

Composition of Movement Primitives

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1 ProMPs

1.1 Recap

From (Paraschos et al., 2013, 2018):

$$\Phi_t = \begin{bmatrix} \phi_1 & \dot{\phi}_1 \\ \vdots & \vdots \\ \phi_n & \dot{\phi}_n \end{bmatrix} \quad (1)$$

Table 1: Notation

q_t	joint angle over time
\dot{q}_t	joint velocity over time
$\tau = \{q_t\}_{t=0\dots T}$	trajectory
\mathbf{w}	weight vector of a single trajectory $[n \times 1]$
ϕ_t	basis function
n	number of basis functions
$\Phi_t = [\phi_t, \dot{\phi}_t]$	$n \times 2$ dimensional time-dependent basis matrix
$z(t)$	monotonically increasing phase variable
$\epsilon_y \sim \mathcal{N}(\mathbf{0}, \Sigma_y)$	zero-mean i.i.d. Gaussian noise

$$\mathbf{y}_t = \begin{bmatrix} q_t \\ \dot{q}_t \end{bmatrix} = \Phi_t^\top \mathbf{w} + \epsilon_y \quad (2)$$

$$p(\tau|\mathbf{w}) = \prod_t \mathcal{N}(\mathbf{y}_t | \Phi_t^\top \mathbf{w}, \Sigma_y) \quad (3)$$

$$p(\tau; \boldsymbol{\theta}) = \int p(\tau|\mathbf{w}) \cdot p(\mathbf{w}; \boldsymbol{\theta}) d\mathbf{w} \quad (4)$$

1.2 Coupling between joints

$$p(\mathbf{y}_t|\mathbf{w}) = \mathcal{N}\left(\begin{bmatrix} \mathbf{y}_{1,t} \\ \vdots \\ \mathbf{y}_{d,t} \end{bmatrix} \middle| \begin{bmatrix} \Phi_t^\top & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \Phi_t^\top \end{bmatrix} \mathbf{w}, \Sigma_y\right) = \mathcal{N}(\mathbf{y}_t | \Psi_t \mathbf{w}, \Sigma_y) \quad (5)$$

with:

Table 2: Notation

$\mathbf{w} = [\mathbf{w}_1^\top, \dots, \mathbf{w}_n^\top]^\top$	combined weight vector $[n \times n]$
Ψ_t	block-diagonal basis matrix containing the basis functions and their derivatives for each dimension
$\mathbf{y}_{i,t} = [q_{i,t}, \dot{q}_{i,t}]^\top$	joint angle and velocity for the i^{th} joint

1.3 Hierarchical Bayesian Model

The Hierarchical Bayesian Model used in ProMPs is illustrated in Fig. 1.

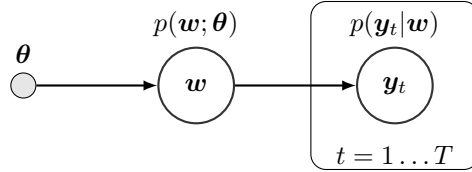


Figure 1: Hierarchical Bayesian Model used in ProMPs.

$$p(\mathbf{y}_t; \boldsymbol{\theta}) = \int \mathcal{N}(\mathbf{y}_t | \Psi_t^\top \mathbf{w}, \Sigma_y) \cdot p(\mathbf{w}; \boldsymbol{\theta}) d\mathbf{w} \quad (6)$$

$$= \int \mathcal{N}(\mathbf{y}_t | \Psi_t^\top \mathbf{w}, \Sigma_y) \cdot \mathcal{N}(\mathbf{w} | \boldsymbol{\mu}_w, \Sigma_w) d\mathbf{w} \quad (7)$$

$$= \mathcal{N}(\mathbf{y}_t | \Psi_t^\top \boldsymbol{\mu}_w, \Psi_t^\top \Sigma_w \Psi_t + \Sigma_y) \quad (8)$$

See Appendix A for the proof.

Table 3: Notation

$\boldsymbol{\theta} = \{\boldsymbol{\mu}_w, \Sigma_w\}$	parameters
$p(\mathbf{w}; \boldsymbol{\theta}) = \mathcal{N}(\mathbf{w} \boldsymbol{\mu}_w, \Sigma_w)$	prior over the weight vector \mathbf{w} , with parameters $\boldsymbol{\theta}$, assumed to be Gaussian

Table 4: Notation

$\mathbf{x}_t^* = [\mathbf{y}_t^*, \boldsymbol{\Sigma}_t^*]$	desired observation
\mathbf{y}_t^*	desired position and velocity vector at time t
$\boldsymbol{\Sigma}_t^*$	accuracy of the desired observation

1.4 Via-Points Modulation

Using Bayes rule:

$$p(\mathbf{w}|\mathbf{x}_t^*) = \frac{p(\mathbf{x}_t^*|\mathbf{w}) \cdot p(\mathbf{w})}{p(\mathbf{x}_t^*)} \quad (9)$$

$$p(\mathbf{w}|\mathbf{x}_t^*) \propto \mathcal{N}(\mathbf{y}_t^*|\boldsymbol{\Psi}_t^\top \mathbf{w}, \boldsymbol{\Sigma}_t^*) \cdot \mathcal{N}(\mathbf{w}|\boldsymbol{\mu}_w, \boldsymbol{\Sigma}_w) \quad (10)$$

$$\boldsymbol{\mu}_w^{[new]} = \boldsymbol{\mu}_w + \boldsymbol{\Sigma}_w \boldsymbol{\Psi}_t \left(\boldsymbol{\Sigma}_y^* + \boldsymbol{\Psi}_t^\top \boldsymbol{\Sigma}_w \boldsymbol{\Psi}_t \right)^{-1} (\mathbf{y}_t^* - \boldsymbol{\Psi}_t^\top \boldsymbol{\mu}_w) \quad (11)$$

$$\boldsymbol{\Sigma}_w^{[new]} = \boldsymbol{\Sigma}_w - \boldsymbol{\Sigma}_w \boldsymbol{\Psi}_t \left(\boldsymbol{\Sigma}_y^* + \boldsymbol{\Psi}_t^\top \boldsymbol{\Sigma}_w \boldsymbol{\Psi}_t \right)^{-1} \boldsymbol{\Psi}_t^\top \boldsymbol{\Sigma}_w \quad (12)$$

See Appendix B for the proof.

1.4.1 Do we actually get the desired mean by applying the conditioning update?

Proof that the posterior mean equals the observed mean.

$$\mathbb{E}[\mathbf{y}_t|\mathbf{x}_t^*] = \boldsymbol{\mu}_{\mathbf{y}_t|\mathbf{x}_t^*} = \boldsymbol{\Psi}_t^\top \boldsymbol{\mu}_w|\mathbf{x}_t^* = \boldsymbol{\Psi}_t^\top \boldsymbol{\mu}_w + \boldsymbol{\Psi}_t^\top \boldsymbol{\Sigma}_w \boldsymbol{\Psi}_t \left(\boldsymbol{\Sigma}_t^* + \boldsymbol{\Psi}_t^\top \boldsymbol{\Sigma}_w \boldsymbol{\Psi}_t \right)^{-1} (\mathbf{y}_t^* - \boldsymbol{\Psi}_t^\top \boldsymbol{\mu}_w) \quad (13)$$

We set the observed covariance $\boldsymbol{\Sigma}_t^*$ to 0 so as to have perfect accuracy around our observed position.

$$\boldsymbol{\Psi}_t^\top \boldsymbol{\mu}_w|\mathbf{x}_t^* = \boldsymbol{\Psi}_t^\top \boldsymbol{\mu}_w + \cancel{\boldsymbol{\Psi}_t^\top \boldsymbol{\Sigma}_w \boldsymbol{\Psi}_t} \left(\cancel{\boldsymbol{\Psi}_t^\top \boldsymbol{\Sigma}_w \boldsymbol{\Psi}_t} \right)^{-1} (\mathbf{y}_t^* - \boldsymbol{\Psi}_t^\top \boldsymbol{\mu}_w) \quad (14)$$

$$= \cancel{\boldsymbol{\Psi}_t^\top \boldsymbol{\mu}_w} + \mathbf{y}_t^* - \cancel{\boldsymbol{\Psi}_t^\top \boldsymbol{\mu}_w} \quad (15)$$

$$= \mathbf{y}_t^* \quad (16)$$

□

1.4.2 Multi via-points

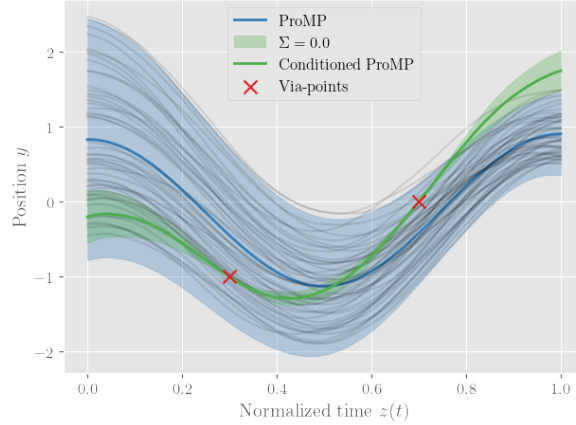


Figure 2: Example of ProMP with two via-points.

1. For the first via-point conditioning update with the observed via-point $\mathbf{x}_{t_1}^* = [\mathbf{y}_{t_1}^*, \Sigma_{t_1}^*]$, we can directly apply Eq. (11) and (12), with Ψ_{t_1} the observation matrix at time t_1 :

$$\mu_{w|\mathbf{x}_{t_1}^*} = \mu_w + \Sigma_w \Psi_{t_1} \left(\Sigma_{t_1}^* + \Psi_{t_1}^\top \Sigma_w \Psi_{t_1} \right)^{-1} (\mathbf{y}_{t_1}^* - \Psi_{t_1}^\top \mu_w) \quad (17)$$

$$\Sigma_{w|\mathbf{x}_{t_1}^*} = \Sigma_w - \Sigma_w \Psi_{t_1} \left(\Sigma_{t_1}^* + \Psi_{t_1}^\top \Sigma_w \Psi_{t_1} \right)^{-1} \Psi_{t_1}^\top \Sigma_w \quad (18)$$

2. For the second via-point conditioning update with the observed via-point $\mathbf{x}_{t_2}^* = [\mathbf{y}_{t_2}^*, \Sigma_{t_2}^*]$, the prior is the posterior from the first via-point, *i.e.*, $\mathbf{w} \sim \mathcal{N}(\mu_{w|\mathbf{x}_{t_1}^*}, \Sigma_{w|\mathbf{x}_{t_1}^*})$, the likelihood is $\mathbf{y}_{t_2}^* \sim \mathcal{N}(\Psi_{t_2}^\top \mathbf{w}, \Sigma_{t_2}^*)$, with Ψ_{t_2} the observation matrix at time t_2 , and the posterior update becomes:

$$\mu_{w|\mathbf{x}_{t_1}^*, \mathbf{x}_{t_2}^*} = \mu_{w|\mathbf{x}_{t_1}^*} + \Sigma_{w|\mathbf{x}_{t_1}^*} \Psi_{t_2} \left(\Sigma_{t_2}^* + \Psi_{t_2}^\top \Sigma_{w|\mathbf{x}_{t_1}^*} \Psi_{t_2} \right)^{-1} (\mathbf{y}_{t_2}^* - \Psi_{t_2}^\top \mu_{w|\mathbf{x}_{t_1}^*}) \quad (19)$$

$$\Sigma_{w|\mathbf{x}_{t_1}^*, \mathbf{x}_{t_2}^*} = \Sigma_{w|\mathbf{x}_{t_1}^*} - \Sigma_{w|\mathbf{x}_{t_1}^*} \Psi_{t_2} \left(\Sigma_{t_2}^* + \Psi_{t_2}^\top \Sigma_{w|\mathbf{x}_{t_1}^*} \Psi_{t_2} \right)^{-1} \Psi_{t_2}^\top \Sigma_{w|\mathbf{x}_{t_1}^*} \quad (20)$$

3. For the k^{th} via-point conditioning update with the observed via-point $\mathbf{x}_{t_k}^* = [\mathbf{y}_{t_k}^*, \Sigma_{t_k}^*]$, the prior is the posterior after conditioning on the previous $k-1$ via-points, *i.e.*, $\mathbf{w} \sim \mathcal{N}(\mu_{w|\mathbf{x}_{t_1}^*, \dots, \mathbf{x}_{t_{k-1}}^*}, \Sigma_{w|\mathbf{x}_{t_1}^*, \dots, \mathbf{x}_{t_{k-1}}^*})$, the likelihood is $\mathbf{y}_{t_k}^* \sim \mathcal{N}(\Psi_{t_k}^\top \mathbf{w}, \Sigma_{t_k}^*)$, with Ψ_{t_k} the observation matrix at time t_k , and the posterior update becomes:

$$\begin{aligned} \mu_{w|\mathbf{x}_{t_1}^*, \dots, \mathbf{x}_{t_k}^*} &= \mu_{w|\mathbf{x}_{t_1}^*, \dots, \mathbf{x}_{t_{k-1}}^*} \\ &+ \Sigma_{w|\mathbf{x}_{t_1}^*, \dots, \mathbf{x}_{t_{k-1}}^*} \Psi_{t_k} \left(\Sigma_{t_k}^* + \Psi_{t_k}^\top \Sigma_{w|\mathbf{x}_{t_1}^*, \dots, \mathbf{x}_{t_{k-1}}^*} \Psi_{t_k} \right)^{-1} (\mathbf{y}_{t_k}^* - \Psi_{t_k}^\top \mu_{w|\mathbf{x}_{t_1}^*, \dots, \mathbf{x}_{t_{k-1}}^*}) \end{aligned} \quad (21)$$

$$\begin{aligned} \Sigma_{w|\mathbf{x}_{t_1}^*, \dots, \mathbf{x}_{t_k}^*} &= \Sigma_{w|\mathbf{x}_{t_1}^*, \dots, \mathbf{x}_{t_{k-1}}^*} \\ &- \Sigma_{w|\mathbf{x}_{t_1}^*, \dots, \mathbf{x}_{t_{k-1}}^*} \Psi_{t_k} \left(\Sigma_{t_k}^* + \Psi_{t_k}^\top \Sigma_{w|\mathbf{x}_{t_1}^*, \dots, \mathbf{x}_{t_{k-1}}^*} \Psi_{t_k} \right)^{-1} \Psi_{t_k}^\top \Sigma_{w|\mathbf{x}_{t_1}^*, \dots, \mathbf{x}_{t_{k-1}}^*} \end{aligned} \quad (22)$$

Alternative Batch Formulation Instead of iterative updates, we could condition on all via-points simultaneously by stacking the observations:

$$\mathbf{y}^* = \begin{bmatrix} \mathbf{y}_{t_1}^* \\ \vdots \\ \mathbf{y}_{t_k}^* \end{bmatrix}, \quad \Psi = \begin{bmatrix} \Psi_{t_1} \\ \vdots \\ \Psi_{t_k} \end{bmatrix}, \quad \Sigma^* = \text{diag}(\Sigma_{t_1}^*, \dots, \Sigma_{t_k}^*) \quad (23)$$

$$\mu_{w|\{\mathbf{x}_{t_k}^*\}_{k=1}^K} = \mu_w + \Sigma_w \Psi \left(\Sigma^* + \Psi^\top \Sigma_w \Psi \right)^{-1} (\mathbf{y}^* - \Psi^\top \mu_w) \quad (24)$$

$$\Sigma_{w|\{\mathbf{x}_{t_k}^*\}_{k=1}^K} = \Sigma_w - \Sigma_w \Psi \left(\Sigma^* + \Psi^\top \Sigma_w \Psi \right)^{-1} \Psi^\top \Sigma_w \quad (25)$$

2 Gaussian mixture modeling (GMM)/Gaussian mixture regression (GMR) recap

2.1 Gaussian Mixture Modeling (GMM)

$$p(\mathbf{x}|\boldsymbol{\theta}) = \sum_{k=1}^K \pi_k \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_k, \Sigma_k) \quad (26)$$

$$0 \leq \pi_k \leq 1, \quad \sum_{k=1}^K \pi_k = 1 \quad (27)$$

$$r_{nk} := \frac{\pi_k \mathcal{N}(\mathbf{x}_n|\boldsymbol{\mu}_k, \Sigma_k)}{\sum_{j=1}^K \pi_j \mathcal{N}(\mathbf{x}_n|\boldsymbol{\mu}_j, \Sigma_j)} \quad (28)$$

Table 5: Notation

π_k	mixture weights
$\boldsymbol{\theta} := \{\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k, \pi_k : k = 1, \dots, K\}$	collection of all parameters of the model
r_{nk}	responsibility of the k^{th} mixture component for the n^{th} data point
N	number of data points
$N_k := \sum_{n=1}^N r_{nk}$	total responsibility of the k^{th} mixture component for the entire dataset

Update of the GMM means:

$$\boldsymbol{\mu}_k^{\text{new}} = \frac{\sum_{n=1}^N r_{nk} \mathbf{x}_n}{\sum_{n=1}^N r_{nk}} \quad (29)$$

Update of the GMM covariances:

$$\boldsymbol{\Sigma}_k^{\text{new}} = \frac{1}{N_k} \sum_{n=1}^N r_{nk} (\mathbf{x}_n - \boldsymbol{\mu}_k)(\mathbf{x}_n - \boldsymbol{\mu}_k)^\top \quad (30)$$

Update of the GMM mixture weights:

$$\pi_k^{\text{new}} = \frac{N_k}{N}, \quad k = 1, \dots, K \quad (31)$$

2.2 Gaussian Mixture Regression (GMR)

Requires to add a series of timesteps $t = (t_1, \dots, t_j, \dots, t_m)$ to divide each demonstration path evenly, and the points of each path can be re-written as $[(t_1, p_1), \dots, (t_j, p_j), \dots, (t_m, p_m)]$, so that each path has the same number of points for better alignment between demonstrations.

Joint probability $\mathcal{P}(t, \mathbf{x})$ learned with GMM:

$$\mathcal{P}(t_j, x_j) = \sum_{i=1}^K \pi_i \cdot \mathcal{N}_i(x_j | t_j; m_i(t_j), \text{cov}_i) \cdot \mathcal{N}_i(t_j | \boldsymbol{\mu}_{it}, \boldsymbol{\Sigma}_{it}) \quad (32)$$

$$\boldsymbol{\mu}_i = \begin{bmatrix} \boldsymbol{\mu}_{it} \\ \boldsymbol{\mu}_{ix} \end{bmatrix}, \quad \boldsymbol{\Sigma}_i = \begin{bmatrix} \boldsymbol{\Sigma}_{itt} & \boldsymbol{\Sigma}_{itx} \\ \boldsymbol{\Sigma}_{ixt} & \boldsymbol{\Sigma}_{ixx} \end{bmatrix} \quad (33)$$

$$(34)$$

$$m_i(t_j) = \boldsymbol{\mu}_{ix} + \boldsymbol{\Sigma}_{ixt} \cdot \boldsymbol{\Sigma}_{itt}^{-1} \cdot (t_j - \boldsymbol{\mu}_{it}) \quad (35)$$

$$\text{cov}_i = \boldsymbol{\Sigma}_{ixx} - \boldsymbol{\Sigma}_{ixt} \cdot \boldsymbol{\Sigma}_{itt}^{-1} \cdot \boldsymbol{\Sigma}_{itx} \quad (36)$$

Marginal probability $P(t_j)$:

$$P(t_j) = \int P(t_j, x_j) dx = \sum_{i=1}^K \pi_i \cdot \mathcal{N}_i(t_j | \boldsymbol{\mu}_{it}, \boldsymbol{\Sigma}_{it}) \quad (37)$$

Retrieve the conditional probability $\mathcal{P}(\mathbf{x} | t)$ with GMR for each timestep:

$$P(x_j | t_j; m_i(t_j), \text{cov}_i) = \frac{P(t_j, x_j)}{P(t_j)} \quad (38)$$

$$= \frac{\sum_{i=1}^K \pi_i \cdot \mathcal{N}_i(x_j | t_j; m_i(t_j), \text{cov}_i) \cdot \mathcal{N}_i(t_j | \boldsymbol{\mu}_{it}, \boldsymbol{\Sigma}_{it})}{\sum_{i=1}^K \pi_i \cdot \mathcal{N}_i(t_j | \boldsymbol{\mu}_{it}, \boldsymbol{\Sigma}_{it})} \quad (39)$$

$$= \sum_{i=1}^K r_{nk} \cdot \mathcal{N}_i(x_j | t_j; m_i(t_j), \text{cov}_i) \quad (40)$$

Algorithm 1: EXPECTATION MAXIMIZATION (EM) algorithm for a Gaussian mixture model

Input : Initial model parameters $\{\boldsymbol{\mu}_k\}, \{\boldsymbol{\Sigma}_k\}, \{\pi_k\}$

Input : Data set $\{\mathbf{x}_1, \dots, \mathbf{x}_N\}$

Output: Final model parameters $\{\boldsymbol{\mu}_k\}, \{\boldsymbol{\Sigma}_k\}, \{\pi_k\}$

repeat

 // E-step

for $n \in \{1, \dots, N\}$ **do**

for $k \in \{1, \dots, K\}$ **do**

$$r_{nk} \leftarrow \frac{\pi_k \mathcal{N}(\mathbf{x}_n | \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)}{\sum_{j=1}^K \pi_j \mathcal{N}(\mathbf{x}_n | \boldsymbol{\mu}_j, \boldsymbol{\Sigma}_j)}$$

end

end

 // M-step

for $k \in \{1, \dots, K\}$ **do**

$$N_k = \sum_{n=1}^N r_{nk}$$

$$\boldsymbol{\mu}_k = \frac{1}{N_k} \sum_{n=1}^N r_{nk} \mathbf{x}_n$$

$$\boldsymbol{\Sigma}_k = \frac{1}{N_k} \sum_{n=1}^N r_{nk} (\mathbf{x}_n - \boldsymbol{\mu}_k)(\mathbf{x}_n - \boldsymbol{\mu}_k)^\top$$

$$\pi_k = \frac{N_k}{N}$$

end

 // Log likelihood

$$\mathcal{L} \leftarrow \sum_{n=1}^N \ln \left\{ \sum_{k=1}^K \pi_k \mathcal{N}(\mathbf{x}_n | \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k) \right\}$$

until *convergence*

return $\{\boldsymbol{\mu}_k\}, \{\boldsymbol{\Sigma}_k\}, \{\pi_k\}$

Table 6: Notation

p_j	position of a constructive point from demonstration, $j = 1, \dots, m$
m	total number of points in all the demonstrations

Regression function (Eq. (41)) and conditional variance (Eq. (42)):

$$m(x) = \mathbb{E}(x_j|t_j) = \sum_{i=1}^K r_{nk} \cdot m_i(t_j) \quad (41)$$

$$var(x) = \sum_{j=1}^K r_{nk} \cdot cov_i \quad (42)$$

3 Composition of MPs

3.1 Stitching

The main issue with stitching is the smoothness of the mean and covariance between ProMPs, see Fig. 3.

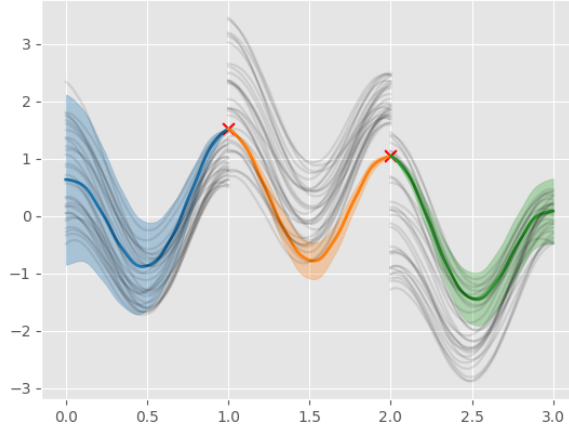


Figure 3: Stitching three ProMPs.

3.2 Piecewise Gaussian Process

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A Hierarchical Bayesian Model proof

Proof of Eq. (8). From (Deisenroth et al., 2020), we have the joint distribution:

$$p(\mathbf{x}_a, \mathbf{x}_b) = \mathcal{N}\left(\begin{bmatrix} \boldsymbol{\mu}_a \\ \boldsymbol{\mu}_b \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_{aa} & \boldsymbol{\Sigma}_{ab} \\ \boldsymbol{\Sigma}_{ba} & \boldsymbol{\Sigma}_{bb} \end{bmatrix}\right) \quad (43)$$

and the marginal distribution $p(\mathbf{x}_a)$ of a joint Gaussian distribution $p(\mathbf{x}_a, \mathbf{x}_b)$:

$$p(\mathbf{x}_a) = \int p(\mathbf{x}_a, \mathbf{x}_b) d\mathbf{x}_b = \mathcal{N}(\mathbf{x}_a | \boldsymbol{\mu}_a, \boldsymbol{\Sigma}_{aa}) \quad (44)$$

Since \mathbf{y}_t and \mathbf{w} are jointly Gaussian, we have:

$$\begin{bmatrix} \mathbf{y}_t \\ \mathbf{w} \end{bmatrix} \sim \mathcal{N} \left(\begin{bmatrix} \boldsymbol{\mu}_{\mathbf{y}_t} \\ \boldsymbol{\mu}_{\mathbf{w}} \end{bmatrix}, \begin{bmatrix} \text{Cov}[\mathbf{y}_t, \mathbf{y}_t] & \text{Cov}[\mathbf{y}_t, \mathbf{w}] \\ \text{Cov}[\mathbf{w}, \mathbf{y}_t] & \text{Cov}[\mathbf{w}, \mathbf{w}] \end{bmatrix} \right) \quad (45)$$

$$\boldsymbol{\mu}_{\mathbf{y}_t} = \mathbb{E}[\mathbf{y}_t] \quad (46)$$

$$= \mathbb{E}[\boldsymbol{\Psi}_t^\top \mathbf{w} + \boldsymbol{\epsilon}_y] \quad (47)$$

$$= \boldsymbol{\Psi}_t^\top \mathbb{E}[\mathbf{w}] + \mathbb{E}[\boldsymbol{\epsilon}_y] \quad (48)$$

$$= \boldsymbol{\Psi}_t^\top \boldsymbol{\mu}_{\mathbf{w}} + 0 \quad (49)$$

$$= \boldsymbol{\Psi}_t^\top \boldsymbol{\mu}_{\mathbf{w}} \quad (50)$$

$$\text{Cov}[\mathbf{y}_t, \mathbf{y}_t] = \text{Cov}[\boldsymbol{\Psi}_t^\top \mathbf{w} + \boldsymbol{\epsilon}_y] \quad (51)$$

$$= \text{Cov}[\boldsymbol{\Psi}_t^\top \mathbf{w}] + \text{Cov}[\boldsymbol{\epsilon}_y] \quad (52)$$

$$= \boldsymbol{\Psi}_t^\top \text{Cov}[\mathbf{w}] \boldsymbol{\Psi}_t + \boldsymbol{\Sigma}_y \quad (53)$$

$$= \boldsymbol{\Psi}_t^\top \boldsymbol{\Sigma}_{\mathbf{w}} \boldsymbol{\Psi}_t + \boldsymbol{\Sigma}_y \quad (54)$$

$$\begin{bmatrix} \mathbf{y}_t \\ \mathbf{w} \end{bmatrix} \sim \mathcal{N} \left(\begin{bmatrix} \boldsymbol{\Psi}_t^\top \boldsymbol{\mu}_{\mathbf{w}} \\ \boldsymbol{\mu}_{\mathbf{w}} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Psi}_t^\top \boldsymbol{\Sigma}_{\mathbf{w}} \boldsymbol{\Psi}_t + \boldsymbol{\Sigma}_y & \boldsymbol{\Psi}_t^\top \boldsymbol{\Sigma}_{\mathbf{w}} \\ \boldsymbol{\Sigma}_{\mathbf{w}} \boldsymbol{\Psi}_t & \boldsymbol{\Sigma}_{\mathbf{w}} \end{bmatrix} \right) \quad (55)$$

$$p(\mathbf{y}_t; \boldsymbol{\theta}) = \mathcal{N}(\mathbf{y}_t | \boldsymbol{\Psi}_t^\top \boldsymbol{\mu}_{\mathbf{w}}, \boldsymbol{\Psi}_t^\top \boldsymbol{\Sigma}_{\mathbf{w}} \boldsymbol{\Psi}_t + \boldsymbol{\Sigma}_y) \quad (56)$$

□

B Via-Points conditioning proof

Proof of Eq. (11) and Eq. (12). With the joint distribution $p(\mathbf{x}_a, \mathbf{x}_b)$ in Eq. (43), and from (Bishop and Bishop, 2024), the parameters of a conditional multivariate Gaussian $p(\mathbf{x}_a | \mathbf{x}_b) = \mathcal{N}(\boldsymbol{\mu}_{a|b}, \boldsymbol{\Sigma}_{a|b})$ are the following:

$$\boldsymbol{\mu}_{a|b} = \boldsymbol{\mu}_a + \boldsymbol{\Sigma}_{ab} \boldsymbol{\Sigma}_{bb}^{-1} (\mathbf{x}_b - \boldsymbol{\mu}_b) \quad (57)$$

$$\boldsymbol{\Sigma}_{a|b} = \boldsymbol{\Sigma}_{aa} - \boldsymbol{\Sigma}_{ab} \boldsymbol{\Sigma}_{bb}^{-1} \boldsymbol{\Sigma}_{ba} \quad (58)$$

We want the posterior $p(\mathbf{w} | \mathbf{x}_t^*)$, knowing the likelihood $\mathbf{x}_t^* | \mathbf{w} \sim \mathcal{N}(\mathbf{y}_t^* | \boldsymbol{\Psi}_t^\top \mathbf{w}, \boldsymbol{\Sigma}_t^*)$, and the prior $\mathbf{w} \sim \mathcal{N}(\mathbf{w} | \boldsymbol{\mu}_{\mathbf{w}}, \boldsymbol{\Sigma}_{\mathbf{w}})$.

$$\begin{bmatrix} \mathbf{w} \\ \mathbf{x}_t^* \end{bmatrix} \sim \mathcal{N} \left(\begin{bmatrix} \boldsymbol{\mu}_{\mathbf{w}} \\ \boldsymbol{\Psi}_t^\top \boldsymbol{\mu}_{\mathbf{w}} \end{bmatrix}, \begin{bmatrix} \text{Cov}[\mathbf{w}, \mathbf{w}] & \text{Cov}[\mathbf{w}, \mathbf{x}_t^*] \\ \text{Cov}[\mathbf{x}_t^*, \mathbf{w}] & \text{Cov}[\mathbf{x}_t^*, \mathbf{x}_t^*] \end{bmatrix} \right) \quad (59)$$

$\text{Cov}[\mathbf{x}_t^*, \mathbf{x}_t^*]$ follows from Eq. (54).

$$\text{Cov}[\mathbf{w}, \mathbf{x}_t^*] = \text{Cov}[\mathbf{w}, \Psi_t^\top \mathbf{w} + \epsilon_y] \quad (60)$$

$$= \text{Cov}[\mathbf{w}, \Psi_t^\top \mathbf{w}] \quad (\text{Cov}[\mathbf{w}, \epsilon_y] = 0 \text{ since } \epsilon_y \text{ is independent of } \mathbf{w}) \quad (61)$$

$$= \mathbb{E}[(\mathbf{w} - \mu_w)(\Psi_t^\top \mathbf{w} - \Psi_t^\top \mu_w)^\top] \quad (62)$$

$$= \mathbb{E}[(\mathbf{w} - \mu_w)(\mathbf{w} - \mu_w)^\top \Psi_t] \quad (63)$$

$$= \text{Cov}[\mathbf{w}, \mathbf{w}] \cdot \Psi_t \quad (64)$$

$$= \Sigma_w \Psi_t \quad (65)$$

$$\begin{bmatrix} \mathbf{w} \\ \mathbf{x}_t^* \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu_w \\ \Psi_t^\top \mu_w \end{bmatrix}, \begin{bmatrix} \Sigma_w & \Sigma_w \Psi_t \\ \Psi_t^\top \Sigma_w & \Psi_t^\top \Sigma_w \Psi_t + \Sigma_t^* \end{bmatrix}\right) \quad (66)$$

Using Eq. (57) we get:

$$\mu_{w|x_t^*} = \mu_w + \Sigma_w \Psi_t \left(\Sigma_t^* + \Psi_t^\top \Sigma_w \Psi_t \right)^{-1} (y_t^* - \Psi_t^\top \mu_w) \quad (67)$$

Using Eq. (58) we get:

$$\Sigma_{w|x_t^*} = \Sigma_w - \Sigma_w \Psi_t \left(\Sigma_t^* + \Psi_t^\top \Sigma_w \Psi_t \right)^{-1} \Psi_t^\top \Sigma_w \quad (68)$$

□