Fourier Optics

Diffraction Recap

- general integral: $\vec{E}(\vec{r}) = C \iint \vec{E}(x', y', 0)\, \vec{U}(k, \vec{R})\, dx'dy'$
- scalar approximation: $\vec{E} \longrightarrow E$
- Fresnel-Kirchoff: $E(r) = \frac{-i}{\lambda} \iint E(x', y', 0) \frac{e^{ikR}}{R} \left[\frac{1 + \cos(r, z)}{2}\right] dx'dy'$
- Fresnel: 1) θ' is small 2) $z^2 \gg (x-x')^2 + (y-y')^2$

$$\Rightarrow R = z\sqrt{1 + \frac{(x-x')^2 + (y-y')^2}{z^2}} \simeq z\left(1 + \frac{(x-x')^2 + (y-y')^2}{2z^2}\right)$$

$$\Rightarrow \frac{1}{R} \simeq \frac{1}{z} \quad \& \quad e^{ikR} \simeq e^{ikz} e^{ik\frac{(x-x')^2 + (y-y')^2}{2z}}$$

$$\Rightarrow E(x,y,z) = \frac{-i}{\lambda z} e^{ikz} e^{i\frac{k}{2z}(x^2+y^2)} \iint E(x', y') e^{i\frac{k}{2z}(x'^2 + y'^2)} e^{-i\frac{k}{z}(xx' + yy')} dx'dy'$$

- Fraunhofer: $z = \frac{a^2}{\lambda}$ (far-field)

$$\Rightarrow E(x,y,z) = \phi(x,y,z) \iint E(x', y') e^{-i\frac{k}{z}(xx' + yy')} dx'dy'$$

Array Theorem: a very nice way to calculate the diffraction pattern caused by N identical apertures with $E_{aperture}(x', y', 0)$

- assume each aperture has a center-of-mass located at (x_n', y_n')

$$E(x,y,z) = \frac{-i}{\lambda z} e^{ikz} e^{i\frac{k}{2z}(x^2+y^2)} \sum_{n=1}^{N} \iint E_{ap}(x'-x_n', y'-y_n') \cdot e^{i\frac{k}{z}(xx'-yy')} dx' dy'$$

- and then make a change of variables

$$\longrightarrow \quad x'' = x' - x_n' \quad \& \quad y'' = y' - y_n'$$

$$E(x,y,z) = \frac{-i}{\lambda z} e^{ikz} e^{i\frac{k}{2z}(x^2+y^2)} \sum_{n=1}^{N} \iint E_{ap}(x'', y'') e^{-i\frac{k}{z}[x(x''+x_n')+y(y''+y_n')]} dx'' dy''$$

- now parts involving x_n & y_n can be pulled out

$$E(x,y,z) = \left[\sum_{n=1}^{N} e^{-i\frac{k}{z}(xx_n'+yy_n')}\right]\left[\frac{-i}{\lambda z} e^{ikz} e^{i\frac{k}{2z}(x^2+y^2)} \iint E_{ap}(x'', y'') e^{i\frac{k}{z}(xx''+yy'')} dx'' dy''\right]$$

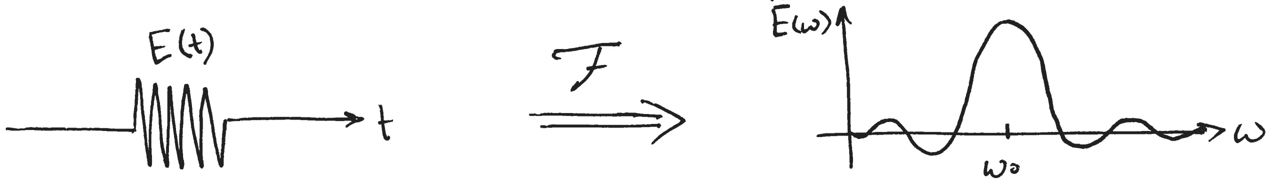
$$= \left[\begin{matrix}\text{array} \\ \text{interference term}\end{matrix}\right]\left[\text{Single aperture diffraction}\right]$$

$$\sim \left[\frac{\sin N\alpha}{\sin\alpha}\right]\left[\frac{\sin\beta}{\beta}\right] \qquad \text{(mutiple slit interference)}$$

Fourier Optics

- Fourier Analysis gives a powerful way of describing optical fields passing through a system
  - describes optical field in real space as fn. of position, $f(x,y)$
  - OR describe the field in terms of spacial frequency, which is related to the so-called angular spectrum of the field

- We've already used F.T.'s in the context of time-frequency

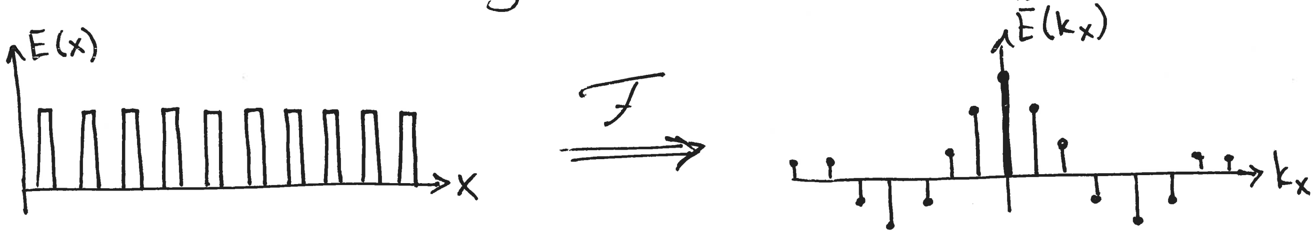


$$\mathcal{F}[E(t)] = \tilde{E}(\omega)$$

- $\tilde{E}(\omega)$ and $E(t)$ <u>both</u> contain the <u>same information</u> about the field, just written in terms of different (conjugate) variables

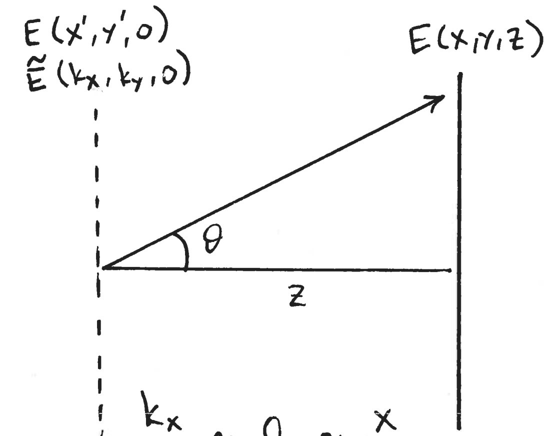
- Now, consider signals in space, not time.

Imagine a series of slits along $\hat{x}$, illuminated by a plane wave



- What does the spatial freq tell us, practically?

<u>A:</u> How much light is propagating in a certain direction after the aperture/grating.



- So, this gives exactly what the intensity pattern at the screen will be for the far-field

$$\frac{k_x}{k} \approx \theta_x \approx \frac{x}{z}$$

$$\frac{k_y}{k} \approx \theta_y \approx \frac{y}{z}$$

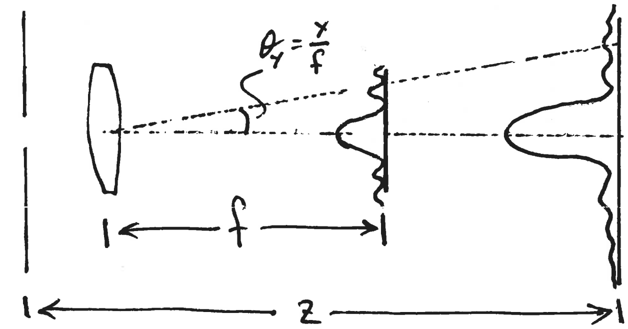
- The far-field E-field is the 2D F.T. of the E-field at the aperture

$$E(x,y,z) = \frac{-i}{\lambda z} e^{ikz} e^{i\frac{k}{2z}(x^2+y^2)} \iint_{\text{aperture}} E(x',y',0)\, e^{-i\frac{k}{z}(xx'+yy')}\, dx'dy'$$

- where the spatial freq. components corresponding to a given k_x represent plane waves propagating at an angle
  $\longrightarrow \theta_x = \frac{k_x}{k}$ (for distance far from slit, $z > a^2/\lambda$)

Ex:
$a = 2mm$
$\lambda = 500nm$
$z = 8m$

- however, place a lens immediately after the aperture and get image at infinity brought to focal plane of lens



- Why is this a useful thing?
  - F.T. is comparatively simple
  - F.T. is linear, so can build up complicated functions from simple ones

- rather than writing out the full integral expression for F.T. each time, is more convenient to write as operators: $\mathcal{F}[\ldots]$ & $\mathcal{F}^{-1}[\ldots]$
  (fourier transform) (its inverse)

$$\Longrightarrow f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x')\, e^{-ikx'}\,dx'\right]dk$$

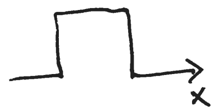
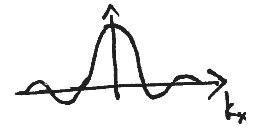
$$= \mathcal{F}^{-1}\left[\mathcal{F}\left[f(x)\right]\right]$$

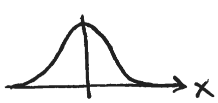
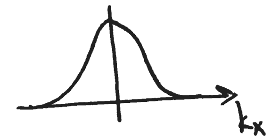
$$= \mathcal{F}^{-1}\left[F(k)\right]$$

$$= f(x)$$

- and this is linear: $\mathcal{F}[f(x) + g(x)] = \mathcal{F}[f(x)] + \mathcal{F}[g(x)] = F(k_x) + G(k_x)$

Examples:

- $f(x) = \mathrm{rect}(x)$  | $\mathcal{F}[\mathrm{rect}(x)] = \mathrm{sinc}\left(\frac{k_x}{2}\right) = F(k_x)$ 

- $f(x) = e^{-x^2/2}$  | $\mathcal{F}[e^{-x^2/2}] = e^{-k_x^2/2} = F(k_x)$ 

- What about the limiting cases?

  - no aperture: $\mathcal{F}[C] = \frac{C}{2\pi}\delta(k_x - k_0)$

  - infinitesimal slit: $\mathcal{F}\left[\frac{C}{2\pi}\delta(x)\right] = C$

the delta function:

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

$$\delta(x - x_0) = \begin{cases} 0 & x \neq x_0 \\ \infty & x = x_0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

- the delta function here is kind of special and very useful as a <u>sifting function</u>

$$f(x) = \int_{-\infty}^{\infty} f(x')\,\delta(x - x')\,dx'$$

- can actually derive the integral definition of $\delta(x)$ from the fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x')\,e^{-ikx'}\,dx'\right]dx$$

$$= \int_{-\infty}^{\infty} f(x')\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ik(x-x')}\,dk\right]dx'$$

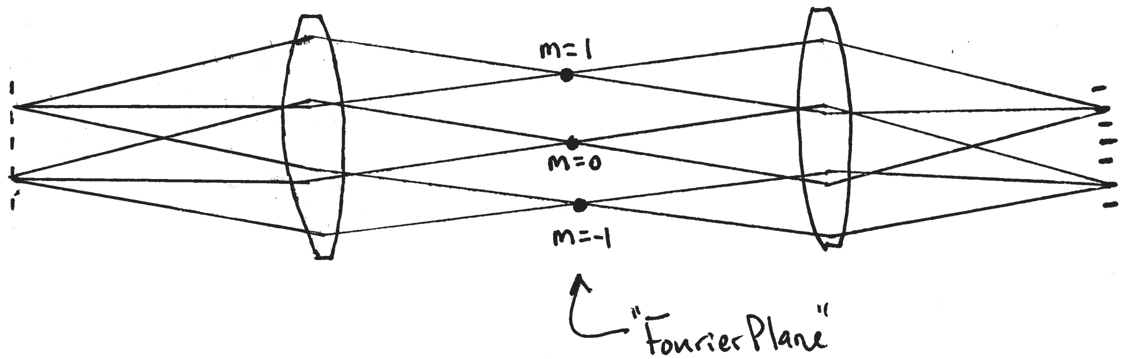
rearrange the order of integration

$$\Longrightarrow \delta(x - x') = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ik(x-x')}\,dk$$

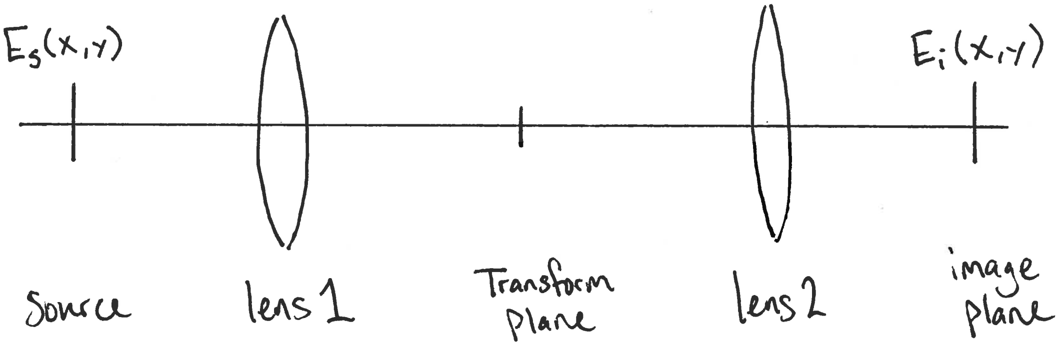
which is a very interesting result – the only way the δ-function can be infinitely sharp is to be a uniform superposition of <u>all</u> <u>freq.</u> components!

Spatial Filtering

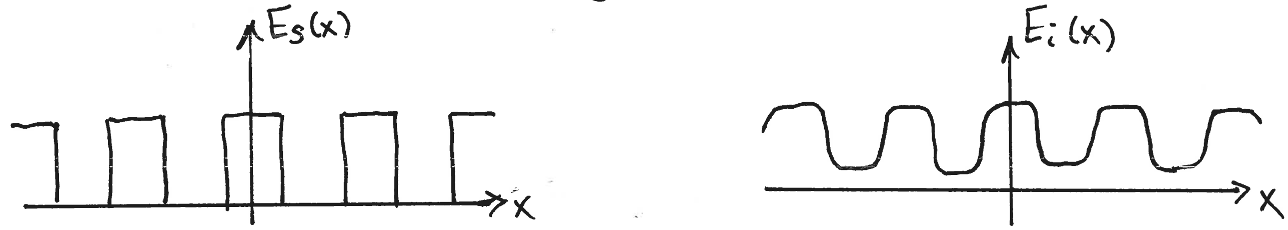
- We've discussed the idea of spatial frequency content within an image, now let's consider this concept in real applications
- Ex: Diffraction by a grating, refocused by a lens



- What happens when the lens is too small?
  - get fewer orders at the Fourier plane
- What happens if the grating period is increased?
  - spots move further apart (so would need larger diameter lens)
- What happens when the first lens is placed closer and closer to the grating?
  - still get diffraction orders at a distance f behind the lens, but the overall pattern is <u>diverging</u>
  - so if we want to re-image the diffraction pattern with another lens, it's most convenient to place the first lens f away from grating ⟶ leads to so-called "4-f" imaging setup



- if the lenses of the 4-f setup are finite diameter, the higher diffraction orders are lost → the pattern at the image plane is not as sharp as the original (loss of contrast)



so, the more diffraction orders retained, the sharper edges will be

- a similar (but not identical) effect occurs if we obstruct the beam in the Fourier plane ⟶ closing down an iris in that plane cuts off information about higher spatial frequencies [essentially forming a low-pass filter]

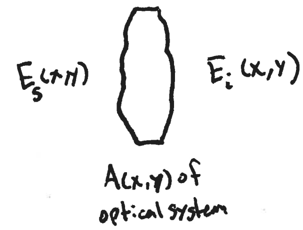
- Ex: have a laser with a "messy" non-uniform intensity profile across its beam cross-section. Can "clean" the beam by focusing it through a pinhole (of carefully chosen diameter) to reject high spatial freq's

- this is all very qualitative so far, need to formalize these ideas

Optical Transfer Functions (general)

- what we want is a formalism which specifies how different spatial frequencies will be handled by an optical system

- imagine a plane wave incident on some real, physical, non-ideal optical element (eg. a lens with defects & smudges)
  - → results in a modified amplitude and phase
  - → can write this as a transfer function

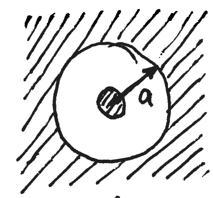


$$A(x,y) = \underbrace{A_o(x,y)}_{\text{amplitude}}\,\underbrace{e^{i\phi(x,y)}}_{\text{phase}}$$

Aperture Function or **Pupil Function**

$$\Rightarrow E_A = E_o\, A_o(x,y)\, e^{i\phi(x,y)}$$

- Why does this matter? Many optical instruments have a non-trivial aperture function (Ex: Cassegrain telescope)



$$A(\rho) = \begin{cases} 1 & \epsilon a < \rho < a \\ 0 & \text{elsewhere} \end{cases}$$

Point Spread Function:

- a point at the source plane will always result in a characteristic diffraction pattern at the imaging plane

- we've seen that diffraction from some aperture is related by a Fourier Transform as the transfer function, which here we call the "Point Spread Function" of diffracted intensities

$$S(x,y) = \left| \mathcal{F}\left[A_o(x,y)\, e^{i\phi(x,y)}\right]\right|^2$$

- we'll only consider <u>real-valued</u> aperture functions, for simplicity

$$S(x,y) = \mathcal{F}\left[A_o(x,y)\right]^2$$

- We've solved for diffraction of fairly simple apertures (circular, square), but what about more complicated (i.e. realistic) ones like the annular aperture of a reflecting telescope → Babinet's Principle!

Ex: simple circular aperture:

$$\mathcal{F}\left[A_{\epsilon=0}\right] = 2\pi a^2 \frac{J_1(ap)}{ap}$$

annular aperture:

$$\mathcal{F}\left[A_\epsilon\right] = 2\pi a^2\left(\frac{J_1(ap)}{ap}\right) - 2\pi\epsilon^2 a^2\left(\frac{J_1(ap)}{\epsilon ap}\right)$$

[using Babinet's principle]

$$= 2\pi a^2\left[\frac{J_1(ap)}{ap} - \epsilon^2\frac{J_1(\epsilon ap)}{\epsilon ap}\right]$$

then the point spread function is the modulus squared

$$S(x,y) = \mathcal{F}\left[A_\epsilon\right]^2 = (2\pi a^2)^2\left[\left(\frac{J_1(ap)}{ap}\right) - \epsilon^2\left(\frac{J_1(\epsilon ap)}{\epsilon ap}\right)\right]^2$$

- <u>Note</u>: you can't do this with the intensities, just the fields. the cross terms are important

Convolution Function:

- okay, so how to treat more complicated images?
  - $\longrightarrow$ simply add up a collection of points describing the image

- the image of an extended, non-point object is the convolution of the ideal image with the point-spread function

$$I(x,y) = \iint I_0(x',y')\, S(x-x', y-y')\, dx'\, dy'$$

$$= \underbrace{I_0(x',y')}_{\text{object irradiance}} \underset{\uparrow \text{ convolution}}{\circledast} \underbrace{S(x-x', y-y')}_{\text{point-spread function}}$$

- So, the main idea of convolution is that it's a special "product" of functions:

$$g(x) = \int_{-\infty}^{\infty} f(x')\, h(x-x')\, dx'$$

- important property of convolution is that it commutes

$$g(x) = \int_{-\infty}^{\infty} f(x')\, h(x-x')\, dx' = \int_{-\infty}^{\infty} f(x-x')\, h(x')\, dx'$$

change of variable $x \to x'+x$

$$= f \circledast h = h \circledast f$$

- another important property is that convolution in spatial representation is the same as simple multiplication in freq. representation

Convolution Theorem:

$$\mathcal{F}[f \circledast h] = \mathcal{F}[f] \cdot \mathcal{F}[h]$$

Proof:

$$\tilde{g}(k) = \mathcal{F}[f \circledast h] = \mathcal{F}\left[\int_{-\infty}^{\infty} f(x')\,h(x-x')\,dx'\right]$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x')\,h(x-x')\,dx'\right] e^{ikx}\,dx$$

Since the limits are at infinity, makes no difference whether we integrate over x or x' first

$$\Rightarrow \tilde{g}(x) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(x-x')\,e^{ikx}\,dx\right] f(x')\,dx'$$

then introduce the multiplicative factor $e^{-ikx'}\,e^{ikx'} = 1$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(x-x')\,e^{ikx}\,e^{-ikx'}\,dx\right] f(x')\,e^{ikx'}\,dx'$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(x-x')\,e^{ik(x-x')}\,dx\right] f(x')\,e^{ikx'}\,dx'$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(\chi)\,e^{ik\chi}\,d\chi\right] f(x')\,e^{ikx'}\,dx'$$, with a variable subs $\chi = x - x'$ & $d\chi = dx$

integrals are now independent

$$= \int_{-\infty}^{\infty} f(x')\,e^{ikx'}\,dx' \cdot \int_{-\infty}^{\infty} h(\chi)\,e^{ik\chi}\,d\chi$$

$$= \mathcal{F}[f(x)] \cdot \mathcal{F}[h(x)] = \underline{\underline{\tilde{f}(k) \cdot \tilde{h}(k)}}$$

Cross-Correlation & Autocorrelation

- the idea of convolution can be extended

$$g(x) = \int_{-\infty}^{\infty} f(x')\, h(x-x')\, dx' = f(x) \circledast h(x) \qquad [\text{Convolution}]$$

- a related integral transform operation is called <u>Cross-Correlation</u>

$$c(x) = \int_{-\infty}^{\infty} f^*(x')\, h(x+x')\, dx' = f(x) \odot h(x)$$

- the cross-correlation is a way of comparing two (possibly complex-valued) functions to see how similar they are to one another
  - also known as a "sliding dot product" since it measures simularity as a function of relative displacement between $f(x)$ & $h(x)$

- some important properties of cross-correlation
  - $f \odot h = f^*(-x) \circledast h$
  - if $f^*(x) = f(-x)$ (i.e. f is Hermetian), then $f \odot h = f \circledast h$
  - if both f and h are Hermetian, then $f \odot h = h \odot f$
  - in general, though: $(f \odot h) \odot (f \odot h) = (f \odot f) \odot (h \odot h)$ (cross-correl. w/itself?)

    $g \odot (f \circledast h) = (g \odot f) \circledast h$
  - and similar to the convolution theorem: $\mathcal{F}[f \odot h] = \mathcal{F}[f]^* \cdot \mathcal{F}[h]$

- there is also the notion of taking the cross-correlation [8-13] of a function with itself $\longrightarrow$ called <u>autocorrelation</u>

$$a(x) = \int_{-\infty}^{\infty} f^*(x')\, f(x+x') = f(x) \odot f(x)$$

- why would you do this? it's really useful for finding periodic, repeating patterns in a complicated signal / wavefunction

Back to Optical Transfer Functions

- okay, so we saw that the image formed by an (imperfect) imaging system with a given point-spread function, $S(x,y)$, is

$$I(x,y) = I_0(x,y) \circledast S(x,y) \qquad \text{[Convolution]}$$

- by using the convolution theorem here, we have

$$\mathcal{F}[I(x,y)] = \mathcal{F}[I_0(x,y)] \cdot \mathcal{F}[S(x,y)]$$

$$\Longrightarrow \tilde{I}(k_x,k_y) = \tilde{I}_0(k_x,k_y) \cdot T(k_x,k_y)$$

the part that's actually called <u>Optical Transfer Function</u>

$$\begin{bmatrix}\text{spatial freq.}\\ \text{image spectrum}\end{bmatrix} = \begin{bmatrix}\text{spatial freq.}\\ \text{object spectrum}\end{bmatrix} \cdot \begin{bmatrix}\text{spatial freq.}\\ \text{transfer function}\end{bmatrix}$$

<u>Optical Transfer Fn:</u> $T(k_x,k_y) = \mathcal{F}[S(x,y)] = \underbrace{M(k_x,k_y)}_{\text{Modulation Transfer Fn.}}\, \underbrace{e^{i\Phi(k_x,k_y)}}_{\text{Phase Transfer Function}}$

- So, what does the OTF, T, tell us about the system?

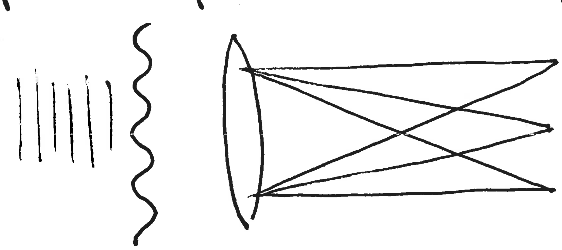
  → if the source contains information at some spatial freq (k_x, k_y), then $T(k_x, k_y)$ gives how much of that information is <u>transferred</u> from the source to the image

- again, we will simplify by not concerning ourselves with the phase transfer function (commonly called just 'PTF')

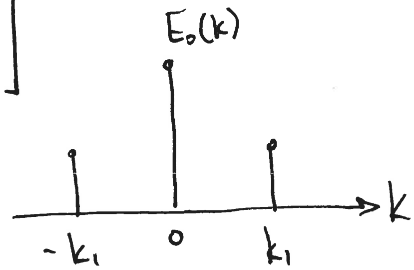
- However, the MTF (modulation transfer function) is very important, as it defines the loss in contrast from object to image vs. frequency

  → $M \equiv$ ratio of image modulation to object modulation at all spatial frequencies

- a simple example to illustrate this: [sinusoidal amplitude transmission mask]

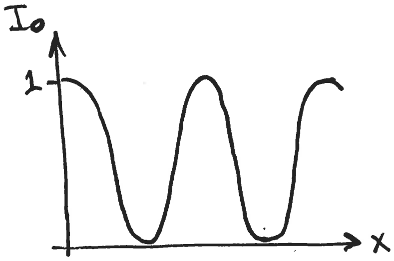


$\Longrightarrow$



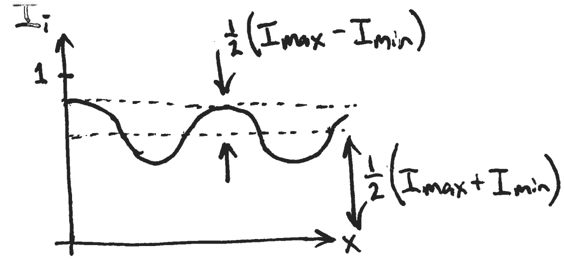
- for a <u>finite</u> aperture lens, the larger k_1 is, the less light will go through

- Since it's the interference of light at $\pm k_1$ that gives the modulation:



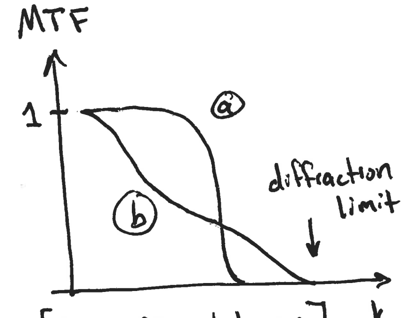
[Object Space]

$I_o \propto 1 + \cos(k_1 x)$



[Image Space]

$I_i \propto 1 + \epsilon \cos(k_1 x)$



[two different lenses]

- Modulation transfer function normally just called "modulation" or "contrast"

$$M = \frac{I_{max} - I_{min}}{I_{max} + I_{min}}$$

- Note: if there was a phase offset between I_o & I_i, then would be due to effects from Phase Transfer Function

- Contrast always falls of at higher spatial frequencies $\rightarrow$ Diffraction Limit!

- as indicated above, MTF is related to PSF (and, hence, aperture)

$$M(k_x, k_y) = \mathcal{F}[S(x,y)]$$

$$= \mathcal{F}\left[\,\left|\mathcal{F}[A(x,y)]\right|^2\,\right]$$

$$= A(x,y) \odot A(x,y)$$

[autocorrelation of the aperture fn. ... obviously.]

- okay, I'm glossing over details like coherent vs. incoherent illumination. The proof of this last step requires wrangling of integrals, variable substitutions, and an interesting observation that $\mathcal{F}[\mathcal{F}[f(x,y)]] = f(-x,-y)$, which is HW 8-A1. Also note that $A(x,y) \odot A(x,y) = A(x,y) \circledast A(x,y)$, iff $A(x,y)$ is even/Hermetian

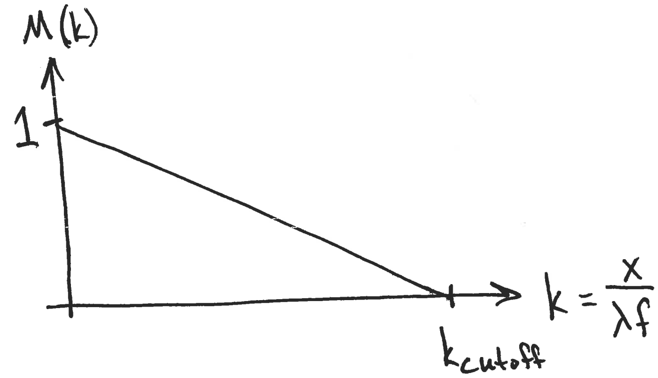
- each of these transfer functions is usually normalized (aperture, PSF, MTF, etc.)

$$S_n(x,y) = \frac{S(x,y)}{\iint_{-\infty}^{\infty} S(x,y)\,dx\,dy} \qquad [\text{Area} = 1]$$

- this way, $M(k_x=0, k_y=0) = 1$

- two common apertures:

1) Square aperture



2) Circular aperture