Section 2

More on semantics

Previously . . .

- Propositional logic
- Propositional formulas
- Connectives: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$
- Truth tables
- Boolean semantics

Formal definition of interpretation of propositional variables and connectives

Definition

- Let S be a set of values that propositional formulas can take
- A **valuation** is a function

$$\mathrm{v} : \{P_1, P_2, \ldots, P_n\} \longrightarrow S$$

  that assigns values to the list $P_1, P_2, \ldots, P_n$ of propositional variables
- Assign interpretations in S to connectives

Interpretation of a propositional formula

Definition

Let A is any propositional formula that contains the propositional variables $P_1, P_2, \ldots, P_n$. Suppose v is a valuation for these propositional variables. The **interpretation** of A **relative to valuation** v is the value of the formula obtained by

- replacing each P_i by $\mathrm{v}\, P_i$, and
- evaluating the assignment to the connectives

In the Boolean semantics

George Boole (1815–1864)

... the values are $S = \{\mathbf{1}, \mathbf{0}\}$
S is known as the **two element Boolean algebra** $\mathbb{B}$

Thus

- A **Boolean valuation** v is a function

$$\mathrm{v} : \{P_1, P_2, \ldots, P_n\} \longrightarrow \{\mathbf{1}, \mathbf{0}\}$$

  that assigns $\mathbf{1}$ or $\mathbf{0}$ to each propositional variable $P_1, P_2, \ldots, P_n$
- The **Boolean interpretation** of a formula A relative to v is the value of the formula obtained by replacing each P_i by $\mathrm{v}\,P_i$ and calculating the truth value of A in accordance with the standard truth tables for the connectives

Boolean interpretation of a formula

Example

Consider

$$A = (P \wedge Q) \rightarrow (Q \vee \neg P)$$

Suppose v is the valuation, mapping

$$P \text{ to } \mathbf{1} \qquad \text{and} \qquad Q \text{ to } \mathbf{0}$$

Interpretation of A relative to v

$$(\mathrm{v}\,P \wedge \mathrm{v}\,Q) \rightarrow (\mathrm{v}\,Q \vee \neg\,\mathrm{v}\,P)$$
$$=$$

Alternatively . . .

Read off interpretation of A relative to our specific v from

P	Q	$(P \wedge Q) \rightarrow (Q \vee \neg P)$
1	**1**	**1**
1	**0**	**1**
0	**1**	**1**
0	**0**	**1**

Note

- Each line in a truth table gives the interpretation of a formula for a given valuation of its propositional variables
- Truth tables display all possible interpretations of a formula
- Computing the interpretation of a formula for a given valuation is a **model checking** task

en.wikipedia.org

Tautology

Definition

A propositional formula A is **tautology**, if the interpretation of A is **1**, for *all possible* valuations v of the propositional variables occurring in A

P_1	...	P_n	A
$\star$	...	$\star$	**1**
$\star$	...	$\star$	**1**
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\star$	...	$\star$	**1**

Example

$(P \wedge Q) \rightarrow (Q \vee \neg P)$ is a tautology

Satisfiability, contradiction

Definition

A propositional formula A is **satisfiable**, if the interpretation of A is $\mathbf{1}$ for *some* valuation v
Otherwise, it is a **contradiction** (we also say it is **unsatisfiable**)

Example

- Suppose $A' = (P \vee Q) \wedge \neg(P \wedge Q)$,
  $\mathrm{v}\,P = \mathbf{1}$ and $\mathrm{v}\,Q = \mathbf{0}$
- We know how to calculate the interpretation of A' for v:

$$(\mathrm{v}\,P \vee \mathrm{v}\,Q) \wedge \neg(\mathrm{v}\,P \wedge \mathrm{v}\,Q)$$
$$= (\mathbf{1} \vee \mathbf{0}) \wedge \neg(\mathbf{1} \wedge \mathbf{0}) = \mathbf{1} \wedge \neg\mathbf{0} = \mathbf{1} \wedge \mathbf{1} = \mathbf{1}$$

- Thus, A' is satisfiable.

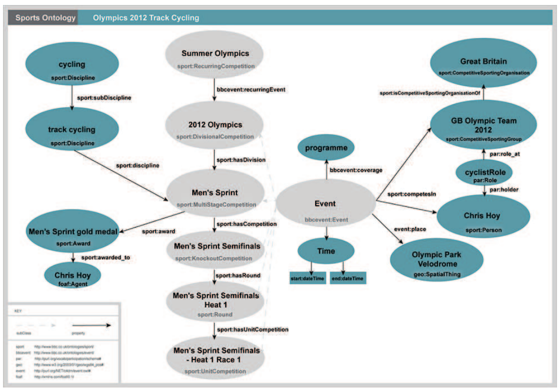
Is A' a contradiction? Is it a tautology?

Power set semantics

Historically, Boole actually worked with the **power set semantics of propositional logic**

Provides the foundation of

- **algebra of logic**
- **logic-based systems used in AI**
  knowledge representation, ontology-based knowledge processing



www.bbc.co.uk/ontologies/

In the power set semantics

. . . the values that propositional formulas can take are subsets of X, where X is an arbitrary (but fixed) non-empty set

Definition

- $S = \mathcal{P}X$
- A valuation v is a function

$$\mathrm{v} : \{P_1, P_2, \dots, P_n\} \longrightarrow \mathcal{P}X$$

  that assigns subsets of X to the list $P_1, P_2, \dots, P_n$
- To obtain the interpretation of a formula A relative to v

for conjunction $\wedge$	use	intersection $\cap$
for disjunction $\vee$	use	union $\cup$
for negation $\neg$	use	complement $X \setminus -$
for $B \to B'$	use	$(X \setminus S_B) \cup S_{B'}$

Computing the interpretation of a formula

Example

- Reconsider $A' = (P \vee Q) \wedge \neg(P \wedge Q)$
  Suppose

$$X = \{a, b, c\}, \qquad \mathrm{v}\,P = \{a, b\} \quad \text{and} \quad \mathrm{v}\,Q = \{b, c\}$$

- Power set interpretation of A' for v:

Tautology, satisfiability, contradiction in the power set semantics

Definition

In the power set semantics:

- A propositional formula A is a **tautology**, if the interpretation of A is X, for *all possible* valuations v and *any non-empty set* X
- A is **satisfiable**, if the interpretation of A is a non-empty set, for *some* valuation v and *some non-empty set* X
- Otherwise, it is a **contradiction**

Is A' from the previous slide a tautology ?
satisfiable ?
a contradiction ?

Tautology, satisfiability, contradiction in the power set semantics

Definition

In the power set semantics:

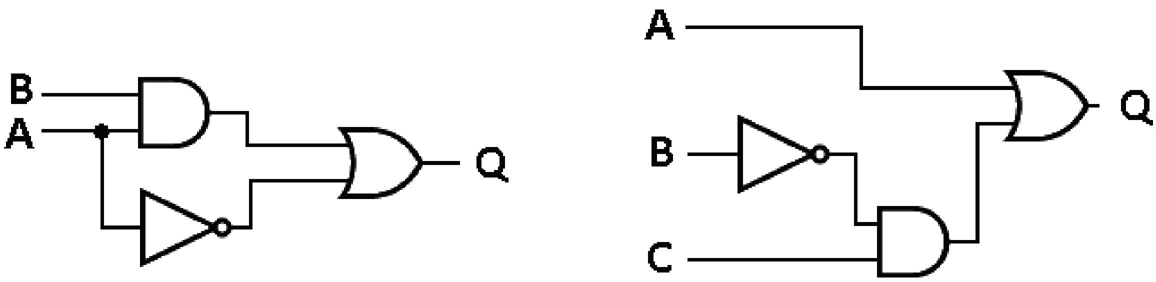
- A propositional formula A is a **tautology**, if the interpretation of A is X, for *all possible* valuations v and *any non-empty set* X
- A is **satisfiable**, if the interpretation of A is a non-empty set, for *some* valuation v and *some non-empty set* X
- Otherwise, it is a **contradiction**

Is A' from the previous slide a tautology ?
satisfiable ?
a contradiction ?

Note: There is no need to calculate further assignments to establish A' is not a tautology

Electronic switching circuits

Propositional logic has direct applications in hardware design



Electronic switching circuits are built from logic gates

Wires can carry two voltage levels:

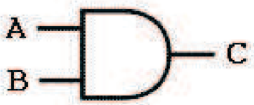
high voltage   or   low voltage

Source: http://www.nandland.com/articles/boolean-algebra-using-look-up-tables-lut.html

The logic circuit semantics

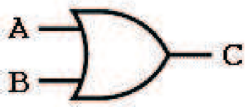
Logic gates:

AND



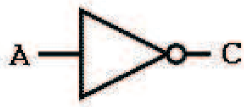
Inputs		Output
A	B	C
0	0	0
0	1	0
1	0	0
1	1	1

OR



Inputs		Output
A	B	C
0	0	0
0	1	1
1	0	1
1	1	1

NOT



Input	Output
A	C
0	1
1	0

1 represents high voltage   **0** represents low voltage

In the **logic circuit semantics**: Each input wire of the circuit represents a propositional variable, and $\wedge$, $\vee$ and $\neg$ are respectively assigned to an AND-gate, an OR-gate and a NOT-gate

Thus: Propositional logic can be used to test if two circuits are equivalent

Source: http://newstudent.groups.et.byu.net/Labs/Logic%20Gates/LogicGates.html