, \quad R = \tfrac{1}{2}(R_{\parallel} + R_\perp)$$

[$p \leftrightarrow \parallel$] [$s \leftrightarrow \perp$]

[See plots of r, t, R, T in Unit 2 Class slides]

Phase Shift at the Interface

the phase of a wave can change phase upon interacting with an interface

$$\Delta\phi_{ti} = \phi_t - \phi_i \qquad \Delta\phi_{ri} = \phi_r - \phi_i$$

these phase differences aren't necessarily zero

Normal Incidence

$$\Delta\phi_{ti} = 0 \qquad t = \frac{2n_i}{n_i + n_t} \text{ (always positive)}$$

$$\Delta\phi_{ri} = \begin{cases} 0 & \text{if } n_i > n_t \\ \pi & \text{if } n_i < n_t \end{cases} \qquad r = \frac{n_t - n_i}{n_t + n_i} \text{ (can change sign)}$$

Arbitrary Incidence

Two new issues appear: 1/ $\theta_i = \theta_{Brewster}$ 2/ $\theta_i > \theta_{crit}$

1) for S-polarized ($\perp$) light, phase flips 180° when θ_i goes through $\theta_{Brewster}$ (projection of dipole oscillation changes)

2) when $\theta_i > \theta_{crit}$, the phases of reflected and "transmitted" (actually evanescent) waves will vary with angle (θ_i)

Generally: $[0 < \Delta\phi_{ri} < \pi]$ & $[0 < \Delta\phi_{ti} < \pi/2]$

Total Internal Reflection (TIR)

Now, for the case where $n_i > n_t$, can see in plot of reflection amplitude coefficients that, as θ_i increases, there is a $\theta_{crit.}$ beyond which $r_{\parallel} = r_{\perp} = 1$.

What is this angle, θ_{crit}? (Ans: the θ_i which causes $\theta_t = 90°$)

Snell's Law: $n_i \sin\theta_i = n_t \sin\theta_t$

$$\Longrightarrow \quad \sin\theta_i = \frac{n_t}{n_i}\sin\theta_t = \frac{n_t}{n_i}\cancelto{1}{\sin \pi/2}$$

$$\Longrightarrow \quad \theta_{crit} = \theta_i\Big|_{\theta_t = \pi/2} = \sin^{-1}(n_t/n_i)$$

So, for $\theta_i \geq \theta_{crit}$, get perfect reflection ($I_r = I_i$ and $I_t = 0$)

Evanescent Waves

What does it mean to have no transmitted wave at interface? BC's still need to be met! ($E_{\parallel}^{(1)} = E_{\parallel}^{(2)}$ & $\epsilon_1 E^{(1)} = \epsilon_2 E^{(2)}$)

Note: The full treatment here is a bit involved, so we'll just take a peak!

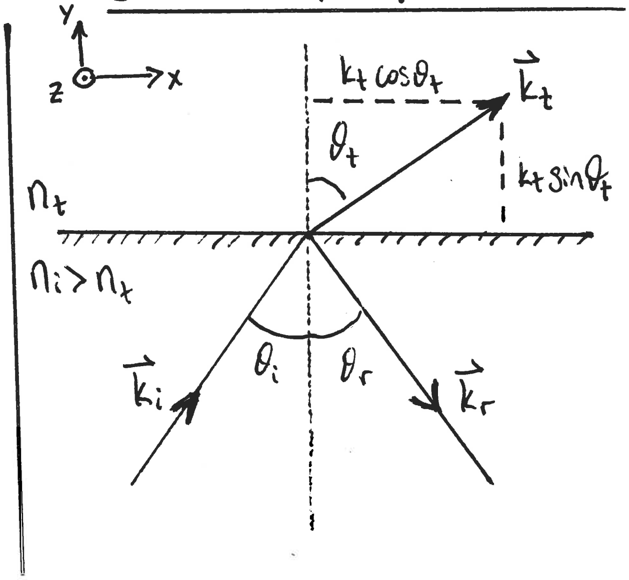
Assume a wavefunction for the transmitted wave

$$\vec{E}_t = \vec{E}_{0t}\, e^{i(\vec{k}_t \cdot \vec{r} - \omega t)}$$

where $\vec{k}_t \cdot r = k_{tx} x + k_{ty} y$

and $k_{tx} = k_t \sin\theta_t$

$k_{ty} = k_t \cos\theta_t$



use Snell's Law to rewrite in terms of θ_i $(n_i \sin\theta_i = n_t \sin\theta_t)$ [2-18]

$$k_{tx} = k_t \sin\theta_t = k_t \frac{n_i}{n_t} \sin\theta_i$$

$$k_{ty} = k_t \cos\theta_t = \pm k_t \sqrt{1-\sin^2\theta_t} = \pm k_t \sqrt{1-\left(\frac{n_i}{n_t}\right)^2 \sin^2\theta_i}$$

but at θ_{crit} have $\sin\theta_{crit} = n_t/n_i$, so when $\theta_i > \theta_{crit}$, then $\sin\theta_i > n_t/n_i$

$$\Longrightarrow k_{ty} = \pm i k_t \sqrt{\left(\frac{n_i}{n_t}\right)^2 \sin^2\theta_i - 1} \equiv \pm i\beta$$ (β: "attenuation coefficient")

Now $\vec{E}_t = \vec{E}_{ot}\, e^{\mp\beta y}\, e^{i(k_t x \frac{n_i}{n_t}\sin\theta_i - \omega t)}$ (Note: $+\beta$ solution is nonphysical)

And have a surface (evanescent) wave traveling on the interface (x-direction),
BUT decays exponentially in the y-direction across the interface

So, how far does the evanescent field penetrate into n_t?
Can use β as a metric since $k = 2\pi/\lambda$

[Note: Remember for free-space $v = c$ and $c = \nu\lambda$,
while for dielectrics $v = c/n$, so $\lambda = c/n\nu = \lambda_o/n$ (λ_o: vacuum wavelength)]

$$\Longrightarrow \beta = \frac{2\pi n_t}{\lambda_o}\sqrt{\left(\frac{n_i}{n_t}\right)^2 \sin^2\theta_i - 1}$$

Define a <u>penetration depth</u> to be where field drops off by 1/e (~65%)

$$\delta = 1/\beta$$

Optical Properties of Metals

What's the key difference between a conductor and an insulator?
Mobile charges! ($\vec{J}$) [Note: these are in addition to the bound charges]

Previously only considered bound charges in a polarizable medium, but now with free charges we can drive actual currents in the material

So, what changes? $\longrightarrow$ Maxwell's Eqn's!

i) $\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon}\rho_{free}$ iii) $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

ii) $\vec{\nabla}\cdot\vec{B} = 0$ iv) $\vec{\nabla}\times\vec{B} = \mu_0\epsilon\frac{\partial\vec{E}}{\partial t} + \mu_0\vec{J}_{free}$ $(\vec{J} = \sigma\vec{E})$

Have to now consider ρ (free charge density) and $\vec{J}$ (free current density)

First, use the continuity eqn. to make a sanity check

$$\frac{\partial\rho}{\partial t} = -\vec{\nabla}\cdot\vec{J} = -\frac{\sigma}{\epsilon}\rho \implies \text{sol'n of form } \rho(t) = \rho(0)e^{-(\sigma t/\epsilon)}$$

and see that initial density, $\rho(0)$, dissipates within time $\tau = \epsilon/\sigma$

So we'll limit ourselves to very good conductors: $\sigma\to\infty$ & $\tau\to 0$

then i) $\vec{\nabla}\cdot\vec{E} = 0$, and we only take care for terms with $\vec{J}$ (Ampere's Law)

Just as before, to get wave eqn's for $\vec{E}$ & $\vec{B}$: $\nabla\times(\nabla\times E)$ & $\nabla\times(\nabla\times B)$

$$\nabla^2\vec{E} = \mu_0\epsilon\frac{\partial^2\vec{E}}{\partial t^2} + \mu_0\sigma\frac{\partial\vec{E}}{\partial t} \quad \& \quad \nabla^2\vec{B} = \mu_0\epsilon\frac{\partial^2\vec{B}}{\partial t^2} + \mu_0\sigma\frac{\partial B}{\partial t}$$

where $\tilde{E}(z,t) = \tilde{E}_0 e^{i(\tilde{k}z-\omega t)}$ & $\tilde{B}(z,t) = \tilde{B}_0 e^{i(\tilde{k}z-\omega t)}$

Why write the wavefunctions in a fully complex form?

Let's check by plugging the wavefunctions into the wave equation

$$\Rightarrow -k^2 \cancel{\tilde{E}(z,t)} = -\mu_0 \epsilon \omega^2 \cancel{\tilde{E}(z,t)} - i\mu_0 \sigma \omega \cancel{\vec{E}(z,t)}$$

$$\Rightarrow k = \pm\sqrt{\mu_0 \epsilon \omega^2 + i\mu_0\sigma\omega} = \pm\omega\sqrt{\mu_0\epsilon}\sqrt{1 + i\sigma/\epsilon\omega}$$

So see that k is complex here and can separate into real/imag components

$$\tilde{k} = k + i\kappa \quad , \quad |\tilde{k}|^2 = k^2 + \kappa^2$$

where the expressions for k and κ get a bit messy

$$Re(\tilde{k}) = k = \omega\sqrt{\frac{\mu_0\epsilon}{2}}\left[\sqrt{\sigma^2/\epsilon\omega + 1} + 1\right]^{1/2}$$

$$Im(\tilde{k}) = \kappa = \omega\sqrt{\frac{\mu_0\epsilon}{2}}\left[\sqrt{\sigma^2/\epsilon\omega + 1} - 1\right]^{1/2}$$

then the wavefunctions become: (after choosing a polarization, applying BCs, $B_0 \to \frac{k}{\omega}E_0$)

$$\tilde{E}(z,t) = \tilde{E}_0 e^{-\kappa z} e^{i(kz - \omega t)}\hat{x} \quad , \quad \tilde{B}(z,t) = \frac{\tilde{k}}{\omega}\tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)}\hat{y}$$

A few important points here:

- Waves continue to propagate into the metal!
- BUT decay exponentially
  - the characteristic depth here, where amplitude decays by 1/e, is called the <u>skin depth</u> (or penetration depth): $\boxed{d = 1/\kappa}$
- Since $\tilde{k}$ is complex, also implies other quantites have real/imag component
  - $\tilde{k} = \frac{\tilde{n}\omega}{c} \longrightarrow \tilde{n} = n_R - i n_I$
  - $\tilde{n} = \sqrt{\tilde{\epsilon}/\epsilon_0} \longrightarrow \tilde{\epsilon} = \epsilon_R - i\epsilon_I$

Okay, now let's figure out the amplitudes

Start by representing $\tilde{k}$ as: $\tilde{k} = |\tilde{k}| e^{i\phi}$

where $|\tilde{k}| = \sqrt{k^2 + \kappa^2}$ & $\phi = \tan^{-1}(\kappa/k)$

and remember that the complex amplitudes $\tilde{E}_0$, $\tilde{B}_0$ can absorb phase offset

$$\tilde{E}_0 = E_0 e^{i\delta_E} \quad , \quad \tilde{B}_0 = B_0 e^{i\delta_B}$$

but $\tilde{B}_0$ also equals: $\tilde{B}_0 = \frac{\tilde{k}}{\omega}\tilde{E}_0 = \frac{|k|e^{i\phi}}{\omega} E_0 e^{i\delta_E}$

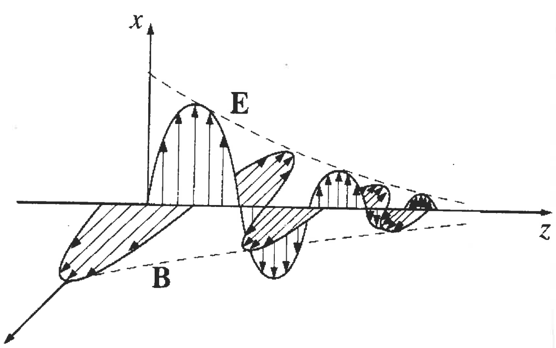
$$\Longrightarrow B_0 e^{i\delta_B} = \frac{|k|e^{i\phi}}{\omega} E_0 e^{i\delta_E} = \frac{|k|}{\omega} E_0 e^{i(\phi + \delta_E)}$$

$$\Longrightarrow \delta_B - \delta_E = \phi \quad (!)$$

Wow! So $\vec{E}$ and $\vec{B}$ are no longer in phase ($\vec{B}$ lags $\vec{E}$)

This was impossible for EM waves in free space and linear dielectrics.

Dispersion in Metals



Remember: defined metals as a collection of both free and bound charges

Have already seen derivation of $n(\omega)$ for bound charges as following from driven-damped oscillator

$$F_{tot} = F_{drive} - F_{damp} - F_{restore} \Longrightarrow \frac{d^2x}{dt^2} = \frac{qE_0}{m} e^{i\omega t} - \gamma_b \frac{dx}{dt} - \omega_0^2 x$$

where $x(t) = \dfrac{q^2/m}{(\omega_0^2 - \omega^2 + i\gamma_b\omega)} E(t)$

and since $P(t) = qNx(t)$ and $\epsilon = \epsilon_0 + \dfrac{\vec{P}}{\vec{E}} = \epsilon_0 + \dfrac{q^2 N/m}{(\omega_0^2 - \omega^2 + i\gamma_b\omega)}$

then $n^2_{bound} = \frac{\epsilon}{\epsilon_0} = 1 + \frac{Nq^2}{\epsilon_0 m}\left(\frac{1}{\omega_0^2 - \omega^2 + i\gamma_b\omega}\right)$

but the free e^- gas of a metal has no restoring force ($\omega_0 \to 0$)

$$\Rightarrow n^2_{free} = 1 + \frac{Nq^2}{\epsilon_0 m}\left(\frac{1}{-\omega^2 + i\gamma_e\omega}\right) = 1 - \frac{Nq^2}{\epsilon_0 m}\left(\frac{1}{\omega^2 - i\gamma_e\omega}\right)$$

two items of note here: 1) pick up a phase change (180° out of phase)

2) γ_e is damping within e^- gas (e^--e^- collisions, etc)

assume for a moment that EM freq. ω is large compared to collision rate ($\omega^2 \gg \gamma\omega$)

$$\Rightarrow n^2_{plasma} = 1 - \frac{Nq^2}{\epsilon_0 m\omega^2} = 1 - \omega_p^2/\omega^2$$

where $\omega_p = \sqrt{\frac{Nq^2}{\epsilon_0 m}}$ is called the <u>Plasma frequency</u>

can see for $\omega < \omega_p$: n is complex valued (material <u>reflects</u>)

$\omega > \omega_p$: n is real valued (material is <u>transparent</u>)

for metals, in general, though

$$\epsilon_{Tot} = \epsilon_{vac} + \epsilon_{bound} + \epsilon_{free} \qquad (\epsilon_{vac} = \epsilon_0)$$

$$\Rightarrow n^2_{metal} = \frac{\epsilon}{\epsilon_0} = 1 + \frac{Nq^2}{\epsilon_0 m}\left[\frac{1}{-\omega^2 + i\gamma_e\omega} + \frac{1}{\omega_0^2 - \omega^2 + i\gamma_b\omega}\right]$$

$$= 1 - \frac{\omega_p}{\omega^2 - i\gamma_e\omega} + \frac{\omega_p}{\omega_0^2 - \omega^2 + i\gamma_b\omega}$$