1. Mathematical Logic

Proposition:

It is a declarative sentence which can be either true (or) false but not both.

Example

1. "The sun raises in the east" is a proposition having truth value T
2. " $2+3=6$ " is a proposition whose truth value is F
3. " $3+5=8$ " is a proposition whose truth value is T
4. "what is time now?" is not a proposition, because it is a question.
5. " $x+5=7$ " is not a proposition because no value is assigned for x

Note:

1. The propositions are denoted by P, Q, R .... (or) p, q, r .... etc

Propositional logic:

The area of logic that deals with propositions is called propositional logic

Negation of a proposition:

The negation of a proposition 'p' is denoted by $\neg p$ (or) $\sim p$ which means "not happening of p". which is also a proposition.

Example

p: "Today is monday"

$\sim p$: "Today is not a monday"

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Truth table for nagation:

P	$\neg P$
T	F
F	T

Connectives:

For constructing compound proposition we need Connectives such as Conjunction ($\wedge$), Disjunction($\vee$), implication($\rightarrow$), Biimplication($\leftrightarrow$)

1. Conjunction ($\wedge$):

Conjunction of two prepositions p, q is denoted by "$p \wedge q$" which is also a proposition having Truth value T whenever both p and q are ~~two~~ true otherwise false.

Example:

"Today is Monday" and "I have attended DM class Today"

Truth table for Conjunction:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

2. Disjunction($\vee$):

Disjunction of two propositions p, q is denoted by ($p \vee q$) which is also a proposition whose truth value is F whenever p and q are false, otherwise it is ~~true~~ T

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Example

"Today is monday" or "I have attended DM class Today"

Truth table for Disjunction

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

3. Exclusive OR (XOR) (⊕):

The XOR of two propositions p and q is denoted by $p \oplus q$ which is also a proposition whose truth value is T only when p and q have opposite truth values.

Example

"In today evening session "attend the class" or "go to a movie"

Truth table:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

4. Implication (→) (conditional statement):

If p and q are any two prepositions then the preposition "$p \to q$" (if p then q) is called an implication (or) Conditional statement whose truth value is false (F) only when p is true and q is false.

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Example

If "there is a flood" then "The crop will be destroyed"

Truth table

P	q	$P \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Problems

1. show that empty set is a subset of every set

Sol Let A be any set

claim: $\phi \subseteq A$

To prove this we have to prove

$x \in \phi \Rightarrow x \in A$

P : "$x \in \phi$" which is always False (F)

q : "$x \in A$" which may be true (T) (or) false (F)

Since, P is false, from implication table $P \to q$ is always true (T)

i.e $x \in \phi \Rightarrow x \in A$ is true

$\therefore \phi \subseteq A$

Hypothesis → conclusion

P	q	$P \to q$
T	T	T
T	F	F
F	T	T
F	F	T

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2. show that every set is a subset of itself.

sol let A be any set

claim : $A \subseteq A$

To prove this we have to prove

$x \in A \Rightarrow x \in A$

p : "$x \in A$" which may be true (T) (or) false (F)

q : "$x \in A$" which may be true (T) (or) false (F)

since, p and q having same truth values from

From implication truth table $p \rightarrow q$ is true (T)

i.e $x \in A \Rightarrow x \in A$ is true

$\therefore A \subseteq A$

5. Biimplication ($\leftrightarrow$)

If p and q be any two propositions then the proposition $p \leftrightarrow q$ (P if and only if q) is called Biimplication (or) Biconditional statement whose truth value is true only when both p and q have same truth values.

Example

"Election results are declared" if and only if "The cm candidate is elected"

Truth table for Biimplication:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Note:

$p \leftrightarrow \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

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Construction of truth tables:

Problems

1. Construct the truth table for the following formulas (or) Compound propositions (or) Expressions.

① $P \vee \neg P$

② $P \wedge \neg P$

③ $\neg P \wedge \neg q$

④ $(P \wedge q) \vee r$

sol ① $P \vee \neg P$

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

② $P \wedge \neg P$

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

③ $\neg P \wedge \neg q$

P	q	$\neg P$	$\neg q$	$\neg P \wedge \neg q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

④ $(P \wedge q) \vee r$

P	q	r	$P \wedge q$	$(P \wedge q) \vee r$
T	T	T	T	T
T	T	F	T	T
T	F	T	F	T
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	F	T
F	F	F	F	F

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2. Construct the truth tables for the following

i. $[q \wedge (p \rightarrow q)] \rightarrow p$

ii. $\neg(p \vee (q \wedge r)) \leftrightarrow (p \vee q) \wedge (p \vee r)$

i. $[q \wedge (p \rightarrow q)] \rightarrow p$

p	q	$p \rightarrow q$	$q \wedge (p \rightarrow q)$	$[q \wedge (p \rightarrow q)] \rightarrow p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

ii. $\neg(p \vee (q \wedge r)) \overset{b}{\leftrightarrow} \overset{a}{(p \vee q) \wedge (p \vee r)}$

p	q	r	$(p \vee q)$	$q \wedge r$	$p \vee (q \wedge r)$	$\neg(p \vee (q \wedge r))$	$(p \vee r)$	a	b
T	T	T	T	T	T	F	T	T	F
T	T	F	T	F	T	F	T	T	F
T	F	T	T	F	T	F	T	T	F
T	F	F	T	F	T	F	T	T	F
F	T	T	T	T	T	F	T	T	F
F	T	F	T	F	F	T	F	F	F
F	F	T	F	F	F	T	T	F	F
F	F	F	F	F	F	T	F	F	F

3. Construct truth table for the following statements.

i. $(p \rightarrow q) \leftrightarrow (\neg p \vee q)$

ii. $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$

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i) $(p \to q) \leftrightarrow (\neg p \vee q)$

p	q	$p \to q$	$\neg p$	$\neg p \vee q$	$(p \to q) \leftrightarrow (\neg p \vee q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

ii) $(p \to q) \wedge (q \to r) \to (p \to r)$

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$(p \to q) \wedge (q \to r) \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

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Tautology:

A compound proposition that is always true irrespective of truth values occur in it is called a tautology.

Example

$P \vee \neg P$ is a Tautology. Because it is always true for any possible truth values of P ~~and ¬P~~

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

Contradiction:

A compound proposition which is always false (F) irrespective of truth values occur in it is called contradiction

Example $P \wedge \neg P$

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

Contingency:

A compount proposition that is neither a tautology not a contradiction is called contingency

Example $P \rightarrow \neg P$

P	$\neg P$	$P \rightarrow \neg P$
T	F	F
F	T	T

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Problems

1. Indicate which of the following formulas are tautologies, (or) contradiction.

i) $p \rightarrow p \vee q$

ii) $(p \rightarrow \neg p) \rightarrow \neg p$

iii) $((\neg q \wedge p) \wedge q)$

iv) $(p \rightarrow (q \rightarrow r)) \rightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)]$

v) $(\neg p \rightarrow q) \rightarrow (q \rightarrow p)$

vi) $((p \wedge q) \leftrightarrow p)$

i)

p	q	$p \vee q$	$p \rightarrow p \vee q$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

It is a Tautology

ii) $(p \rightarrow \neg p) \rightarrow \neg p$

p	$\neg p$	$p \rightarrow \neg p$	$(p \rightarrow \neg p) \rightarrow \neg p$
T	F	F	T
F	T	T	T

It is a Tautology

iii) $((\neg q \wedge p) \wedge q)$

p	q	$\neg q$	$\neg q \wedge p$	$(\neg q \wedge p) \wedge q$
T	T	F	F	F
T	F	T	T	F
F	T	F	F	F
F	F	T	F	F

It is a contradiction.

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$(p \to (q \to r)) \to [(p \to q) \to (p \to r)]$

iv,

p	q	r	$q \to r$	$p \to (q \to r)$	$p \to q$	$p \to r$	a	b
T	T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F	T
T	F	T	T	T	F	T	T	T
T	F	F	T	T	F	F	T	T
F	T	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

It is a Tautology.

v, $(\neg p \to q) \to (q \to p)$

p	q	$\neg p$	$\neg p \to q$	$q \to p$	$(\neg p \to q) \to (q \to p)$
T	T	F	T	T	T
T	F	F	T	T	T
F	T	T	T	F	F
F	F	T	F	T	T

It is a Condingency.

vi, $((p \wedge q) \leftrightarrow p)$

p	q	$p \wedge q$	$(p \wedge q) \leftrightarrow p$
T	T	T	T
T	F	F	F
F	T	F	T
F	F	F	T

It is a Condingency.

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Equivalence formulas:

Two compound propositions are said to be equivalent if both have same truth values in any posibility.

Example

1. If two compound prepositions $\neg(p \wedge q)$ and $(\neg p \vee \neg q)$ both are equivalent

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$\neg(p \vee q)$
T	T	T	F	F	F	F	F
T	F	F	T	F	T	T	T
F	T	F	T	T	F	T	T
F	F	F	T	T	T	T	T

2. show that the following are equivalent.

* i) $p \to q \equiv \neg p \vee q$

ii) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

iii) $(p \to q) \wedge (q \to r) \equiv p \vee r$

p	q	$\neg p$	$p \to q$	$\neg p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

$p \to q$ is equivalent to $\neg p \vee q$

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ii,

p	q	r	$q \wedge r$	$p \vee q$	$p \vee r$	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	T	F	F	F
F	F	T	F	F	T	F	F
F	F	F	F	F	F	F	F

It is equivalent.

iii,

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \vee r$
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	F	T	F	T
T	F	F	F	T	F	T
F	T	T	T	T	T	T
F	T	F	T	F	F	F
F	F	T	T	T	T	T
F	F	F	T	T	T	F

It is not equivalent

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Equivalent Formulas:

1. Idempotent law:

$$P \vee P \equiv P$$

$$P \wedge P \equiv P$$

2. Commutative law:

$$(P \vee q) \equiv (q \vee P)$$

$$(P \wedge q) \equiv (q \wedge P)$$

3. Assosiative law:

$$(P \vee q) \vee r \equiv P \vee (q \vee r)$$

$$(P \wedge q) \wedge r \equiv P \wedge (q \wedge r)$$

4. Distributive law:

$$P \vee (q \wedge r) \equiv (P \vee q) \wedge (P \vee r)$$

$$P \wedge (q \vee r) \equiv (P \wedge q) \vee (P \wedge r)$$

5. Demorgan's law:

$$\neg (P \wedge q) \equiv \neg P \vee \neg q$$

$$\neg (P \vee q) \equiv \neg P \wedge \neg q$$

6. Absorption law:

$$P \vee (P \wedge q) \equiv P$$

$$P \wedge (P \vee q) \equiv P$$

7. Identity law:

$$P \wedge T \equiv P$$

$$P \vee F \equiv P$$

8. Domination law:

$$P \vee T \equiv T$$

$$P \wedge F \equiv F$$

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9. Double negotation law:

$$\neg(\neg p) \equiv p$$

10. Negation law:

$$p \vee \neg p \equiv T$$

$$p \wedge \neg p \equiv F$$

Problems

1. show that

i) $p \vee (p \wedge q) \equiv p$

ii) $p \wedge (p \vee q) \equiv p$

i)

p	q	$p \wedge q$	$p \vee (p \wedge q)$
T	T	T	T
T	F	F	T
F	T	F	F
F	F	F	F

$\therefore p \vee (p \wedge q) = p$

ii) $p \wedge (p \vee q)$

p	q	$p \vee q$	$p \wedge (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

$\therefore p \wedge (p \vee q) = p$

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2. show that i, $P \to (q \to r) \Leftrightarrow (P \wedge q) \to r$

ii, $\neg(P \vee (\neg P \wedge q)) \equiv \neg P \wedge \neg q$ without using truth table.

Sol i $P \to (q \to r) \equiv (P \wedge q) \to r$

$P \to (q \to r)$

$\equiv \neg P \vee (q \to r)$ $(\because P \to q \equiv \neg P \vee q)$

$\equiv \neg P \vee (\neg q \vee r)$ $(\because P \to q \equiv \neg P \vee q)$

$\equiv (\neg P \vee \neg q) \vee r$ $(\because$ Assosiative law)

$\equiv \neg(P \wedge q) \vee r$ $(\because$ By demorgan's law)

$\equiv (P \wedge q) \to r$ $(\because P \to q \equiv \neg P \vee q)$

ii, $\neg(P \vee (\neg P \wedge q)) \equiv \neg P \wedge \neg q$

$\neg(P \vee (\neg P \wedge q))$

$\equiv \neg P \wedge \neg(\neg P \wedge q)$ $(\because$ Demorgans law)

$\equiv \neg P \wedge (\neg(\neg P) \vee \neg q)$ $(\because$ Demorgans law)

$\equiv \neg P \wedge (P \vee \neg q)$ $(\because$ Double nagation)

$\equiv (\neg P \wedge P) \vee (\neg P \wedge \neg q)$ $(\because$ Distributive)

$\equiv F \vee (\neg P \wedge \neg q)$ $(\because$ nagation law)

$\equiv \neg P \wedge \neg q$

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iii, $(\neg p \wedge (\neg q \wedge r)) \vee (q \wedge r) \vee (p \wedge r) \equiv r$

$(\neg p \wedge (\neg q \wedge r)) \vee (q \wedge r) \vee (p \wedge r)$

$\equiv ((\neg p \wedge \neg q) \wedge r) \vee (q \wedge r) \vee (p \wedge r)$ (∵ Associative law)

$\equiv (\neg (p \vee q) \wedge r) \vee (q \wedge r) \vee (p \wedge r)$ (∵ Demorgans law)

$\equiv (\neg (p \vee q) \wedge r) \vee ((q \vee p) \wedge r)$ (∵ Distributive law)

$\equiv (\neg (p \vee q) \vee (q \vee p)) \wedge r$ (∵ Distributive law)

$\equiv T \wedge r$ (∵ Nagation law)

$\equiv r$ (∵ Identity law)

3. show that i, $\neg (p \wedge q) \rightarrow (\neg p \vee (\neg p \vee q)) \Leftrightarrow (\neg p \vee q)$

ii, $(p \vee q) \wedge (\neg p \wedge (\neg p \wedge q)) \Leftrightarrow (\neg p \wedge q)$

Sol i. $\neg (p \wedge q) \rightarrow (\neg p \vee (\neg p \vee q)) \Leftrightarrow (\neg p \vee q)$

$\neg (p \wedge q) \rightarrow (\neg p \vee (\neg p \vee q))$

$\equiv \neg (\neg (p \wedge q)) \vee (\neg p \vee (\neg p \vee q))$

$\equiv (p \wedge q) \vee (\neg p \vee (\neg p \vee q))$

$\equiv (p \wedge q) \vee ((\neg p \vee \neg p) \vee q)$

$\equiv (p \wedge q) \vee (\neg p \vee q)$

$\equiv ((p \wedge q) \vee \neg p) \vee q$

$\equiv ((p \vee \neg p) \wedge (q \vee \neg p)) \vee q$

$\equiv (T \wedge (q \vee \neg p)) \vee q$

$\equiv (q \vee \neg p) \vee q$

$\equiv (\neg p \vee q) \vee q$

$\equiv \neg p \vee (q \vee q) \equiv \neg p \vee q$

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ii) $(p \vee q) \wedge (\neg p \wedge (\neg p \wedge q))$

$\equiv (p \vee q) \wedge ((\neg p \wedge \neg p) \wedge q)$ ($\because$ Associate rule)

$\equiv (p \vee q) \wedge (\neg p \wedge q)$ ($\because$ Idempotent rule)

$\equiv ((p \vee q) \wedge \neg p) \wedge q$ ($\because$ Associative rule)

$\equiv ((p \wedge \neg p) \vee (q \wedge \neg p)) \wedge q$ ($\because$ Distributive rule)

$= (F \vee (q \wedge \neg p)) \wedge q$ ($\because$ Negation law)

$= (q \wedge \neg p) \wedge q$

$= (\neg p \wedge q) \wedge q$ ($\because$ Commutative law)

$= \neg p \wedge (q \wedge q)$ ($\because$ Associative law)

$= \neg p \wedge q$

(or)

<u>Method-2</u>

From ① we have

$\neg(p \wedge q) \rightarrow (\neg p \vee (\neg p \vee q)) \Leftrightarrow \neg p \vee q$

$\neg(\neg(p \wedge q)) \vee (\neg p \vee (\neg p \vee q)) \Leftrightarrow \neg p \vee q$

$(p \wedge q) \vee (\neg p \vee (\neg p \vee q)) \Leftrightarrow \neg p \vee q$

Now by duality rule replace $\wedge, \vee$ with $\vee, \wedge$

$(p \vee q) \wedge (\neg p \wedge (\neg p \wedge q)) \Leftrightarrow \neg p \wedge q$

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1. show that $p \wedge q \rightarrow p \vee q$ is tautology.

$p \wedge q \rightarrow p \vee q$ $\quad (\because p \rightarrow q = \neg p \vee q)$

$\equiv \neg(p \wedge q) \vee (p \vee q)$ $\quad (\because$ Demorgan's law$)$

$\equiv (\neg p \vee \neg q) \vee (p \vee q)$ $\quad (\because$ Associative law$)$

$\equiv ((\neg p \vee \neg q) \vee p) \vee q$ $\quad (\because$ Commutative law$)$

$\equiv ((\neg q \vee \neg p) \vee p) \vee q$ $\quad (\because$ Associative law$)$

$\equiv (\neg q \vee (\neg p \vee p)) \vee q$ $\quad (\because$ Identity law$)$

$\equiv (\neg q \vee T) \vee q$ $\quad (\because$ Identity law$)$

$\equiv \neg q \vee q$

$\equiv T$

2. S.T $((p \vee q) \wedge \neg(\neg p \wedge (\neg q \vee \neg r))) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$ is tautology.

$((p \vee q) \wedge \neg(\neg p \wedge (\neg q \vee \neg r))) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$ — ①

let us consider

$\neg(\neg p \wedge (\neg q \vee \neg r))$

$\neg(\neg p) \vee \neg(\neg q \vee \neg r)$

$p \vee \neg(\neg(q \wedge r))$

$p \vee (q \wedge r)$

$(p \vee q) \wedge (p \vee r)$ — ②

from ① and ②.

$(p \vee q) \wedge ((p \vee q) \wedge (p \vee r)) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$

$[((p \vee q) \wedge (p \vee q)) \wedge (p \vee r)] \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$

$[(p \vee q) \wedge (p \vee r)] \vee (\neg(p \vee q) \vee \neg(p \vee r))$

$\cancel{[(p \vee q) \wedge \neg(p \vee q)] \wedge [(p \vee r) \vee (p \vee q)] \vee \neg(p \vee r)}$

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$$\cancel{T \wedge ((p \vee r) \vee \neg(p \vee q)) \vee \neg(p \vee r)}$$

$$((p \vee r) \vee \neg(p \vee q)) \vee \neg(p \vee r)$$

$$((p \vee q) \wedge (p \vee r)) \vee \neg((p \vee q) \wedge (p \vee r))$$

$$T$$

3. write the following formulas equivalent to formulas that involved $\neg, \wedge$ only

i) $\neg(p \leftrightarrow (q \rightarrow (r \vee p)))$

ii) $((p \vee q) \wedge r) \rightarrow p \vee r$

$$\neg(p \leftrightarrow (q \rightarrow (r \vee p)))$$

$$\neg[[p \rightarrow (q \rightarrow (r \vee p))] \wedge [(q \rightarrow (r \vee p)) \rightarrow p]]$$

$$\neg[[\neg p \vee (q \rightarrow (r \vee p))] \wedge [(\neg q \vee (r \vee p)) \vee p]]$$

$$\neg[[\neg p \vee \neg q \vee (r \vee p)] \wedge [\neg(\neg q \vee (r \vee p)) \vee p]]$$

$$\neg[[(\neg p \vee p) \vee \neg q \vee r] \wedge [\neg(\neg q) \wedge \neg(r \vee p) \vee p]]$$

$$\neg[[T \vee (\neg q \vee r)] \wedge [q \wedge (\neg r \wedge \neg p) \vee p]]$$

$$\neg[T \wedge [q \wedge (\neg r \wedge \neg p) \vee p]]$$

$$\neg[[q \wedge (\neg r \wedge \neg p)] \vee p]$$

$$\neg(q \wedge (\neg r \wedge \neg p)) \wedge \neg p$$

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ii, $((p \vee q) \wedge r) \rightarrow (p \vee r)$

$\neg((p \vee q) \wedge r) \vee (p \vee r)$

$\neg((p \wedge r) \vee (q \wedge r)) \vee (p \vee r)$

$[\neg(p \wedge r) \wedge \neg(q \wedge r)] \vee (p \vee r)$

$(\neg p \vee \neg r) \wedge (\neg q \vee \neg r) \vee (p \vee r)$

$[(\neg p \vee \neg r) \vee (p \vee r)] \wedge [(\neg q \vee \neg r) \vee (p \vee r)]$

$(\neg p \vee p) \vee (\neg r \vee r) \wedge [(\neg q \vee \neg r) \vee (p \vee r)]$

$T \vee T \wedge [(\neg r \vee r) \vee \neg q \vee p]$

$T \wedge [T \vee (\neg q \vee p)]$

$T \wedge T$

T.

Normal forms:

Elementary product:

The product of the variables and their negation in a formula is called elementary product.

For example: let p and q be any two atomic variables then p, $\neg p \wedge q$, $p \wedge \neg r$, $r \wedge \neg p$, $p \wedge r$ are elementary product

Elementary sum:

The sum of the variables and their negation in a formula is called elementary sum

For example: let p and q be any two atomic variables then p, $\neg p \vee q$, $p \vee \neg r$, $r \vee \neg p$, $p \vee r$ are elementary sum

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Factor:

Any part of any elementary sum (or) product which is itself an elementary sum (or) product is called a factor of the original elementary sum or product

Example

$\neg q$, p, $\neg p$, $\neg q \wedge \neg p$, $p \wedge \neg p$, $\neg p \wedge \neg q$ are some of the factors of $\neg q \wedge p \wedge \neg p$

Disjunctive Normal Form (DNF) (SOP):

Example: $(p \wedge q) \vee (p \wedge \neg q) \vee p$

Problems

1. obtain a DNF of $p \wedge (p \rightarrow q)$

$p \wedge (p \rightarrow q)$

$\equiv p \wedge (\neg p \vee q)$

$\equiv (p \wedge \neg p) \vee (p \wedge q)$ $\Leftrightarrow F \vee (p \wedge q) \Rightarrow p \wedge q \Rightarrow (p \wedge q) \vee (p \wedge q)$

Disjunctive normal form

2. obtain a DNF of ~~$(p \wedge q)$~~ $\neg(p \vee q) \leftrightarrow p \wedge q$

$\neg(p \vee q) \leftrightarrow p \wedge q$

$\equiv [\neg(p \vee q) \rightarrow (p \wedge q)] \wedge [(p \wedge q) \rightarrow \neg(p \vee q)]$

$\equiv \neg(\neg(p \vee q)) \vee (p \wedge q) \wedge [\neg(p \wedge q) \vee \neg(p \vee q)]$

$\equiv (p \vee q) \vee (p \wedge q) \wedge [(\neg p \vee \neg q) \vee (\neg p \wedge \neg q)$

$= ((p \vee q) \vee p) \wedge ((p \vee q) \vee q) \wedge [((\neg p \vee \neg q) \vee \neg p) \wedge (\neg p \vee \neg$

$= (p \wedge q) \wedge (\neg p \wedge \neg q)$

$= (p \wedge q) \wedge \neg(p \vee q)$

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$\equiv T$

$\equiv p \vee \neg p$

(or)

$\neg(p \vee q) \Leftrightarrow p \wedge q$

$[\neg(p \vee q) \wedge (p \wedge q)] \vee [\neg(\neg(p \vee q) \wedge \neg(p \wedge q)]$

$(\because p \Leftrightarrow q = (p \wedge q) \vee (\neg p \wedge \neg q))$

$[\neg(p \vee q) \wedge (p \wedge q)] \vee [(p \vee q) \wedge \neg(p \wedge q)]$

$[(\neg p \wedge \neg q) \wedge (p \wedge q)] \vee [(p \vee q) \wedge (\neg p \vee \neg q)]$

$[(\neg p \wedge p) \wedge (\neg q \wedge q)] \vee [(p \vee q) \wedge (\neg p \vee \neg q)]$

$F \wedge F \vee [(p \vee q) \wedge (\neg p \vee \neg q)]$

$F \vee [(p \vee q) \wedge (\neg p \vee \neg q)]$

$(p \vee q) \wedge (\neg p \vee \neg q)$

$((p \vee q) \wedge \neg p) \vee ((p \vee q) \wedge \neg q)$

$(p \wedge \neg p) \vee (q \wedge \neg p) \vee (p \wedge \neg q) \vee (q \wedge \neg q)$

$F \vee (q \wedge \neg p) \vee (p \wedge \neg q) \vee F$

$(q \wedge \neg p) \vee (p \wedge \neg q)$

$\boxed{p \leftrightarrow q = (p \wedge q) \vee (\neg p \wedge \neg q)}$

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3. obtain DNF of $P \to ((P \to q) \wedge \neg(\neg q \vee \neg P))$

$P \to ((P \to q) \wedge \neg(\neg q \vee \neg P))$

$P \to ((P \to q) \wedge \neg(\neg(q \wedge P))$

$P \to ((P \to q) \wedge (P \wedge q))$

$\neg P \vee ((P \to q) \wedge (P \wedge q))$

$\neg P \vee ((\neg P \vee q) \wedge (P \wedge q))$

$\neg P \vee ((\neg P \wedge P \wedge q) \vee (q \wedge P \wedge q))$

$\neg P \vee ((F \wedge q) \vee (P \wedge q))$

$\neg P \vee (\cancel{F \vee q})\ (P \wedge q)$

$\neg P \vee (q \wedge P)$

$(\neg P \wedge \neg P) \vee (q \wedge P)$

Conjunctive Normal form:

A formula which is equivalent to a given formula and consists of a product of elementary sum is called a Conjunctive normal form of the given formula

Example

$(P \vee q) \wedge (\neg P \vee q) \wedge (\neg P \vee r)$

Problem

1. obtain a CNF of $P \wedge (P \to q)$

$P \wedge (P \to q)$

$P \wedge (\neg P \vee q)$

$(q \wedge P) \wedge (\neg P \vee q)$

$(P \wedge \neg P) \vee (P \wedge q)$

$F \vee (P \wedge q)$

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$(p \wedge q)$

$(p \vee p) \wedge (q \vee q)$

2. obtain a CNF of $\neg(p \vee q) \leftrightarrow (p \wedge q)$

$\neg(p \vee q) \leftrightarrow (p \wedge q)$

$\equiv (\neg(p \vee q) \wedge (p \wedge q)) \vee (\neg(\neg(p \vee q)) \wedge \neg(p \wedge q))$

$\equiv (\neg p \wedge \neg q \wedge p \wedge q) \vee ((p \vee q) \wedge (\neg p \vee \neg q))$

$\equiv (p \wedge \neg p \wedge q \wedge \neg q) \vee (((p \vee q) \wedge \neg p) \vee (p \vee q) \wedge \neg q)$

$\equiv F \vee ((p \vee q) \wedge \neg p \vee (p \vee q) \wedge \neg q)$

$\equiv (p \wedge \neg p) \vee (q \wedge \neg p) \vee (p \wedge \neg q) \vee (q \wedge \neg q)$

$\equiv F \vee (q \wedge \neg p) \vee (p \wedge \neg q) \vee F$

$\equiv ((q \wedge \neg p) \vee p) \wedge ((q \wedge \neg p) \vee \neg p)$

$\equiv (q \vee p) \wedge (\neg p \vee p) \wedge (q \vee \neg q) \wedge (\neg p \vee \neg q)$

$\equiv (q \vee p) \wedge T \wedge T \wedge (\neg p \vee \neg q)$

$\equiv (q \vee p) \wedge (\neg p \vee \neg q)$

(or)

$\neg(p \vee q) \leftrightarrow (p \wedge q)$

$\equiv (\neg(p \vee q) \rightarrow p \wedge q) \wedge ((p \wedge q) \rightarrow \neg(p \vee q))$

$\equiv (\neg(\neg(p \vee q)) \vee (p \wedge q)) \wedge (\neg(p \wedge q) \vee \neg(p \vee q))$

$\equiv (p \vee q) \vee (p \wedge q) \wedge ((\neg p \vee \neg q) \vee (\neg p \wedge \neg q))$

$\equiv p \vee (q \vee (p \wedge q)) \wedge (\neg p \vee (\neg q \vee (\neg p \wedge \neg q)))$

$\equiv (p \vee q) \wedge (\neg p \vee \neg q)$

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3. obtain a CNF of $P \to (P \to q) \wedge \neg(\neg q \vee \neg P)$

$P \to (P \to q) \wedge \neg(\neg q \vee \neg P)$

$P \to (P \to q) \wedge \neg(\neg(q \wedge P))$

$P \to (P \to q) \wedge (q \wedge P)$

$\neg P \vee (P \to q) \wedge (q \wedge P)$

$\neg P \vee (\neg P \vee q) \wedge (q \wedge P)$

$(\neg P \vee \neg P \vee q) \wedge (q \wedge P)$

$(\neg P \vee q) \wedge (q \wedge P)$

$((\neg P \vee q) \wedge q) \wedge ((\neg P \vee q) \wedge P)$

$q \wedge ((\neg P \wedge P) \vee (q \wedge P))$

$q \wedge (F \vee (q \wedge P))$

$q \wedge q \wedge P$

$P \wedge q$

4. obtain a CNF of $P \to ((P \to q) \wedge \neg(\neg q \vee \neg P))$

$\neg P \vee ((P \to q) \wedge \neg(\neg(q \wedge P)))$

$\neg P \vee ((\neg P \vee q) \wedge (q \wedge P))$

$\neg P \vee (((\neg P \vee q) \wedge q) \wedge ((\neg P \vee q) \wedge P))$

$\neg P \vee (q \wedge (\neg P \wedge P) \vee (q \wedge P))$

$\neg P \vee ((q \wedge F) \vee (q \wedge P))$

$\neg P \vee F \vee (q \wedge P)$

$\neg P \vee (q \wedge P)$

$(\neg P \vee q) \wedge (\neg P \vee P)$

$\cancel{\neg P \vee q \wedge T}$ $T \wedge (\neg P \vee q)$

$(\neg P \vee q) \equiv (\neg P \vee q) \wedge (\neg P \vee q)$

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Principal disjunctive normal form:

For a given formula an equivalent formula consisting of disjunction of min terms containing conjunctions in which each variable (or) its nagation ~~but~~ not both appears only once

Example $(p \wedge q) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge q)$ for PDNF

example for not a PDNF is

$$(\neg p \wedge p) \vee (\neg p \wedge q)$$

1. obtain PDNF of $\neg p \vee q$

$$\neg p \vee q$$

$$\equiv \cancel{(\neg p \wedge \neg p) \vee (q \wedge q)}$$

$$\equiv (\neg p \wedge T) \vee (q \wedge T)$$

$$\equiv (\neg p \wedge (q \vee \neg q)) \vee (q \wedge (p \vee \neg p))$$

$$= (\neg p \wedge q) \vee (\neg p \wedge \neg q) \vee (q \wedge p) \vee (q \wedge \neg p)$$

$$= (\neg p \wedge q) \vee (\neg p \wedge \neg q) \vee (q \wedge p)$$

2. obtain PDNF of nagation $p \vee (\neg p \wedge \neg q \wedge r)$

$$\neg (p \vee (\neg p \wedge \neg q \wedge r))$$

$$\neg (p \vee \neg (p \vee q) \wedge r)$$

$$\cancel{\neg} \neg p \wedge \neg ((p \vee q) \wedge r)$$

$$\neg p \wedge \neg (\neg (p \vee q)) \vee \neg r$$

$$\neg p \wedge (p \vee q) \vee \neg r$$

$$\neg p \wedge ((p \vee q) \vee \neg r) \Rightarrow (\neg p \wedge (p \vee q)) \vee (\neg p \wedge \neg r)$$

$$(\neg p \wedge p) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg r)$$

$$F \vee (\neg p \wedge q) \vee (\neg p \wedge \neg r)$$

$$(\neg p \wedge q) \vee (\neg p \wedge \neg r)$$

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3. obtain PDNF of the formulas

i) $P \to q$

ii) $p \vee q$

iii) $\neg(p \wedge q)$

i) $P \to q$

$\neg p \vee q$

$\neg p \wedge F \vee q \wedge F$

$\cancel{\neg p \vee (q \wedge \neg q) \vee q \vee (p \wedge \neg p)}$

$(\neg p \wedge (q \vee \neg q)) \vee (q \wedge (p \vee \neg p))$

$(\neg p \wedge q) \vee (\neg p \wedge \neg q) \vee (q \wedge p) \vee (q \wedge \neg p)$

$(\neg p \wedge q) \vee (\neg p \wedge \neg q) \vee (q \wedge p)$

ii) $p \vee q$

$(p \wedge T) \vee (q \wedge T)$

$p \wedge (q \vee \neg q) \vee (q \wedge (p \vee \neg p))$

$(p \wedge q) \vee (p \vee \neg q) \vee (q \wedge p) \vee (q \vee \neg p)$

$(p \wedge q) \vee (p \vee \neg q) \vee (q \vee \neg p)$

iii) $\neg(p \wedge q)$

$\cancel{\neg((p \vee T) \wedge ($}$

$\neg p \vee \neg q$

$(\neg p \wedge T) \vee (\neg q \wedge T)$

$\neg p \wedge (q \vee \neg q) \vee (\neg q \wedge (p \vee \neg p))$

$(\neg p \wedge q) \vee (\neg p \wedge \neg q) \vee (\neg q \wedge p) \vee (\neg p \wedge \neg q)$

$(\neg p \wedge q) \vee (\neg p \wedge \neg q) \vee (\neg q \wedge p)$

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4. obtain PDNF of $P \to [(P \to q) \wedge \neg(\neg q \vee \neg P)]$

$P \to [(P \to q) \wedge \neg(\neg q \vee \neg P)]$

$\neg P \vee [(P \to q) \wedge \neg(\neg(q \wedge P))]$

$\neg P \vee [(\neg P \vee q) \wedge (q \wedge P)]$

$\cancel{\neg P \vee [(\neg P \vee q) \wedge q}$

$\neg P \vee [\neg P \wedge (q \wedge P) \vee q \wedge (q \wedge P)]$

$\neg P \vee [(\neg P \wedge P) \wedge q \vee (P \wedge q)]$

$\neg P \vee [(F \wedge q) \vee (P \wedge q)]$

$\neg P \vee [F \vee (P \wedge q)]$

$\neg P \vee (P \wedge q)$

$(\neg P \wedge F) \vee (P \wedge q)$

$(\neg P \wedge (q \vee \neg q)) \vee (P \wedge q)$

$(\neg P \wedge q) \vee (\neg P \wedge \neg q) \vee (P \wedge q)$

5. show that the following

i. $P \vee (P \wedge q) \Leftrightarrow P$

ii. $P \vee (\neg P \wedge q) \Leftrightarrow P \vee q$

Sol i. PDNF of $P \vee (P \wedge q)$

$\cancel{P \vee (P \wedge q)}$ $\quad P \vee (P \wedge q)$

$\cancel{(P \vee P) \wedge (P \vee q)}$ $\quad (P \wedge T) \vee (P \wedge q)$

$\cancel{P \wedge (P \vee q)}$ $\quad P \wedge (q \vee \neg q) \vee (P \wedge q)$

$\cancel{(P \wedge T)}$ $\quad (P \wedge q) \vee (P \wedge \neg q) \vee (P \wedge q)$

$(P \wedge q) \vee (P \wedge \neg q)$

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PDNF of p

$p \wedge T$

$p \wedge (q \vee \neg q)$

$(p \wedge q) \vee (p \wedge \neg q)$

$\therefore$ PDNF of $p \vee (p \wedge q) \equiv$ PDNF of p

i.e $(p \wedge q) \vee (p \wedge \neg q) \equiv (p \wedge q) \vee (p \wedge \neg q)$

ii) $p \vee (\neg p \wedge q) \Leftrightarrow p \vee q$

PDNF of $p \vee (\neg p \wedge q)$

$(p \wedge T) \vee (\neg p \wedge q)$

$p \wedge (q \vee \neg q) \vee (\neg p \wedge q)$

$(p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge q)$

PDNF of $p \vee q$

$(p \wedge T) \vee (q \wedge T)$

$p \wedge (q \vee \neg q) \vee (q \wedge (p \vee \neg p))$

$(p \wedge q) \vee (p \wedge \neg q) \vee (q \wedge p) \vee (q \wedge \neg p)$

$(p \wedge q) \vee (p \wedge \neg q) \vee (q \wedge \neg p)$

$\therefore$ Both have same PDNF

$\therefore$ The above ~~two~~ equation are equivalent

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PCNF

Principle Conjunctive Normal form :

For a given formula an equivalent formula consisting of conjunction of min terms consting of disjunction in which each variable (or) its nagation but not both appears only once.

Problems

1. Obtain PCNF of $(\neg P \rightarrow r) \wedge (r \leftrightarrow P)$

$(\neg P \rightarrow r) \wedge (r \leftrightarrow P)$

$(\neg(\neg P) \vee r) \wedge ((r \rightarrow P) \wedge (P \rightarrow r))$

$(P \vee r) \wedge (\neg r \vee P) \wedge (\neg P \vee r)$

2. Obtain PCNF of the following formulas.

i. $\neg(P \vee r)$

ii. $\neg(P \rightarrow r)$

iii. $\neg(P \leftrightarrow r)$

Sol i. $\neg(P \vee r)$

$\neg P \wedge \neg r$

$\neg P \vee F \wedge \neg r \vee F$

$\neg P \vee (r \wedge \neg r) \wedge \neg r \vee (P \wedge \neg P)$

$(\neg P \vee r) \wedge (\neg P \vee \neg r) \wedge (\neg r \vee P) \wedge (\neg r \wedge \neg P)$

$(\neg P \vee r) \wedge (\neg P \vee \neg r) \wedge (\neg r \vee P)$

ii. $\neg(P \rightarrow r)$

$\neg(\neg P \vee r)$

$\neg(\neg P) \wedge \neg r$

$P \wedge \neg r$

$(P \vee F) \wedge (\neg r \vee F)$

$(P \vee (r \wedge \neg r)) \wedge (\neg r \vee (P \wedge \neg P))$

$(P \vee r) \wedge (P \vee \neg r) \wedge (\neg r \vee P) \wedge (\neg P \vee \neg r) \Rightarrow (P \vee r) \wedge (\neg r \vee P) \wedge (\neg P \vee \neg r)$

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iii, $\neg(p \Leftrightarrow q)$

$\neg((p \to q) \wedge (q \to p))$

$\neg((\neg p \vee q) \wedge (\neg q \vee p))$

$\neg(\neg p \vee q) \vee \neg(\neg q \vee p)$

$(\neg(\neg p) \wedge \neg q) \vee (\neg(\neg q) \wedge \neg p)$

$(p \wedge \neg q) \vee (q \wedge \neg p)$

$((p \wedge \neg q) \vee q) \wedge ((p \wedge \neg q) \vee \neg p)$

$(p \vee q) \wedge (\neg q \vee q) \wedge ((p \vee \neg p) \wedge (\neg q \vee \neg p)$

$(p \vee q) \wedge T \wedge T \wedge (\neg q \vee \neg p)$

$(p \vee q) \wedge (\neg q \vee \neg p)$

(or)

$\neg(p \Leftrightarrow q)$

$\neg((p \wedge q) \vee (\neg p \wedge \neg q))$

$\neg(p \wedge q) \wedge \neg(\neg p \wedge \neg q)$

$(\neg p \vee \neg q) \wedge (\neg(\neg p) \vee (\neg(\neg q))$

$(\neg p \vee \neg q) \wedge (p \vee q)$

3. Obtain the product of sums canonical forms of the following

i, $(p \wedge q \wedge r) \vee (\neg p \wedge r \wedge q) \vee (\neg p \wedge \neg q \wedge \neg r)$

ii, $p \to [(p \to q) \wedge \neg(\neg q \vee \neg p)]$

sol

i, $(p \wedge q \wedge r) \vee (\neg p \wedge r \wedge q) \vee (\neg p \wedge \neg q \wedge \neg r)$

$[(p \wedge q) \vee (\neg p \wedge r \wedge q) \wedge (r \vee (\neg p \wedge r \wedge q))] \vee (\neg p \wedge \neg q \wedge \neg r)$

$(p \wedge q \vee \neg p) \wedge (p \wedge q \vee r) \wedge (p \wedge q \vee q)$

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i) $(p \wedge q \wedge r) \vee (\neg p \wedge r \wedge q) \vee (\neg p \wedge \neg q \wedge \neg r)$

$[(p \vee \neg p) \wedge (q \wedge r)] \vee (\neg p \wedge \neg q \wedge \neg r)$

$T \wedge (q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)$

$(q \wedge r \vee \neg p) \wedge ((q \wedge r) \vee (\neg q \wedge \neg r))$

$(q \wedge r \vee \neg p) \wedge ((q \wedge r) \vee \neg q) \wedge ((q \wedge r) \vee \neg r)$

$(\neg p \vee q) \wedge (\neg p \vee r) \wedge (q \vee \neg q) \wedge (r \vee \neg q) \wedge F \wedge (q \vee \neg r)$

$(\neg p \vee q) \wedge (\neg p \vee r) \wedge T \wedge (r \vee \neg q) \wedge (q \vee \neg r)$

$(\neg p \vee q) \wedge (\neg p \vee r) \wedge (r \vee \neg q) \wedge (q \vee \neg r)$

ii) $p \to [(p \to q) \wedge \neg(\neg q \vee \neg p)]$

$p \to [(\neg p \vee q) \wedge (\neg(\neg(p \wedge q))]$

$p \to [(\neg p \vee q) \wedge (p \wedge q)]$

$p \to [\cancel{(\neg p \wedge p) \wedge} (\neg p \vee q) \wedge p \wedge (\neg p \vee q) \wedge q]$

$p \to [(\neg p \wedge p) \vee (q \wedge p) \wedge (\neg p \wedge q) \vee (q \wedge q)]$

$p \to [F \vee (q \wedge p) \wedge (\neg p \wedge q) \vee q]$

$p \to [(q \wedge p) \wedge q]$

$p \to [p \wedge q] \Rightarrow p \to [F \vee (p \wedge q)]$

$p \to [(q \wedge \neg q) \vee (p \wedge q)]$

$p \to [(q \vee (p \wedge q)) \wedge (\neg q \vee (p \wedge q)]$

$p \to [q \wedge (\neg q \vee q) \wedge (\neg q \vee p)]$

$p \to [q \wedge (\neg q \vee p)]$

$p \to [(q \wedge (\neg q)) \vee (q \wedge p)]$

$p \to (p \wedge q)$

$\neg p \vee (p \wedge q) \Rightarrow (\neg p \vee p) \wedge (\neg p \wedge q) \Rightarrow T \wedge (\neg p \wedge q)$

$\cancel{\neg p \vee p}$ $\cancel{(\neg p \vee p) \wedge (\neg p \wedge q)}$ $(q \vee \neg q) \wedge (\neg p \wedge q)$

$((q \vee \neg q) \wedge \neg p) \wedge ((q \vee \neg q) \wedge q)$

$\cancel{(\neg p \wedge (p \wedge q)) \vee (p \wedge (\neg p \wedge q))}$

$(\neg p \wedge q) \cancel{\vee (}$ $(q \wedge \neg p) \vee (\neg p \wedge \neg q) \wedge q$

$((q \wedge \neg p) \vee \neg p) \wedge ((q \wedge \neg p) \vee \neg q) \wedge q$

$(q \vee \neg p) \wedge (\neg p \vee \neg p) \wedge (q \vee \neg q) \wedge (\neg p \vee \neg q) \wedge q$

$(q \vee \neg p) \wedge \cancel{(\neg p} \wedge T \wedge (T) \wedge (\neg p \vee \neg q) \wedge q$

$((q \vee \neg p) \wedge q) \wedge (\neg p \vee \neg q) \wedge (T \wedge q)$

$q \vee (\neg p \wedge q)$

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Theory of Inference for statement calculus :

The main function of logic is to provide rules of inference (principles of reasoning). The theorey associated with such rules is known as inference theorey. Because it is concerned with the inference of a conclusion from certain premisies.

when a conclusion is derived from certain premisies by using accepted rules of reasoning the process of derivation is called a deduction (or) formal proof.

validity using truth table:

Let $\{H_1, H_2, H_3, \ldots, H_m\}$ be a set of premisis and a conclusion c follows logically $H_1 \wedge H_2 \wedge \ldots \wedge H_m \Rightarrow c$ if and only if whenever all H_i's are true then c should be true in that row

Problems

1. Determine whether the conclusion c follows logically from H_1 and H_2 in the following

i) $H_1 : \neg P$, $H_2 : P \leftrightarrow Q$; $c = \neg(P \wedge Q)$

ii) $H_1 : P \rightarrow Q$, $H_2 : P$; $c = Q$

iii) $H_1 : P \rightarrow Q$, $H_2 : \neg P$: $c : Q$

iv) $H_1 : P \rightarrow Q$, $H_2 : \neg(P \wedge Q)$; $c : \neg P$

v) $H_1 : P \rightarrow Q$; $H_2 : Q$; $c : P$

i) Truth table for $H_1 : \neg P$; $H_2 : P \leftrightarrow Q$; $c = \neg(P \wedge Q)$

P	Q	$\neg P (H_1)$	$P \leftrightarrow Q (H_2)$	$(P \wedge Q)$	$\neg(P \wedge Q) (c)$
T	T	F	T	T	F
T	F	F	F	F	T
F	T	T	F	F	T
F	F	T	T	F	T

Here only H_1 and H_2 are both true in last row and also

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C is also true in that row so the statement $[P \wedge (P \Leftrightarrow r)] \Rightarrow \neg(P \wedge r)$ is valid.

ii, $H_1 : P \rightarrow r$; $H_2 : P$; $C = r$

$P(H_2)$	$r(C)$	$P \rightarrow r$ (H_1)
T	T	T
T	F	F
F	T	T
F	F	T

Here only H_1 and H_2 are both true in first row and also C is also true in that row so the statement $[(P \rightarrow r) \wedge P] \Rightarrow r$ is valid

iii, $H_1 : P \rightarrow r$; $H_2 : \neg P$; $C : r$

P	$r(C)$	$\neg P(H_2)$	$P \rightarrow r$ (H_1)
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

~~Here only H_1 and H_2 are true in third row and C is also true in that row so the statement $((P \rightarrow r) \wedge \neg P) \Rightarrow r$~~

i

iv, $H_1 : P \rightarrow r$, H_2 ; $\neg(P \wedge r)$; $C : \neg P$

P	r	$\neg P(C)$	$P \rightarrow r(H_1)$	$P \wedge r$	$\neg(P \wedge r)$ (H_2)
T	T	F	T	T	F
T	F	F	F	F	T
F	T	T	T	F	T
F	F	T	T	F	T

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ii) $H_1 : P \to q$ $H_2 : q$, $C : P$

P (c)	q (q)	$P \to q$ (H_1)
T	T	T
T	F	F
F	T	T
F	F	T

Here H_1, H_2 are both true in first row and also c is also true. so the statement is valid.

Rules of inferences:

Rule P: A premise may be introduced at any point in the derivation.

Rule T: A formula S may be introduced in a derivation if 'S' is tautologically implied by one or more of the preceding formulas in the derivation.

Implication formulas:

1. $P \wedge q \Rightarrow P$
   $P \wedge q \Rightarrow q$ } (simplification)

2. $P \Rightarrow P \vee q$
   $q \Rightarrow P \vee q$ } (addition)

3. $\neg P \Rightarrow P \to q$
   $q \Rightarrow P \to q$

4. $P \to q \Rightarrow \neg q \to \neg P$

5. $\neg(P \to q) \Rightarrow P$
   $\neg(P \to q) \Rightarrow \neg q$

6. $P, q \Rightarrow P \wedge q$

7. $\neg P, P \vee q \Rightarrow q$

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8. $p, p \to q \Rightarrow q$

9. $\neg q, p \to q \Rightarrow \neg p$

10. $p \to q, q \to r \Rightarrow p \to r$

11. $p \vee q, p \to q, q \to r \Rightarrow r$

Problems

1. Demonstrate that r is a valid inference from the premises $p \to q, q \to r$ and p

Sol

$\{1\}$	(1)	$p \to q$	rule-p
$\{2\}$	(2)	p	rule-p
$\{1,2\}$	(3)	q	rule T
$\{4\}$	(4)	$q \to r$	rule p
~~$\{5\}$~~			
$\{1,2,4\}$	(5)	r	rule-T

2. show that $r \vee s$ follows logically from the premises $c \vee d$, $c \vee d \to \neg h$, $\neg h \to (a \wedge \neg b)$ and $(a \wedge \neg b) \to (r \vee s)$

Sol

$\{1\}$	(1)	$c \vee d \to \neg h$	rule-P
$\{2\}$	(2)	$\neg h \to (a \wedge \neg b)$	rule-P
$\{1,2\}$	(3)	$c \vee d \to (a \wedge \neg b)$	rule-T
$\{4\}$	(4)	$(a \wedge \neg b) \to (r \vee s)$	rule-P
$\{1,2,4\}$	(5)	$c \vee d \to (r \vee s)$	rule-T
$\{6\}$	(6)	$c \vee d$	rule-P
$\{1,2,4,6\}$	(7)	$r \vee s$	rule T

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3. show that $s \vee r$ is tortologically implied by ~~(p∨q)~~ $(p \vee q) \wedge (p \to r) \wedge (q \to s)$

{1}	(1)	$p \vee q$	rule-P
{1}	(2)	$\neg p \to q$	rule-T
{3}	(3)	$p \to r$	rule-P
{3}	(4)	$\neg r \to \neg p$	rule-T
{1,3}	(5)	$\neg r \to q$	rule-T
{6}	(6)	$q \to s$	rule-P
{1,3,6}	(7)	$\neg r \to s$	rule-T
{1,3,6}	(8)	$r \vee s$	rule-T

4. show that $R \wedge (P \vee Q)$ is a valid Conclusion from the premises $P \vee Q$, $Q \to R$, $P \to M$ and $\sim M$

{1}	(1)	$P \to M$	rule-P
{2}	(2)	$\neg M$	rule-P
{1,2}	(3)	$\neg P$	rule-T
{4}	(4)	$P \vee Q$	rule-P
{4}	(5)	$\neg P \to Q$	rule-T
{6}	(6)	$Q \to R$	rule-P
{4,6}	(7)	$\neg P \to R$	rule-T
{4,6,1,2}	(8)	R	Rule-T
{1,2,4,6,4}	(9)	$R \wedge (P \vee Q)$	rule-T

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5. show that $\neg q, p \to q \Rightarrow \neg p$

{1}	(1)	$\neg q$	Rule -P
{2}	(2)	$p \to q$	Rule -P
{2}	(3)	$\neg q \to \neg p$	Rule -T
{1,2}	(4)	$\neg p$	Rule -T

6. show that $r \to s$ can be derived from the premises $p \to (q \to s)$, $\neg r \vee p$, and q

sol

Instead of proving $r \to s$ we can include r as an additional premise and finally we show s

{1}	(1)	r	Rule -P
{2}	(2)	$\neg r \vee p$	Rule -P
{2}	(3)	$r \to p$	Rule -T
{1,2}	(4)	p	Rule -T
{5}	(5)	$p \to (q \to s)$	Rule -P
{1,2,5}	(6)	$q \to s$	Rule -T
{7}	(7)	q	Rule -P
{1,2,5,7}	(8)	s	Rule -T
{1,2,5,7}	(9)	$r \to s$	Rule -CP (conditional proof)

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7. Test the validity of the following argument:

"Sonia is watching T.V", "if sonia is watching TV then she is not studying", "if she is not studying then her father will not by her a scooty". Therefore, "Sonia's father will not buy a scooty.

Sol let us indicate the statements as follows

p : sonia is watching TV

q : sonia is studying

r : sonia's father will buy a scooty.

Now, we convert the statements into logical statements as follows

Hypothesis (or) premisses : p, $p \to \neg q$, $\neg q \to \neg r$

and the Conclusion : $\neg r$

{1}	(1)	p	rule-P
{2}	(2)	$p \to \neg q$	rule-P
{1,2}	(3)	$\neg q$	rule-T
{4}	(4)	$\neg q \to \neg r$	rule-P
{1,2,4}	(5)	$\neg r$	rule-T

Hence the result is $\neg r$

8. "If there was a ball game, then travelling was difficult", "If they arrived on time then travelling was not difficult". "They arrived on time". Therefore there was no ball game. show that these statements Constitute a valid argument.

Sol let us indicate the statements as follows.

p : There was a ball game.

q : Travelling was difficult

r : They arrived on time.

~~(Now, we convert to)~~

Hence the given premisses are $p \to q$, $r \to \neg q$, r

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the conclusion is $\neg p$

$\{1\}$	(1)	r	rule-p
$\{2\}$	(2)	$r \to \neg q$	rule-p
$\{1,2\}$	(3)	$\neg q$	rule-T
$\{4\}$	(4)	$p \to q$	rule-p
$\{1,2,4\}$	(5)	$\neg p$	rule-T

Hence The Conclusion is $\neg p$

9. By using method of derivation show that the following statements constitute a valid argument:
"If A works hard then either B or C will enjoy",
"If B enjoys then A will not work hard",
"If D enjoys then C will not". Therefore,
"If A works hard then D will not enjoy".

Sol Let us indicate the statements as follows.

p : A works hard
q : B will enjoy
r : C will enjoy
s : D will enjoy

Hence the given premises are $p \to (q \vee r)$, $q \to \neg p$, $s \to \neg r$, the conclusion is $p \to \neg s$.

{1}

Instead of proving $p \to \neg s$ we can include p as an additional premises and finally shows $\neg s$

$\{1\}$	(1)	p	rule-p
$\{2\}$	(2)	$p \to (q \vee r)$	rule-p
$\{1,2\}$	(3)	$q \vee r$	rule-T
$\{4\}$	(4)	p	rule-p

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$\{5\}$ (5) $q \to \neg p$ rule-P

$\{5\}$ (6) $p \to \neg q$ rule-T

$\{4,5\}$ (7) $\neg q$ rule-T

$\{1,2\}$ (8) $\neg q \to r$ rule-T

$\{4,5,1,2\}$ (9) r rule-T

$\{10\}$ (10) $s \to \neg r \Rightarrow r \to \neg s$ rule-P

$\{4,5,1,2,10\}$ (11) $\neg s$ rule T

$\{4,5,1,2,10\}$ (12) $p \to \neg s$ rule CP

Indirect method of proof:

To prove $p \to q$ we prove $\neg q \to \neg p$

$(\because p \to q \equiv \neg q \to \neg p)$

Example

1. If $x = 2$ then $x^2 = 4$

Sol Direct proof: let $x = 2$

then $x^2 = 2^2 = 4$

Indirect proof:

To prove $x = 2 \to x^2 = 4$ by using indirect proof we prove $x^2 \neq 4 \to x \neq 2$

Suppose $x^2 \neq 4$

$\Rightarrow x \neq \pm 2$

which is contradiction because $x = 2$

Therefore our supposition is wrong and hence $x^2 = 4$

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Problems

1. show that $\neg(p \wedge q)$ follows from $\neg p \wedge \neg q$

(1) $\neg(\neg(p \wedge q))$

(2) $p \wedge q$

(3) p

(4) $\neg p \wedge \neg q$

(5) $\neg p$

(6) $p \wedge \neg p$

(7) F

2. using the indirect method show that $p \to q$, $q \to r$, $\neg(p \wedge r)$, $p \vee r \Leftrightarrow r$

{1} (1) $\neg r$ rule p (assumed)

{2} (2) $q \to r$ rule p

{1, 2} (3) $\neg q$ rule T

{3} (4) $p \to q$ rule p

{4} (5) $\neg p$ rule T

~~(5)~~ ~~$\neg(p \wedge r)$~~ ~~rule p~~

~~(6)~~ ~~$\neg p \vee \neg r$~~ ~~rule T~~

~~(7)~~ ~~$p \to \neg r$~~

{5} (6) $p \vee r$ rule-p

{4} (7) r rule-T

(4, 5, 1) (8) $\neg r \wedge r \equiv F$

Hence our assumption is wrong

$\therefore$ we can conclude with r

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Consistency of premises:

1. show that the premises $a \to (b \to c)$, $d \to (b \wedge \neg c)$, and a are inconsistent.

$\{1\}$	(1)	a	rule-P
$\{1\}$	(2)	a	rule-T
$\{3\}$	(3)	$a \to (b \to c)$	rule-P
$\{1,3\}$	(4)	$b \to c$	rule-T
$\{5\}$	(5)	$d \to (b \wedge \neg c)$	rule-P
$\{1\}$	(6)	$a \wedge d$	rule-P
$\{1\}$	(7)	d	rule-T
$\{1,5\}$	(8)	$b \wedge \neg c$	rule-T
$\{1,5\}$	(9)	b	rule-T
$\{1,3,1,5\}$	(10)	c	rule-T
$\{1,3,1,5,1,5\}$	(11)	$E \wedge \neg C \equiv F$	rule-T

$\therefore$ The premises are inconsistent.

2. show that the following set of premisses is inconsistent
"If the contract is valid then John is liable for penality"
"If john in liable for penality he will go bankrupt"
"If the bank will loan him money then he will not go bankrupt".
"As a matter of fact the contract is valid and the bank will loan him money".

Sol Let us indicate the statements as follows.

p : The contract is valid

v : John is liable for penality

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r : he will go bankrupt

s : The bank will loan him money

The premises are

$p \to q,\ q \to r,\ s \to \neg r,\ p \wedge s$

{1}	(1)	$p \to q$	rule-P
{2}	(2)	$q \to r$	rule-P
{1,2}	(3)	$p \to r$	rule-T
{4}	(4)	$s \to \neg r$	rule-P
{4}	(5)	$r \to \neg s$	rule-T
{1,2,4}	(6)	$p \to \neg s$	rule-T
{7}	(7)	$p \wedge s$	rule-P
{7}	(8)	s	rule-T
{1,2,4,7}	(9)	$\neg p$	rule-T
{7}	(10)	p	rule-P
{1,2,4,7,7}	(11)	$p \wedge \neg p \equiv F$	

$\therefore$ The premises are inconsistent.

3. show that the following premises are inconsistent.

"If Jack misses many classes because of illness, then he fails high school"

"if jack fails high school, then he is uneducated"

"if jack reads a lot of books, then he is not uneducated"

"Jack misses many classes because of illness and reads a lot of books".

Sol let us indicate the statements as follows

p : Jack misses many classes because of illness

q : he is uneducated

r : Jack reads a lot of book

s : he fails high school

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$P \to q,\ q \to r,\ s \to \neg r;\ P \wedge S$

$\{1\}$ (1) $P \to q$

$\{2\}$ (2) $q \to r$

$\{1,2\}$ (3) $P \to r$

$\{4\}$ (4) $P \wedge S$

$\{4\}$ (5) P

$\{1,2,4\}$ (6) r

$\{7\}$ (7) $S \to \neg r$

$\{8\}$ (8) $P \wedge S$

$\{8\}$ (9) S

$\{7,8\}$ $(S \to \neg r) \wedge S \equiv \neg r$

$\{7,8,1,2,4\}$ $r \wedge \neg r \equiv F$

Predicate:

Consider the following statements

i) "$x > 3$"

ii) "$x + y = 3$"

iii) "Ramesh is a boy"

iv) "Ramesh is taller than suresh".

If you observe above statements in

In i) x is the subject and "greater than 3" is predicate.

In ii, x, y are the subjects and ~~equal~~ "$x + y = 3$" is predicate.

in iii, Ramesh is the subject and "~~R~~ is a boy" is predicate.

in iv, Ramesh and Suresh ~~is~~ are the subject and "~~Ramesh~~ taller than" is predicate.

To convert above non propositions into propositions by defining predicates as follows.

i) $P(x)$: "x is greater than 3"

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ii, $Q(x,y)$: "$x+y=3$"

iii, $R(x)$: "x is a boy"

iv, $S(x,y)$: "x is taller than y"

we can make given non proposition into proposition by assigning values to the variables. as follows.

$P(4)$: "4 greater than 3" which is true, a proposition

$P(2)$: "2 greater than 3" which is false

$Q(1,2)$: "$2+1=3$" is true which is a proposition

$Q(3,2)$: is false which is a proposition.

S(Pavan, surosh): is a proposition.

Quantifier:

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UNIT-2

Set theory and Relations

The collection of well-defined objects is called a set. The object in a set can be numbers, people, letters etc.

Any object belonging to a set is called a member or element of that set.

well-defined set:

~~It is~~

If it is possible to determine whether any given object is a member of the set.

Example

The set of ~~vowles~~ vowels of alphabet $= \{a, e, i, o, u\} = A$

The set of all +ve integers $< 10 = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

The set of people living on earth

Note:

i. we denote the sets with capital letters such as A, B, C, X, Y, Z and elements with lower cases a, b, c, x, y, z

ii. If an element x is in 'A', then we write $x \in A$ otherwise $x \notin A$

iii. The no. of elements of 'A' is denoted by $n(A)$ or $|A|$

Representation of set:

① Tabular form

② set builder form

① A set can be defined by listing its members

Example $= A = \{1, 2, 3, 4, 5\}$

② A set builder form is ~~re~~ defined by stating the characteristics of elements of the set.

Example

$$B = \{x / x \neq y,\ y \in Z\}$$

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NULL set:

A set consisting of no elements is called NULL set and is denoted by ϕ (or) $\{\}$.

Example

The set of integers between 2 and 3 is an empty set

$$\{x / x \text{ is an integer } \& \ 2 < x < 3\}$$

Universal set:

A set is said to be universal set. If it contains every set under discussion

Universal set is also known as universe of discourse and is denoted by u

Example

In a dice throwing experiment, the universal set is $u = \{1, 2, 3, 4, 5, 6\}$

Subset:

Let A and B are two sets, we say A is subset of B $(A \subseteq B)$. If every element of A should be an element of B

i.e $x \in A \Rightarrow x \in B$

Proper subset:

A is said to be proper subset of B if $A \subseteq B$ and $A \neq B$.

we denote it by $A \subseteq B$

Example let $x = \{1, 3, 5\}$ and $y = \{1, 2, 3, 4, 5\}$. Is x is a proper subset of y?

Sol Yes $x \subseteq y$ and $x \neq y$

Equality of two sets:

A and B are said to be equal $(A = B)$.

$\Leftrightarrow A \subseteq B$ and $B \subseteq A$

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Eg: ① $x = \{1,2,3,4,5\}$ and $x = \{3,5,1,2,4\}$. Are they equal?

Sol Yes, Because $x \subseteq y$ and $y \subseteq x$

② $x = \{1,2,3\}$ & $y = \{1,1,2,3\}$

Sol $x = y$

③ $x = \{\{1,2\}, 3\}$ & $y = \{1,2,3\}$

Sol $x \neq y$

Finite set:

If the number of elements in a set x is finite or countable then the set 'x' is called finite set otherwise it is infinite set.

Example set of all +ve $z < 5 = \{1,2,3,4\} \rightarrow$ finite set

of all +ve $z = \{1,2,3,4 \cdots\} \rightarrow$ infinite.

Power set:

The set of all subsets of a set A is called powerset of A and is denoted by P(A)

Example

Find the power set of $A = \{1,2,3\}$

$P(A) = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{3,1\}, \{1,2,3\}, \{\}\}$

Note:

If a set A consists of n elements then its power set consists of 2^n elements.

Disjoint set: Two sets A and B are said to be disjoint set if $A \cap B = \phi$

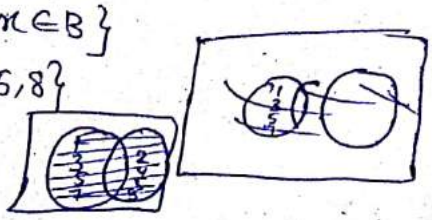
Example $A = \{1,2,3\}$ & $B = \{4,5,7\}$

Operations on sets:

Union: $A \cup B = \{x : x \in A \text{ or } x \in B\}$

Eg: $A = \{1,3,5,7\}$ $B = \{2,4,5,8\}$

$A \cup B = \{1,2,3,4,5,7,8\}$



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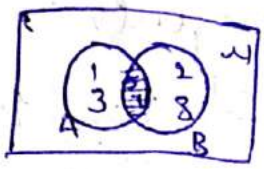
Intersection: $A \cap B = \{x : x \in A \text{ and } x \in B\}$

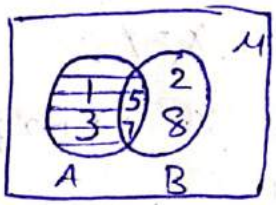
Example $A = \{1, 3, 5, 7\}$ $B = \{2, 5, 7, 8\}$



$A \cap B = \{5, 7\}$

set difference:

$A - B = \{x : x \in A \text{ and } x \notin B\}$



$A = \{1, 3, 5, 7\}$ $B = \{2, 5, 7, 8\}$

$A - B = \{1, 3\}$

Problems

1. Given $A = \{x : x \in Z \ \& \ 1 \le x \le 5\}$

$A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 17\}$, $C = \{1, 2, 3, 4 \cdots\}$

Find $A \cap B$, $A \cap C$, $A \cup B$, $A \cup C$

sol

$A \cap B = \{3, 4, 5\}$ $A \cup C = \{1, 2, 3 \cdots\} = C$

$A \cap C = \{3, 4, 5, 17\}$ $A \cup B = \{1, 2, 3, 4, 5, 17\}$

2. show that $A \subseteq A \cup B$ & $A \cap B \subseteq A$

sol let $x \in A \Rightarrow x \in A \cup B$

$\therefore A \subseteq A \cup B$

$x \in A \cap B = x \in A \ \& \ x \in B \Rightarrow x \in A$

$\therefore A \cap B \subseteq A$

3. $A = \{1, 2, 3, 4, 5, 6\}$ $B = \{3, 5, 7, 9\}$. Find $A - B$, $B - A$, $(A - B) \cup (B - A)$, $(A - B) \cap (B - A)$

Sol $A - B = \{1, 2, 4, 6\}$ $B - A = \{5, 9\}$

$(A - B) \cup (B - A) = \{1, 2, 4, 5, 6, 9\}$

$(A - B) \cap (B - A) = \{\ \} = \phi$

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4. show that i) $(A \cap B) \cap C = A \cap (B \cap C)$

ii) $(A \cup B) \cup C = A \cup (B \cup C)$

iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Sol: i) $(A \cap B) \cap C = A \cap (B \cap C)$

let $x \in (A \cap B) \cap C$

$\Rightarrow x \in A \cap B$ and $x \in C$

$\Rightarrow x \in A$ & $x \in B$ & $x \in C$

$\Rightarrow x \in A$ and $x \in (B \cap C)$

$\Rightarrow x \in A \cap (B \cap C) \Rightarrow A \cap (B \cap C) = (A \cap B) \cap C$

ii) $(A \cup B) \cup C = A \cup (B \cup C)$

let $x \in (A \cup B) \cup C$

$\Rightarrow x \in A \cup B$ or $x \in C$

$\Rightarrow x \in A$ or $x \in B$ or $x \in C$

$\Rightarrow x \in A$ or $x \in B \cup C$

$\Rightarrow x \in A \cup (B \cup C)$

$\therefore (A \cup B) \cup C = A \cup (B \cup C)$

iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

let $x \in A \cup (B \cap C)$

$\Rightarrow x \in A$ or $x \in B$ and C

$\Rightarrow x \in A$ or $[x \in B$ and $x \in C]$

$\Rightarrow x \in A$ or $x \in B$ and $x \in A$ or $x \in C$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$

$\therefore A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

5. show that $A - (A \cap B) = A - B$

let $x \in A - (A \cap B)$

$\Rightarrow x \in A$ and $x \notin A \cap B$

$\Rightarrow x \in A$ and $x \notin A$ and $x \notin B$

$\Rightarrow x \in A$ and $x \notin B$

$\Rightarrow A - B$

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Compliment of a set:

Let A be a set then compliment of set A is denoted by A' or $A^c = \{x / x \in \mu \ \& \ x \{A\} = \mu - A\}$

Ex: $\mu = \{1,2,3,4,5,6\}$ $A = \{2,4,6\}$

$A' = A^c = \mu - A = \{1,3,5\}$

Principle of inclusion and exclusion:

Let A and B be any 2 minor sets then the number of elements in the union of 2 sets A and B is the sum of number of elements in the sets

i.e $|A \cup B| = |A| + |B| - |A \cap B|$

Q. show that for any three ~~sets~~ disjoint sets A, B, C we have $|A| + |B| + |C| - (A \cap B) - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

Sol For two sets A and B we have $|A \cup B| = |A| + |B| - |A \cap B|$

$$|A \cup B \cup C| = |A \cup B| + |C| - |(A \cup B) \cap C|$$

$$= |A \cup B| + |C| - [\cancel{|A \cup B| + |C| - |A \cap (A \cap C) \cup (B \cap C)|}$$

$$= |A \cup B| + |C| - [|A \cap C| + |B \cap C| - |A \cap C \cap B \cap C|]$$

$$= |A| + |B| - |A \cap B| + |C| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

* The general form of inclusion and exclusion principle is given by $|A_1 \cup A_2 \cup \ldots \cup A_n| = \cancel{|A_1| + |A_2| + \ldots + |A_n| -}$

$$\sum_{i=1}^{n} |A_i| - \sum_{\substack{1 \le i,j \le n \\ i \ne j}} |A_i \cap A_j| + \sum_{1 \le i,j,k \le n} |A_i \cap A_j \cap A_k| - \ldots$$

$$+ (-1)^{n+1} |A_1 \cap A_2 \cap A_2 \ldots \cap A_n|$$

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Problem

1. In a class of 50 students 20 students play football and 16 students play hockey it is found that 10 students play both the games. Find the number of students who play neither the games.

Sol: let A be the set of students playing and B be the set of students who plays hockey.

The number of players who plays either hockey or foot ball. $= |A \cup B| = |A| + |B| - |A \cap B|$

$= 20 + 16 - 10$

$= 36 - 10$

$|A \cup B| = 26$ total no. of students $\mu = 50$

The number of students who play neither games is

$\mu - |A \cup B| = 50 - 26$

$= 24$

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2(b) RELATION

* Relations:

* Relation
* properties of Binary relation in a set
* Representation of relation : Relation matrix, Digraph
** Equivalence relation
* partial ordering relation
* Hasse diagram
* Lattice and its properties

Relation:

A relation R on a set A is ~~set~~ of ~~all~~ subsets of $A \times A$

Example:

Let R_i be a relation on a set $A = \{1,2,3\}$

R_1 be a relation such that (a,b) belongs to R_1 if and only if $a = b$

① $(a,b) \in R_1 \Leftrightarrow a = b$

② $(a,b) \in R_2 \Leftrightarrow a < b$

③ $(a,b) \in R_3 \Leftrightarrow a >= b$

④ $(a,b) \in R_4 \Leftrightarrow a/b$ (a divides b)

⑤ $(a,b) \in R_5 \Leftrightarrow$ a is congurent to b mod|2| i.e $2/a-b$

Given $A = \{1,2,3\}$

$A \times A = \{(1,1)\ (1,2)\ (1,3)\ (2,2)\ (2,1)\ (2,3)\ (3,3)\ (3,2)\ (3,1)\}$

$R_1 = \{(1,1)\ (2,2)\ (3,3)$

$R_2 = \{(1,2)\ (2,3)\ (1,3)\}$

$R_3 = \{(1,1)\ (2,1)\ (3,1)\ (3,2)\ (2,2)\ (3,3)\}$

$R_4 = \{(1,1)\ (2,2)\ (3,3)\ (1,2)\ (1,3)\}$

$R_5 = \{(1,1)\ (2,2)\ (3,3)\ (1,3)\ (3,1)\}$

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* R is said to be a relation from A to B then R is the subset of $A \times B$.

Example:

$A = \{1,2\}$, $B = \{1,4,5\}$

Let R be a relation from A to B such that $(a,b) \in R \Leftrightarrow$ both a and b are even or odd.

Given $A = \{1,2\}$ $B = \{1,4,5\}$

$A \times B = \{(1,1)(1,4)(1,5)(2,1)(2,4)(2,5)\}$

$R = \{(1,1)(1,5)(2,4)\}$

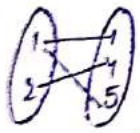
Domain and Range of a Relation:

Let R be a binary relation then the domain of R is the set of all first elements of the order pairs and is denoted by 'D'. The range of R is the set of all second elements of the order pairs in relation R and is denoted by R(D)

Example

In the above relation the domain $D = \{1,2\}$

range $R(D) = \{1,4,5\}$



(Operations on Relations)

Properties of Relations:

Let ~~properties of~~ R be relation on a set A. R is said to be

① Reflexive: $(a,a) \in R$, $\forall a \in A$ (or) aRa, $\forall a \in A$

② Symmetric: $(a,b) \in R \Rightarrow (b,a) \in R$ (or) $aRb \Rightarrow bRa$

③ Transitive: $(a,b)(b,c) \in R \Rightarrow (a,c) \in R$ (or) $aRb, bRc \Rightarrow aRc$

④ Antisymmetric: $(a,b)(b,a) \in R \Rightarrow a = b$ (or) $aRb, bRa \Rightarrow a = b$

Example

check various properties of relations R_1, R_2, R_3, R_4, R_5 on $A = \{1,2,3\}$ defined in above example.

Sol R_5: $R_5 = \{(1,1)(1,3)(2,2)(2,1)(3,3)\}$

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① R_5 is reflexive ($\because (a,a) \in R \ \forall a \in A$)

② R_5 is symmetric ($\because (a,b) \in R \Rightarrow (b,a) \in R \ \forall (a,b) \in A$)

③ R_5 is transitive ($\because (a,b),(b,c) \in R_5 \Rightarrow (a,c) \in R_5$)

R_1: $R_1 = \{(1,1)(2,2)(3,3)\}$

① R_1 is reflective

② R_1 is symmetric

③ R_1 is transitive

④ R_1 is antisymmetric

R_2: $R_2 = \{(1,2)(1,3)(2,3)\}$

R_2 is not reflective

R_2 is not symmetric

R_2 is transitive

R_3: $R_3 = \{(1,1)(2,1)(2,2)(3,1)(3,2)(3,3)\}$

R_3 is reflective

R_3 is not symmetric

R_3 is transitive

R_3 is Antisymmetric

R_4: $R_4 = \{(1,1)(1,2)(1,3)(2,2)(2,3)\}$

R_4 is reflective

R_4 is not symmetric

R_4 is transitive.

Q- let R be the relation from $X = \{2,3,4,5\}$ to $Y = \{3,6,7,10\}$ which is defined by "x divides y". write the relation and verify its properties.

Sol Given $X = \{2,3,4,5\}$ $Y = \{3,6,7,10\}$

$R = \{(2,6)(2,10), (3,3), (3,6), (5,10)\}$.

R_6 is not reflexive

R is not symmetric

R is transitive

R is Anti symmetric

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2. Let R be the relation in $X = \{2,3,4,5,6\}$ defined by xRy $\Leftrightarrow$ $|x-y|$ is divisible by 3 (ie $3|(x-y)$)

Sol Given $X = \{2,3,4,5,6\}$

$R = \{\cancel{(2,3)}\ (2,5)\ \cancel{(3,2)}\ \cancel{(3,4)}\ (3,6)\ \cancel{(4,3)}\ \cancel{(4,5)}\ (5,2)\ \cancel{(5,4)}$
$\cancel{(5,6)}\ (6,3)\ \cancel{(6,5)}\ (2,2)\ (3,3)\ (4,4)\ (5,5)\ (6,6)\}$

R is ~~not~~ clearly reflexive (because $(a,a) \in R \ \forall a \in A$)

R is symmetric (because $(a,b) \in R \Rightarrow (b,a) \in R \ \forall (a,b) \in A$

R is transitive ($\because (a,b) \in R, (b,c) \in R \Rightarrow (a,c) \in R$)

3. If $R = \{(x,y) \mid x \in N, y \in N, x+3y = 12\}$ where N is set of all natural numbers.

Sol

$R = \{(3,3)\ (6,2)\ (9,1)\}$

R is not reflexive

R is not symmetric

R is transitive

R is Antisymmetric

Irreflexive relation

A Relation R on a set A is said to be irreflexive if for every $x \in A$, $(x,x) \notin R$

Example

R_2 in the above example is irreflexive

Problem

1. Given an example of a relation that is neither reflexive nor irreflexive

Sol Let R be a relation on $A = \{1,2,3\}$

$R = \{(1,1), (2,3), (3,1)\}$ is neither reflexive nor irreflexive

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2. Given an example of a relation that is bothsymmetric and Antisymmetric.

Sol

let R be a relation on $A = \{1,2,3\}$

$R = \{(1,1)\ (2,2)\ (3,3)\}$

3. Give an example of a relation which is ~~both~~ irreflexive and transitive.

sol let R be a relation on $A = \{1,2,3\}$ then this R is defined by $aRb \Rightarrow a<b$

$\therefore R = \{(1,2)\ (2,3)\ (1,3)\}$ is clearly transitive and irreflexive.

4. Given an example of a relation that is reflexive, antisymmetric and transitive.

Sol

let R be a relation on $A = \{1,2,3\}$ then this R is defined by $aRb \Rightarrow a \leq b$

$R = \{(1,1)\ (2,2)\ (3,3)\ (1,2)\ (1,3)\ (2,3)\}$ is clearly reflexive antisymmetric and transitive.

5. Given $S = \{1,2,3,\ldots,10\}$ and a relation R on S where $R = \{(x,y) / x+y = 10\}$ what are the properties of the relation ~~as~~ R.

Sol

$R = \{(1,9)\ (2,8)\ (3,7)\ (4,6)\ (5,5)\ (6,4)\ (7,3)\ (8,2)\ (9,1)\}$

R is not reflexive because $(a,a) \notin R\ \forall a \in S$

R is symmetric because $(a,b) \in R$

$\Rightarrow a+b = 10$

$\Rightarrow b+a = 10$

$\Rightarrow (b,a) \in R$

R is not transitive because $(2,8) \in R$, $(8,2) \in R$ but

$(2,2) \notin R$

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R is not antisymmetric because $(2,8), (8,2) \in R$ but $2 \neq 8$

R is not irreflexive because $(5,5) \in R$

6. let R be a relation on $P(S)$ defined by ARB if and only if $A \subseteq B$. check the properties of Relation.

Sol i) Reflexive: let $A \in P(S)$

$A \subseteq A, \forall A \in P(S)$

$ARA, \forall A \in P(S)$

ii) symmetric: let ARB

$\Rightarrow A \subseteq B$

$A = \{1,2\} \subseteq \{1,2,3\} = B$

$\{1,2,3\} \nsubseteq \{1,2\}$

$\therefore$ R is not symmetric

iii) Antisymmetric: let $AB \in P(S) \ni$

$\Rightarrow ARB$ and BRA

$\Rightarrow A \subseteq B$ and $B \subseteq A$

$\Rightarrow A = B$

$\therefore$ It is antisymmetric

iv) Transitive: let $A, B, C \in P(S) \ni$

$\Rightarrow ARB, BRC, A\cancel{RC}$

$\Rightarrow A \subseteq B$ and $B \subseteq C$

$\Rightarrow A \subseteq B \subseteq C$

$\Rightarrow A \subseteq C$

$\therefore$ It is transitive.

v) R is not irreflexive

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Representation of a Relation:

We can represent a relation in two ways

1. Relation matrix
2. Digraph

Relation matrix:

Let R be a relation from a set $A = \{a_1, a_2, a_3 \ldots, a_m\}$ to $B = \{b_1, b_2, b_3 \ldots b_n\}$ be finite sets containing m and n elements then R can be represented by a 0-1 matrix of order $m \times n$ which is defined as $M_R = [r_{ij}]$ where $r_{ij} = 1$

$$r_{ij} = \begin{cases} 1 & ; \; a_i R b_j \\ 0 & , \; a_i \not R \, b_j \end{cases}$$

$$\begin{array}{c|c} & b_1 \quad b_2 \ldots b_j \ldots b_n \\ \hline \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_i \\ \vdots \\ a_m \end{array} & \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & 1/0 & \\ & & & \end{bmatrix} \end{array}$$

Example

1. $A = \{1, 2, 3, 4\}$; $B = \{b_1, b_2, b_3\}$ consider the relation $\{(1, b_2), (1, b_3), (3, b_2), (4, b_1), (4, b_3)\}$ determine the matrix of relation R.

Sol $A = \{1, 2, 3, 4\}$, $B = \{b_1, b_2, b_3\}$

Given $R = \{(1, b_2), (1, b_3), (3, b_2), (4, b_1), (4, b_3)\}$

$$M_R = \begin{array}{c|ccc} & b_1 & b_2 & b_3 \\ \hline 1 & 0 & 1 & 1 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 4 & 1 & 0 & 1 \end{array}$$

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2. Let $A = \{1, 2, 3, 4\}$. Find the relation R on A determined by the matrix

$$M_R = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$$M_R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 4 & 1 & 1 & 0 & 1 \end{array}$$

$R = \{(1,1), (1,3), (2,3), (3,1), (4,1), (4,2), (4,4)\}$

Lattice:

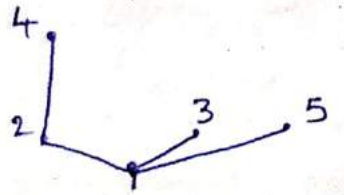
A partial ordered set $(S, \leq)$ is said to be a lattice if every pair of elements in S has greatest lower bound and least upper bound

Example

Determine whether the following posets are lattices

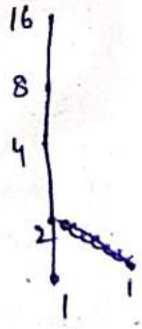
① $(\{1, 2, 3, 4, 5\}, /)$ ② $(\{1, 2, 4, 8, 16\} / 16)$

① $(\{1, 2, 3, 4, 5\}, /)$



is not a lattice, because there is no upper bound for (2,3) and there is no upper bound for (3,5)

② $(\{1, 2, 4, 8, 16\} / 16)$



It is chain and it is clearly lattice

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UNIT-3

Algebraic structures (or) systems

① Properties of binary operations:

1. closure: $a, b \in S \Rightarrow a * b \in S$
2. Associative: $a, b, c \in S \Rightarrow (a*b)*c = a*(b*c)$
3. Identity: let $a \in S \Rightarrow \exists e \in S \ni a*e = a = e*a$
4. Inverse: let $a \in S \Rightarrow \exists b \in S \ni a*b = b*a = e$
5. Abelian: $a*b = b*a$; $\forall a, b \in S$

closure → Binary

② semi group → closure, Assosiative

monoid → closure, Assosiative, identity

group → " " " , Inverse.

abelian group → " " " , " , abelian (or binary)

Properties:

let S be a non empty set and '*' be a operation on S then (S, *) ~~will~~ may have the above properties

above ①

above ②

Example

① Semi group, monoid and group are all algebraic systems with one binary operation.

Example

① $(N, +)$

It is semigroup ~~and~~ but it is not a monoid because, identity of N is '0' but it does not belongs to N $(0 \notin N)$

② $(N, *)$

It is a monoid with identity '1' belongs to N but not a group because inverse of 2 is $1/2$ but it does not belongs to N $(\frac{1}{2} \notin N)$

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⑧ $(R,+)$, $(R,*)$

~~is a~~

$(R,+)$ and $(R,*)$ are ~~groups~~ abelian groups.

⑨ $(Z,+)$

$(Z,+)$ is also an abelian group.

Problems

1. show that $(Z,*)$ is a group, where $*$ is defined by $a*b = a+b+1$

Sol Given $a*b = a+b+1$

1. closure: let $a,b \in Z$

$\Rightarrow a+b \in Z$

$\Rightarrow a+b+1 \in Z$

$\Rightarrow a*b \in Z$

Z is closed under $*$

2. Associative:

let $a,b,c \in Z$

$(a*b)*c = (a+b+1)*c$

$= (a+b+1+c+1)$

$(a*b)*c = (a+b+c+2)$

$a*(b*c) = a*(b+c+1)$

$= (a+b+c+1+1)$

$a*(b*c) = (a+b+c+2)$

$\therefore (a*b)*c = a*(b*c)$

$\therefore Z$ is associative under $*$

3. Identity: let $a \in Z \ni a*e = a$

$(a+e+1) = a$

$e = -1 \in Z$

Hence $e = -1 \in Z$ is identity under $*$

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4. Inverse : let $a \in Z \ni a * b = e$

$a+b+1 = e$

$\Rightarrow b = e-a-1$

$\Rightarrow b = -1-a-1$

$\Rightarrow b = -a-2 \in Z$

$\therefore$ It is inverse under $*$

$\therefore (Z, *)$ is a group

5. Abelian group: let $a, b \in Z$

$a * b = a+b+1 = b+a+1$

$= b * a$

$\therefore Z, *$ is an abelian group.

2. show that Set Q of all rational numbers other than '1' with operation $*$ defined by $a*b = a+b-ab$ constitutes an abelian group

Sol Given $a*b = a+b-ab$, Given $(Q-\{1\}, *)$

1. closure: let $a, b \in Q$

$\Rightarrow a+b, ab \in Q$

$\Rightarrow a+b-ab \in Q$

$\Rightarrow a*b \in Q$

$\therefore Z$ is closed under $*$

2. Associative: let $a, b, c \in Z$

$\Rightarrow (a*b)*c = (a+b-ab)*c$

$= a+b-ab+c-(a+b-ab)c$

$(a*b)*c = a+b+c-ab-ac-bc+abc$

$a*(b*c) = a*(b+c-bc)$

$= a+b+c-bc-(b+c-bc)a$

$a*(b*c) = a+b+c-bc-ab-ac+abc$

$(a*b)*c = a*(b*c)$

$\therefore Z$ is associative.

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3. <u>Identity</u>: let $a \in Q-\{1\} \ni a * e = a$

$$a+e-ae = a$$

$$a+e(1-a) = a$$

$$e = \frac{0}{1-a} \in Q-\{1\}$$

$$e = 0 \in Q-\{1\}$$

Hence $e=0$ is identity

<u>Inverse</u>: let $a \in Q-\{1\} \ni a*b = e$

$$\Rightarrow a+b-ab = 0$$

$$\Rightarrow a+b(1-a) = 0$$

$$\Rightarrow b(1-a) = -a$$

$$\Rightarrow b = \frac{-a}{1-a}$$

$$\Rightarrow b = \frac{a}{a-1} \in Q-\{1\}$$

It is inverse

<u>Abelian</u>: let $a, b \in Q-\{1\}$

$$\Rightarrow a*b = a+b-ab$$

$$= b+a-ba$$

$$a*b = b*a$$

$\therefore Q-\{1\}, *$ is abelian group

3. In a group G show that $(a*b)^{-1} = b^{-1}*a^{-1}$ for all $a,b \in G$

<u>Sol</u> <u>Proof</u>:

let $(G,*)$ be a group.

let $(a,b) \in G$]

$$\Rightarrow a*a^{-1} = a^{-1}*a = e$$

$$\Rightarrow b*b^{-1} = b^{-1}*b = e$$

claim: $(a*b)^{-1} = b^{-1}*a^{-1}$

$$(a*b)*(b^{-1}*a^{-1})$$

$$\Rightarrow a*(b*b^{-1})*a^{-1}$$

$$\Rightarrow a*e*a^{-1}$$

$$\Rightarrow a*a^{-1}$$

$$\Rightarrow e$$

$\therefore (a*b)^{-1} = b^{-1}*a^{-1}$

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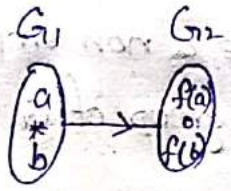
Semi group homomorphism:

Let $(G_1, *)$, $(G_2, \circ)$ be two groups then a mapping $f: G_1 \to G_2$ is said to be a homomorphism if $f(a*b) = f(a) \circ f(b)$



Example

let $f:(R,+) \to (R,\cdot)$ defined by $f(a+b) = ab$

$$f(a+b) = a+b$$
$$= f(a) + f(b)$$

Problem

∴ In a group G, $(ab)^2 = a^2b^2$ if and only if G is an abelian group.

Sol

Given $(G,*)$ is a group

suppose G is abelian group

claim :- $(a*b)^2 = a^2*b^2$

let $a, b \in G$

$$(a*b)^2 = (a*b)(a*b)$$
$$= ((a*b)*a)*b$$
$$= (a*(b*a))*b \quad (\because G \text{ is abelian group})$$
$$= (a*(a*b))*b \quad (\because a*b = b.a, \forall a,b \in G)$$
$$= ((a*a)*b)*b$$
$$= (a*a)(b*b)$$
$$(a*b)^2 = a^2*b^2$$

Conversely suppose

$$(a*b)^2 = a^2*b^2$$
$$(a*b)*(a*b) = (a*a)*(b*b)$$
$$a*(b*(a*b)) = a*(a*(b*b))$$
$$b*(a*b) = a*(b*b) \quad (\because G \text{ is a group by cancellation law})$$
$$(b*a)*b = (a*b)*b \quad (\text{Associative law}$$

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$b*a = a*b$ (∵ cancellation law)

$a*b = b*a$

$(G, *)$ is a abelian

subgroups:

Let G be a group ~~and~~ then let H be a non empty subset of a group G is said to be a sub group of G if it is a group itself

Example

let $G = (\{1, -1, i, -i\}, \times)$ is a group and let $H_1 = (\{1, -1\}, \times)$

$H_2 = (\{1, i\}, \times)$

clearly, H_1 is a group itself. So H_1 is a sub group of G but H_2 is not a group itself Because, $i \times i = -1$ does not belongs to H_2

$\times$	1	-1
1	1	-1
-1	-1	1

H_1

$\times$	1	i
1	1	i
i	i	-1

H_2 Here $-1 \notin H_2$

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