Sets, Relations and Functions

1 The Axiom of Extension

Sets (or collections) have elements (or members). Any object (concrete or otherwise), may be an element of a set. This extends to sets, which may be an element of some other set. In fact, it is generally sufficient to only discuss sets of sets, as the properties will extend to all other sets. There are some ways to relate sets: belonging, inclusion and equality.

If an object x is an element of set A, we say that it *belongs* to A, or "x is contained in A". Denote this as $x \in A$.

If a set A is *included* in some set B, then A is a subset of B. Denote this as $A \subset B$ or $B \supset A$. Some related definitions: set inclusion is *reflexive* since a set is included in itself ($A \subset A$)[1]; set inclusion is *transistive* as $A \subset B$ and $B \subset C$ implies $A \subset C$. (Conversely, belonging is neither reflexive or transistive).

Sets can also be related by equality ($A = B$), as defined by the Axiom of Extension.

Axiom 1 (Axiom of Extension)**.** *Two sets are equal if and only if they have the same elements.*

In terms of inclusion, this can be rewritten as "$A \subset B$ and $B \subset A \Leftrightarrow A = B$".

A set is said to be a *proper* subset of another if $A \subset B$ and $A \neq B$; that is, all subsets are proper except from the set itslf, which is a subset but not proper.

Equality is *symmetric*, that is $A = B \Leftrightarrow B = A$, whereas set inclusion is *antisymmetric* (that is, $A \subset B$ and $B \subset A \Rightarrow A = B$)[2].

2 The Axiom of Specification

The basis of most Set Theoretic principles is to construct new sets from old sets. The most important such principle is the so called "*Aussonderungsaxiom*":

Axiom 2 (Axiom of Specification)**.** *To every set A and every condition S(x) there correponds a set B whose elements are exactly those elements x of A for which S(x) holds.*

That is to say, anything one can say about a the elements of a set defines a subset. The condition must form a valid *sentence*. A sentence is formed using the atomic sentences of belonging ($x \in A$) and equality ($A = B$), which are used to create more specific sentences using the following logical operators:

- and
- or
- if and only if
- not
- if - then - (equivalently *implies*)
- for some (equivalently *there exists*)
- for all.

[1]Clearly, $A \subset B$ is the same as $A \subseteq B$ in this notation.

[2]Antisymmetry is defined by "if aRb and bRa, then $a = b$" for some binary relation R.

There are a few general rules for sentence construction[3]:

1. *and*, *or* and *if and only if* are placed between two sentences
2. *not* can be placed before a sentence enclosed by parentheses
3. the dashes in *if* - *then* - are replaced with sentences
4. follow *for some* or *for all* by a letter, in turn followed by a sentence

We can write the Axiom of Specification symbolically as: $B = \{x \epsilon A : S(x)\}$. This set B is uniquely specified due to the Axiom of Extension.

Remark 1 (Russell paradox)**.** *Nothing contains everything (or equivalently, there is no universe)*

Proof. Assume there is a set A which contains everything.

Consider the condition: not $(x \epsilon x)$. Let us rewrite this as $(x \epsilon' x)$ for ease of use.

Constructing the set B from A where this condition holds:

$$B = \{x \epsilon A : x \epsilon' x\} \tag{1}$$

Following from the Axiom of Extension,

$$y \epsilon B \text{ if and only if } (y \epsilon A \text{ and } y \epsilon' y) \tag{2}$$

For A to contain everything, this must hold for any y. Consider $y = B$. If $B \epsilon A$, then $B \epsilon B \Leftrightarrow B \epsilon' B$. This is clearly a contradiction so A does not contain B and by extension, A cannot contain everything. $\square$

[3]Some general notes: the set of a single object is not the same as that object itself; "for some y $(x \epsilon A)$" is equivalent to "$x \epsilon A$"; "for some x $(x \epsilon A)$" and "for some y $(y \epsilon A)$" are equivalent.