Superposition of Waves

- A very simple concept (but extremely powerful) → [Leads to: - polarization - interference - diffraction]

- Remember the wave equation? $\nabla^2 \psi = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$
  - if $\psi_1(r,t)$ is a solution, and $\psi_2(r,t)$ is a solution, then $\psi_1 + \psi_2$ is also a solution

- Example: Plane Waves $\vec{E}(\vec{r},t) = \vec{E}_0 \sin(kx - \omega t + \phi)$ $\left[\frac{\omega}{k} = \frac{c}{n}\right]$
  - any linear combination of these is still a solution to wave eqn.

$$\Longrightarrow \sum_{i=1}^{N} E_{0i} \sin(k_i x - \omega_i t + \phi) \qquad \left[\frac{\omega_i}{k_i} = \frac{c}{n_i}\right]$$

- Let's consider a few cases to see some of the consequences

Case 1: two waves where $\omega_1 = \omega_2$, but $k_i x + \phi_i \equiv \alpha_i$ may differ

Algebraic Method (Hecht 7.1)

$$E \sin(\omega t + \alpha) = E_1 \sin(\omega t + \alpha_1) + E \sin(\omega t + \alpha_2)$$

apply the sum-angle trig identity:

$$= E_1(\sin\omega t \cos\alpha_1 + \cos\omega t \sin\alpha_1) + E_2(\sin\omega t \cos\alpha_2 + \cos\omega t \sin\alpha_2)$$

$$= (E_1\cos\alpha_1 + E_2\cos\alpha_2)\sin\omega t + (E_1\sin\alpha_1 + E_2\sin\alpha_2)\cos\omega t$$

Complex/Phasor Method (Hecht 7.1.2)

$$E e^{i(\omega t - \alpha)} = E_1 e^{i(\omega t - \alpha_1)} + E_2 e^{i(\omega t - \alpha_2)}$$

* want amplitude E → square both sides

$$E^2 = \left(E_1 e^{i(\omega t - \alpha_1)} + E_2 e^{i(\omega t - \alpha_2)}\right) \times \left(E_1 e^{-i(\omega t - \alpha_1)} + E_2 e^{-i(\omega t - \alpha_2)}\right)$$

$$= E_1^2 + E_1E_2 e^{i(\alpha_1 - \alpha_2)} + E_1E_2 e^{i(\alpha_2 - \alpha_1)} + E_2^2$$

$$= E_1^2 + E_2^2 + E_1E_2\left(e^{i(\alpha_1-\alpha_2)} + e^{-i(\alpha_1-\alpha_2)}\right)$$

$$= E_1^2 + E_2^2 + 2E_1E_2\cos(\alpha_1 - \alpha_2) \checkmark$$

where the parenthetical quantities are constant in time $\longrightarrow$ these are the parts we want to find amplitude E_0

Just call each $(\cdots) = E_0 \cos\alpha,\ E_0 \sin\alpha$, respectively, then square and add them

$$E_0^2 \cos^2\alpha = E_1^2 \cos^2\alpha_1 + 2E_1E_2 \cos\alpha_1 \cos\alpha_2 + E_2^2 \cos^2\alpha_2$$

$$E_0^2 \sin^2\alpha = E_1^2 \sin^2\alpha_2 + 2E_1E_2 \sin\alpha_1 \sin\alpha_2 + E_2^2 \sin^2\alpha_2$$

$$\Rightarrow E_0^2 = E_0^2 \cos^2\alpha + E_0^2 \sin^2\alpha$$

$$\Rightarrow \boxed{E_0^2 = E_1^2 + 2E_1E_2 \cos(\alpha_2 - \alpha_1) + E_2^2}$$

Now have amplitude of the field, What about α?

$$\Rightarrow \boxed{\tan\alpha = \frac{E_0 \sin\alpha}{E_0 \cos\alpha} = \frac{E_1 \sin\alpha_1 + E_2 \sin\alpha_2}{E_1 \cos\alpha_1 + E_2 \cos\alpha_2}}$$

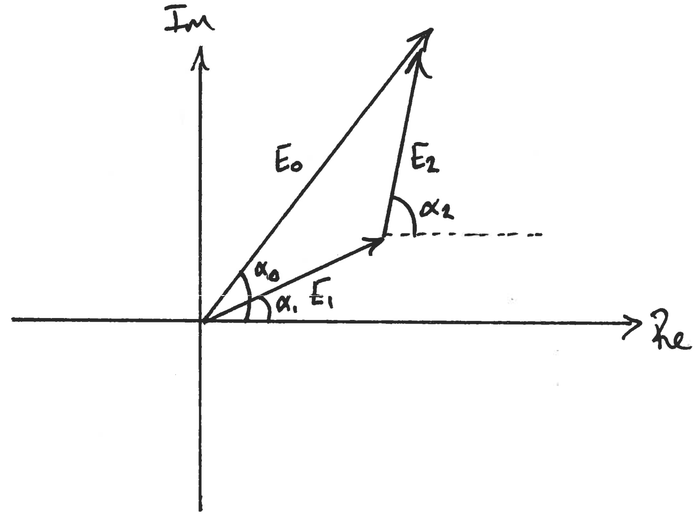
So, finally get

$$E = E_0 \cos\alpha \sin\omega t + E_0 \sin\alpha \cos\omega t$$

$\boxed{E = E_0 \sin(\omega t + \alpha)}$, where we can use these expressions to find E_0 & α

Complex can be much easier!

Using Phasors is quite handy, too:

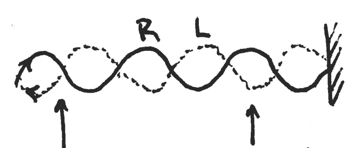


Case 2: two waves where $\omega_1 = -\omega_2$, but $k_1 = k_2$ and phases independant

- this scenario can be used to describe standing waves from reflecting surface

$$E(x,t) = E_L(x,t) + E_R(x,t) \quad \begin{cases} E_L = E_{0L} \sin(kx + \omega t + \phi_L) \\ E_R = E_{0R} \sin(kx - \omega t + \phi_R) \end{cases}$$

if $\delta = \phi_L + \phi_R = \frac{\pi}{2}$, then have situation like:



$$\text{then } E(x,t) = E_{0R}\big(\sin(kx + \omega t) + \sin(kx - \omega t)\big)$$
$$= 2E_{0R} \sin kx \cos\omega t$$

by $\sin\alpha + \sin\beta = 2\sin\frac{1}{2}(\alpha+\beta) \times \cos\frac{1}{2}(\alpha-\beta)$

Case 3: two waves where $\omega_1 \neq \omega_2$ & $k_1 \neq k_2$

Here $E(x,t) = \underbrace{E_0 \cos(k_1 x - \omega_1 t)}_{E_1} + \underbrace{E_0 \cos(k_2 x - \omega_2 t)}_{E_2}$

$$= 2E_0 \cos\tfrac{1}{2}\left[(k_1+k_2)x - (\omega_1+\omega_2)t\right] \times \cos\tfrac{1}{2}\left[(k_1-k_2)x - (\omega_1-\omega_2)t\right]$$

(using trig identity: $\cos\alpha + \cos\beta = 2\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$)

And now see these average & difference frequencies & wavenumbers

avg. angular freq.: $\bar{\omega} = \frac{1}{2}(\omega_1 + \omega_2)$

avg. wavenumber: $\bar{k} = \frac{1}{2}(k_1 + k_2)$

modulation freq.: $\Delta\omega = \frac{1}{2}(\omega_1 - \omega_2)$

modulation wavenumber: $\Delta k = \frac{1}{2}(k_1 - k_2)$

given these definitions can always find:

$\omega_1 = \bar{\omega} + \Delta\omega$

$\omega_2 = \bar{\omega} - \Delta\omega$

$k_1 = \bar{k} + \Delta k$

$k_2 = \bar{k} - \Delta k$

Now $E = 2E_0 \underbrace{\cos(\Delta k x - \Delta\omega t)}_{\text{slow "beat" or "envelope" function}} \underbrace{\cos(\bar{k}x - \bar{\omega}t)}_{\text{fast "carrier" wave}}$

[See Slide, or Fig. 7.11e Hecht]

- So far, no big surprises.
- However, in general (ie: in a dielectric), waves with different frequencies travel at different velocities.
- May even already suspect that the carrier & envelopes have their own characteristic velocities

Velocity of single monochromatic wave : $\left[\psi_1 \propto \cos(k_1 x - \omega_1 t)\right]$

- have already seen this (phase velocity)

$$\left(\frac{\partial x}{\partial t}\right) = \frac{-(\partial\psi_1/\partial t)}{(\partial\psi_1/\partial x)} = \frac{\omega_1}{k_1} = v_p$$

Velocity of the "carrier" : $\left[\bar{\psi} \propto \cos(\bar{k}x - \bar{\omega}t)\right]$

$$\bar{v} = \frac{-(\partial\bar{\psi}/\partial t)}{(\partial\bar{\psi}/\partial x)} = \frac{\bar{\omega}}{\bar{k}} = \frac{\omega_1 + \omega_2}{k_1 + k_2}$$

Velocity of the "envelope": $\left[\Delta\psi \propto \cos(\Delta k x - \Delta\omega t)\right]$

$$v_g = \frac{-(\partial\Delta\psi/\partial t)}{(\partial\Delta\psi/\partial x)} = \frac{\Delta\omega}{\Delta k} = \frac{\omega_1 - \omega_2}{k_1 - k_2}$$

- for EM waves in vacuum, everything travels at c

$$\frac{\omega_1}{k_1} = \frac{\omega_2}{k_2} = c = \bar{v} = v_g$$

- in general, though, refractive index varies with frequency

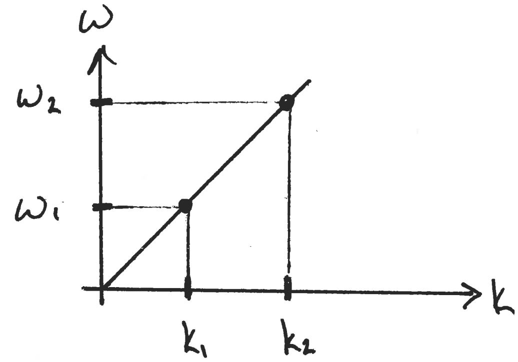
$$v_1 = \frac{\omega_1}{k_1} = \frac{c}{n_1} \neq \frac{c}{n_2} = \frac{\omega_2}{k_2} = v_2$$

- ultimately, we'll take this to the infinitesimal limit and see that $v_g = \left(\frac{d\omega}{dk}\right)_{\bar{\omega}}$ ← group velocity of a wave

Dispersion Relations

- In physics, we'll often encounter relations between energy and momentum – so-called dispersion relations.
- These are essentially a relation between frequency and wavelength of a particle/field/whatever (In quantum have: $E = \hbar\omega$ & $p = \hbar k$)

Ex: Linear Dispersion (photons in vacuum)

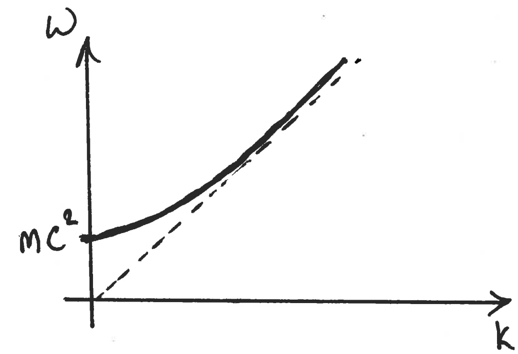


$$\omega(k) = kc$$

- Could even define indices of refraction for phase & group velocity components

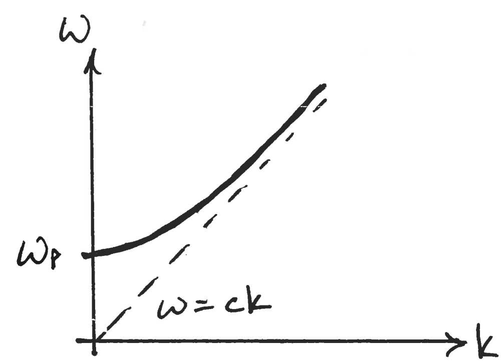
$$n = \frac{c}{v} \Longrightarrow n_\phi = c/(\omega/k) \quad \& \quad n_g = c/\left(\frac{\partial\omega}{\partial k}\right) \longrightarrow \text{both are 1 in vac.}$$

Ex: Massive Free Particle (ex: electron)



$$\omega(k) = \frac{1}{\hbar}\left(m^2c^4 + \frac{\hbar^2k^2}{2m}\right)^{\frac{1}{2}}$$

Ex: Dispersion of EM waves in a plasma



- neglecting Ohmic losses (damping)

$$\omega(k) = (\omega_p^2 + c^2k^2)$$

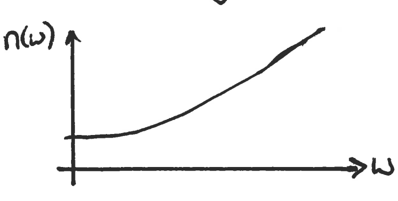
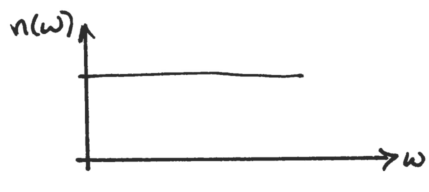
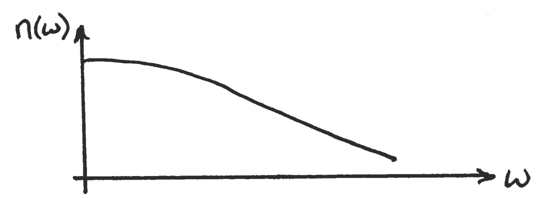
remember: $\omega_p = \sqrt{\frac{Nq^2}{\epsilon_0 m}}$

Group Velocity

- ω is always the same regardless of the material
  - $\rightarrow$ λ (and therefore $k = \frac{2\pi}{\lambda}$) and velocity change, though
- $k(\lambda)$ depends on the refractive index, n, which depends on ω
  - $\rightarrow$ $v = \frac{\omega}{k} = \frac{c}{n} \implies k = \frac{\omega}{c} n(\omega)$
- above, we defined the group velocity as $v_g = \frac{\partial \omega}{\partial k}$
  - since we usually have $n(\omega)$, this isn't always the most useful form
- More useful: $\frac{\partial k}{\partial \omega} = \frac{\partial}{\partial \omega}\left(\frac{\omega}{c} n(\omega)\right) = \frac{1}{c}\left(n(\omega) + \omega \frac{\partial n(\omega)}{\partial \omega}\right)$

$$\implies \boxed{V_g = \frac{c}{n + \omega \frac{\partial n}{\partial \omega}} = \frac{c/n}{1 + \frac{\omega}{n}\frac{\partial n}{\partial \omega}} = \frac{V_{phase}}{1 + \frac{\omega}{n}\frac{\partial n}{\partial \omega}}}$$

- in general, $V_g \neq V_{phase}$ (and can be <u>very</u> different)
- kind of saw this coming - infinitesimal limit of $V_g = \frac{\Delta \omega}{\Delta k}$, where $\Delta \omega = \frac{\omega_1 - \omega_2}{2}$
  and $\Delta k = \frac{k_1 - k_2}{2} = \frac{1}{2}\left(\frac{\omega_1 n(\omega_1)}{c} - \frac{\omega_2 n(\omega_2)}{c}\right.$
- 3 cases to consider:

"Normal" Dispersion	Zero Dispersion	"Anomalous" Dispersion
$\frac{\partial n}{\partial \omega} > 0$ & $V_g < V_{phase}$	$\frac{\partial n}{\partial \omega} = 0$ & $V_g = V_{phase}$	$\frac{\partial n}{\partial \omega} < 0$ & $V_g > V_{phase}$
		
envelope velocity < carrier vel.	envelope vel. = carrier vel.	envelope vel. > carrier vel. (!)

Fourier Transform Methods

- the Fourier transform is a way of rewriting a (potentially) complicated function in a (potentially) more manageable or useful form

- the core concept here: <u>conjugate variables</u>

$$\begin{bmatrix} f(t) & \longleftrightarrow & f(\omega) \\ \text{(time)} & & \text{(frequency)} \end{bmatrix} \quad \begin{bmatrix} f(x) & \longleftrightarrow & f(k) \\ \text{(space)} & & \text{("spatial freq.")} \end{bmatrix}$$

- for now, we'll stick to time/freq. as the transform variables and limit dimensions to 1D, but we'll expand this to 2D space/spatial freq.

- techniques/expressions for converting between representations varies a bit in their details depending on things like:
  - is the function periodic?
  - is it real or complex valued?

- we'll cover these different cases, but is essentially all the same trick!

- What's the trick?
  - Every well behaved function can be written as a sum of harmonic fn's
  - "well behaved" here is itself a topic of study within Fourier Analysis (we'll stay within very comfortable bounds for our needs, though)

- let's take a look at our "building blocks"

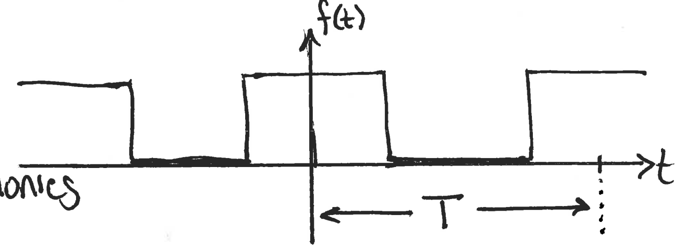
  time: $\{ \sin(\omega t), \ \cos(\omega t), \ e^{i\omega t} \}$

  space: $\{ \sin(kx), \ \cos(kx), \ e^{ikx} \}$

Periodic Functions

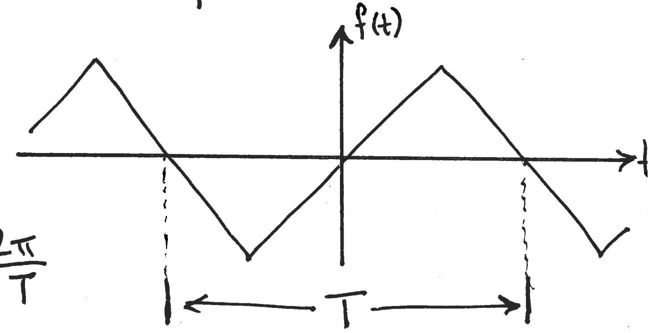
Consider a function $f(t)$ that happens to be periodic with period T

- if it's even, we can express the function as a sum of cosine harmonics



$$f(t) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos(m\omega_0 t), \quad \omega_0 = \frac{2\pi}{T}$$

- if it's odd, we use sine harmonics



$$f(t) = \sum_{m=1}^{\infty} B_m \sin(m\omega_0 t), \quad \omega_0 = \frac{2\pi}{T}$$

[Note: B_0 is always zero for an odd function]

Some general function that isn't particularly even or odd can be expressed as

$$f(t) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos(m\omega_0 t) + \sum_{m=1}^{\infty} B_m \sin(m\omega_0 t)$$

(synthesis fn.)

How accurately does this mimic the 'actual' function?

→ Depends on how many terms we keep in the sum.

(A very important issue for data compression!)

- So, can add up harmonics to produce complicated functions
- Can we do the reverse?
  → given some function, how to find the constituent harmonic components (how do we find each A_m & B_m?)
- Obviously need some mathematical "measure" of how much of a certain harmonic appears in a given function (becomes a calculus problem!)

What to do mathematically:

Use the property that the functions $\sin(m\omega_0 t)$ form an "orthogonal basis set" →

"Kronecker delta"

$$\int_0^T \sin(n\omega_0 t)\sin(m\omega_0 t)\,dt = \delta_{nm} = \begin{cases} \frac{T}{2} & \text{if } n=m \\ 0 & \text{if } n\neq m \end{cases}$$

then can write: $f(t) = \sum_{m=1}^{\infty} B_m \sin(m\omega_0 t)$

$$\Rightarrow f(t)\sin(n\omega_0 t) = \sum_{m=1}^{\infty} B_m \sin(n\omega_0 t)\sin(m\omega_0 t)$$

$$\Rightarrow \int_0^T f(t)\sin(n\omega_0 t)\,dt = \int_0^T \sum_{m=1}^{\infty} B_m \sin(n\omega_0 t)\sin(m\omega_0 t)\,dt$$

$$\Rightarrow \int_0^T f(t)\sin(n\omega_0 t)\,dt = B_n \frac{T}{2}$$

[Orthogonality: $= T/2$ if $n=m$, zero otherwise]

$$\Rightarrow B_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega_0 t)\,dt$$

and same for: $A_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega_0 t)\,dt$

$\Big\}$ (analysis fn's) Fourier Coefficients

What we just did mathematically:

① take the function we want to decompose and multiply it by the harmonic whose coefficient we want (basically a measure of the "overlap" b/w fn's)

② integrate the product over one period of the function → gives a measure of how much of that harmonic function is present in our periodic function

- so have reduced the problem of finding the coefficients to a calculus problem (sometimes a not-so-simple calc. problem)
- why go through the trouble? many times it's easier to solve a problem in the freq. domain and then convert back to time domain

The Complex Fourier Transform (not in Hecht, but should be)

- seen many times the inconvenience of trig calculations (same is true here)
- start with the same synthesis relation

$$f(t) = A_0 + 2\sum_{m=1}^{\infty}\left(A_m\cos(m\omega_0 t) + B_m\sin(m\omega_0 t)\right)$$

- but then let the sum run from $-\infty$ to ∞ instead. What happens?

Remember: $\cos(m\omega t) = \cos(-m\omega t)$ so $A_m = A_{-m}$ [even]

$\sin(m\omega t) = -\sin(-m\omega t)$ so $B_m = -B_{-m}$ [odd]

- using Euler, this would now be equivalent to

$$f(t) = A_0 + B_0 + \sum_{m=1}^{\infty} A_m\left(e^{im\omega_0 t} + e^{-im\omega_0 t}\right) - iB_m\left(e^{im\omega_0 t} - e^{-im\omega_0 t}\right)$$

- So, in general, for m over the entire range $-\infty < m < \infty$

$$f(t) = \sum_{m=-\infty}^{\infty} A_m e^{-im\omega_0 t} - iB_m e^{-im\omega_0 t}$$

$$= \sum_{m=-\infty}^{\infty} C_m e^{-im\omega_0 t}, \quad \text{where } C_m = A_m - iB_m$$

- the corresponding analysis relation is:

$$C_m = \frac{1}{T}\int_0^T f(t)\, e^{im\omega_0 t}\, dt$$

$$A_m = \frac{C_m + C_m^*}{2} = \frac{C_m + C_{-m}}{2}$$

$$B_m = \frac{C_m - C_m^*}{2} = \frac{C_m - C_{-m}}{2}$$

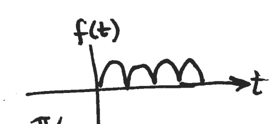
- Bonus: Can handle $\tilde{f}(t)$ when it's complex, too.

Evaluating Fourier Sums

- make use of even/odd symmetries when setting things up
- pay attention to the domain (period T), sometimes can be simplified by "folding" the integration domain
- is always best to find closed-form expressions for Fourier Coeff's (if able)
- the form of the Fourier Sum is not necessarily unique
  - different but equivalent forms by choosing trig or exp expansions, or shifting the sum's starting point (ex $m=0$ or $m=1$)

Example: Fourier Sum for a rectified sine wave $f(t) = |\sin \omega t|$

- note that the effective frequency is now 2ω, not ω, $T = \pi/\omega$



$$\Longrightarrow f(t) = |\sin \omega_0 t| = \sum_{n=-\infty}^{\infty} C_n e^{-in(2\omega)t}$$

- What coefficients are needed to construct this function?

$$C_n = \frac{1}{T}\int_0^{T} |\sin \omega t|\, e^{in(2\omega_0)t}\, dt$$

- but see that $|\sin \omega_0 t| = \sin \omega_0 t = \dfrac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i}$ over interval $[0, \frac{\pi}{\omega}]$

$$\Rightarrow C_n = \frac{1}{T}\int_0^T \frac{1}{2i}\left(e^{i\omega_0 t} - e^{-i\omega_0 t}\right) e^{in(2\omega_0)t}\, dt$$

$$= \frac{1}{2iT}\int_0^T \left(e^{i(2n+1)\omega_0 t} - e^{-i(2n-1)\omega_0 t}\right) dt$$

Change of variables
$\tau = \omega_0 t$
$d\tau = \omega_0 dt$
$\Rightarrow dt = \frac{1}{\omega_0} d\tau = \frac{T}{\pi} d\tau$

$$= \frac{1}{2\pi i}\int_0^\pi \left(e^{i(2n+1)\tau} - e^{i(2n-1)\tau}\right) d\tau$$

$$= \frac{1}{2\pi i}\left[\frac{e^{i(2n+1)\pi} - 1}{i(2n+1)} - \frac{e^{i(2n-1)\pi} - 1}{i(2n-1)}\right]$$

- here, we're lucky since $e^{i(2n\pm1)} = 1$ for all integer n

$$\Rightarrow C_n = \frac{1}{2\pi i}\left[\frac{(-2)}{i(2n+1)} - \frac{(-2)}{i(2n-1)}\right] = \frac{2}{\pi(1-4n^2)}$$

- So, finally have $f(t) = |\sin \omega_0 t| = \displaystyle\sum_{n=-\infty}^{\infty} \frac{2}{\pi(1-4n^2)} e^{-in(2\omega_0)t}$ ✓

- notice here that $C_n = C_{-n} = C_n^*$. This is because the function is <u>real</u> and <u>even</u> which is a relatively simple case

- not everything of interest is a periodic signal  or 

- to consider a finite signal, consider it to be periodic on an infinite time scale! $T \to \infty$.

- what changes then?

- Remember had:

$$f(t) = \sum_{m=-\infty}^{\infty} C_m e^{-im\omega_0 t} \quad \& \quad C_m = \frac{1}{T}\int_0^T f(t) e^{im\omega_0 t}\, dt$$

- for the coefficients, can choose our bounds to be $\int_{-T/2}^{T/2}$ instead
- given a finite signal, nothing exists outside these bounds, so can just as well allow $T \to \infty$ so we have

$$C_m = \frac{1}{T}\int_{-\infty}^{\infty} f(t) e^{im\omega_0 t}\, dt \Rightarrow TC_m = \int_{-\infty}^{\infty} f(t) e^{im\omega_0 t}\, dt$$

- but as $T \to \infty$, TC_m no longer describes a discrete set of integer multiples of the fundamental freq: $m\omega_0 \longrightarrow \omega$ (continuous)

$$\Rightarrow TC_m\Big|_{T\to\infty} \equiv \tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t}\, dt$$

[Fourier Transform]

- also, can then write $f(t)$ as:

$$f(t) = \sum_{m=-\infty}^{\infty} TC_m e^{-im\omega t} \frac{1}{T}$$

$$= \sum_{m=-\infty}^{\infty} TC_m e^{-im\omega t} \frac{\Delta\omega}{2\pi}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega) e^{-i\omega t}\, d\omega$$

[Inverse Fourier Transform]

$T = 2\pi/\Delta\omega$

as $T \to \infty$, $\Delta\omega \to 0$

$\Rightarrow \Delta\omega \to d\omega$

$\Rightarrow \sum_{m=-\infty}^{\infty} \frac{1}{T} \longrightarrow \int_{-\infty}^{\infty} d\omega/2\pi$

- Here we have the Fourier Transform pair: $f(t) \overset{\mathcal{F}}{\Longleftrightarrow} \tilde{f}(\omega)$

[time domain] [freq domain]

$$\mathcal{F}^{-1}\left[\mathcal{F}\left[f(t)\right]\right] = f(t)$$

- note that normalization conventions may vary b/w sources.

For us, though: $\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt\right] e^{-i\omega t}\, d\omega = f(t)$

where the $\frac{1}{2\pi}$ needs to be there to balance $d\omega$ & dt

<u>But</u>, could also write in terms of $\nu = \frac{\omega}{2\pi}$, then

$$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\, e^{i2\pi\nu t}\, dt\right] e^{-i2\pi\nu t}\, d\nu$$

- in terms of trig fn's, these would look like

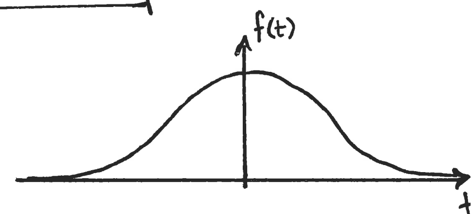
$$f(t) = \frac{1}{\pi}\left[\int_{0}^{\infty} A(\omega)\cos(\omega t)\, d\omega + \int_{0}^{\infty} B(\omega)\sin(\omega t)\, d\omega\right]$$

where $A(\omega) = \int_{-\infty}^{\infty} f(t)\cos(\omega t)\, dt$ & $B(\omega) = \int_{-\infty}^{\infty} f(t)\sin(\omega t)\, dt$

Wave Packets

- Hecht goes over examples of 2 important cases: the square pulse and cosine wave packet.
- Another <u>very</u> common pulse form is the <u>Gaussian pulse</u>

gaussian function: $f(t) = A e^{-\alpha t^2}$



- Apply Fourier Transform:

$$\mathcal{F}[f(t)] = \tilde{f}(\omega) = \int_{-\infty}^{\infty} A e^{-\alpha t^2 + i\omega t}\, dt$$

- to evaluate this integral, need to use a trick:

Complete the square! where in general

$$at^2 + bt + c = a\left(t + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

So, with $a = -\alpha$, $b = i\omega$, and $c = 0$, the exponent is

$$\int_{-\infty}^{\infty} A e^{-\alpha\left(t - \frac{i\omega}{2\alpha}\right)^2 + 0 - \frac{\omega^2}{4\alpha}} dt = A e^{-\frac{\omega^2}{4\alpha}} \int_{-\infty}^{\infty} e^{-\alpha\left(t - \frac{i\omega}{2\alpha}\right)^2} dt$$

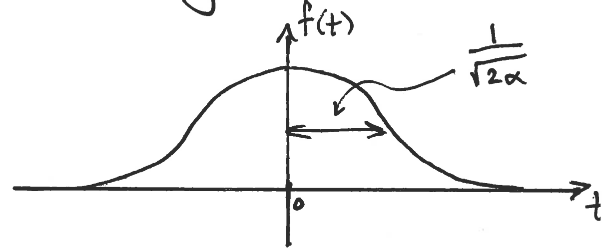
and (another trick) see that integral is the same if we substitute

$t \rightarrow t + b$ (okay since limits are $\pm\infty$)

$$\Rightarrow \tilde{f}(\omega) = A e^{-\omega^2/4\alpha} \int_{-\infty}^{\infty} e^{-\alpha t^2} dt = A e^{-\omega^2/4\alpha} \sqrt{\pi/\alpha}$$

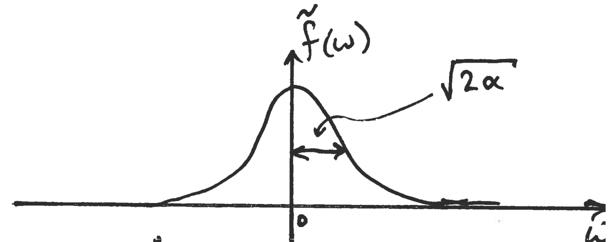
So the Fourier Transform of a gaussian is a gaussian!

$$A e^{-\alpha t^2} \xleftrightarrow{\mathcal{F}} A\sqrt{\frac{\pi}{\alpha}}\, e^{-\omega^2/4\alpha}$$



temporal width of gaussian: $\sigma_t = \frac{1}{\sqrt{2\alpha}}$

spectral width of gaussian: $\sigma_\omega = \sqrt{2\alpha}$



Spectral & temporal widths are inversely related!

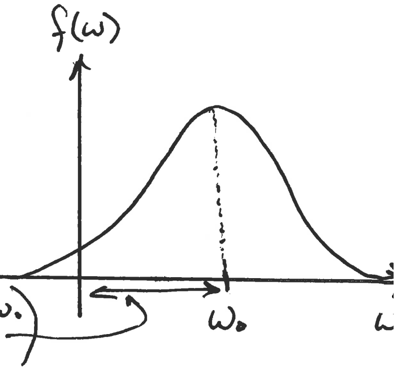
Note: Negative freq's!

What about off-center gaussians?

$$f(t) = A e^{-\frac{t^2}{2\tau^2} - i\omega_0 t}$$

$$\tilde{f}(t) = \frac{A}{\sqrt{2\pi}} \sqrt{\frac{\pi}{\tau^2/2}}\; e^{-\frac{(\omega - \omega_0)^2}{4(1/2\tau^2)}}$$

(gaussian shifted by ω_0)



Coherence of Optical Fields

- in general, a non-zero spectral width (σ_ω) of a physical optical source means the field isn't perfectly coherent (we'll formalize the idea of coherence a bit more later)

<u>Coherence time</u>: $\delta t_c = \frac{1}{\delta \nu}$

<u>Coherence length</u>: $\delta l_c = c\,\delta t_c$

$\nu = \frac{c}{\lambda}$, but $\delta\nu \neq \frac{c}{\delta\lambda}$

$\Rightarrow \delta\nu = \frac{c\,\delta\lambda}{\lambda^2}$