SUPERPOSITION

P47 – Optics: Unit 4

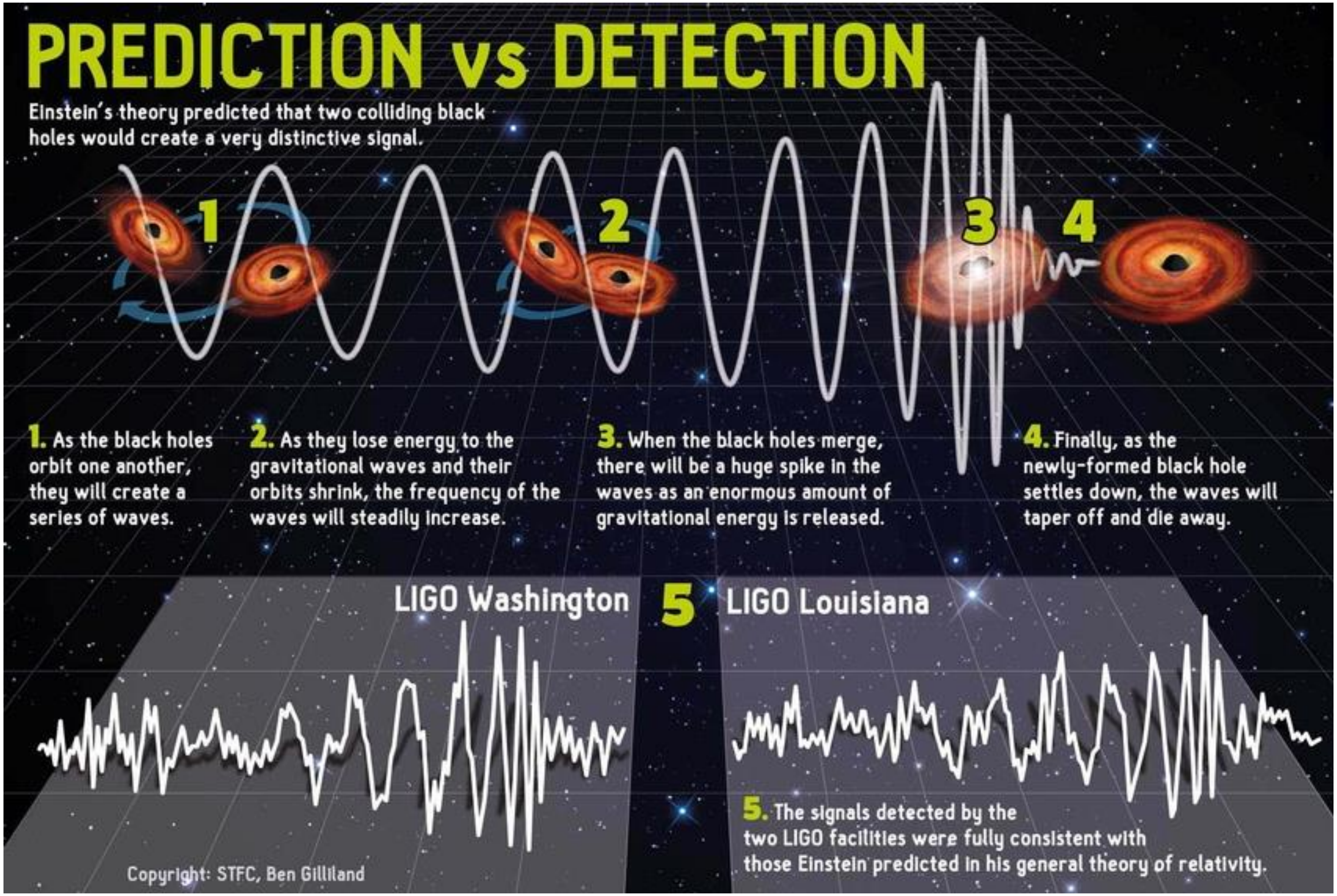
Course Outline

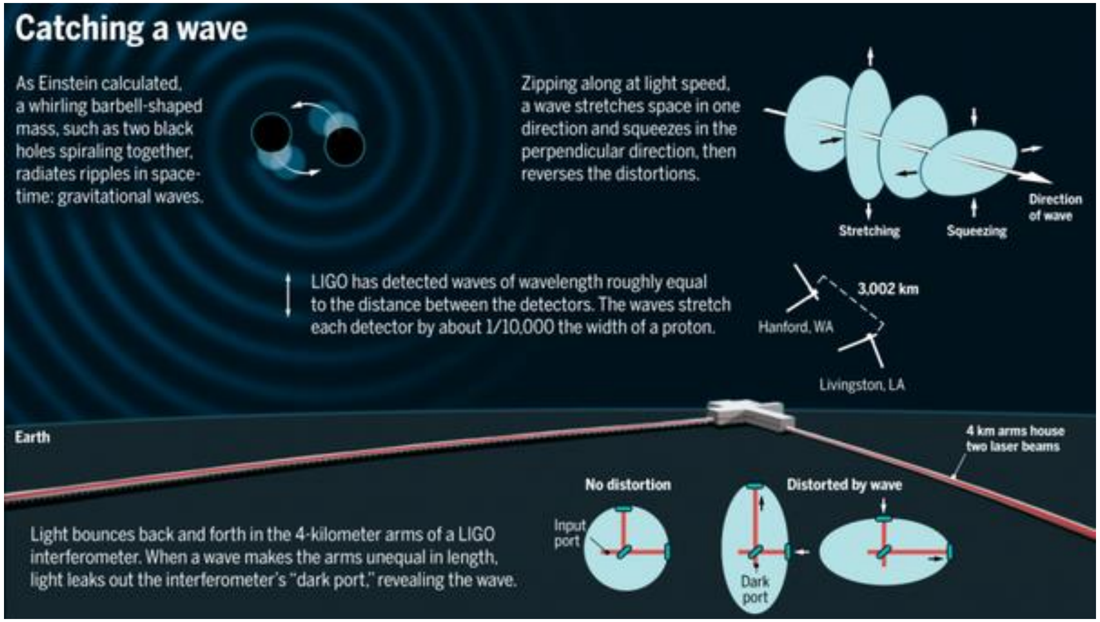
~~Unit 1: Electromagnetic Waves~~
~~Unit 2: Interaction with Matter~~
~~Unit 3: Geometric Optics~~

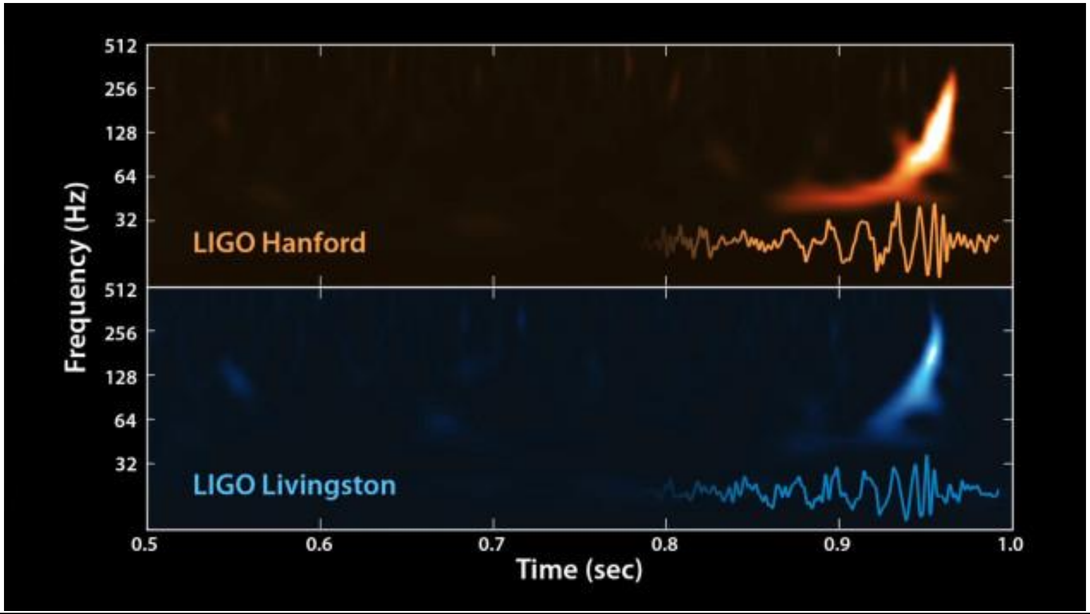
Unit 4: Superposition of Waves
Unit 5: Polarization
Unit 6: Interference

Unit 7: Diffraction
Unit 8: Fourier Optics
Unit 9: Modern Optics

How to describe waves like this? → Superposition!




Catching a wave
As Einstein calculated, a whirling barbell-shaped mass, such as two black holes spiraling together, radiates ripples in space-time: gravitational waves.
Zipping along at light speed, a wave stretches space in one direction and squeezes in the perpendicular direction, then reverses the distortions.
Stretching
Squeezing
Direction of wave
LIGO has detected waves of wavelength roughly equal to the distance between the detectors. The waves stretch each detector by about 1/10,000 the width of a proton.
3,002 km
Hanford, WA
Livingston, LA
Earth
4 km arms house two laser beams
No distortion
Distorted by wave
Input port
Dark port
Light bounces back and forth in the 4-kilometer arms of a LIGO interferometer. When a wave makes the arms unequal in length, light leaks out the interferometer's "dark port," revealing the wave.


Frequency (Hz)
512
256
128
64
32
LIGO Hanford
LIGO Livingston
0.5
0.6
0.7
0.8
0.9
1.0
Time (sec)

Superposition Principle

- Assume we have a set of elementary solutions (e.g. plane wave) to the wave equation: $\longrightarrow$ $\psi_i(\boldsymbol{r}, t)$

- We can express complicated waves as sum of these elementary waves: $\longrightarrow$ $\psi(\mathbf{r}, \mathrm{t}) = \sum_{i=1}^{n} C_i \psi_i(\boldsymbol{r}, t)$

- The result is still a solution of the wave equation because it's a *linear* differential equation (no powers of **r** or t)

Superposition of two scalar monochromatic waves

$$Ee^{i\phi} = E_1 e^{i\phi_1} + E_2 e^{i\phi_2}$$

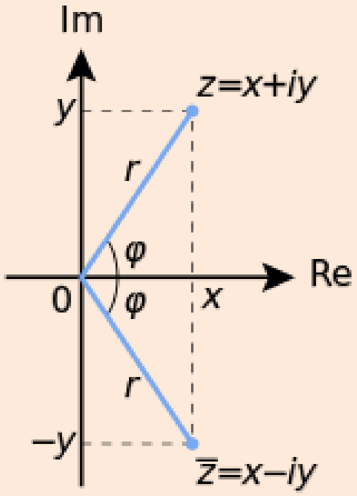
$$e^{i\phi} = \cos\phi + i\sin\phi$$

$$\sin\phi = \frac{e^{i\phi} - e^{-i\phi}}{2i} \qquad \cos\phi = \frac{e^{i\phi} + e^{-i\phi}}{2}$$

Complex number representation

$$z = re^{i\phi} = x + iy \qquad \mathrm{Re}\{z\} = x = (z + z^*)/2$$

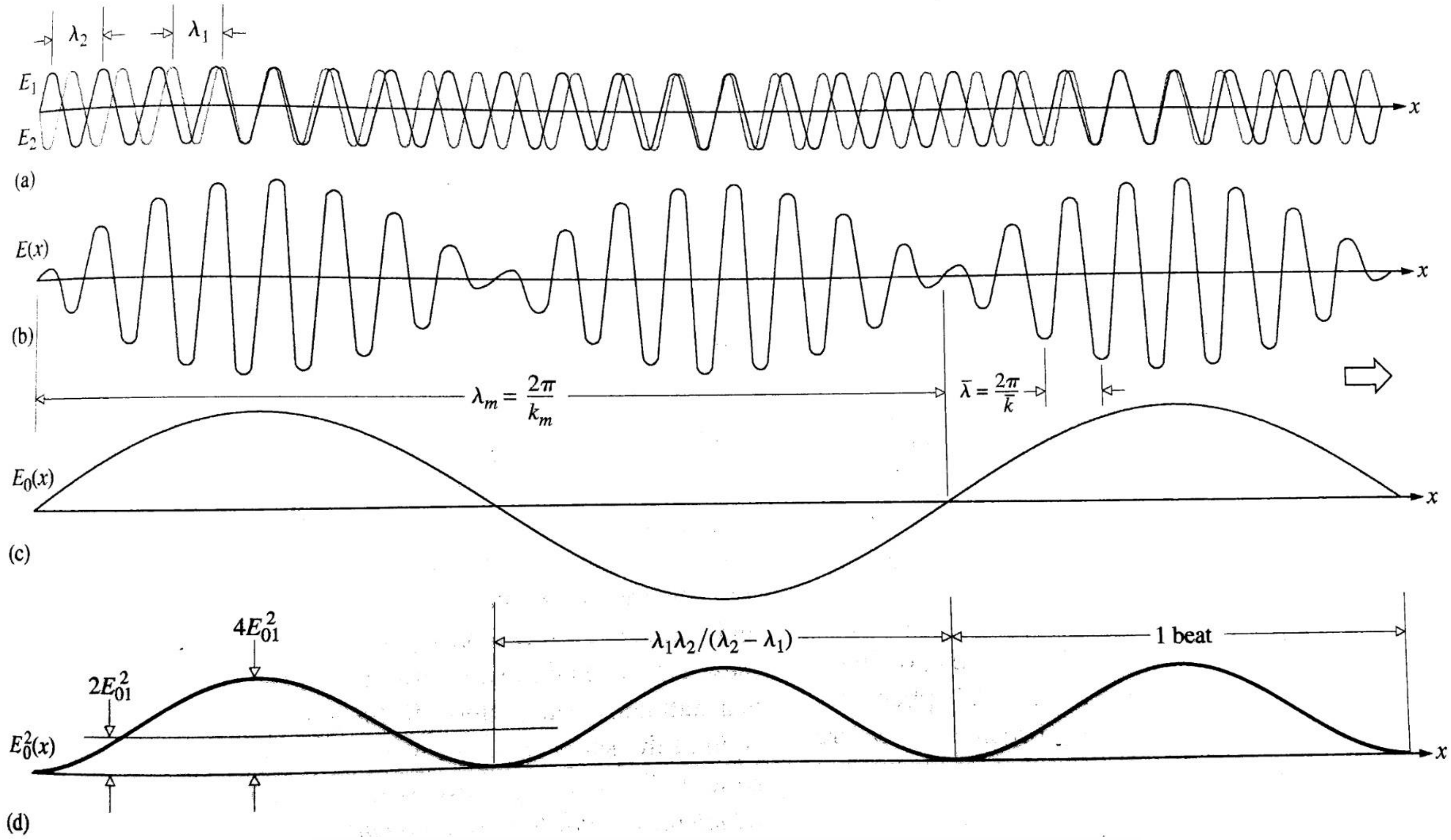
$$z^* = re^{-i\phi} = x - iy \qquad \mathrm{Im}\{z\} = y = (z - z^*)/2i$$



Modulus of a complex value: $r = |z| = (zz^*)^{1/2}$

Argument of a complex value: $\phi = tan^{-1}\left(\frac{y}{x}\right) = tan^{-1}\left(-i\frac{z - z^*}{z + z^*}\right)$

Superposition of two scalar monochromatic waves



$$E = 2E_{01} \underbrace{\cos(k_m x - \omega_m t)}_{\text{modulation/beat/envelope}} \underbrace{\cos(\bar{k}x - \bar{\omega}t)}_{\text{carrier}}$$

Superposition of two scalar monochromatic waves

$$E e^{i\phi} = E_1 e^{i\phi_1} + E_2 e^{i\phi_2}$$

$$E^2 = {E_1}^2 + {E_2}^2 + 2E_1 E_2 \cos(\phi_2 - \phi_1)$$

$$\phi = \tan^{-1}\left(\frac{E_1 \sin\phi_1 + E_2 \sin\phi_2}{E_1 \cos\phi_1 + E_2 \cos\phi_2}\right)$$

Note: detectors measure irradiance, not field amplitude

$$I = \frac{c\epsilon}{2} E^2 \longrightarrow I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos(\phi_2 - \phi_1)$$

Sometimes it's convenient to define a scaled field

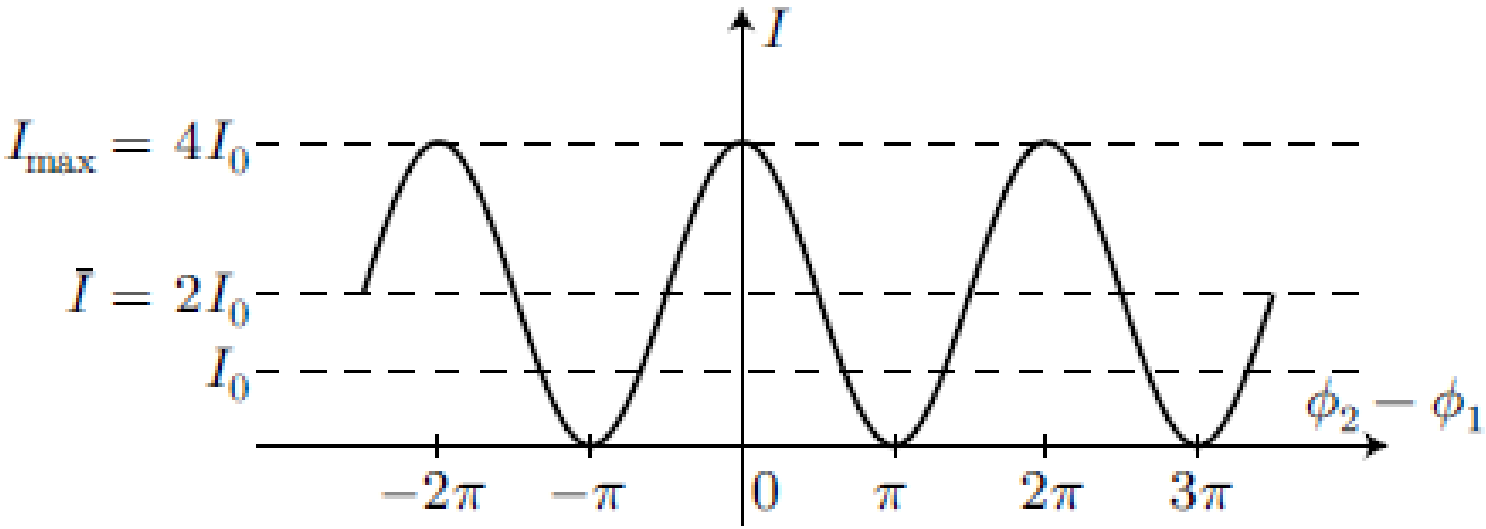
$$U = \sqrt{\frac{c\epsilon}{2}}\, E$$

Superposition of two scalar monochromatic waves

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos(\phi_2 - \phi_1)$$

If we take $I_0 = I_1 = I_2$ then

$$I = 2I_0[1 + \cos(\phi_2 - \phi_1)]$$



Superposition of many scalar monochromatic waves

$$E e^{i\phi} = \sum_n E_n e^{i\phi_n}$$

n=2 case:

$$E^2 = {E_1}^2 + {E_2}^2 + 2E_1E_2\cos(\phi_2 - \phi_1)$$

$$\phi = \tan^{-1}\left(\frac{E_1\sin\phi_1 + E_2\sin\phi_2}{E_1\cos\phi_1 + E_2\cos\phi_2}\right)$$

Summing up a lot of sources, the result depends a lot of the relative phase, or *coherence* of the source elements

$$E^2 = \sum_n {E_n}^2 + 2\sum_{l>n}\sum_n E_nE_l\cos(\phi_n - \phi_l)$$

$$\phi = \tan^{-1}\left(\frac{\sum_n E_n\sin\phi_n}{\sum_n E_n\cos\phi_n}\right)$$

For N sources with random phase, this last term averages to zero!

$$E^2 = N{E_0}^2$$

If all fields are coherent at a single point in space:

$$E^2 = N^2{E_0}^2$$

Standing Waves

Superposition of opposite-direction traveling waves creates a **standing wave**

$$E = Re\{E_R e^{i(kx-\omega t+\delta)} + E_L e^{i(kx+\omega t+\delta)}\}$$

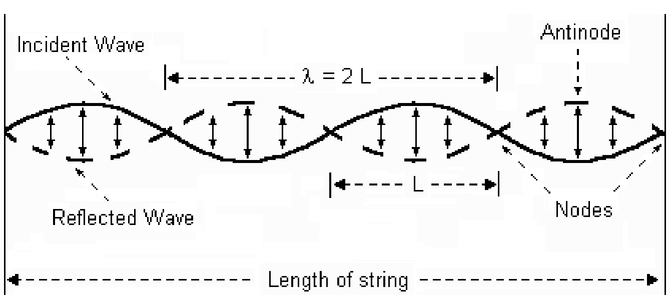
Assume equal field amplitudes…

$$E = E_0\ Re\{e^{i(kx+\delta)}(e^{i\omega t} + e^{-i\omega t})\}$$

Assume δ=π/2…

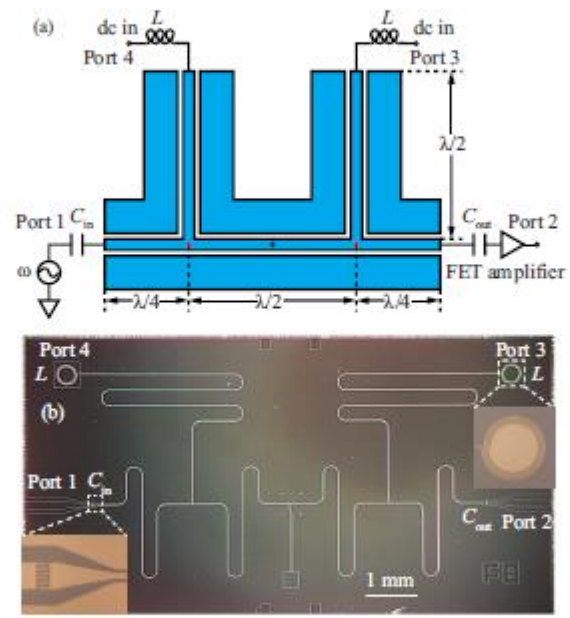
$$E = 2E_0 \sin kx \cos \omega t$$

Still oscillates in time, but has a fixed sinusoidal envelope.



Research Application

Rimberg group: inject DC current at the nodes of a microwave cavity without causing power leaks!



Video: [Argonne Natl. Labs Acoustic levitation!]

Phase & Group Velocity

Can define many different "propagation" velocities

- Phase velocity
- Group velocity

We'll just focus on these two

- Front velocity
- Energy propagation velocity
- Signal velocity

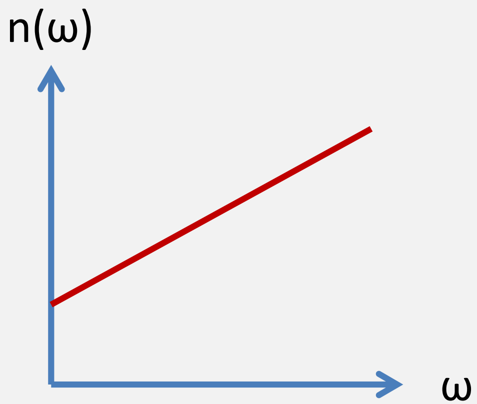
Group Velocity

$$v_g = \frac{v_{\text{phase}}}{1 + \frac{\omega}{n}\frac{dn}{d\omega}}$$

“Normal” Dispersion

$$\frac{dn}{d\omega} > 0$$

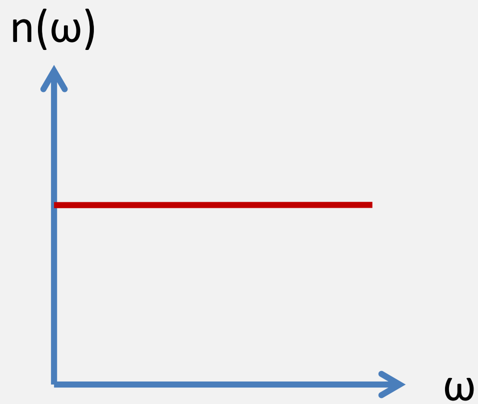
$$v_g < v_{\text{phase}}$$



Zero Dispersion

$$\frac{dn}{d\omega} = 0$$

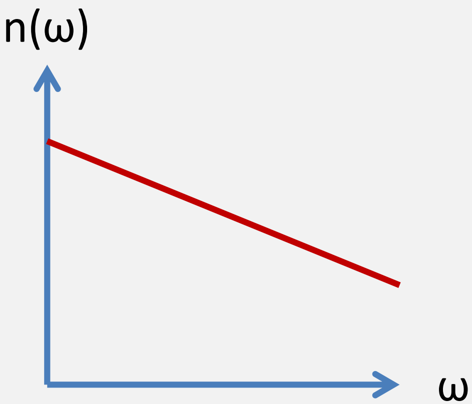
$$v_g = v_{\text{phase}}$$



“Anomalous” Dispersion

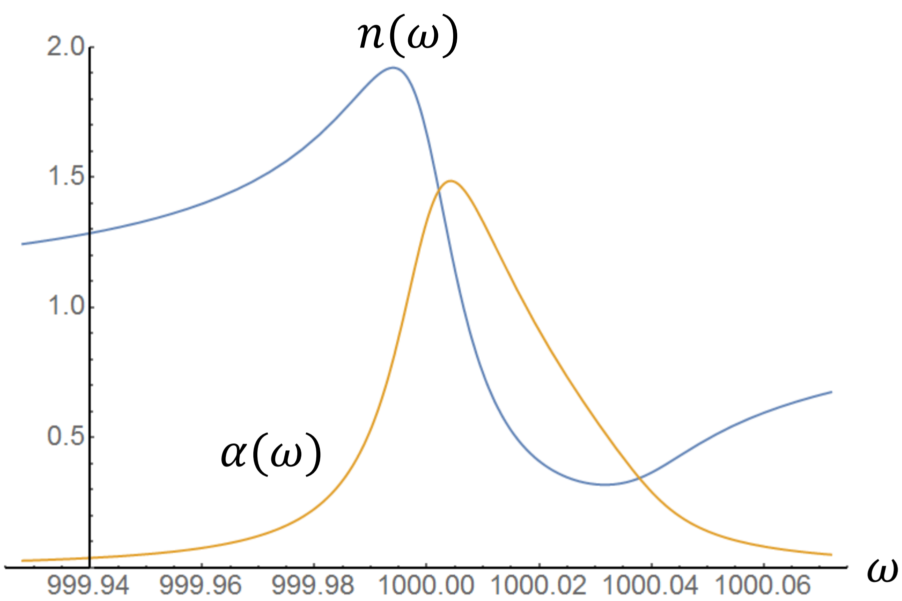
$$\frac{dn}{d\omega} < 0$$

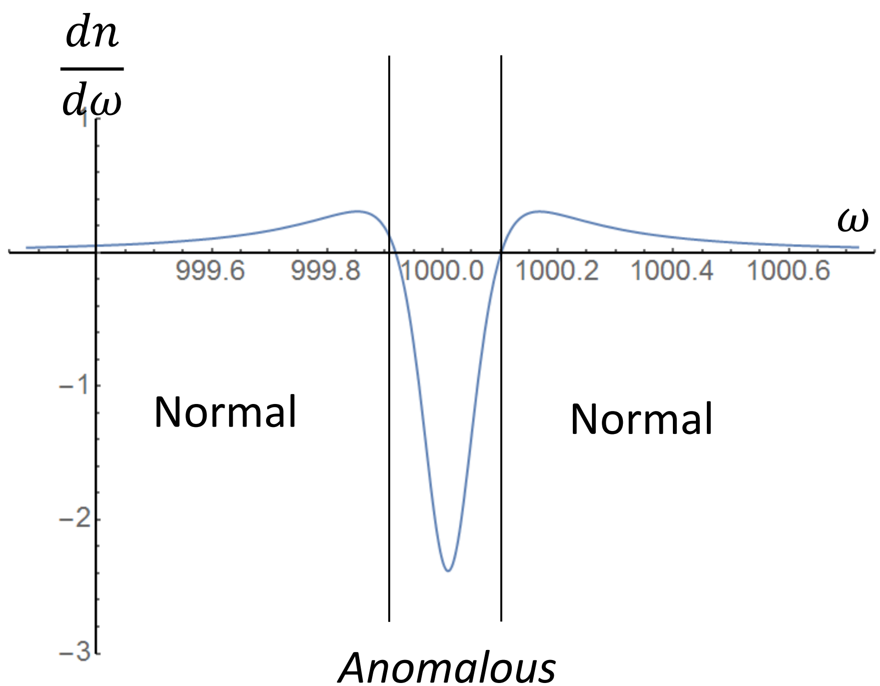
$$v_g > v_{\text{phase}}$$



Dispersion Near a Strong Resonance

$$n(\omega)^2 = 1 + \frac{Ne^2}{\epsilon_0 m}\frac{1}{\omega_0^2 - \omega^2 + i\gamma\omega}$$

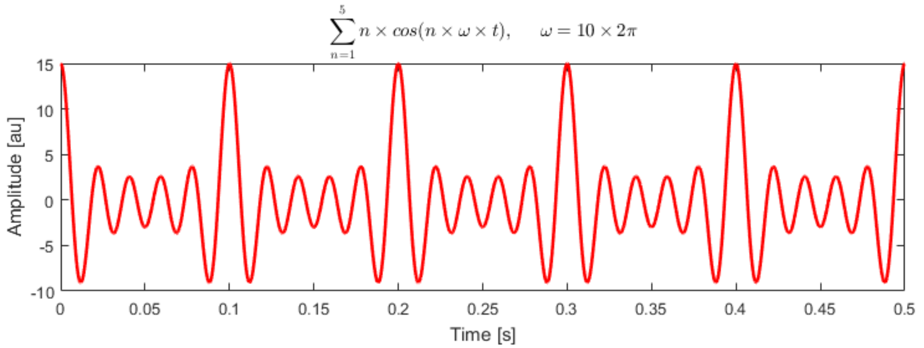


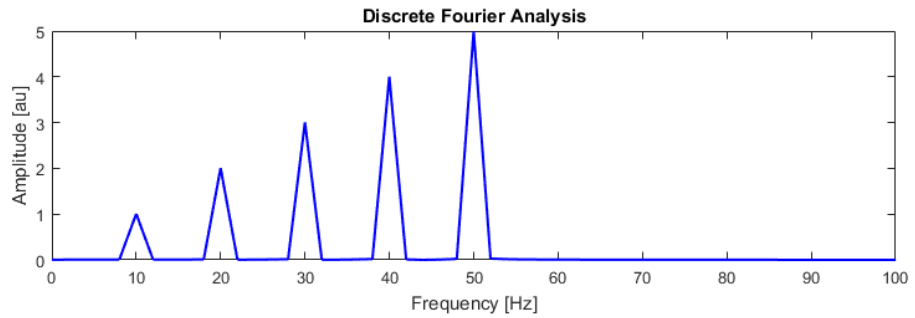


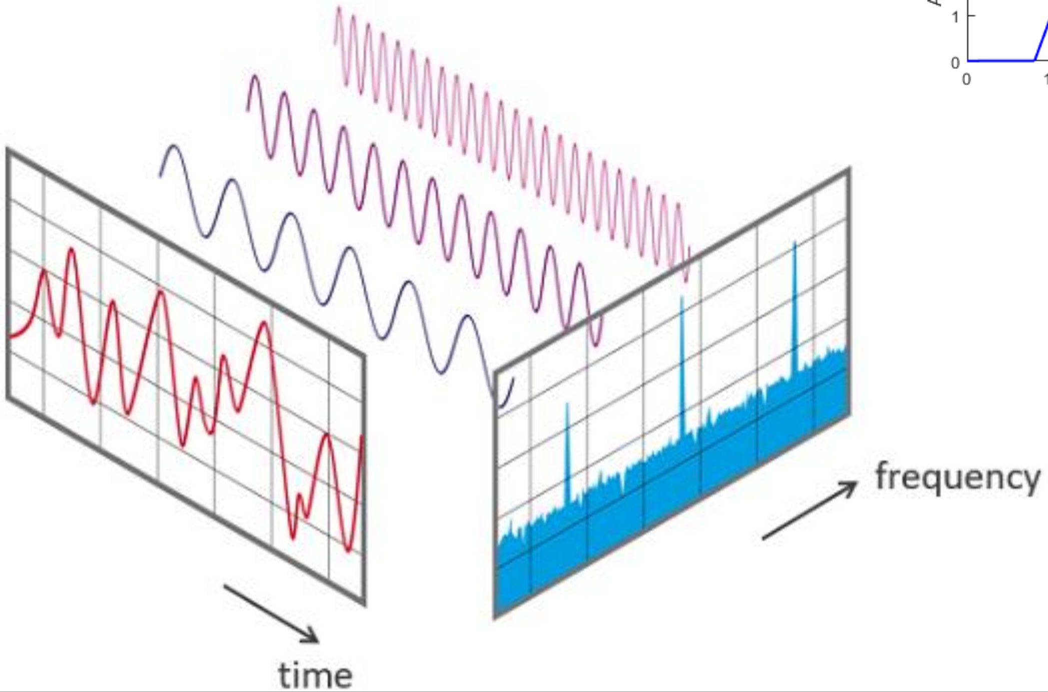
Fourier Analysis

Used to:

- convert from time-domain to frequency-domain
- define power/energy of a waveform
- classify different signal types
- define convolution







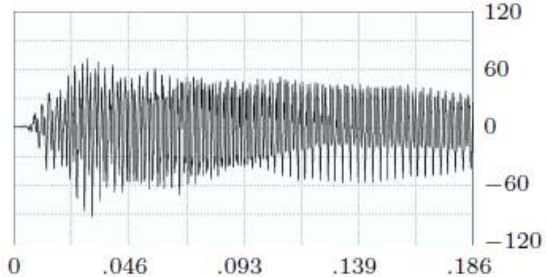
Fourier Analysis

The information in a waveform can be represented in a number of different ways:

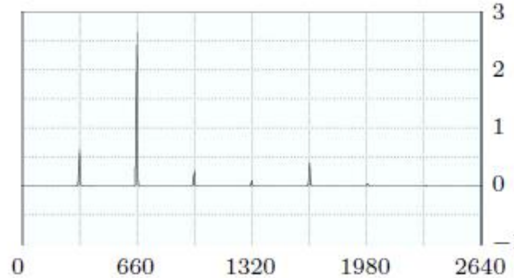
Most instruments acquire data in either the time domain (oscilloscope) or frequency domain (spectrometer)

Sometimes it's convenient to look at the data in a mixed representation (spectrogram)

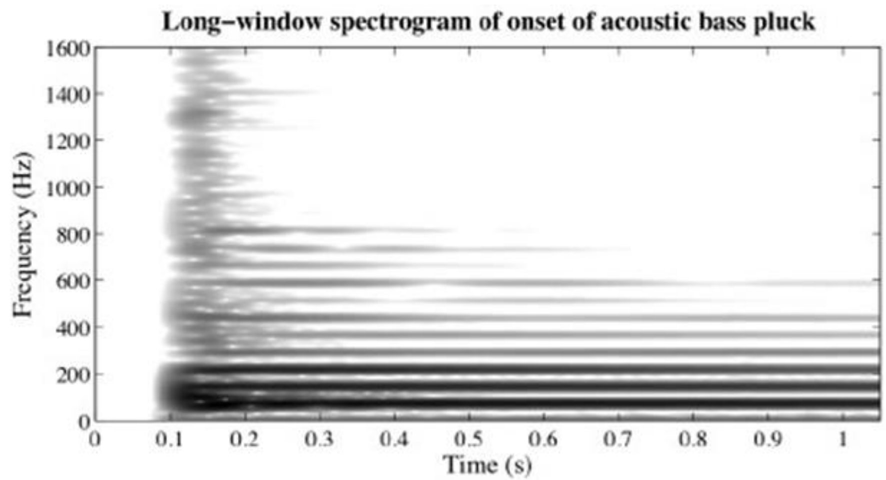
Piano playing E4 (329.63 Hz)



Time domain signal

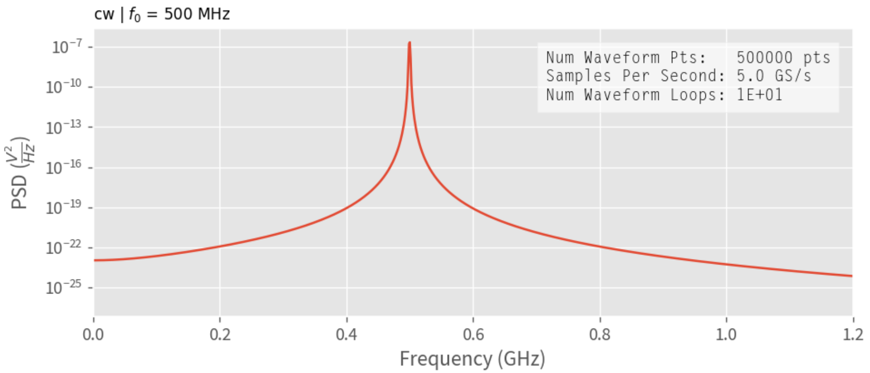


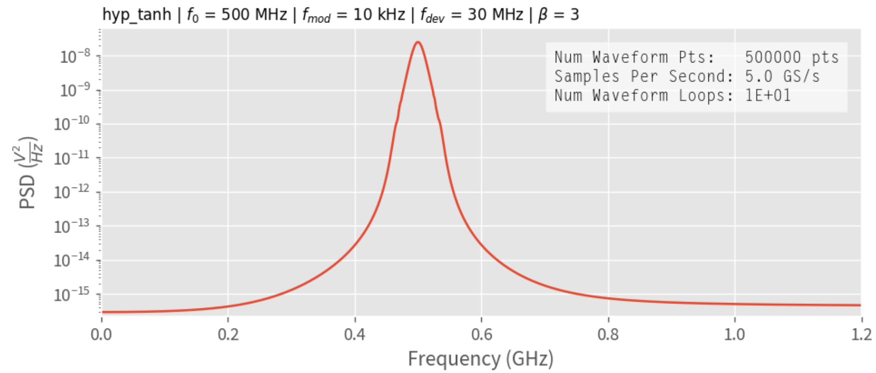
Frequency domain signal

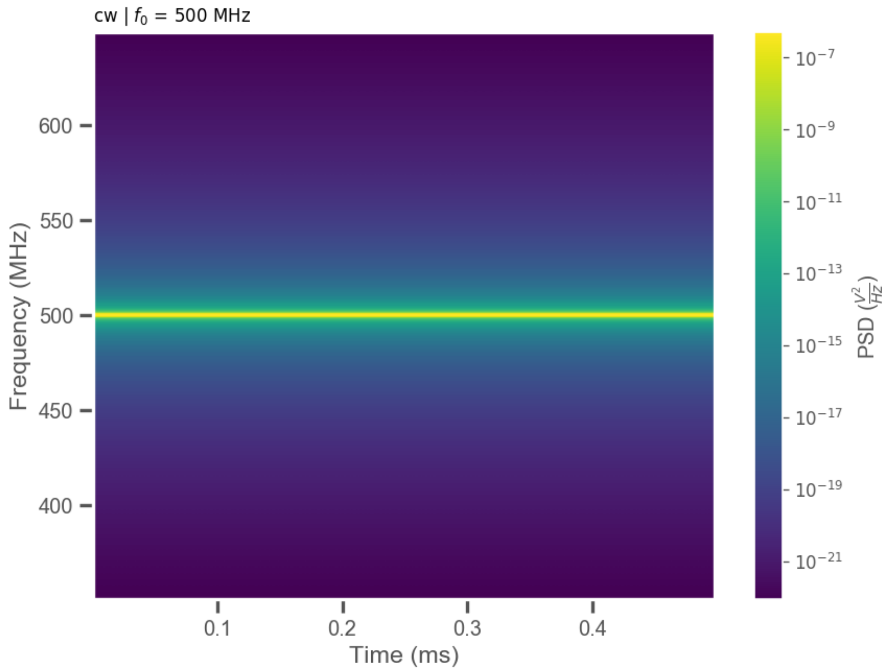


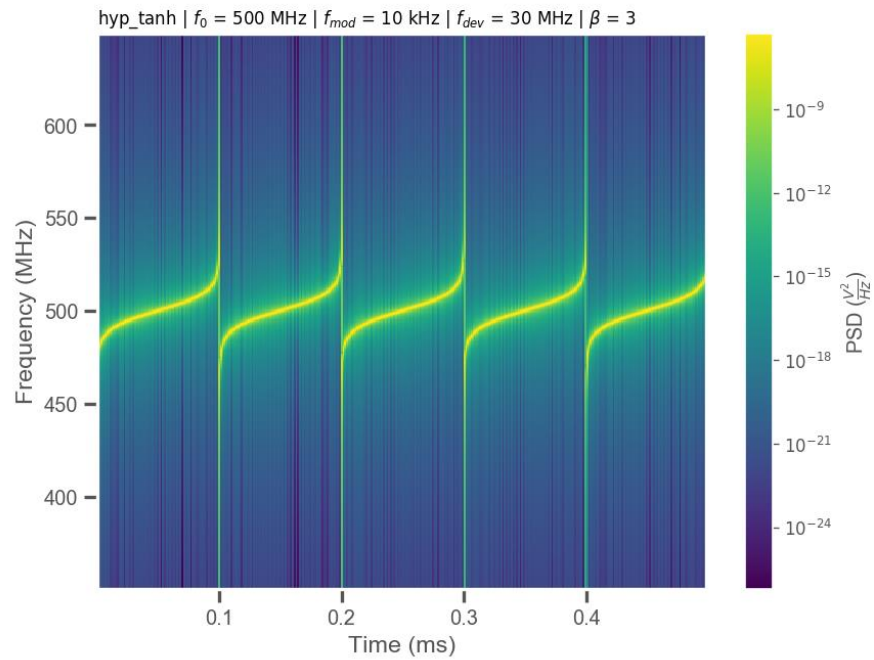
Spectrogram: Time and Frequency
(Have to specify a window function)

Fourier Analysis - Spectrogram

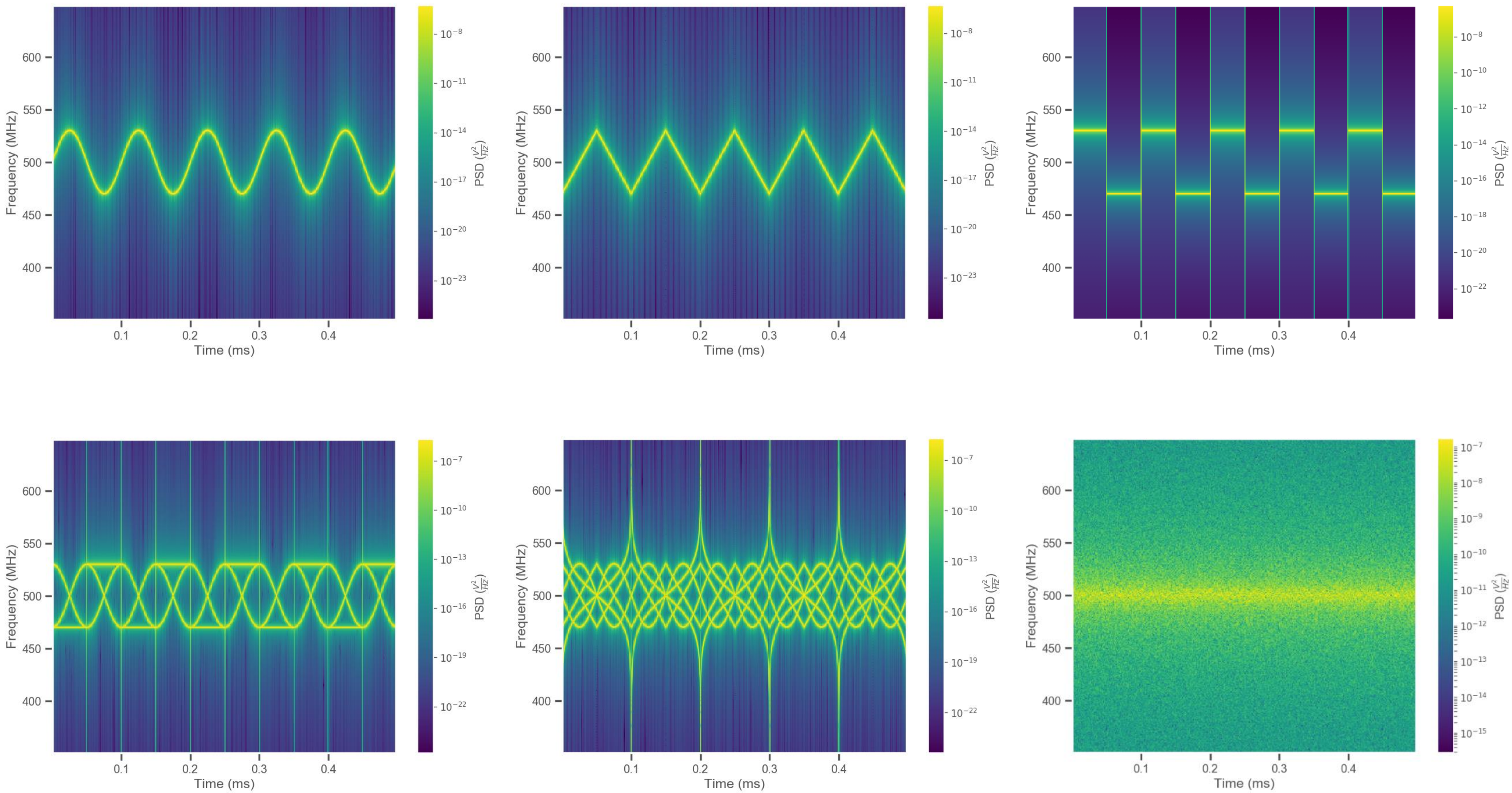








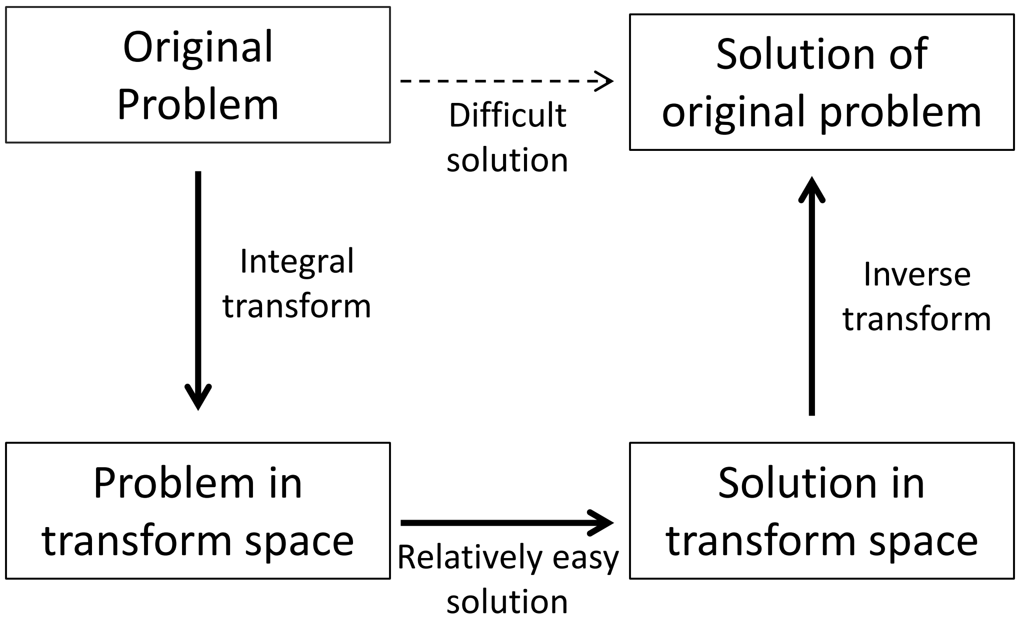
Fourier Analysis - Spectrogram



Fourier Analysis

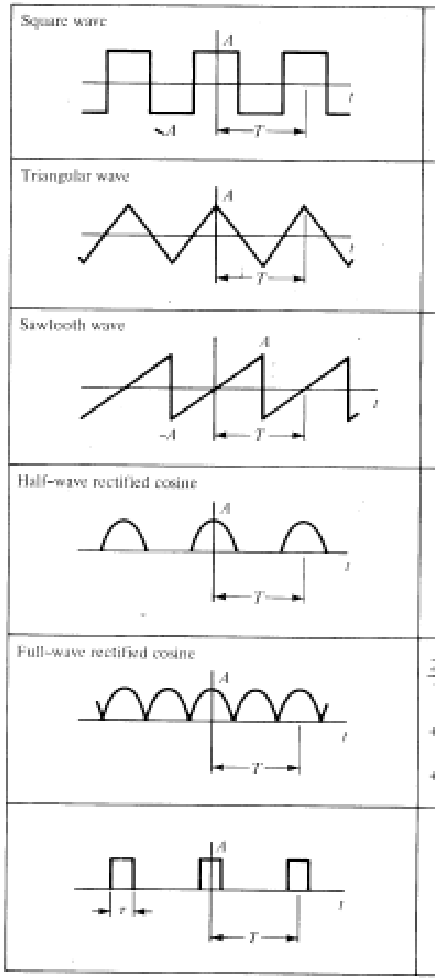
Many transforms:

- Fourier Transform
- Laplace Transform
- Hilbert transform
- And many more…

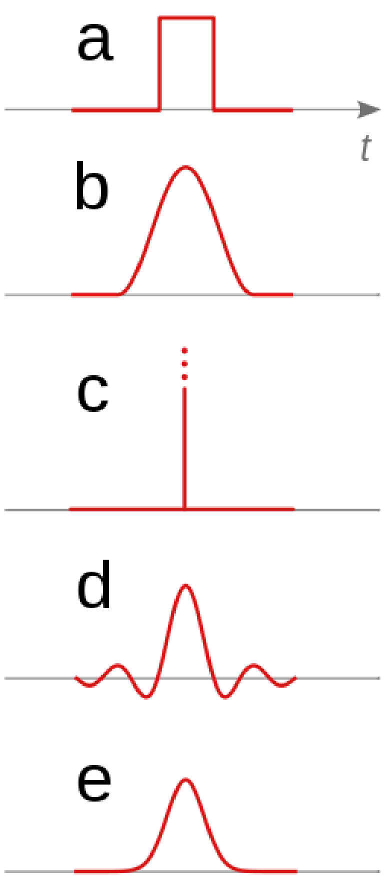


Fourier Analysis

Fourier Series
(periodic waveforms)



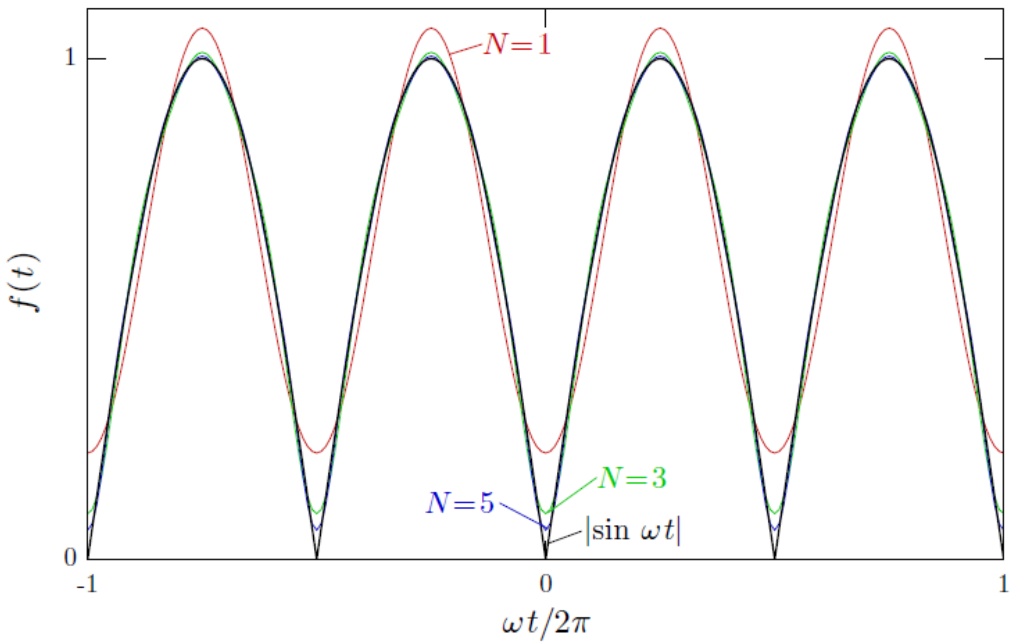
Fourier Transform
(non-periodic waveforms - pulses)



a) Rectangular
b) Cosine squared
c) Dirac delta function
d) Sinc pulse
e) **Gaussian pulse**

Fourier Analysis – Ex: Rectified Sine Wave

$$f(t) = |\sin \omega t|$$



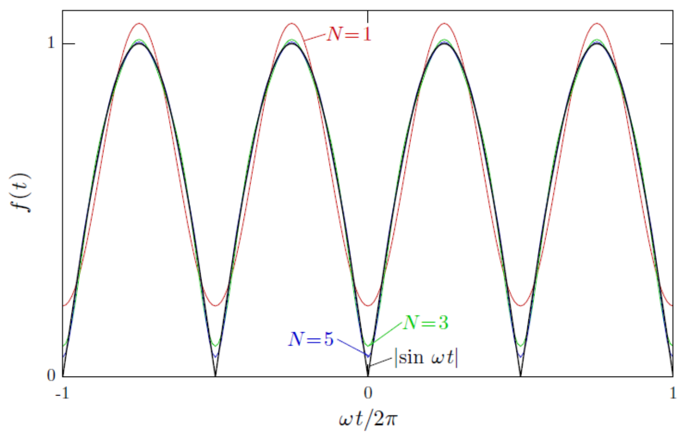
Even function with period π/ω, so only even harmonics allowed, right from the start…

Fourier Analysis – Ex: Rectified Sine Wave

$$\mathrm{f}(t) = |\sin \omega t|$$

Even function with period π/ω, so only even harmonics allowed, right from the start…

$$\begin{aligned}
c_n &= \frac{1}{T}\int_0^T |\sin \omega t| e^{in(2\omega)t}\, dt \\
&= \frac{1}{T}\int_0^T \frac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right) e^{in(2\omega)t}\, dt \\
&= \frac{1}{2iT}\int_0^T \left(e^{i(2n+1)\omega t} - e^{i(2n-1)\omega t}\right) dt \\
&= \frac{1}{2\pi i}\int_0^\pi \left(e^{i(2n+1)x} - e^{i(2n-1)x}\right) dx \\
&= \frac{1}{2\pi i}\left[\frac{e^{i(2n+1)\pi} - 1}{i(2n+1)} - \frac{e^{i(2n-1)\pi} - 1}{i(2n-1)}\right] \\
&= \frac{1}{\pi(2n+1)} - \frac{1}{\pi(2n-1)} = \frac{2}{\pi(1-4n^2)}.
\end{aligned}$$



$e^{i(2n+1)\pi} = -1$ for all values of n!

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} \frac{2}{\pi(1-4n^2)} e^{-i2n\omega t}}$$

Note that $c_n = c_{-n} = c_n^*$

Because the function is real and even

In general we usually only have $c_n = c^*_{-n}$