Sets, Relations and Functions

1 The Axiom of Extension

Sets (or collections) have elements (or members). Any object (concrete or otherwise), may be an element of a set. This extends to sets, which may be an element of some other set. In fact, it is generally sufficient to only discuss sets of sets, as the properties will extend to all other sets. There are some ways to relate sets: belonging, inclusion and equality.

If an object x is an element of set A, we say that it *belongs* to A, or "x is contained in A". Denote this as $x \epsilon A$.

If a set A is *included* in some set B, then A is a subset of B. Denote this as $A \subset B$ or $B \supset A$. Some related definitions: set inclusion is *reflexive* since a set is included in itself $(A \subset A)$[1]; set inclusion is *transistive* as $A \subset B$ and $B \subset C$ implies $A \subset C$. (Conversely, belonging is neither reflexive or transistive).

Sets can also be related by equality $(A = B)$, as defined by the Axiom of Extension.

Axiom 1 (Axiom of Extension)**.** *Two sets are equal if and only if they have the same elements.*

In terms of inclusion, this can be rewritten as "$A \subset B$ and $B \subset A \Leftrightarrow A = B$".

A set is said to be a *proper* subset of another if $A \subset B$ and $A \neq B$; that is, all subsets are proper except from the set itslf, which is a subset but not proper.

Equality is *symmetric*, that is $A = B \Leftrightarrow B = A$, whereas set inclusion is *antisymmetric* (that is, $A \subset B$ and $B \subset A \Rightarrow A = B$)[2].

As an aside, the Axiom of Extension implies that all elements of a set are distinct; that is to say, $\{1, 2, 2\} = \{1, 2\}$ (using notation introduced later). There is no element x that belongs to the left hand side for which x belonging to the right hand side isn't also true.

2 The Axiom of Specification

The basis of most Set Theoretic principles is to construct new sets from old sets. The most important such principle is the so called "*Aussonderungsaxiom*":

Axiom 2 (Axiom of Specification)**.** *To every set A and every condition S(x) there correponds a set B whose elements are exactly those elements x of A for which S(x) holds.*

That is to say, anything one can say about a the elements of a set defines a subset. The condition must form a valid *sentence*. A sentence is formed using the atomic sentences of belonging $(x \epsilon A)$ and equality $(A = B)$, which are used to create more specific sentences using the following logical operators:

- and
- or
- if and only if
- not
- if - then - (equivalently *implies*)

[1] Clearly, $A \subset B$ is the same as $A \subseteq B$ in this notation.

[2] Antisymmetry is defined by "if aRb and bRa, then $a = b$" for some binary relation R.

- for some (equivalently *there exists*)
- for all.

There are a few general rules for sentence construction[3]:

1. *and*, *or* and *if and only if* are placed between two sentences
2. *not* can be placed before a sentence enclosed by parentheses
3. the dashes in *if* - *then* - are replaced with sentences
4. follow *for some* or *for all* by a letter, in turn followed by a sentence

We can write the Axiom of Specification symbolically as: $B = \{x \epsilon A : S(x)\}$. This set B is uniquely specified due to the Axiom of Extension.

Remark 1 (Russell paradox)**.** *Nothing contains everything (or equivalently, there is no universe)*

Proof. Assume there is a set A which contains everything.
Consider the condition: not $(x \epsilon x)$. Let us rewrite this as $(x \epsilon' x)$ for ease of use.
Constructing the set B from A where this condition holds:

$$B = \{x \epsilon A : x \epsilon' x\}$$

Following from the Axiom of Extension,

$$y \epsilon B \text{ if and only if } (y \epsilon A \text{ and } y \epsilon' y)$$

For A to contain everything, this must hold for any y. Consider $y = B$. If $B \epsilon A$, then $B \epsilon B \Leftrightarrow B \epsilon' B$. This is clearly a contradiction so A does not contain B and by extension, A cannot contain everything. □

The Russell Paradox gives us an example of a condition that produces an illegal set. Some texts name these illegal sets as "classes".

3 Unordered Pairs

In order to make any progress it is necessary for us to make an assumption: *there exists a set*. This will be formulated more carefully later on but for now this is sufficient.

Given the assumption that there exists a set, one can easily construct the *empty set* using the Axiom of Specification, using the condition $x \neq x$. Clearly there are no elements of such a set. This set is unique by the Axiom of Extension. Denote this as $\{x \epsilon A : x \neq x\} = \varnothing$.

For any set A, $\varnothing \subset A$ as every element in $\varnothing$ is in A. This is a vacuous truth.[4]

Axiom 3 (Axiom of Pairing)**.** *For any two sets there exists a set that they both belong to.*

[3]Some general notes: the set of a single object is not the same as that object itself; "for some y $(x \epsilon A)$" is equivalent to "$x \epsilon A$"; "for some x $(x \epsilon A)$" and "for some y $(y \epsilon A)$" are equivalent.

[4]The arguement can also be made from the other direction: the only case where $\varnothing \subset A$ is false is where there is some element in $\varnothing$ that is not contained in A. Since $\varnothing$ is empty, this doesn't make any sense and so the statement can never be false.

That is, for some sets a and b, there exists a set A such that $a \epsilon A$ and $b \epsilon A$.

Consider the set that contains a and b and nothing else[5]. This can be constructed from the Axiom of Specification using the condition $S(x) =$ "$x = a$ or $x = b$". By the Axiom of Extension, this set is unique. We call this the *pair* or, more specifically, the *unordered pair* formed by a and b. Denote this set as

$$\{x \epsilon A : x = a \text{ or } x = b\} = \{a, b\}. \tag{1}$$

Note that this isn't the same as the set of all elements in a or b but the set containing only two elements: a and b.

The *singleton* of a is a special case unordered pair $\{a, a\}$, denoted by $\{a\}$, where the set has a as its only element. $a \epsilon A$ is equivalent to $\{a\} \subset A$.

Considering the unordered pair of any set and the empty set, it follows from the Axiom of Pairing that every set is an element of some other set. We can also infer the existance of infinitely many sets by construction: first consider the singletons $\varnothing, \{\varnothing\}, \{\{\varnothing\}\}$ etc.; now consider the pairs of such sets, and the pairs of these and the singletons and so on. Note that all of these sets are unique by the Axiom of Extension.

If $S(x)$ is a condition on x, then the set of all x that satisfy this condition is given by $\{x : S(x)\}$, granted that this is a valid set. Rewriting the generic set constructed by the Axiom of Specification,

$$\{x \epsilon A : S(x)\} = \{x : x \epsilon A \text{ and } S(x)\}.$$

4 Unions and Intersections

Axiom 4 (Axiom of Unions)**.** *For every collection of sets there exists a set that contains all the elements that belong to at least one set of the given collection.*

In other words, let $x \epsilon U$ only if $x \epsilon X$ for some X in collection $\mathcal{C}$[6]. Currently, this set U could contain elements not belonging to any of the sets in the collection. To construct a set that only contains elements belonging to some set in $\mathcal{C}$, use the Axiom of Specification with the condition:

$$\{x \epsilon U : x \epsilon X \text{ for some } X \text{ in } \mathcal{C}\}$$

Change the notation and name this new set U. This set is the *union* of the collection of sets (that is the set of sets) $\mathcal{C}$, which is unique by the Axiom of Extension. Denote this as

$$\bigcup \{X : X \epsilon \mathcal{C}\}. \tag{2}$$

We shall now introduce various facts regarding unions (proofs are in Appendix A). The facts regarding singletons are trivial.

Remark 2. $\bigcup \{X : X \epsilon \varnothing\} = \varnothing$

Remark 3. $\bigcup \{X : X \epsilon \{A\}\} = A$

There is special notation for the union of pairs of sets:

$$\bigcup \{X : X \epsilon \{A, B\}\} = A \cup B. \tag{3}$$

We may now introduce properties of pairs.

[5] The existence of such a set is actually equivalent to the Axiom of Pairing: if there is some set A such that $a \epsilon A$ and $b \epsilon A$, then we can construct a set with only a and b in using the sentence "$x = a$ or $x = b$".

[6] Set and collection can be used interchangably but both are used here to make it easier to discern what is being referred to.

Remark 4. *$x \epsilon A \cup B$ if and only if $x \epsilon A$ or $x \epsilon B$*

Thus, we can write:

$$A \cup B = \{x : x \epsilon A \text{ or } x \epsilon B\}. \tag{4}$$

Using this, we prove the following elementary properties of the union of pairs:

Remark 5. $A \cup \varnothing = A$

Remark 6 (Commutativity)**.** $A \cup B = B \cup A$

Remark 7 (Associativity)**.** $A \cup (B \cup C) = (A \cup B) \cup C$

Remark 8 (Idempotence)**.** $A \cup A = A$

Remark 9. *$A \subset B$ if and only if $A \cup B = B$*

Remark 10. $\{a\} \cup \{b\} = \{a, b\}$

Define $\{a, b, c\} = \{a\} \cup \{b\} \cup \{c\}$.

Remark 11. $\{a, b, c\} = \{x : x = a \textit{ or } x = b \textit{ or } x = c\}$

This remark means that, for every three sets, there exists a unique set that contains the three sets and nothing else. This is the *unordered triple* formed by them. Clearly, one can extend this arguement to *quadruples* and so on.

We develop a the topic of intersections in a way analagous to that of unions.

Remark 12. *For each non-empty collection of sets there exists a set that contains exactly those elements that belong to every set of the given collection.*

Proof. We can prove this by simply constructing such a set for a general non-empty collection using the Axiom of Specification. Given that the collection $\mathcal{C}$ is non-empty, let A be some set in $\mathcal{C}$. Thus we can write

$$V = \{x \epsilon A : x \epsilon X \text{ for every } X \text{ in } \mathcal{C}\}.$$

Given that A is included in the statement "for all X in $\mathcal{C}$", we can in fact write:

$$V = \{x : x \epsilon X \text{ for every } X \text{ in } \mathcal{C}\}. \tag{5}$$

□

The set V is the *intersection* of the collection $\mathcal{C}$, unique by the Axiom of Extension. Denote this by:

$$\bigcap \{X : X \epsilon \mathcal{C}\} \tag{6}$$

The following notation is used to denote the intersection of sets A and B (that is, of a pair):

$$A \cap B = \{x : x \epsilon A \text{ and } x \epsilon B\} \tag{7}$$

Using this, we prove the following elementary properties of the intersection of pairs (proofs in Appendix A):

Remark 13. $A \cap \varnothing = \varnothing$

Remark 14 (Commutativity)**.** $A \cap B = B \cap A$

Remark 15 (Associativity)**.** $A \cap (B \cap C) = (A \cap B) \cap C$

Remark 16 (Idempotence)**.** $A \cap A = A$

Remark 17. *$A \subset B$ if and only if $A \cap B = A$*

In the case where $A \cap B = \varnothing$, the pair of sets are called *disjoint*. For a collection of sets, if any two distinct sets in the collection are disjoint, the collection is called *pairwise disjoint*.

We shall now prove the *distributive laws*. As with many Set Theoretic proofs, to prove $A = B$, we show $A \subset B$ and $A \supset B$.

Remark 18. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof. If x belongs to the left side, then x belongs to A and to either B or C. If x is in B, then x is in $A \cap B$ and, if x is in C, then x is in $A \cap C$. Thus, x is in either $A \cap B$ or $A \cap C$ and so is in the right side in either case.

If x belongs to the right side, it belongs to either A and B or to A and C. In both cases, it belongs to A. It either belongs to B or C so x belongs to $B \cup C$. Thus, x is also in the left side in either case.

This is true for all x and so we can conclude equality by the Axiom of Extension. □

Remark 19. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof. If x belongs to the left side, then x belongs either to A or to both B and C. If x is in A, then x is in both $A \cup B$ and $A \cup C$. If x is in both B and C, then x is, once again, in both $A \cup B$ and $A \cup C$. Thus, in any case, x belongs to the right side. Thus the right side includes the left.

The reverse inclusion is proved by observing that if x belongs to both $A \cup B$ and $A \cup C$, then x belongs either to A or to both B and C. □

Here is a further proof to provide more examples of Set Theoretic proofs.

Remark 20. *A necessary and sufficient condition that $(A \cap B) \cup C = A \cap (B \cup C)$ is that $C \subset A$. Also, observe that the condition has nothing to do with the set B.*

Proof. First, we prove that $C \subset A$ implies $(A \cap B) \cup C = A \cap (B \cup C)$.

If $C \subset A$, then $x \in C$ implies that $x \in A$.

If $x \in B$, then $x \in (A \cap B)$ and $x \in C$. Thus, $x \in (A \cap B) \cup C$. Also, $x \in A$ and $x \in C$ so $x \in A \cap (B \cup C)$.

If $x \notin B$, then $x \notin A \cap B$ but $x \in C$ so $x \in (A \cap B) \cup C$. Also, $x \in A$ and $x \in B \cup C$ so $x \in A \cap (B \cup C)$.

This is true for all cases of x so by the Axiom of Extension, $(A \cap B) \cup C = A \cap (B \cup C)$. This is true independent of the set B.

Next, we prove that $(A \cap B) \cup C = A \cap (B \cup C)$ implies $C \subset A$.

Assume $(A \cap B) \cup C = A \cap (B \cup C)$.

If $x \in (A \cap B) \cup C$, either both $x \in A$ and $x \in B$ or $x \in C$. If $A \cap (B \cup C)$, either both $x \in A$ and $x \in B$ or both $x \in A$ and $x \in C$.

Thus, if $x \in C$, for the right side to be true, $x \in A$ must also be true. Thus, $x \in C$ implies $x \in A$ so $C \subset A$.

Once again, this argument is independent of the set B. □

A Proofs of Union and Intersection properties

Remark 21. $\bigcup \{X : X \epsilon \varnothing\} = \varnothing$

Proof. $\bigcup\{X : X \in \varnothing\}$ is the set of all elements of the sets that are, in turn, elements of $\varnothing$. This set has no elements, so by extension the set of these non-existant sets is also empty. □

Remark 22. $\bigcup\{X : X \in \{A\}\} = A$

Proof. Similarly, $\bigcup\{X : X \in \{A\}\}$ is the set of all elements in each set in the collection $\{A\}$. As this is a singleton, this is just the set A, by definition. □

Remark 23. *$x \in A \cup B$ if and only if $x \in A$ or $x \in B$*

Proof. If $x \in A$ or $x \in B$, by definition $x \in \{A, B\}$ so $x \in A \cup B$ immediately follows.

If $x \in A \cup B$, then the definition of the union states that $x \in X$ for some X in $\{A, B\}$. This is equivalent to saying $x \in A$ or $x \in B$. □

Thus, we can write:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}. \tag{8}$$

Using this, we prove the following elementary properties of the union of pairs:

Remark 24. $A \cup \varnothing = A$

Proof.

$$A \cup \varnothing = \{x : x \in A \text{ or } x \in \varnothing\}$$

$x \in \varnothing$ is always false so

$$A \cup \varnothing = \{x : x \in A\} = A$$

□

Remark 25 (Commutativity)**.** $A \cup B = B \cup A$

Proof.

$$\begin{aligned} A \cup B &= \{x : x \in A \text{ or } x \in B\} \\ &= \{x : x \in B \text{ or } x \in A\} \\ &= B \cup A \end{aligned}$$

where we use the fact that the logical operator *or* is commutative. □

Remark 26 (Associativity)**.** $A \cup (B \cup C) = (A \cup B) \cup C$

Proof.

$$\begin{aligned} A \cup (B \cup C) &= A \cup (\{x : x \in B \text{ or } x \in C\}) \\ &= \{x : x \in A \text{ or } (x \in B \text{ or } x \in C)\} \\ &= \{x : (x \in A \text{ or } x \in B) \text{ or } x \in C\} \\ &= (A \cup B) \cup C \end{aligned}$$

where we used the fact that the logical operator *or* is associative. □

Remark 27 (Idempotence)**.** $A \cup A = A$

Proof.

$$\begin{aligned} A \cup A &= \{x : x \in A \text{ or } x \in A\} \\ &= \{x : x \in A\} \\ &= A \end{aligned}$$

□

Remark 28. *$A \subset B$ if and only if $A \cup B = B$*

Proof. If $A \cup B = \{x : x \epsilon A \text{ or } x \epsilon B\} = B$, then there must be no case where $x \epsilon A$ where $x \epsilon B$ is not also true. Thus, any element $x \epsilon A$ must also be in B, so $A \subset B$ by definition.

If $A \subset B$, every element in A is also in B. Thus,

$$\begin{aligned} A \cup B &= \{x : x \epsilon A \text{ or } x \epsilon B\} \\ &= \{x : x \epsilon B\} \\ &= B. \end{aligned}$$

□

Remark 29. $\{a\} \cup \{b\} = \{a, b\}$

Proof.

$$\begin{aligned} \{a\} \cup \{b\} &= \{x : x \epsilon \{a\} \text{ or } x \epsilon \{b\}\} \\ &= \{x : x = a \text{ or } x = b\} \\ &= \{a, b\} \end{aligned}$$

□

Define $\{a, b, c\} = \{a\} \cup \{b\} \cup \{c\}$.

Remark 30. $\{a, b, c\} = \{x : x = a \textit{ or } x = b \textit{ or } x = c\}$

Proof.

$$\begin{aligned} \{a, b, c\} &= \{a\} \cup \{b\} \cup \{c\} \\ &= \{x : x = a \text{ or } x = b\} \cup \{c\} \\ &= \{x : x \epsilon \{x : x = a \text{ or } x = b\} \text{ or } x \epsilon \{c\}\} \\ &= \{x : x = a \text{ or } x = b \text{ or } x = c\}. \end{aligned}$$

□

The following are the proofs for the analagous properties of intersections of pairs.

Remark 31. $A \cap \varnothing = \varnothing$

Proof.

$$A \cap \varnothing = \{x : x \epsilon A \text{ and } x \epsilon \varnothing\}$$

$x \epsilon \varnothing$ is always false so there are no elements in this set. Thus, $A \cap \varnothing = \varnothing$ □

Remark 32 (Commutativity)**.** $A \cap B = B \cap A$

Proof.

$$\begin{aligned} A \cap B &= \{x : x \epsilon A \text{ and } x \epsilon B\} \\ &= \{x : x \epsilon B \text{ and } x \epsilon A\} \\ &= B \cap A \end{aligned}$$

where we use the fact that the logical operator *and* is commutative. □

Remark 33 (Associativity)**.** $A \cap (B \cap C) = (A \cap B) \cap C$

Proof.

$$\begin{aligned} A \cap (B \cap C) &= A \cap (\{x : x \epsilon B \text{ and } x \epsilon C\}) \\ &= \{x : x \epsilon A \text{ and } (x \epsilon B \text{ and } x \epsilon C)\} \\ &= \{x : (x \epsilon A \text{ and } x \epsilon B) \text{ and } x \epsilon C\} \\ &= (A \cap B) \cap C \end{aligned}$$

where we used the fact that the logical operator *and* is associative. □

Remark 34 (Idempotence)**.** $A \cap A = A$

Proof.

$$\begin{aligned} A \cap A &= \{x : x \epsilon A \text{ and } x \epsilon A\} \\ &= \{x : x \epsilon A\} \\ &= A \end{aligned}$$

□

Remark 35. *$A \subset B$ if and only if $A \cap B = A$*

Proof. If $A \cap B = \{x : x \epsilon A \text{ and } x \epsilon B\} = A$, then every element which belongs to A must also belong to B, as $x \epsilon B$ must be true for all such elements. Thus, any element $x \epsilon A$ must also be in B, so $A \subset B$ by definition.

Another way to see this is that $A \cap B \subset B$ by definition. Thus, by the initial assumption, $A \subset B$.

If $A \subset B$, every element for all $x \epsilon A$, $x \epsilon B$ is also true. Thus,

$$\begin{aligned} A \cap B &= \{x : x \epsilon A \text{ and } x \epsilon B\} \\ &= \{x : x \epsilon A\} \\ &= A. \end{aligned}$$

□