Sets, Relations and Functions

1 The Axiom of Extension

Sets (or collections) have elements (or members). Any object (concrete or otherwise), may be an element of a set. This extends to sets, which may be an element of some other set. In fact, it is generally sufficient to only discuss sets of sets, as the properties will extend to all other sets. There are some ways to relate sets: belonging, inclusion and equality.

If an object x is an element of set A, we say that it *belongs* to A, or "x is contained in A". Denote this as $x \epsilon A$.

If a set A is *included* in some set B, then A is a subset of B. Denote this as $A \subset B$ or $B \supset A$. Some related definitions: set inclusion is *reflexive* since a set is included in itself $(A \subset A)$[1]; set inclusion is *transistive* as $A \subset B$ and $B \subset C$ implies $A \subset C$. (Conversely, belonging is neither reflexive or transistive).

Sets can also be related by equality $(A = B)$, as defined by the Axiom of Extension.

Axiom 1 (Axiom of Extension)**.** *Two sets are equal if and only if they have the same elements.*

In terms of inclusion, this can be rewritten as "$A \subset B$ and $B \subset A \Leftrightarrow A = B$".

A set is said to be a *proper* subset of another if $A \subset B$ and $A \neq B$; that is, all subsets are proper except from the set itslf, which is a subset but not proper.

Equality is *symmetric*, that is $A = B \Leftrightarrow B = A$, whereas set inclusion is *antisymmetric* (that is, $A \subset B$ and $B \subset A \Rightarrow A = B)$[2].

As an aside, the Axiom of Extension implies that all elements of a set are distinct; that is to say, $\{1, 2, 2\} = \{1, 2\}$ (using notation introduced later). There is no element x that belongs to the left hand side for which x belonging to the right hand side isn't also true.

2 The Axiom of Specification

The basis of most Set Theoretic principles is to construct new sets from old sets. The most important such principle is the so called "*Aussonderungsaxiom*":

Axiom 2 (Axiom of Specification)**.** *To every set A and every condition S(x) there correponds a set B whose elements are exactly those elements x of A for which S(x) holds.*

That is to say, anything one can say about a the elements of a set defines a subset. The condition must form a valid *sentence*. A sentence is formed using the atomic sentences of belonging $(x \epsilon A)$ and equality $(A = B)$, which are used to create more specific sentences using the following logical operators:

- and
- or
- if and only if
- not
- if - then - (equivalently *implies*)

[1]Clearly, $A \subset B$ is the same as $A \subseteq B$ in this notation.

[2]Antisymmetry is defined by "if aRb and bRa, then $a = b$" for some binary relation R.

- for some (equivalently *there exists*)
- for all.

There are a few general rules for sentence construction[3]:

1. *and*, *or* and *if and only if* are placed between two sentences
2. *not* can be placed before a sentence enclosed by parentheses
3. the dashes in *if* - *then* - are replaced with sentences
4. follow *for some* or *for all* by a letter, in turn followed by a sentence

We can write the Axiom of Specification symbolically as: $B = \{x \in A : S(x)\}$. This set B is uniquely specified due to the Axiom of Extension.

Remark 1 (Russell paradox)**.** *Nothing contains everything (or equivalently, there is no universe)*

Proof. Assume there is a set A which contains everything.

Consider the condition: not $(x \in x)$. Let us rewrite this as $(x \in' x)$ for ease of use.

Constructing the set B from A where this condition holds:

$$B = \{x \in A : x \in' x\}$$

Following from the Axiom of Extension,

$$y \in B \text{ if and only if } (y \in A \text{ and } y \in' y)$$

For A to contain everything, this must hold for any y. Consider $y = B$. If $B \in A$, then $B \in B \Leftrightarrow B \in' B$. This is clearly a contradiction so A does not contain B and by extension, A cannot contain everything. □

The Russell Paradox gives us an example of a condition that produces an illegal set. Some texts name these illegal sets as "classes".

3 Unordered Pairs

In order to make any progress it is necessary for us to make an assumption: *there exists a set*. This will be formulated more carefully later on but for now this is sufficient.

Given the assumption that there exists a set, one can easily construct the *empty set* using the Axiom of Specification, using the condition $x \neq x$. Clearly there are no elements of such a set. This set is unique by the Axiom of Extension. Denote this as $\{x \in A : x \neq x\} = \varnothing$.

For any set A, $\varnothing \subset A$ as every element in $\varnothing$ is in A. This is a vacuous truth.[4]

Axiom 3 (Axiom of Pairing)**.** *For any two sets there exists a set that they both belong to.*

[3]Some general notes: the set of a single object is not the same as that object itself; "for some y $(x \in A)$" is equivalent to "$x \in A$"; "for some x $(x \in A)$" and "for some y $(y \in A)$" are equivalent.

[4]The arguement can also be made from the other direction: the only case where $\varnothing \subset A$ is false is where there is some element in $\varnothing$ that is not contained in A. Since $\varnothing$ is empty, this doesn't make any sense and so the statement can never be false.

That is, for some sets a and b, there exists a set A such that $a \epsilon A$ and $b \epsilon A$.

Consider the set that contains a and b and nothing else[5]. This can be constructed from the Axiom of Specification using the condition $S(x) =$ "$x = a$ or $x = b$". By the Axiom of Extension, this set is unique. We call this the *pair* or, more specifically, the *unordered pair* formed by a and b. Denote this set as

$$\{x \epsilon A : x = a \text{ or } x = b\} = \{a, b\}. \tag{1}$$

Note that this isn't the same as the set of all elements in a or b but the set containing only two elements: a and b.

The *singleton* of a is a special case unordered pair $\{a, a\}$, denoted by $\{a\}$, where the set has a as its only element. $a \epsilon A$ is equivalent to $\{a\} \subset A$.

Considering the unordered pair of any set and the empty set, it follows from the Axiom of Pairing that every set is an element of some other set. We can also infer the existance of infinitely many sets by construction: first consider the singletons $\varnothing, \{\varnothing\}, \{\{\varnothing\}\}$ etc.; now consider the pairs of such sets, and the pairs of these and the singletons and so on. Note that all of these sets are unique by the Axiom of Extension.

If $S(x)$ is a condition on x, then the set of all x that satisfy this condition is given by $\{x : S(x)\}$, granted that this is a valid set. Rewriting the generic set constructed by the Axiom of Specification,

$$\{x \epsilon A : S(x)\} = \{x : x \epsilon A \text{ and } S(x)\}.$$

4 Unions and Intersections

Axiom 4 (Axiom of Unions)**.** *For every collection of sets there exists a set that contains all the elements that belong to at least one set of the given collection.*

In other words, let $x \epsilon U$ only if $x \epsilon X$ for some X in collection $\mathcal{C}$[6]. Currently, this set U could contain elements not belonging to any of the sets in the collection. To construct a set that only contains elements belonging to some set in $\mathcal{C}$, use the Axiom of Specification with the condition:

$$\{x \epsilon U : x \epsilon X \text{ for some } X \text{ in } \mathcal{C}\}$$

Change the notation and name this new set U. This set is the *union* of the collection of sets (that is the set of sets) $\mathcal{C}$, which is unique by the Axiom of Extension. Denote this as

$$\bigcup \{X : X \epsilon \mathcal{C}\}. \tag{2}$$

We shall now introduce various facts regarding unions (proofs are in Appendix A). The facts regarding singletons are trivial.

Remark 2. $\bigcup \{X : X \epsilon \varnothing\} = \varnothing$

Remark 3. $\bigcup \{X : X \epsilon \{A\}\} = A$

There is special notation for the union of pairs of sets:

$$\bigcup \{X : X \epsilon \{A, B\}\} = A \cup B. \tag{3}$$

We may now introduce properties of pairs.

[5]The existence of such a set is actually equivalent to the Axiom of Pairing: if there is some set A such that $a \epsilon A$ and $b \epsilon A$, then we can construct a set with only a and b in using the sentence "$x = a$ or $x = b$".

[6]Set and collection can be used interchangably but both are used here to make it easier to discern what is being referred to.

Remark 4. *$x \in A \cup B$ if and only if $x \in A$ or $x \in B$*

Thus, we can write:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}. \tag{4}$$

Using this, we prove the following elementary properties of the union of pairs:

Remark 5. $A \cup \varnothing = A$

Remark 6 (Commutativity)**.** $A \cup B = B \cup A$

Remark 7 (Associativity)**.** $A \cup (B \cup C) = (A \cup B) \cup C$

Remark 8 (Idempotence)**.** $A \cup A = A$

Remark 9. *$A \subset B$ if and only if $A \cup B = B$*

Remark 10. $\{a\} \cup \{b\} = \{a, b\}$

Define $\{a, b, c\} = \{a\} \cup \{b\} \cup \{c\}$.

Remark 11. $\{a, b, c\} = \{x : x = a \textit{ or } x = b \textit{ or } x = c\}$

This remark means that, for every three sets, there exists a unique set that contains the three sets and nothing else. This is the *unordered triple* formed by them. Clearly, one can extend this arguement to *quadruples* and so on.

We develop a the topic of intersections in a way analagous to that of unions.

Remark 12. *For each non-empty collection of sets there exists a set that contains exactly those elements that belong to every set of the given collection.*

Proof. We can prove this by simply constructing such a set for a general non-empty collection using the Axiom of Specification. Given that the collection $\mathcal{C}$ is non-empty, let A be some set in $\mathcal{C}$. Thus we can write

$$V = \{x \in A : x \in X \text{ for every } X \text{ in } \mathcal{C}\}.$$

Given that A is included in the statement "for every X in $\mathcal{C}$", we can in fact write:

$$V = \{x : x \in X \text{ for every } X \text{ in } \mathcal{C}\}. \tag{5}$$

□

The set V is the *intersection* of the collection $\mathcal{C}$, unique by the Axiom of Extension. Denote this by:

$$\bigcap \{X : X \in \mathcal{C}\} \tag{6}$$

The following notation is used to denote the intersection of sets A and B (that is, of a pair):

$$A \cap B = \{x : x \in A \text{ and } x \in B\} \tag{7}$$

Using this, we prove the following elementary properties of the intersection of pairs (proofs in Appendix A):

Remark 13. $A \cap \varnothing = \varnothing$

Remark 14 (Commutativity)**.** $A \cap B = B \cap A$

Remark 15 (Associativity)**.** $A \cap (B \cap C) = (A \cap B) \cap C$

Remark 16 (Idempotence). $A \cap A = A$

Remark 17. $A \subset B$ *if and only if* $A \cap B = A$

In the case where $A \cap B = \varnothing$, the pair of sets are called *disjoint*. For a collection of sets, if any two distinct sets in the collection are disjoint, the collection is called *pairwise disjoint*.

We shall now prove the *distributive laws*. As with many Set Theoretic proofs, to prove $A = B$, we show $A \subset B$ and $A \supset B$.

Remark 18. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof. If x belongs to the left side, then x belongs to A and to either B or C. If x is in B, then x is in $A \cap B$ and, if x is in C, then x is in $A \cap C$. Thus, x is in either $A \cap B$ or $A \cap C$ and so is in the right side in either case.

If x belongs to the right side, it belongs to either A and B or to A and C. In both cases, it belongs to A. It either belongs to B or C so x belongs to $B \cup C$. Thus, x is also in the left side in either case.

This is true for all x and so we can conclude equality by the Axiom of Extension. □

Remark 19. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof. If x belongs to the left side, then x belongs either to A or to both B and C. If x is in A, then x is in both $A \cup B$ and $A \cup C$. If x is in both B and C, then x is, once again, in both $A \cup B$ and $A \cup C$. Thus, in any case, x belongs to the right side. Thus the right side includes the left.

The reverse inclusion is proved by observing that if x belongs to both $A \cup B$ and $A \cup C$, then x belongs either to A or to both B and C. □

Here is a further proof to provide more examples of Set Theoretic proofs.

Remark 20. *A necessary and sufficient condition that* $(A \cap B) \cup C = A \cap (B \cup C)$ *is that* $C \subset A$. *Also, observe that the condition has nothing to do with the set* B.

Proof. First, we prove that $C \subset A$ implies $(A \cap B) \cup C = A \cap (B \cup C)$.

If $C \subset A$, then $x \in C$ implies that $x \in A$.

Assume that $x \in C$. If $x \in B$, then $x \in (A \cap B)$ and $x \in C$. Thus, $x \in (A \cap B) \cup C$. Also, $x \in A$ and $x \in C$ so $x \in A \cap (B \cup C)$.

If $x \notin B$, then $x \notin A \cap B$ but $x \in C$ so $x \in (A \cap B) \cup C$. Also, $x \in A$ and $x \in B \cup C$ so $x \in A \cap (B \cup C)$.

This is true for all cases of x so by the Axiom of Extension, $(A \cap B) \cup C = A \cap (B \cup C)$. This is true independent of the set B.

Next, we prove that $(A \cap B) \cup C = A \cap (B \cup C)$ implies $C \subset A$.

Assume $(A \cap B) \cup C = A \cap (B \cup C)$.

If $x \in (A \cap B) \cup C$, either both $x \in A$ and $x \in B$ or $x \in C$. If $A \cap (B \cup C)$, either both $x \in A$ and $x \in B$ or both $x \in A$ and $x \in C$.

Thus, if $x \in C$, for the right side to be true, $x \in A$ must also be true. Thus, $x \in C$ implies $x \in A$ so $C \subset A$.

Once again, this argument is independent of the set B. □

5 Complements and Powers

The *difference* between sets A and B, or *relative complement* of B in A, is defined by

$$A - B = \{x : x \in A \text{and} x \notin B\}. \tag{8}$$

Consider some set E and its collection of subsets. We can then consider the complement of one of these subsets (with respect to E) as the absolute complement. Denote the (temporarily) absolute complement of A as A'. With this notation, we can state various basic facts (proofs in Appendix B):

Remark 21. $(A')' = A$

Remark 22. $A \subset B$ *if and only if* $B' \subset A'$

The following properties are known as the *De Morgan laws*:

$$(A \cup B)' = A' \cap B' \tag{9}$$

$$(A \cap B)' = A' \cup B' \tag{10}$$

It is common for set theory properties to come in pairs. That is, replacing every set by its complement, reversing inclusions and swapping unions and intersetions produces another theorem. This is known as the *principle of duality.* The De Morgan laws are an example of this.

Remark 23. $A - B = A \cap B'$

Proof.

$$\begin{aligned} A - B &= \{x : x \in A \text{ and } x \notin B\} \\ &= \{x : x \in A \text{ and } x \in B'\} \\ &= A \cap B' \end{aligned}$$

□

The *symmetric difference* or *Boolean sum* of two sets, A and B, is defined by $A + B = (A - B) \cup (B - A)$. This is both commutative and associative.

Let us now formalise the set of all subsets of some set:

Axiom 5 (Axiom of powers)**.** *For each set there exists a collection of sets that contains among its elements all the subsets of the given set.*

Note that collection has been used instead of set to aid clarity. Symbolically, if E is some set, there exists some set $\mathcal{P}$ such that if $X \subset E$, then $X \in \mathcal{P}$. In order to construct a set that *only* contains the subsets of a given set, apply the axiom of specification with the condition that $X \subset E$. That is, the set is $\{X \in \mathcal{P} : X \subset E\}$. Given that any set that satisfies the condition also satisfies $X \in \mathcal{P}$ by definition, change the notation such that $\mathcal{P} = \{X : X \subset E\}$. This set is called the *power set*, more specifically notated as $\mathcal{P}(E)$ to highlight that it contains all the subsets of E.

How many elements does a power set contain? First, let's consider some examples. The set $\mathcal{P}(\emptyset) = \{\emptyset\}$. The power set of some singleton is $\mathcal{P}(\{a\}) = \{\emptyset, \{a\}\}$, while $\mathcal{P}(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. One may notice a pattern and assert the following (which we will consequently prove):

Remark 24. *The power set of a finite set with n-elements has* 2^n *elements.*

Proof. Base case: a finite set with zero elements (that is, the empty set), has $2^0 = 1$ element, that being $\{\emptyset\}$.

Assume the power set of a finite set with n-elements has 2^n elements.

The power set of a finite set with one additional element to the n-element set (that is, one containing $n+1$ elements) contains all of the 2^n subsets in the assumed case, along with the pairs of the new elements with each element. Thus, there are an additional 2^n elements in this power set compared with the n-element set. Thus, the $n+1$ element set contains $2^n + 2^n = 2 \times 2^n = 2^{n+1}$. This completes the inductive argument. □

We shall prove some more useful remarks.

Remark 25. $\mathcal{P}(E) \cap \mathcal{P}(F) = \mathcal{P}(E \cap F)$

Proof. Let $X \in \mathcal{P}(E)$ and $X \in \mathcal{P}$, such that $X \in \mathcal{P}(E) \cap \mathcal{P}(F)$. Thus, $X \subset E$ and $X \subset F$, so $x \in X \implies x \in E$ and $x \in F$. Given that $E \cap F = \{x : x \in E \text{ and } x \in F\}$, $x \in E \cap F$ so $X \subset E \cap F$ and thus, $X \in \mathcal{P}(E \cap F)$.

Now prove the opposite direction. Let $X \in \mathcal{P}(E \cap F)$, such that $X \subset E \cap F$. Thus, $x \in X \implies x \in E$ and $x \in F$. Thus, $X \subset E$ and $X \subset F$, or equivalently, $X \in \mathcal{P}(E)$ and $X \in \mathcal{P}$. Thus $X \in \mathcal{P}(E) \cap \mathcal{P}(F)$. □

Remark 26. $\mathcal{P}(E) \cup \mathcal{P}(F) \subset \mathcal{P}(E \cup F)$

Proof. Let $X \in \mathcal{P}(E)$. Thus, $X \in \mathcal{P}(E) \cup \mathcal{P}(F)$. Let $x \in X$. $X \subset E \implies x \in E$ so $x \in E \cup F$. Thus, $X \subset E \cup F$ so $X \in \mathcal{P}(E \cup F)$.

Let $Y \in \mathcal{P}(F)$. We can immediately conclude that $Y \in \mathcal{P}(E \cup F)$ by symmetry. Thus, $\mathcal{P}(E) \cup \mathcal{P}(F) \subset \mathcal{P}(E \cup F)$. □

Remark 27 (Monoticity of power sets)**.** *If $E \subset F$, then $\mathcal{P}(E) \subset \mathcal{P}(F)$.*

Proof. If $E \subset F$, then $x \in E \implies x \in F$. Recall that $\mathcal{P}(A) = \{X : X \subset A\}$. Let $X \in \mathcal{P}(E)$. For all $x \in X$, the definition of the power set implies that $x \in E$ and thus, $x \in F$. Therefore, $X \subset F$ so $X \in \mathcal{P}(F)$. Thus, $E \subset F \implies \mathcal{P}(E) \subset \mathcal{P}(F)$ □

The general forms of the De Morgan Laws can be written with the following notation:

$$\left(\bigcup_{X \in \mathcal{C}} X\right)' = \bigcap_{X \in \mathcal{C}} X' \tag{11}$$

$$\left(\bigcap_{X \in \mathcal{C}} X\right)' = \bigcup_{X \in \mathcal{C}} X' \tag{12}$$

Proof. We shall only prove the former, as both proofs are similar.

Let $\bigcup_{X \in \mathcal{C}} X = \{x : x \in X \text{ for some } X \in \mathcal{C}\}$ and $\bigcap_{X \in \mathcal{C}} X' = \{x : x \in X' \text{ for every } X \in \mathcal{C}\}$, where $\mathcal{C}$ is a collection of subsets of some set E.

If $x \in \left(\bigcup_{X \in \mathcal{C}} X\right)'$ then $x \notin \bigcup_{X \in \mathcal{C}} X$. Thus, there is no X in $\mathcal{C}$ for which $x \in X$. Thus, $x \in X'$ for every X in $\mathcal{C}$, and so we conclude that $x \in \bigcap_{X \in \mathcal{C}} X'$.

For the reverse direction, let $x \in \bigcap_{X \in \mathcal{C}} X'$. Thus, $x \in X'$ for every X in $\mathcal{C}$, or equivalently, there is no X in $\mathcal{C}$ which contains x. This is equivalent to $x \notin \bigcup_{X \in \mathcal{C}} X$, or $x \in (\bigcup_{X \in \mathcal{C}} X)'$.

Thus, we have shown that each side is a subset of the other and thus we have proven equality. □

6 Ordered Pairs

Consider an informal construction of the idea of "order", without having an intrinsic understanding of what order means. Consider the set of elements $\{a, b, c, d\}$ in the "order" (this has no intrinsic meaning as of yet) $c\ b\ d\ a$. Let the collection $\{\{c\}, \{c, b\}, \{c, b, d\}, \{c, b, d, a\}\}$ capture this order. Note that the elements in this set can be "reordered" without losing the ability to reconstruct the original order.

With the above in mind, we define the *ordered pair* of a and b to be $(a, b) = \{\{a\}, \{a, b\}\}$ where a is the *first coordinate* and b is the *second coordinate*. We must ensure that this definition uniquely determines the ordered pair.

Remark 28. *If (a, b) and (x, y) are ordered pairs and if $(a, b) = (x, y)$, then $a = x$ and $b = y$.*

Proof. Assert that, if $a = b$, then (a, b) is the singleton $\{a\}$. Conversely, if (a, b) is a singleton, then $\{a\} = \{a, b\}$ and so $b \in \{a\}$ by the axiom of extension. Thus, $a = b$.

Now consider the case where $a \neq b$. Both (a, b) and (x, y) contain one singleton and one pair respectively. If they are equal, they have the same elements so the singletons must be the same. Thus $\{a\} = \{x\}$ and thus $a = x$. Similarly, the pairs must be equal so $\{a, b\} = \{x, y\}$. Thus $b \in \{x, y\}$. If $b = x$, then $a = b$ and we create a contradiction. Thus $b = y$. □

We shall now construct a set that contains all the ordered pairs (a, b) where $a \in A$ and $b \in B$. Given that $\{a\} \subset A$ and $\{b\} \subset B$, then $\{a, b\} \subset A \cup B$. That is, $\{a\} \in \mathcal{P}(A \cup B)$ and $\{a, b\} \in \mathcal{P}(A \cup B)$. Thus $\{\{a\}, \{a, b\}\} \subset \mathcal{P}(A \cup B)$, or equivalently $\{\{a\}, \{a, b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B))$. Applying the axiom of extension and the axiom of specification allows us to construct a unique set that only contains the ordered pairs (a, b) where $a \in A$ and $b \in B$. This is the *Cartesian product* of A and B, denoted by:

$$A \times B = \{x : x = (a, b) \text{ for some } a \text{ in } A \text{ and for some } b \text{ in } B\}$$

The Cartesian product and its subsets are sets of ordered pairs. It is also true that every set of ordered pairs is a subset of some Cartesian product of two sets. We shall now prove this.

Remark 29. *If R is a set such that every element of R is an ordered pair, then there exist two sets A and B such that $R \subset A \times B$.*

Proof. Let $x \in R$ where $x = \{\{a\}, \{a, b\}\}$. We wish to extract a and b, so we take the union of R: $\{a, b\} \in \bigcup R$. Thus, $a \in \bigcup\bigcup R$ and $b \in \bigcup\bigcup R$. Taking both A and B to be $\bigcup\bigcup R$ satisfies the statement, that is, $R \subset \bigcup\bigcup R \times \bigcup\bigcup R$. □

Apply the axiom of specification to construct the smallest possible A and B, as below. These are the *projections* of R onto the first and second coordinates respectively.

$$A = \{a : \text{ for some } b((a, b) \in R)\}$$
$$B = \{b : \text{ for some } a((a, b) \in R)\}$$

7 Relations

A *relation* captures some connection between objects. A *binary relation* captures some connection between two objects. Specifically, a (binary) relation is a set of ordered pairs. For example, if X is the set of men and Y is the set of women, the relation could be the set of married couples, which is a subset of $X \times Y$. Looking at this from the opposite direction, a set is a relation if all of its elements are ordered pairs. We now introduce new notation: if $(x, y) \in R$, we write xRy.

The simplest (and least interesting) example of a relation is the empty set. This is a set of ordered pairs as it doesn't contain any elements that are *not* ordered pairs. Another example is the set of pairs (x, y) in $X \times X$ where $x = y$. This is simply the relation of equality in a set. The set of elements (x, A) of $X \times \mathcal{P}(X)$ (that is, $A \subset X$) where $x \in A$ is the relation of belonging between elements of a set and its subsets.

For relations, the projections onto the first and second coordinates are called the *domain* and the *range* respectively. Thus, in the example of married couple, the domain is the set of married men and the range is the set of married women. In the relation of equality, the range and domain are equal: ran $R =$ dom $R = X$. In the relation of belonging, ran $R = X$ and dom $R = \mathcal{P}(X) - \{\emptyset\}$. The empty set has been removed as it doesn't contain any element b for aRb. With the new notation,

$$\begin{aligned} dom\ R &= \{a : \text{ for some } b\ (aRb)\} \\ ran\ R &= \{b : \text{ for some } a\ (aRb)\} \end{aligned}$$

If a relation is a subset of $X \times Y$, it is said to be *from* X *to* Y. If a relation is a subset of $X \times X$, it is said to be *in* X.

For a relation R in X, it is *reflexive* if xRx for every $x \in X$. It is *symmetric* if $xRy \implies yRx$. It is *transitive* if xRy and $yRz \implies xRz$. A relation for which all three properties hold is an *equivalence relataion.* The simplest example is equality in X: clearly $x = x$ for every $x \in X$ so xRx for all x. $x = y \implies y = x$ so $xRy \implies yRx$. $x = y$ and $y = z \implies x = z$ so xRy and $yRz \implies xRz$. Another example is the Cartesian product $X \times Y$. Note that the empty set is vacuously an equivalence relation. An example of a relation that is not an equivalence relation is the set of ordered pairs (x, y) in $\mathbb{R}$ such that $x \leq y$. Clearly, $x \leq x$ for all $x \in X$; $x \leq y$ and $y \leq z \implies x \leq z$ is also obvious. However, $x \leq y$ doesn't imply $y \leq x$ (only when equality holds does this implication hold). Thus, this relation isn't symmetric and, as such, isn't an equivalence relation.

Given an equivalence relation R in X, for some $x \in X$, the set of elements y such that xRy is called an *equivalence class.* Note that, due to relflexivity, x is always in this set. Thus, every element in X is in some equivalence class. Denote the equivalence class of x as x/R. Denote the set of all equivalence classes of R as X/R. The set of equivalence classes can be seen to be a valid set by applying the axiom of specification to the power set of X, $\mathcal{P}(X)$:

$$X/R = \{Y \in \mathcal{P}(X) : \exists\, x \in X \text{ such that } (\forall y \in Y, xRy) \text{ and } (\forall y \in X,\ \text{if } xRy,\ \text{then } y \in Y)\}$$

That is, every element in X/R is a subset Y for which all the elements in Y are in a equivalence class with some x, and if an element in X is in an equivalence class with x, it is necessarily also in the subset Y.

A *partition* of X is a collection (or set) $\mathcal{C}$ of disjoint subsets whose union is exactly X. That is, every element of X is contained in exactly one of the subsets in the partition. Such a collection can be said to induce an equivalence relation: the relation is the set $x\ X/\mathcal{C}\ y$ where x and y are in the same subset in $\mathcal{C}$.

Remark 30. *Given an equivalence relation R in X, the set of equivalence classes of R is a partition $\mathcal{C}$ of X which induces the same equivalence relation. Conversely, for some partition $\mathcal{C}$ of X, the relation induced is an equivalence relation with the set of its equivalence classes being the same as the initial partition $\mathcal{C}$.*

Proof. We shall prove the second statement first, as this is trivial. For a partition $\mathcal{C}$, consider the induced relation: it induces $x\ X/\mathcal{C}\ y$ where x and y are in the same subset in $\mathcal{C}$. Thus,

$x\ X/\mathcal{C}\ x$ as an element always belongs to the same subset as itself, so the induced relation is reflexive. Similarly, if $x\ X/\mathcal{C}\ y$, then both x and y are in the same subset. Thus, $y\ X/\mathcal{C}\ x$ is also true, and the induced relation is symmetric. If $x\ X/\mathcal{C}\ y$ and $y\ X/\mathcal{C}\ z$, then x, y and z are all in the same subset in $\mathcal{C}$. Thus, $x\ X/\mathcal{C}\ z$ so the induced relation is transitive. Therefore the induced relation is an equivalence relation. By construction, the set of equivalence classes is exactly $\mathcal{C}$ (as two elements are only related if they are in the same subset).

Now for the first statement. Consider the set of equivalence classes of some equivalence relation R in X. The equivalence class of x is the set of elements y such that xRy, so every element is in at least one equivalence class (recall that reflexivity implies that the equivalence class of x contains at least x). Let x be in the equivalence class of some other element, $x_1 \in X$. Thus, x_1Ry. Due to the symmetry of equivalence relations, xRx_1 so the equivalence class of x and x_1 are the same. Thus, each element in X is also in exactly one equivalence class. Thus, the set of equivalence classes is a partition of X. If two elements are in the set in this collection, then by definition, they stand in relation to each other. Thus we can conclude that this partition induces the equivalence relation R. $\square$

Thus, R and $X/\mathcal{C}$ represent the same relation, and X/R and $\mathcal{C}$ represent the same set of equivalence classes.

A Proofs of Union and Intersection properties

Remark 31. $\bigcup \{X : X \in \varnothing\} = \varnothing$

Proof. $\bigcup \{X : X \in \varnothing\}$ is the set of all elements of the sets that are, in turn, elements of $\varnothing$. This set has no elements, so by extension the set of these non-existant sets is also empty. $\square$

Remark 32. $\bigcup \{X : X \in \{A\}\} = A$

Proof. Similarly, $\bigcup \{X : X \in \{A\}\}$ is the set of all elements in each set in the collection $\{A\}$. As this is a singleton, this is just the set A, by definition. $\square$

Remark 33. *$x \in A \cup B$ if and only if $x \in A$ or $x \in B$*

Proof. If $x \in A$ or $x \in B$, by definition $x \in \{A, B\}$ so $x \in A \cup B$ immediately follows.

If $x \in A \cup B$, then the definition of the union states that $x \in X$ for some X in $\{A, B\}$. This is equivalent to saying $x \in A$ or $x \in B$. $\square$

Thus, we can write:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}. \tag{13}$$

Using this, we prove the following elementary properties of the union of pairs:

Remark 34. $A \cup \varnothing = A$

Proof.

$$A \cup \varnothing = \{x : x \in A \text{ or } x \in \varnothing\}$$

$x \in \varnothing$ is always false so

$$A \cup \varnothing = \{x : x \in A\} = A$$

$\square$

Remark 35 (Commutativity)**.** $A \cup B = B \cup A$

Proof.

$$\begin{aligned} A \cup B &= \{x : x \epsilon A \text{ or } x \epsilon B\} \\ &= \{x : x \epsilon B \text{ or } x \epsilon A\} \\ &= B \cup A \end{aligned}$$

where we use the fact that the logical operator *or* is commutative. □

Remark 36 (Associativity)**.** $A \cup (B \cup C) = (A \cup B) \cup C$

Proof.

$$\begin{aligned} A \cup (B \cup C) &= A \cup (\{x : x \epsilon B \text{ or } x \epsilon C\}) \\ &= \{x : x \epsilon A \text{ or } (x \epsilon B \text{ or } x \epsilon C)\} \\ &= \{x : (x \epsilon A \text{ or } x \epsilon B) \text{ or } x \epsilon C\} \\ &= (A \cup B) \cup C \end{aligned}$$

where we used the fact that the logical operator *or* is associative. □

Remark 37 (Idempotence)**.** $A \cup A = A$

Proof.

$$\begin{aligned} A \cup A &= \{x : x \epsilon A \text{ or } x \epsilon A\} \\ &= \{x : x \epsilon A\} \\ &= A \end{aligned}$$

□

Remark 38. $A \subset B$ *if and only if* $A \cup B = B$

Proof. If $A \cup B = \{x : x \epsilon A \text{ or } x \epsilon B\} = B$, then there must be no case where $x \epsilon A$ where $x \epsilon B$ is not also true. Thus, any element $x \epsilon A$ must also be in B, so $A \subset B$ by definition.

If $A \subset B$, every element in A is also in B. Thus,

$$\begin{aligned} A \cup B &= \{x : x \epsilon A \text{ or } x \epsilon B\} \\ &= \{x : x \epsilon B\} \\ &= B. \end{aligned}$$

□

Remark 39. $\{a\} \cup \{b\} = \{a, b\}$

Proof.

$$\begin{aligned} \{a\} \cup \{b\} &= \{x : x \epsilon \{a\} \text{ or } x \epsilon \{b\}\} \\ &= \{x : x = a \text{ or } x = b\} \\ &= \{a, b\} \end{aligned}$$

□

Define $\{a, b, c\} = \{a\} \cup \{b\} \cup \{c\}$.

Remark 40. $\{a, b, c\} = \{x : x = a \textit{ or } x = b \textit{ or } x = c\}$

Proof.

$$\begin{aligned}\{a,b,c\} &= \{a\}\cup\{b\}\cup\{c\}\\ &= \{x : x = a \text{ or } x = b\}\cup\{c\}\\ &= \{x : x \,\epsilon\, \{x : x = a \text{ or } x = b\} \text{ or } x \,\epsilon\, \{c\}\}\\ &= \{x : x = a \text{ or } x = b \text{ or } x = c\}.\end{aligned}$$

□

The following are the proofs for the analagous properties of intersections of pairs.

Remark 41. $A \cap \varnothing = \varnothing$

Proof.

$$A \cap \varnothing = \{x : x \,\epsilon\, A \text{ and } x \,\epsilon\, \varnothing\}$$

$x \,\epsilon\, \varnothing$ is always false so there are no elements in this set. Thus, $A \cap \varnothing = \varnothing$ □

Remark 42 (Commutativity)**.** $A \cap B = B \cap A$

Proof.

$$\begin{aligned}A \cap B &= \{x : x \,\epsilon\, A \text{ and } x \,\epsilon\, B\}\\ &= \{x : x \,\epsilon\, B \text{ and } x \,\epsilon\, A\}\\ &= B \cap A\end{aligned}$$

where we use the fact that the logical operator *and* is commutative. □

Remark 43 (Associativity)**.** $A \cap (B \cap C) = (A \cap B) \cap C$

Proof.

$$\begin{aligned}A \cap (B \cap C) &= A \cap (\{x : x \,\epsilon\, B \text{ and } x \,\epsilon\, C\})\\ &= \{x : x \,\epsilon\, A \text{ and } (x \,\epsilon\, B \text{ and } x \,\epsilon\, C)\}\\ &= \{x : (x \,\epsilon\, A \text{ and } x \,\epsilon\, B) \text{ and } x \,\epsilon\, C\}\\ &= (A \cap B) \cap C\end{aligned}$$

where we used the fact that the logical operator *and* is associative. □

Remark 44 (Idempotence)**.** $A \cap A = A$

Proof.

$$\begin{aligned}A \cap A &= \{x : x \,\epsilon\, A \text{ and } x \,\epsilon\, A\}\\ &= \{x : x \,\epsilon\, A\}\\ &= A\end{aligned}$$

□

Remark 45. *$A \subset B$ if and only if $A \cap B = A$*

Proof. If $A \cap B = \{x : x \,\epsilon\, A \text{ and } x \,\epsilon\, B\} = A$, then every element which belongs to A must also belong to B, as $x \,\epsilon\, B$ must be true for all such elements. Thus, any element $x \,\epsilon\, A$ must also be in B, so $A \subset B$ by definition.

Another way to see this is that $A \cap B \subset B$ by definition. Thus, by the initial assumption, $A \subset B$.

If $A \subset B$, every element for all $x \,\epsilon\, A$, $x \,\epsilon\, B$ is also true. Thus,

$$\begin{aligned}A \cap B &= \{x : x \,\epsilon\, A \text{ and } x \,\epsilon\, B\}\\ &= \{x : x \,\epsilon\, A\}\\ &= A.\end{aligned}$$

□

B Proofs of Complements and Powers properties

Remark 46. *$A \subset B$ if and only if $B' \subset A'$*

Proof. If $x \in B' \implies x \in A'$, then there is no case where x isn't in B where it is in A. Thus, if $x \in A$, it must be in B also so $x \in A \implies x \in B$.

For the reverse direction, if $x \in A \implies x \in B$ there is no case where x is in A where it isn't also in B. Thus, if x isn't in B, it can't be in A either. Thus, $x \in B' \implies x \in A'$. □

The following are the proofs of the De Morgan Laws:

Remark 47. $(A \cup B)' = A' \cap B'$

Proof. We shall prove this by showing that each side is a subset of the other side respectively.

Let $x \in (A \cup B)'$. That is, $x \notin A \cup B$. $A \cup B = \{y : y \in A \text{ or } y \in B\}$ so both $x \notin A$ and $x \notin B$. Thus, $x \in A'$ and $x \in B'$, which implies that $x \in A' \cap B'$. Thus, $(A \cup B)' \subset A' \cap B'$.

Let $x \in A' \cap B'$. Assume that $x \in A \cup B$. $x \in A' \cap B' \implies x \notin A$ and $x \notin B$ are both true. Thus there is no x for which $x \in A$ or $x \in B$ and so we have found a contradiction. Thus $x \in (A \cup B)'$ and we can conlude $(A \cup B)' \supset A' \cap B'$ □

Remark 48. $(A \cap B)' = A' \cup B'$

Proof. We shall prove this by showing that each side is a subset of the other side respectively.

Let $x \in (A \cap B)'$. That is, $x \notin A \cap B$. $A \cap B = \{y : y \in A \text{ and } y \in B\}$ so either $x \notin A$ or $x \notin B$ or both, which implies that $x \in A' \cup B'$ (as $A' \cup B' = \{x : x \in A' \text{ or } x \in B'\}$). Thus, $(A \cap B)' \subset A' \cup B'$.

Let $x \in A' \cup B'$. Assume that $x \in A \cap B$. $x \in A' \cup B' \implies x \notin A$ and $x \notin B$, which contradicts the assumption. Thus $x \in (A \cap B)'$ and we can conlude $(A \cap B)' \supset A' \cup B'$ □

Remark 49 (Commutativity)**.** $A + B = B + A$

Proof.

$$\begin{aligned} A + B &= (A - B) \cup (B - A) \\ &= \{x : x \in (A - B) \text{ or } x \in (B - A)\} \\ &= \{x : x \in (B - A) \text{ or } x \in (A - B)\} \\ &= B + A \end{aligned}$$

□

Remark 50 (Associativity)**.** $A + (B + C) = (A + B) + C$

Proof. We shall prove this by considering every case where some x belongs to some combination of A, B and C. There are eight such cases (draw a Venn diagram to convince yourself that these are all the cases):

- $x \in A, x \in B, x \in C$
- $x \in A, x \in B, x \notin C$
- $x \in A, x \notin B, x \in C$
- $x \notin A, x \in B, x \in C$
- $x \in A, x \notin B, x \notin C$
- $x \notin A, x \in B, x \notin C$

- $x \notin A, x \notin B, x \in C$
- $x \notin A, x \notin B, x \notin C$

If we show associativity holds for each case exhaustively, then it must hold generally.

Consider $x \in A, x \in B, x \in C$. $(B + C) = (B - C) \cup (C - B) = (B \cap C') \cup (C \cap B')$ We know that $x \notin C'$ so $x \notin B \cap C'$. Similarly, we know that $x \notin B'$ so $x \notin C \cap B'$. Thus $x \notin (B + C)$.

Let $D = (B + C)$. Thus, $A + (B + C) = A + D = (A \cap D') \cup (D \cap A')$. From above, $x \notin D$ so $x \in (A \cap D')$ and thus $x \in (A \cap D') \cup (D \cap A')$. Therefore, $x \in A + (B + C)$

We can make a similar argument to show that $x \in (A + B) + C$ but given the symmetry of this case (note that symmetric difference is commutative), we can immediately conclude this (see this by rewriting $A + (B + C) = (C + B) + A$).

Now consider $x \notin A, x \in B, x \in C$.

We know that $x \notin C'$ so $x \notin B \cap C'$. Similarly, we know that $x \notin B'$ so $x \notin C \cap B'$. Thus $x \notin (B + C)$ (or $x \notin D$).

$x \notin A \implies x \notin A \cap D'$. Similarly, $x \notin D \implies x \notin D \cap A'$. Thus, $x \notin A + D$ so $x \notin A + (B + C)$.

We know that $x \in A'$ so $x \in B \cap A'$. Thus $x \in (A + B)$.

Let $E = A + B$, such that $E + C = (E \cap C') \cup (C \cap E')$. We know that $x \notin C'$ so $x \notin E \cap C'$. Similarly, $x \in (A + B) \implies x \in E$ (or equivalently, $x \notin E'$) so $x \notin (C \cap E')$. Thus $x \notin E + C$ so $x \notin (A + B) + C$.

We can consider the rest of the cases by similar arguments (you should do this to complete the proof). In doing so, every value of x is exhaustively considered. There is no element belonging to $A + (B + C)$ that does not also belong to $(A + B) + C$ and vice versa.

Thus $A + (B + C) = (A + B) + C$ by the axiom of extension. $\square$