

# Composition of Movement Primitives

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## 1 ProMPs

### 1.1 Recap

From (Paraschos et al., 2013, 2018):

- $q_t$ : joint angle over time
- $\dot{q}_t$ : joint velocity over time
- $\tau = \{q_t\}_{t=0\dots T}$ : trajectory
- $\mathbf{w}$ : weight vector of a single trajectory  $[n \times 1]$
- $\phi_t$ : basis function
- $n$ : number of basis functions
- $\Phi_t = [\phi_t, \dot{\phi}_t]$ :  $n \times 2$  dimensional time-dependent basis matrix
- $z(t)$ : monotonically increasing phase variable
- $\epsilon_y \sim \mathcal{N}(\mathbf{0}, \Sigma_y)$ : zero-mean i.i.d. Gaussian noise

$$\Phi_t = \begin{bmatrix} \phi_1 & \dot{\phi}_1 \\ \vdots & \vdots \\ \phi_n & \dot{\phi}_n \end{bmatrix} \quad (1)$$

$$\mathbf{y}_t = \begin{bmatrix} q_t \\ \dot{q}_t \end{bmatrix} = \Phi_t^\top \mathbf{w} + \epsilon_y \quad (2)$$

$$p(\tau|\mathbf{w}) = \prod_t \mathcal{N}(\mathbf{y}_t | \Phi_t^\top \mathbf{w}, \Sigma_y) \quad (3)$$

$$p(\tau; \theta) = \int p(\tau|\mathbf{w}) \cdot p(\mathbf{w}; \theta) d\mathbf{w} \quad (4)$$

## 1.2 Coupling between joints

$$p(\mathbf{y}_t|\mathbf{w}) = \mathcal{N}\left(\begin{bmatrix} \mathbf{y}_{1,t} \\ \vdots \\ \mathbf{y}_{d,t} \end{bmatrix} \middle| \begin{bmatrix} \Phi_t^\top & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \Phi_t^\top \end{bmatrix} \mathbf{w}, \Sigma_y\right) = \mathcal{N}(\mathbf{y}_t | \Psi_t \mathbf{w}, \Sigma_y) \quad (5)$$

with:

- $\mathbf{w} = [\mathbf{w}_1^\top, \dots, \mathbf{w}_n^\top]^\top$ : combined weight vector  $[n \times n]$
- $\Psi_t$ : block-diagonal basis matrix containing the basis functions and their derivatives for each dimension
- $\mathbf{y}_{i,t} = [q_{i,t}, \dot{q}_{i,t}]^\top$ : joint angle and velocity for the  $i^{\text{th}}$  joint

## 1.3 Hierarchical Bayesian Model

The Hierarchical Bayesian Model used in ProMPs is illustrated in Fig. 1.

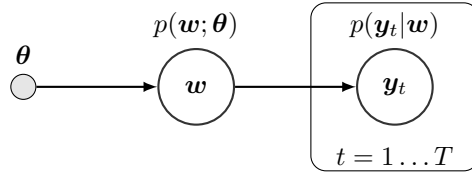


Figure 1: Hierarchical Bayesian Model used in ProMPs.

- $\theta = \{\mu_w, \Sigma_w\}$
- $p(\mathbf{w}; \theta) = \mathcal{N}(\mathbf{w} | \mu_w, \Sigma_w)$ : prior over the weight vector  $\mathbf{w}$ , with parameters  $\theta$ , assumed to be Gaussian

$$p(\mathbf{y}_t; \theta) = \int \mathcal{N}(\mathbf{y}_t | \Psi_t^\top \mathbf{w}, \Sigma_y) \cdot p(\mathbf{w}; \theta) d\mathbf{w} \quad (6)$$

$$= \int \mathcal{N}(\mathbf{y}_t | \Psi_t^\top \mathbf{w}, \Sigma_y) \cdot \mathcal{N}(\mathbf{w} | \mu_w, \Sigma_w) d\mathbf{w} \quad (7)$$

$$= \mathcal{N}(\mathbf{y}_t | \Psi_t^\top \mu_w, \Psi_t^\top \Sigma_w \Psi_t + \Sigma_y) \quad (8)$$

See Appendix A for the proof.

## 1.4 Via-Points Modulation

- $\mathbf{x}_t^* = [\mathbf{y}_t^*, \Sigma_t^*]$ : desired observation
- $\mathbf{y}_t^*$ : desired position and velocity vector at time  $t$
- $\Sigma_t^*$ : accuracy of the desired observation

Using Bayes rule:

$$p(\mathbf{w}|\mathbf{x}_t^*) = \frac{p(\mathbf{x}_t^*|\mathbf{w}) \cdot p(\mathbf{w})}{p(\mathbf{x}_t^*)} \quad (9)$$

$$p(\mathbf{w}|\mathbf{x}_t^*) \propto \mathcal{N}(\mathbf{y}_t^*|\Psi_t^\top \mathbf{w}, \Sigma_t^*) \cdot \mathcal{N}(\mathbf{w}|\mu_w, \Sigma_w) \quad (10)$$

$$\mu_w^{[new]} = \mu_w + \Sigma_w \Psi_t \left( \Sigma_y^* + \Psi_t^\top \Sigma_w \Psi_t \right)^{-1} (\mathbf{y}_t^* - \Psi_t^\top \mu_w) \quad (11)$$

$$\Sigma_w^{[new]} = \Sigma_w - \Sigma_w \Psi_t \left( \Sigma_y^* + \Psi_t^\top \Sigma_w \Psi_t \right)^{-1} \Psi_t^\top \Sigma_w \quad (12)$$

See Appendix B for the proof.

#### 1.4.1 Do we actually get the desired mean by applying the conditioning update?

$$\mu_{w|\mathbf{x}_t^*} = \mu_w + \Sigma_w \Psi_t \left( \Sigma_t^* + \Psi_t^\top \Sigma_w \Psi_t \right)^{-1} (\mathbf{y}_t^* - \Psi_t^\top \mu_w) \quad (13)$$

Let us set the observed covariance  $\Sigma_t^*$  to 0 so as to have perfect accuracy around our observed position.

$$\mu_{w|\mathbf{x}_t^*} = \mu_w + \Sigma_w \Psi_t \left( \Psi_t^\top \Sigma_w \Psi_t \right)^{-1} (\mathbf{y}_t^* - \Psi_t^\top \mu_w) \quad (14)$$

$$= \mu_w + \Sigma_w \Psi_t \left( \Psi_t^\top \Sigma_w \Psi_t \right)^{-1} (\mathbf{y}_t^* - \mu_{y_t}) \quad (\text{which does not simplify further}) \quad (15)$$

$$\neq \mathbf{y}_t^* \quad (16)$$

#### 1.4.2 Multi via-points

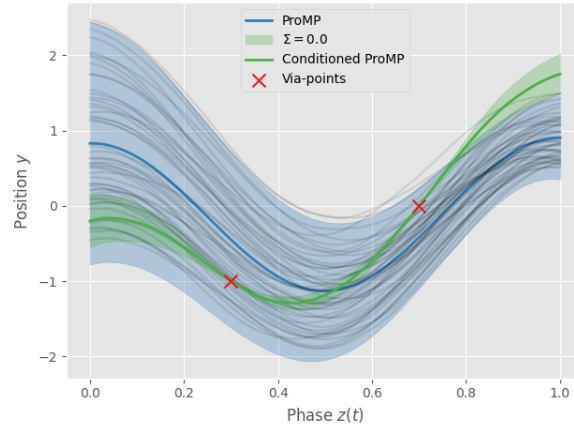


Figure 2: Example of ProMP with two via points.

Two points conditional update derivation

## 2 Composition of MPs

### 2.1 Blending

### 2.2 Stitching

## References

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## A Hierarchical Bayesian Model proof

*Proof of Eq. (8).* From (Deisenroth et al., 2020), we have the joint distribution:

$$p(\mathbf{x}_a, \mathbf{x}_b) = \mathcal{N}\left(\begin{bmatrix} \boldsymbol{\mu}_a \\ \boldsymbol{\mu}_b \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_{aa} & \boldsymbol{\Sigma}_{ab} \\ \boldsymbol{\Sigma}_{ba} & \boldsymbol{\Sigma}_{bb} \end{bmatrix}\right) \quad (17)$$

and the marginal distribution  $p(\mathbf{x}_a)$  of a joint Gaussian distribution  $p(\mathbf{x}_a, \mathbf{x}_b)$ :

$$p(\mathbf{x}_a) = \int p(\mathbf{x}_a, \mathbf{x}_b) d\mathbf{x}_b = \mathcal{N}(\mathbf{x}_a | \boldsymbol{\mu}_a, \boldsymbol{\Sigma}_{aa}) \quad (18)$$

Since  $\mathbf{y}_t$  and  $\mathbf{w}$  are jointly Gaussian, we have:

$$\begin{bmatrix} \mathbf{y}_t \\ \mathbf{w} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \boldsymbol{\mu}_{\mathbf{y}_t} \\ \boldsymbol{\mu}_{\mathbf{w}} \end{bmatrix}, \begin{bmatrix} \text{Cov}[\mathbf{y}_t, \mathbf{y}_t] & \text{Cov}[\mathbf{y}_t, \mathbf{w}] \\ \text{Cov}[\mathbf{w}, \mathbf{y}_t] & \text{Cov}[\mathbf{w}, \mathbf{w}] \end{bmatrix}\right) \quad (19)$$

$$\boldsymbol{\mu}_{\mathbf{y}_t} = \mathbb{E}[\mathbf{y}_t] \quad (20)$$

$$= \mathbb{E}[\boldsymbol{\Psi}_t^\top \mathbf{w} + \boldsymbol{\epsilon}_y] \quad (21)$$

$$= \boldsymbol{\Psi}_t^\top \mathbb{E}[\mathbf{w}] + \mathbb{E}[\boldsymbol{\epsilon}_y] \quad (22)$$

$$= \boldsymbol{\Psi}_t^\top \boldsymbol{\mu}_{\mathbf{w}} + 0 \quad (23)$$

$$= \boldsymbol{\Psi}_t^\top \boldsymbol{\mu}_{\mathbf{w}} \quad (24)$$

$$\text{Cov}[\mathbf{y}_t, \mathbf{y}_t] = \text{Cov}[\boldsymbol{\Psi}_t^\top \mathbf{w} + \boldsymbol{\epsilon}_y] \quad (25)$$

$$= \text{Cov}[\boldsymbol{\Psi}_t^\top \mathbf{w}] + \text{Cov}[\boldsymbol{\epsilon}_y] \quad (26)$$

$$= \boldsymbol{\Psi}_t^\top \text{Cov}[\mathbf{w}] \boldsymbol{\Psi}_t + \boldsymbol{\Sigma}_y \quad (27)$$

$$= \boldsymbol{\Psi}_t^\top \boldsymbol{\Sigma}_{\mathbf{w}} \boldsymbol{\Psi}_t + \boldsymbol{\Sigma}_y \quad (28)$$

$$\begin{bmatrix} \mathbf{y}_t \\ \mathbf{w} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \boldsymbol{\Psi}_t^\top \boldsymbol{\mu}_{\mathbf{w}} \\ \boldsymbol{\mu}_{\mathbf{w}} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Psi}_t^\top \boldsymbol{\Sigma}_{\mathbf{w}} \boldsymbol{\Psi}_t + \boldsymbol{\Sigma}_y & \boldsymbol{\Psi}_t^\top \boldsymbol{\Sigma}_{\mathbf{w}} \\ \boldsymbol{\Sigma}_{\mathbf{w}} \boldsymbol{\Psi}_t & \boldsymbol{\Sigma}_{\mathbf{w}} \end{bmatrix}\right) \quad (29)$$

$$p(\mathbf{y}_t; \boldsymbol{\theta}) = \mathcal{N}(\mathbf{y}_t | \boldsymbol{\Psi}_t^\top \boldsymbol{\mu}_{\mathbf{w}}, \boldsymbol{\Psi}_t^\top \boldsymbol{\Sigma}_{\mathbf{w}} \boldsymbol{\Psi}_t + \boldsymbol{\Sigma}_y) \quad (30)$$

□

## B Via-Points conditioning proof

*Proof of Eq. (11) and Eq. (12).* With the joint distribution  $p(\mathbf{x}_a, \mathbf{x}_b)$  in Eq. (17), and from (Bishop and Bishop, 2024), the parameters of a conditional multivariate Gaussian  $p(\mathbf{x}_a|\mathbf{x}_b) = \mathcal{N}(\boldsymbol{\mu}_{a|b}, \boldsymbol{\Sigma}_{a|b})$  are the following:

$$\boldsymbol{\mu}_{a|b} = \boldsymbol{\mu}_a + \boldsymbol{\Sigma}_{ab}\boldsymbol{\Sigma}_{bb}^{-1}(\mathbf{x}_b - \boldsymbol{\mu}_b) \quad (31)$$

$$\boldsymbol{\Sigma}_{a|b} = \boldsymbol{\Sigma}_{aa} - \boldsymbol{\Sigma}_{ab}\boldsymbol{\Sigma}_{bb}^{-1}\boldsymbol{\Sigma}_{ba} \quad (32)$$

We want the posterior  $p(\mathbf{w}|\mathbf{x}_t^*)$ , knowing the likelihood  $\mathbf{x}_t^*|\mathbf{w} \sim \mathcal{N}(\mathbf{y}_t^*|\boldsymbol{\Psi}_t^\top \mathbf{w}, \boldsymbol{\Sigma}_t^*)$ , and the prior  $\mathbf{w} \sim \mathcal{N}(\mathbf{w}|\boldsymbol{\mu}_w, \boldsymbol{\Sigma}_w)$ .

$$\begin{bmatrix} \mathbf{w} \\ \mathbf{x}_t^* \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \boldsymbol{\mu}_w \\ \boldsymbol{\Psi}_t^\top \boldsymbol{\mu}_w \end{bmatrix}, \begin{bmatrix} \text{Cov}[\mathbf{w}, \mathbf{w}] & \text{Cov}[\mathbf{w}, \mathbf{x}_t^*] \\ \text{Cov}[\mathbf{x}_t^*, \mathbf{w}] & \text{Cov}[\mathbf{x}_t^*, \mathbf{x}_t^*] \end{bmatrix}\right) \quad (33)$$

$\text{Cov}[\mathbf{x}_t^*, \mathbf{x}_t^*]$  follows from Eq. (28).

$$\text{Cov}[\mathbf{w}, \mathbf{x}_t^*] = \text{Cov}[\mathbf{w}, \boldsymbol{\Psi}_t^\top \mathbf{w} + \boldsymbol{\epsilon}_y] \quad (34)$$

$$= \text{Cov}[\mathbf{w}, \boldsymbol{\Psi}_t^\top \mathbf{w}] \quad (\text{Cov}[\mathbf{w}, \boldsymbol{\epsilon}_y] = 0 \text{ since } \boldsymbol{\epsilon}_y \text{ is independent of } \mathbf{w}) \quad (35)$$

$$= \mathbb{E}[(\mathbf{w} - \boldsymbol{\mu}_w)(\boldsymbol{\Psi}_t^\top \mathbf{w} - \boldsymbol{\Psi}_t^\top \boldsymbol{\mu}_w)^\top] \quad (36)$$

$$= \mathbb{E}[(\mathbf{w} - \boldsymbol{\mu}_w)(\mathbf{w} - \boldsymbol{\mu}_w)^\top \boldsymbol{\Psi}_t] \quad (37)$$

$$= \text{Cov}[\mathbf{w}, \mathbf{w}] \cdot \boldsymbol{\Psi}_t \quad (38)$$

$$= \boldsymbol{\Sigma}_w \boldsymbol{\Psi}_t \quad (39)$$

$$\begin{bmatrix} \mathbf{w} \\ \mathbf{x}_t^* \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \boldsymbol{\mu}_w \\ \boldsymbol{\Psi}_t^\top \boldsymbol{\mu}_w \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_w & \boldsymbol{\Sigma}_w \boldsymbol{\Psi}_t \\ \boldsymbol{\Psi}_t^\top \boldsymbol{\Sigma}_w & \boldsymbol{\Psi}_t^\top \boldsymbol{\Sigma}_w \boldsymbol{\Psi}_t + \boldsymbol{\Sigma}_t^* \end{bmatrix}\right) \quad (40)$$

Using Eq. (31) we get:

$$\boldsymbol{\mu}_{w|\mathbf{x}_t^*} = \boldsymbol{\mu}_w + \boldsymbol{\Sigma}_w \boldsymbol{\Psi}_t \left( \boldsymbol{\Sigma}_t^* + \boldsymbol{\Psi}_t^\top \boldsymbol{\Sigma}_w \boldsymbol{\Psi}_t \right)^{-1} (\mathbf{y}_t^* - \boldsymbol{\Psi}_t^\top \boldsymbol{\mu}_w) \quad (41)$$

Using Eq. (32) we get:

$$\boldsymbol{\Sigma}_{w|\mathbf{x}_t^*} = \boldsymbol{\Sigma}_w - \boldsymbol{\Sigma}_w \boldsymbol{\Psi}_t \left( \boldsymbol{\Sigma}_t^* + \boldsymbol{\Psi}_t^\top \boldsymbol{\Sigma}_w \boldsymbol{\Psi}_t \right)^{-1} \boldsymbol{\Psi}_t^\top \boldsymbol{\Sigma}_w \quad (42)$$

□