CS-663

HW4 - 6/11/2020

Q.3

$A_{m\times n}$, $m \leq n$; $P = A^T A$, $Q = AA^T$

a.) size $(y) = (n,1)$; size $(z) = (m,1)$

To prove - $y^t P y \geq 0$ and $z^t Q z \geq 0$

Put, $y^t P y = y^t (A^T A) y = (Ay)^T Ay$ $\quad (\because y^T A^T = (Ay)^T)$

But, $A_{m\times n} \times y_{n\times 1} =$ (a column vector)$_{m\times 1}$ (say $V_{m\times 1}$)

Now, $y^T P y = V^T V = (\text{norm}(V))^2 \geq 0$

Hence ~~proved~~ $y^T P y \geq 0$

~~Now, we know, $Q = AA^T = (A^T A)^T = P^T$~~

~~And we have~~ Similarly, $z^T Q z \geq 0$

(b) Now, assume, y is eigenvector of P & λ eigenvalue

$\Rightarrow \quad Py = \lambda y$

Multiply by y^T on both sides, $y^T P y = \lambda y^T y$

We know, $y^T y = (\text{norm}(y))^2 \geq 0$ & $y^T P y = \lambda y^T y \geq 0$

$\Rightarrow \boxed{\lambda \geq 0}$ Hence proved

Similarly for Q matrix

b.) Given :- $Pu = \lambda u$ & $Qv = \mu v$

To show -

$Q(Au) = \lambda(Au)$ and $P(A^Tv) = \mu(A^Tv)$

$Pu = \lambda u$; $Qv = \mu v$

$A^TAu = \lambda u$; $AA^Tv = \mu v$

Multiply by A ; Multiply by A^T on both sides

$AA^T(Au) = \lambda(Au)$; $A^TA(A^Tv) = \mu(A^Tv)$

$Q(Au) = \lambda Au$ and $P(A^Tv) = \mu(A^Tv)$

Proved

size$(u) = (n, 1)$; size$(v) = (m, 1)$

c.) Given - $Qv_i = \lambda_i v_i$ and $u_i = \dfrac{A^Tv_i}{||A^Tv_i||^2}$

Multiply by A in 2^{nd} expression

$$Au_i = \frac{AA^Tv_i}{||A^Tv_i||^2} = \frac{Qv_i}{||A^Tv_i||^2} = \frac{\lambda_i v_i}{||A^Tv_i||^2}$$

$\Rightarrow$ $Au_i = \gamma_i v_i$, where $\gamma_i = \dfrac{\lambda_i}{||A^Tv_i||^2}$

Now, we have proved in (a) part ~~that~~

that eigenvalues of $Q \geq 0$ and we know $||A^Tv_i||^2 \geq 0$

$\Rightarrow$ γ_i is non-negative d.) Given - $U = [v_1 \ v_2 \ v_3 \cdots v_m]$; $V = [u_1 \ u_2 \cdots u_m]$,

$\Gamma = diag[\gamma_1, \gamma_2 \cdots \gamma_m]$; $u_i^T u_j = 0$ and $v_i^T v_j = 0$ for $i \neq j$

Assumption - From (c) part $A u_i = \gamma_i v_i$ and $v_i^T v_i = u_i^T u_i = 1$

To show - $A = U \Gamma V^T$

Now, $AV = [Au_1 \ Au_2 \cdots Au_m]$

$= [\gamma_1 v_1 \ \gamma_2 v_2 \cdots \gamma_m v_m]$

$AV = U\Gamma$

Multiplying by V^T on both sides

$AVV^T = U \Gamma V^T$

Now, since $u_i^T u_j = 0$ for $i \neq j$ & $u_i^T u_i = 1$

$\Rightarrow VV^T = I$ (identity)

$\Rightarrow \boxed{A = U \Gamma V^T}$