Therefore, equation (2) becomes

Hint $\omega_0 T = 2\pi$

$\omega_0 (4T_1) = 2\pi$

$\omega_0 T_1 = \frac{\pi}{2}$

$$a_0 = \frac{2T_1}{4T_1} = \frac{1}{2}, \ k = 0$$

Problem 5.5 Determine the Fourier series coefficients (exponential representation) of the given signal.

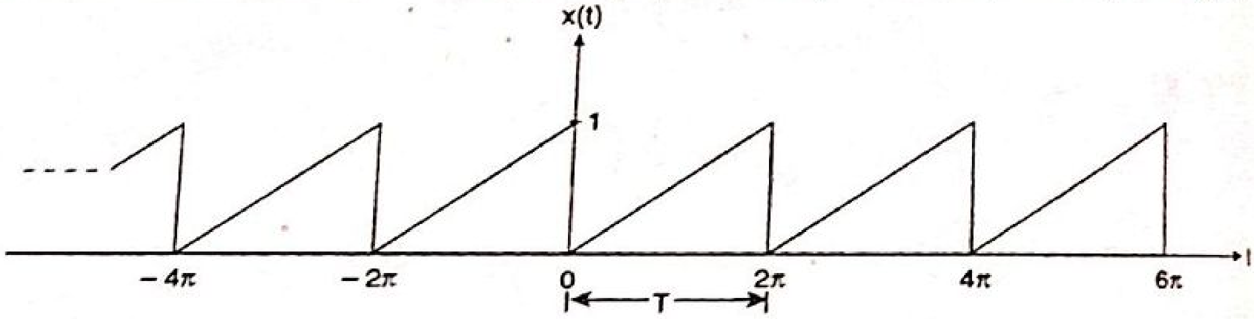


Fig. 5.3 Sawthooth Waveform of Period 2π

Solution The equation of straight line is

$$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

$$\frac{x(t) - 0}{t - 0} = \frac{0 - 1}{0 - 2\pi}$$

$$x(t) = \frac{t}{2\pi}, \ 2\pi > t > 0$$

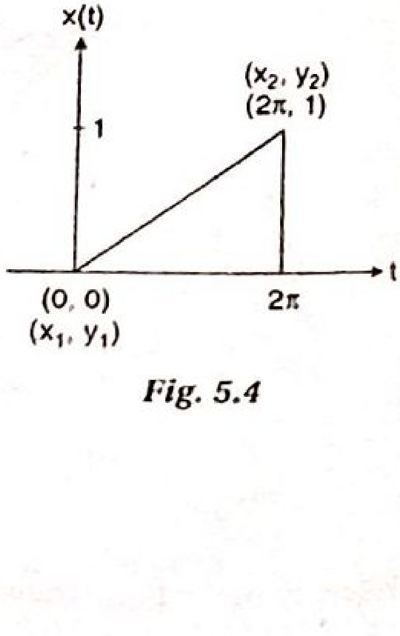


Fig. 5.4

Therefore, the equation for the given signal waveform is

$$x(t) = \left\{ \frac{t}{2\pi}, 2\pi > t \geq 0 \right.$$

The Fourier series representation (exponential) of $x(t)$ is

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} \, dt \qquad (1)$$

The fundamental period, $T = 2\pi$.

The fundamental frequency is

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

Equation (1) becomes

$$a_k = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{t}{2\pi} \right) e^{-jk\omega_0 t} \, dt$$

Since $\omega_0 = 1$,

$$a_k = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{t}{2\pi}\right)e^{-jkt}\,dt \quad (2)$$

For $k = 0$,

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{t}{2\pi}\right)dt = \frac{1}{(2\pi)^2}\int_0^{2\pi} t\,dt$$

$$a_0 = \frac{1}{4\pi^2}\frac{t^2}{2}\Bigg|_0^{2\pi} = \frac{1}{8\pi^2}\left[(2\pi)^2 - 0\right] = \frac{1}{2} \quad (3)$$

For $k \neq 0$,

$$a_k = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{t}{2\pi}\right)e^{-jkt}\,dt = \frac{1}{4\pi^2}\int_0^{2\pi} te^{-jkt}\,dt$$

$$a_k = \frac{1}{4\pi^2}\left\{\frac{1}{(-jk)^2}e^{-jkt}[(-jkt)-1]\Bigg|_0^{2\pi}\right\} \qquad \textbf{Hint} \int_a^b te^{\alpha t}\,dt = \frac{1}{\alpha^2}e^{\alpha t}(\alpha t - 1)\Bigg|_a^b$$

$$a_k = \frac{1}{4\pi^2 k^2}\left[jkt\,e^{-jkt}\Big|_0^{2\pi} + e^{-jkt}\Big|_0^{2\pi}\right]$$

$$a_k = \frac{1}{(2\pi k)^2}\left[(jk(2\pi)e^{-jk(2\pi)} - 0) + (e^{-jk(2\pi)} - 1)\right]$$

$$a_k = \frac{1}{(2\pi k)^2}\left[j2\pi k\,e^{-j2\pi k} + e^{-j2\pi k} - 1\right]$$

Problem 5.6 Determine the Fourier series coefficients (exponential representation) of the signal $x(t)$ given below.

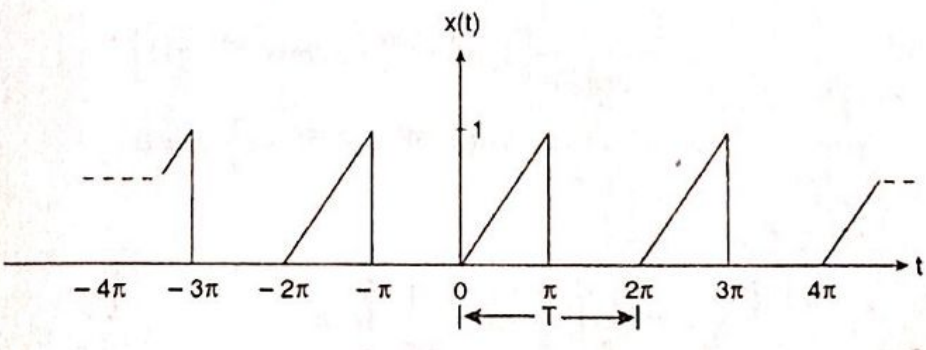


Fig. 5.5 Sawtooth Waveform

Solution The equation of straight line is given by

$$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

$$\frac{x(t) - 0}{t - 0} = \frac{0 - 1}{0 - \pi}$$

$$x(t) = \frac{t}{\pi}$$

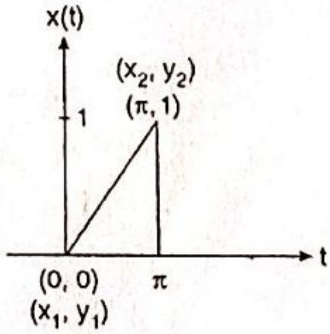


Fig. 5.6

The equation for the given signal waveform is

$$x(t)=\begin{cases}\dfrac{t}{\pi}, & \pi \geq t \geq 0\\ 0, & 2\pi > t > \pi\end{cases}$$

The Fourier series representation (exponential) of $x(t)$ is

$$a_k=\frac{1}{T}\int_T x(t)\,e^{-jk\omega_0 t}\,dt$$

The fundamental period, $T=2\pi$.

The fundamental frequency, $\omega_0=\dfrac{2\pi}{T}=1$

$$a_k=\frac{1}{T}\left[\int_0^{\pi}\left(\frac{t}{\pi}\right)e^{-jk\omega_0 t}\,dt+\int_0^{\pi}0\cdot e^{-jk\omega_0 t}\,dt\right]$$

Since $\omega_0=1$ and $T=2\pi$,

$$a_k=\frac{1}{2\pi^2}\int_0^{\pi}te^{-jk\omega_0 t}\,dt$$

Hint $\displaystyle\int_a^b te^{\alpha t}\,dt=\frac{1}{\alpha^2}\left[e^{\alpha t}(\alpha t-1)\right]$

For $k \neq 0$,

$$a_k=\frac{1}{2\pi^2}\left\{\frac{1}{(-jk)^2}\left[e^{-jkt}(-jkt-1)\right]_0^{\pi}\right\}$$

$$a_k=\frac{1}{2(\pi k)^2}\left[jkt\,e^{-jkt}\Big|_0^{\pi}+e^{-jkt}\Big|_0^{\pi}\right]$$

$$a_k=\frac{1}{2(\pi k)^2}\left[(j\pi k\,e^{-j\pi k}-0)+(e^{-j\pi k}-1)\right]$$

$$a_k=\frac{1}{2(\pi k)^2}\left[j\pi k\,e^{-j\pi k}+e^{-j\pi k}-1\right],\quad k\neq 0$$

For $k=0$,

$$a_0=\frac{1}{2\pi}\int_0^{\pi}\left(\frac{t}{\pi}\right)dt=\frac{1}{2\pi^2}\int_0^{\pi}t\,dt$$

$$a_0=\frac{1}{2\pi^2}\frac{t^2}{2}\Big|_0^{\pi}$$

$$a_0=\frac{1}{4\pi^2}\left[(\pi^2-0)\right]=\frac{1}{4},\quad k=0$$

Problem 5.7 Determine the Fourier series coefficients (exponential representation) of the signal $x(t)$ given below.

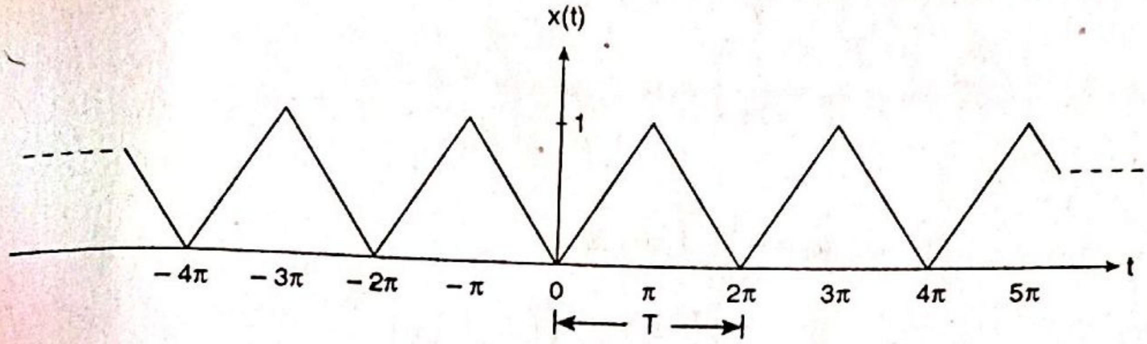
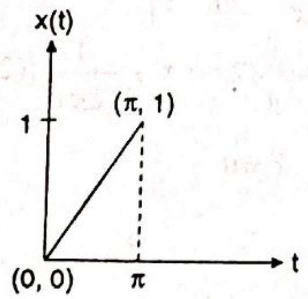


Fig. 5.7 Triangular Waveform of Period 2π

Solution The straight line equation is given by

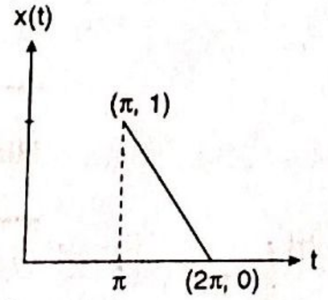
$$\frac{y-y_1}{x-x_1}=\frac{y_1-y_2}{x_1-x_2}$$

For



$$\frac{x(t)-0}{t-0}=\frac{0-1}{0-\pi}$$

$$x(t)=\frac{t}{\pi}$$



$$\frac{x(t)-1}{t-\pi}=\frac{1-0}{\pi-2\pi}$$

$$x(t)=\left(2-\frac{t}{\pi}\right)$$

Fig. 5.8

The equation for the given signal waveform,

$$x(t)=\begin{cases}\left(\dfrac{t}{\pi}\right), & \pi>t\geq 0\\[2ex] \left[2-\left(\dfrac{t}{\pi}\right)\right], & 2\pi>t\geq\pi\end{cases}$$

The Fourier series representation of $x(t)$ is given by

$$a_k=\frac{1}{T}\int_{T}x(t)e^{-jk\omega_0 t}\,dt$$

The fundamental period, $T = 2\pi$.

The fundamental frequency, $\omega_0 = \frac{2\pi}{T} = 1$

$$a_k = \frac{1}{2\pi}\int_0^{2\pi} x(t)e^{-jkt}\,dt$$

For $k = 0$,

$$a_0 = \frac{1}{(2\pi)}\int_0^{2\pi} x(t)\,dt$$

$$a_0 = \frac{1}{2\pi}\left[\int_0^{\pi}\left(\frac{t}{\pi}\right)dt + \int_{\pi}^{2\pi}\left(2-\frac{t}{\pi}\right)dt\right]$$

$$a_0 = \frac{1}{2\pi^2}\int_0^{\pi} t\,dt + \frac{1}{\pi}\int_{\pi}^{2\pi} dt - \frac{1}{2\pi^2}\int_{\pi}^{2\pi} t\,dt$$

$$a_0 = \frac{1}{2\pi^2}\frac{t^2}{2}\Bigg|_0^{\pi} + \frac{1}{\pi}t\Bigg|_{\pi}^{2\pi} - \frac{1}{2\pi^2}\frac{t^2}{2}\Bigg|_{\pi}^{2\pi}$$

$$a_0 = \frac{1}{(2\pi)^2}(\pi^2 - 0) + \frac{1}{\pi}(2\pi - \pi) - \frac{1}{(2\pi)^2}[(2\pi)^2 - (\pi)^2]$$

$$a_0 = \frac{1}{4} + 1 - \frac{3}{4} = +\frac{1}{2},\ k = 0$$

For $k \neq 0$,

$$a_k = \frac{1}{2\pi}\left[\int_0^{\pi}\left(\frac{t}{\pi}\right)e^{-jkt}\,dt + \int_0^{2\pi}\left(2-\frac{t}{\pi}\right)e^{-jkt}\,dt\right]$$

$$a_k = \frac{1}{2\pi^2}\int_0^{\pi} te^{-jkt}\,dt + \frac{1}{\pi}\int_{\pi}^{2\pi} e^{-jkt}\,dt - \frac{1}{2\pi^2}\int_{\pi}^{2\pi} te^{-jkt}\,dt$$

Hint $\int_a^b te^{\alpha t}\,dt = \frac{1}{\alpha^2}\left[e^{\alpha t}(\alpha t - 1)\right]_a^b$

$$a_k = \frac{1}{2(\pi k)^2}\left[e^{-j\pi k}(j\pi k + 1) - 1\right] - \frac{1}{j\pi k}(e^{-j2\pi k} - e^{-j\pi k}) - \frac{1}{2(\pi k)^2}\left[e^{-j2\pi k}(j2\pi k - 1) - e^{-j\pi k}(j\pi k - 1)\right]$$

$$a_k = \frac{1}{2(\pi k)^2}[e^{-j\pi k+1} - 1] - \frac{1}{j\pi k}(e^{-j2\pi k} - e^{-j\pi k}) - \frac{1}{2(\pi k)^2}[e^{-j2\pi k}(j2\pi k - 1) - e^{-j\pi k}(j\pi k - 1)]$$

$$a_k = \frac{1}{2(\pi k)^2}(2e^{-j\pi k} - e^{-j2\pi k} - 1)$$

Problem 5.8 Determine the Fourier series coefficients (exponential representation) of the given signal $x(t)$.

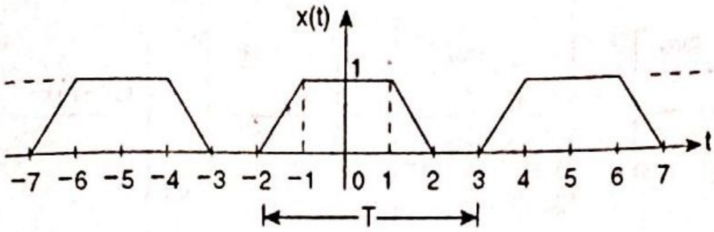


Fig. 5.9

Solution

$$x(t)=\begin{cases}(t+2), & -1>t\geq -2\\ 1, & 1>t\geq -1\\ (2-t), & 2>t\geq 1\\ 0, & 3>t\geq 2\end{cases}$$

The Fourier series representation of the given signal $x(t)$,

$$a_k=\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt$$

The fundamental period, $T=5$

The fundamental frequency, $\omega_0=\frac{2\pi}{T}=\frac{2}{5}\pi$

$$a_k=\frac{1}{5}\int_T x(t)e^{-j\frac{2\pi k}{5}t}\,dt$$

For $k=0$,

$$a_0=\frac{1}{5}\int_T x(t)\,dt=\frac{1}{5}\left[\int_{-2}^{-1}(t+2)\,dt+\int_{-1}^{1}1\,dt+\int_{1}^{2}(2-t)\,dt\right]$$

$$a_0=\frac{1}{5}\left[\frac{t^2}{2}\Big|_{-2}^{-1}+2t\Big|_{-2}^{-1}+t\Big|_{-1}^{1}+2t\Big|_{1}^{2}-\frac{t^2}{2}\Big|_{1}^{2}\right]$$

$$a_0=\frac{1}{10}(1-4)+\frac{2}{5}(-1+2)+\frac{1}{5}(1+1)+\frac{2}{5}(2-1)-\frac{1}{10}(4-1)$$

$$a_0=\frac{-3}{10}+\frac{2}{5}+\frac{2}{5}+\frac{2}{5}-\frac{3}{10}=\frac{3}{5}$$

a_0 is the average or DC value of the signal.

For $k\neq 0$,

$$a_k=\frac{1}{5}\int_T x(t)e^{-j\frac{2\pi k}{5}t}\,dt$$

$$a_k=\frac{1}{5}\left[\int_{-2}^{-1}(t+2)e^{-j\frac{2\pi kt}{5}}\,dt+\int_{-1}^{1}e^{-j\frac{2\pi kt}{5}}\,dt+\int_{1}^{2}(2-t)e^{-j\frac{2\pi kt}{5}}\,dt\right]$$

Applying Bernoulli's formula,

$$a_k = \frac{1}{5}\left\{\left[\frac{(t+2)e^{-j\frac{2\pi kt}{5}}}{-j\frac{2\pi kt}{5}}\right]_{-2}^{-1} - \left[\frac{e^{-j\frac{2\pi kt}{5}}}{\left(-j\frac{2\pi kt}{5}\right)^2}\right]_{-2}^{-1} + \left[\frac{e^{-j\frac{2\pi kt}{5}}}{-j\frac{2\pi kt}{5}}\right]_{-1}^{1} + \left[\frac{(2-t)e^{-j\frac{2\pi kt}{5}}}{-j\frac{2\pi kt}{5}}\right]_{1}^{2} + \left[\frac{e^{-j\frac{2\pi kt}{5}}}{\left(-j\frac{2\pi kt}{5}\right)^2}\right]_{1}^{2}\right\}$$

$$a_k = \frac{1}{5}\left[\frac{e^{-j\frac{2\pi k}{5}}}{-j\frac{2\pi k}{5}} + \left(\frac{e^{j\frac{2\pi k}{5}} - e^{j\frac{4\pi k}{5}}}{\left(\frac{2\pi k}{5}\right)^2}\right) - \left(\frac{e^{-j\frac{2\pi k}{5}} - e^{j\frac{2\pi k}{5}}}{j\frac{2\pi k}{5}}\right) + \frac{e^{-j\frac{2\pi k}{5}}}{j\frac{2\pi k}{5}} - \left(\frac{e^{-j\frac{4\pi k}{5}} + e^{-j\frac{2\pi k}{5}}}{\left(\frac{2\pi k}{5}\right)^2}\right)\right]$$

$$a_k = \frac{1}{5}\left[\left(\frac{e^{j\frac{2\pi k}{5}} + e^{-j\frac{2\pi k}{5}}}{\left(\frac{2\pi k}{5}\right)^2}\right) - \left(\frac{e^{j\frac{4\pi k}{5}} + e^{-j\frac{4\pi k}{5}}}{\left(\frac{2\pi k}{5}\right)^2}\right)\right]$$

$$a_k = \frac{5}{2(\pi k)^2}\left[\cos\left(\frac{2\pi k}{5}\right) - \cos\left(\frac{4\pi k}{5}\right)\right]$$

5.1.4 Trigonometric Representation of Continuous-time Fourier Series

The periodic signal $x(t)$ can be expressed as trigonometric series, i.e. in terms of sine and cosine terms, i.e.

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \qquad (5.17)$$

where a_0 = Average or DC components of the signal $x(t)$

a_n, b_n = Coefficients of signal, constant

To evaluate a_0 integrate equation (5.17) over one period, say t_0 to $(t_0 + T)$ at an arbitrary time t_0.

$$\int_{t_0}^{t_0+T} x(t)\,dt = a_0 \int_{t_0}^{t_0+T} dt + \int_{t_0}^{t_0+T} \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)\,dt$$

$$\int_{t_0}^{t_0+T} x(t)\,dt = a_0 T + \sum_{n=1}^{\infty}\left[a_n \int_{t_0}^{t_0+T} \cos n\omega_0 t\,dt + b_n \int_{t_0}^{t_0+T} \sin n\omega_0 t\,dt\right] \qquad (5.18)$$