Section 3

Equivalent formulas

Semantically equivalent formulas

Definition

We say two formulas A and B are **semantically equivalent**, and write $A \equiv B$ in this case, iff their interpretations are the same, for any valuation v

I.e., in the Boolean semantics,

$$A \equiv B, \qquad \text{if } A \text{ and } B \text{ have the same truth tables}$$

Examples of semantically equivalent formulas

$P \equiv \neg\neg P$

P	$\neg P$	$\neg\neg P$

double negation / involution

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idempotence

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$P \;\equiv\; P \wedge P$

P	$P \wedge P$

idempotence

$P \;\equiv\; P \vee P$

P	$P \vee P$
1	**1**
0	**0**

idempotence

Proving formulas are semantically equivalent

$P \to Q \;\equiv\; \neg P \vee Q$

P	Q	$P \to Q$	$\neg P$	$\neg P \vee Q$
1	**1**	**1**	**0**	**1**
1	**0**	**0**	**0**	**0**
0	**1**	**1**	**1**	**1**
0	**0**	**1**	**1**	**1**

This means: $\to$ can be expressed in terms of $\neg$ and $\vee$

Fundamental semantic equivalences

$P \wedge P \equiv P$		idempotence
$P \vee P \equiv P$		idempotence
$P \wedge Q \equiv Q \wedge P$		commutative
$P \vee Q \equiv Q \vee P$		commutative
$P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$		associative
$P \vee (Q \vee R) \equiv (P \vee Q) \vee R$		associative
$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$		distributive
$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$		distributive
$P \wedge (P \vee Q) \equiv P$		absorption
$P \vee (P \wedge Q) \equiv P$		absorption

Fundamental semantic equivalences (cont'd)

$\neg\neg P \equiv P$	double negation/involution
$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$	De Morgan's law
$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$	De Morgan's law
$P \rightarrow Q \equiv \neg P \vee Q$	
$P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$	
$P \rightarrow Q \equiv \neg Q \rightarrow \neg P$	

Augustus De Morgan (1806–1871)
`https://en.wikipedia.org/`

Substitution for propositional variables

Substitution means uniformly replacing propositional variables within a formula by other formulas

Substituting formulas $C_1, \dots, C_n$ for $P_1, \dots, P_n$ in

$$A(P_1, \dots, P_n)$$

gives

$$A(C_1, \dots, C_n)$$

Example

$$A(P, Q) = P \lor \neg(Q \to P)$$

implies $A(P \land Q, \neg R) =$

Substitution Theorem

Theorem

If

$$A(P_1, \dots, P_n) \equiv B(P_1, \dots, P_n)$$

then

$$A(C_1, \dots, C_n) \equiv B(C_1, \dots, C_n)$$

Example

From De Morgan's law:

$$\neg(P \wedge Q) \;\equiv\; \neg P \vee \neg Q$$

we get

$$\neg(\underline{(P \vee \neg R)} \wedge \underline{(P \to S)}) \;\equiv\; \neg\underline{(P \vee \neg R)} \vee \neg\underline{(P \to S)}$$

In fact, we get

$$\neg(A \wedge B) \;\equiv\; \neg A \vee \neg B \qquad \text{for any propositional formulas } A, B$$

Similarly, for the other fundamental semantic equivalences

Replacement of a formula

Assume A is a subformula of C

$$C(\ldots A \ldots)$$

Replacing one (!) occurrence of A in C by B results in the formula

$$C(\ldots B \ldots)$$

Replacing the second occurrence of $P \wedge Q$ in

$$(P \wedge Q) \to (R \leftrightarrow \underline{(P \wedge Q)})$$

by $Q \wedge P$ results in

$$(P \wedge Q) \to (R \leftrightarrow \underline{(Q \wedge P)})$$

Equivalent Replacement Theorem

Theorem

Given *are propositional formulas* A, B, C, *and suppose* A *is a subformula of* C

Then

$$A \equiv B \qquad \textit{implies} \qquad C(\ldots A \ldots) \ \equiv \ C(\ldots B \ldots)$$

$C(\ldots B \ldots)$ *is obtained by replacement from* $C(\ldots A \ldots)$

From

$$\neg(P \wedge Q) \ \equiv \ \neg P \vee \neg Q$$

we get by equivalent replacement

$$\underline{\neg(P \wedge Q)} \vee \neg Q \ \equiv \ \underline{(\neg P \vee \neg Q)} \vee \neg Q$$

Simplifying propositional formulas

Example

Simplify

$$\neg(P \wedge Q) \vee \neg Q$$

Solution:

$\neg(P \wedge Q) \vee \neg Q$

$\equiv$	$(\neg P \vee \neg Q) \vee \neg Q$	by equiv. replacement using De Morgan $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
$\equiv$	$\neg P \vee (\neg Q \vee \neg Q)$	by equiv. repl. using general assoc. of $\vee$
$\equiv$	$\neg P \vee \neg Q$	by equiv. repl. using $A \vee A \equiv A$

This shows

$$\neg(P \wedge Q) \vee \neg Q \ \equiv \ \neg P \vee \neg Q$$

The logical constants $\top$ and $\bot$

It is useful to extend the language with the logical constants

$\top$ **top/truth** $\qquad$ $\bot$ **bottom/falsum**

Definition

- The interpretation of $\top$ is $\mathbf{1}$, for any interpretation (i.e., relative to any valuation of the propositional variables)
- The interpretation of $\bot$ is $\mathbf{0}$, for any interpretation

Think of $\top$ and $\bot$ as abbreviations of $P \vee \neg P$ and $P \wedge \neg P$, since:

P	$\neg P$	$P \vee \neg P$	$P \wedge \neg P$
$\mathbf{1}$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{0}$
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$

$\top$ is a tautology $\qquad$ $\bot$ is a contradiction

Fundamental properties of $\top$ and $\bot$

Theorem

- *If A is a tautology then* $A \equiv \top$
- *If A is a contradiction then* $A \equiv \bot$

$$A \vee \neg A \equiv \top \qquad \text{excluded middle}$$

$$A \wedge \neg A \equiv \bot \qquad \text{contradiction}$$

$$A \to A \equiv \top$$

$$A \leftrightarrow A \equiv \top$$

$$\neg\top \equiv \bot \qquad\qquad \neg\bot \equiv \top$$

$$A \vee \top \equiv \top \equiv \top \vee A \qquad\qquad A \wedge \top \equiv A \equiv \top \wedge A$$

$$A \vee \bot \equiv A \equiv \top \vee A \qquad\qquad A \wedge \bot \equiv \bot \equiv \bot \wedge A$$

$$A \to \bot \equiv \neg A \qquad\qquad A \leftrightarrow \top \equiv A$$

$$\bot \to A \equiv \top \qquad\qquad A \leftrightarrow \bot \equiv \neg A$$

Example

Simplify

$$((P \wedge Q) \rightarrow (Q \vee \neg P)) \wedge (P \vee \neg P)$$

Solution:

$$\begin{array}{lll} \multicolumn{3}{l}{\underline{((P \wedge Q) \rightarrow (Q \vee \neg P))} \wedge (P \vee \neg P)} \\ \equiv & \top \wedge (P \vee \neg P) & (P \wedge Q) \rightarrow (Q \vee \neg P) \text{ is a tautology,} \\ & & \text{see Slide 31, and first part of Theorem} \\ \equiv & \top \wedge \top & \text{excluded middle } A \vee \neg A \equiv \top \\ \equiv & \top & \top \wedge A \equiv A \end{array}$$

Summary

- Using truth tables we can check whether formulas are tautologies, contradictions, satisfiable, semantically equivalent
- Limitations of truth tables:
  Already for a modest number of variables truth tables are unacceptably large!
- Alternative to establishing equivalence:
  Using the fundamental laws, along with the substitution into propositional variables and equivalent replacement
- This is useful for simplifying formulas

Semantically equivalent formulas in the other semantics

Definition (as before)

$A \equiv B$ iff their interpretations are the same, for any valuation v.

In the Boolean semantics

$A \equiv B$ if A and B have the same truth tables

In the power set semantics

$A \equiv B$ if $S_A = S_B$,
where S_A denotes the assignment to A

In the logic circuit semantics

$A \equiv B$ if the voltage output of either circuit is the same
for all possible input voltages

Note: To test semantic equivalence it suffices to do this in one model
We don't have the test equivalence in all three models!