Diffraction

Essentially, there's no real distinction between interference and diffractio

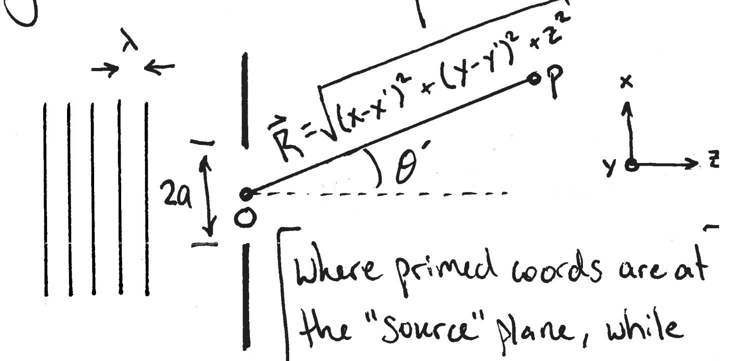
- "interference" : normally when considering only a few waves (and phase shifts)
- "diffraction" : when considering many waves and/or amplitude effects

Ex: a) multiple-beam interference in Fabry-Perot } Physically & mathematically
b) diffraction from a grating } essentially the same thing

Huygens-Fresnel Principle:

- remember Huygens model of re-radiating "wavelets" making up wavefronts?
- We can model the propagation of light as a sum of spherical wavelets

$$E(r) = C\iint \underset{\substack{\uparrow\\ \text{incident light}}}{\vec{E}(x',y',z=0)}\,\underset{\substack{\uparrow\\ \text{diffracted wavelet}}}{\vec{u}(k,\vec{R})}\,dx'dy'$$



[Where primed coords are at the "source" plane, while others are at some arbitrary plane of interest]

Qualitative Diffraction:

- light bends around edges of physical object/aperture into region of geometric shadow due to interference of wavelets making up the wavefront
- if the "source" (object/aperture) is small compared to λ → plane waves spread out alo
- if the source is large compared to λ → light spreads out (comparitively) little

- the full details of diffraction theory can get <u>very</u> complicated [7-2]
  - vector vs scalar fields (include polarization or not)
  - appropriate modeling of the spherical wavelets
  - near-field vs. far-field effects (Fraunhofer vs Fresnel) $\underline{\underline{R > \frac{a^2}{\lambda}}}$

Single Slit Diffraction

- take the scenario pictured above with the slit
- reduce this to a 1D problem → slit ∞ long in $\hat{x}$, ∞ short in $\hat{z}$
- then by Huygens-Fresnel, each point w/in the aperture (z=0, along 2a) can be considered a set of pt. emitters of spherical waves that add up
- by now have: $E(x,y,z) = C \iint E(x',y',z=0)\, U(k,R)\, dx'dy'$

$$(1D) \Rightarrow \quad E(x,z) = C \int_{-a}^{a} E(x',z=0)\, U(k,R)\, dx'$$

where the outgoing wavelets are: $U(k,R) = \underbrace{\frac{E_0}{R} e^{ikR}}\underbrace{\left[\frac{1+\cos(\theta')}{2}\right]}$

- we recognize the first factor from spherical emitters, but where does that second factor come from?
- "<u>Obliquity" factor</u> basically placed by hand (by Huygens & Fresnel anyway) to limit spread in $\hat{x}$ and negate backwards propagation in $\hat{z}$
- $\underbrace{\text{this general integral}}$ is <u>very</u> difficult to solve → let's simplify!

  ↳ called <u>Fresnel-Kirchoff Diffraction</u>

Simplification 1: $\boxed{\theta' \text{ is small}}$ (Paraxial approx. → Fresnel Diffract.)

- now $\cos\theta' \simeq 1$ and the obliquity factor disappears

$$\Longrightarrow E(x,z) = C\int_{-a}^{a} E(x', z=0)\,\frac{e^{ikR}}{R}\,dx'$$

$\theta' \simeq 0$

$R = \sqrt{(x-x')^2+z^2}$

Simplification 2: $\boxed{z^2 \gg (x-x')^2}$

- now the R in the factor $1/R$ can be expanded to the approximation

$$\Longrightarrow R = \sqrt{z^2+(x-x')^2} = z\sqrt{1+\frac{(x-x')^2}{z^2}} \simeq z\left(1+\frac{(x-x')^2}{2z^2}\right) \simeq z$$

small factors

- BUT this last step won't work in the exponent (too sensitive to phase) since differences of λ are important even if z is large

$$\Longrightarrow E(x,z) = C\int_{-a}^{a} E(x', z=0)\,\frac{1}{z}\,e^{ikz}\,e^{ik\left(\frac{(x-x')^2}{2z^2}\right)}\,dx'$$

$$= \underbrace{C\,\frac{e^{ikz}\,e^{ik\frac{x^2}{2z}}}{z}}_{}\int_{-a}^{a} E(x', z=0)\,e^{ik\frac{x'^2}{2z}}\,e^{-ik\frac{xx'}{z}}\,dx'$$

- have pulled out the overall phase factors that don't depend on x'.

but what is this?

- phase is not uniform over span of "screen"
- phase increases with z
- phase increases quadratically with x (important for multiple apertures, stay tuned...)

- what about the constant, C? $\Longrightarrow C = -i/\lambda$
  - spherical wave amplitudes are scaled by $1/\lambda$
  - if $a \to \infty$, then must get back our plane wave, so phase must match. not shown, but Huygens' wavelets are 90° out-of-phase with plane waves → $-i$

Simplification 3: $\boxed{z >> \frac{k}{2} a^2} \Rightarrow e^{ik\frac{x'^2}{2z}} \simeq 1$

- in words, if the screen is far enough away from aperture, then the change in phase across the aperture isn't to large an effect

[Fraunhofer or Far-Field Limit]

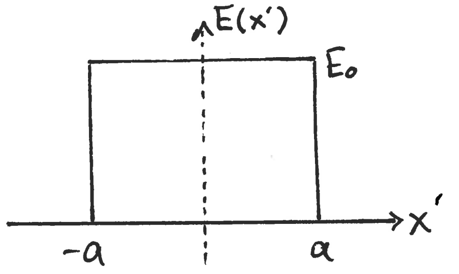
$$\Rightarrow E(x,z) = \frac{c}{z} e^{ikz} e^{ik\frac{x^2}{2z}} \int_{-a}^{a} E(x', z=0)\, e^{-ik\frac{xx'}{z}} dx'$$

$$= \frac{c}{z} e^{i\phi(x,z)} \int_{-a}^{a} E(x', z=0)\, e^{-i\left(\frac{kx}{z}\right)x'} dx'$$

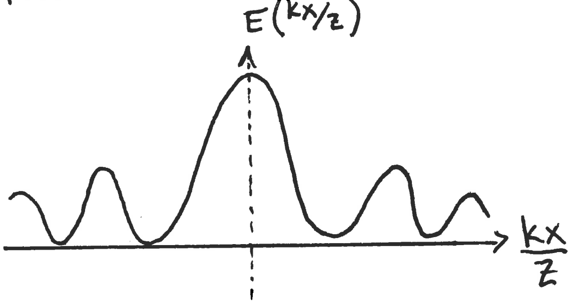
- here's where the magic comes in: Looks like a Fourier Transform!

(but the transform variables look abit different → more on that later)

- for now, let's look at the solution



[at the slit]



[at the screen]

$$\Rightarrow E(x,z) = \frac{c}{z} e^{i\phi(x,z)} \cdot \frac{\sin\left(\frac{kax}{z}\right)}{kax/z} = \frac{c}{z} e^{i\phi(x,z)} \frac{\sin\beta}{\beta}$$

look! sinc!

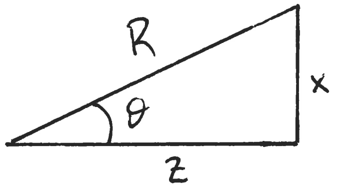
where $\beta = \frac{kax}{z} = ka \sin\theta$

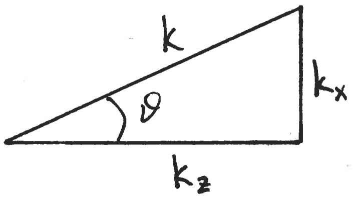
and in general $E(r) = C \frac{(\text{phase})}{r} \operatorname{sinc}\beta$

$$\underline{\underline{I(r) \propto \frac{1}{r^2} \operatorname{sinc}^2 \beta}}$$

- is sometimes useful to recast this answer as a function of θ, or of x

- Can use $\beta = \frac{kax}{z} = ka\sin\theta$ in the paraxial limit for coordinate interchangeability





$$\Longrightarrow \theta = \frac{x}{z} = \frac{k_x}{k} \qquad \Longrightarrow \quad E_z(x) = \tilde{E}\left(k\frac{x}{z}\right) \qquad \left[\underline{\text{spatial}}\ \text{frequency}\right]$$

(can replace k_x with $k\frac{x}{z}$)

- just like when dealing with <u>temporal</u> signals, a <u>short</u> square pulse is composed of a broader spectrum of frequencies. Here, a narrow slit relsuts in a transmitted field having a lot of different spatial frequencies

- Okay, let's extend this to 2D.

The Square Aperture:

- essentially, there's no difference from 1D, just more coords.

Fresnel - Kirchoff:

$$E(x,y,z) = \frac{-i}{\lambda} \iint_{\text{aperture}} E(x',y',z=0)\,\frac{e^{ikr}}{r}\left[\frac{1+\cos\theta}{2}\right]dx'dy'$$

Fresnel:

$$E(x,y,z) = \frac{-i}{\lambda z}\, e^{ikz}\, e^{i\frac{k}{2z}(x^2+y^2)} \iint E(x',y',0)\, e^{i\frac{k}{2z}(x'^2+y'^2)}\, e^{-i\frac{k}{z}(xx'+yy')}\, dx'dy'$$

Fraunhofer:

$$E(x,y,z) = \frac{-i}{\lambda z} e^{ikz} e^{i\frac{k}{2z}(x^2+y^2)} \iint E(x',y',0)\, e^{-i\frac{k}{z}(xx'+yy')}\, dx'dy'$$

if $E_0 = E(x',y',0)$
$$= \frac{-iE_0}{\lambda z} e^{ikz} \left[e^{i\frac{k}{2z}x^2} \int e^{-i\frac{k}{z}xx'} dx' \right] \left[e^{i\frac{k}{2z}y^2} \int e^{-i\frac{k}{z}yy'} dy' \right]$$

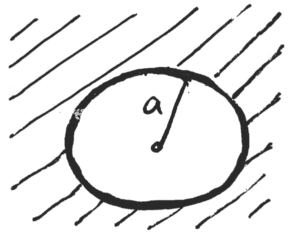
$$= \frac{-iE_0}{\lambda z} e^{ikz} \left[e^{i\frac{k}{2z}x^2} \int_{-a_x}^{a_x} e^{-ik_x x'} dx' \right] \left[e^{i\frac{k}{2z}y^2} \int_{-a_y}^{a_y} e^{-ik_y y'} dy' \right]$$

$$= \frac{-iE}{\lambda z} e^{ikz} e^{i\frac{k}{2z}(x^2+y^2)}\, \text{sinc}\left(\frac{ka_x x}{z}\right) \text{sinc}\left(\frac{ka_y y}{z}\right)$$

and then $I \propto I_{max}\, \text{sinc}^2(\beta_x)\, \text{sinc}^2(\beta_y)$

[just the product of 1D results in $\hat{x}, \hat{y}$ dirs]

The Circular Aperture:



- end up going through exactly the same steps as above, except convert to cylindrical coords.

$$\Longrightarrow E(\rho,z) = \frac{-i}{\lambda z} e^{ikz} e^{i\frac{k\rho^2}{2z}} \int_0^{2\pi}\int_0^a E(\rho',0)\, e^{i\frac{k\rho'^2}{2z}} e^{-i\frac{k\rho\rho'}{z}\cos(\phi-\phi')} \rho' d\rho' d\phi'$$

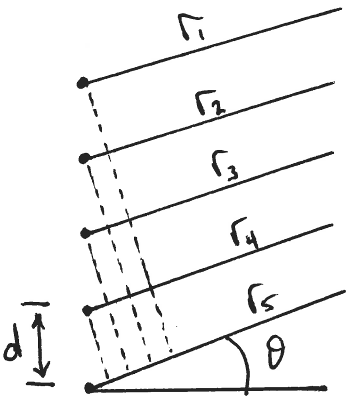
$$= \frac{-i}{\lambda z} 2\pi e^{ikz} e^{i\frac{k\rho^2}{2z}} \int_0^a J_0\left(\frac{k\rho\rho'}{z}\right) \rho' d\rho'$$

- where J_0 is a Bessel function (order 0) and integral has form of Hankel Transform

$$\Longrightarrow I(\rho,z) = I_0 \left(\frac{\pi a^2}{\lambda z}\right)^2 \left[2\frac{J_1(ka\rho/z)}{ka\rho/z} \right]^2$$

[Airy disk or Airy rings]

Array Theorem:



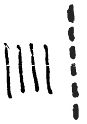
[linear array of in-phase coherent oscillators]

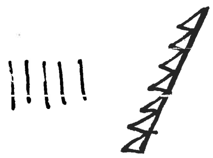
- consider case of several coheret oscillators

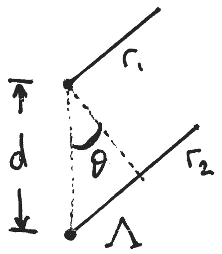
$$E = E_0(r)\left[e^{i(kr_1-\omega t)} + e^{i(kr_2-\omega t)} + \cdots + e^{i(kr_N-\omega t)}\right]$$

$$= E_0(r)\, e^{i\omega t}\, e^{ikr_1}\left[1 + e^{ik(r_2-r_1)} + e^{ik(r_3-r_1)} + \cdots\right]$$

- this array of oscillators has many physically equivalent representations
  - plane wave incident on N slits (transmission grating)
  - plane wave incident on N mirrors (reflection grating)
  - any regularly-spaced dispersive element which introduces a fixed phase offset between oscillators







- the phase difference between individual waves is

$$\delta = k_0\Lambda = k_0 n d \sin\theta$$

$$\Rightarrow E = E_0(r)\, e^{-i\omega t}\, e^{ikr_1}\left[1 + e^{i\delta} + e^{2i\delta} + \cdots + e^{i(N-1)\delta}\right]$$

$$= E_0(r)\, e^{-i\omega t}\, e^{ikr_1}\left(\frac{e^{i\delta N}-1}{e^{i\delta}-1}\right)$$

where
$$\left(\frac{e^{iN\delta}-1}{e^{i\delta}-1}\right) = \frac{e^{iN\delta/2}\left(e^{iN\delta/2} - e^{-iN\delta/2}\right)}{e^{i\delta/2}\left(e^{i\delta/2} - e^{-i\delta/2}\right)} = e^{i\frac{(N-1)}{2}\delta}\left(\frac{\sin N\delta/2}{\sin \delta/2}\right)$$

$$\Rightarrow E = E_0(r)\, e^{-i\omega t}\, e^{i\left[kr_1 + \frac{(N-1)}{2}\delta\right]}\, \frac{\sin(N\delta/2)}{\sin(\delta/2)}$$

and can say the "average" distance to our point of interest is [7-8]

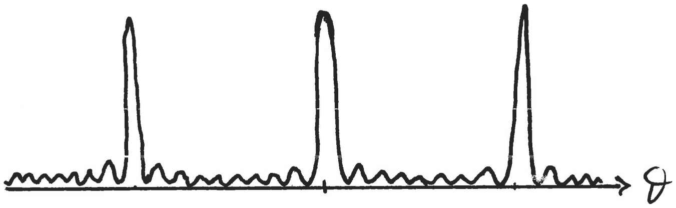
$$R = \frac{1}{2}(N-1)\, d \sin\theta$$

$$\Rightarrow E = E_0(r)\, e^{i(kR-\omega t)} \frac{\sin N\delta/2}{\sin \delta/2}$$

$$\Rightarrow I \propto E E^* \quad \Rightarrow I = I_0 \frac{\sin^2(N\delta/2)}{\sin^2(\delta/2)}$$

then go back and explicitly write in $\delta = k_0 n d \sin\theta$

$$\Rightarrow I = I_0 \frac{\sin^2[N(kd/2)\sin\theta]}{\sin^2[kd/2 \sin\theta]} \quad \begin{array}{l} \leftarrow \text{Varies } \underline{\text{rapidly}} \\ \leftarrow \text{Varies slowly} \end{array}$$



- Maxima in intensity occur where $\delta = 2m\pi = kd\sin\theta \Rightarrow \boxed{d \sin\theta_m = m\lambda}$

  the "grating" equation

- m is the order parameter (integer → 1st Order, 2nd Order, etc.)

- there are N-1 minima between each maxima

- represents a <u>very important tool</u> for spectroscopy since the diffraction angle can be made strongly dependent on λ

- need to be more realistic, though → slits are not infinitesimally narrow

- define a re-scaled angle to account for center-to-center spacing of slits with finite width, b.

$$\Longrightarrow \alpha = \frac{kd}{2}\sin\theta \qquad \Longrightarrow I(\alpha) = I_0\left(\frac{\sin(N\alpha)}{\sin\alpha}\right)^2$$

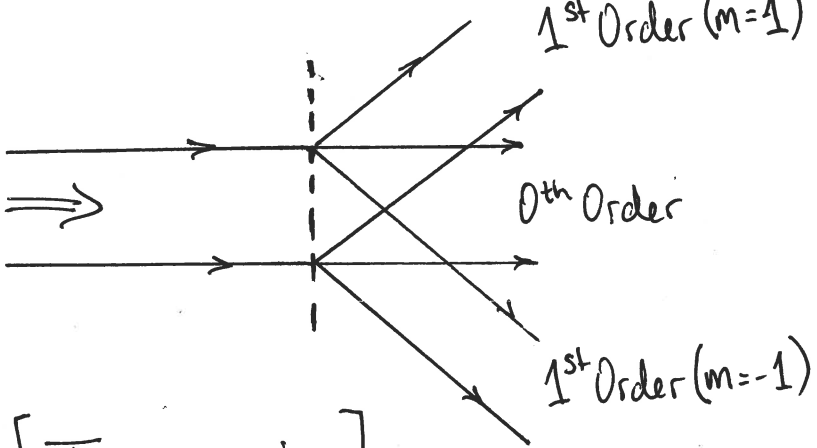
maxima at:
$\alpha = m\pi$
minima at:
$\alpha = \pm\frac{\pi}{N}, \pm\frac{2\pi}{N}\ldots$

- as we saw before for the single slit of finite width, b, the intensity distribution at the screen follows an envelope

$$\Longrightarrow I(\alpha,\beta) = \frac{I_0}{N^2}\left(\frac{\sin\beta}{\beta}\right)^2\left(\frac{\sin N\alpha}{\sin\alpha}\right)^2 \quad , \beta = \frac{kb}{2}\sin\theta$$

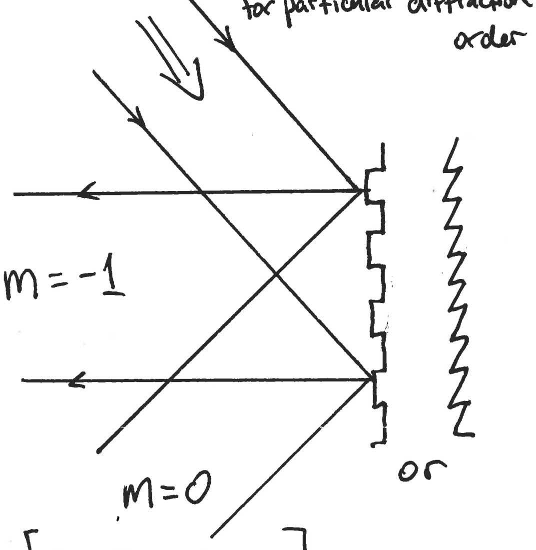
$$= \frac{I_0}{N^2}\,\mathrm{sinc}^2\beta\left(\frac{\sin N\alpha}{\sin\alpha}\right)^2$$

Different Grating Types:



[Transmission]

Blazed gratings:
Surface profile optimized for particular diffraction order



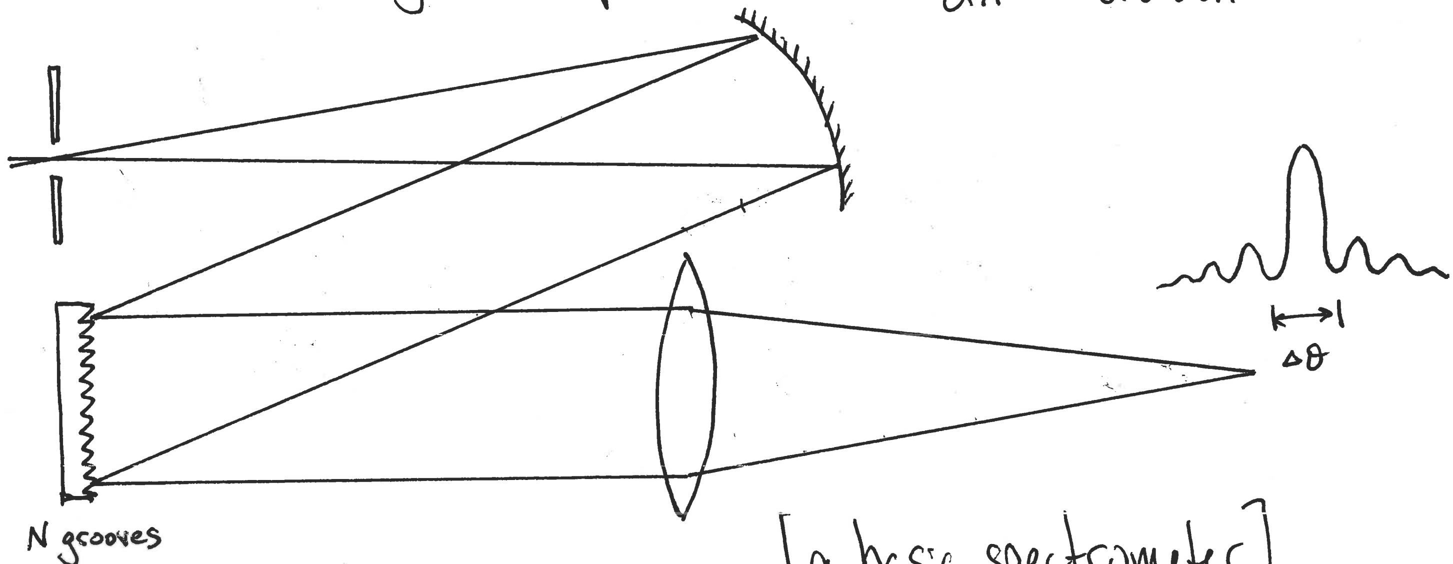
[Reflection]

- can also make gratings acting only on amplitude or phase across wavefront

Grating Spectroscopy:

[7-10]

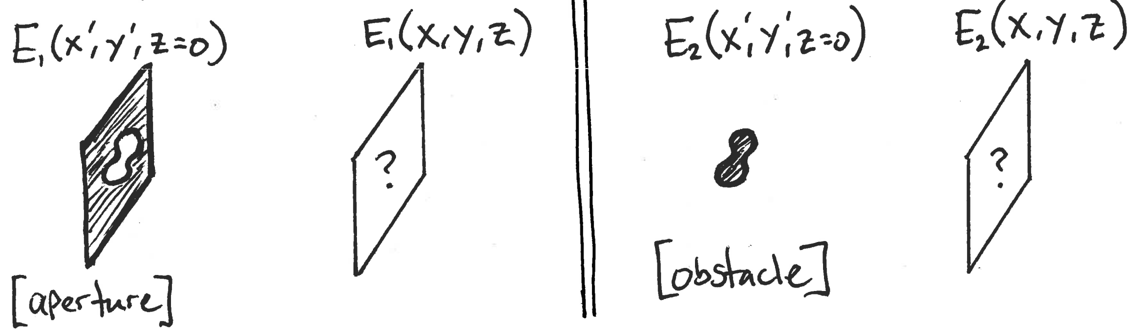
- saw above that θ depends on λ $(d \sin\theta = m\lambda)$ for a dispersive grating
- can define an angular dispersion: $D = \frac{d\theta}{d\lambda} = \frac{m}{d\cos\theta_m}$



[a basic spectrometer]

- for a given λ, the angular width of a diffracted beam will be $\Delta\theta = \frac{2\lambda}{Nd\cos\theta_m}$ [where Nd is the grating width]

Babinet's Principle:



- Consider these 2 different diffraction problems, with plane waves $E_0 e^{i(kz-\omega t)}$ incident from the left side
- What diffraction patterns do we observe at the screens?

- as a consequence of wave function linearity and the linearity of the Fourier transform operator, <u>they are complimentary!</u>

- after looking at the problem for a moment it becomes obvious
  → the sum of radiation patterns from each scenario must equal the the radiation pattern of the unobstucted beam

- <u>Caveat</u>: these two fields are not necessarily equal in amplitude! <u>Except</u>, wherever the original beam would not have reached, then the radiated fields are equal amplitude, just opposite phase.

- So, the fields after the apertures add up so that (neglecting time dep.),

$$E_1(x', y', z=0) + E_2(x', y', z=0) = E_0$$

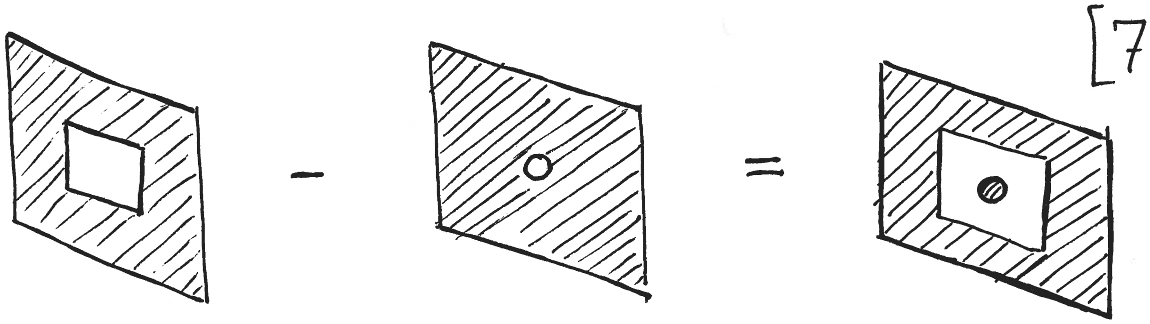
and the diffraction operation, which is just a Fourier transform, is also linear, so

$$D[E_1(x', y')] + D[E_2(x', y')] = E_0$$

where, in general, have : $E_1(x, y, z) + E_2(x, y, z) = E_0$

- <u>Result</u>: if you solve the diffraction problem for an aperture, you automatically earn the solution for its <u>conjugate</u> obstruction

- also, get to build-up more complicated apertures from simple ones!



Note: Babinet's Principle only applies to the electric fields, not the intensities. (Phase is lost when you take EE^*!)

Ex/ Let's reconsider the diffraction pattern of a circular aperture → there's a pretty neat effect here!

- in 2D cartesian we'll have:

$$E(R) = C\iint E(x',y',z=0)\, U(k,R)\, dx'dy' \quad , \text{ with } R=\sqrt{(x-x')^2+(y-y')^2+z^2}$$

- but makes much more sense to convert to cylindrical symmetry

$$\Longrightarrow x=\rho\cos\phi \ , \ y=\rho\sin\phi \ \ \& \ \ x'=\rho'\cos\phi' \ , \ y'=\rho'\sin\phi'$$

$$E(\rho,z) = \frac{-i}{\lambda z} e^{ikz} e^{i\frac{k\rho^2}{2z}} \int_0^{2\pi}\int_0^{\ell/2} E(\rho',0)\, e^{i\frac{k\rho'^2}{2z}} e^{-i\frac{k}{z}(\rho\rho'\cos\phi\cos\phi'+\rho\rho'\sin\phi\sin\phi')} \rho' d\rho' d\phi$$

- trig: use $\rho\rho'(\cos\phi\cos\phi'+\sin\phi\sin\phi') = \rho\rho'\cos(\phi'-\phi)$

$$E(\rho,z) = \frac{-i}{\lambda z} e^{ikz} e^{\frac{ik\rho^2}{2z}} \int_0^{\ell/2} E(\rho',0)\, e^{i\frac{k\rho'^2}{2z}} \left[\int_0^{2\pi} e^{-i\frac{k\rho\rho'}{z}\cos(\phi'-\phi)} d\phi'\right] \rho' d\rho'$$

where the bracketed portion happens to be a zeroth-order Bessel function of first kind

$$E(\rho,z) = \frac{-2\pi i}{\lambda z} e^{ikz} e^{i\frac{k\rho^2}{2z}} \int_0^{\ell/2} E(\rho',0)\, e^{i\frac{k\rho'^2}{2z}} J_0\left(\frac{k\rho\rho'}{z}\right) \rho' d\rho'$$

- in the Fraunhofer limit (far-field), we take $e^{i\frac{k\rho'^2}{2z}} = 1$ [7-13]

$$E(\rho,z) = -\frac{2\pi i}{\lambda z} e^{ikz} e^{i\frac{k\rho^2}{2z}} \int_0^{\ell/2} E(\rho',0)\, J_0\!\left(\frac{k\rho\rho'}{z}\right)\rho'\, d\rho'$$

- this kind of looks like a Fourier transform → it's not! [Hankel transform of order ν]

$$F_\nu(k) = \int f(\rho)\, J_\nu(k\rho)\,\rho\, d\rho$$

Bessel functions are the sine/cosine of cylindrical geometry (beyond the scope of *this* course, but kind of neat.)

- what's important here is that the integral has an analytical solution

$$E(\rho,z) = -E_0\left(\frac{\pi\ell^2}{4\lambda z}\right) e^{i\frac{k\rho^2}{2z}} \cdot \frac{2J_1\left(\frac{k\ell\rho}{2z}\right)}{\left(k\ell\rho/2z\right)}$$

[Airy function]

(cylindrical generalization of the sinc function)

- remember, this is the far-field limit solution. What if instead, we are close to the circular aperture? (Fresnel limit)

→ the general solution would need to be computed numerically, *but* we can analytically solve for the on-axis field! Which is just solving the above for $\rho = 0$

→ Also, where's the "neat effect"?

An Historical Aside

- 1818 - Ongoing debate between wave & corpuscular theory of light [7-14]
- Corpuscular (particle) theory of light winning popularity polls

  → it's hard to discount Newton's double prism experiments
- diffraction is still an outstanding problem, though
- enter: The French Academy of Science

  - organized a contest to overcome this last obstacle
- many minor contributions submitted... one major contribution by Fresnel on the wave nature of light (extending Huygens' work)
- two people on judges' panel: Siméon Poisson & François Arago
- Poisson (staunch believer in corpuscular theory), took Fresnel's work, applied it to the shadow cast by a circular disk, and showed that it predicted a bright spot generated in exact center of shadow!

  → attempted logical fallacy: reductio ad absurdum
- Arago saw Poisson's argument, setup the experiment, observed the spot!
- Fresnel wins the prize
- few years later, corpuscular theory nearly totally abandoned

---

- Okay. back to the on-axis solution

$$E(\rho=0,z) = \frac{-i}{\lambda}\int_0^{\ell/2}\int_0^{2\pi} E(\rho',0)\,\frac{e^{ik\sqrt{\rho'^2+z^2}}}{\sqrt{\rho'^2+z^2}}\,\rho' d\rho' d\phi'$$

$$= -\frac{2\pi i E_0}{\lambda}\cdot\frac{e^{ik\sqrt{\rho'^2+z^2}}}{ik}\Bigg|_0^{\ell/2}$$

$$= -E_0\left(e^{ik\sqrt{(\ell/2)^2+z^2}} - e^{ikz}\right) \qquad \left[e^{-i\pi/2} = -i\right]$$

$$I(\rho=0,z) = 2|E_0|^2\left[1-\cos\left(k\sqrt{(\ell/2)^2+z^2}-kz\right)\right]$$

☆ see plot on slide

- So what does this have to do with Babinet's Principle?

→ we actually solved for an open aperture, but

remember: Aperture + Obstacle = Plane Wave

$$\Rightarrow E_0 e^{ikz} = E^{ap}(z) + E^{ob}(z)$$

$$\Rightarrow E^{ob}(z) = E_0 e^{ikz} - E^{ap}(z)$$

$$= E_0 e^{ikz} - -E_0\left(e^{ik\sqrt{(\ell/2)^2+z^2}} - e^{ikz}\right)$$

$$= E_0 e^{ik\sqrt{(\ell/2)^2+z^2}}$$

$$\Rightarrow \underline{\underline{I^{ob}(z) = E_z E_z^* = |E_0|^2}}$$

[the intensity at the center of the spot will be the same as if the obstacle wasn't even there]

<u>Note:</u> You would get the wrong answer if you had tried to apply Babinet's Principle to the intensities instead

$$\Longrightarrow \quad I^{ob}(z) \neq I_0 - I^{ap}(z)$$

- notice that this formula predicts constant intensity on-axis right up to the point behind the circular obstacle

  $\rightarrow$ not actually true! what's wrong? Omitted obliquity factor

$$u(k,R) = \frac{1}{R} e^{ikR} \underbrace{\left[\frac{1-\cos\theta}{2}\right]}$$

- a more realistic calculation with ↗ gives a more physical answer

$$\longrightarrow \quad I(\rho=0, z) = \frac{z^2}{z^2 + (\ell/2)^2} I_0$$