Composition of Movement Primitives

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Contents

1 ProMPs

1.1 Recap

From (Paraschos et al., 2013, 2018):

- q_t: joint angle over time
- $\dot{q}_t$: joint velocity over time
- $\boldsymbol{\tau} = \{q_t\}_{t=0\dots T}$: trajectory
- $\boldsymbol{w}$: weight vector of a single trajectory
- ϕ_t: basis function
- $\boldsymbol{\Phi}_t = [\phi_t, \dot{\phi_t}]$: $n \times 2$ dimensional time-dependent basis matrix
- $z(t)$: monotonically increasing phase variable
- $\boldsymbol{\epsilon}_y \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_y)$: zero-mean i.i.d. Gaussian noise

$$\boldsymbol{y}_t = \begin{bmatrix} q_t \\ \dot{q}_t \end{bmatrix} = \boldsymbol{\Phi}_t^\top \boldsymbol{w} + \boldsymbol{\epsilon}_y \tag{1}$$

$$p(\boldsymbol{\tau}|\boldsymbol{w}) = \prod_t \mathcal{N}\Big(\boldsymbol{y}_t | \boldsymbol{\Phi}_t^\top \boldsymbol{w}, \boldsymbol{\Sigma}_y\Big) \tag{2}$$

$$p(\boldsymbol{\tau}; \boldsymbol{\theta}) = \int p(\boldsymbol{\tau}|\boldsymbol{w}) \cdot p(\boldsymbol{w}; \boldsymbol{\theta}) d\boldsymbol{w} \tag{3}$$

1.2 Coupling between joints

$$p(\boldsymbol{y}_t|\boldsymbol{w}) = \mathcal{N}\left(\begin{bmatrix} \boldsymbol{y}_{1,t} \\ \vdots \\ \boldsymbol{y}_{d,t} \end{bmatrix} \middle| \begin{bmatrix} \boldsymbol{\Phi}_t^\top & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \boldsymbol{\Phi}_t^\top \end{bmatrix} \boldsymbol{w}, \boldsymbol{\Sigma}_y \right) = \mathcal{N}\Big(\boldsymbol{y}_t|\boldsymbol{\Psi}_t\boldsymbol{w}, \boldsymbol{\Sigma}_y\Big) \tag{4}$$

with:

- $\boldsymbol{w} = [\boldsymbol{w}_1^\top, \ldots, \boldsymbol{w}_n^\top]^\top$: combined weight vector
- $\boldsymbol{\Phi}_t$: block-diagonal basis matrix containing the basis functions and their derivatives for each dimension
- $\boldsymbol{y}_{i,t} = [q_{i,t}, \dot{q}_{i,t}]^\top$: joint angle and velocity for the i^{th} joint

1.3 Hierarchical Bayesian Model

The Hierarchical Bayesian Model used in ProMPs is illustrated in Fig. 1.

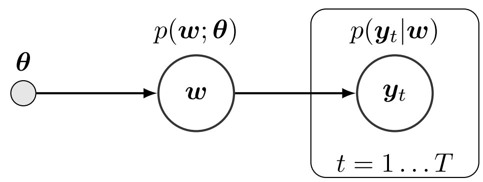

Figure 1: Hierarchical Bayesian Model used in ProMPs.

- $\boldsymbol{\theta} = \{\boldsymbol{\mu}_w, \boldsymbol{\Sigma}_w\}$
- $p(\boldsymbol{w};\boldsymbol{\theta}) = \mathcal{N}(\boldsymbol{w}|\boldsymbol{\mu}_w, \boldsymbol{\Sigma}_w)$: prior over the weight vector $\boldsymbol{w}$, with parameters $\boldsymbol{\theta}$, assumed to be Gaussian

$$p(\boldsymbol{y}_t;\boldsymbol{\theta}) = \int \mathcal{N}\Big(\boldsymbol{y}_t|\boldsymbol{\Psi}_t^\top\boldsymbol{w}, \boldsymbol{\Sigma}_y\Big) \cdot p(\boldsymbol{w};\boldsymbol{\theta})\; d\boldsymbol{w} \tag{5}$$

$$= \int \mathcal{N}\Big(\boldsymbol{y}_t|\boldsymbol{\Psi}_t^\top\boldsymbol{w}, \boldsymbol{\Sigma}_y\Big) \cdot \mathcal{N}\Big(\boldsymbol{w}|\boldsymbol{\mu}_{\boldsymbol{w}}, \boldsymbol{\Sigma}_{\boldsymbol{w}}\Big)\; d\boldsymbol{w} \tag{6}$$

$$= \mathcal{N}\Big(\boldsymbol{y}_t|\boldsymbol{\Psi}_t^\top\boldsymbol{\mu}_{\boldsymbol{w}}, \boldsymbol{\Psi}_t^\top\boldsymbol{\Sigma}_{\boldsymbol{w}}\boldsymbol{\Psi}_t + \boldsymbol{\Sigma}_y\Big) \tag{7}$$

See Appendix A for the proof.

1.4 Via-Points Modulation

- $\boldsymbol{x}_t^\star = [\boldsymbol{y}_t^\star, \boldsymbol{\Sigma}_t^\star]$: desired observation
- $\boldsymbol{y}_t^\star$: desired position and velocity vector at time t
- $\boldsymbol{\Sigma}_t^\star$: accuracy of the desired observation

Using Bayes rule:

$$p(\boldsymbol{w}|\boldsymbol{x}_t^\star) = \frac{p(\boldsymbol{x}_t^\star|\boldsymbol{w}) \cdot p(\boldsymbol{w})}{p(\boldsymbol{x}_t^\star)} \tag{8}$$

$$p(\boldsymbol{w}|\boldsymbol{x}_t^\star) \propto \mathcal{N}\Big(\boldsymbol{y}_t^\star|\boldsymbol{\Psi}_t^\top\boldsymbol{w}, \boldsymbol{\Sigma}_t^\star\Big) \cdot \mathcal{N}(\boldsymbol{w}|\boldsymbol{\mu}_w, \boldsymbol{\Sigma}_w) \tag{9}$$

$$\boldsymbol{\mu}_{\boldsymbol{w}}^{[new]} = \boldsymbol{\mu}_{\boldsymbol{w}} + \boldsymbol{\Sigma}_{\boldsymbol{w}}\boldsymbol{\Psi}_t\Big(\boldsymbol{\Sigma}_y^\star + \boldsymbol{\Psi}_t^\top\boldsymbol{\Sigma}_{\boldsymbol{w}}\boldsymbol{\Psi}_t\Big)^{-1}(\boldsymbol{y}_t^\star - \boldsymbol{\Psi}_t^\top\boldsymbol{\mu}_{\boldsymbol{w}}) \tag{10}$$

$$\boldsymbol{\Sigma}_{\boldsymbol{w}}^{[new]} = \boldsymbol{\Sigma}_{\boldsymbol{w}} - \boldsymbol{\Sigma}_{\boldsymbol{w}}\boldsymbol{\Psi}_t\Big(\boldsymbol{\Sigma}_y^\star + \boldsymbol{\Psi}_t^\top\boldsymbol{\Sigma}_{\boldsymbol{w}}\boldsymbol{\Psi}_t\Big)^{-1}\boldsymbol{\Psi}_t^\top\boldsymbol{\Sigma}_{\boldsymbol{w}} \tag{11}$$

See Appendix B for the proof.

1.4.1 Do we actually get the desired mean by applying the conditioning update?

$$\boldsymbol{\mu}_{\boldsymbol{w}}^{[new]}(t) = \boldsymbol{\mu}_{\boldsymbol{w}} + \boldsymbol{\Sigma}_{\boldsymbol{w}}\boldsymbol{\Psi}_t\left(\boldsymbol{\Sigma}_t^{\star}\boldsymbol{\Psi}_t^{\top}\boldsymbol{\Sigma}_{\boldsymbol{w}}\boldsymbol{\Psi}_t\right)^{-1}(\boldsymbol{y}_t^{\star} - \boldsymbol{\Psi}_t^{\top}\boldsymbol{\mu}_{\boldsymbol{w}}) \tag{12}$$

$$\stackrel{?}{=} \boldsymbol{y}_t^{\star} \tag{13}$$

ToDo

1.4.2 Does the mean accuracy change if the number of via-points increase?

2 Composition of MPs

2.1 Blending

2.2 Stitching

References

A. Paraschos, C. Daniel, J. R. Peters, and G. Neumann, "Probabilistic Movement Primitives," in *Advances in Neural Information Processing Systems*, vol. 26. Curran Associates, Inc., 2013. [Online]. Available: https://proceedings.neurips.cc/paper/2013/hash/e53a0a2978c28872a4505bdb51db06dc-Abstract.html

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A Hierarchical Bayesian Model proof

Proof of Eq. (7). From (Deisenroth et al., 2020), we have the joint distribution:

$$p(\mathbf{x}_a, \mathbf{x}_b) = \mathcal{N}\left(\begin{bmatrix}\boldsymbol{\mu}_a \\ \boldsymbol{\mu}_b\end{bmatrix}, \begin{bmatrix}\boldsymbol{\Sigma}_{aa} & \boldsymbol{\Sigma}_{ab} \\ \boldsymbol{\Sigma}_{ba} & \boldsymbol{\Sigma}_{bb}\end{bmatrix}\right) \tag{14}$$

and the marginal distribution $p(\mathbf{x}_a)$ of a joint Gaussian distribution $p(\mathbf{x}_a, \mathbf{x}_b)$:

$$p(\mathbf{x}_a) = \int p(\mathbf{x}_a, \mathbf{x}_b) d\mathbf{x}_b = \mathcal{N}(\mathbf{x}_a | \boldsymbol{\mu}_a, \boldsymbol{\Sigma}_{aa}) \tag{15}$$

Since $\boldsymbol{y}_t$ and $\boldsymbol{w}$ are jointly Gaussian, we have:

$$\begin{bmatrix}\boldsymbol{y}_t \\ \boldsymbol{w}\end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix}\boldsymbol{\mu}_{\boldsymbol{y}_t} \\ \boldsymbol{\mu}_{\boldsymbol{w}}\end{bmatrix}, \begin{bmatrix}\mathrm{Cov}[\boldsymbol{y}_t, \boldsymbol{y}_t] & \mathrm{Cov}[\boldsymbol{y}_t, \boldsymbol{w}] \\ \mathrm{Cov}[\boldsymbol{w}, \boldsymbol{y}_t] & \mathrm{Cov}[\boldsymbol{w}, \boldsymbol{w}]\end{bmatrix}\right) \tag{16}$$

$$
\begin{aligned}
\boldsymbol{\mu}_{\boldsymbol{y}_t} &= \mathbb{E}[\boldsymbol{y}_t] && (17)\\
&= \mathbb{E}[\boldsymbol{\Psi}_t^\top \boldsymbol{w} + \boldsymbol{\epsilon}_y] && (18)\\
&= \boldsymbol{\Psi}_t^\top \mathbb{E}[\boldsymbol{w}] + \mathbb{E}[\boldsymbol{\epsilon}_y] && (19)\\
&= \boldsymbol{\Psi}_t^\top \boldsymbol{\mu}_{\boldsymbol{w}} + 0 && (20)\\
&= \boldsymbol{\Psi}_t^\top \boldsymbol{\mu}_{\boldsymbol{w}} && (21)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Cov}[\boldsymbol{y}_t, \boldsymbol{y}_t] &= \mathrm{Cov}[\boldsymbol{\Psi}_t^\top \boldsymbol{w} + \boldsymbol{\epsilon}_y] && (22)\\
&= \mathrm{Cov}[\boldsymbol{\Psi}_t^\top \boldsymbol{w}] + \mathrm{Cov}[\boldsymbol{\epsilon}_y] && (23)\\
&= \boldsymbol{\Psi}_t^\top \mathrm{Cov}[\boldsymbol{w}] \boldsymbol{\Psi}_t + \boldsymbol{\Sigma}_y && (24)\\
&= \boldsymbol{\Psi}_t^\top \boldsymbol{\Sigma}_{\boldsymbol{w}} \boldsymbol{\Psi}_t + \boldsymbol{\Sigma}_y && (25)
\end{aligned}
$$

$$
\begin{bmatrix} \boldsymbol{y}_t \\ \boldsymbol{w} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \boldsymbol{\Psi}_t^\top \boldsymbol{\mu}_{\boldsymbol{w}} \\ \boldsymbol{\mu}_{\boldsymbol{w}} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Psi}_t^\top \boldsymbol{\Sigma}_{\boldsymbol{w}} \boldsymbol{\Psi}_t + \boldsymbol{\Sigma}_y & \boldsymbol{\Psi}_t^\top \boldsymbol{\Sigma}_{\boldsymbol{w}} \\ \boldsymbol{\Sigma}_{\boldsymbol{w}} \boldsymbol{\Psi}_t & \boldsymbol{\Sigma}_{\boldsymbol{w}} \end{bmatrix} \right) \tag{26}
$$

$$
p(\boldsymbol{y}_t; \boldsymbol{\theta}) = \mathcal{N}\left(\boldsymbol{y}_t | \boldsymbol{\Psi}_t^\top \boldsymbol{\mu}_{\boldsymbol{w}}, \boldsymbol{\Psi}_t^\top \boldsymbol{\Sigma}_{\boldsymbol{w}} \boldsymbol{\Psi}_t + \boldsymbol{\Sigma}_y \right) \tag{27}
$$

□

B Via-Points conditioning proof

Proof of Eq. (10) *and Eq.* (11). With the joint distribution $p(\mathrm{x}_a, \mathrm{x}_b)$ in Eq. (14), and from (Bishop and Bishop, 2024), the parameters of a conditional multivariate Gaussian $p(\mathrm{x}_a|\mathrm{x}_b) = \mathcal{N}(\boldsymbol{\mu}_{a|b}, \boldsymbol{\Sigma}_{a|b})$ are the following:

$$
\begin{aligned}
\boldsymbol{\mu}_{a|b} &= \boldsymbol{\mu}_a + \boldsymbol{\Sigma}_{ab} \boldsymbol{\Sigma}_{bb}^{-1} (\mathrm{x}_b - \boldsymbol{\mu}_b) && (28)\\
\boldsymbol{\Sigma}_{a|b} &= \boldsymbol{\Sigma}_{aa} - \boldsymbol{\Sigma}_{ab} \boldsymbol{\Sigma}_{bb}^{-1} \boldsymbol{\Sigma}_{ba} && (29)
\end{aligned}
$$

We want the posterior $p(\boldsymbol{w}|\boldsymbol{x}_t^\star)$, knowing the likelihood $\boldsymbol{x}_t^\star|\boldsymbol{w} \sim \mathcal{N}\left(\boldsymbol{y}_t^\star | \boldsymbol{\Psi}_t^\top \boldsymbol{w}, \boldsymbol{\Sigma}_t^\star\right)$, and the prior $\boldsymbol{w} \sim \mathcal{N}(\boldsymbol{w}|\boldsymbol{\mu}_w, \boldsymbol{\Sigma}_w)$.

$$
\begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{x}_t^\star \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \boldsymbol{\mu}_{\boldsymbol{w}} \\ \boldsymbol{\Psi}_t^\top \boldsymbol{\mu}_{\boldsymbol{w}} \end{bmatrix}, \begin{bmatrix} \mathrm{Cov}[\boldsymbol{w}, \boldsymbol{w}] & \mathrm{Cov}[\boldsymbol{w}, \boldsymbol{x}_t^\star] \\ \mathrm{Cov}[\boldsymbol{x}_t^\star, \boldsymbol{w}] & \mathrm{Cov}[\boldsymbol{x}_t^\star, \boldsymbol{x}_t^\star] \end{bmatrix} \right) \tag{30}
$$

$\mathrm{Cov}[\boldsymbol{x}_t^\star, \boldsymbol{x}_t^\star]$ follows from Eq. (25).

$$
\begin{aligned}
\mathrm{Cov}[\boldsymbol{w}, \boldsymbol{x}_t^\star] &= \mathrm{Cov}[\boldsymbol{w}, \boldsymbol{\Psi}_t^\top \boldsymbol{w} + \boldsymbol{\epsilon}_y] && (31)\\
&= \mathrm{Cov}[\boldsymbol{w}, \boldsymbol{\Psi}_t^\top \boldsymbol{w}] \qquad (\mathrm{Cov}[\boldsymbol{w}, \boldsymbol{\epsilon}_{\boldsymbol{y}}] = 0 \text{ since } \boldsymbol{\epsilon}_{\boldsymbol{y}} \text{ is independent of } \boldsymbol{w}) && (32)\\
&= \mathbb{E}[(\boldsymbol{w} - \boldsymbol{\mu}_{\boldsymbol{w}})(\boldsymbol{\Psi}_t^\top \boldsymbol{w} - \boldsymbol{\Psi}_t^\top \boldsymbol{\mu}_{\boldsymbol{w}})^\top] && (33)\\
&= \mathbb{E}[(\boldsymbol{w} - \boldsymbol{\mu}_{\boldsymbol{w}})(\boldsymbol{w} - \boldsymbol{\mu}_{\boldsymbol{w}})^\top \boldsymbol{\Psi}_t] && (34)\\
&= \mathrm{Cov}[\boldsymbol{w}, \boldsymbol{w}] \cdot \boldsymbol{\Psi}_t && (35)\\
&= \boldsymbol{\Sigma}_{\boldsymbol{w}} \boldsymbol{\Psi}_t && (36)
\end{aligned}
$$

$$
\begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{x}_t^\star \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \boldsymbol{\mu}_{\boldsymbol{w}} \\ \boldsymbol{\Psi}_t^\top \boldsymbol{\mu}_{\boldsymbol{w}} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_{\boldsymbol{w}} & \boldsymbol{\Sigma}_{\boldsymbol{w}} \boldsymbol{\Psi}_t \\ \boldsymbol{\Psi}_t^\top \boldsymbol{\Sigma}_{\boldsymbol{w}} & \boldsymbol{\Psi}_t^\top \boldsymbol{\Sigma}_{\boldsymbol{w}} \boldsymbol{\Psi}_t + \boldsymbol{\Sigma}_t^\star \end{bmatrix} \right) \tag{37}
$$

Using Eq. (28) we get:

$$\boldsymbol{\mu}_{\boldsymbol{w}|\boldsymbol{x}_t^\star} = \boldsymbol{\mu}_{\boldsymbol{w}} + \boldsymbol{\Sigma}_{\boldsymbol{w}} \boldsymbol{\Psi}_t \Big(\boldsymbol{\Sigma}_t^\star + \boldsymbol{\Psi}_t^\top \boldsymbol{\Sigma}_{\boldsymbol{w}} \boldsymbol{\Psi}_t \Big)^{-1} (\boldsymbol{y}_t^\star - \boldsymbol{\Psi}_t^\top \boldsymbol{\mu}_{\boldsymbol{w}}) \tag{38}$$

Using Eq. (29) we get:

$$\boldsymbol{\Sigma}_{\boldsymbol{w}|\boldsymbol{x}_t^\star} = \boldsymbol{\Sigma}_{\boldsymbol{w}} - \boldsymbol{\Sigma}_{\boldsymbol{w}} \boldsymbol{\Psi}_t \Big(\boldsymbol{\Sigma}_t^\star + \boldsymbol{\Psi}_t^\top \boldsymbol{\Sigma}_{\boldsymbol{w}} \boldsymbol{\Psi}_t \Big)^{-1} \boldsymbol{\Psi}_t^\top \boldsymbol{\Sigma}_{\boldsymbol{w}} \tag{39}$$

□