1.1. PERIODIC FUNCTIONS

A function $f(x)$ which satisfies the relation $f(x + T) = f(x)$ for all x is called a periodic function. The *smallest positive number* T, for which this relation holds, is called the **period of** $f(x)$.

If T is the period of $f(x)$, then $f(x) = f(x + T) = f(x + 2T) = \ldots\ldots = f(x + nT) = \ldots\ldots$

Also $f(x) = f(x - T) = f(x - 2T) = \ldots\ldots = f(x - nT) = \ldots\ldots$

$\therefore$ $f(x) = f(x \pm nT)$, where n is a positive integer.

Thus, $f(x)$ repeats itself after periods of T.

For example, $\sin x$, $\cos x$, $\sec x$ and $\operatorname{cosec} x$ are periodic functions with period 2π while $\tan x$ and $\cot x$ are periodic functions with period π. The functions $\sin nx$ and $\cos nx$ are periodic with period $\frac{2\pi}{n}$.

The sum of a number of periodic functions is also periodic. If T_1 and T_2 are the periods of $f(x)$ and $g(x)$, then the period of $a\,f(x) + b\,g(x)$ is the least common multiple of T_1 and T_2.

For example, $\cos x$, $\cos 2x$, $\cos 3x$ are periodic functions with periods 2π, π and $\frac{2\pi}{3}$ respectively.

$\therefore$ $f(x) = \cos x + \frac{1}{2}\cos 2x + \frac{1}{3}\cos 3x$ is also periodic with period 2π, the L.C.M. of 2π, π and $\frac{2\pi}{3}$.

1.2. FOURIER SERIES

Periodic functions are of common occurrence in many physical and engineering problems ; for example, in conduction of heat and mechanical vibrations. It is useful to express these functions in a series of sines and cosines. Most of the single valued functions which occur in applied mathematics can be expressed in the form $\frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \ldots\ldots$ $+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots\ldots$
within a desired range of values of the variable. Such a series is known as *Fourier Series*. Thus, any function $f(x)$ defined in the interval $c_1 \le x \le c_2$ can be expressed in the Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where a_0, a_n, b_n $(n = 1, 2, 3, \ldots\ldots)$ are constants, called the Fourier co-efficients of $f(x)$.

Note. To determine a_0, a_n and b_n, we shall need the following results : (m and n are integers)

(i) $\int_c^{c+2\pi} \sin nx\, dx = -\left[\frac{\cos nx}{n}\right]_c^{c+2\pi} = 0$, $\int_c^{c+2\pi} \cos nx\, dx = \left[\frac{\sin nx}{n}\right]_c^{c+2\pi} = 0, n \neq 0$

(ii) $\int_c^{c+2\pi} \sin mx \cos nx\, dx = \frac{1}{2}\int_c^{c+2\pi} [\sin (m+n)\,x + \sin (m-n)\,x]\, dx$

$$= -\frac{1}{2}\left[\frac{\cos (m+n)\,x}{m+n} + \frac{\cos (m-n)x}{m-n}\right]_c^{c+2\pi} = 0, m \neq n$$

(iii) $\int_c^{c+2\pi} \cos mx \cos nx\, dx = \frac{1}{2}\int_c^{c+2\pi} [\cos (m+n)\,x + \cos (m-n)\,x]\, dx$

$$= \frac{1}{2}\left[\frac{\sin (m+n)x}{m+n} + \frac{\sin (m-n)x}{m-n}\right]_c^{c+2\pi} = 0, m \neq n$$

(iv) $\int_c^{c+2\pi} \sin mx \sin nx\, dx = \frac{1}{2}\int_c^{c+2\pi} [\cos (m-n)\,x - \cos (m+n)\,x]\, dx$

$$= \frac{1}{2}\left[\frac{\sin (m-n)x}{m-n} - \frac{\sin (m+n)x}{m+n}\right]_c^{c+2\pi} = 0, m \neq n$$

(v) $\int_c^{c+2\pi} \cos^2 nx\, dx = \left[\frac{x}{2} + \frac{\sin 2nx}{4n}\right]_c^{c+2\pi} = \pi$, $\int_c^{c+2\pi} \sin^2 nx\, dx = \left[\frac{x}{2} - \frac{\sin 2nx}{4n}\right]_c^{c+2\pi} = \pi, n \neq 0$

(vi) $\int_c^{c+2\pi} \sin nx \cos nx\, dx = \frac{1}{2}\int_c^{c+2\pi} \sin 2nx\, dx = -\frac{1}{2}\left[\frac{\cos 2nx}{2n}\right]_c^{c+2\pi} = 0, n \neq 0$

(vii) To integrate the product of two functions, one of which is a positive integral power of x, we apply the *generalised rule of integration by parts*. If dashes denote differentiation and suffixes denote integration w.r.t. x, the rule can be stated as follows :

$\int uv\, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\ldots$ where u and v are functions of x. *i.e.*, Integral of the product of two functions

= 1st function × integral of 2nd – go on differentiating 1st, integrating 2nd, signs alternately + ve and – ve.

[Simplification should be done only when the integration is over.]

For example, $\int x^3 e^{-2x}\, dx = x^3\left(\frac{e^{-2x}}{-2}\right) - 3x^2\left[\frac{e^{-2x}}{(-2)^2}\right] + 6x\left[\frac{e^{-2x}}{(-2)^3}\right] - 6\left[\frac{e^{-2x}}{(-2)^4}\right]$

$$= e^{-2x}\left[-\frac{1}{2}x^3 - \frac{3}{4}x^2 - \frac{3}{4}x - \frac{3}{8}\right] = -\frac{1}{8}e^{-2x}(4x^3 + 6x^2 + 6x + 3)$$

$\int x^2 \cos nx\, dx = x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)$

$$= \frac{x^2}{n}\sin nx + \frac{2x}{n^2}\cos nx - \frac{2}{n^3}\sin nx.$$

(viii) $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$

$\sin\left(n + \frac{1}{2}\right)\pi = (-1)^n$ and $\cos\left(n + \frac{1}{2}\right)\pi = 0$, where n is an integer.

(*ix*) **Even and Odd Functions**

A function $f(x)$ is said to be *even* if $f(-x) = f(x)$ *e.g.*, x^2, $\cos x$, $\sin^2 x$ are even functions.

The graph of an even function is symmetrical about the y-axis.

A function $f(x)$ is said to be *odd* if $f(-x) = -f(x)$ *e.g.*, x^3, $\sin x$, $\tan^3 x$ are odd functions.

The graph of an odd function is symmetrical about the origin.

The product of two even functions or two odd functions is an even function while the product of an even function and an odd function is an odd function.

Also, $\int_{-c}^{c} f(x)\,dx = 0$, when $f(x)$ is an odd function

and $\int_{-c}^{c} f(x)\,dx = 2\int_{0}^{c} f(x)\,dx$, when $f(x)$ is an even function.

1.3. EULER'S FORMULAE

The Fourier series for the function $f(x)$ in the interval $c < x < c + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad ...(1)$$

In finding the co-efficients a_0, a_n and b_n, we assume that the series on the right hand side of (1) is uniformly convergent for $c < x < c + 2\pi$ and it can be integrated term by term in the given interval.

To find a_0, Integrate both sides of (1) w.r.t. x, between the limits c to $c + 2\pi$.

$$\int_c^{c+2\pi} f(x)\,dx = \frac{a_0}{2}\int_c^{c+2\pi} dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)dx$$

$$= \frac{a_0}{2}(c + 2\pi - c) + 0 + 0 \qquad \text{[by formulae } (i) \text{ above]}$$

$$= a_0\pi$$

$$\therefore \quad \boxed{a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx}$$

To find a_n, multiply both sides of (1) by $\cos nx$ and integrate w.r.t. x, between the limits c to $c + 2\pi$.

$$\int_c^{c+2\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_c^{c+2\pi} \cos nx\,dx$$

$$+ \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)\cos nx\,dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)\cos nx\,dx$$

$$= 0 + a_n\pi + 0 \qquad \text{[by formulae } (i), (v) \text{ and } (vi)]$$

$$= a_n\pi$$

$$\therefore \quad \boxed{a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx}$$

To find b_n, multiply both sides of (1) by $\sin nx$ and integrate w.r.t. x between the limits c to $c+2\pi$.

$$\int_c^{c+2\pi} f(x)\sin nx\,dx = \frac{a_0}{2}\int_c^{c+2\pi}\sin nx\,dx + \int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} a_n\cos nx\right)\sin nx\,dx$$

$$+\int_c^{c+2\pi}\left(\sum_{n=1}^{\infty} b_n\sin nx\right)\sin nx\,dx$$

$$= 0 + 0 + b_n\pi \qquad \text{[by formulae } (i), (vi) \text{ and } (v)]$$

$$= b_n\pi$$

$$\therefore \qquad \boxed{b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx}$$

Hence $a_0 = \dfrac{1}{\pi}\displaystyle\int_c^{c+2\pi} f(x)\,dx$; $a_n = \dfrac{1}{\pi}\displaystyle\int_c^{c+2\pi} f(x)\cos nx\,dx$; and $b_n = \dfrac{1}{\pi}\displaystyle\int_c^{c+2\pi} f(x)\sin nx\,dx$ ...(I)

These values of a_0, a_n and b_n are called Euler's formulae.

Cor. 1. *If $c = 0$, the interval becomes $0 < x < 2\pi$ and the formulae I reduce to*

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx\,; \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx\,; \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

Cor. 2. *If $c = -\pi$, the interval becomes $-\pi < x < \pi$, and the formulae I reduce to*

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx\,; a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx\,; b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Cor. 3. *When $f(x)$ is an odd function* $\quad a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = 0$

Since $\cos nx$ is an even function, therefore, $f(x)\cos nx$ is an odd function.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$$

Since $\sin nx$ is an odd function, therefore, $f(x)\sin nx$ is an even function.

$$\therefore \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Hence, if a periodic function $f(x)$ is odd, its Fourier expansion contains only sine terms,

i.e.,

$$f(x) = \sum_{n=1}^{\infty} b_n\sin nx, \quad \text{where} \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

When $f(x)$ is an even function $\quad a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\,dx$

Since $\cos nx$ is an even function, therefore, $f(x)\cos nx$ is an even function.

$$\therefore \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

Since $\sin nx$ is an odd function, therefore, $f(x) \sin nx$ is an odd function.

$$\therefore \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0$$

Hence, if a periodic function $f(x)$ is even, its Fourier expansion contains only cosine terms,

i.e.,
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \text{ where } a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx \text{ and } a_0 = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Obtain the Fourier series to represent* $f(x) = \frac{1}{4}(\pi - x)^2, 0 < x < 2\pi$.

Hence obtain the following relations :

$$(i) \quad \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$(ii) \quad \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$$

$$(iii) \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

Sol. Let $\quad f(x) = \frac{1}{4}(\pi - x)^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

By Euler's formulae, we have

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi - x)^2 \, dx = \frac{1}{4\pi}\left[\frac{(\pi - x)^3}{-3}\right]_0^{2\pi} = -\frac{1}{12\pi}[-\pi^3 - \pi^3] = \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx \, dx = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi - x)^2 \cos nx \, dx$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2 \frac{\sin nx}{n} - \{-2(\pi - x)\}\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(0 + \frac{2\pi \cos 2n\pi}{n^2} + 0\right) - \left(0 - \frac{2\pi \cos 0}{n^2} + 0\right)\right] = \frac{1}{4\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right] = \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi - x)^2 \sin nx \, dx$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2\left(-\frac{\cos nx}{n}\right) - \{-2(\pi - x)\}\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(-\frac{\pi^2 \cos 2n\pi}{n} - 0 + \frac{2 \cos 2n\pi}{n^3}\right) - \left(-\frac{\pi^2}{n} - 0 + \frac{2 \cos 0}{n^3}\right)\right]$$

$$= \frac{1}{4\pi}\left[\left(-\frac{\pi^2}{n}+\frac{2}{n^3}\right)-\left(-\frac{\pi^2}{n}+\frac{2}{n^3}\right)\right]=0$$

$$f(x)=\frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}=\frac{\pi^2}{12}+\frac{\cos x}{1^2}+\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}+\ldots\ldots \qquad \ldots(1)$$

Deductions

(i) Putting $x = 0$ in equation (1), we get

$$f(0)=\frac{\pi^2}{12}+\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots\right)$$

$$\Rightarrow \quad \frac{\pi^2}{4}=\frac{\pi^2}{12}+\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots\right)$$

$$\Rightarrow \quad \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots=\frac{\pi^2}{6} \qquad \ldots(2)$$

(ii) Putting $x = \pi$ in equation (1), we get

$$f(\pi)=\frac{\pi^2}{12}+\left[\left(\frac{-1}{1^2}\right)+\frac{1}{2^2}+\left(\frac{-1}{3^2}\right)+\frac{1}{4^2}+\ldots\ldots\right]$$

$$\Rightarrow \quad 0=\frac{\pi^2}{12}-\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots$$

$$\Rightarrow \quad \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots\ldots=\frac{\pi^2}{12} \qquad \ldots(3)$$

(iii) Adding (2) and (3), we get

$$2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\right)=\frac{\pi^2}{6}+\frac{\pi^2}{12}$$

$$\Rightarrow \quad \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots=\frac{1}{2}\left(\frac{\pi^2}{4}\right)=\frac{\pi^2}{8}.$$

Example 2. *Expand $f(x) = x \sin x$, $0 < x < 2\pi$ as a Fourier series.*

Sol. Let $f(x)=x\sin x=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$

By Euler's formulae, we have $a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)\,dx=\frac{1}{\pi}\int_0^{2\pi}x\sin x\,dx$

$$=\frac{1}{\pi}\Big[x(-\cos x)-1.(-\sin x)\Big]_0^{2\pi}=\frac{1}{\pi}[-2\pi]=-2$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{2\pi}x\sin x\cos nx\,dx=\frac{1}{2\pi}\int_0^{2\pi}x(2\cos nx\sin x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[\sin(n+1)x - \sin(n-1)x]\,dx$$

$$= \frac{1}{2\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - 1.\left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left\{-\frac{\cos 2(n+1)\pi}{n+1} + \frac{\cos 2(n-1)\pi}{n-1}\right\}\right]$$

$$= -\frac{1}{n+1} + \frac{1}{n-1} = \frac{2}{n^2-1},\ n \neq 1$$

When $n = 1$, we have $a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x\,dx = \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x\,dx$

$$= \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - 1.\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi} = \frac{1}{2\pi}[-\pi] = -\frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2\sin nx \sin x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} x[\cos(n-1)x - \cos(n+1)x]\,dx$$

$$= \frac{1}{2\pi}\left[x\left\{\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right\} - 1.\left\{-\frac{\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{\cos 2(n+1)\pi}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right] = 0,\ n \neq 1$$

When $n = 1$, we have $b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin x\,dx = \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x)\,dx$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - 1.\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi(2\pi) - \frac{4\pi^2}{2} - \frac{1}{4} + \frac{1}{4}\right] = \frac{1}{2\pi}(2\pi^2) = \pi$$

$$\therefore \quad f(x) = \frac{a_0}{2} + a_1\cos x + b_1 \sin x + \sum_{n=2}^{\infty} a_n \cos nx + \sum_{n=2}^{\infty} b_n \sin nx$$

$$= -1 - \frac{1}{2}\cos x + \pi \sin x + \sum_{n=2}^{\infty} \frac{2}{n^2-1}\cos nx + 0$$

$$= -1 + \pi \sin x - \frac{1}{2}\cos x + \frac{2}{2^2-1}\cos 2x + \frac{2}{3^2-1}\cos 3x + \ldots\ldots$$

Example 3. *Find a Fourier series to represent $x-x^2$ from $x=-\pi$ to $x=\pi$. Hence show that*

$$\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots\ldots=\frac{\pi^2}{12}.$$

Sol. Let $x-x^2=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx+\sum_{n=1}^{\infty} b_n \sin nx$

By Euler's formulae, we have $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\,dx=\frac{1}{\pi}\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_{-\pi}^{\pi}$

$$=\frac{1}{\pi}\left[\left(\frac{\pi^2}{2}-\frac{\pi^3}{3}\right)-\left(\frac{\pi^2}{2}+\frac{\pi^3}{3}\right)\right]=-\frac{2\pi^2}{3}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos nx\,dx$$

$$=\frac{1}{\pi}\left[(x-x^2)\frac{\sin nx}{n}-(1-2x)\left(-\frac{\cos nx}{n^2}\right)+(-2)\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left[(1-2\pi)\frac{\cos n\pi}{n^2}-(1+2\pi)\frac{\cos n\pi}{n^2}\right]=\frac{1}{\pi}\left(-4\pi\,.\frac{\cos n\pi}{n^2}\right)$$

$$=-4\frac{(-1)^n}{n^2} \qquad [\because \cos n\pi=(-1)^n]$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx\,dx$$

$$=\frac{1}{\pi}\left[(x-x^2)\left(-\frac{\cos nx}{n}\right)-(1-2x)\left(-\frac{\sin nx}{n^2}\right)+(-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left[(\pi^2-\pi)\frac{\cos n\pi}{n}-2\frac{\cos n\pi}{n^3}+(-\pi-\pi^2)\frac{\cos n\pi}{n}+2\frac{\cos n\pi}{n^3}\right]$$

$$=\frac{1}{\pi}\left[-2\pi\,.\frac{\cos n\pi}{n}\right]=-2\frac{(-1)^n}{n}$$

$$\therefore \quad x-x^2=-\frac{\pi^2}{3}-4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

$$=-\frac{\pi^2}{3}-4\left[-\frac{\cos x}{1^2}+\frac{\cos 2x}{2^2}-\frac{\cos 3x}{3^2}+\ldots\ldots\right]$$

$$-2\left[\frac{-\sin x}{1}+\frac{\sin 2x}{2}-\frac{\sin 3x}{3}+\ldots\ldots\right]$$

$$=-\frac{\pi^2}{3}+4\left[\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\ldots\ldots\right]+2\left[\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\ldots\ldots\right]$$

Putting $x=0$, we get $0=-\frac{\pi^2}{3}+4\left(\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots\ldots\right)$

$$\Rightarrow \quad \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots..=\frac{\pi^2}{12}.$$

Example 4. *Obtain the Fourier series for the function $f(x) = x^2$, $-\pi < x < \pi$. Hence show that*

$$(i)\ \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

$$(ii)\ \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots\ldots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}=\frac{\pi^2}{12}$$

$$(iii)\ \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}.$$

Sol. Since $f(-x) = (-x)^2 = x^2 = f(x)$.

$\therefore$ $f(x)$ is an even function and hence $b_n = 0$

Let $$f(x) = x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Then $$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{3}\pi^2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi} = \frac{2}{\pi}\left[2\pi\,.\,\frac{\cos n\pi}{n^2}\right] = 4\,\frac{(-1)^n}{n^2}$$

$$\therefore \quad x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$$= \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \ldots\ldots\right) \quad \ldots(1)$$

Putting $x = \pi$ in (1), we get

$$\pi^2 = \frac{\pi^2}{3} - 4\left(-\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\right) \Rightarrow \frac{2\pi^2}{3} = 4\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots\right)$$

$$\therefore \quad \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots=\frac{\pi^2}{6} \qquad \text{[Result } (i)]$$

Putting $x = 0$ in (1), we get $0 = \dfrac{\pi^2}{3} - 4\left(\dfrac{1}{1^2}-\dfrac{1}{2^2}+\dfrac{1}{3^2}-\dfrac{1}{4^2}+\ldots\ldots\right)$

$$\therefore \quad \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots\ldots=\frac{\pi^2}{12} \qquad \text{[Result } (ii)]$$

Adding (*i*) and (*ii*), we get $2\left(\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\ldots\ldots\right)=\dfrac{\pi^2}{4}$

$$\therefore \quad \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots=\frac{\pi^2}{8}. \qquad \text{[Result } (iii)]$$

Example 5. *Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$.*

Sol. Let $$f(x) = e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Then $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx = \frac{1}{\pi}\Big[-e^{-x}\Big]_0^{2\pi} = \frac{1-e^{-2\pi}}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-\cos nx + n\sin nx)\right]_0^{2\pi}$$

$$\left[\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$$

$$= \frac{1-e^{-2\pi}}{\pi(1+n^2)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-\sin nx - n\cos nx)\right]_0^{2\pi} = \frac{1-e^{-2\pi}}{\pi}\cdot\frac{n}{1+n^2}$$

$$\left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$\therefore \quad e^{-x} = \frac{1-e^{-2\pi}}{2\pi} + \frac{1-e^{-2\pi}}{\pi}\sum_{n=1}^{\infty}\frac{\cos nx}{1+n^2} + \frac{1-e^{-2\pi}}{\pi}\sum_{n=1}^{\infty}\frac{n\sin nx}{1+n^2}$$

$$= \frac{1-e^{-2\pi}}{\pi}\left[\frac{1}{2} + \left(\frac{1}{2}\cos x + \frac{1}{5}\cos 2x + \frac{1}{10}\cos 3x + \ldots\ldots\right) + \left(\frac{1}{2}\sin x + \frac{2}{5}\sin 2x + \frac{3}{10}\sin 3x + \ldots\ldots\right)\right]$$

Example 6. *Find the Fourier series to represent e^{ax} in the interval $-\pi < x < \pi$. Hence derive series for $\frac{\pi}{\sinh \pi}$.*

Sol. Let $$f(x) = e^{ax} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Then $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\,dx = \frac{1}{\pi}\left[\frac{e^{ax}}{a}\right]_{-\pi}^{\pi} = \frac{1}{a\pi}(e^{a\pi} - e^{-a\pi}) = \frac{2\sinh a\pi}{a\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{ax}}{a^2+n^2}(a\cos nx + n\sin nx)\right]_{-\pi}^{\pi} = \frac{1}{\pi(a^2+n^2)}[ae^{a\pi}\cos n\pi - ae^{-a\pi}\cos n\pi]$$

$$= \frac{a\cos n\pi\,(e^{a\pi} - e^{-a\pi})}{\pi(a^2+n^2)} = \frac{2a(-1)^n\sinh a\pi}{\pi(a^2+n^2)}$$

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Similarly, $b_n = \dfrac{2n(-1)^n \sinh a\pi}{\pi(a^2+n^2)}$

$$\therefore \quad e^{ax} = \frac{\sinh a\pi}{a\pi} + \sum_{n=1}^{\infty} \frac{2a(-1)^n \sinh a\pi}{\pi(a^2+n^2)} \cos nx + \sum_{n=1}^{\infty} \frac{2n(-1)^n \sinh a\pi}{\pi(a^2+n^2)} \sin nx$$

$$= \frac{2\sinh a\pi}{\pi}\left[\frac{1}{2a} - a\left(\frac{\cos x}{a^2+1^2} - \frac{\cos 2x}{a^2+2^2} + \frac{\cos 3x}{a^2+3^2} - \ldots\ldots\right) - \left(\frac{\sin x}{a^2+1^2} - \frac{2\sin 2x}{a^2+2^2} + \frac{3\sin 3x}{a^2+3^2} - \ldots\ldots\right)\right]$$

Deduction. Putting $x = 0$ and $a = 1$, we get

$$1 = \frac{2\sinh \pi}{\pi}\left[\frac{1}{2} - \left(\frac{1}{1+1^2} - \frac{1}{1+2^2} + \frac{1}{1+3^2} - \frac{1}{1+4^2} + \ldots\ldots\right)\right]$$

$$\Rightarrow \quad \frac{\pi}{\sinh \pi} = 2\left(\frac{1}{1+2^2} - \frac{1}{1+3^2} + \frac{1}{1+4^2} - \ldots\ldots\right)$$

Example 7. *Express $f(x) = |x|, -\pi < x < \pi$, as Fourier series.* *(M.G.U. Dec. 2007)*

Sol. Since $f(-x) = |-x| = |x| = f(x)$

$\therefore$ $f(x)$ is an even function and hence $b_n = 0$

Let $\quad f(x) = |x| = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos nx$

Then $\quad a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\,dx = \dfrac{2}{\pi}\int_0^{\pi} |x|\,dx = \dfrac{2}{\pi}\int_0^{\pi} x\,dx = \dfrac{2}{\pi}\left[\dfrac{x^2}{2}\right]_0^{\pi} = \pi$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} |x|\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1.\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}[(-1)^n - 1] = \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{-4}{\pi n^2}, & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore \quad |x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right)$$

Note. Putting $x = 0$ in the above result, we get $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots\ldots = \dfrac{\pi^2}{8}$.

Example 8. *Expand the function $f(x) = x \sin x$ as a Fourier series in the interval $-\pi \le x \le \pi$.*

Deduce that $\dfrac{1}{1.3} - \dfrac{1}{3.5} + \dfrac{1}{5.7} - \dfrac{1}{7.9} + \ldots\ldots = \dfrac{\pi - 2}{4}$.

Sol. Since $x \sin x$ is an even function of x, $b_n = 0$

Let $\qquad f(x) = x \sin x = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Then $\qquad a_0 = \dfrac{2}{\pi}\int_0^{\pi} x \sin x \, dx = \dfrac{2}{\pi}\Big[x(-\cos x) - 1.(-\sin x)\Big]_0^{\pi} = \dfrac{2}{\pi}(-\pi \cos \pi) = 2$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx \, dx = \frac{1}{\pi}\int_0^{\pi} x(2 \cos nx \sin x)\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x[\sin (n+1)x - \sin (n-1)x]\, dx$$

$$= \frac{1}{\pi}\left[x\left\{-\frac{\cos (n+1)x}{n+1} + \frac{\cos (n-1)x}{n-1}\right\} - 1.\left\{-\frac{\sin (n+1)x}{(n+1)^2} + \frac{\sin (n-1)x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\left\{-\frac{\cos (n+1)\pi}{n+1} + \frac{\cos (n-1)\pi}{n-1}\right\}\right], n \neq 1$$

$$= \frac{\cos (n-1)\pi}{n-1} - \frac{\cos (n+1)\pi}{n+1}$$

When n is odd, $n \neq 1$, $n - 1$ and $n + 1$ are even

$\therefore \qquad a_n = \dfrac{1}{n-1} - \dfrac{1}{n+1} = \dfrac{2}{n^2-1}$

When n is even, $n - 1$ and $n + 1$ are odd

$\therefore \qquad a_n = \dfrac{-1}{n-1} + \dfrac{1}{n+1} = \dfrac{-2}{n^2-1}$

When $\qquad n = 1$, we have $a_1 = \dfrac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx = \dfrac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - 1.\left(-\frac{\sin 2x}{4}\right)\right]_0^{\pi} = \frac{1}{\pi}\left[-\frac{\pi \cos 2\pi}{2}\right] = -\frac{1}{2}$$

$\therefore \qquad x \sin x = 1 - \dfrac{1}{2}\cos x - 2\left(\dfrac{\cos 2x}{2^2-1} - \dfrac{\cos 3x}{3^2-1} + \dfrac{\cos 4x}{4^2-1} - \dfrac{\cos 5x}{5^2-1} + \ldots\ldots\right)$

Putting $\qquad x = \dfrac{\pi}{2}$, we get $\dfrac{\pi}{2} = 1 - 2\left(\dfrac{-1}{2^2-1} + \dfrac{1}{4^2-1} - \dfrac{1}{6^2-1} + \ldots\ldots\right)$

$\Rightarrow \qquad \dfrac{\pi}{2} - 1 = 2\left(\dfrac{1}{2^2-1} - \dfrac{1}{4^2-1} + \dfrac{1}{6^2-1} - \ldots\ldots\right) \quad \Rightarrow \quad \dfrac{\pi-2}{4} = \dfrac{1}{1.3} - \dfrac{1}{3.5} + \dfrac{1}{5.7} - \ldots\ldots$

Example 9. *Show that for* $-\pi < x < \pi$,

(M.G.U. May 2009)

$$\sin ax = \frac{2 \sin a\pi}{\pi}\left(\frac{\sin x}{1^2 - a^2} - \frac{2 \sin 2x}{2^2 - a^2} + \frac{3 \sin 3x}{3^2 - a^2} - \ldots\ldots\right).$$

Sol. Since $\sin ax$ is an odd function of x, $a_0 = 0$ and $a_n = 0$

Let $\quad \sin ax = \sum_{n=1}^{\infty} b_n \sin nx$

Then $\quad b_n = \frac{2}{\pi}\int_0^\pi \sin ax \sin nx \, dx = \frac{1}{\pi}\int_0^\pi [\cos(n-a)x - \cos(n+a)x]\, dx$

$$= \frac{1}{\pi}\left[\frac{\sin(n-a)x}{n-a} - \frac{\sin(n+a)x}{n+a}\right]_0^\pi = \frac{1}{\pi}\left[\frac{\sin(n-a)\pi}{n-a} - \frac{\sin(n+a)\pi}{n+a}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n(-\sin a\pi)}{n-a} - \frac{(-1)^n \sin a\pi}{n+a}\right] = -\frac{(-1)^n \sin a\pi}{\pi}\left[\frac{1}{n-a} + \frac{1}{n+a}\right]$$

$$= (-1)^{n+1} \cdot \frac{2n \sin a\pi}{\pi(n^2 - a^2)}$$

$$\therefore \quad \sin ax = \frac{2\sin a\pi}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 - a^2}\sin nx$$

$$= \frac{2\sin a\pi}{\pi}\left(\frac{\sin x}{1^2 - a^2} - \frac{2\sin 2x}{2^2 - a^2} + \frac{3\sin 3x}{3^2 - a^2} - \ldots\ldots\right).$$

Example 10. *Obtain Fourier series for the function f(x) given by*

$$f(x) = \begin{cases} 1 + \dfrac{2x}{\pi}, & -\pi \le x \le 0 \\ 1 - \dfrac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$$

Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$.

Sol. When $\quad -\pi \le x \le 0, \qquad 0 \le -x \le \pi$

$$\therefore \quad f(-x) = 1 - \frac{2(-x)}{\pi} = 1 + \frac{2x}{\pi} = f(x)$$

When $\quad 0 \le x \le \pi, \quad -\pi \le -x \le 0$

$$\therefore \quad f(-x) = 1 + \frac{2(-x)}{\pi} = 1 - \frac{2x}{\pi} = f(x)$$

$\Rightarrow$ $f(x)$ is an even function of x in $[-\pi, \pi]$. This is also clear from its graph which is symmetrical above the y-axis.

$\therefore \quad b_n = 0$

Let $\quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

then $\quad a_0 = \frac{2}{\pi}\int_0^\pi f(x)\, dx = \frac{2}{\pi}\int_0^\pi \left(1 - \frac{2x}{\pi}\right) dx = \frac{2}{\pi}\left[x - \frac{x^2}{\pi}\right]_0^\pi = 0$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx \, dx = \frac{2}{\pi}\int_0^\pi \left(1 - \frac{2x}{\pi}\right)\cos nx \, dx$$

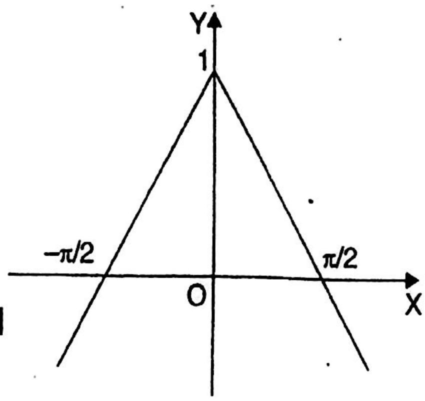
$$= \frac{2}{\pi}\left[\left(1 - \frac{2x}{\pi}\right)\frac{\sin nx}{n} - \left(-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right]_0^\pi$$

$\int_{-\pi}^{0} 1 + \frac{2x}{\pi}$

$\left(x + \frac{2x^2}{2\pi}\right)$

$-\pi + \pi$

$$= \frac{2}{\pi}\left[-\frac{2\cos n\pi}{\pi n^2} + \frac{2}{\pi n^2}\right] = \frac{4}{\pi^2 n^2}[1-(-1)^n]$$



$$\therefore \quad f(x) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}[1-(-1)^n]\frac{\cos nx}{n^2}$$

$$= \frac{4}{\pi^2}\left(\frac{2\cos x}{1^2} + \frac{2\cos 3x}{3^2} + \frac{2\cos 5x}{5^2} + \ldots\ldots\right)$$

$[\because \ 1-(-1)^n = 0$ when n is even$]$

$$= \frac{8}{\pi^2}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right)$$

Putting $x = 0$, we get $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$, since $f(0) = 1$.

TEST YOUR KNOWLEDGE

1. Expand in a Fourier series the function $f(x) = x$ in the interval $0 < x < 2\pi$.
2. Express $f(x) = \frac{1}{2}(\pi - x)$ in a Fourier series in the interval $0 < x < 2\pi$.
3. Find the Fourier series for the function $f(x) = x + x^2$ in the interval $-\pi < x < \pi$. Hence show that

   (i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{12}$. (ii) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{6}$

4. If $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in the interval $0 < x < 2\pi$, show that $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$

   Hence obtain the following relations :

   (i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{6}$ (ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{12}$

   (iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$.

5. Prove that for all values of x between $-\pi$ and π, $\frac{1}{2}x = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots\ldots$
6. Obtain the Fourier series to represent e^x in the interval $0 < x < 2\pi$.
7. Find the Fourier series to represent e^x in the interval $-\pi < x < \pi$.
8. Find the Fourier series to represent the function $f(x) = |\sin x|, -\pi < x < \pi$.
9. Expand $f(x) = |\cos x|$ as a Fourier series in the interval $-\pi < x < \pi$.
10. Prove that in the interval $-\pi < x < \pi$, $x \cos x = -\frac{1}{2}\sin x + 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\sin nx$.
11. Prove that for $-\pi < x < \pi$, $\frac{x(\pi^2 - x^2)}{12} = \frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \frac{\sin 4x}{4^3} + \ldots\ldots$
12. (a) Obtain a Fourier expansion for $\sqrt{1 - \cos x}$ in the interval $-\pi < x < \pi$.

    [**Hint.** For all integral values of n, $\cos\left(n + \frac{1}{2}\right)\pi = 0 = \cos\left(n - \frac{1}{2}\right)\pi$.]

(b) Obtain a Fourier series for $\sqrt{1-\cos x}$ in the interval $(0, 2\pi)$ and hence find the value of

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots\ldots$$

13. Express $f(x) = \cos wx, -\pi < x < \pi$, where w is a fraction, as a Fourier series. Hence prove that

$$\cot\theta = \frac{1}{\theta}+\frac{2\theta}{\theta^2-\pi^2}+\frac{2\theta}{\theta^2-4\pi^2}+\ldots\ldots$$

14. Find the Fourier series for $f(x)$ in the interval $(-\pi, \pi)$ when

$$f(x) = \pi + x, \qquad -\pi < x < 0$$
$$= \pi - x, \qquad 0 < x < \pi.$$

15. Obtain a Fourier series to represent e^{-ax} from $x = -\pi$ to $x = \pi$. Hence derive series for $\frac{\pi}{\sinh \pi}$.

16. Prove that in the range $-\pi < x < \pi$, $\cosh ax = \frac{2a}{\pi}\sinh a\pi\left[\frac{1}{2a^2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+a^2}\cos nx\right]$.

17. Given $f(x) = \begin{cases} -x+1 & \text{for } -\pi \le x \le 0 \\ x+1 & \text{for } 0 \le x \le \pi \end{cases}$

Is the function even or odd ? Find the Fourier series for $f(x)$ and deduce the value of

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots$$

18. Find the Fourier series for $f(x) = x + x^2; -\pi < x < \pi$.

19. Obtain the F.S. of the function $f(x) = x; -\pi < x < \pi$ and deduce that

$$\frac{\pi}{4} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$$

Answers

1. $f(x) = \pi - 2\sum_{n=1}^{\infty}\frac{\sin nx}{n}$ 2. $f(x) = \sum_{n=1}^{\infty}\frac{\sin nx}{n}$

3. $f(x) = \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\ldots\ldots\right)+2\left(\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\ldots\ldots\right)$

6. $e^x = \frac{e^{2\pi}-1}{2\pi}+\frac{e^{2\pi}-1}{\pi}\sum_{n=1}^{\infty}\left(\frac{\cos nx}{1+n^2}-\frac{n}{1+n^2}\sin nx\right)$

7. $e^x = \frac{2\sinh\pi}{\pi}$

$$\left[\frac{1}{2}-\left(\frac{1}{2}\cos x-\frac{1}{5}\cos 2x+\frac{1}{10}\cos 3x-\ldots\ldots\right)-\left(\frac{1}{2}\sin x-\frac{2}{5}\sin 2x+\frac{3}{10}\sin 3x-\ldots\ldots\right)\right]$$

8. $|\sin x| = \frac{2}{\pi}-\frac{4}{\pi}\left(\frac{\cos 2x}{3}+\frac{\cos 4x}{15}+\ldots\ldots+\frac{\cos 2nx}{4n^2-1}+\ldots\ldots\right)$

9. $|\cos x| = \frac{2}{\pi}+\frac{4}{\pi}\left(\frac{\cos 2x}{3}-\frac{\cos 4x}{15}+\ldots\ldots\right)$

12. (a) $\sqrt{1-\cos x} = \frac{2\sqrt{2}}{\pi}-\frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{\cos nx}{4n^2-1}$ (b) Same as in part (a) ; $\frac{1}{2}$

13. $\cos wx = \dfrac{2w \sin w\pi}{\pi}\left(\dfrac{1}{2w^2} + \dfrac{\cos x}{1^2 - w^2} - \dfrac{\cos 2x}{2^2 - w^2} + \dfrac{\cos 3x}{3^2 - w^2} - \ldots\ldots\right)$

14. $f(x) = \dfrac{\pi}{2} + \dfrac{4}{\pi}\left(\dfrac{\cos x}{1^2} + \dfrac{\cos 3x}{3^2} + \dfrac{\cos 5x}{5^2} + \ldots\ldots\right)$

15. $e^{-ax} = \dfrac{2 \sinh a\pi}{\pi}\left[\left(\dfrac{1}{2a} - \dfrac{a \cos x}{1^2 + a^2} + \dfrac{a \cos 2x}{2^2 + a^2} - \ldots\ldots\right) - \left(\dfrac{\sin x}{1^2 + a^2} - \dfrac{2 \sin 2x}{2^2 + a^2} + \dfrac{3 \sin 3x}{3^2 + a^2} - \ldots\ldots\right)\right]$

$\dfrac{\pi}{\sinh \pi} = 2\left[\dfrac{1}{2^2 + 1} - \dfrac{1}{3^2 + 1} + \dfrac{1}{4^2 + 1} - \ldots\ldots\right]$

17. Even $f(x) = \dfrac{\pi}{2} + 1 - \dfrac{4}{\pi}\left(\cos x + \dfrac{\cos 3x}{3^2} + \dfrac{\cos 5x}{5^2} + \ldots\ldots\right); \dfrac{\pi^2}{8}$

18. $\dfrac{\pi^2}{3} + \sum_{n=1}^{\infty} \dfrac{4(-1)^n}{n^2} \cos nx + \sum_{n=1}^{\infty} \dfrac{2}{n}(-1)^{n+1} \sin nx$ 19. $x = \sum_{n=1}^{\infty} \dfrac{2(-1)^{n+1}}{n} \sin nx.$

1.4. DIRICHLET'S CONDITIONS

The sufficient conditions for the uniform convergence of a Fourier series are called Dirichlet's conditions (after Dirichlet, a German mathematician). All the functions that normally arise in engineering problems satisfy these conditions and hence they can be expressed as a Fourier series.

Any function $f(x)$ can be expressed as a Fourier series $\dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ *where a_0, a_n, b_n are constants, provided*

(i) $f(x)$ is periodic, single valued and finite

(ii) $f(x)$ has a finite number of finite discontinuities in any one period.

(iii) $f(x)$ has a finite number of maxima and minima.

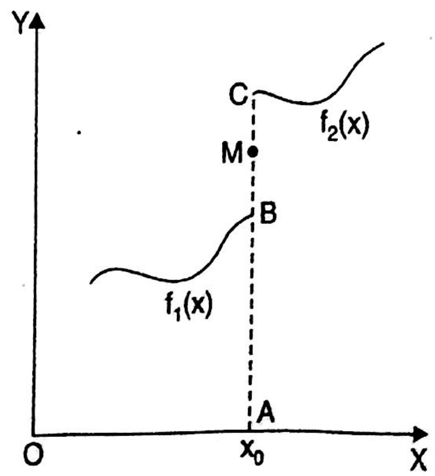
When these conditions are satisfied, the Fourier series converges to $f(x)$ at every point of continuity. At a point of discontinuity, the sum of the series is equal to the mean of the limits on the right and left

i.e., $$\tfrac{1}{2}[f(x+0) + f(x-0)]$$

where $f(x + 0)$ and $f(x - 0)$ denote the limit on the right and the limit on the left respectively.

1.5. FOURIER SERIES FOR DISCONTINUOUS FUNCTIONS

In 1.3, we derived Euler's formulae for a_0, a_n, b_n on the assumption that $f(x)$ is continous in $(c, c + 2\pi)$. However, if $f(x)$ has finitely many points of finite discontinuity, even then it can be expressed as a Fourier series. The integrals for a_0, a_n, b_n are to be evaluated by breaking up the range of integration.



Let $f(x)$ be defined by $f(x) = f_1(x);\ c < x < x_0$

$= f_2(x),\ x_0 < x < c + 2\pi$

where x_0 is the point of finite discontinuity in the interval $(c, c + 2\pi)$.

The values of a_0, a_n, b_n are given by

$$a_0 = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\,dx + \int_{x_0}^{c+2\pi} f_2(x)\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\cos nx\,dx + \int_{x_0}^{c+2\pi} f_2(x)\cos nx\,dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\sin nx\,dx + \int_{x_0}^{c+2\pi} f_2(x)\sin nx\,dx\right]$$

At $x = x_0$, there is a finite jump in the graph of the function. Both the limits $f(x_0 - 0)$ and $f(x_0 + 0)$ exist but are unequal. The sum of the Fourier series $= \frac{1}{2}[f(x_0 - 0) + f(x_0 + 0)] = \frac{1}{2}$ (AB + AC) = AM, where M is the mid-point of BC.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the Fourier series to represent the function f(x) given by*

$$f(x) = x \quad \text{for } 0 \le x \le \pi$$

and

$$= 2\pi - x \quad \text{for } \pi \le x \le 2\pi.$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$. (M.G.U. June 2006)

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

Then $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi}(2\pi - x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\left|\frac{x^2}{2}\right|_0^{\pi} + \left|2\pi x - \frac{x^2}{2}\right|_{\pi}^{2\pi}\right] = \frac{1}{\pi}\left[\frac{\pi^2}{2} + (4\pi^2 - 2\pi^2) - \left(2\pi^2 - \frac{\pi^2}{2}\right)\right] = \pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left|x\,.\frac{\sin nx}{n} - 1.\left(-\frac{\cos nx}{n^2}\right)\right|_0^{\pi} + \left|(2\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right) + \left(-\frac{\cos 2n\pi}{n^2} + \frac{\cos n\pi}{n^2}\right)\right]$$

$$= \frac{1}{\pi n^2}[(-1)^n - 1 - 1 + (-1)^n] = \frac{2}{\pi n^2}[(-1)^n - 1] = \begin{cases} -\frac{4}{\pi n^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\sin nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left| x\left(-\frac{\cos nx}{n}\right) - 1.\left(-\frac{\sin nx}{n^2}\right)\right|_0^\pi + \left|(2\pi - x)\times\left(-\frac{\cos nx}{n}\right) - 1.\left(-\frac{\sin nx}{n^2}\right)\right|_\pi^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi\cos n\pi}{n} + \frac{\pi\cos n\pi}{n}\right] = 0$$

$$\therefore \quad f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right)$$

Putting $x = 0$, we get $0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right) \Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$

Example 2. *If* $f(x) = 0, \quad -\pi \le x \le 0$
$\quad = \sin x, \quad 0 \le x \le \pi,$

prove that $f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2 - 1}.$

Hence show that

(i) $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots\ldots = \frac{1}{2}$ (ii) $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\ldots = \frac{\pi - 2}{4}$.

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

Then $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^\pi \sin x\,dx\right] = \frac{2}{\pi}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^\pi \sin x\cos nx\,dx\right]$$

$$= \frac{1}{2\pi}\int_0^\pi 2\cos nx\sin x\,dx = \frac{1}{2\pi}\int_0^\pi [\sin(n+1)x - \sin(n-1)x]\,dx$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^\pi, \; n \ne 1$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \begin{cases} \frac{1}{2\pi}\left(-\frac{1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right), & \text{when } n \text{ is odd} \\ \frac{1}{2\pi}\left(\frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right), & \text{when } n \text{ is even} \end{cases}$$

$$= \begin{cases} 0, \text{ when } n \text{ is odd, } i.e., n = 3, 5, 7, ... \\ -\dfrac{2}{\pi(n^2-1)}, \text{ when } n \text{ is even} \end{cases}$$

When $n = 1$, we have $a_1 = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \sin x \cos x \, dx = \dfrac{1}{2\pi}\int_0^{\pi} \sin 2x \, dx = \dfrac{1}{2\pi}\left[-\dfrac{\cos 2x}{2}\right]_0^{\pi} = 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \, dx + \int_0^{\pi} \sin x \sin nx \, dx\right]$$

$$= \frac{1}{2\pi}\int_0^{\pi} 2 \sin nx \sin x \, dx = \frac{1}{2\pi}\int_0^{\pi} [\cos (n-1) x - \cos (n+1) x] \, dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin (n-1)x}{n-1} - \frac{\sin (n+1)x}{n+1}\right]_0^{\pi} = 0, n \neq 1$$

When $n = 1$, we have

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \sin x \, dx = \frac{1}{2\pi}\int_0^{\pi} (1 - \cos 2x) \, dx = \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2}$$

$$\therefore \quad f(x) = \frac{1}{\pi} - \frac{2}{\pi}\left[\frac{\cos 2x}{2^2 - 1} + \frac{\cos 4x}{4^2 - 1} + \frac{\cos 6x}{6^2 - 1} + ...\right] + \frac{1}{2}\sin x$$

$$= \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{(2n)^2 - 1} \quad ...(1)$$

Putting $x = 0$ in (1), we have $0 = \dfrac{1}{\pi} - \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{4n^2 - 1}$

$$\Rightarrow \quad \frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + ...$$

Putting $x = \dfrac{\pi}{2}$ in (1), we have $1 = \dfrac{1}{\pi} + \dfrac{1}{2} - \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty} \dfrac{\cos n\pi}{4n^2 - 1}$

$$\Rightarrow \quad \frac{1}{2} - \frac{1}{\pi} = -\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}$$

$$\Rightarrow \quad \frac{\pi - 2}{4} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)(2n+1)} = -\left(-\frac{1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + ...\right)$$

$$\Rightarrow \quad \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - ...... = \frac{\pi - 2}{4}.$$

TEST YOUR KNOWLEDGE

1. Find the Fourier series to represent the function

$$f(x) = -k \quad \text{when} \quad -\pi < x < 0$$
$$= k \quad \text{when} \quad 0 < x < \pi$$

Also deduce that $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + ...$

2. Develop $f(x)$ in a Fourier series in the interval $(-\pi, \pi)$ if $f(x) = 0$ when $-\pi < x < 0$
$= 1$ when $0 < x < \pi$.

3. Find the Fourier series expansion for $f(x)$, if $f(x) = -\pi, -\pi < x < 0$
$= x, \quad 0 < x < \pi$

Deduce that $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots\ldots = \dfrac{\pi^2}{8}$.

[**Hint.** For the deduction, put $x = 0$ in the expansion of $f(x)$.

$$f(0-0) = -\pi \text{ and } f(0+0) = 0 \quad \therefore \quad f(0) = \frac{1}{2}[f(0-0) + f(0+0)] = -\frac{\pi}{2}\Bigg]$$

4. Find the Fourier expansion of the function defined in one period by the relations

$$f(x) = 1 \text{ for } 0 < x < \pi$$
$$= 2 \text{ for } \pi < x < 2\pi$$

and deduce that $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \ldots\ldots$

5. Find the Fourier series of the following function :

$$f(x) = x^2, \qquad 0 \le x \le \pi$$
$$= -x^2, \qquad -\pi \le x \le 0.$$

6. An alternating current after passing through a rectifier has the form

$$i = I_0 \sin x \qquad \text{for} \qquad 0 \le x \le \pi$$
$$= 0 \qquad \text{for} \qquad \pi \le x \le 2\pi$$

where I_0 is the maximum current and the period is 2π. Express i as a Fourier series.

7. Obtain Fourier series for the function

$$f(x) = \begin{cases} x & \text{for } \; -\pi < x < 0 \\ -x & \text{for } \; 0 < x < \pi \end{cases}$$

and hence show that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$$

8. Find the Fourier series for the function

$$f(x) = \begin{cases} -1 & \text{for } -\pi < x < -\dfrac{\pi}{2} \\ 0 & \text{for } -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ 1 & \text{for } \dfrac{\pi}{2} < x < \pi \end{cases}$$

Answers

1. $f(x) = \dfrac{4k}{\pi}\left(\sin x + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5} + \ldots\ldots\right)$

2. $f(x) = \dfrac{1}{2} + \dfrac{2}{\pi}\left(\sin x + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5} + \ldots\ldots\right)$

3. $f(x) = -\dfrac{\pi}{4} - \dfrac{2}{\pi}\left(\cos x + \dfrac{\cos 3x}{3^2} + \dfrac{\cos 5x}{5^2} + \ldots\ldots\right) + \left(3 \sin x - \dfrac{\sin 2x}{2} + \sin 3x - \dfrac{\sin 4x}{4} + \ldots\ldots\right)$

4. $f(x) = \dfrac{3}{2} - \dfrac{2}{\pi}\left(\sin x + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5} + \ldots\ldots\right)$

5. $f(x) = 2\left(\pi - \frac{4}{\pi}\right)\sin x - \pi \sin 2x + \frac{2}{3}\left(\pi - \frac{4}{9\pi}\right)\sin 3x - \frac{\pi}{2}\sin 4x + \ldots\ldots$

6. $i = \frac{I_0}{2} + \frac{I_0}{2}\sin x - \frac{2I_0}{\pi}\left(\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \ldots\ldots\right)$

7. $f(x) = -\frac{\pi}{2} + \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right)$

8. $f(x) = \frac{2}{\pi}\left(\sin x - \sin 2x + \frac{\sin 3x}{3} - \ldots\ldots\right)$

1.6. CHANGE OF INTERVAL

In many engineering problems, it is desired to expand a function in a Fourier series over an interval of length $2l$ and not 2π. In order to apply foregoing theory, this interval must be transformed into an interval of length 2π. This can bé achieved by a transformation of the variable.

Consider a periodic function $f(x)$ defined in the interval $c < x < c + 2l$. To change the interval into one of length 2π, we put

$$\frac{x}{l} = \frac{z}{\pi} \quad \text{or} \quad z = \frac{\pi x}{l} \quad \text{so that when } x = c,\ z = \frac{\pi c}{l} = d \text{ (say)}$$

and when $x = c + 2l$, $\quad z = \frac{\pi(c+2l)}{l} = \frac{\pi c}{l} + 2\pi = d + 2\pi.$

Thus the function $f(x)$ of period $2l$ in $(c, c + 2l)$ is transformed to the function $f\left(\frac{lz}{\pi}\right) = F(z)$, say, of period 2π in $(d, d + 2\pi)$ and the latter function can be expressed as the Fourier series

$$F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz + \sum_{n=1}^{\infty} b_n \sin nz \qquad \ldots(1)$$

where $a_0 = \frac{1}{\pi}\int_d^{d+2\pi} F(z)\,dz$; $a_n = \frac{1}{\pi}\int_d^{d+2\pi} F(z)\cos nz\,dz$; and $b_n = \frac{1}{\pi}\int_d^{d+2\pi} F(z)\sin nz\,dz$ $\quad \ldots(2)$

Now making the inverse substitution $z = \frac{\pi x}{l}$, $dz = \frac{\pi}{l}\,dx$

When $z = d$, $x = c$ and when $z = d + 2\pi$, $x = c + 2l$.

The expression (1) becomes $F(z) = F\left(\frac{\pi x}{l}\right) = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ and the co-efficients a_0, a_n, b_n from (2) reduce to

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx\ ; \quad a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx\ ; \quad \text{and} \quad b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Hence the Fourier series for f(x) in the interval c < x < c + 2l is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where $a_0 = \int_c^{c+2l} f(x)\,dx,\quad a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$ and $b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx.$

Cor. 1. If we put c = 0, the interval becomes $0 < x < 2l$ and the above results reduce to

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx\,;\quad a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx\,;\quad \text{and}\quad a_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

Cor. 2. If we put c = – l, the interval becomes $-l < x < l$ and the above results reduce to

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx,\quad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx\quad\text{and}\quad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

If f(x) is an even function, we have

$$a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx,\quad a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx\quad\text{and}\quad b_n = 0$$

If f(x) is an odd function, we have $a_0 = 0,\ a_n = 0$

and
$$b_n = \frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}\,dx.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Find Fourier expansion for the function $f(x) = x - x^2,\ -1 < x < 1$.*

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$

Then
$$a_0 = \int_{-1}^{1}(x - x^2)\,dx = \int_{-1}^{1} x\,dx - \int_{-1}^{1} x^2\,dx = 0 - 2\int_0^1 x^2\,dx = -2\left[\frac{x^3}{3}\right]_0^1 = -\frac{2}{3}$$

$$a_n = \int_{-1}^{1}(x - x^2)\cos n\pi x\,dx = \int_{-1}^{1} x\cos n\pi x\,dx - \int_{-1}^{1} x^2\cos n\pi x\,dx$$

$$= 0 - 2\int_0^1 x^2\cos n\pi x\,dx = -2\left[x^2\cdot\frac{\sin n\pi x}{n\pi} - 2x\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + 2\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= -2\left[\frac{2\cos n\pi}{n^2\pi^2}\right] = \frac{-4(-1)^n}{n^2\pi^2} = \frac{4(-1)^{n+1}}{n^2\pi^2}$$

$$b_n = \int_{-1}^{1}(x - x^2)\sin n\pi x\,dx = \int_{-1}^{1} x\sin n\pi x\,dx - \int_{-1}^{1} x^2\sin n\pi x\,dx$$

$$= 2\int_0^1 x\sin n\pi x\,dx - 0 = 2\left[x\left(-\frac{\cos n\pi x}{n\pi}\right) - 1.\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$= 2\left[-\frac{\cos n\pi}{n\pi}\right] = \frac{-2(-1)^n}{n\pi} = \frac{2(-1)^{n+1}}{n\pi}$$

$$\therefore \quad x - x^2 = -\frac{1}{3} + \frac{4}{\pi^2}\left(\frac{\cos \pi x}{1^2} - \frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} - \ldots\ldots\right)$$

$$+ \frac{2}{\pi}\left(\frac{\sin \pi x}{1} - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3^2} - \ldots\ldots\right)$$

Example 2. *Find the Fourier series to represent $f(x) = x^2 - 2$, when $-2 \le x \le 2$.*

Sol. Since $f(x)$ is an even function, $b_n = 0$.

Let $\quad f(x) = x^2 - 2 = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2}$

Then $\quad a_0 = \dfrac{2}{2}\displaystyle\int_0^2 (x^2 - 2)\, dx = \left[\frac{x^3}{3} - 2x\right]_0^2 = \frac{8}{3} - 4 = -\frac{4}{3}$

$$a_n = \frac{2}{2}\int_0^2 (x^2 - 2) \cos \frac{n\pi x}{2}\, dx$$

$$= \left[(x^2 - 2) . \frac{\sin \frac{n\pi x}{2}}{\frac{n\pi}{2}} - 2x\left(-\frac{\cos \frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right) + 2\left(-\frac{\sin \frac{n\pi x}{2}}{\frac{n^3\pi^3}{8}}\right)\right]_0^2$$

$$= \frac{16 \cos n\pi}{n^2\pi^2} = \frac{16(-1)^n}{n^2\pi^2}$$

$$\therefore \quad x^2 - 2 = -\frac{2}{3} - \frac{16}{\pi^2}\left(\cos \frac{\pi x}{2} - \frac{1}{4}\cos \pi x + \frac{1}{9}\cos \frac{3\pi x}{2} - \ldots\ldots\right).$$

Example 3. *Expand $f(x) = e^{-x}$ as a Fourier series in the interval $(-l, l)$.*

Sol. Let $f(x) = e^{-x} = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

Then $\quad a_0 = \dfrac{1}{l}\displaystyle\int_{-l}^{l} e^{-x}\, dx = \frac{1}{l}\left[-e^{-x}\right]_{-l}^{l} = \frac{1}{l}(e^l - e^{-l}) = \frac{2 \sinh l}{l}$

$$a_n = \frac{1}{l}\int_{-l}^{l} e^{-x} \cos \frac{n\pi x}{l}\, dx = \frac{1}{l}\left[\frac{e^{-x}}{1 + \left(\frac{n\pi}{l}\right)^2}\left(-\cos \frac{n\pi x}{l} + \frac{n\pi}{l}\sin \frac{n\pi x}{l}\right)\right]_{-l}^{l}$$

$$\left[\because \int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx)\right]$$

$$= \frac{l}{l^2 + (n\pi)^2}[-e^{-l} \cos n\pi + e^l \cos n\pi] = -\frac{2l \cos n\pi}{l^2 + (n\pi)^2}\left(\frac{e^l - e^{-l}}{2}\right) = \frac{2l(-1)^n \sinh l}{l^2 + (n\pi)^2}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} e^{-x} \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{1}{l}\left[\frac{e^{-x}}{1+\left(\frac{n\pi}{l}\right)^2}\left(-\sin\frac{n\pi x}{l} - \frac{n\pi}{l}\cos\frac{n\pi x}{l}\right)\right]_{-l}^{l}$$

$$\left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$= -\frac{1}{l^2+(n\pi)^2}\left[\frac{n\pi}{l}(e^{-l}-e^{l})\cos n\pi\right] = \frac{2n\pi\cos n\pi}{l^2+(n\pi)^2}\left(\frac{e^{l}-e^{-l}}{2}\right) = \frac{2n\pi(-1)^n\sinh l}{l^2+(n\pi)^2}$$

$$\therefore \quad e^{-x} = \sinh l\left[\frac{1}{l} - 2l\left(\frac{1}{l^2+\pi^2}\cos\frac{\pi x}{l} - \frac{1}{l^2+2^2\pi^2}\cos\frac{2\pi x}{l} + \frac{1}{l^2+3^2\pi^2}\cos\frac{3\pi x}{l} - \ldots\ldots\right)\right.$$

$$\left. - 2\pi\left(\frac{1}{l^2+\pi^2}\sin\frac{\pi x}{l} - \frac{2}{l^2+2^2\pi^2}\sin\frac{2\pi x}{l} + \frac{3}{l^2+3^2\pi^2}\sin\frac{3\pi x}{l} - \ldots\ldots\right)\right].$$

Example 4. *Obtain Fourier series for the function* $f(x) = \pi x, \quad 0 \le x \le 1$
$= \pi(2-x), \quad 1 \le x \le 2.$

(M.G.U. June 2006)

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$

Then $a_0 = \int_0^2 f(x)\,dx = \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx = \pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[2x - \frac{x^2}{2}\right]_1^2$

$$= \pi\left(\frac{1}{2}\right) + \pi\left[(4-2) - \left(2-\frac{1}{2}\right)\right] = \pi$$

$$a_n = \int_0^2 f(x)\cos n\pi x\,dx = \int_0^1 \pi x\cos n\pi x\,dx + \int_1^2 \pi(2-x)\cos n\pi x\,dx$$

$$= \left[\pi x\cdot\frac{\sin n\pi x}{n\pi} - \pi\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[\pi(2-x)\cdot\frac{\sin n\pi x}{n\pi} - (-\pi)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= \left[\frac{\cos n\pi}{n^2\pi} - \frac{1}{n^2\pi}\right] + \left[-\frac{\cos 2n\pi}{n^2\pi} + \frac{\cos n\pi}{n^2\pi}\right] = \frac{2}{n^2\pi}(\cos n\pi - 1) = \frac{2}{n^2\pi}[(-1)^n - 1]$$

$= 0$ or $-\frac{4}{n^2\pi}$ according as n is even or odd.

$$b_n = \int_0^2 f(x)\sin n\pi x\,dx = \int_0^1 \pi x\sin n\pi x\,dx + \int_1^2 \pi(2-x)\sin n\pi x\,dx$$

$$= \left[\pi x\left(-\frac{\cos n\pi x}{n\pi}\right) - \pi\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[\pi(2-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-\pi)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= \left[-\frac{\cos n\pi}{n}\right] + \left[\frac{\cos n\pi}{n}\right] = 0$$

$$\therefore \quad f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots\ldots\right).$$

Note. Putting $x = 0$, we have $f(0) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right)$

or $$0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right)$$

$$\therefore \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}.$$

TEST YOUR KNOWLEDGE

1. Find a Fourier series for $f(t) = 1 - t^2$ when $-1 \le t \le 1$.
2. Expand $f(x)$ in Fourier series in the interval $(-2, 2)$ when $f(x) = 0, \quad -2 < x < 0$
   $= 1, \quad 0 < x < 2.$
3. Develop $f(x)$ in a Fourier series in the interval $(0, 2)$ if $f(x) = x, \quad 0 < x < 1$
   $= 0, \quad 1 < x < 2.$
4. Find the Fourier expansion for $f(x) = \pi x$ from $x = -c$ to $x = c$.
5. Find the Fourier expansion for the function $f(x) = x - x^3$ in the interval $-1 < x < 1$.
6. Find the Fourier series for the function given by $f(t) = t, \quad 0 < t < 1$
   $= 1 - t, \quad 1 < t < 2.$
7. Find a Fourier series to represent x^2 in the interval $(-l, l)$. *(M.G.U., May 2009)*
8. Expand $f(x) = e^{-x}$ as a Fourier series in the interval $(-2, 2)$.
9. Expand : $$f(x) = \begin{cases} \frac{1}{4} - x, & 0 < x < \frac{1}{2} \\ x - \frac{3}{4}, & \frac{1}{2} < x < 1 \\ 1, & 1 < x < \frac{3}{2} \\ x - 1, & \frac{3}{2} < x < 2 \end{cases}$$
   as a Fourier series.
10. Find the Fourier series for the function $f(x) = \begin{cases} 0 & \text{when} \quad -2 < x < -1 \\ k & \text{when} \quad -1 < x < 1 \\ 0 & \text{when} \quad 1 < x < 2. \end{cases}$
11. A sinusoidal voltage E sin ωt is passed through a half-wave rectifier which clips the negative portion of the wave.

    Expand the resulting periodic function $u(t) = \begin{cases} 0 & \text{when } -\frac{T}{2} < t < 0 \\ E \sin \omega t & \text{when } 0 < t < \frac{T}{2} \end{cases}$

    and $T = \frac{2\pi}{\omega}$, in a Fourier series.

Answers

1. $1 - t^2 = \frac{2}{3} + \frac{4}{\pi^2}\left(\cos \pi t - \frac{\cos 2\pi t}{2^2} + \frac{\cos 3\pi t}{3^2} - \ldots\ldots\right)$

2. $f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\sin \frac{\pi x}{2} + \frac{1}{3}\sin \frac{3\pi x}{2} + \frac{1}{5}\sin \frac{5\pi x}{2} + \ldots\ldots\right)$

3. $f(x) = \frac{1}{4} - \frac{2}{\pi^2}\left(\cos \pi x + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots\ldots\right) + \frac{1}{\pi}\left(\sin \pi x + \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} + \ldots\ldots\right)$

4. $f(x) = 2c\left[\sin\left(\frac{\pi x}{c}\right) - \frac{1}{2}\sin\left(\frac{2\pi x}{c}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{c}\right) - \ldots\ldots\right]$

5. $f(x) = \frac{12}{\pi^3}\left(\sin \pi x - \frac{\sin 2\pi x}{2^3} + \frac{\sin 3\pi x}{3^3} - \ldots\ldots\right)$

6. $f(t) = -\frac{4}{\pi^2}\left(\cos \pi t + \frac{\cos 3\pi t}{3^2} + \frac{\cos 5\pi t}{5^3} + \ldots\ldots\right) + \frac{2}{\pi}\left(\sin \pi t + \frac{\sin 3\pi t}{3} + \ldots\ldots\right)$

7. $x^2 = \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left(\frac{\cos \pi x/l}{1^2} - \frac{\cos 2\pi x/l}{2^2} + \frac{\cos 3\pi x/l}{3^2} - \frac{\cos 4\pi x/l}{4^2} + \ldots\ldots\right)$

8. $$e^{-x} = \sinh 2\left[\frac{1}{2} - 4\left(\frac{1}{2^2+\pi^2}\cos\frac{\pi x}{2} - \frac{1}{2^2+2^2\pi^2}\cos \pi x + \frac{1}{2^2+3^2\pi^2}\cos\frac{3\pi x}{2} - \ldots\ldots\right)\right.$$
$$\left. - 2\pi\left(\frac{1}{2^2+\pi^2}\sin\frac{\pi x}{2} - \frac{2}{2^2+2^2\pi^2}\sin \pi x + \frac{3}{2^2+3^2\pi^2}\sin\frac{3\pi x}{2} - \ldots\ldots\right)\right]$$

9. $f(x) = \frac{7}{16} + \frac{1}{\pi}\left(\frac{1}{\pi} + \frac{1}{4}\right)\cos \pi x - \frac{3}{\pi}\left(\frac{1}{\pi} + \frac{1}{2}\right)\sin \pi x + \ldots\ldots$

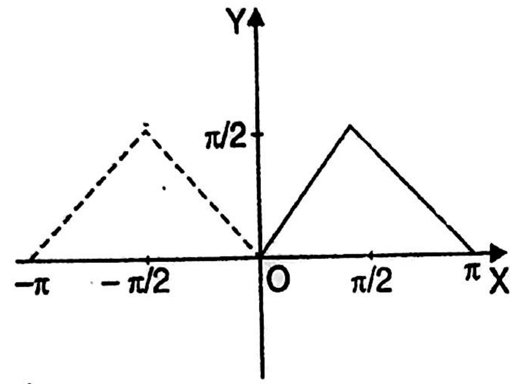
10. $f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi t}{2} - \frac{1}{3}\cos\frac{3\pi t}{2} + \frac{1}{5}\cos\frac{5\pi t}{2} - \ldots\ldots\right)$

11. $u(t) = \frac{E}{\pi} + \frac{E}{2}\sin \omega t - \frac{2E}{\pi}\left(\frac{\cos 2\omega t}{1.3} + \frac{\cos 4\omega t}{3.5} + \frac{\cos 6\omega t}{5.7} + \ldots\ldots\right)$

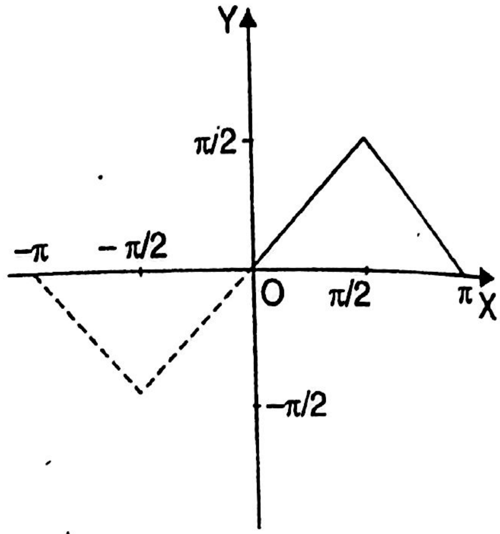
For example, consider the function

$$f(x) = x, \qquad 0 < x < \frac{\pi}{2}$$

$$= \pi - x \qquad \frac{\pi}{2} < x < \pi$$



(Reflection in the y-axis)



(Reflection in the origin)

Hence a function $f(x)$ defined over the interval $0 < x < l$ is capable of two distinct half-range series.

The half-range cosine series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$

where $a_0 = \frac{2}{l} \int_0^l f(x)\, dx$; $a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l}\, dx$.

The half-range sine series is $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$, where $b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$.

Cor. If the range is $0 < x < \pi$, then

(*i*) The half-range cosine series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

where $a_0 = \frac{2}{\pi} \int_0^\pi f(x)\, dx$; $a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx\, dx$.

(*ii*) The half-range sine series is $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$, where $b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx\, dx$

ILLUSTRATIVE EXAMPLES

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(-\frac{\cos nx}{n}\right) - (\pi - 2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[-\frac{2\cos n\pi}{n^3} + \frac{2}{n^3}\right] = \frac{4}{\pi n^3}[1 - (-1)^n]$$

$$= 0 \quad \text{or} \quad \frac{8}{\pi n^3} \text{ according as } n \text{ is even or odd.}$$

$$\therefore \qquad \pi x - x^2 = \frac{8}{\pi}\left(\sin x + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \ldots\ldots\right).$$

Example 2. *If* $f(x) = x, \quad 0 < x < \frac{\pi}{2}$

$$= \pi - x, \quad \frac{\pi}{2} < x < \pi$$

show that (i) $f(x) = \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \ldots\ldots\right]$

(ii) $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \ldots\ldots\right].$

Sol. (*i*) **For the half-range sine series.**

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

Then $b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x\sin nx\,dx + \int_{\pi/2}^{\pi}(\pi - x)\sin nx\,dx\right]$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1.\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2} + \frac{2}{\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right] + \frac{2}{\pi}\left[\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\sin\frac{n\pi}{2}\right] = \frac{4}{\pi n^2}\sin\frac{n\pi}{2}$$

When n is even, $b_n = 0$.

$$\therefore \quad f(x) = \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \ldots\ldots\right]$$

(*ii*) **For the half-range cosine series.**

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Then $a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi}(\pi - x)\,dx\right]$

$$= \frac{2}{\pi}\left[\left|\frac{x^2}{2}\right|_0^{\pi/2} + \left|\pi x - \frac{x^2}{2}\right|_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{8} + \left(\pi^2 - \frac{\pi^2}{2}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right] = \frac{2}{\pi}\left[\frac{\pi^2}{4}\right] = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi}(\pi - x)\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[x\cdot\frac{\sin nx}{n} - 1.\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi/2} + \frac{2}{\pi}\left[(\pi - x)\cdot\frac{\sin nx}{n} - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2}\right] + \frac{2}{\pi}\left[-\frac{\cos n\pi}{n^2} - \frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\cos\frac{n\pi}{2} - \frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}\left[2\cos\frac{n\pi}{2} - \cos n\pi - 1\right]$$

$\therefore$ $$a_1 = 0,\ a_2 = \frac{2}{\pi.2^2}(2\cos\pi - \cos 2\pi - 1) = \frac{-2}{\pi.1^2},$$

$$a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ a_6 = \frac{2}{\pi.6^2}(2\cos 3\pi - \cos 6\pi - 1) = \frac{-2}{\pi.3^2},$$

$$a_7 = 0,\ a_8 = 0,\ a_9 = 0,\ a_{10} = \frac{2}{\pi.10^2}(2\cos 5\pi - \cos 10\pi - 1) = \frac{-2}{\pi.5^2},\ \ldots\ldots$$

Hence $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \ldots\ldots\right]$.

Example 3. *Find a series of cosines of multiples of x which will represent x sin x in the interval* $(0, \pi)$ *and show that* $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\ldots = \frac{\pi - 2}{4}$.

Sol. Since $x\sin x$ is an even function of x, $b_n = 0$

Let $$x\sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx$$

Then $$a_0 = \frac{2}{\pi}\int_0^{\pi} x\sin x\,dx = \frac{2}{\pi}\Big[x(-\cos x) - 1.(-\sin x)\Big]_0^{\pi} = \frac{2}{\pi}[-\pi\cos\pi] = 2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} x\,(2\cos nx\sin x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\,[\sin(n+1)x - \sin(n-1)x]\,dx$$

$$= \frac{1}{\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - 1.\left\{-\frac{\sin(n+1)x}{(n+1)^2} - \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\pi\cos(n+1)\pi}{n+1} + \frac{\pi\cos(n-1)\pi}{n-1}\right],\quad \text{when } n \neq 1$$

$$= -\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} = (-1)^{n-1}\left[\frac{1}{n-1} - \frac{1}{n+1}\right] = \frac{2(-1)^{n-1}}{(n-1)(n+1)}$$

When $n = 1$, we have

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx = \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx = \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - 1.\left(-\frac{\sin 2x}{2^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\pi \cos 2\pi}{2}\right] = -\frac{1}{2}$$

$$\therefore \quad x \sin x = 1 - \frac{1}{2}\cos x - 2\left(\frac{\cos 2x}{1.3} - \frac{\cos 3x}{2.4} + \frac{\cos 4x}{3.5} - \ldots\ldots\right).$$

Putting $x = \frac{\pi}{2}$, we get $\frac{\pi}{2} = 1 - 2\left(\frac{-1}{1.3} + \frac{1}{3.5} + \frac{-1}{5.7} - \ldots\ldots\right)$

$$\Rightarrow \quad 1 + \frac{2}{1.3} - \frac{2}{3.5} + \frac{2}{5.7} - \ldots\ldots = \frac{\pi}{2}$$

$$\Rightarrow \quad \frac{2}{1.3} - \frac{2}{3.5} + \frac{2}{5.7} - \ldots\ldots = \frac{\pi}{2} - 1$$

Hence $$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\ldots = \frac{\pi - 2}{4}.$$

Example 4. *Obtain the half-range sine series for e^x in $0 < x < 1$.*

Sol. Let $e^x = \sum_{n=1}^{\infty} b_n \sin n\pi x$, (since $l = 1$)

Then $$b_n = 2\int_0^1 e^x \sin n\pi x \, dx = 2\left[\frac{e^x}{1+(n\pi)^2}(\sin n\pi x - n\pi \cos n\pi x)\right]_0^1$$

$$\left[\because \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)\right]$$

$$= 2\left[\frac{e}{1+(n\pi)^2}(-n\pi \cos n\pi) - \frac{1}{1+(n\pi)^2}(-n\pi)\right]$$

$$= \frac{2}{1+n^2\pi^2}[-en\pi(-1)^n + n\pi] = \frac{2n\pi}{1+n^2\pi^2}[1 - e(-1)^n]$$

Hence $$e^x = 2\pi \sum_{n=1}^{\infty} \frac{n[1-e(-1)^n]}{1+n^2\pi^2}$$

$$= 2\pi\left[\frac{1+e}{1+\pi^2}\sin \pi x + \frac{2(1-e)}{1+4\pi^2}\sin 2\pi x + \frac{3(1+e)}{1+9\pi^2}\sin 3\pi x + \ldots\ldots\right].$$

Example 5. *Develop $\sin\left(\frac{\pi x}{l}\right)$ in half-range cosine series in the range $0 < x < l$.*

(M.G.U. May 2009)

Sol. Let $\sin\left(\frac{\pi x}{l}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$

then $$a_0 = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\,dx = \frac{2}{l}\left[-\frac{\cos\frac{\pi x}{l}}{\frac{\pi}{l}}\right]_0^l = -\frac{2}{\pi}[\cos\pi - 1] = \frac{4}{\pi}$$

$$a_n = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_0^l \left[\sin(n+1)\frac{\pi x}{l} - \sin(n-1)\frac{\pi x}{l}\right]dx$$

$$= \frac{1}{l}\left[-\frac{\cos(n+1)\frac{\pi x}{l}}{(n+1)\frac{\pi}{l}} + \frac{\cos(n-1)\frac{\pi x}{l}}{(n-1)\frac{\pi}{l}}\right]_0^l$$

$$= \frac{1}{\pi}\left[\left\{-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right\} - \left\{-\frac{1}{n+1} + \frac{1}{n-1}\right\}\right]$$

$$= \frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

When n is odd, $a_n = \frac{1}{\pi}\left[-\frac{1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] = 0$

When n is even, $a_n = \frac{1}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$

$$= \frac{2}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1}\right] = -\frac{4}{\pi(n+1)(n-1)}$$

$$\therefore \quad \sin\left(\frac{\pi x}{l}\right) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos\frac{2\pi x}{l}}{1.3} + \frac{\cos\frac{4\pi x}{l}}{3.5} + \frac{\cos\frac{6\pi x}{l}}{5.7} + \ldots\ldots\right].$$

Example 6. *Obtain a half-range cosine series for*

$$f(x) = kx \qquad \text{for} \quad 0 \le x \le \frac{l}{2}$$

$$= k(l - x) \qquad \text{for} \quad \frac{l}{2} \le x \le l.$$

Deduce the sum of the series $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$

then $$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\left[\int_0^{l/2} kx\,dx + \int_{l/2}^{l} k(l-x)\,dx\right]$$

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$$= \frac{2}{l}\left[\left|\frac{kx^2}{2}\right|_0^{l/2} + \left|k\left(lx - \frac{x^2}{2}\right)\right|_{l/2}^{l}\right]$$

$$= \frac{2}{l}\left[\frac{kl^2}{8} + k\left(l^2 - \frac{l^2}{2}\right) - k\left(\frac{l^2}{2} - \frac{l^2}{8}\right)\right] = \frac{2}{l}\left(\frac{kl^2}{4}\right) = \frac{kl}{2}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\left[\int_0^{l/2} kx\,.\cos\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} k(l-x)\,.\cos\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left[\left|kx\,.\frac{1}{n\pi}\sin\frac{n\pi x}{l} + k\,.\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right|_0^{l/2}\right.$$

$$\left. + \left|k(l-x)\,.\frac{1}{n\pi}\sin\frac{n\pi x}{l} - k\,.\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right|_{l/2}^{l}\right]$$

$$= \frac{2}{l}\left[\left|\frac{kl^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{kl^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)\right|\right.$$

$$\left. + \left|\frac{-kl^2}{n^2\pi^2}\cos n\pi - \frac{kl^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{kl^2}{n^2\pi^2}\cos\frac{n\pi}{2}\right|\right]$$

$$= \frac{2}{l}\left[\frac{2kl^2}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{kl^2}{n^2\pi^2} - \frac{kl^2}{n^2\pi^2}\cos n\pi\right] = \frac{2kl}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right]$$

When n is odd, $\cos\frac{n\pi}{2} = 0$ and $\cos n\pi = -1 \quad \therefore \quad a_n = 0 \;\Rightarrow\; a_1 = a_3 = a_5 = \ldots\ldots = 0$

When n is even, $a_2 = \frac{2kl}{2^2\pi^2}\,[2\cos\pi - 1 - \cos 2\pi] = -\frac{8kl}{2^2\pi^2}$;

$$a_4 = \frac{2kl}{4^2\pi^2}\,[2\cos 2\pi - 1 - \cos 4\pi] = 0$$

$$a_6 = \frac{2kl}{6^2\pi^2}\,[2\cos 3\pi - 1 - \cos 6\pi]$$

$$= \frac{2kl}{6^2\pi^2}(-2-1-1) = -\frac{8kl}{6^2\pi^2} \text{ and so on.}$$

$$\therefore \qquad f(x) = \frac{kl}{4} - \frac{8kl}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} + \ldots\ldots\right) \qquad \ldots(1)$$

Putting $\quad x = l,\ f(l) = 0$

$\therefore$ From (1), we have $\quad 0 = \frac{kl}{4} - \frac{8kl}{\pi^2}\left(\frac{1}{2^2} + \frac{1}{6^2} + \ldots\ldots\right)$

$$\Rightarrow \qquad \frac{1}{2^2} + \frac{1}{6^2} + \ldots\ldots = \frac{\pi^2}{32} \;\Rightarrow\; \frac{1}{2^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \ldots\ldots\right] = \frac{\pi^2}{32}$$

Hence $\quad \frac{1}{1^2} + \frac{1}{3^2} + \ldots\ldots = \frac{\pi^2}{8}$.

Example 7. *Express $f(x) = x(\pi - x)$; $0 < x < \pi$ as Fourier sine series in period 2π.*
(M.G.U. Dec., 2007)

Sol. The Fourier sine series expansion of $f(x)$ is $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

where
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx.$$

Now,
$$b_n = \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2) \sin nx \, dx$$
$$= \frac{2}{\pi}\left[(\pi x - x^2)\frac{-\cos nx}{n} - (\pi - 2x)\frac{-\sin nx}{n^2} + (-2)\frac{\cos nx}{n^3}\right]_0^{\pi}$$
$$= \frac{2}{\pi}\frac{(-2)\cos n\pi}{n^3} - \frac{2}{\pi}\frac{(-2)}{n^3}$$
$$= \frac{-4}{\pi n^3}(\cos n\pi - 1)$$
$$= \frac{-4}{n^3\pi}[(-1)^n - 1]$$

Therefore the series expansion is $f(x) = \sum_{n=1}^{\infty} \frac{-4}{n^3\pi}[(-1)^n - 1] \sin nx.$

Hence the result.

Example 8. *Find Fourier cosine series of* $f(x) = \begin{cases} x^2 & in \quad 0 < x < 1 \\ 2 - x & in \quad 1 < x < 2 \end{cases}$
(M.G.U. Dec., 2007)

Sol. Let
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$$

where
$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx.$$
$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx$$

Now
$$a_0 = \int_0^2 f(x)\,dx = \int_0^1 x^2\,dx + \int_1^2 (2 - x)\,dx$$
$$= \left(\frac{x^3}{3}\right)_0^1 + \left(2x - \frac{x^2}{2}\right)_1^2 = \frac{1}{3} + \left(2 - \frac{3}{2}\right) = \frac{5}{6}$$
$$a_n = \int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$$
$$= \int_0^1 x^2\cos\frac{n\pi x}{2}\,dx + \int_1^2 (2 - x)\cos\frac{n\pi x}{2}\,dx$$

$$= \left[x^2 \frac{\sin \frac{n\pi}{2} x}{\frac{n\pi}{2}} - 2x \frac{-\cos \frac{n\pi}{2} x}{\left(\frac{n\pi}{2}\right)^2} + 2 \frac{-\sin \frac{n\pi}{2} x}{\left(\frac{n\pi}{2}\right)^3} \right]_0^1$$

$$+ \left[(2-x) \frac{\sin \frac{n\pi}{2} x}{\left(\frac{n\pi}{2}\right)} + \frac{-\cos \frac{n\pi}{2} x}{\left(\frac{n\pi}{2}\right)^2} \right]_1^2$$

$$= \frac{8}{n^3\pi^3} \sin\left(\frac{n\pi}{2}\right) + \frac{12}{n^2\pi^2} \cos\left(\frac{n\pi}{2}\right) - \frac{4}{n^2\pi^2} \cos(n\pi).$$

Hence the half range cosine series of $f(x)$ is

$$f(x) = \frac{5}{12} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2}$$

$$= \frac{5}{12} + \sum_{n=1}^{\infty} \left[\frac{8}{n^3\pi^3} \sin\left(\frac{n\pi}{2}\right) + \frac{12}{n^2\pi^2} \cos\left(\frac{n\pi}{2}\right) - \frac{4}{n^2\pi^2} \cos(n\pi) \right] \cos\left(\frac{n\pi x}{2}\right).$$

TEST YOUR KNOWLEDGE

1. (*a*) Obtain cosine and sine series for $f(x) = x$ in the interval $0 \le x \le \pi$. Hence show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$. (*M.G.U. May 2009*)

   (*b*) Prove that for $0 < x < l$

$$x = \frac{l}{2} - \frac{4l}{\pi^2} \left(\cos \frac{\pi x}{l} + \frac{1}{3^2} \cos \frac{3\pi x}{l} + \frac{1}{5^2} \cos \frac{5\pi x}{l} + \ldots\ldots \right)$$

2. Find the half-range cosine series for the function $f(x) = x^2$ in the range $0 \le x \le \pi$.
3. Find the half-range cosine series for the function $f(x) = (x-1)^2$ in the interval $0 < x < 1$. (*M.G.U. May 2009 ; M.D.U.*)

   Hence show that

   (*i*) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{6}$ (*ii*) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{12}$

   (*iii*) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\ldots = \frac{\pi^2}{8}$.
4. (*a*) Express $\sin x$ as a cosine series in $0 < x < \pi$.

   (*b*) Show that a constant function c can be expanded in an infinite series $\frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots \right)$ in the range $0 < x < \pi$.

5. If
$$f(x) = \frac{\pi}{3}, \quad 0 \le x \le \frac{\pi}{3}$$
$$= 0, \quad \frac{\pi}{3} \le x \le \frac{2\pi}{3}$$
$$= -\frac{\pi}{3}, \quad \frac{2\pi}{3} \le x \le \pi$$
then show that $f(x) = \frac{2}{\sqrt{3}}\left[\cos x - \frac{\cos 5x}{5} + \frac{\cos 7x}{7} - \ldots\ldots\right]$

6. If
$$f(x) = mx, \quad 0 \le x \le \frac{\pi}{2}$$
$$= m(\pi - x), \quad \frac{\pi}{2} \le x \le \pi$$
then show that $f(x) = \frac{4m}{\pi}\left[\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \ldots\ldots\right]$

7. Express $f(x) = x$ as a half-range.
   (*i*) sine series in $0 < x < 2$. (*ii*) cosine series in $0 < x < 2$.

8. Show that the series $\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{l}$ represents $\frac{1}{2}l - x$ when $0 < x < l$.

9. Find the half-range sine series for $f(x) = \frac{1}{4} - x, \quad 0 < x < \frac{1}{2}$
$$= x - \frac{3}{4}, \quad \frac{1}{2} < x < 1.$$

10. Represent the following function by Fourier sine series
$$f(x) = 1 \quad \text{when} \quad 0 < x < \frac{l}{2}$$
$$= 0 \quad \text{when} \quad \frac{l}{2} < x < l.$$

11. Find the half-range sine series for the function $f(t) = t - t^2$, $0 < t < 1$.

12. Prove that for $0 < x < \pi$,
$$x(\pi - x) = \frac{\pi^2}{6} - \left(\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \ldots\ldots\right)$$

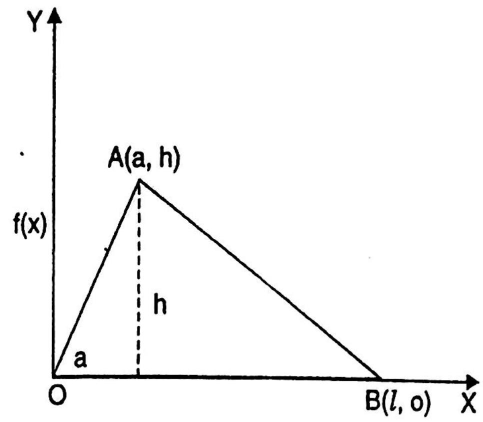
13. Let $f(x) = \begin{cases} \omega x, & \text{when } 0 \le x \le \frac{l}{2} \\ \omega(l - x), & \text{when } \frac{l}{2} \le x \le l \end{cases}$

Show that $f(x) = \frac{4\omega l}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\frac{(2n+1)\pi x}{l}$

Hence obtain the sum of the series
$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

14. If $f(x) = \begin{cases} \sin x, & \text{for } 0 \le x < \frac{\pi}{4} \\ \cos x, & \text{for } \frac{\pi}{4} \le x \le \frac{\pi}{2} \end{cases}$ expand $f(x)$ in a series of sines.

15. For the function defined by the graph OAB, find the half-range Fourier sine series.



16. Find the half-range sine series for $f(x) = \frac{l}{2} - x, \quad 0 < x < l.$ *(M.G.U. June 2006)*

17. Find the half-range cosine series for $f(x) = x \sin x,\ 0 < x < \pi.$ *(M.G.U. June 2006)*

Answers

1. (a) $\frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right)$; $2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots\ldots\right)$

2. $\frac{\pi^2}{3} - 4\left[\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \ldots\ldots\right]$

3. $\frac{1}{3} + \frac{1}{\pi^2}\left(\cos \pi x + \frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} + \ldots\ldots\right)$ **4.** (a) $\frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2x}{1.3} + \frac{\cos 4x}{3.5} + \frac{\cos 6x}{5.7} + \ldots\ldots\right]$

7. (i) $\frac{4}{\pi}\left(\sin \frac{\pi x}{2} - \frac{1}{2}\sin \frac{2\pi x}{2} + \frac{1}{3}\sin \frac{3\pi x}{2} - \frac{1}{4}\sin \frac{4\pi x}{2} + \ldots\ldots\right)$

(ii) $1 - \frac{8}{\pi^2}\left[\cos \frac{\pi x}{2} + \frac{1}{3^2}\cos \frac{3\pi x}{2} + \frac{1}{5^2}\cos \frac{5\pi x}{2} + \ldots\ldots\right]$

9. $f(x) = \left(\frac{1}{\pi} - \frac{4}{\pi^2}\right)\sin \pi x + \left(\frac{1}{3\pi} - \frac{4}{3^2\pi^2}\right)\sin 3\pi x + \left(\frac{1}{5\pi} - \frac{4}{5^2\pi^2}\right)\sin 5\pi x + \ldots$

10. $f(x) = \frac{2}{\pi}\left[\sin \frac{\pi x}{l} + \sin \frac{2\pi x}{l} + \frac{1}{3}\sin \frac{3\pi x}{l} + \frac{1}{5}\sin \frac{5\pi x}{l} + \ldots\ldots\right]$

11. $\frac{8}{\pi^3}\left(\frac{\sin \pi t}{1^3} + \frac{\sin 3\pi t}{3^3} + \frac{\sin 5\pi t}{5^3} + \ldots\ldots\right)$ **14.** $\frac{4\sqrt{2}}{\pi}\left(\frac{\sin 2x}{1.3} - \frac{\sin 6x}{5.7} + \frac{\sin 10x}{9.11} \ldots\ldots\right)$

15. $\dfrac{2l^2h}{a(l-a)\pi^2}\sum_{n=1}^{\infty}\dfrac{1}{n^2}\sin\dfrac{n\pi a}{l}\sin\dfrac{n\pi x}{l}$.

16. $f(x)=\sum_{n=1}^{\infty}\dfrac{l}{n\pi}\left[(-1)^n+1\right]\sin\left(\dfrac{n\pi x}{l}\right)$

17. $f(x)=1+\left(-\dfrac{1}{2}\right)\cos x+\sum_{n=2}^{\infty}\dfrac{2(-1)^{n+1}}{n^2-1}\cos nx.$

1.8. PARSEVAL'S THEOREM ON FOURIER CONSTANTS

If the Fourier series of f(x) over an interval $c < x < c + 2l$ is given as

$$\mathbf{f(x)}=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left\{a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right\}$$

then $$\frac{1}{2l}\int_c^{c+2l}[f(x)]^2\,dx=\frac{a_0^{\,2}}{4}+\frac{1}{2}\sum_{n=1}^{\infty}(a_n^{\,2}+b_n^{\,2}).$$

Proof. The Fourier series of $f(x)$ in $c < x < c + 2l$ is given as

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left\{a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right\} \quad ...(1)$$

where $a_0=\dfrac{1}{l}\displaystyle\int_c^{c+2l}f(x)\,dx\ ;\ a_n=\dfrac{1}{l}\int_c^{c+2l}f(x)\cos\dfrac{n\pi x}{l}\,dx\ ;\ b_n=\dfrac{1}{l}\int_c^{c+2l}f(x)\sin\dfrac{n\pi x}{l}\,dx$ ...(2)

Multiplying both sides of (1) by $f(x)$, we have

$$[f(x)]^2=\frac{a_0}{2}f(x)+\sum_{n=1}^{\infty}a_nf(x)\cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty}b_nf(x)\sin\frac{n\pi x}{l}$$

Integrating both sides w.r.t. x, between the limits c to $c + 2l$, we have

$$\int_c^{c+2l}[f(x)]^2\,dx=\frac{a_0}{2}\int_c^{c+2l}f(x)\,dx+\sum_{n=1}^{\infty}a_n\int_c^{c+2l}f(x)\cos\frac{n\pi x}{l}\,dx$$

$$+\sum_{n=1}^{\infty}b_n\int_c^{c+2l}f(x)\sin\frac{n\pi x}{l}\,dx$$

$$=\frac{a_0}{2}\,.\,la_0+\sum_{n=1}^{\infty}a_n(la_n)+\sum_{n=1}^{\infty}b_n(lb_n) \quad \text{[Using (2)]}$$

$$\Rightarrow \quad \int_c^{c+2l}[f(x)]^2\,dx=\frac{la_0^{\,2}}{2}+l\sum_{n=1}^{\infty}(a_n^{\,2}+b_n^{\,2})$$

$$\therefore \quad \frac{1}{2l}\int_c^{c+2l}[f(x)]^2\,dx=\frac{1}{2l}\left\{\frac{la_0^{\,2}}{2}+l\sum_{n=1}^{\infty}(a_n^{\,2}+b_n^{\,2})\right\}$$

or $$\frac{1}{2l}\int_c^{c+2l}[f(x)]^2\,dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$ (Parseval's identity)

Hence the proof.

Note. *Parseval's identities in different cases :*

(*i*) If $c = 0$, the interval becomes $0 < x < 2l$ and Parseval's identity reduces to

$$\frac{1}{2l}\int_0^{2l}[f(x)]^2\,dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

If $c = -l$, the interval becomes $-l < x < l$ and Parseval's identity reduces to

$$\frac{1}{2l}\int_{-l}^{l}[f(x)]^2\,dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

(*ii*) If $f(x)$ is an even function in $(-l, l)$ then $\frac{2}{l}\int_0^{l}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$

(*iii*) If $f(x)$ is an odd function in $(-l, l)$ then $\frac{2}{l}\int_0^{l}[f(x)]^2\,dx = \sum_{n=1}^{\infty} b_n^2$

(*iv*) If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$ in $(0, l)$ then $\frac{2}{l}\int_0^{l}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$

(*v*) If $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ in $(0, l)$ then $\frac{2}{l}\int_0^{l}[f(x)]^2\,dx = \sum_{n=1}^{\infty} b_n^2$.

1.9. ROOT MEAN SQUARE (R.M.S.) VALUVE

The root mean square value of the function $f(x)$ over an interval (a, b) is defined as

$$\left[f(x)\right]_{\text{r.m.s.}} = \sqrt{\frac{\int_a^b [f(x)]^2\,dx}{b-a}}$$

The r.m.s. value of a function is also known as the effective value of the function.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the Fourier sine series for unity in $0 < x < \pi$ and hence show that*

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\ldots = \frac{\pi^2}{8}.$$

Sol. We require half-range Fourier sine series for 1 in $(0, \pi)$

Let $$1 = \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots(1)$$

Then $$b_n = \frac{2}{\pi}\int_0^{\pi} (1) \sin nx\, dx = \frac{2}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi} = -\frac{2}{n\pi}(\cos n\pi - 1)$$

$$= \frac{2}{n\pi}[1-(-1)^n] \qquad [\because \cos n\pi = (-1)^n]$$

Now $b_n = 0$ when n is even ; and $b_n = \dfrac{4}{n\pi}$ when n is odd.

Substituting in (1), we get

$$\therefore\ 1 = \sum_{m=1}^{\infty} \frac{4}{(2m-1)\pi} \sin(2m-1)x \quad \text{or} \quad 1 = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right) \qquad \ldots(2)$$

Now from Parseval's theorem on Fourier constants

$$\int_c^{c+2l} [f(x)]^2\, dx = 2l\left[\frac{{a_0}^2}{4} + \frac{l}{2}\sum_{n=1}^{\infty}({a_n}^2 + {b_n}^2)\right] \qquad \ldots(3)$$

Applying (3) to half-range sine series for 1 in $(0, \pi)$

$$c = 0,\ 2l = \pi,\ f(x) = 1,\ a_0 = 0,\ a_n = 0, \text{ and } b_n = \frac{4}{(2m-1)\pi},\ m = 1, 2 \ldots\ldots$$

We get, $$\int_0^{\pi} (1)^2\, dx = \pi \cdot \frac{1}{2}\sum_{m=1}^{\infty} \frac{16}{(2m-1)^2} \cdot \pi^2$$

$$\Rightarrow \qquad \left[x\right]_0^{\pi} = \frac{8}{\pi}\left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right\} \quad \text{or} \quad \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

Hence the result.

Example 2. *Find Fourier series of x^2 in $(-\pi, \pi)$. Use Parseval's identity to prove that*

$$\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\ldots$$

Sol. The Fourier series of x^2 in $(-\pi, \pi)$ is

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx \qquad \ldots(1)$$

Here $$a_0 = \frac{2\pi^2}{3},\ a_n = \frac{4(-1)^n}{n^2},\ b_n = 0,\ f(x) = x^2$$

Now by Parseval's identity from (1), we get

$$\int_{-\pi}^{\pi}(x^2)^2\,dx = 2\pi\left[\frac{\pi^4}{9}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4}\right]$$

$$\Rightarrow \quad \left[\frac{x^5}{5}\right]_{-\pi}^{\pi} = \frac{2\pi^5}{9}+\pi\sum_{n=1}^{\infty}\frac{16}{n^4} \quad \text{or} \quad \frac{2\pi^5}{5}-\frac{2\pi^5}{9} = \pi\sum_{n=1}^{\infty}\frac{16}{n^4}$$

or

$$\frac{\pi^4}{90} = \sum_{n=1}^{\infty}\frac{1}{n^4} \quad \text{or} \quad 1+\frac{1}{2^4}+\frac{1}{3^4}+\ldots\ldots = \frac{\pi^4}{90}.$$

TEST YOUR KNOWLEDGE

1. If $f(x)$ has the Fourier series expansion

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right) \text{ in } a \le x \le a+2l$$

show that $\displaystyle\int_a^{a+2l}[f(x)]^2\,dx = 2l\left[\frac{{a_0}^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}({a_n}^2+{b_n}^2)\right].$

2. If $f(x) = \dfrac{a_0}{2}+\displaystyle\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}$ in $0 < x < l$, then show that

$$\int_0^l [f(x)]^2\,dx = \frac{l}{2}\left[\frac{{a_0}^2}{2}+\sum_{n=1}^{\infty}{a_n}^2\right].$$

3. If $f(x) = \displaystyle\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$ in $(0, l)$, then show that $\displaystyle\int_0^l [f(x)]^2\,dx = \frac{l}{2}\sum_{n=1}^{\infty}{b_n}^2$.

4. Prove that in the range $(0, l)$, $x = \dfrac{l}{2}-\dfrac{4l}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2m-1)^2}\cos\frac{(2m-1)\pi x}{l}$ and deduce that

$$\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots\ldots = \frac{\pi^4}{96}.$$

5. Show that for $0 < x < \pi$,

$$x(\pi - x) = \frac{\pi^2}{6}-\left(\frac{\cos 2x}{1^2}+\frac{\cos 4x}{2^2}+\frac{\cos 6x}{3^2}+\ldots\ldots\right)$$

and hence evaluate $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4}$.

1.10. TYPICAL WAVEFORMS

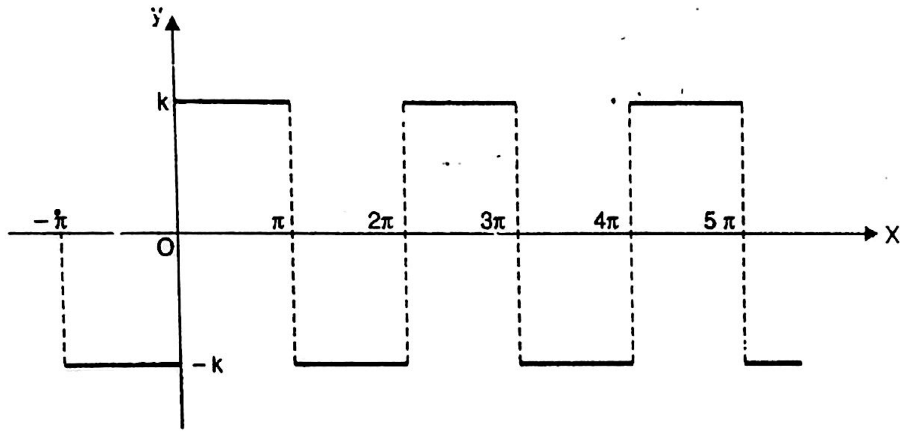
A periodic waveform is a waveform that repeats a basic pattern. It is a single-valued periodic function. Therefore it can be developed as a Fourier series.

We give below some typical waveforms usually met with in communication engineering and also the corresponding Fourier series. The student is urged to construct the Fourier series in each case.

I. Square (or Rectangular) Waveform

It is a periodic function of the form given below.

(*i*) $$f(x) = \begin{cases} -k & \text{for } -\pi < x < 0 \\ k & \text{for } \ 0 < x < \pi \end{cases}, \quad f(x + 2\pi) = f(x)$$

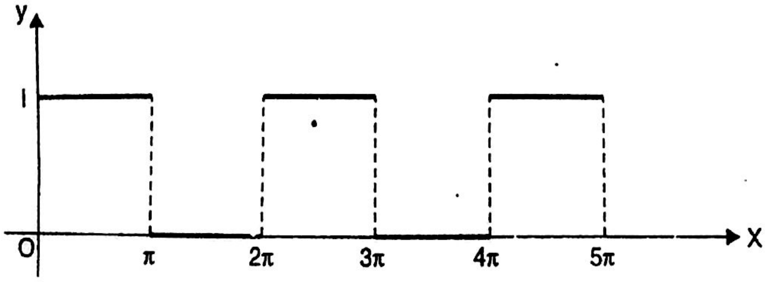


Its Fourier expansion is

$$f(x) = \frac{4k}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right]$$

[See Question 1 in Test Your Knowledge on Page No. 19]

(*ii*) $$f(x) = \begin{cases} 1 & \text{when } \ 0 < x < \pi \\ 0 & \text{when } \pi < x < 2\pi \end{cases}, \quad f(x + 2\pi) = f(x)$$



Its Fourier expansion is

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right)$$

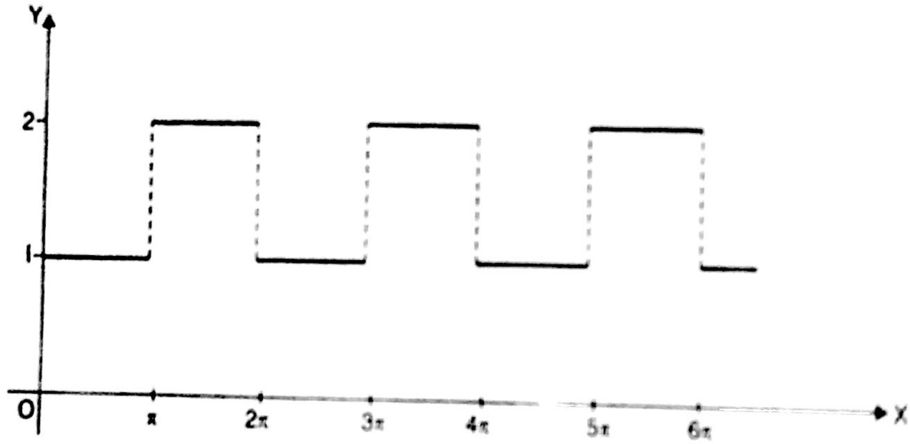
(*iii*) $$f(x) = \begin{cases} 0 & \text{for } -\pi < x < 0 \\ 1 & \text{for } \quad 0 < x < \pi \end{cases}, f(x + 2\pi) = f(x)$$

Its Fourier expansion is

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right)$$

[See Question 2 in Test Your Knowledge on Page 20]

(*iv*) $$f(x) = \begin{cases} 1 & \text{for } \ 0 < x < \pi \\ 2 & \text{for } \pi < x < 2\pi \end{cases}, f(x + 2\pi) = f(x)$$



Its Fourier expansion is

$$f(x) = \frac{3}{2} - \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right)$$

(*v*) $$f(x) = \begin{cases} -\dfrac{\pi}{4} & \text{for } -\pi < x < 0 \\ \dfrac{\pi}{4} & \text{for } 0 < x < \pi \end{cases}, f(x + 2\pi) = f(x)$$

Its Fourier expansion is

$$f(x) = \frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots$$

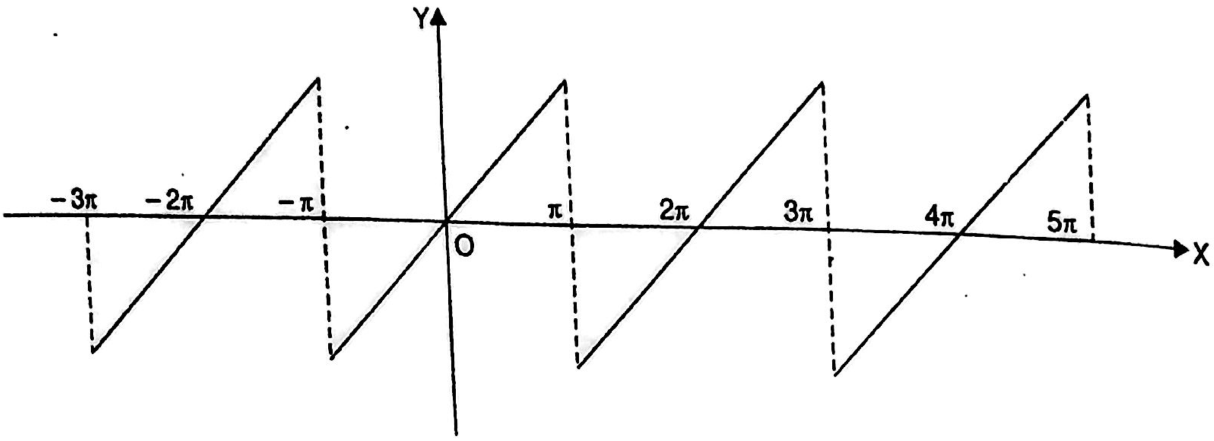
II. Saw-toothed Waveform

It is a periodic function of the form given below.

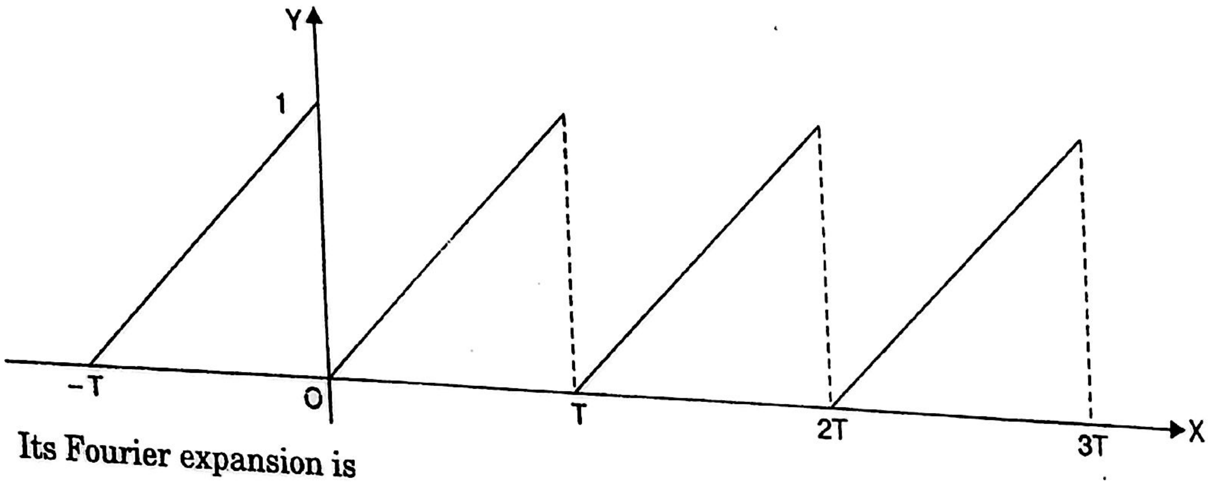
(*i*) $$f(x) = x, -\pi < x < \pi \quad \text{and} \quad f(x + 2\pi) = f(x)$$

Its Fourier expansion is

$$f(x) = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots\ldots\right)$$



(ii) $$f(x) = \frac{1}{T}x \quad \text{when} \quad 0 < x < T \quad \text{and} \quad f(x+T) = f(x)$$



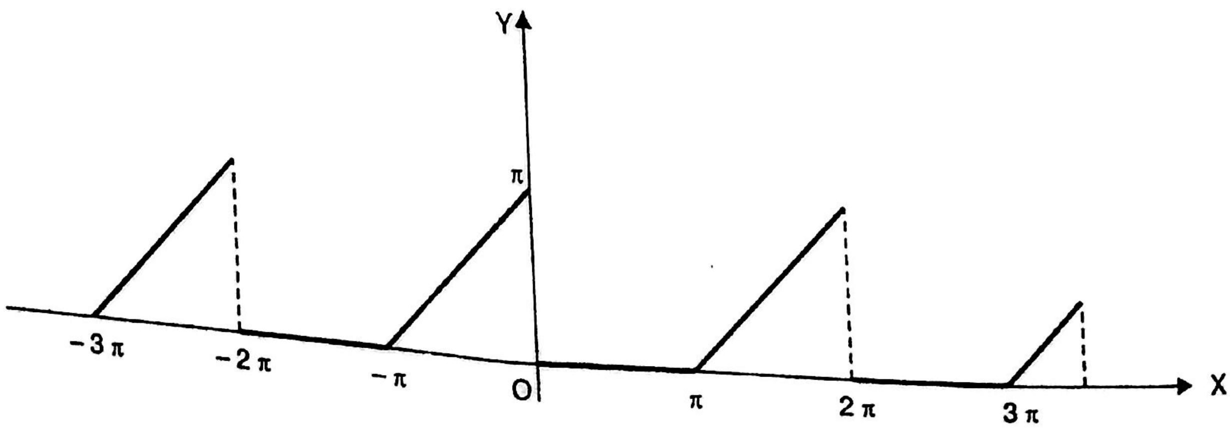
Its Fourier expansion is

$$f(x) = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\omega x}{n}, \qquad \text{where } \omega = \frac{2\pi}{T}.$$

III. Modified Saw-toothed Waveform

It is a periodic function of the form given below.

$$f(x) = \begin{cases} \pi + x & \text{for } -\pi < x < 0 \\ 0 & \text{for } \ 0 \le x < \pi \end{cases}, \ f(x + 2\pi) = f(x)$$



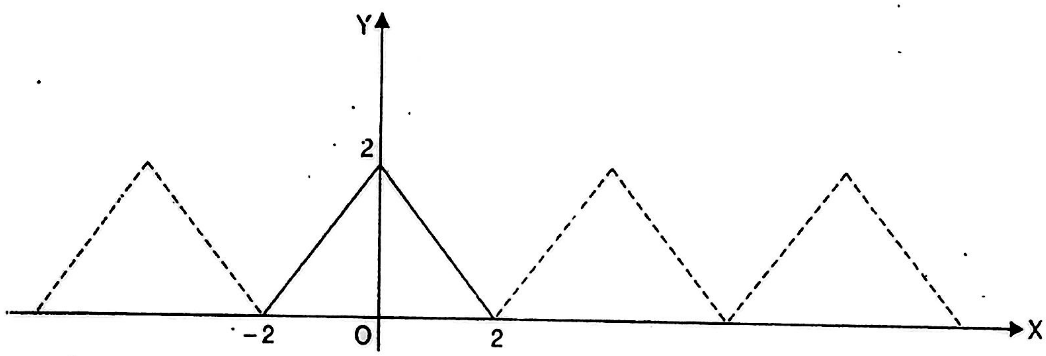
Its Fourier expansion is

$$f(x) = \frac{\pi}{4} + \frac{2}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \ldots\ldots\right) - \left(\frac{\sin x}{1} + \frac{\sin 2x}{2} + \ldots\ldots\right)$$

IV. Triangular Waveform

It is a periodic function of the form given below.

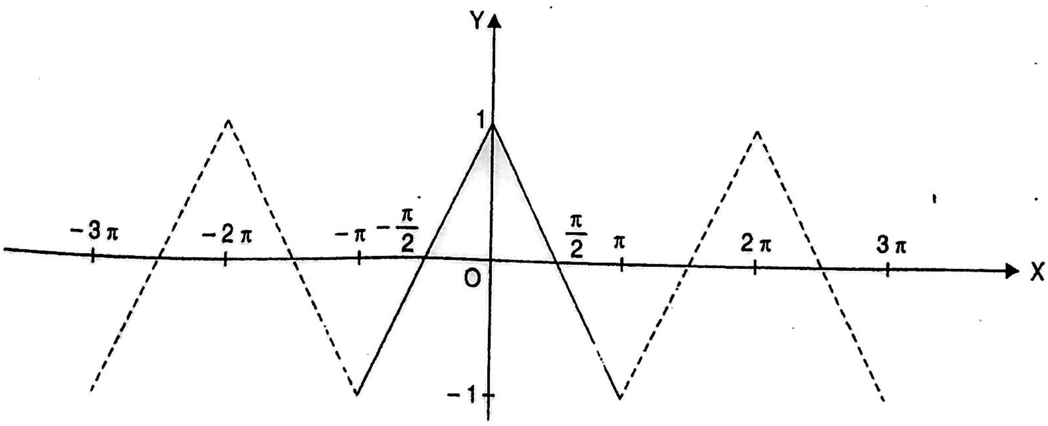
(*i*) $$f(x) = \begin{cases} 2+x & \text{for } -2 \le x \le 0 \\ 2-x & \text{for } \quad 0 < x \le 2 \end{cases}, f(x+4) = f(x)$$



Its Fourier expansion is

$$f(x) = 1 + \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left[(2n-1)\frac{\pi x}{2}\right]$$

(*ii*) $$f(x) = \begin{cases} 1+\dfrac{2x}{\pi} & \text{for } -\pi \le x \le 0 \\ 1-\dfrac{2x}{\pi} & \text{for } \quad 0 < x \le \pi \end{cases}, f(x+2\pi) = f(x)$$



Its Fourier expansion is

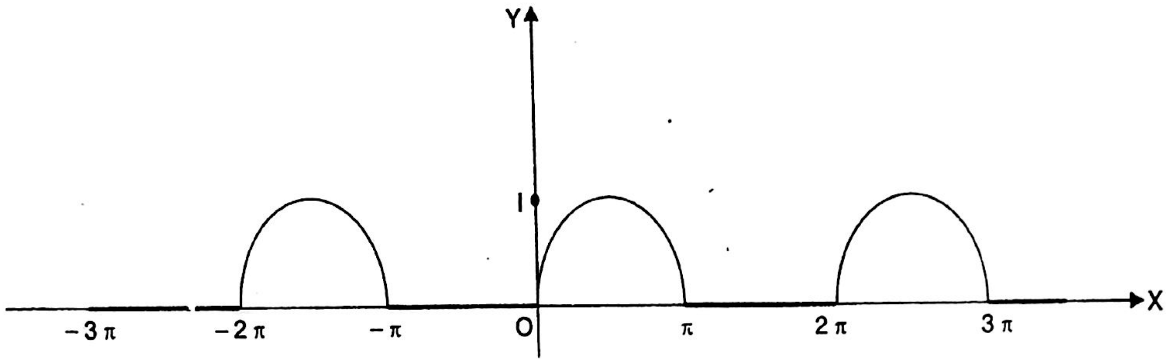
$$f(x) = \frac{8}{\pi^2}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right)$$

[See Example 10 before Test Your Knowledge on Page 14]

V. Half Rectified Waveform

It is a periodic function of the form given below.

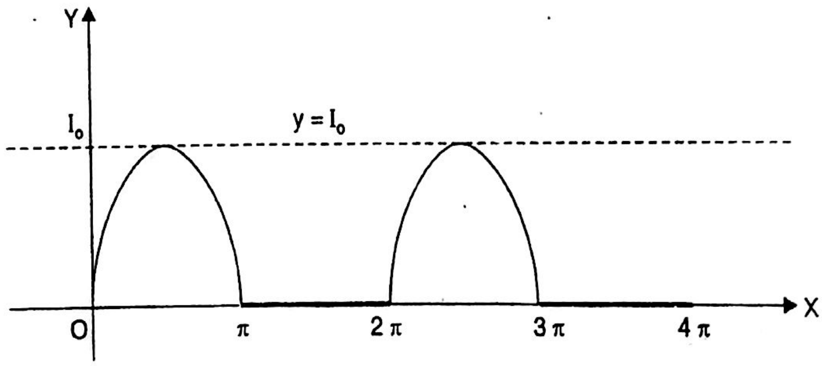
(*i*) $$f(x) = \begin{cases} 0 & \text{for } -\pi \le x \le 0 \\ \sin x & \text{for } 0 \le x \le \pi \end{cases}$$



Its Fourier expansion is

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1}$$

(*ii*) $$f(x) = \begin{cases} I_0 \sin x & \text{for } 0 \le x \le \pi \\ 0 & \text{for } \pi \le x \le 2\pi \end{cases}, \; f(x + 2\pi) = f(x)$$



Its Fourier expansion is

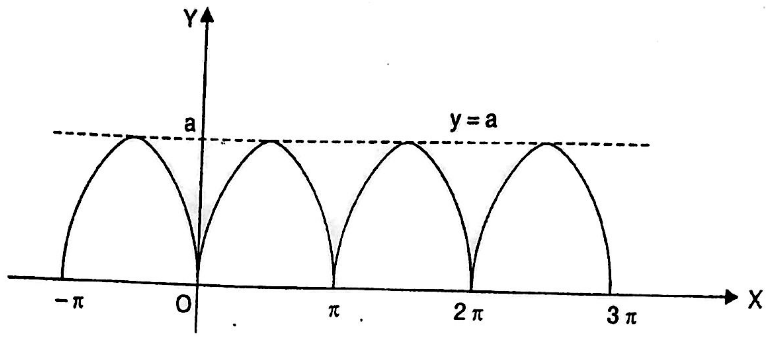
$$f(x) = \frac{I_0}{\pi} + \frac{1}{2} I_0 \sin x - \frac{2I_0}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1}$$

[See Question 6 in Test Your Knowledge on Page 19]

VI. Full Rectified Waveform

It is a periodic function of the form given below.

$$f(x) = a \sin x \text{ for } 0 \le x \le \pi,\ f(x+\pi) = f(x)$$



Its Fourier expansion is

$$f(x) = \frac{4a}{\pi}\left[\frac{1}{2} - \frac{\cos 2x}{1.3} - \frac{\cos 4x}{3.5} - \frac{\cos 6x}{5.7} - \ldots\ldots\right]$$

1.11. COMPLEX FORM OF FOURIER SERIES

We know that in the interval $c < x < c + 2l$, the Fourier series expansion of $f(x)$ can be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right) \qquad \ldots(1)$$

We also know that $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

Substituting the exponential equivalents for the trigonometric terms in (1), we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\left(\frac{e^{\frac{in\pi x}{l}} + e^{-\frac{in\pi x}{l}}}{2}\right) + \sum_{n=1}^{\infty} b_n\left(\frac{e^{\frac{in\pi x}{l}} - e^{-\frac{in\pi x}{l}}}{2i}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{a_n - ib_n}{2}\right)e^{\frac{in\pi x}{l}} + \sum_{n=1}^{\infty}\left(\frac{a_n + ib_n}{2}\right)e^{-\frac{in\pi x}{l}} \qquad \left[\because \frac{1}{i} = -i\right]$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{\frac{in\pi x}{l}} + \sum_{n=1}^{\infty} c_{-n}\, e^{-\frac{in\pi x}{l}} = \sum_{n=1}^{\infty} e^{\frac{in\pi x}{l}}$$

where $c_0 = \frac{a_0}{2},\ c_n = \frac{a_n - ib_n}{2},\ c_{-n} = \frac{a_n + ib_n}{2}$

Hence the *complex* or *exponential* form of the Fourier series is

$$\mathbf{f(x)} = \sum_{n=-\infty}^{\infty} \mathbf{c_n e^{\frac{in\pi x}{l}}} \qquad \ldots(2)$$

Now multiplying on both sides of (2) by $e^{-\frac{in\pi x}{l}}$ and integrating from c to $c + 2l$ w.r.t. x, we get

$$\int_c^{c+2l} f(x).e^{-\frac{in\pi x}{l}}\,dx = c_n \int_c^{c+2l} e^{\frac{in\pi x}{l}} e^{-\frac{in\pi x}{l}}\,dx$$

$$= c_n \int_c^{c+2l} dx = c_n [x]_c^{c+2l} = c_n\,(2l)$$

$$\therefore \quad \mathbf{c_n = \frac{1}{2l}\int_c^{c+2l} f(x)\, e^{-\frac{in\pi x}{l}}\,dx.}$$

ILLUSTRATIVE EXAMPLES

Example 1. *Find the complex form of Fourier series $f(x) = e^{ax}$ $(-\pi < x < \pi)$ in the form*

$$e^{ax} = \frac{\sinh a\pi}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{a+in}{a^2+n^2} e^{inx}$$

and hence prove that $\dfrac{\pi}{a \sinh a\pi} = \displaystyle\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2+a^2}$.

Sol. The complex form of Fourier series of $f(x)$ in $c < x < c + 2l$ is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}} \qquad \text{where } c_n = \frac{1}{2l}\int_c^{c+2l} f(x)\, e^{-\frac{in\pi x}{l}}\,dx$$

Here $2l = 2\pi \Rightarrow l = \pi \quad \therefore \quad f(x) = \displaystyle\sum_{n=-\infty}^{\infty} c_n e^{inx}$ ...(1)

Now
$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-\frac{in\pi x}{\pi}}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}\,.\,e^{-inx}\,dx \qquad [\because \; f(x) = e^{ax}]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(a-in)x}\,dx = \frac{1}{2\pi}\left[\frac{e^{(a-in)x}}{(a-in)}\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\left[\frac{e^{(a-in)\pi} - e^{-(a-in)\pi}}{a-in}\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{a\pi}(\cos n\pi - i\sin n\pi) - e^{-a\pi}(\cos n\pi + i \sin n\pi)}{a-in}\right]$$

$(e^{\pm in\pi} = \cos n\pi \pm i \sin n\pi)$

$$= \frac{1}{2\pi(a-in)}(e^{a\pi} - e^{-a\pi})\cos n\pi = \frac{1}{2\pi(a-in)}(2\sinh a\pi)\cos n\pi$$

$$\left(\because \frac{e^{a\pi} - e^{-a\pi}}{2} = \sinh a\pi\right)$$

$$= \frac{(-1)^n \sinh a\pi}{\pi(a-in)} \qquad [\because \cos n\pi = (-1)^n]$$

or $$c_n = \frac{(-1)^n (a+in)\sinh a\pi}{\pi(a^2+n^2)}$$

Substituting in (1), we get

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n (a+in)\sinh a\pi}{\pi(a^2+n^2)} e^{inx}$$

i.e., $$e^{ax} = \frac{\sinh a\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n (a+in)}{a^2+n^2} e^{inx} \qquad ...(2)$$

Putting $x = 0$ in (2), we get $1 = \dfrac{\sinh a\pi}{\pi} \displaystyle\sum_{n=-\infty}^{\infty} \frac{(-1)^n (a+in)}{a^2+n^2}$

or $$\frac{\pi}{\sinh a\pi} = \sum_{n=-\infty}^{\infty} (-1)^n \left\{\frac{a}{a^2+n^2} + i\frac{n}{a^2+n^2}\right\}$$

Equating the real part, we get

$$\frac{\pi}{\sinh a\pi} = \sum_{n=-\infty}^{\infty} (-1)^n \frac{a}{a^2+n^2} \text{ or } \frac{\pi}{a\sinh a\pi} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2+n^2}$$

which proves the result.

Example 2. *Find the complex form of the Fourier series of*

$f(x) = \cos ax$ in $-\pi < x < \pi$.

Sol. The complex form of Fourier series in $-\pi$ to π is given by

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx} \qquad ...(1) \quad \text{where} \quad C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos ax \,.\, e^{-inx}\, dx = \frac{1}{2\pi}\left[\frac{e^{-inx}}{(-in)^2+a^2}\{-in\cos ax + a\sin ax\}\right]_{-\pi}^{\pi}$$

$$\left[\because \int e^{ax}\cos(bx+c) = \frac{e^{ax}}{a^2+b^2}\{a\cos(bx+c) + b\sin(bx+c)\}\right]$$

$$= \frac{1}{2\pi(a^2-n^2)}[e^{-inx}(-in\cos a\pi + a\sin a\pi) - e^{inx}(-in\cos a\pi - a\sin a\pi)]$$

$$= \frac{1}{2\pi(a^2 - n^2)} [\cos n\pi \{- in \cos a\pi + a \sin a\pi + in \cos a\pi + a \sin a\pi\}]$$

$$[\because e^{\pm inx} = \cos n\pi \pm i \sin n\pi = \cos n\pi = (-1)^n]$$

$$\therefore \quad C_n = \frac{(-1)^n a \sin a\pi}{\pi(a^2 - n^2)}$$

Substituting in (1)

$$f(x) = \cos ax = \sum_{n=-\infty}^{\infty} \frac{(-1)^n a \sin a\pi}{\pi(a^2 - n^2)} e^{inx} = \frac{a \sin a\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2 - n^2} e^{inx}.$$

TEST YOUR KNOWLEDGE

1. Derive the complex form (or exponential form) of Fourier series for $f(x)$ in $-\pi < x < \pi$.
2. Find the complex form of Fourier series for e^x in $-\pi < x < \pi$.
3. Find the complex form of Fourier series for e^{-x} in $-1 < x < 1$.
4. Find the complex form of the Fourier series for the function e^{ax} in $-\pi < x < \pi$, given that α is a real constant. Deduce that

   (*i*) When α is a constant other than an integer $\cos \alpha x = \dfrac{\sin \pi\alpha}{\pi} \displaystyle\sum_{n=-\infty}^{\infty} (-1)^n \frac{\alpha}{\alpha^2 - n^2} e^{inx}$ where $-\pi < x < \pi$ and

   (*ii*) $\displaystyle\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 + a^2} = \frac{\pi}{a \sinh a\pi}$.

Answers

1. $f(x) = \displaystyle\sum_{n=-\infty}^{\infty} c_n e^{-inx}$; $c_n = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} f(x) e^{-inx} \, dx$ 2. $e^x = \dfrac{\sinh \pi}{\pi} \displaystyle\sum_{n=-\infty}^{\infty} (-1)^n \left(\frac{1 + in}{1 + n^2}\right) e^{inx}$

3. $e^{-x} = \displaystyle\sum_{n=-\infty}^{\infty} (-1)^n (\sinh 1) \left(\frac{1 - in\pi}{1 + n^2 \pi^2}\right) e^{in\pi x}$.

1.12. PRACTICAL HARMONIC ANALYSIS

When the function $f(x)$ is not given by an analytical expression, rather by its graph or a table of corresponding values at a number of equi-spaced points, we cannot evaluate the integrals for Fourier co-efficients. However, using the rules of approximate integration, we can find approximate values of the first few terms of the Fourier expansions.

Let the Fourier series for $y = f(x)$ in $(0, 2\pi)$ by

$$y = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \ldots\ldots + b_1 \sin x + b_2 \sin 2x + \ldots\ldots$$

where $a_0 = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} y \, dx$, $a_n = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} y \cos nx \, dx$; $b_n = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} y \sin nx \, dx$.

Let the range $(0, 2\pi)$ be divided into m equal parts by the points $x = x_0, x_1, x_2 \ldots\ldots, x_m = 2\pi$ so that each sub-interval is of the length $\frac{2\pi}{m}$. Let the ordinates at these points be denoted by $y_0, y_1, y_2, \ldots\ldots, y_m$.

Using the trapezoidal rule of approximate integration, we get

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} y\,dx = \frac{1}{\pi}\cdot\frac{2\pi}{m}(y_0 + y_1 + y_2 + \ldots\ldots + y_{m-1}) = 2 \times \frac{1}{m}\sum_{r=0}^{m-1} y_r$$

$$= 2\,[\text{mean value of } y \text{ in } (0, 2\pi)]$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} y \cos nx\,dx$$

$$\approx \frac{1}{\pi}\cdot\frac{2\pi}{m}(y_0 \cos nx_0 + y_1 \cos nx_1 + \ldots\ldots + y_{m-1}\cos nx_{m-1})$$

$$= 2 \times \frac{1}{m}\sum_{r=0}^{m-1} y_r \cos nx_r = 2\,[\text{mean value of } y \cos nx \text{ in } (0, 2\pi)]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} y \sin nx\,dx = \frac{1}{\pi}\cdot\frac{2\pi}{m}(y_0 \sin nx_0 + y_1 \sin nx_1 + \ldots\ldots + y_{m-1}\sin nx_{m-1})$$

$$= 2 \times \frac{1}{m}\sum_{r=0}^{m-1} y_r \sin nx_r$$

$$= 2\,[\text{mean value of } y \sin nx \text{ in } (0, 2\pi)]$$

The process of finding the Fourier series for a function given by numerical values is known as *harmonic analysis*.

The term $(a_1 \cos x + b_1 \sin x)$ is called the *fundamental* or *first harmonic*, the term $(a_2 \cos 2x + b_2 \sin 2x)$ is called the *second harmonic* and so on.

The number of ordinates used should not be less than twice the number of highest harmonic to be found.

ILLUSTRATIVE EXAMPLES

Example 1. *Analyse harmonically the data given below and express y in Fourier series upto the third harmonic :*

x :	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
y :	1.0	1.4	1.9	1.7	1.5	1.2	1.0

Sol. Since the last value of y is a repetition of the first, only the first six values will be used. The length of the interval is 2π.

Let $$y = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + (a_3 \cos 3x + b_3 \sin 3x)$$

The values of $x, y, \sin x, \cos x, \sin 2x, \cos 2x, \sin 3x, \cos 3x$ are tabulated below :

x	y	$\cos x$	$\sin x$	$\cos 2x$	$\sin 2x$	$\cos 3x$	$\sin 3x$
0	1.0	1	0	1	0	1	0
$\frac{\pi}{3}$	1.4	0.5	0.866	– 0.5	0.866	– 1	0
$\frac{2\pi}{3}$	1.9	– 0.5	0.866	– 0.5	– 0.866	1	0
π	1.7	– 1	0	1	0	– 1	0
$\frac{4\pi}{3}$	1.5	– 0.5	0.866	– 0.5	0.866	1	0
$\frac{5\pi}{3}$	1.2	0.5	– 0.866	– 0.5	– 0.866	– 1	0

Using the values in the above table, we have

$$a_0 = \frac{2}{n} \sum y = \frac{2}{6} (8.7) = 2.9$$

$$a_1 = \frac{2}{n} \sum y \cos x = \frac{2}{6} [1(1.0 - 1.7) + 0.5 (1.4 - 1.9 - 1.5 + 1.2)] = \frac{1}{3} (-1.1) \approx -0.37$$

$$b_1 = \frac{2}{n} \sum y \sin x = \frac{2}{6} [0.866 (1.4 + 1.9 - 1.5 - 1.2)] = \frac{1}{3} [0.866 (.6)] \approx 0.17$$

$$a_2 = \frac{2}{n} \sum y \cos 2x = \frac{2}{6} [1(1.0 + 1.7) - 0.5(1.4 + 1.9 - 1.5 + 1.2)] = \frac{1}{3} [2.7 - 3] = 0.1$$

$$b_2 = \frac{2}{n} \sum y \sin 2x = \frac{2}{6} [0.866 (1.4 - 1.9 + 1.5 - 1.2)] = \frac{1}{3} [0.866 (-2)] = -0.06$$

$$a_3 = \frac{2}{n} \sum y \cos 3x = \frac{2}{6} [1.0 - 1.4 + 1.9 - 1.7 + 1.5 - 1.2] = \frac{1}{3} (0.1) = 0.03$$

$$b_3 = \frac{2}{n} \sum y \sin 3x = 0$$

$$\therefore \quad y = 1.45 + (-0.37 \cos x + 0.17 \sin x) - (0.1 \cos 2x + 0.06 \sin 2x) + 0.03 \cos 3x.$$

Example 2. *The following values of y give the displacement in inches of a certain machine part for the rotation x of the flywheel. Expand y in the form of a Fourier series :*

x :	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$
y :	0	9.2	14.4	17.8	17.3	11.7

Sol. Here length of interval is π, *i.e.*, $l = \frac{\pi}{2}$

Let
$$y = \frac{a_0}{2} + \left(a_1 \cos \frac{\pi x}{\pi/2} + b_1 \sin \frac{\pi x}{\pi/2}\right) + \left(a_2 \cos \frac{2\pi x}{\pi/2} + b_2 \sin \frac{\pi x}{\pi/2}\right) + \ldots\ldots$$

$$= \frac{a_0}{2} + (a_1 \cos 2x + b_1 \sin 2x) + (a_2 \cos 4x + b_2 \sin 4x) + \ldots\ldots$$

The values of x, y, $\sin 2x$, $\cos 2x$, $\sin 4x$, $\cos 4x$ are tabulated below :

x	y	$\cos 2x$	$\sin 2x$	$\cos 4x$	$\sin 4x$
0	0	1	0	1	0
$\frac{\pi}{6}$	9 2	.5	.87	– .5	.87
$\frac{2\pi}{6}$	14.4	– .5	.87	– .5	– .87
$\frac{3\pi}{6}$	17.8	– 1	0	1	0
$\frac{4\pi}{6}$	17.3	– .5	– .87	– .5	.87
$\frac{5\pi}{6}$	11.7	.5	– .87	– .5	– 8.7

Using the values in the above table, we have

$$a_0 = \frac{2}{n}\sum y = \frac{2}{6}(70.4) = 23.46$$

$$a_1 = \frac{2}{n}\sum y\cos 2x = \frac{2}{6}[.5(9.2 - 14.4 - 17.3 + 11.7) - 17.8]$$

$$= \frac{1}{3}(-23.2) = -7.73$$

$$b_1 = \frac{2}{n}\sum y\sin 2x = \frac{2}{6}[.87(9.2 + 14.4 - 17.3 - 11.7)]$$

$$= \frac{1}{3}[.87(-5.4)] = -1.566$$

$$a_2 = \frac{2}{n}\sum y\cos 4x = \frac{2}{6}[-.5(9.2 + 14.4 + 17.3 + 11.7) + 17.8]$$

$$= \frac{1}{3}[-26.3 + 17.8] = \frac{1}{3}(-8.5) = -2.83$$

$$b_2 = \frac{2}{n}\sum y\sin 4x = \frac{2}{6}[87(9.2 - 14.4 + 17.3 - 11.7)] = \frac{1}{3}[.87(4)] = .116$$

$\therefore$ $y = 11.73 - (7.73\cos 2x + 1.566\sin 2x) + (-2.83\cos 4x + .116\sin 4x)$.

Example 3. *Obtain the constant term and the co-efficients of the first sine and* [illegible] *terms in the Fourier expansion of y as given in the following* [illegible]

x : 0

The values of $x, y, \sin\frac{\pi x}{3}, \cos\frac{\pi x}{3}$ are tabulated below :

x	y	$\cos\frac{\pi x}{3}$	$\sin\frac{\pi x}{3}$
0	9	1	0
1	18	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
2	24	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
3	28	-1	0
4	26	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
5	20	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$

Using the values in the above table, we have

$$a_0 = \frac{2}{n}\sum y = \frac{2}{6}(125) = 41.67$$

$$a_1 = \frac{2}{n}\sum y\cos\frac{\pi x}{3} = \frac{2}{6}\left[9 - 28 + \tfrac{1}{2}(18 - 24 - 26 + 20)\right] = \tfrac{1}{3}[-19 - 6] = -8.33$$

$$b_1 = \frac{2}{n}\sum y\sin\frac{\pi x}{3} = \frac{2}{6}\left[\frac{\sqrt{3}}{2}(18 + 24 - 26 - 20)\right] = \frac{\sqrt{3}}{6}(-4) = -\frac{2\sqrt{3}}{3} = -1.15.$$

Example 4. *The following table gives the variations of periodic current over a period :*

t (secs.) :	*0*	*T/6*	*T/3*	*T/2*	*2T/3*	*5T/6*	*T*
A (amp.) :	*1.98*	*1.30*	*1.05*	*1.30*	*−0.88*	*−0.25*	*1.98*

Show, by numerical analysis, that there is a direct current part of 0.75 amp. in the variable current and obtain the amplitude of the first harmonic.

Sol. Hence length of interval is T *i.e.*, $l = \frac{T}{2}$

Let $$A = \frac{a_0}{2} + a_1\cos\frac{2\pi t}{T} + b_1\sin\frac{2\pi t}{T} + \ldots\ldots$$

where $\frac{a_0}{2}$ represents the direct current part and $\sqrt{a_1^2 + b_1^2}$ gives the amplitude of the first harmonic.

The values of t, A, $\sin \frac{2\pi t}{T}$, $\cos \frac{2\pi t}{T}$ are tabulated below :

t	A	$\cos \frac{2\pi t}{T}$	$\sin \frac{2\pi t}{T}$
0	1.98	1	0
T/6	1.30	0.5	0.866
T/3	1.05	– 0.5	0.866
T/2	1.30	– 1	0
2T/3	– 0.88	– 0.5	– 0.866
5T/6	– 0.25	0.5	– 0.866

Using the values in the above table, we have

$$a_0 = \frac{2}{n} \Sigma A = \frac{2}{6} (4.5) = 1.50$$

$$a_1 = \frac{2}{n} \Sigma A \cos \frac{2\pi t}{T} = \frac{2}{6} [1.98 - 1.30 + 0.5 (1.30 - 1.05 + 0.88 - 0.25)]$$

$$= \frac{1}{3} [0.68 + 0.5 (.88)] = \frac{1}{3} (1.12) = 0.373$$

$$b_1 = \frac{2}{n} \Sigma A \sin \frac{2\pi t}{T} = \frac{2}{6} [0.866 (1.30 + 1.05 + 0.88 + 0.25)]$$

$$= \frac{1}{3} [0.866 \times 3.48] = 1.005$$

$$\therefore \quad A = 0.75 + 0.373 \cos \frac{2\pi t}{T} + 1.005 \sin \frac{2\pi t}{T} + \ldots\ldots$$

Direct current part $= \frac{a_0}{2} = 0.75$ amp.

The amplitude of the first harmonic $= \sqrt{a_1^2 + b_1^2} = \sqrt{(0.373)^2 + (1.005)^2} = 1.072$.

Example 5. *The turning moment T units of the crank shaft of a steam engine is given for a series of values of the crank-angle θ in degrees :*

θ :	*0*	*30*	*60*	*90*	*120*	*150*	*180*
T :	*0*	*5224*	*8097*	*7850*	*5499*	*2626*	*0*

Find the first four terms in a series of sines to represent T. Also calculate T when θ = 75°.

Sol. Let the half-range sine series to represent T be

$$T = b_1 \sin \theta + b_2 \sin 2\theta + b_3 \sin 3\theta + b_4 \sin 4\theta$$

The values of θ, T, sinθ, sin 2θ, sin 3θ, sin 4θ are tabulated below :

θ	T	$\sin \theta$	$\sin 2\theta$	$\sin 3\theta$	$\sin 4\theta$
0	0	0	0	0	0
30	5224	0.5	0.866	1	0.866
60	8097	0.866	0.866	0	– 0.866
90	7850	1	0	– 1	0
120	5499	0.866	– 0.866	0	0.866
150	2626	0.5	– 0.866	1	– 0.866

Using the values in the above table, we have

$$b_1 = \frac{2}{n}\Sigma T \sin\theta$$

$$= \frac{2}{6}[0.5\,(5224 + 2626) + 0.866\,(8097 + 5499) + 7850]$$

$$= \frac{1}{3}(3925 + 11774 + 7850) = \frac{1}{3}(23549) = 7850$$

$$b_2 = \frac{2}{n}\Sigma T \sin 2\theta = \frac{2}{6}[0.866\,(5224 + 8097 - 5499 - 2626)]$$

$$= \frac{1}{3}[0.866 \times 5196] = 1500$$

$$b_3 = \frac{2}{n}\Sigma T \sin 3\theta = \frac{2}{6}[5224 - 7850 + 2626] = 0$$

$$b_4 = \frac{2}{n}\Sigma T \sin 4\theta = \frac{2}{6}[0.866\,(5224 - 8097 + 5499 - 2626)] = 0$$

$\therefore$ $T = 7850 \sin\theta + 1500 \sin 2\theta$

When $\theta = 75°$, $T = 7850 \sin 75° + 1500 \sin 150°$

$$= 7850 \times 0.9659 + 1500 \times 0.5 = 8332.$$

TEST YOUR KNOWLEDGE

1. The following values of y give the displacement of a certain machine part for the rotation x of the flywheel

x :	0°	60°	120°	180°	240°	300°	360°
y :	1.98	2.15	2.77	− 0.22	− 0.31	1.43	1.98

Express y in Fourier series upto the third harmonic.

2. The displacement y of a part of a mechanism is tabulated with corresponding angular movement $x°$ of the crank. Express y as a Fourier series neglecting the harmonics above the third.

$x°$:	0	30	60	90	120	150	180	210	240	270	300	330
y :	1.80	1.10	0.30	0.16	0.50	1.30	2.16	1.25	1.30	1.52	1.76	2.00

3. Determine the first two harmonics of the Fourier series for the following values :

$x°$:	30	60	90	120	150	180	210	240	270	300	330	360
y :	2.34	3.01	3.68	4.15	3.69	2.20	0.83	0.51	0.88	1.09	1.19	1.64

4. In a machine the displacement y of a given point is given for a certain angle θ as follows :

$\theta°$ =	0	30	60	90	120	150	180	210	240	270	300	330
y =	7.9	8.0	7.2	5.6	3.6	1.7	0.5	0.2	0.9	2.5	4.7	6.8

Find the co-efficient of sin 2θ in the Fourier series representing the above variation.

5. Obtain the first three co-efficients in the Fourier cosine series for y, where y is given in the following table :

x :	0	1	2	3	4	5
y :	4	8	15	7	6	2

6. The turning moment T on the crank-shaft of a steam engine for the crank angle θ degrees is given as follows :

θ :	0	15	30	45	60	75	90	105	120	135	150	165	180
T :	0	2.7	5.2	7.0	8.1	8.3	7.9	6.8	5.5	4.1	2.6	1.2	0

Expand T in a series of sines upto third harmonics.

Answers

1. $y = 1.3 + (0.92 \cos x + 1.097 \sin x) - (0.42 \cos 2x + 0.681 \sin 2x) + 0.36 \cos 3x$
2. $y = 1.26 + (0.04 \cos x - 0.63 \sin x) + (0.53 \cos 2x - 0.23 \sin 2x) + (-0.1 \cos 3x + 0.085 \sin 3x)$
3. $y = 2.102 - 0.283 \cos x + 1.60 \sin x - 0.18 \cos 2x - 0.49 \sin 2x$
4. -0.072
5. $y = 7 - 2.8 \cos\theta - 1.5 \cos 2\theta + 2.7 \cos 3\theta$
6. $T = 7.8 \sin\theta + 1.5 \sin 2\theta - 0.03 \sin 3\theta$