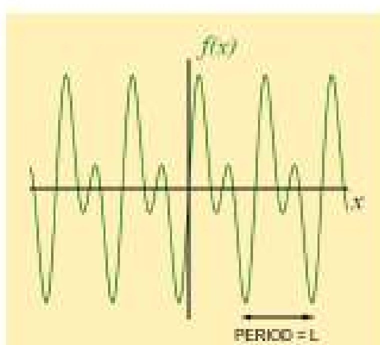
CS010 404

Continuous Time Fourier Series

Dept. of Computer Science and Engineering

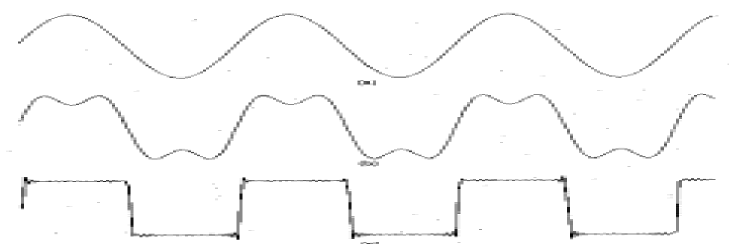
Periodic Signals

- A graph of **periodic** function $f(x)$ that has period L exhibits the same pattern every L units along the x-axis, so that $f(x + L) = f(x)$ for every value of x. If we know what the function looks like over one complete period, we can thus sketch a graph of the function over a wider interval of x (that may contain many periods)



Time Domain and Frequency Domain

- In 1807, Jean Baptiste Joseph Fourier showed that any periodic signal could be represented by a series of sinusoidal functions



In picture: the composition of the first two functions gives the bottom one

Continuous Time Fourier Series - 1

- Fourier (or frequency domain) analysis constitutes a tool of great usefulness
- Accomplishes decomposition of broad classes of signals using complex exponentials
- Allows for insightful characterization of signals and systems
- The Fourier series represents a signal as a linear combination of *complex sinusoids*
- Fourier found that any periodic waveform can be expressed as a series of harmonically related sinusoids, i.e., sinusoids whose frequencies are multiples of a *fundamental frequency (or first harmonic). For example, a* series of sinusoids with frequencies 1 MHz,2 MHz,3MHz , and so on, contains the fundamental frequency of 1 MHz, a second harmonic of 2 MHz, a third harmonic of 3 MHz, and so on.

CTFS - Basics

- In this Tutorial, we consider working out Fourier series for functions $f(x)$ with period $L = 2\pi$. Their fundamental frequency is then $k = \frac{2\pi}{L} = 1$, and their Fourier series representations involve terms like

$$\begin{aligned} &a_1 \cos x \ , \quad b_1 \sin x \\ &a_2 \cos 2x \ , \quad b_2 \sin 2x \\ &a_3 \cos 3x \ , \quad b_3 \sin 3x \end{aligned}$$

We also include a constant term $a_0/2$ in the Fourier series. This allows us to represent functions that are, for example, entirely above the $x-$axis. With a sufficient number of harmonics included, our approximate series can exactly represent a given function $f(x)$

$$\begin{aligned} f(x) = a_0/2 \quad &+ \quad a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + ... \\ &+ \quad b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + ... \end{aligned}$$

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Continuous Time Fourier Series - 2

In general, any periodic waveform can be expressed as

$$\begin{aligned} f(t) = \frac{1}{2}a_0 &+ a_1 cos\,\omega t + a_2 cos\,2\omega t + a_3 cos\,3\omega t + a_4 cos\,4\omega t + \ldots \\ &+ b_1 sin\,\omega t + b_2 sin\,2\omega t + b_3 sin\,3\omega t + b_4 sin\,4\omega t + \ldots \end{aligned}$$

OR

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n cos\,n\omega t + b_n sin\,n\omega t)$$

Where the first term $a_0/2$ is a constant and it represents DC (Average) component of f(t) . If f(t) represent some voltage v(t) or current i(t), the term $a_0/2$ is average value of v(t) or i(t). The terms with coefficients a1 and b1 represent the fundamental frequency component ω, and terms with coefficients a2 and b2 represent the 2nd harmonic 2ω and so on

$$k_1 cos\,\omega t + k_2 sin\,\omega t = k cos(\omega t + \theta) \text{ where } \theta \text{ is a constant.}$$

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Continuous Time Fourier Series - 3

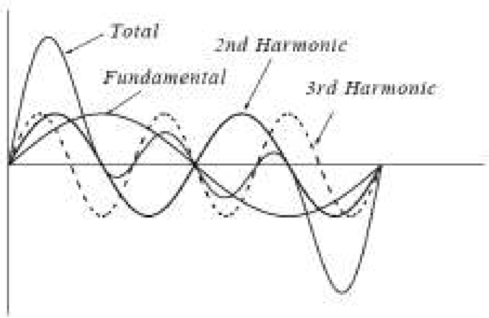
The coefficients a_0, a_n, and b_n are found from the following relations.

One can even approximate a square-wave pattern with a suitable sum that involves a fundamental sine-wave plus a combination of harmonics of this fundamental frequency. This sum is called a Fourier series

$$\frac{1}{2}a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt\,dt$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt\,dt$$



DC, the fundamental, the second harmonic, and so on, must produce the waveform . F(t). Generally, the sum of two or more sinusoids of different frequencies produce a waveform that is not a sinusoid as shown in Figure.

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CTFS – Evaluation steps

A more compact way of writing the Fourier series of a function $f(x)$, with period 2π, uses the variable subscript $n = 1, 2, 3, \ldots$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$$

- We need to work out the **Fourier coefficients** (a_0, a_n and b_n) for given functions $f(x)$. This process is broken down into three steps

STEP ONE $$a_0 = \frac{1}{\pi}\int_{2\pi} f(x)\,dx$$

STEP TWO $$a_n = \frac{1}{\pi}\int_{2\pi} f(x)\cos nx\,dx$$

STEP THREE $$b_n = \frac{1}{\pi}\int_{2\pi} f(x)\sin nx\,dx$$

where integrations are over a single interval in x of $L = 2\pi$

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Convergence of Fourier Series

-Periodic function x(t) with discontinuities) → infinite number of Fourier coefficients

- Convergence
  – Does Fourier series converge?
  – In what sense does it converge?

Will converge if Dirichlet conditions are satisfied

– Dirichlet conditions

1. $x(t)$ is absolutely integrable over one period

$$\int_T |x(t)|\, dt < \infty \; .$$

2. $x(t)$ has a finite number of maxima and minima during one period.
3. $x(t)$ has a finite number of discontinuities in one period.

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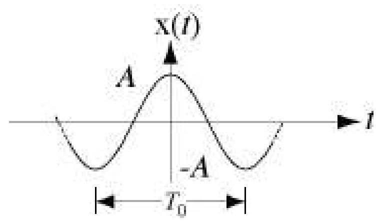
Negative Frequency

This signal is obviously a sinusoid. How is it described mathematically?



It could be described by

$$x(t) = A\cos\left(\frac{2\pi t}{T_0}\right) = A\cos(2\pi f_0 t)$$

But it could also be described by

$$x(t) = A\cos(2\pi(-f_0)t)$$

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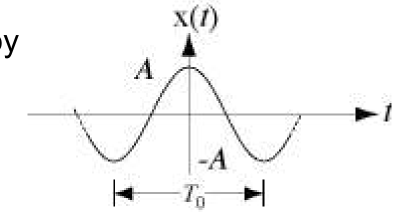
Negative Frequency

x(t) could also be described by



$$x(t) = A\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}$$

or

$$x(t) = A_1 \cos(2\pi f_0 t) + A_2 \cos(2\pi(-f_0)t) \;,\; A_1 + A_2 = A$$

and probably in a few other different-looking ways. So who is to say whether the frequency is positive or negative? For the purposes of signal analysis, it does not matter.

CTFS – Alternate form

- Fourier series of a periodic signal with fundamental period T and fundamental frequency $\omega_0 = 2\pi/T$

$$\boxed{\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} \\ a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}\,dt \end{aligned}}$$

$\int_T$: integrate over *any* interval of length T $\boxed{x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k}$

- a_k: Fourier series coefficients or spectral coefficients
- $a_{(\pm k)}e^{j(\pm k)\omega_0 t}$: kth harmonic components
- a_0: dc component
- $a_{(\pm 1)}e^{j(\pm 1)\omega_0 t}$: fundamental component

CTFS - Example

1. Periodic time function

$$x(t) = 1 + \sin\omega_0 t + 2\cos\omega_0 t + \cos(2\omega_0 t + \pi/4)$$

– Expand $x(t)$

$$\begin{aligned} x(t) &= 1 + \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) \\ &\quad + \left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) + \frac{1}{2}\left(e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}\right) \\ &= 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} \\ &\quad + \left(\frac{1}{2}e^{j\pi/4}\right)e^{2j\omega_0 t} + \left(\frac{1}{2}e^{-j\pi/4}\right)e^{-2j\omega_0 t} \end{aligned}$$

– Fourier series coefficients

$$\begin{aligned} a_0 &= 1 \\ a_1 &= \left(1 + \frac{1}{2j}\right) \\ a_{-1} &= \left(1 - \frac{1}{2j}\right) \\ a_2 &= \frac{1}{2}e^{j\pi/4} \\ a_{-2} &= \frac{1}{2}e^{-j\pi/4} \\ a_k &= 0,\ |k| > 2 \end{aligned}$$

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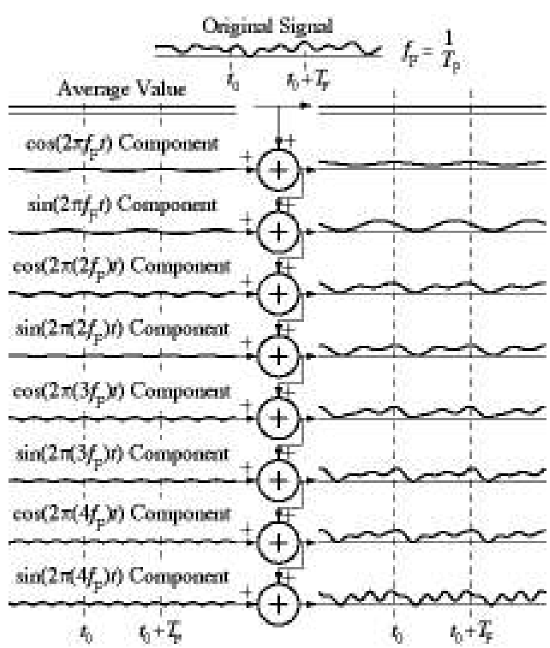
Continuous-Time Fourier Series Concept



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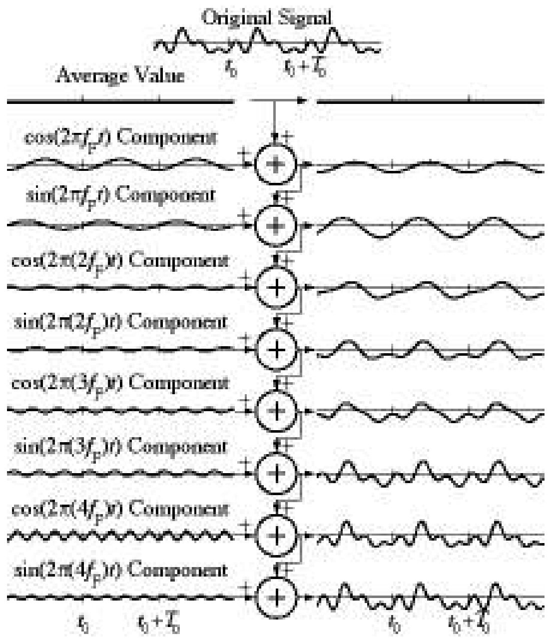
Continuous-Time Fourier Series Concept



Fourier Series Properties

- Linearity
- Time Shift
- Time scaling
- Time-reversal
- Conjugate Symmetry
- Multiplication
- Parseval’s Theorem

Properties of CTFS - Linearity

- *Linearity*
  - Let
$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$$
$$y(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} b_k$$
    two periodic signals with period T, then
    $z(t) = Ax(t) + By(t)$ is periodic with T and
$$\boxed{z(t) = Ax(t) + By(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} c_k = Aa_k + Bb_k}$$

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Properties of CTFS – Time Shift

- *Time Shifting*
  - Let
$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$$
    then
$$\boxed{x(t-t_0) \stackrel{\mathcal{FS}}{\longleftrightarrow} b_k = \mathrm{e}^{-\mathrm{j}k\omega_0 t_0} a_k = \mathrm{e}^{-\mathrm{j}k(2\pi/T)t_0} a_k}$$
  - Observe: same magnitudes $|b_k| = |a_k|$

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Properties of CTFS – Time Scaling

- *Time scaling*
  - $t \to \alpha t$, $\alpha > 0$
    - $\Rightarrow$ period changes $T \to T/\alpha$
    - $\Rightarrow$ fundamental frequency changes $\omega_0 \to \alpha\omega_0$
  - Fourier series representation

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k \mathrm{e}^{\mathrm{j}k(\alpha 2\pi/T)t} = \sum_{k=-\infty}^{\infty} a_k \mathrm{e}^{\mathrm{j}k(\alpha\omega_0)t}$$

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Properties of CTFS – Time Reversal

- *Time Reversal*
  - Let

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

then

$$\boxed{x(-t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k = a_{-k}}$$

  - Observe:
    - $x(t)$ even $\Rightarrow$ Fourier series coefficients even, $a_{-k} = a_k$
    - $x(t)$ odd $\Rightarrow$ Fourier series coefficients odd, $a_{-k} = -a_k$

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Properties of CTFS – Conjugate Symmetry

- *Conjugation and Conjugate Symmetry*
  - Let
    $$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$
    then
    $$x^*(t) \xleftrightarrow{\mathcal{FS}} a^*_{-k}$$
  - Implication:
    - Real $x(t) = x^*(t) \Rightarrow$ Fourier series coefficients are *conjugate symmetric* $a_k = a^*_{-k}$
    - Imaginary $x(t) = -x^*(t) \Rightarrow a_k = -a^*_{-k}$
  - Conjugation + time reversal properties

$$\boxed{\begin{array}{ccccccccc} x(t) & = & \mathrm{Re}\{\mathrm{Ev}\{x(t)\}\} & + & \mathrm{Re}\{\mathrm{Od}\{x(t)\}\} & + & \mathrm{jIm}\{\mathrm{Ev}\{x(t)\}\} & + & \mathrm{jIm}\{\mathrm{Od}\{x(t)\}\} \\ a_k & = & \mathrm{Re}\{\mathrm{Ev}\{a_k\}\} & + & \mathrm{Re}\{\mathrm{Od}\{a_k\}\} & + & \mathrm{jIm}\{\mathrm{Ev}\{a_k\}\} & + & \mathrm{jIm}\{\mathrm{Od}\{a_k\}\} \end{array}}$$

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Properties of CTFS - Multiplication

- *Multiplication*
  - Let
    $$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$
    $$y(t) \xleftrightarrow{\mathcal{FS}} b_k$$
    two periodic signals with period T, then
    $z(t) = x(t)y(t)$ is periodic with T and
    $$\boxed{z(t) = x(t)y(t) \xleftrightarrow{\mathcal{FS}} c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}}$$
  - Observe: $\sum_{l=-\infty}^{\infty} a_l b_{k-l}$ = discrete–time convolution of a_k and b_k

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Properties of CTFS – Parseval's Theorem

- *Parseval's Relation*
  - If $x(t) \xleftrightarrow{\mathcal{FS}} a_k$, then

$$\boxed{\frac{1}{T}\int_T |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |a_k|^2}$$

  - Observe that

$$\frac{1}{T}\int_T |a_k e^{jk\omega_0 t}|^2\,dt = \frac{1}{T}\int_T |a_k|^2\,dt = |a_k|^2$$

  - Hence: Average total power of the signal is sum of the average powers in all of its harmonic components.

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