Chapter : 1 Review of Network Analysis

There are two techniques:

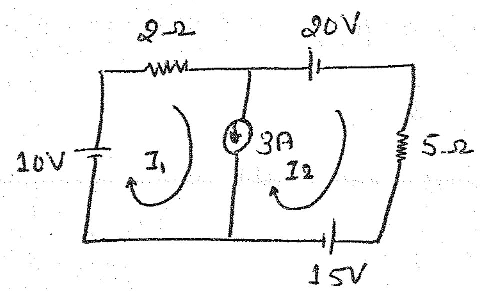
① Mesh analysis/ loop analysis

uses KVL

② Nodal Analysis

Uses KCL

Q.1 Finding the current I_1 and I_2



Soln: From given circuit,

$I_1 - I_2 = 3A$

$\Rightarrow I_1 = I_2 + 3$ ...... ①

Now, applying KVL to the supermesh corresponding to the current source, we get

$10 - 2I_1 - 20 - 5I_2 - 15 = 0$

or, $-2I_1 - 5I_2 - 25 = 0$

or, $-2I_1 - 5I_2 = 25$

or, $-2(I_2+3) - 5I_2 = 25$ [∵ From eqn. ①, $I_1 = I_2 + 3$]

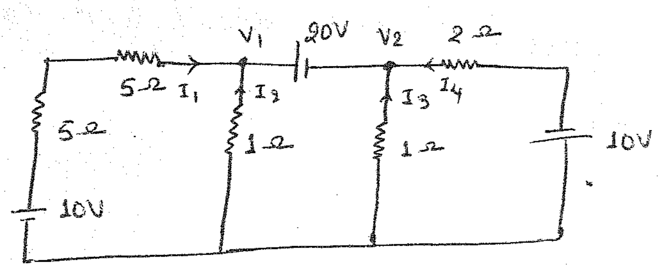
or, $-2I_2 - 6 - 5I_2 = 25$

or, $-7I_2 = 31$

or, $I_2 = -\frac{31}{7} = -4.43A$

and, $I_1 = -4.43 + 3 = -1.43A$

Q.2 Find V_1 and V_2.



Soln: From given circuit

$\Rightarrow V_1 - V_2 = 20V$

$V_1 = 20 + V_2$ ..... ①

Now, applying KCL to the supernode corresponding to the voltage source, we get

$$\frac{10-V_1}{10} + \frac{0-V_1}{1} + \frac{0-V_2}{1} + \frac{-10-V_2}{2} = 0$$

or, $\frac{10-V_1}{10} - V_1 - V_2 + \left(\frac{-10-V_2}{2}\right) = 0$

or, $10 - V_1 - 10V_1 - 10V_2 - 50 - 5V_2 = 0$

or, $-11V_1 - 15V_2 = 40$

or, $-11(20+V_2) - 15V_2 = 40$ .... [from eq^n. ① , $V_1 = V_2 + 20$]

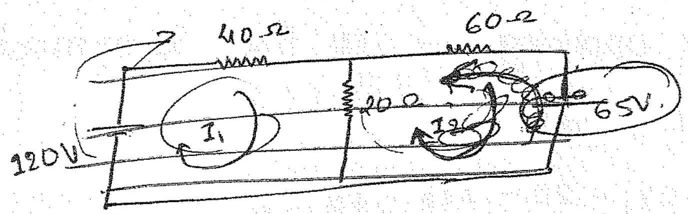
or, $-220 - 11V_2 - 15V_2 = 40$

or, $-26V_2 = 260$

$\therefore V_2 = -10V.$

and, $V_1 = -10 + 20 = 10V$

Q.3. Find I_1 and I_2



soln:

From loop 1st,

$120 - 40I_1 - 20(I_1 - I_2) = 0$

or, $120 - 40I_1 - 20I_1 + 20I_2 = 0$

or, $60I_1 - 20I_2 = 120$

or, $3I_1 - I_2 = 6$ ... ①

$\therefore I_2 = 3I_1 - 6$

Now, from loop 2nd,

$20(I_2 - I_1) + 60I_2 + 65 = 0$

or, $20I_2 - 20I_1 + 60I_2 - 65 = 0$

or, $80I_2 - 20I_1 = 65$

or, $80(3I_1 - 6) - 20I_1 = 65$ (from eq^n. ① , $I_2 = 3I_1 - 6$)

or, $240I_1 - 20I_1 = 65 + 480$

or, $I_1 = \frac{545}{220} = 2.48A$

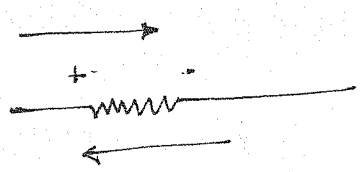
and, $I_2 = 3 \times 2.48 - 6 = 1.43$ A.

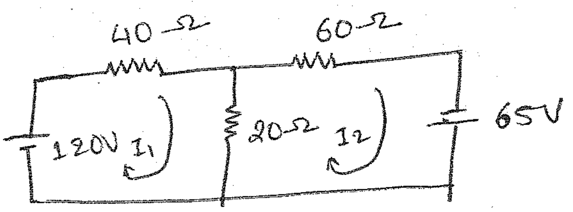
$120 - 40I_1 - 20(I_1 - I_2) = 0$

$-60I_2 - 65 - 20(I_2 - I_1) = 0$

$65 - 60I_2 - 20(I_2 - I_1)$

(−)
(+)





Chapter: 2 Circuit differential equations (Formulation and solutions)

Electronic circuit

Electronic circuit is the combination of the electronic components connected to form a network providing desired output interms of voltage or current. Three major components of circuit are:

i) Sources of energy
ii) Circuit elements
iii) Destination or load [can also be included in circuit elements]

classification of elements:

Circuit elements can be classified as:

i) Active and passive
ii) Linear and non-linear
iii) Unilateral and Bilateral

i) Active and passive elements:

Active:

With the ability to control flow of electron (diode, transistor, vacuum tubes, Silicon controlled rectifiers)

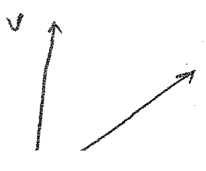
Passive:

passive electronic components are those that don't have the ability to control current by means of another electrical signal.

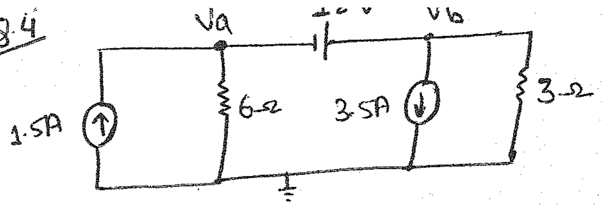
Examples: capacitors, resistors, inductors etc.

ii) Linear and non-Linear:

Linear:



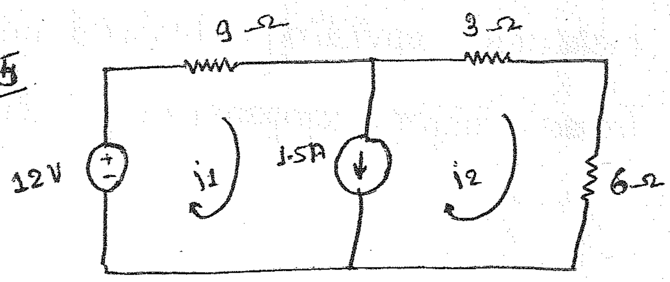
Q.4



Determine the values of the node voltages, V_a and V_b.

[Ans: $V_a = -12V$, $V_b = 0V$]

Q.5



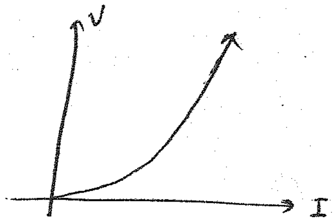
Find i_1 and i_2 [Ans: $I_1 = 1.4167A$ and $I_2 = -83.3mA$]

The characteristics of these components or voltage which
relationship is linear i.e. current or voltage increases in a constant proportion with respect to time. It follows ohms law ($v = IR$)

examples:

- Resistance, capacitor, inductor

Non-linear:



The output characteristics is not linear. It does not follow the ohm law.

examples:

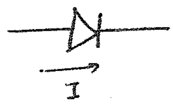
- diode, transistor, transformer, inductor (when the core is saturated)

iii) Unilateral and Bilateral

unilateral:

current flows in only one direction.

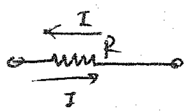
Eg: diode



bilateral

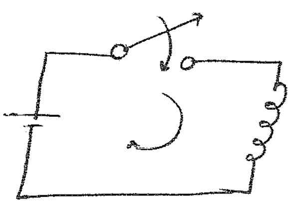
Current flows in both direction

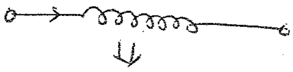
eg: Resistance



Inductance and capacitance behaviour:

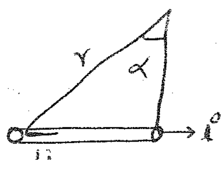
A. Inductance Behaviour:





N ⇒ No. of turns of coil

It produces magnetic field



Magnetic field, $dB = \frac{\mu i \cos\alpha \, dt}{}$ where,

$t = 0^-$ open

$t = 0$

$t = 0^+$ closed

We know,

$$V = \frac{d\psi}{dt}, \quad \psi = \text{Magnetic Flux}, \quad v = \text{induced voltage}$$

NOTE: Permeability, the degree of magetization of a material in response to a magnetic field]

where, Magnetic Flux,

$$\psi = Li \quad \ldots \text{①}$$

L is inductance of inductor

Now,

$$v = \frac{d(Li)}{dt} = L\frac{di}{dt} \quad (\because \text{from } i)$$

$$i = \frac{1}{L}\int_{-\infty}^{t} v\,dt$$

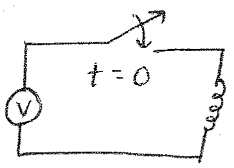
$$= \frac{1}{L}\int_{-\infty}^{0} v\,dt + \frac{1}{L}\int_{0}^{t} v\,dt$$

$$i = i(0^-) + i(t>0)$$

Thus, closing the switch at the instant $t=0$, maintains the same flux linkage before and after the switch is changed. Thus, " current can not changed instanteneously across inductor."

i.e. At, $t=0^-$ and $t=0^+$, current remains same.

For example,

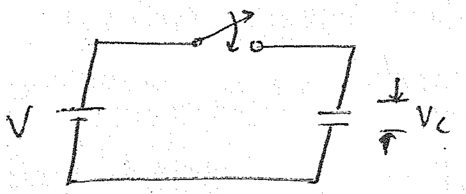


For above circuit, Switch is open at, $t=0^-$ and is closed at $t=0^+$

i.e. $i(0^-) \Rightarrow$ current at time $t=0^- = 0$ (zero)

Then, $t=0^+$

B. Capacitive behaviour:



We have, relationship between charge and current

$$i = \frac{dq}{dt} \quad \ldots \text{①}$$

Also, $q = CV$ where, C = capacitance

Differentiating on both side w.r.t. t,

$$\frac{dq}{dt} = \frac{d}{dt}(CV) = C\frac{dV}{dt}$$

Thus, $\frac{dV}{dt} = \frac{i}{C}$ ($\because$ from (i))

or, $V = \frac{1}{C}\int_{-\infty}^{t} i\,dt \Rightarrow \frac{1}{C}\int_{-\infty}^{0} i\,dt + \frac{1}{C}\int_{0}^{t} i\,dt$

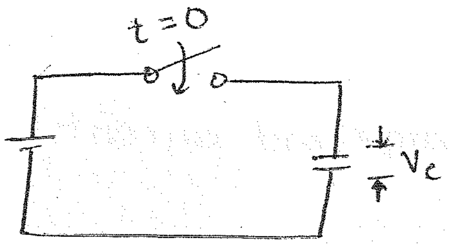
$V = V_c(0^-) + V_c(t>0)$

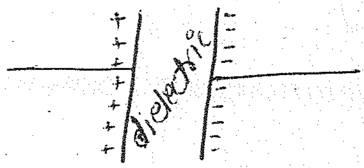
So, "Voltage can not change instanteneously ~~across~~ across the capacitor"

i.e. $V_c(0^-) = V_c(0^+)$

Capacitive Behaviour:

* current and charge relationship : $i = \frac{dq}{dt}$
* Voltage and charge relationship : $q = cV$





electric field produces

For capacitor, we have

$$q = cV$$

Differentiating with respect to t, we get

$$\frac{dq}{dt} = c\frac{dv}{dt}$$

or, $i = c\frac{dv}{dt} \Rightarrow dv = \frac{i}{c}dt$ ......①

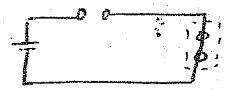
For, instanteneous change. $dt = 0$ i.e. $(t = 0^-) - (t = 0^+) = 0$

Now, equation ① becomes, $dv = 0$

Thus, "Voltage across capacitor can not change instanteneously

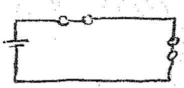
i.e. $V(t = 0^-) = V(t = 0^+)$.

At, $t = 0^-$, Let switch is open



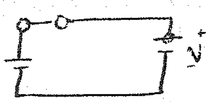
short circuit

and, at $t = 0$, switch closes



at $t = 0^+$, switch closes

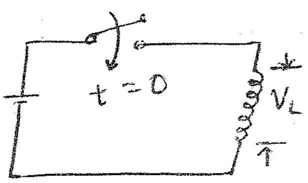
at $t = \infty$ (fully energised)



open circuit

As t increases voltage across capacitor also increases and acts as voltage source. At $t = \infty$, capacitance is fully charged and acts as open circuit i.e like voltage source.

Inductive Behaviour:



induced magnetic field

For inductor, flux ϕ is related with voltage by the relation

Where,

$\psi = Li$ Where, L = inductance

For $K = 1$, $v = \frac{d\psi}{dt}$

$v = \frac{d(Li)}{dt} = L\frac{di}{dt}$

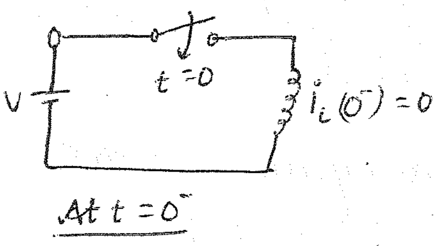
or, $di = \frac{v}{L}dt$ ...... ①

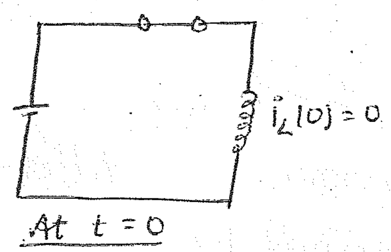
For instanteneous change, $dt = 0$, and if $dt = 0$, $di = 0$.

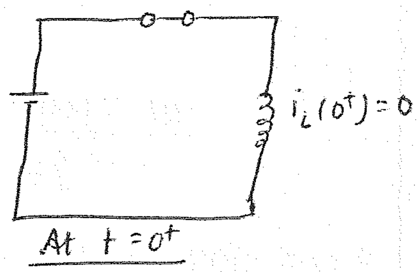
Thus, " Current across inductor can not change instanteneously

i.e. $i(t=0^-) = i(t=0^+)$ "

Let, switch is open at $t = 0^-$





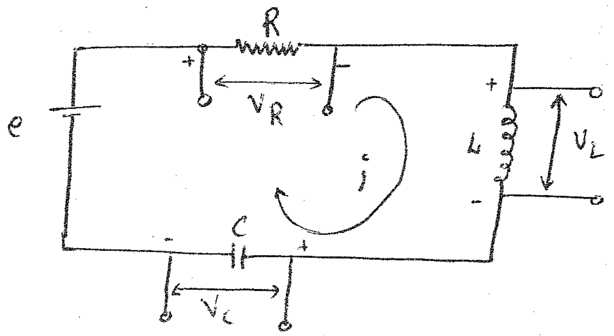


As t increases, $i_L(t)$ also increases and acts as a current source.

And, at $t = \infty$, inductor is fully energised and acts as short circuit.

Circuit differential equation

Let us consider a circuit with resistor of resistance R, inductor of inductance L and capacitor of capacitance C.



Here, $V_R = iR$, $V_L = L\frac{di}{dt}$ and, $V_C = \frac{1}{C}\int^{\infty} i dt$

Now, apply KVL in above circuit, we get

$$e = V_R + V_L + V_C$$

$$\Rightarrow e = iR + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{\infty} i\,dt$$

This equation is known as integro-differential equation.

Differentiating above equation with respect to t, we get

$$\Rightarrow 0 = R\frac{di}{dt} + L\frac{d^2i}{dt} + \frac{1}{C}i$$

$$\Rightarrow \boxed{L\frac{d^2i}{dt} + R\frac{di}{dt} + \frac{i}{C} = 0} \quad \text{.....} \; (a)$$

which is called differential equation.

Differential operator / P operator:

It changes the differential equation to the algebric expression and is defined by,

$$P(\;) = \frac{d}{dt}(\;) \quad \text{...} \; (i)$$

For any function f(t)

$$P f(t) = \frac{d}{dt} f(t) \quad \text{......} \; (ii)$$

Similarly,
$$\frac{d^n f(n)}{dt^n} = p^n f(t) \quad \text{....} \; (iii)$$

The above differential equation (a) becomes,

$$\boxed{L p^2 i + R p i + \frac{i}{C} = 0}$$

Similarly

$$\int_0^t i\,dt = \frac{i}{P} \quad \text{(only for finite)}$$

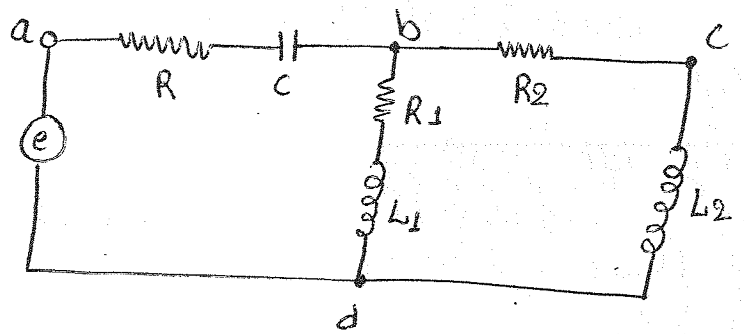
NOTE: Inverse operator is defined for finite integral only.

i.e. $\int^{t} i\,dt \neq \frac{i}{}$

Driving point operational impedance:

The total impedance of circuit viewed from voltage source is called driving point operational impedance.

1. Write the equation for the operational impedance for each branch of circuit and determine the expression for driving point operational impedance through terminal pair a-d.



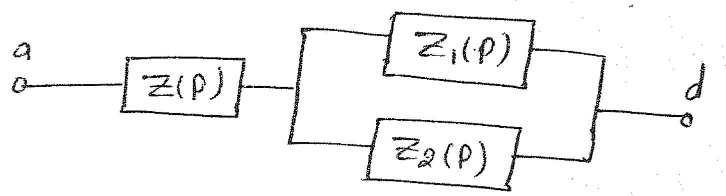
Sol^n:

For branch ab $\rightarrow$ $Z(P) = R + \frac{1}{PC}$

branch bd $\rightarrow$ $Z_1(P) = R_1 + PL_1$

branch bcd $\rightarrow$ $Z_2(P) = R_2 + L_2 P$

Now,



Thus, driving point impedance across terminal ad is,

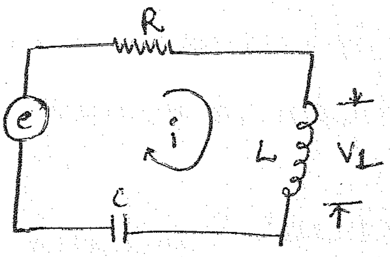
$$Z_{ad} = Z_p + Z_1(P) \,//\, Z_2(P)$$

$$= R + \frac{1}{PC} + \left\{(R_1 + L_1 P) // (R_2 + L_2 P)\right\}$$

$$= R + \frac{1}{PC} + \frac{(R_1 + L_1 P)(R_2 + L_2 P)}{R_1 + L_1 P + R_2 + L_2 P}$$

$$\boxed{Z_{ad} = R + \frac{1}{PC} + \frac{R_1 R_2 + R_1 L_2 P + R_2 L_1 P + L_1 L_2 P^2}{R_1 + R_2 + L_1 P + L_2 P}}$$

Q.2 Find the differential equation that relate V_L and e in the following circuit:



Sol^n: Method I

Applying KVL in the above circuit:

$$e = V_R + V_L + V_C$$

$$\Rightarrow e = iR + V_L + \frac{1}{C}\int i\,dt \quad \cdots ①$$

We have, $i = \frac{1}{L}\int V_L\,dt$

Now, equation ① becomes

$$e = \frac{R}{L}\int V_L\,dt + V_L + \frac{1}{LC}\int\int (V_L\,dt)\,dt \quad \cdots ⑪$$

Differentiating eq^n ⑪ w.r.to t

$$\frac{de}{dt} = \frac{R}{L}V_L + \frac{dV_L}{dt} + \frac{1}{LC}\int V_L\,dt$$

Again, differentiating w.r.to t

$$\frac{d^2e}{dt^2} = \frac{R}{L}\frac{dV_L}{dt} + \frac{d^2V_L}{dt} + \frac{1}{LC}V_L$$

$$\Rightarrow \boxed{\frac{d^2e}{dt^2} = \frac{d^2V_L}{dt^2} + \frac{R}{L}\frac{dV_L}{dt} + \frac{V_L}{LC}}$$, which is the required differential equation.

Method II:

We have, $Z_R = R$, $Z_L = LP$, $Z_C = \frac{1}{PC}$

Now, using voltage dividing rule, we get

$$V_L = \frac{e * Z_L}{Z_L + Z_R + Z_C}$$

6

or, $V_L = \dfrac{e \times LP}{R + LP + \frac{1}{PC}}$

or, $V_L = \dfrac{eLP \times PC}{RPC + LCP^2 + 1} = \dfrac{P^2 eLC}{P^2LC + PRC + 1}$

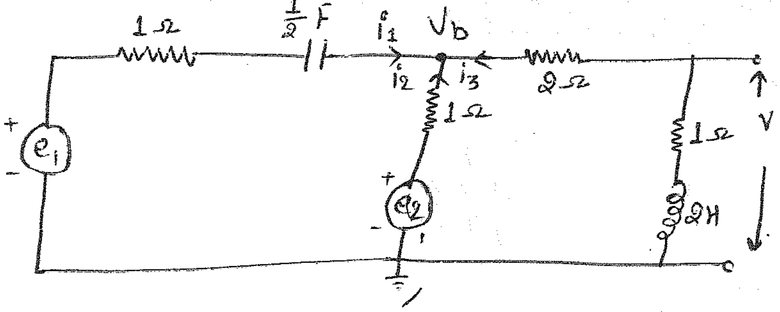
or; $P^2LCV_L + PRCV_L + V_L = P^2 eLC$

or, $P^2V_L + \frac{R}{L}PV_L + \frac{1}{LC}V_L = P^2 e$

Replacing, P by $\frac{d}{dt}()$ we get

$$\boxed{\frac{d^2(V_L)}{dt^2} + \frac{R}{L}\frac{dV_L}{dt} + \frac{1}{LC}V_L = \frac{d^2(e)}{dt^2}}$$, which is required differential equation,

Q3. Find the differential equation that relate V_o to the source voltage e_1 and e_2.



Soln: Applying nodal analysis at node b

$i_1 + i_2 + i_3 = 0$

$$\frac{e_1 - V_b}{1 + \frac{2}{P}} + \frac{e_2 - V_b}{1} + \frac{0 - V_b}{3 + 2P} = 0$$

$$\Rightarrow \frac{P(e_1 - V_b)}{P + 2} + (e_2 - V_b) + \left(\frac{-V_b}{3 + 2P}\right) = 0$$

$\Rightarrow P(e_1 - V_b)(3+2P) + (P+2)(3+2P)(e_2 - V_b) - V_b(P+2) = 0$

$\Rightarrow (Pe_1 - PV_b)(3+2P) + (P+2)(3e_2 - 3V_b + 2Pe_2 - 2PV_b) - V_bP - 2V_b = 0$

$\Rightarrow 3Pe_1 + 2P^2e_1 - 3PV_b - 2P^2V_b + 3Pe_2 - 3PV_b + 2P^2e_2 - 2P^2V_b + 6e_2 -$

$6V_b + 4Pe_2 - 4PV_b - V_bP - 2V_b = 0$

$\Rightarrow -3pV_b - 2p^2V_b - 3pV_b - 2p^2V_b - 6V_b - 4pV_b - V_bp - 2V_b = -3pe_1 - 2p^2e_1$
$-3pe_2 - 2p^2e_2 - 6e_2 + 4pe_2$

$\Rightarrow -4p^2V_b - 11pV_b - 8V_b = -e_1(2p^2+3p) - e_2(2p^2+7p+6)$

$$\Rightarrow V_b = \frac{e_1(2p^2+3p) + e_2(2p^2+7p+6)}{4p^2+11p+8} = \frac{e_1p(3+2p) + e_2(3+2p)(p+2)}{4p^2+11p+8}$$

Now, using voltage dividing rule,

$$V_o = \frac{V_b \times (1+2p)}{(3+2p)} = \frac{e_1p(3+2p) + e_2(3+2p)(p+2)}{(3+2p)\times(4p^2+11p+8)} \times (1+2p)$$

$$V_o = \frac{e_1p(1+2p) + e_2(p+2)(1+2p)}{4p^2+11p+8}$$

or, $4p^2V_o + 11pV_o + 8V_o = 2p^2e_1 + pe_1 + e_2(2p^2+5p+2)$

or, $4p^2V_o + 11pV_o + 8V_o = 2p^2e_1 + pe_1 + 2p^2e_2 + 5pe_2 + 2e_2$

Now, Replacing p by $\frac{d}{dt}()$

$$\boxed{4\frac{d^2}{dt^2}(V_o) + 11\frac{d}{dt}(V_o) + 8V_o = 2\frac{d^2}{dt^2}e_1 + \frac{de_1}{dt} + 2\frac{d^2e_2}{dt^2} + 5\frac{de_2}{dt} + 2e_2}$$

General formulation of Differential equation

General equation for differential equation is defined as,

$$a_0\frac{d^n}{dt^n} + a_1\frac{d^{n-1}}{dt^{n-1}} + a_2\frac{d^{n-2}}{dt^{n-2}} + \dots\dots = c$$

Where, n = order of circuit and
if $c=0$, the differential equation is called homogeneous equation.
and, if $c \neq 0$, the differential equation is called non homogeneous equation

For any electric circuit,

Response function = Network function * Source function

i.e. $y(t) = G(p) \cdot F(t)$

$= \frac{N(P)}{D(P)} \cdot f(t)$, where $y(t)$ = response function

$G(P)$ = network function

$N(P)$ = Numerator function

$D(P)$ = Denominator function

Solution of non-homogeneous differential equations:

General differential non-homogeneous equation is,

$y(t) = G(P) \cdot f(t)$

$\Rightarrow y(t) = \frac{N(P)}{D(P)} \cdot f(t)$

$\Rightarrow y(t) \cdot D(P) = N(P) \cdot f(t)$

Its solution consists of two parts:

① Particular solution / forced solution / steady - state response

② Complementary solution / Transient solution / Natural solution / Homogeneous solution.

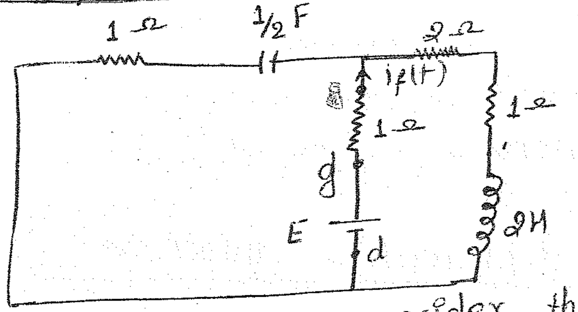
① Forced solution or particular solution

Let $f(t)$ be the applied source function to the network and $y(t)$ be its forced response. It is termed as forced because it remains in existance as long as the source function is applied. It is also called steady state response since it remains in the circuit long after the transient part disappears.

Math review:

Form of $f(t)$		Form of $y(t)$
Exponential	Ae^{st}	$D'e^{st}$
Sinusoidal	$A\sin wt$ $A\cos wt$	$D'\sin wt + B\cos wt$
constant	K	K'
Polynomial	t^n	$a_0 + a_1t_1 + a_2t_2 + \cdots + a_nt_n$

Force response to the constant source:



Let us consider the given circuit. Here, E is constant source and let us find force battery current $i_f(t)$, corresponding to the constant source E. The driving point impedance $Z_{gd}(P)$ is found by looking into the circuit through terminal g and d.

$$Z_{gd}(P) = \left[\left(1+\frac{2}{P}\right) // (3+2P)\right] + 1$$

$$= \frac{\left(\frac{P+2}{P}\right) * (3+2P)}{\frac{P+2}{P} + 3 + 2P} + 1$$

$$= \frac{3P + 2P^2 + 6 + 4P}{P + 2 + 3P + 2P^2} + 1$$

$$= \frac{2P^2 + 7P + 6 + 2P^2 + 4P + 2}{2P^2 + 4P + 2} = \frac{4P^2 + 11P + 8}{2P^2 + 4P + 2}$$

Then, $i_f(t) = \frac{E}{Z_{gd}(P)} = \frac{2P^2 + 4P + 2}{4P^2 + 11P + 8} E$

$\Rightarrow (4P^2 + 11P + 8)\, i_f(t) = (2P^2 + 4P + 2) E$

Replace P by $\frac{d}{dt}(\ )$ we get

$$4\frac{d^2}{dt^2} i_f(t) + 11\frac{d}{dt} i_f(t) + 8 i_f(t) = 2\frac{d^2E}{dt^2} + 4\frac{dE}{dt} + 2E$$

$\because$ E is constant so,

$$4\frac{d^2}{dt^2} i_f(t) + 11\frac{d}{dt} i_f(t) + 8 i_f(t) = 2E \quad \cdots ①$$

Let, the solution of differential equation is K, i.e. $i_f(t) = K$

Then equation ① becomes

$$4\frac{d^2}{dt^2}K + 11\frac{d}{dt}K + 8K = 2E$$

$\Rightarrow K = \frac{E}{4}$ $\quad \therefore i_f(t) = \frac{E}{4}$

Alternative - method :

We have,

$$i_f(t) = \frac{2p^2+4p+2}{4p^2+11p+8} E ,$$

Let, $p = 0$, then $i_f(t) = \frac{E}{4}$

Conclusion :

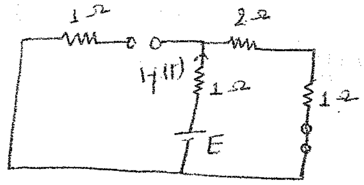
Thus, The force response due to constant source can be found as.

$y(t) = [G(p)]_{p=0} f(t)$

Concept :

As, $t \to \infty$ capacitor ⇒ Open circuit
inductor ⇒ short circuit

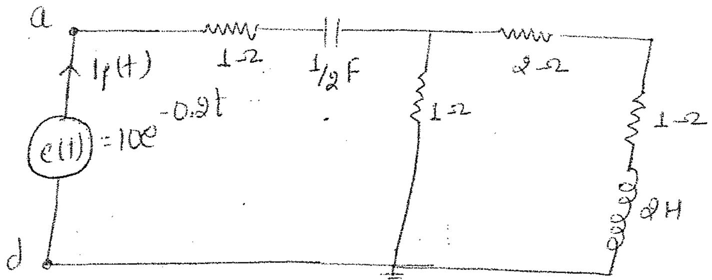
at, $t \to \infty$, the above circuit becomes.



$i_f(t) = \frac{E}{4}$

Forced response of the exponential source :

Let us consider the following circuit:



Here, $e(t) = 10e^{-0.2t}$ is the exponential source defined by equation, $e(t) = Ae^{st}$ ... ①

Differentiating ① with respect to t

$$\frac{d\,e(t)}{dt} = \frac{d}{dt} Ae^{st} = sAe^{st} = se(t)$$

∴ $pe(t) = se(t)$

i.e. differentiating to exponential function leads to algebric multiplication by s. As, integration is the inverse operation of differentiati

Now, we have

$y(t) = G(p) \cdot f(t)$

$\Rightarrow i_f(t) = \dfrac{e(t)}{Z_{ad}(p)}$ ...... (ii)

and,

$Z_{ad}(p) = \left(1+\frac{2}{p}\right) + \{1 \,//\, (3+2p)\} = \dfrac{4p^2+11p+8}{2p^2+4p}$ ...... (iii).

from (ii) and (iii)

$i_f(t) = \left(\dfrac{2p^2+4p}{4p^2+11p+8}\right) e(t) = \left(\dfrac{2p^2+4p}{4p^2+11p+8}\right) 10e^{-0.2t}$

or, $(4p^2+11p+8)\, i_f(t) = (2p^2+4p)\, 10e^{-0.2t}$

Now, replace p by $\frac{d}{dt}()$

$4\dfrac{d^2}{dt^2} i_f(t) + 11\dfrac{d}{dt} i_f(t) + 8 i_f(t) = 2\dfrac{d^2}{dt^2}(10e^{-0.2t}) + 4\dfrac{d}{dt}(10e^{-0.2t})$

~~We have~~, Let, $i_f(t) = Ae^{-0.2t}$ be the forced response.

Then,

$4\dfrac{d^2}{dt^2}(Ae^{-0.2t}) + 11\dfrac{d}{dt}(Ae^{-0.2t}) + 8Ae^{-0.2t} = 20\times(-0.2)\dfrac{d}{dt}e^{-0.2t} + 4\times(-0.2)\times10\, e^{-0.2}$

or, $4A(-0.2)(-0.2)e^{-0.2t} + 11A(-0.2)e^{-0.2t} + 8Ae^{-0.2t} = (-4\times0.2)e^{-0.2t} + (-0.8)\times10e^{-0.2t}$

or, $0.16Ae^{-0.2t} - 2.2Ae^{-0.2t} + 8Ae^{-0.2t} = 0.8e^{-0.2t} - 8e^{-0.2t}$

or, $5.96Ae^{-0.2t} = -7.2e^{-0.2t}$

$\therefore A = -1.21$

Thus, Required solution is, $i_f(t) = -1.21e^{-0.2t}$

Alternative method,

We have, $i_f(t) = \dfrac{2p^2+4p}{4p^2+11p+8} \times 10e^{-0.2t}$

replace p by s and put $s = -0.2$

$i_f(t) = \dfrac{2s^2+4s}{4s^2+11s+8}\, 10e^{-0.2t} = \dfrac{2\times(-0.2)^2+4\times(-0.2)}{4\times(0.2)^2+11(-0.2)+8} \times 10e^{-0.2t}$

$$= \frac{0.08 - 0.8}{0.16 - 2.2 + 8} \times 10 e^{-0.2t}$$

$$= \frac{-0.72 \times 10}{5.96} \times e^{-0.2t}$$

$$\therefore i_f(t) = -1.21 e^{-0.2t}$$

Conclusion :

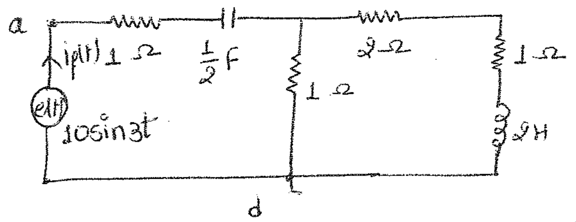
In general way, if source function is exponential i.e. $f(t) = Ae^{S_g t}$, then corresponding response function $y_f(t)$ is given by

$$y_f(t) = [G(P)]\, f(t)$$

$$y_f(t) = G(S)\Big|_{S=S_g} f(t)$$

Forced response to sinusoidal source:

Let us consider the following circuit:



If $I_f(t)$ be the force response due to sinusoidal function $10\sin 3t$ then $i_f(t) = \dfrac{e(t)}{Z_{ad}(P)}$. As from the circuit,

$$Z_{ad}(P) = \left(1 + \frac{2}{P}\right) + \left\{ 1 \,\|\, (3+2P) \right\} = \frac{4P^2 + 11P + 8}{2P^2 + 4P}$$

$$\text{Now, } i_f(t) = \frac{2P^2 + 4P}{4P^2 + 11P + 8} \cdot e(t) = \frac{2P^2 + 4P}{4P^2 + 11P + 8} * 10\sin 3t$$

$$\text{or, } (4P^2 + 11P + 8)\, i_f(t) = (2P^2 + 4P)\, 10\sin 3t$$

$$\text{Now, } (2P^2 + 4P)\, 10\sin 3t = 2\frac{d^2}{dt^2}(10\sin 3t) + 4\frac{d}{dt}(10\sin 3t)$$

$$= 20 \times 3 \times 3(-\sin 3t) + 40 \times 3\cos 3t$$

$$= -180\sin 3t + 120\cos 3t$$

Thus, we get

$(4p^2 + 11p + 8)\, i_f(t) = -180\sin 3t + 120\cos 3t$ ..... ①

Let, $i_f(t) = A\sin 3t + B\cos 3t$ be the form of solution.

Then, $4p^2 i_f(t) = 4\frac{d^2}{dt^2}(A\sin 3t + B\cos 3t) = -36A\sin 3t - 36B\cos 3t$

$11p\, i_f(t) = 11\frac{d}{dt}(A\sin 3t + B\cos 3t) = 33A\cos 3t - 33B\sin 3t$

Then, equation ① becomes,

or, $-36A\sin 3t - 36B\cos 3t + 33A\cos 3t - 33B\sin 3t + 8A\sin 3t + 8B\cos 3t = -180\sin 3t + 120\cos 3t$

or, $(-36A - 33B + 8A)\sin 3t + (-36B + 33A + 8B)\cos 3t = -180\sin 3t + 120\cos 3t$

comparing coefficient of $\sin 3t$ and $\cos 3t$

$-28A - 33B = -180$ ... ⓐ

$-28B + 33A = 120$ ... ⓑ

solving ⓐ and ⓑ, we get

$A = 4.8$ and $B = 1.38$

Thus,

$i_f(t) = 4.8\sin 3t + 1.38\cos 3t$

Alternative method:

We have, $i_f(t) = \dfrac{2p^2 + 4p}{4p^2 + 11p + 8}\, 10\sin 3t$

$= \left[\dfrac{2p^2 + 4p}{4p^2 + 11p + 8} \times 10e^{j3t}\right]^*$

Now, replacing p by s and $s = 3j$ we get

$i_f^*(t) = \dfrac{2s^2 + 4s}{4s^2 + 11s + 8}\, 10e^{3jt} = \dfrac{2(3j)^2 + 4(3j)}{4(3j)^2 + 11(3j) + 8} \times 10e^{j3t}$

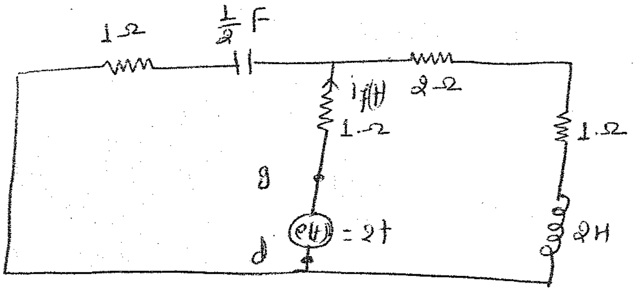
$= \dfrac{-18 + 12j}{-28 + 33j} \times 10e^{3jt} = \dfrac{21.6\angle 146.3}{48.3\angle 130.3} \times 10e^{3jt}$

$= 0.5\angle 16^\circ \times 10\angle 3t = 5e^{j(3t+16)}$

so $i_f(t) = \mathrm{Im}(i_f^*(t)) = 5\sin(3t + 16) = 4.8\sin 3t + 1.38\cos 3t$

Forced response of polynomial source:

Consider the following source:



$$i_f(t) = \frac{e(t)}{Z_{ad}(P)} \quad \cdots \text{①}$$

Where, $Z_{ad}(P) = 1 + \left\{\left(1+\frac{2}{P}\right) // (3+2P)\right\}$

$$= \frac{4P^2+11P+8}{2P^2+4P+2}$$

Now, eq^n. ① becomes

$$i_f(t) = \frac{2P^2+4P+2}{4P^2+11P+8}\, e(t)$$

or, $(4P^2+11P+8)\, i_f(t) = (2P^2+4P+2)\, 2t$

Let, $i_f(t) = A + Bt$, be the forced solution then,

$$\Rightarrow 4\frac{d^2}{dt^2}(A+Bt) + 11\frac{d}{dt}(A+Bt) + 8(A+Bt) = 2\frac{d^2}{dt^2}(2t) + 4\frac{d}{dt}(2t) + 4t$$

or, $0 + 11B + 8A + 8Bt = 0 + 8 + 4t$

or, $(11B+8A) + (8B)t = 8 + (4)t$

comparing constant and coefficient of t

$\therefore 8B = 4 \Rightarrow B = 1/2$

$\therefore 11B + 8A = 8$

or, $5.5 + 8A = 8$

or, $8A = 2.5$

$$A = \frac{2.5}{8} = \frac{5}{16}$$

$\therefore i_f(t) = \frac{5}{16} + \frac{1}{2}t$ is the forced response due to source $e(t)$.

The natural Response or Transient Response:

→ Transient response represents the characteristic of current or voltage for the circuit having energy storing elements. The transient response is obtained as the solution of homogeneous

→ The source is made to zero then the differential equation converts to homogeneous version.

Generalised procedure for finding transient solution:

The general form of differential is given by

$$D(P)\cdot Y(t) = N(P)\cdot F(t) \quad \cdots (I)$$

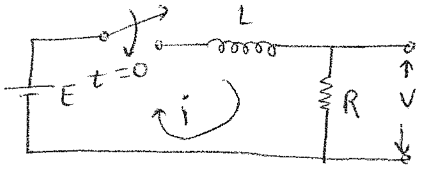
Equation (I) represents the non-homogeneous differential equation in operator form.

The homogeneous version of the differential equation is obtained by setting $f(t)=0$ i.e. $D(P)\cdot Y(t)=0 \quad \cdots (II)$

Since, Only exponential function qualifies as a solution to the differential equation, the p operator can be replaced by the algebric multiplier 's' so, $D(s)\,Y(t)=0 \cdots (III)$

$D(s)$ is the denominator polynomial of the network function $G(s)$ and $D(s)=0$ represents the characteristics equation.

Let us consider the following circuit:



Applying KVL when switch is closed, i.e. at $t=0^+$

$$E = LPi + Ri \quad \cdots (I)$$

Now, $V = \dfrac{E*R}{LP+R} \Rightarrow VLP + VR = ER \Rightarrow \dfrac{L}{R}VP + V = E$

$$\Rightarrow \frac{L}{R}\frac{dv}{dt} + V = E \quad \cdots (II)$$, which is non-homogeneous equation

At steady state (i.e. $t\to\infty$), for constant source, put $p=0$ i.e. $\frac{d()}{dt}=0$

$V_f(t) = E \quad \cdots (III)$

At $t=0^-$, $i_L(0^-)=0=i_L(0^+)$

$V(0^+) = i_L(0^+)*R = 0$, so eq^n (III) is not for time $t=0^+$

Thus, equation (I) is not the solution for all the time.

Now, for transient response, the homogeneous version of differential

$$\frac{L}{R}\frac{dv}{dt} + v = 0 \quad \cdots (iv)$$

Since, only the exponential form satisfies the above condition. so,

Let, $V_t = ke^{st}$ be the transient solution.

Then from eq^n. (iv)

$$\frac{L}{R}\frac{d}{dt}(ke^{st}) + ke^{st} = 0$$

$$\Rightarrow \frac{L}{R}kse^{st} + ke^{st} = 0 \quad \text{or,} \quad ke^{st}\left(\frac{L}{R}s+1\right) = 0$$

i.e. $ke^{st} \neq 0$ so, $\frac{L}{R}s + 1 = 0$

$$\Rightarrow s = -\frac{R}{L}$$

$$\therefore V_t = ke^{-R/L\, t}$$

Then, total solution would be, $V = V_f + V_t$

$$= E + ke^{-R/L\, t} \quad \cdots (v)$$

Here, k is unknown and it can be found by using initial conditions.

Initial condition:

At $(t = 0^-)$

$i(0^-) = 0$, $V(0^-) = 0$ [since switch is open]

At $(t = 0^+)$, $i(0^+) = i(0^-) = 0$ [since current through ~~conductor~~ inductor can not change instanteneously]

$\Rightarrow V(0^+) = 0$

Now, for $t = 0^+$, eq^n. (v) becomes

$$V(0^+) = E + ke^{-R/L(0^+)} \quad \Rightarrow 0 = E + k \quad \therefore k = -E$$

Hence, equation (v) becomes,

$$V = E - Ee^{-R/L\, t} = E(1 - e^{-R/L\, t})$$

This is the required solution.

Initial conditions:

The calculation of current (i) and voltage (v) and their derivative at $t = 0^+$ are the initial conditions.

Negative(−) and positive(+) sign with time are used to differentiate between the time immediately before and immediately after the operation of the switch.

Initial conditions in elements

① Resistor:

$V = IR$

The current through resistance changes instanteneously if voltage across it changes instateneously.

i.e. $i(0^-) \neq i(0^+)$ and, $V(0^-) \neq V(0^+)$

② Inductor:

$V = L\frac{di}{dt}$

The current through the inductor can not change instanteneously. so,

$i(0^-) = i(0^+)$

③ Capacitor:

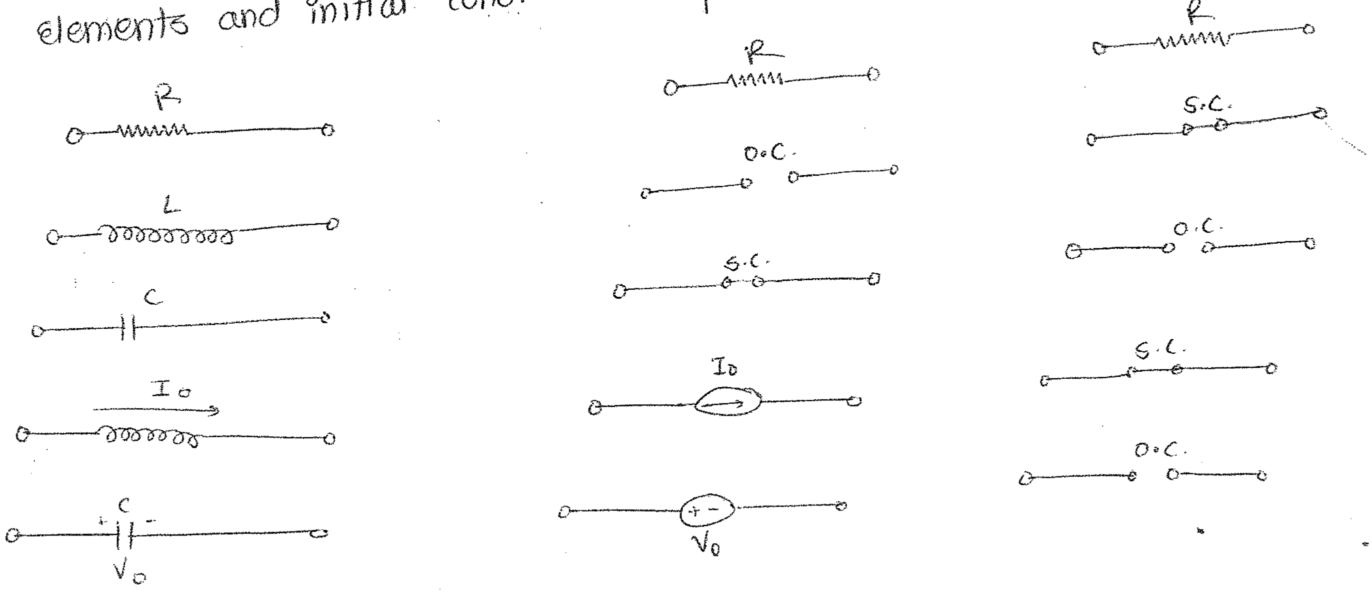
$V = \frac{1}{C}\int_0^t i(t)dt$

The voltage through the capacitor can not change instanteneously

so, $V(0^-) = V(0^+)$.

Summary:

elements and initial conditions — Equivalent at $t=0^+$ — Equivalent at $t=\infty$



procedure to find initial conditions

① Draw an equivalent circuit for $t=0^+$, based on the following rule:

② Replace all inductors by an open circuit if there is no initial current or

)

by current source if there is initial current.

(b) Replace all capacitor by short circuit or by voltage source depending on wheather there is initial voltage or not.

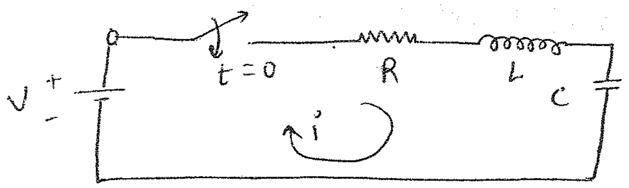
3) Find the values of initial current and voltage i.e. at $t = 0^+$ from equivalent circuits.

) To find higher derivatives of initial values we need to write integral or differential equation using KVL or KCL for original circuit.

umericals:

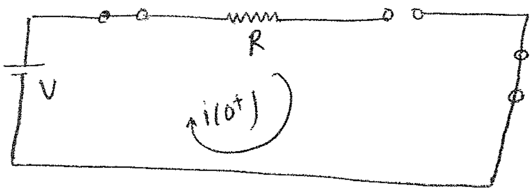
(1) In the circuit shown below: $V = 10V$, $R = 10\Omega$, $L = 1H$, $C = 10\mu F$.

Find $i(0^+)$, $\frac{di(0^+)}{dt}$, $\frac{d^2i(0^+)}{dt^2}$:



soln: At $t = 0^+$, the equivalent circuit is

[Before the switch is closed, $i(0^-) = 0$ & $V_c(0^-) = 0$].



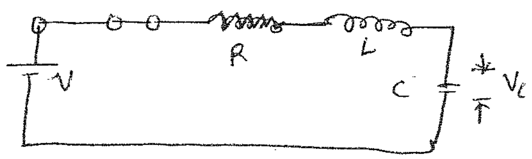
Here,

$$\boxed{i(0^+) = 0}$$

Using, KVL for $t > 0^+$

$$V = iR + L\frac{di}{dt} + V_c \quad \ldots (1)$$

equivalent circuit for $t > 0^+$



At, $t = 0^+$,

$$10 = i(0^+)R + \frac{L\,di(0^+)}{dt} + V_c(0^+)$$

Here, $V_c(0^+) = V_c(0^-) = 0$ since, Voltage across capacitor can not change instanteneously.

Now,

$$10 = \frac{di(0^+)}{dt}$$

$$\boxed{\therefore \frac{di(0^+)}{dt} = 10 \text{ amp/sec.}}$$

equation ① can be written as,

$$v = iR + L\frac{di}{dt} + \frac{1}{C}\int i\,dt \quad \ldots ⑪$$

Differentiating eqⁿ. ⑪ w.r.to t, we get

$$R\frac{di}{dt} + L\frac{d^2 i}{dt^2} + \frac{i}{C} = 0$$

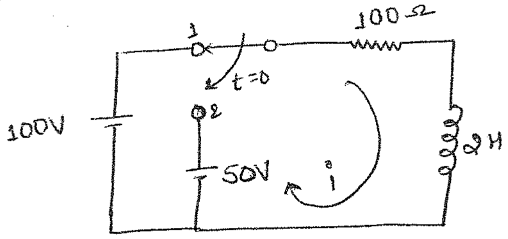
At, $t = 0^+$

$$R\frac{di(0^+)}{dt} + L\frac{d^2 i(0^+)}{dt^2} + \frac{i(0^+)}{C} = 0$$

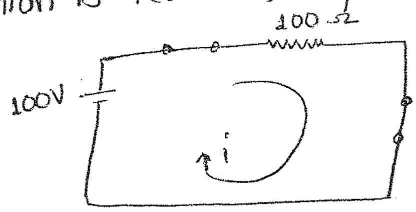
$$10 \times 10 + \frac{d^2 i(0^+)}{dt^2} + 0 = 0$$

$$\boxed{\therefore \frac{d^2 i(0^+)}{dt^2} = -100 \text{ amp/sec}^2}$$

(2) Find the total response for $t>0$ for i. If switch was initially on 1 until steady state is reached and moved to position 2 at time $t=0$.



solⁿ: When the switch is at position 1,
At $t=0^-$ (i.e. steady state condition is reached), equivalent circuit is,

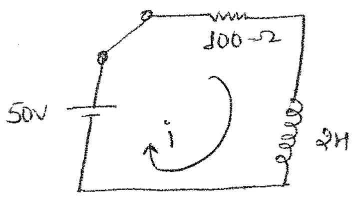


i.e. $100 = 100i \Rightarrow i = 1$ amp

$\therefore i(0^-) = 1$ amp

so, $\boxed{i(0^+) = i(0^-) = 1 \text{amp}}$ since, current through inductor can not change instanteneously.

At $t>0$ i.e. switch is at position 2



Applying KVL,

$$50 = 100i + 2\frac{di}{dt} \quad \ldots ①$$

or, $50 = 100i + 2pi$

$$\Rightarrow i = \frac{50}{100+2p}$$

For, Forced response, since source is constant, replace p by 0

then, $i_f(t) = \frac{50}{\;} = 0.5$ amp

For transient response,

Homogeneous version of equation ① is,

$$100i + 2\frac{di}{dt} = 0$$

Let, $i = Ke^{st}$

or, $100Ke^{st} + 2\frac{d}{dt}Ke^{st} = 0$

or, $100Ke^{st} + 2sKe^{st} = 0$

or, $Ke^{st}(100+2s) = 0$

i.e. $Ke^{st} \neq 0$

so, $100 + 2s = 0 \Rightarrow s = -50$.

$\therefore i_t(t) = Ke^{-50t}$

$\therefore$ The total response would be, $i(t) = i_f(t) + i_t(t)$
$= 0.5 + Ke^{-50t}$

Using initial condition:

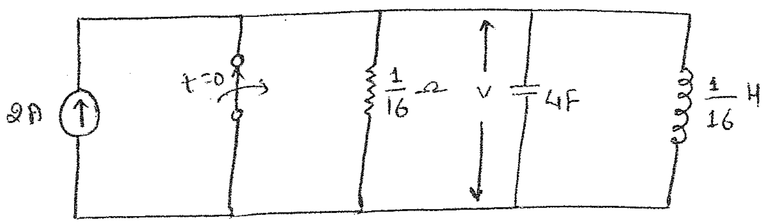
$i(0^+) = 0.5 + Ke^{-50(0^+)}$

or, $1 = 0.5 + K \Rightarrow K = 0.5$

Hence, Total response is, $i(t) = 0.5 + 0.5e^{-50t}$

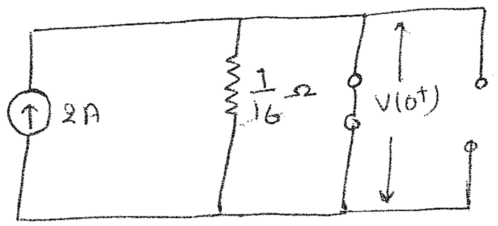
$$\boxed{\therefore i = 0.5(1+e^{-50t})}$$

③ Find $v(0^+)$, $\frac{dv(0^+)}{dt}$ and $\frac{d^2v(0^+)}{dt^2}$ for circuit below: [2011 Fall] [2012 Spring] 14F VC



Sol^n: At $t = 0^-$, switch is closed i.e. $i_R(0^-) = i_c(0^-) = i_L(0^-) = 0$ amp

At, $t = 0^+$, switch is opened. The equivalent circuit is

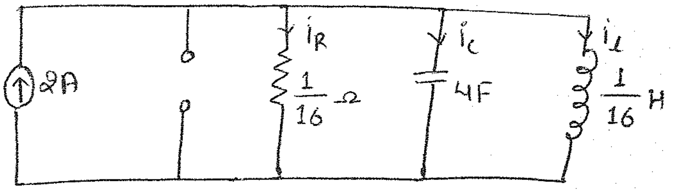


Then,

$v(0^+) = 2A \times \frac{1}{16}\Omega = \frac{1}{8}$ V.

$$\boxed{v(0^+) = \frac{1}{8}V}$$

At $t>0$, circuit is



using KCL;

$$2A = i_R + i_c + i_L$$

or, $2A = 16v + 4\frac{dv}{dt} + i_L \quad \cdots (1)$

For $t=0^+$

$$2 = 16\,V(0^+) + 4\frac{dv(0^+)}{dt} + i_L(0^+)$$

or, $2 = 16 \times \frac{1}{8} + 4\frac{dv(0^+)}{dt} + 0$

or, $\boxed{\frac{dv(0^+)}{dt} = \frac{1}{2}\ V/sec}$

Now, Differentiating (i) with respect to t,

$$0 = 16\frac{dv}{dt} + 4\frac{d^2v}{dt^2} + \frac{v}{L} \quad \left[\because i_L = \frac{1}{L}\int v\,dt\right]$$

At $t=0^+$

$$0 = 16\frac{dv(0^+)}{dt} + 4\frac{d^2v(0^+)}{dt^2} + \frac{v(0^+)}{1/16}$$

or, $0 = \frac{16\times 1}{2} + 4\frac{d^2v(0^+)}{dt^2} + \frac{1}{8}\times\frac{16}{1} \Rightarrow -8 = 4\frac{d^2v(0^+)}{dt^2}$

$\boxed{\therefore \frac{d^2v(0^+)}{dt^2} = -2\ V/s^2}$

Summary:

1. The voltage and current across inductor is given by, $V_L = L\frac{di_L}{dt}$ and $i_L = \frac{1}{L}\int V_L\,dt$
2. The current and voltage across capacitor is given by, $i_c = C\frac{dV_c}{dt}$ and $V_c = \frac{1}{C}\int i_c\,dt$
3. The voltage across capacitor and current across inductor can not change instanteneously.
4. The driving point operational impedance of a network is the overall impedance

expressed interms of p- operator and viewed across source.

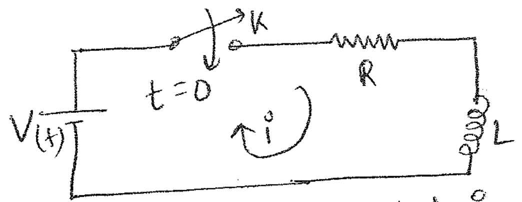
- The circuit equation for network having energy storing elements is non- homogeneous differential equation and the characteristics of circuit (i.e. current and voltage) is determined by solving non-homogeneous differential equation.
- The solution of non homogeneous differential equation consists of two parts i.e. Forced solution and transient solution.
- Forced solution represents the steady state behaviour of the circuit and nature of the source determines the nature of the forced response.
- Transient solution is obtained by solving the homogeneous version of differential equation and represents the changing characteristics of current or voltage.
- Total solution is the summation of forced response and transient response i.e. $i(t) = i_t(t) + i_f(t)$ and the solution is complete when we use initial condition to determine the value of constant.
- Initial condition gives the value of current and voltage and their derivatives at time $t = 0^+$.

Chapter. 3 Circuit Dynamics

It is the study of dynamic behaviour of linear circuits and systems containing one or more energy storing elements. It gives the information about how long it takes to the circuit to respond to a source function. The period of adjustment during which the stored energy changes from some initial level to a commanded final level is called the settling time of the circuit.

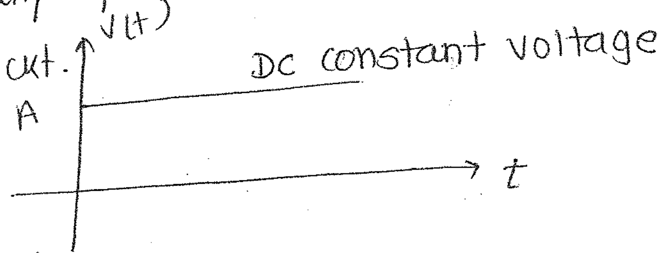
1. Step response to RL circuit

Let us consider the RL circuit as given below:



Here, Switch is initially opened and at $t=0$, it is closed.

We apply step response, to above ckt.



i.e. $V(t) = \begin{cases} 0 & \text{at } t<0 \\ A & \text{otherwise} \end{cases}$

At, $t=0^-$, $i(0^-) = 0$ amp and,

$i(0^+) = i(0^-) = 0$ amp .... (i)

[∵ Since, current through inductor can not change instanteneously.]

Now, at $t>0$ apply KVL

$$V = iR + L\frac{di}{dt} \quad \text{.... (ii)}$$

For forced response, above equation using p-operator is

$$V = iR + LPi$$

or, $$i = \frac{V}{R+LP}$$

For constant source, put $p=0$

$$\Rightarrow i_f(t) = \frac{V}{R} \quad \text{.... (iii)}$$

For transient response,

The characteristic equation for above differential equation is,

$$R + LS = 0$$

$$\Rightarrow S = -\frac{R}{L} \quad \ldots\ldots (iv)$$

And, homogeneous form of equation (ii) is,

$$L\frac{di}{dt} + iR = 0$$

Then,

transient solution, $i_t(t)$ is given by,

$$i_t(t) = Ke^{st} = Ke^{-\frac{R}{L}t}$$

Thus, total response becomes,

$$i(t) = i_p(t) + i_t(t)$$

$$= \frac{V}{R} + Ke^{-R/L\, t} \quad \ldots\ldots (v)$$

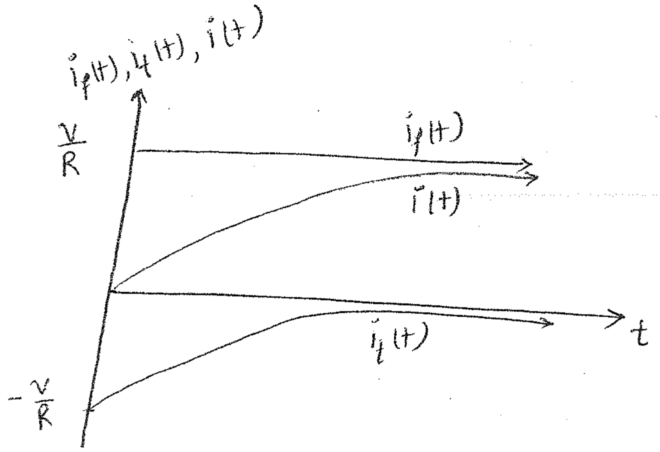
Now, using initial conditions, at $t = 0^+$

$$i(0^+) = \frac{V}{R} + Ke^{-\frac{R}{L}(0^+)}$$

$$\Rightarrow 0 = \frac{V}{R} + K \qquad \Rightarrow K = -\frac{V}{R}$$

Thus, total solution is,

$$\boxed{i(t) = \frac{V}{R} - \frac{V}{R}e^{-\frac{R}{L}t}}$$



Time constant

At, $t = 0$, $i(t) = 0$

at, $t = \infty$, $i(t) = \frac{V}{R}$

at, $t = \frac{L}{R}$, $i(t) = \frac{V}{R} - \frac{V}{R}\cdot e^{-\frac{R}{L}\cdot\frac{L}{R}}$

$$= \frac{V}{R} - \frac{V}{R}\cdot e^{-1} = \frac{V}{R}(1 - e^{-1})$$

$$= 0.632\ \frac{V}{R}$$

→ $\tau = \frac{L}{R}$ is known as the time constant of the circuit and is defined as the interval after which current or voltage changes 63.2% of its total change. OR,

Time constant is defined as time taken by the total response to reach 63.2% of the steady state value. it is represented by τ.

Again for,

$$i_t(t) \text{ at, } t = \frac{L}{R}, \quad i_t(t) = -\frac{V}{R} e^{-\frac{R}{L} \times \frac{L}{R}} = -\frac{V}{R} e^{-1} = -0.37 \frac{V}{R}$$

so, time constant is also defined as time taken by the transient response to decay to 37% of its initial value.

Settling time:

At,

$t = \tau$, $i_t(t) = -0.37 \frac{V}{R}$

$t = 2\tau$, $i_t(t) = -0.135 \frac{V}{R}$

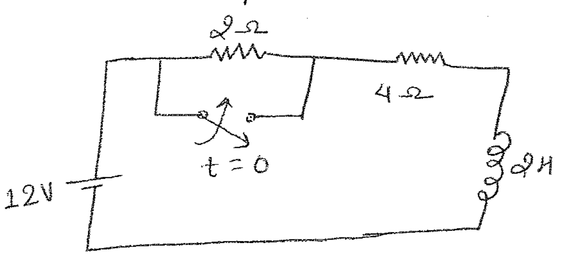
$t = 3\tau$, $i_t(t) = -0.0498 \, V/R$

$t = 4\tau$, $i_t(t) = -0.0183 \, V/R$

$t = 5\tau$, $i_t(t) = -0.0067 \, V/R$

Since, at $t = 5\tau$, transient response has almost decay to less than 1% of initial value and then circuit settles. so, settling time for above circuit is 5τ.

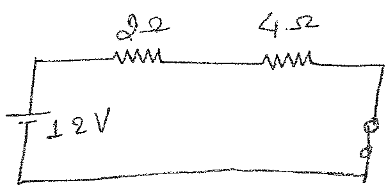
Q.1 Find the expression for $i(t)$ for $t > 0$



Also, find time constant and settling time.

Soln: At $t = 0^-$,

equivalent circuit is,

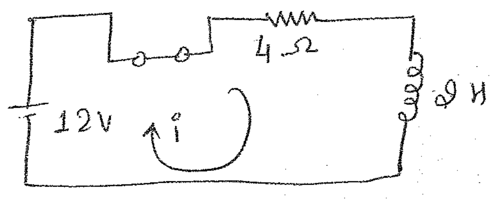


$$i(0^-) = \frac{12V}{6\Omega} = 2 \text{ amp}$$

~~At $t = 0^+$, equivalent circuit is,~~ $\therefore i(0^-) = i(0^+) = 2\text{amp}$ ($\because$ since current ... instantaneously)

)

Using KVL at $t>0$



$$12 - 4i - 2\frac{di}{dt} = 0 \quad \cdots \text{①}$$

In P operator form

$$12 = 4i + 2pi \quad \Rightarrow i = \frac{12}{4+2p}$$

for forced response.

$$i_f(t) = \frac{12}{4+2p}$$

Being DC source of 12V, i.e. constant replace p by 0.

$i_f(t) = 3$ amp.

For transient response,

Homogeneous equation is, $4i + 2\frac{di}{dt} = 0$

characteristic equation is, $4+2s=0$

$$\Rightarrow s = -2$$

Thus, $i_t(t) = Ke^{st} = Ke^{-2t}$

Then, total solution $i(t) = i_f(t) + i_t(t)$

$$= 3 + Ke^{-2t}$$

Now, using initial condition,

At $t = 0^+$

or, $i(0^+) = 3 + Ke^{-2(0^+)}$

$\Rightarrow \quad 2 = 3 + K \qquad \therefore K = -1$

Thus, $\boxed{i(t) = 3 - e^{-2t}}$

Since, $i_t(t) = -e^{-2t}$ compare it with $e^{-\frac{t}{\tau}}$ we get, $\tau = \frac{1}{2}$

$\therefore$ Time constant $(\tau) = 0.5$ sec.

$\therefore$ Settling time $t = 5\tau = 5 \times \frac{1}{2} = 2.5$ sec.

$\Rightarrow \tau = \frac{L}{R}$ is known as the time constant of the circuit and is defined as the interval after which current or voltage changes 63.2% of its total change. OR,

Time constant is defined as time taken by the total response to reach 63.2% of the steady state value. it is represented by τ.

Again for,

$i_t(t)$ at, $t = \frac{L}{R}$, $i_t(t) = -\frac{V}{R} e^{-\frac{R}{L} \times \frac{L}{R}} = -\frac{V}{R} e^{-1} = -0.37 \frac{V}{R}$

so, time constant is also defined as time taken by the transient response to decay to 37% of its initial value.

<u>Settling time:</u>

At, $t = \tau$, $i_t(t) = -0.37 \frac{V}{R}$ =

$t = 2\tau$, $i_t(t) = -0.135 \frac{V}{R}$

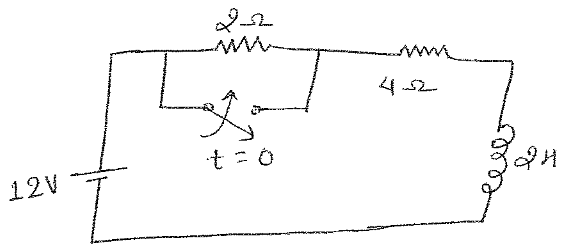
$t = 3\tau$, $i_t(t) = -0.0498\ V/R$

$t = 4\tau$, $i_t(t) = -0.0183\ V/R$

$t = 5\tau$, $i_t(t) = -0.0067\ V/R$

Since, at $t = 5\tau$, transient response has almost decay to less than 1% of initial value and then circuit settles. so, settling time for above circuit is 5τ.

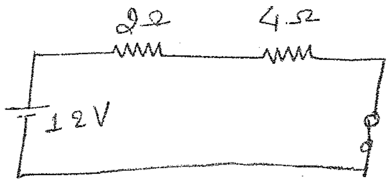
<u>Q.1</u> Find the expression for $i(t)$ for $t > 0$



Also, find time constant and settling time.

Sol^n: At $t = 0^-$,

equivalent circuit is,

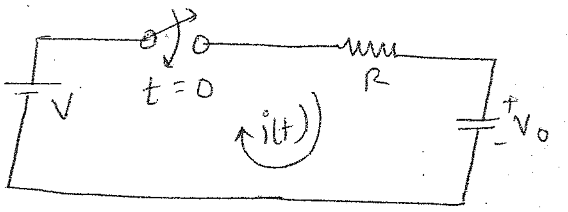


$i(0^-) = \frac{12V}{6\Omega} = 2\text{amp}$

~~At $t = 0^+$, equivalent circuit is,~~ $\therefore i(0^-) = i(0^+) = 2\text{amp}$ ($\because$since current

Step response to series RC circuit:

Consider the following circuit,

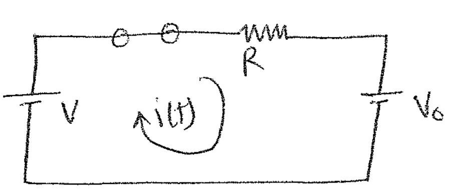


Sol^n: At, $t=0^-$, $V_c(0^-) = V_0$ and $i(0^-) = 0$

At, $t = 0^+$

$V_c(0^+) = V_c(0^-) = V_0$ $\because$ Voltage across capacitor can not change instanteneously.

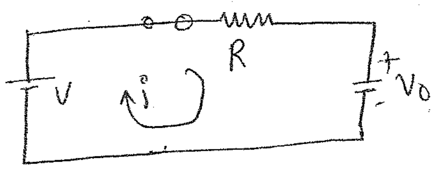
Equivalent circuit is,



$$i(t) = \frac{V - V_0}{R}$$

At, $t > 0$

using KVL



$$V = iR + \frac{1}{C}\int_{-\infty}^{t} i\,dt$$

$$V = iR + \frac{1}{C}\int_{-\infty}^{0} i\,dt + \frac{1}{C}\int_{0}^{t} i\,dt$$

$$V = iR + V_0 + \frac{1}{C}\int_{0}^{t} i\,dt \quad \cdots ①$$

[Here, V is step response i.e. constant DC source]

Differentiating with respect to t,

$$0 = R\frac{di}{dt} + 0 + \frac{i}{C}$$

or, $$R\frac{di}{dt} + \frac{i}{C} = 0$$

using p-operator form,

$$RPi + \frac{i}{C} = 0$$

This is the homogeneous equation.

So, forced response, $i_f(t) = 0$.

Now, for transient solution/response;

Characteristic equation is,

$sR + \frac{1}{C} = 0 \Rightarrow s = -\frac{1}{RC}$

Thus, $i_t(t) = Ke^{st}$

$= Ke^{-\frac{1}{RC}t}$

Note: Here, time constant

$\boxed{\tau = RC}$ at, $t = RC$

$i(t) = \left(\frac{V - V_0}{R}\right) e^{-1}$

$= 0.368 \left(\frac{V - V_0}{R}\right)$

Total response,

$i(t) = i_f(t) + i_t(t)$

$= Ke^{-\frac{1}{RC}t}$

Now, using initial condition i.e. at $t = 0^+$

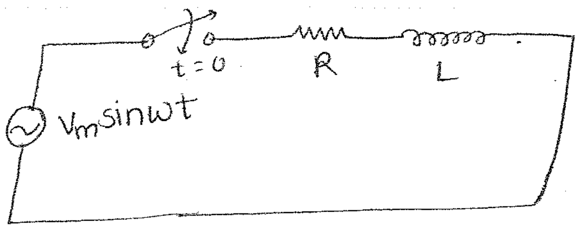
$i(0^+) = Ke^{-\frac{1}{RC}(0^+)}$

or, $\frac{V - V_0}{R} = K$

∴ Thus, total solution, $\boxed{i(t) = \left(\frac{V - V_0}{R}\right) e^{-\frac{1}{RC}t}}$

Response of RL circuit to sinusoidal driving function:

Consider the following circuit:



At, $t = 0^-$, $i(0^-) = 0$ amp

At, $t = 0^+$, $i(0^-) = i(0^+) = 0$ Since current through inductor can not change instanteneously.

Now,

Apply KVL for $t > 0$,

$V_m \sin\omega t = iR + L\frac{di}{dt}$ .... (1)

in p operator form,

$V_m \sin\omega t = iR + LPi$

$$i = \frac{V_m \sin\omega t}{R+LP} \quad \cdots\cdots (2)$$

equation (1) is non-homogeneous equation. Its solution has two parts.

(1) Forced response : $i(t) = \frac{I_m [V_m e^{j\omega t}]}{R+LP}$

Replace p by s and s by $j\omega$ we get

Then,

$$i_f(t) = \frac{I_m [V_m e^{j\omega t}]}{R+Lj\omega} = I_m\left(\frac{V_m e^{j\omega t}}{Z\angle\phi}\right)$$

where, $Z = \sqrt{R^2+\omega^2L^2}$ and, $\phi = \tan^{-1}\left(\frac{\omega L}{R}\right)$

$$= \frac{V_m}{Z} I_m\left(e^{j(\omega t-\phi)}\right)$$

$$\therefore i_f(t) = \frac{V_m}{Z}\sin(\omega t-\phi)$$

(2) Transient response:

Homogeneous form of equation (1) is,

$$iR + L\frac{di}{dt} = 0$$

characteristic equation is,

$$R+LS = 0$$

$$S = -R/L$$

$$\therefore i_t(t) = Ke^{st} = Ke^{-\frac{R}{L}t}$$

Total response, $i(t) = i_f(t) + i_t(t)$

$$i(t) = \frac{V_m}{Z}\sin(\omega t-\phi) + Ke^{-\frac{R}{L}t}$$

using initial values, i.e. at $t=0^+$

$$i(0^+) = \frac{V_m}{Z}\sin(\omega(0^+)-\phi) + Ke^{-\frac{R}{L}(0^+)}$$

or, $0 = \frac{V_m}{Z}(\sin(-\phi)) + K$

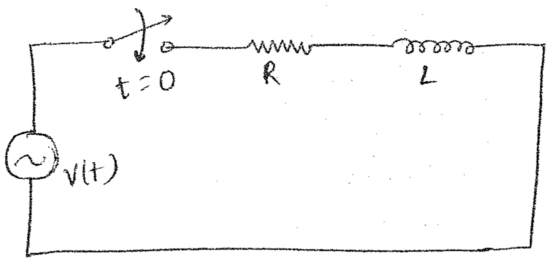
$\therefore K = \frac{V_m}{Z}\sin\phi$

$\therefore$ Thus, total response,

$$\boxed{i(t) = \frac{V_m}{Z}\left\{\sin(\omega t-\phi) + \sin\phi\, e^{-R/L\, t}\right.}$$

8)

8. For series RL circuit $v(t) = 10\sin(10^4 t + \pi/6)$, $R = 2\,\Omega$, and $L = 0.01\,H$. Voltage $v(t)$ is applied at $t = 0$, calculate $i(t)$ for $t > 0$. Assume $i_L(0^-) = 0$.



Sol: At, $t = 0^-$

$i(0^-) = 0$

Now, $i(0^+) = i(0^-) = 0$ $\because$ Current through inductor can not change instanteneously.

Apply KVL, at $t > 0$

$$v(t) = Ri + L\frac{di}{dt}$$

or, $$10\sin\left(10^4 t + \frac{\pi}{6}\right) = 2i + 0.01\frac{di}{dt} \quad \cdots\cdots (1)$$

using p-operator

$$10\sin\left(10^4 t + \frac{\pi}{6}\right) = 2i + 0.01pi \quad \cdots (2)$$

Equation (1) is non-homogeneous equation. So, its solution contains two parts

(1) Forced solution

from equation (2)

$$i = \frac{10\sin\left(10^4 t + \frac{\pi}{6}\right)}{2 + 0.01p}$$

$$= \frac{I_m\left[10e^{j\left(10^4 t + \frac{\pi}{6}\right)}\right]}{2 + 0.01p}$$

Now, Replace p by s and put $s = j10^4$.

Now,

$$i_f(t) = \frac{I_m\left[10e^{j\left(10^4 t + \frac{\pi}{6}\right)}\right]}{}$$

$$= \mathrm{Im}\left\{ \frac{10\, e^{j(10^4 t + \frac{\pi}{6})}}{100 \angle 88.85} \right\}$$

$$= \mathrm{Im}\left\{ 0.1\, e^{j(10^4 t + \frac{\pi}{6} - 88.85)} \right\}$$

$$i_f(t) = 0.1 \sin\left(10^4 t + \frac{\pi}{6} - 88.85\right)$$

② For transient solution

Homogeneous form of equation ① is,

$$2i + 0.01\frac{di}{dt} = 0$$

characteristic equation is,

$$2 + 0.01s = 0$$

$$\Rightarrow s = -200$$

$\therefore$ Transient solution, $i_t(t) = Ke^{st} = Ke^{-200t}$

Thus,

$$i(t) = i_f(t) + i_t(t)$$

$$= 0.1\sin\left(10^4 t + \frac{\pi}{6} - 88.85\right) + Ke^{-200t}$$

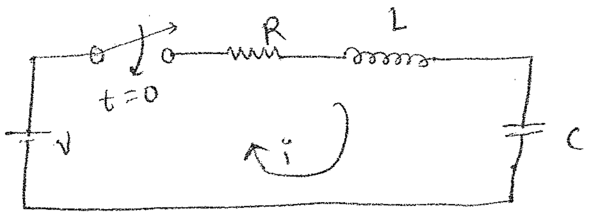
at, $t = 0^+$

$$i(0^+) = 0.1\sin\left(\frac{\pi}{6} - 88.85\right) + K$$

$$\therefore K = 0.08526$$

Thus, total response, $\boxed{i(t) = 0.08526\, e^{-200t} + 0.1\sin\left(10^4 t + \frac{\pi}{6} - 88.85\right)}$

Step response of series RLC circuit



Let us consider a series RLC circuit as shown in figure above.

At $t = 0^-$

$i(0^-) = 0$ amp. at, $t = 0^+$, $i(0^+) = i(0^-) = 0$ ($\because$ current through inductor

Apply KVL for $t>0$.

$$V = iR + L\frac{di}{dt} + \frac{1}{C}\int_0^t i\,dt \quad \cdots\cdots (1)$$

Differentiation with respect to t,

$$0 = R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C}$$

$$\Rightarrow L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = 0$$

$$\Rightarrow \frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0 \quad \cdots\cdots (2)$$

Being homogeneous equation, forced response

$$i_f(t) = 0.$$

For transient response,

characteristic equation is : $s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \quad \cdots\cdots (3)$

equation (3) is quadratic equations and it has two roots. Let, s_1 and s_2 be their roots.

i.e. $$s_1, s_2 = \frac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - 4\cdot\frac{1}{LC}}}{2.1}$$

$$= -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right)} \quad \cdots\cdots (4)$$

Then, transient equation is,

$$i_t(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

$$= K_1 e^{\left\{-\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right)}\right\}t} + K_2 e^{\left\{-\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right\}t} \quad \cdots\cdots (5)$$

The current in equation (5) depends upon the value of s_1 and s_2.

which might be
- Complex - conjugate
- Real and equal

The system which has real and equal roots is called critically damped system. The value of resistance which makes system critically damped is called critical resistance and denoted by R_c.

From equation ④ roots will be equal if,

$$\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right) = 0$$

Then, Critical resistance R_c is given by

$$\left(\frac{R_c}{2L}\right)^2 = \frac{1}{LC}$$

$$R_c^2 = \frac{4L}{C}$$

$$\boxed{\therefore R_c = 2\sqrt{\frac{L}{C}}} \quad \text{......} \quad ⑥$$

In order to put the second order system in standard form, we define two parameters called damping ratio and damped natural frequency.

Damping ratio is defined as the ratio of actual resistance to critical resistance of the circuit and denoted by zeta (ζ) then,

$$\zeta = \frac{R}{R_c} = \frac{R}{2\sqrt{L/C}} = \frac{R}{2}\sqrt{\frac{C}{L}} \quad \text{....} \quad ⑦$$

The second parameter is defined as the frequency at which response oscillates and is denoted by ω_n.

$$\omega_n = \frac{1}{\sqrt{LC}} \quad \text{......} \quad ⑧$$

NOW,

$$2\zeta\omega_n = 2 \times \frac{R}{2}\sqrt{\frac{C}{L}} \times \frac{1}{\sqrt{LC}} = \frac{R}{L} \quad \text{....} \quad ⑨$$

From equation ⑧, ⑨ and ② we can represent second order differential equation in standard form as:

$$\frac{d^2 i}{dt^2} + 2\zeta\omega_n \frac{di}{dt} + \omega_n^2 i = 0 \quad \text{....} \quad ⑩$$

Now, Characteristic equation for above differential equation is, Roots are,

$$S_1, S_2 = \frac{-2\xi\omega_n \pm \sqrt{4(\xi\omega_n)^2 - 4\omega_n^2}}{2} = -\xi\omega_n \pm \sqrt{\omega_n^2(\xi^2-1)}$$

$$= -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1} \quad \ldots\ldots (12)$$

<u>Case I :</u>

<u>If $\xi > 1$</u>

For, $\xi > 1$, roots are real and distinct and such system are called over damped system. Then, $i(t)$ is given by,

$$i(t) = K_1 e^{S_1 t} + K_2 e^{S_2 t}$$

$$= K_1 e^{\{-\xi\omega_n - \omega_n\sqrt{\xi^2-1}\}t} + K_2 e^{\{-\xi\omega_n + \omega_n\sqrt{\xi^2-1}\}t} \quad \ldots\ldots (13)$$

Now, value of K_1 and K_2 are determined by using initial condition,

We have, $i(0^+) = 0$ Amp. $\quad \ldots\ldots (14)$

and, for $\frac{di(0^+)}{dt}$ from equation (1), $V = iR + L\frac{di}{dt} + \frac{1}{C}\int_0^t i\,dt$

at, $t = 0^+$

$$V = i(0^+)R + L\frac{di(0^+)}{dt} + V_C(0^+)$$

$$\therefore \frac{di(0^+)}{dt} = \frac{V}{L} \quad \ldots\ldots (15)$$

From equation (13), at $t = 0^+$

$$i(0^+) = K_1 e^0 + K_2 e^0 \quad \Rightarrow K_1 = -K_2 \quad \ldots\ldots (16)$$

Now, differentiating equation (13) with respect to t, we get

$$\frac{di(t)}{dt} = \left(-\xi\omega_n - \omega_n\sqrt{\xi^2-1}\right) K_1 e^{\{-\xi\omega_n - \omega_n\sqrt{\xi^2-1}\}t} + \left(-\xi\omega_n + \omega_n\sqrt{\xi^2-1}\right) K_2 e^{\{-\xi\omega_n + \omega_n\sqrt{\xi^2-1}\}t}$$

at, $t = 0^+$

$$\frac{di(0^+)}{dt} = \left(-\xi\omega_n - \omega_n\sqrt{\xi^2-1}\right)K_1 e^0 + \left(-\xi\omega_n + \omega_n\sqrt{\xi^2-1}\right)K_2 e^0$$

or, $\frac{V}{L} = -\xi\omega_n K_1 - \omega_n\sqrt{\xi^2-1}\,K_1 + \xi\omega_n K_1 - \omega_n\sqrt{\xi^2-1}\,K_1$ $[\because K_2 = -K_1]$

or, $\frac{V}{L} = K_1\left(-2\omega_n\sqrt{\xi^2-1}\right)$

or, $K_1 = -\frac{V}{2L\omega_n\sqrt{\xi^2-1}} = -K_2$

Therefore,

$$i(t) = -\frac{V}{2L\omega_n\sqrt{\xi^2-1}}\, e^{\left(-\xi\omega_n - \omega_n\sqrt{\xi^2-1}\right)t} + \frac{V}{2L\omega_n\sqrt{\xi^2-1}}\, e^{\left(-\xi\omega_n + \omega_n\sqrt{\xi^2-1}\right)t} \quad \ldots\ldots (17)$$

Case II If $\xi = 1$,

For $\xi = 1$, roots are real and equal. Then $i(t)$ becomes,

$$i(t) = K_1 e^{st} + K_2 t e^{st} = (K_1 + K_2 t)e^{st} \quad \ldots\ldots (18)$$

Now, root will be,

from equation 12,

$s = -\omega_n$

Then, $i(t) = (K_1 + K_2 t)\, e^{-\omega_n t}$ ...... (19)

Differentiating w.r.to t,

$$\frac{di(t)}{dt} = -\omega_n(K_1 + K_2 t)e^{-\omega_n t} + K_2 e^{-\omega_n t} \quad \ldots\ldots (20)$$

using initial condition at $t = 0^+$, from equation (19) and (20) we get

$i(0^+) = \{K_1 + K_2(0^+)\}\, e^{-\omega_n \times 0}$

$0 = K_1$

Then, $\frac{di(0^+)}{dt} = -\omega_n\{K_1 + K_2(0)\}\, e^{-\omega_n \times 0} + K_2 e^{-\omega_n \times 0}$

$\Rightarrow \frac{V}{L} = 0 + K_2 \quad \Rightarrow K_2 = \frac{V}{L}$ ...... (21)

Hence, total solution,

$$i(t) = (K_1 + K_2 t)\, e^{-\omega_n t} = \frac{V}{L} t\, e^{-\omega_n t} \quad \text{----- (22)}$$

Case III : If $\xi < 1$,

For $\xi < 1$, roots are imaginary and complex conjugate. Here

roots are given as,

$$S_1, S_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1} \quad \text{------- (23)}$$

$$= -\xi\omega_n \pm \omega_n\sqrt{-(1-\xi^2)}$$

$$= -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2} \quad \text{------ (24)}$$

$= -\xi\omega_n \pm j\omega_d$ where, $\omega_d = \omega_n\sqrt{1-\xi^2}$ is called

damped frequency of oscillation.

Then,

transient response,

$$i_t(t) = K_1 e^{S_1 t} + K_2 e^{S_2 t}$$

$$= K_1 e^{(-\xi\omega_n + j\omega_d)t} + K_2 e^{(-\xi\omega_n - j\omega_d)t}$$

re,

$$i_t(t) = i(t) = e^{-\xi\omega_n t}\left[K_1 e^{j\omega_d t} + K_2 e^{-j\omega_d t}\right] \quad \text{----- (25)}$$

Differentiating eq^n (25) w.r.to t we get

$$\frac{di_t(t)}{dt} = e^{-\xi\omega_n t}\left[K_1 j\omega_d\, e^{j\omega_d t} + K_2(-j\omega_d)e^{-j\omega_d t}\right] + (-\xi\omega_n)\, e^{-\xi\omega_n t} \times \left[K_1 e^{j\omega_d t} + K_2 e^{-j\omega_d t}\right] \quad \text{(26)}$$

using initial condition, at $t = 0^+$, from eq^n (25) and (26) we get

$$i(0^+) = e^{-\xi\omega_n t}[K_1 + K_2] \quad \Rightarrow e^{-\xi\omega_n} \neq 0 \text{ so}$$

$\Rightarrow K_1 + K_2 = 0$ or $K_1 = -K_2$

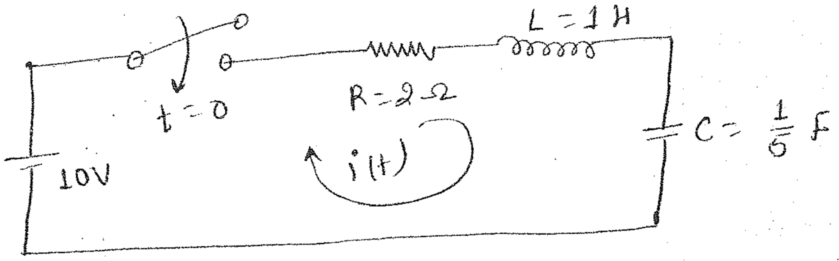
and, $$\frac{di(0^+)}{dt} = e^{-\xi\omega_n t}\left[-K_2 j\omega_d\, e^0 - j\omega_d K_2 e^0\right] - \xi\omega_n \cdot e^0[-K_2 + K_2].$$

$$\frac{V}{L} = e^{-\xi\omega_n \times (0)} \cdot -2K_2 j\omega_d \quad \Rightarrow K_2 = -\frac{V}{2jL\omega_d} = -K_1$$

Hence, $$i(t) = e^{-\xi\omega_n t}\left[\frac{V}{2jL\omega_d} e^{j\omega_d t} - \frac{V}{2jL\omega_d} e^{-j\omega_d t}\right]$$

in the form also $\left\{ e^{-\xi\omega_n t} \frac{V}{\omega_d L} \sin\omega_d t \right\}$

Q: For the given circuit calculate $i(t)$ for $t>0$. Assume initial charges and current zero.



Soln: Here,

$i(0^-) = 0 = i(0^+)$ $\because$ Since current through inductor can not change instanteously.

Now, apply KVL at $t>0$, we have

$$10 = Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt \quad \cdots\cdots (1)$$

Differentiating w.r.to t,

$$0 = R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C} \quad \cdots\cdots (2)$$

Equation (2) is homogeneous second order differential equation. So, forced response becomes zero.

Thus, $i(t) = \cancel{i_f(t)}^{\,0} + i_t(t) = i_t(t)$

Now, For transient response, $Ls^2 + Rs + \frac{1}{C} = 0$

characteristic equation is, $s^2 + 2s + 5 = 0$

$$s_1, s_2 = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4j}{2} = -1 \pm 2j$$

Thus, $i(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$

$$= K_1 e^{(-1+2j)t} + K_2 e^{(-1-2j)t}$$

$$= e^{-t}\left[K_1 e^{2jt} + K_2 e^{-2jt}\right] \quad \cdots\cdots (3)$$

using initial condition, at $t=0^+$

$i(0^+) = e^0\left[K_1 e^0 + K_2 e^0\right] \Rightarrow 0 = K_1 + K_2$

$\Rightarrow K_1 = -K_2 \quad \cdots\cdots (4)$

From equation (5) and (4)

$$i(t) = e^{-t}\left[K_1 e^{2jt} - K_1 e^{-2jt}\right].$$

$$i(t) = K_1 e^{-t}\left[e^{2jt} - e^{-2jt}\right].$$

differentiating w. r. to t,

$$\frac{di(t)}{dt} = K_1 e^{-t}\left[2je^{2jt} + 2je^{-2jt}\right] + (-)K_1 e^{-t}\left[e^{2jt} - e^{-2jt}\right]. \quad \text{.....} (5)$$

Now, from equation (1) at $t = 0^+$

$$10 = Ri(0^+) + L\frac{di(0^+)}{dt} + V_c(0^+)$$

$$10 = 0 + \frac{di(0^+)}{dt}$$

$$\frac{di(0^+)}{dt} = 10 \text{ amp/sec.}$$

at $t = 0^+$ eqn. (5) gives,

$$\frac{di(0^+)}{dt} = K_1\left[2j + 2j\right] - K_1\left[1 - 1\right]$$

$$10 = K_1\left[4j - \cancel{10}\right] \Rightarrow K_1 = \frac{10}{4j} = \frac{10}{4}j$$

$$\therefore K_1 = 2.5j = -K_2$$

Required solution,

$$\boxed{i(t) = e^{-t}\left[-2.5je^{2jt} + 2.5je^{-2jt}\right]}$$

A LAPLACE transform is mathematical tool that is widely used to find the solution of differential equations without exactly solving the differential equation by classical method.

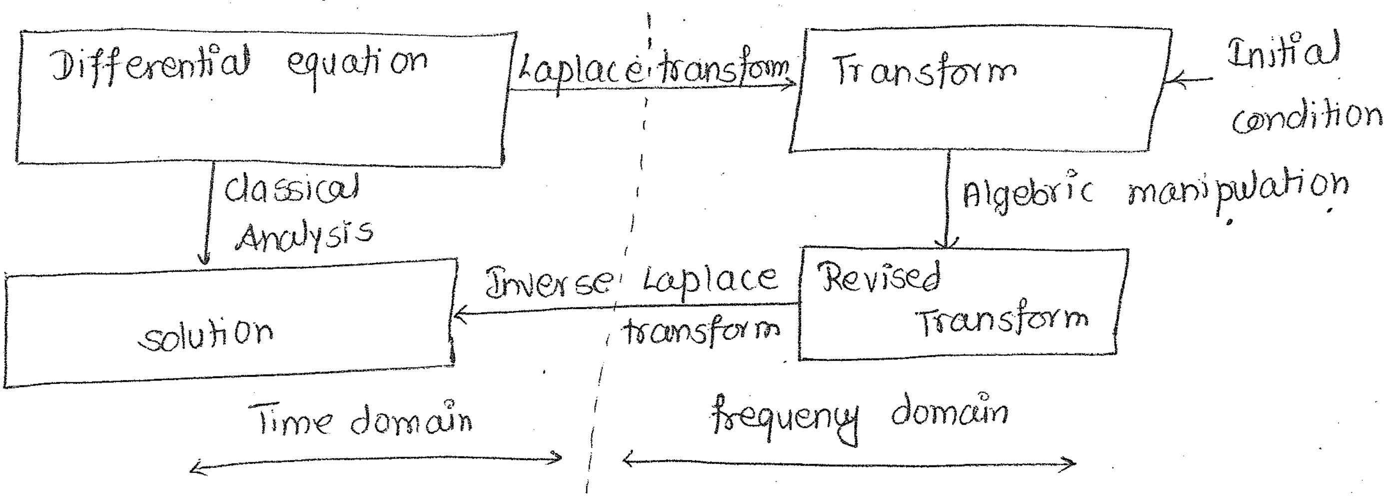
Mathematically, If $f(t)$ be the given function which is defined for $t \geq 0$ the laplace transform is given by

$$L[f(t)] = F(s) = \int_0^{\infty} f(t)\, e^{-st}\, dt \quad \ldots\ldots (1)$$

Where, $s = \sigma + j\omega$, is called complex frequeny. Similorly, Inverse laplac transform is $F(s)$ and is given by,

$$L^{-1}[F(s)] = f(t) = \int_{-\infty}^{\infty} F(s) e^{st}\, ds \quad \ldots\ldots (2)$$

Flowsheet for Laplace transform:



Steps:

1. We start with an integro-differential equation and find corresponding Laplace transform
2. Transform is manipulated algebrically after the initial conditions are applied.
3. We perform an inverse laplace transform to find the complete solution

Advantages:

(1) It yields the complete information about the sustained (steady) and transient solution rather than the three part solution procedure of classical method.

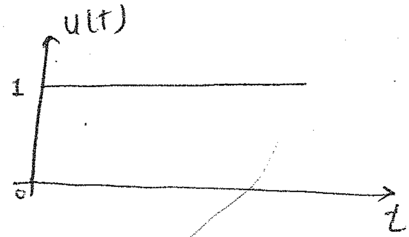
(2) Solution gets highly Simplified.

(3) Gives a Single solution (complete solution) rather than in parts.

) Initial conditions are automatically specified on transformed equation and are applied in first step rather than in last step.

Laplace transform of some basic function :

1) Unit step function



We have, $u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Then,

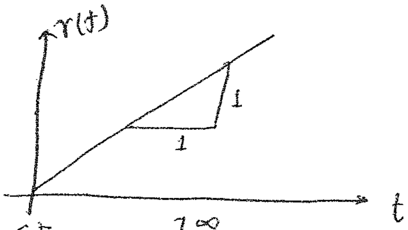
$$L[u(t)] = \int_0^\infty e^{-st} u(t).dt = \int_0^\infty e^{-st} dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}$$

$$\boxed{u(t) \xleftrightarrow{L.T.} \frac{1}{s}}$$

2) Exponential function e^{at} where a is constant :

$$L[e^{at}] = \int_0^\infty e^{-st}.e^{at}.dt = \int_0^\infty e^{-(s-a)t} dt = \left[\frac{e^{-(s-a)}}{-(s-a)}\right]_0^\infty = \frac{1}{s-a}$$

) Ramp function :



We have, $r(t) = \begin{cases} t & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Then, $L[r(t)] = \int_0^\infty e^{-st}\, t\, dt = \left[t \int e^{-st} dt - \int\int e^{-st} dt\, dt\right]_0^\infty$

$$= \left[t\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^\infty = \frac{1}{s^2}$$

) Similarly,

$$t^2 \xleftrightarrow{L.T.} \frac{2}{s^3} \quad \text{and} \quad t^n \xleftrightarrow{L.T.} \frac{n!}{s^{n+1}}$$

) sinωt and cosωt

$$L[f(t)] = \int_0^\infty e^{-st} \sin\omega t\, dt$$

We have, $e^{j\omega t} = \cos\omega t + j\sin\omega t$

Then, $L[e^{j\omega t}] = L[\cos\omega t + j\sin\omega t]$

$= L[\cos\omega t] + j\,L[\sin\omega t]$

or, $\dfrac{s+j\omega}{s^2+\omega^2} = L[\cos\omega t] + jL[\sin\omega t]$

or, $\dfrac{s}{s^2+\omega^2} + j\dfrac{\omega}{s^2+\omega^2} = L[\cos\omega t] + jL[\sin\omega t]$

$\therefore L[\sin\omega t] = \dfrac{\omega}{s^2+\omega^2}$ and, $L[\cos\omega t] = \dfrac{s}{s^2+\omega^2}$

(6) unit impulse function

Impulse function is defined as,

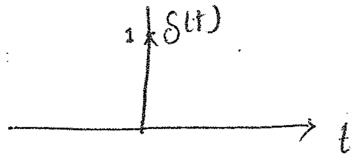
$\delta(t) = 0$ for $t \neq 0$

$\quad\quad \infty$ for $t = 0$

unit impulse function,

$\delta(t) = 0$ for $t \neq 0$

$\quad\quad 1$ for $t = 0$



$L\{\delta(t)\} = \int_0^\infty e^{-st}\delta(t)\,dt = e^{-st}\Big|_{\text{for } t=0} = 1.$

(7) Laplace transform of derivatives:

$L\left[\dfrac{df(t)}{dt}\right] = \int_0^\infty \dfrac{df(t)}{dt} e^{-st} dt = \int_0^\infty u\,dv$

for, $u = e^{-st}$ and, $dv = \dfrac{df(t)}{dt}$ Then, $du = -se^{-st}$ and $v = f(t)$

$\int_a^b u\,dv = [uv]_a^b - [\int v\,du]_a^b$

Then, $\int_0^\infty \dfrac{df(t)}{dt} \cdot e^{-st} dt = \left[e^{-st} f(t)\right]_0^\infty + \left[s\int e^{-st} f(t)dt\right]_0^\infty$

$= sF(s) + [\cancel{e^{-\infty} f(\infty)}]^{\to 0} - [e^0 f(0)]$

$= sF(s) - F(0)$

$= sF(s) - F(0)$

where, $F(0)$ is the value of $f(t)$ at $t = 0^+$ and given by initial conditio of the circuit.

Similarly, $L\left[\dfrac{d^2 f(t)}{dt^2}\right] = s^2 F(s) - sF(0) - F'(0)$

$L\left[\dfrac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1}F(0) - s^{n-2}F'(0) + \cdots F^{n-1}(0)$

8) Laplace transform of integrals

$$L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s} \text{ and}$$

$$L\left[\int_{-\infty}^{t} f(t)\,dt\right] = \int_{-\infty}^{0} f(t)\,dt + \int_0^t f(t)\,dt = \frac{f(0)}{s} + \frac{F(s)}{s}$$

Some formulas :

f(t)	F(s)	f(t)	F(s)
1	$1/s$	te^{at}	$\frac{1}{(s-a)^2}$
t	$1/s^2$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
t^2	$\frac{2!}{s^3}$	$t\sin at$	$\frac{2as}{(s^2+a^2)^2}$
t^n	$\frac{n!}{s^{n+1}}$	$t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
e^{at}	$\frac{1}{s-a}$	$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$\cos at$	$\frac{s}{s^2+a^2}$	$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
$\sin at$	$\frac{a}{s^2+a^2}$	e^{-at}	$\frac{1}{s+a}$
$\sinh at$	$\frac{a}{s^2-a^2}$	te^{-at}	$\frac{1}{(s+a)^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$		

Q1 Solve for $i(t)$ using L.T method.

$$\frac{d^2i}{dt^2} + 4\frac{di}{dt} + 3i = 0,\ i(0) = 3,\ \frac{di(0)}{dt} = 1$$

Soln:

We have,

$$\frac{d^2i}{dt^2} + 4\frac{di}{dt} + 3i = 0$$

Taking Laplace transform on both side,

$$L\left\{\frac{d^2i}{dt^2}\right\} + 4L\left\{\frac{di}{dt}\right\} + 3L(i) = 0$$

or, $[s^2+4s+3]I(s) - s*3 - 1 - 12 = 0$

or, $(s^2+4s+3)\, I(s) = 3s+13$

$\Rightarrow I(s) = \dfrac{3s+13}{s^2+4s+3} = \dfrac{3s+13}{(s+3)(s+1)} = \dfrac{A}{s+3} + \dfrac{B}{s+1}$

$A = \dfrac{3s+13}{(s+3)(s+1)} \times (s+3) \Big|_{s=-3} = \dfrac{3\times -3+13}{-3+1} = \dfrac{4}{-2} = -2$

$B = \dfrac{3s+13}{(s+3)(s+1)} \times (s+1) \Big|_{s=-1} = \dfrac{3\times -1+13}{-1+3} = \dfrac{10}{2} = 5$

$\therefore I(s) = -\dfrac{2}{s+3} + \dfrac{5}{s+1}$

Now, taking inverse L.T

$i(t) = L^{-1}\{I(s)\} = -2e^{-3t} + 5e^{-t}$

Q2. Solve by Laplace transform method

$\dfrac{d^2i}{dt^2} + 2\dfrac{di}{dt} + 5i = 0, \quad i(0) = 2, \; i'(0) = -4$

solⁿ: We have, $\dfrac{d^2i}{dt^2} + 2\dfrac{di}{dt} + 5i = 0$

Taking Laplace transform on both side,

$L\left[\dfrac{d^2i}{dt^2}\right] + 2L\left[\dfrac{di}{dt}\right] + 5L[i] = 0$

or, $[s^2 I(s) - s\,i(0) - i'(0)] + 2[sI(s) - i(0)] + 5I(s) = 0$

or, $s^2 I(s) - 2s + 4 + 2sI(s) - 4 + 5I(s) = 0$

or, $(s^2+2s+5)\, I(s) = 2s$

or, $I(s) = \dfrac{2s}{s^2+2s+5} = \dfrac{2s}{s^2+2s+1+4} = \dfrac{2s}{(s+1)^2+(2)^2}$

$I(s) = \dfrac{2(s+1)}{(s+1)^2+(2)^2} - \dfrac{2}{(s+1)^2+(2)^2}$

Now taking inverse Laplace transform

$i(t) = L^{-1}\{I(s)\} = 2 \cdot e^{-t}\cos 2t - e^{-t}\sin 2t$

33 Solve by L.T method.

$$\frac{d^2 i}{dt^2} + 10\frac{di}{dt} + 25 i = 0 \quad , i(0^+) = 2, \ \frac{di(0^+)}{dt} = 0$$

Taking Laplace transform on both side,

$$L\left[\frac{d^2 i}{dt^2}\right] + 10 L\left[\frac{di}{dt}\right] + 25 L[i] = 0$$

$\Rightarrow s^2 I(s) - s i(0^+) - i'(0^+) + 10 s I(s) - 10 i(0^+) + 25 I(s) = 0$

$\Rightarrow (s^2 + 10s + 25) I(s) - 2s - 20 = 0$

$\Rightarrow (s^2 + 10s + 25) I(s) = 2s + 20$

$\Rightarrow I(s) = \dfrac{2s+20}{(s+5)(s+5)} = \dfrac{A}{(s+5)} + \dfrac{B}{(s+5)^2}$

Now, $B = \left.\dfrac{2s+20}{(s+5)(s+5)} \times (s+5)^2\right|_{s=-5} = 10$

and, $2s + 20 = (s+5)A + B$

or, $2s + 20 = As + 5A + B$

comparing coefficient of s,

$A = 2$

So, $I(s) = \dfrac{2}{s+5} + \dfrac{10}{(s+5)^2}$

Using Inverse Laplace transform, we get

$i(t) = L^{-1}[I(s)] = L^{-1}\left[\dfrac{2}{s+5}\right] + L^{-1}\left[\dfrac{10}{(s+5)^2}\right]$

$$\boxed{i(t) = 2e^{-5t} + 10te^{-5t}}$$

Properties of Laplace transform:

① Linear Combination:

If $L[f_1(t)] = F_1(s)$ and $L[f_2(t)] = F_2(s)$

Then, $L[af_1(t) + bf_2(t)] = aF_1(s) + bF_2(s)$ where a and b are constants.

② Change of scale

If $L[f(t)] = F(s)$ then,

$$L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$$

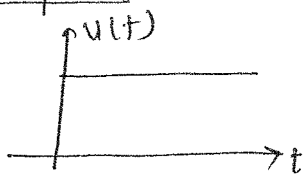
③ Shift in time domain

If $L[F(t)] = F(s)$ then,

$$\cancel{L[f(at)]} \; L[f(t-a)] = e^{-as} F(s)$$

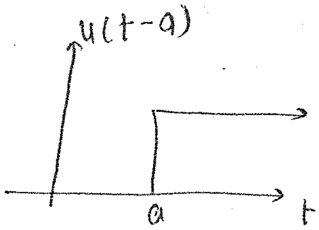
Examples:

①



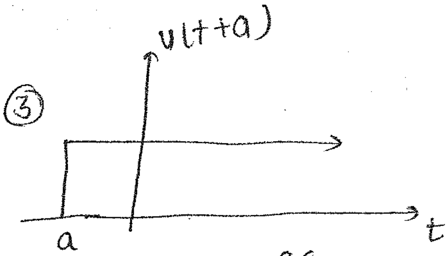
$$L[u(t)] = \frac{1}{s}$$

②



$$L[u(t-a)] = \frac{e^{-as}}{s}$$

③



$$L[u(t+a)] = \frac{e^{as}}{s}$$

④ Shifting in s-domain:

If $f(t) \xleftrightarrow{L.T.} F(s)$ then

$$L\{e^{at} f(t)\} = F(s-a)$$

Eg: Find L.T. of $e^{-4t} \sin 10t$

We have, $L[\sin 10t] = \dfrac{10}{s^2+10^2} = F(s)$

Then, $L[e^{-4t} \sin 10t] = F(s+4) = F(s)\big|_{s=s+4} = \dfrac{10}{(s+4)^2+100}$

⑤ Multiplication by t:

If $L[f(t)] = F(s)$ then,

$$L[t\,f(t)] = -F'(s) \text{ and}$$

$$L[t^n f(t)] = (-1)^n \frac{d^n F(s)}{ds^n}$$

Eg: Find, L.T. of $t \sin 3t$

We have, $L[\sin 3t] = \dfrac{3}{s^2+9}$

Then, $L[t \sin 3t] = (-1)^1 \dfrac{d}{ds}\left(\dfrac{3}{s^2+9}\right) = -1 \cdot \dfrac{3 \cdot (-1) \cdot 2s}{(s^2+9)^2} = \dfrac{6s}{(s^2+9)^2}$

⑥ Initial value theorem and final value theorem:

Initial value theorem

If $f(t)$ and its first derivative $f'(t)$ are Laplace transformable.

then,
$$f(0) = \lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$$

Final value theorem

If $f(t)$ and its first derivative $f'(t)$ are Laplace transformable.

then,
$$f(\infty) = \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

<u>Assignment</u>: State and prove initial and final value theorem.

Q. Find the Laplace transform of $5 + 5e^{-2t} + 10e^{-4t}$. And verify initial and final value theorem.

Sol^n: Here,
$$f(t) = 5 + 5e^{-2t} + 10e^{-4t}$$

Taking Laplace transform on both side, we get
$$L[f(t)] = L(5) + L[5e^{-2t}] + L[10e^{-4t}]$$

$$F(s) = \frac{5}{s} + \frac{5}{s+2} + \frac{10}{s+4}$$

Now, ① <u>initial value theorem</u>,

$$\lim_{t \to 0} f(t) = \lim_{t \to 0} [5 + 5e^{-2t} + 10e^{-4t}] = 5+5+10 = 20$$

$$\lim_{s \to \infty} sF(s) = \lim_{s \to \infty} s\left[\frac{5}{s} + \frac{5}{s+2} + \frac{10}{s+4}\right] = \lim_{s \to \infty} \left(5 + \frac{5}{1+2/s} + \frac{10}{1+4/s}\right)$$
$$= 5 + 5 + 10 = 20$$

Here, $\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$. Hence, initial value theorem is verified.

Again, ⑪ <u>final value theorem</u>,

$$\lim_{t \to \infty} f(t) = \lim_{t \to \infty} [5 + 5e^{-2t} + 10e^{-4t}] = 5 + 5e^{-\infty} + 10e^{-\infty} = 5$$

$$\lim sF(s) = \lim s\left[\frac{5}{s} + \frac{5}{s+2} + \frac{10}{s+4}\right] = \lim \left[5 + \frac{5.s}{s+2} + \frac{10s}{s+4}\right] = 5$$

Here, $\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s)$. Hence, final value theorem is verified.

Inverse Laplace transform :

We use partial fraction expansion to find $f(t)$ from $F(s)$. Laplace transform equation is polynomial of s. i.e

$$F(s) = \frac{N(s)}{D(s)}$$

For partial expansion, order of $N(s)$ should be always less than $D(s)$, otherwise we divide $N(s)$ by $D(s)$.

Next, we factorize $D(s)$ interms of its roots.

$$D(s) = a_0 s^d + a_1 s^{d-1} + \ldots\ldots + a_d$$
$$= a_0(s-s_1)(s-s_2)\ldots\ldots(s-s_d)$$

Case 1

If roots are real and distinct

$$\frac{N(s)}{(s-s_1)(s-s_2)\ldots(s-s_d)} = \frac{K_1}{(s-s_1)} + \frac{K_2}{(s-s_2)} + \ldots + \frac{K_d}{s-s_d}$$

where, $K_d = \left.\frac{N(s)}{D(s)} \times (s-s_d)\right|_{s=s_d}$

Case 2 If roots are equal and real.

$$\frac{N(s)}{(s-s_1)^r} = \frac{K_1}{(s-s_1)} + \frac{K_2}{(s-s_1)^2} + \ldots + \frac{K_n}{(s-s_1)^r}$$

Where, $K_n = \frac{1}{(r-n)!}\left[\frac{d^{r-n}}{ds^{r-n}}\left\{\frac{N(s)}{D(s)}(s-s_1)^r\right\}\right]_{s=s_1}$.

Case 3 if roots are complex conjugate

$$\frac{N(s)}{(s-\alpha+j\beta)(s-\alpha-j\beta)} = \frac{K_1}{(s-\alpha+j\beta)} + \frac{K_2}{(s-\alpha-j\beta)}$$

Where, K_1 or $K_2 = \left.\left[\frac{N(s)}{D(s)} \times (s-\alpha\pm j\beta)\right]\right|_{s=\alpha\pm j\beta}$

Analysis of R, L, C, network using Laplace transform:

General methods:

Step 1: Find initial condition from the circuit

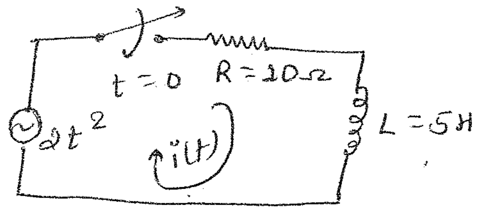
Step 2: Use KVL or KCL to form differential equation

Step 3: Use laplace transform to find polynomial in s-domain

Step 4: Use inverse laplace transform to obtain solution in time domain.

Q.1

Solve for $i(t)$ for the following circuit:



Soln: At, $t = 0^-$

$i(0^-) = 0$

$\therefore i(0^+) = i(0^-) = 0$ [$\because$ Current through inductor can not change instanteneously]

For, $t > 0$, using KVL,

$$2t^2 = 10i(t) + 5\frac{di(t)}{dt}$$

Taking laplace transform on both side,

$$L[2t^2] = L[10i(t)] + L\left[5\frac{di(t)}{dt}\right]$$

$$2 \cdot \frac{2}{s^3} = 10\,I(s) + 5\,[sI(s) - I(0)]$$

or, $$\frac{4}{s^3} = 10I(s) + 5sI(s) - 5\times 0$$

or, $$I(s)[10+5s] = \frac{4}{s^3}$$

or, $$I(s) = \frac{4/5}{s^3(s+2)}$$

using partial fraction,

$$I(s) = \frac{4/5}{s^3(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+2}$$

$$\frac{4}{5} = As^2(s+2) + Bs(s+2) + C(s+2) + Ds^3$$

or, $$\frac{4}{5} = As^3 + 2s^2A + s^2B + 2sB + sC + 2C + s^3D$$

or, $$\frac{4}{5} = (A+D)s^3 + (2A+B)s^2 + (2B+C)s + 2C$$

comparing coefficient,

$2C = \frac{4}{5} \quad \Rightarrow C = \frac{2}{5}$

$2B + C = 0 \quad \Rightarrow B = -\frac{C}{2} = -\frac{1}{5}$

$2A + B = 0 \quad \Rightarrow A = -\frac{B}{2} = \frac{1}{10}$

$A + D = 0 \quad \Rightarrow D = -A = -\frac{1}{10}$

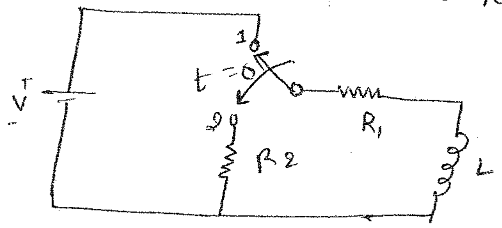
Thus,

$$I(s) = \frac{1}{10s} - \frac{1}{5s^2} + \frac{2}{5s^3} - \frac{1}{10(s+2)}$$

Taking inverse laplace transform,

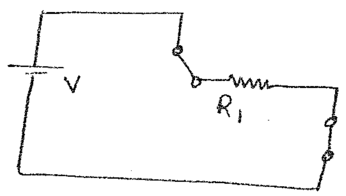
$$\boxed{i(t) = \frac{1}{10} - \frac{1}{5}t + \frac{1}{5}t^2 - \frac{1}{10}e^{-2t}}$$

Q.2 2005 fall 3(b) Solve for i(t) for the following circuit.



sol^n: For $t = 0^-$

Equivalent circuit is,



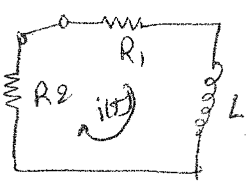
$$i(0^-) = \frac{V}{R_1}$$

$\therefore i(0^+) = i(0^-) = \frac{V}{R_1}$ ($\because$ Current through inductor can not change instanteneously.)

Now, using KVL for $t > 0$.

$$R_1 i(t) + R_2 i(t) + L\frac{di(t)}{dt} = 0$$

At, $t > 0$, equivalent circuit is



or, $$i(t)(R_1 + R_2) + L\frac{di(t)}{dt} = 0$$

Using, Laplace transform we get

$$(R_1 + R_2)I(s) + LsI(s) - LI(0) = 0$$

$$\Rightarrow I(S) = \frac{L\ V}{R_1(R_1+R_2+LS)}$$

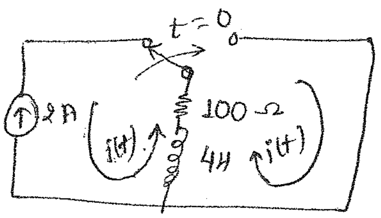
$$\Rightarrow I(S) = \frac{V}{R_1\left(\frac{R_1+R_2}{L}+S\right)}$$

Taking inverse Laplace transform, we get

$$i(t) = \frac{V}{R_1} e^{-\left(\frac{R_1+R_2}{L}\right)t}$$

5 <u>2005 spring</u>

In the circuit shown, the switch is moved from position 1 to 2 at $t = 0$. Find the solution for $i(t)$.



Solⁿ: At, $t = 0^-$,

$i(0^-) = -2$ A

$\therefore i(0^+) = i(0^-) = -2$A [$\because$ Current through inductor can not change instanteneously]

using KVL for $t \cancel{>} 0$, we get

$$4\frac{di(t)}{dt} + 100i(t) = 0$$

Taking Laplace transform, we get

$$4S\,I(S) - 4I(0^+) + 100I(S) = 0$$

$$I(S)(4S+100) = 4\times -2 = -8$$

$$I(S) = \frac{-8}{4(S+25)} = -\frac{2}{S+25}$$

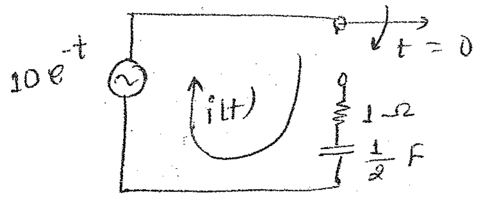
taking inverse Laplace transform, we get

$$i(t) = -2e^{-25t}$$

<u>Q.4</u> <u>2006 Fall</u>

For series circuit having $R = 1\Omega$, $C = \frac{1}{2}$F, with no initial voltage expression for the resulting current in the circuit for $t > 0$. use Laplace transform.

50



Sol. At $t = 0^-$

$$V_c(0^-) = 0$$

$\therefore V_c(0^+) = V_c(0^-) = 0$ ($\because$ Voltage across capacitor can not change instantaneously)

For, $t > 0$, using KVL

$$10e^{-t} = i(t) + \frac{1}{C}\int_{-\infty}^{t} i(t)\,dt$$

$$\Rightarrow 10e^{-t} = i(t) + \frac{1}{C}\int_{-\infty}^{0} i(t)\,dt + \frac{1}{C}\int_{0}^{t} i(t)\,dt$$

$$\Rightarrow 10e^{-t} = i(t) + \cancel{V_c(0^+)}^{0} + \frac{1}{C}\int_{0}^{t} i(t)\,dt$$

Taking Laplace transform, we get

$$\frac{10}{S+1} = I(S) + \frac{1}{1/2}\,\frac{I(S)}{S}$$

$$\Rightarrow \frac{10}{S+1} = I(S)\left[1 + \frac{2}{S}\right]$$

$$\Rightarrow \frac{10}{S+1} = I(S)\left(\frac{S+2}{S}\right)$$

$$\Rightarrow I(S) = \frac{10S}{(S+1)(S+2)}$$

Now, using partial fraction,

$$I(S) = \frac{10S}{(S+1)(S+2)} = \frac{A}{S+1} + \frac{B}{S+2}$$

$$A = \frac{10S}{(S+1)(S+2)} \times (S+1)\Big|_{S=-1} = \frac{-10}{1} = -10$$

$$B = \frac{10S}{(S+1)(S+2)} \times (S+2)\Big|_{S=-2} = \frac{-20}{-1} = 20$$

Then,

$$I(S) = -\frac{10}{S+1} + \frac{20}{S+2}$$

Taking inverse Laplace transform, we get $\boxed{i(t) = -10e^{-t} + 20e^{-2t}}$

Analysis of R, L, C

① For capacitor

Voltage across capacitor is given by,

$$V_C = \frac{1}{C}\int_{-\infty}^{t} i\,dt = \frac{1}{C}\int_{-\infty}^{0} i\,dt + \frac{1}{C}\int_{0}^{t} i\,dt$$

$$= V_C(0^+) + \frac{1}{C}\int_{0}^{t} i\,dt$$

Now, Taking L.T. of V_C, we get

$$L[V_C] = L[V_C(0^+)] + \frac{1}{C}\frac{I(s)}{s}$$

) For inductor

$$V_L = L\frac{di}{dt} \quad \text{and,} \quad L[V_L] = L[sI(s) - I(0^+)]$$

3) For resistance

$$V_R = Ri \quad \text{and,} \quad L[V_R] = RI(s)$$

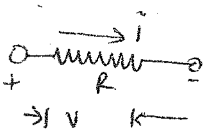
Transform impedance and transformed circuit:

The ratio of voltage and current in s domain is called transformed impedance.

i.e. Transformed impedance, $Z(S) = \frac{V(S)}{I(S)}$

Resistance

In time domain, we have $v = iR$



Taking Laplace transform we have

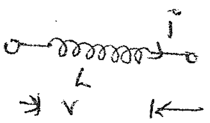
$L[V] = L[iR]$

$V(S) = R\,I(S) \Rightarrow R = \frac{V(S)}{I(S)}$

Inductor

(with no initial current)

In time domain, $v = L\frac{di}{dt}$



Taking L.T., we get

$V(S) = LS\,I(S) - \cancel{L\,I(0^+)} \to 0$

$LS = \frac{V(S)}{I(S)}$

(with initial current)

$V(S) = LS\,I(S) - L\,I(0^+)$

Capacitance

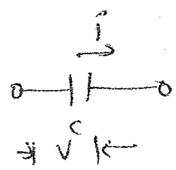
(with no initial charge)

In time domain, $v = \frac{1}{C}\int i\,dt$

$\Rightarrow i = C\frac{dv}{dt}$

Taking L.T we get.

$I(S) = CS\,V(S) - \cancel{C\,V(0^+)} \to 0$



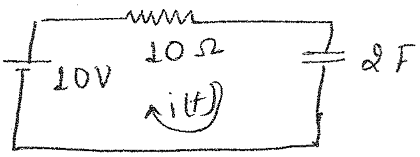
$\frac{V(S)}{I(S)} = \frac{1}{CS}$

(with initial charge)

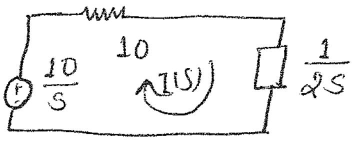
$I(S) = CS\,V(S) - C\,V(0^+)$

When, all the elements in time domain along with source are converted to s-domain then newly formed circuit is called transformed circuit.

Ex 1 Find $i(t)$ by transformed circuit method.



Soln: In s-domain, above circuit becomes



using, KVL

$$\frac{10}{s} = 10\,I(s) + \frac{1}{2s}\,I(s)$$

$$\Rightarrow I(s)\left[10 + \frac{1}{2s}\right] = \frac{10}{s}$$

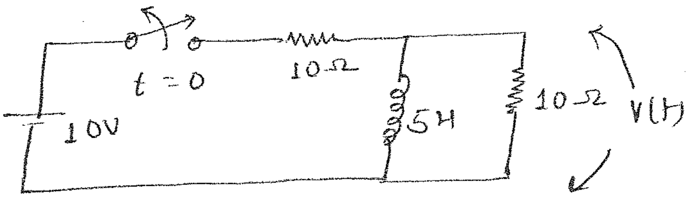
$$\Rightarrow I(s)\left[\frac{20s+1}{2s}\right] = \frac{10}{s}$$

$$\Rightarrow I(s) = \frac{20}{20s+1} = \frac{1}{s+1/20}$$

Taking, inverse L.T., we get

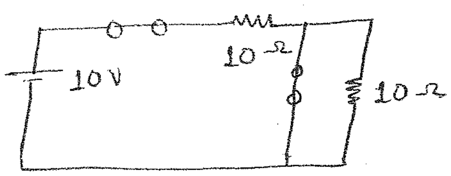
$$i(t) = e^{-\frac{1}{20}t}$$

Ex: 2 Find $v(t)$ using transformed circuit:



Soln:

Here, At, $t = 0^-$

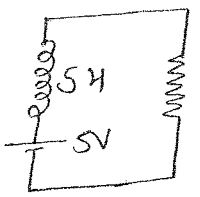


$$i(0^-) = \frac{10}{10} = 1A$$

$\therefore\ i(0^+) = i(0^-) = 1A$ [$\because$ Current through inductor can not change instanteneously.

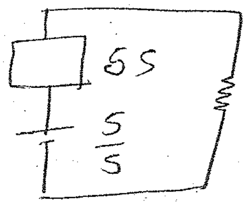
For, $t = 0^+$

Now, Equivalent circuit is,



Now, ~~Taking KVL,~~ In s-domain,

~~$5V = 5\frac{di}{dt} + 10$~~



using voltage dividing rule

$$V(S) = \frac{5}{S} \cdot \frac{10}{10+5S} = \frac{10}{S(S+2)}$$

$$= \frac{5}{S} - \frac{5}{S+2}$$

Taking inverse L.T. we get

$$v(t) = 5 - 5e^{-2t}$$

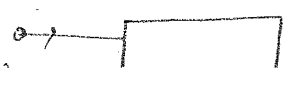
Transfer Function

Transfer function is a mathematical expression in s-domain which relates input and output characteristics of transform circuit. It can also be defined as the ratio of Laplace transform of output to the Laplace transform of the input, under the assumption that all the initial conditions are zero.

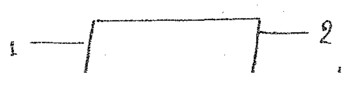
Mathematically, Let $x(t)$ be the input function and $y(t)$ be the output function. Then network function $H(s)$ is given by,

$$H(S) = \frac{L[y(t)]}{L[x(t)]} = \frac{Y(S)}{X(S)}$$

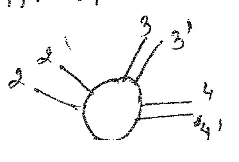
For any electronic circuit two terminals are associated which are called terminal pair or port. In one port network, terminal pair is connected to an energy source which is the driving force for the network, so that pair of terminal is called driving point of the n/w.



1 port n/w



2 port n/w



In two port network, 1-1' port is connected to driving force and port 2-2' is connected to the load. For one port network the function that relates the voltage and current at the same port is called driving point impedance Z(s) or driving point admittance Y(s).

i.e. $Z(s) = \frac{V(s)}{I(s)}$, $Y(s) = \frac{I(s)}{V(s)}$

where, V(s) (voltage) is the response parameter and I(s) (current) is source or vice-versa.

<u>For two port network</u>

There are two pairs of driving point impedance and two pairs of driving point admittance for two port network, given by

$$Z_{11}(s) = \frac{V_1(s)}{I_1(s)} \quad , \quad Y_{11}(s) = \frac{I_1(s)}{V_1(s)}$$

$$Z_{22}(s) = \frac{V_2(s)}{I_2(s)} \quad , \quad Y_{22} = \frac{I_2(s)}{V_2(s)}$$

The network function which relates the transform of a quantity at one port to the transform of another quantity at the another port is called transfer function. Thus, the transfer function which relates voltage and current for two port n/w has the following cases:

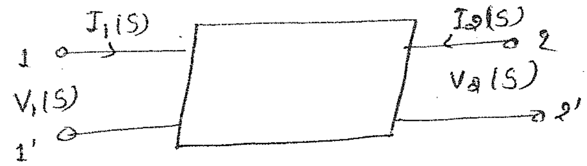


Fig: 2 port n/w.

ⓐ <u>Voltage transfer Function (ratio):</u>

It is defined as the ratio of laplace transform of voltages at two ports and is given by,

$$G_{12}(s) = \frac{V_2(s)}{V_1(s)}$$

ⓑ <u>Current transfer function (ratio):</u>

It is defined as the ratio of laplace transform of currents at two ports and is given by, $\alpha_{12} = \frac{I_2(s)}{\ldots}$

(c) Transfer admittance:

It is defined as the ratio of transform of current at one port to transform of voltage at next ~~point~~ port.

i.e. $Y_{12}(S) = \frac{I_2(S)}{V_1(S)}$ and, $Y_{21}(S) = \frac{I_1(S)}{V_2(S)}$

(d) Transfer impedance:

ratio of transform of voltage at one port to transform of current at next port.

i.e. $Z_{12}(S) = \frac{V_1(S)}{I_2(S)}$ and, $Z_{21}(S) = \frac{V_2(S)}{I_1(S)}$

(e) Driving Impedance

It is the ratio of Laplace transform of voltage and currents at the same ports.

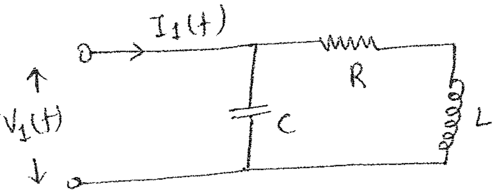
$Z_{11}(S) = \frac{V_1(S)}{I_1(S)}$ and $Z_{22}(S) = \frac{V_2(S)}{I_2(S)}$

(f) Driving admittance

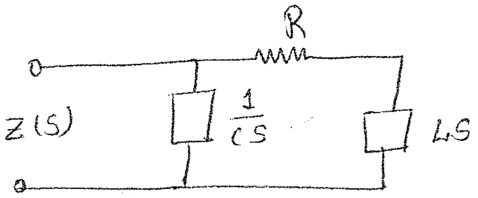
It is the ratio of Laplace transform of current and voltage at the same ports.

$Y_{11}(S) = \frac{I_1(S)}{V_1(S)}$ and, $Y_{22}(S) = \frac{I_2(S)}{V_2(S)}$

Q.1. Find the driving point impedance for the following one port n/w.



soln: The transform circuit is,

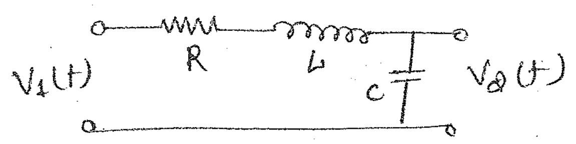


Now, $Z(S) = \frac{1}{CS} // (R+LS)$

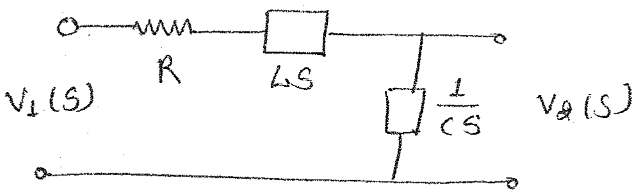
$$= \frac{\frac{1}{CS}(R+LS)}{\frac{1}{CS}+R+LS} = \frac{R+LS}{1+RCS+LCS^2} = \frac{R+LS}{LCS^2+RCS+1}$$

$$\therefore Z(S) = \frac{\frac{R}{LC}+\frac{1}{C}S}{}$$

Q.2 Find the transfer function $\frac{V_2(S)}{V_1(S)}$ in the following circuit.



Sol^n: Transform circuit is,



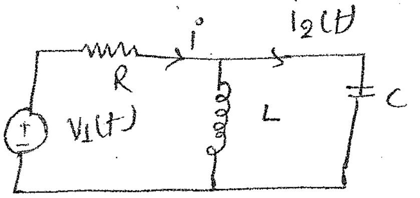
By using voltage dividing rule, we get

$$V_2(S) = \frac{\frac{1}{CS} \times V_1(S)}{R+LS+\frac{1}{CS}}$$

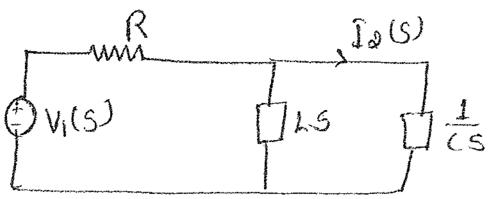
$$\Rightarrow \frac{V_2(S)}{V_1(S)} = \frac{\frac{1}{CS}}{\frac{1}{CS}\left[RCS+LCS^2+1\right]} = \frac{1}{LCS^2+RCS+1}$$

$$\therefore \frac{V_2(S)}{V_1(S)} = \frac{\frac{1}{LC}}{S^2+\frac{R}{L}S+\frac{1}{LC}}$$

Q3 Find transfer admittance $\frac{I_2(S)}{V_1(S)}$ in the following circuit.



Sol^n: The transform circuit is,



$$I(S) = \frac{V_1(S)}{R+(LS \,//\, \frac{1}{CS})}$$

$$= \frac{V_1(S)}{R+\frac{LS \times 1/CS}{LS+\frac{1}{CS}}} = \frac{V_1(S)}{R+\frac{L/C}{\frac{LCS^2+1}{CS}}} = \frac{V_1(S)}{RLCS^2+R+LS} \times (LCS^2+1)$$

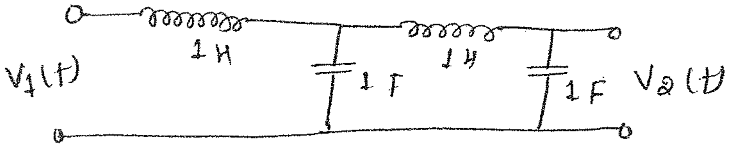
$$= \frac{(LCS^2+1)}{(RLCS^2+LS+R)} V_1(S)$$

NOW, $I_2(S) = \dfrac{LS}{LS + \frac{1}{CS}} \times I(S)$

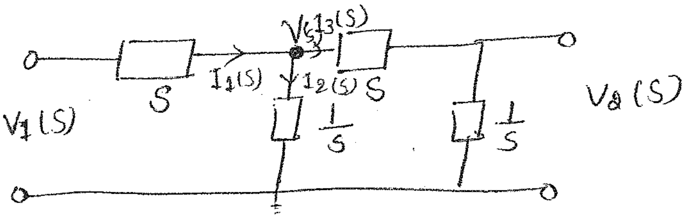
$$= \frac{LCS^2}{(LCS^2+1)} \cdot \frac{(LCS^2+1)V_1(S)}{(RLCS^2+LS+R)}$$

or, $$\frac{I_2(S)}{V_1(S)} = \frac{LCS^2}{RLCS^2+LS+R} = \frac{S^2/R}{S^2 + \frac{1}{RC}S + \frac{1}{LC}}$$

3.4. Find the transfer function $\dfrac{V_2(S)}{V_1(S)}$ of the following circuit.



Sol^n: The transform circuit is,



Using nodal analysis.

$$I_1(S) = I_2(S) + I_3(S)$$

$$\frac{V_1(S) - V(S)}{S} = \frac{V(S) - 0}{1/S} + \frac{V(S) - V_2(S)}{S}$$

$$\frac{V_1(S)}{S} - \frac{V(S)}{S} = SV(S) + \frac{V(S)}{S} - \frac{V_2(S)}{S}$$

$$V(S)\left[\frac{1}{S} + S + \frac{1}{S}\right] = \frac{V_1(S) + V_2(S)}{S}$$

$$V(S)\left[\frac{2+S^2}{S}\right] = \frac{V_1(S) + V_2(S)}{S}$$

$$V(S) = \frac{V_1(S) + V_2(S)}{(S^2+2)}$$

Using voltage dividing rule

$$V_2(S) = \frac{\frac{1}{S} \times V(S)}{S + \frac{1}{S}} = \frac{V_1(S) + V_2(S)}{(S^2+2)} \times \frac{1}{(S^2+1)} = \frac{V_1(S) + V_2(S)}{S^4 + 3S^2 + 2}$$

or, $(s^4+3s^2+2)\, V_2(s) - V_2(s) = V_1(s)$

or, $(s^4+3s^2+1)\, V_2(s) = V_1(s)$

$$\boxed{\therefore \frac{V_2(s)}{V_1(s)} = \frac{1}{s^4+3s^2+1}}$$

Poles and zeros of network functions (Transfer functions)

All the network function can be written as the result of polynomial as a function of s.

Mathematically, $N(s) = \frac{P(s)}{Q(s)} = \frac{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}$

$$= \frac{a_0}{b_0} \frac{(s-z_1)(s-z_2)(s-z_3)\cdots(s-z_n)}{(s-P_1)(s-P_2)(s-P_3)\cdots(s-P_m)}$$

where $\frac{a_0}{b_0}$ is constant, $z_1, z_2, z_3, \ldots z_n$ and $P_1, P_2, P_3, \ldots P_m$ are factors of numerator and denuminator of $N(s)$ and are complex frequencies.

When the variable s has the value equals to $z_1, z_2, \ldots, z_n$ polynomial $N(s)$ equals zero. Then such complex frequencies for which $N(s) = 0$ are called zeros of transfer function $N(s)$.

Similarly, when the variable s has the value equals to $P_1, P_2, \ldots, P_M$, polynomial $N(s)$ equals infinity (∞). Then such complex frequencies for which $N(s) = \infty$ are called poles of transfer function $N(s)$.

For any network functions zeros are represented by (o), and poles are represented by $(*)$.

Examples

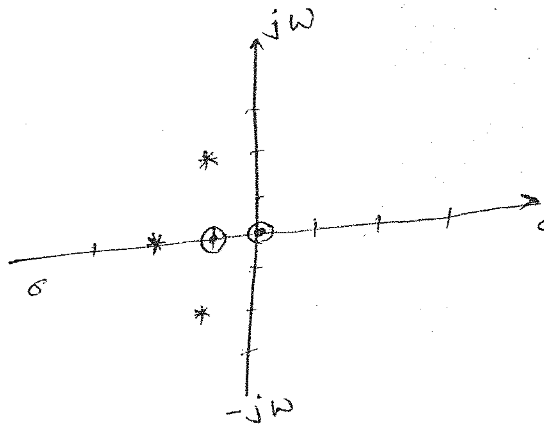
Draw poles and zeros for network function:

① $N(s) = \dfrac{s(s+1)}{(s+2)(s^2+2s+5)}$

Sol^n: Given, $N(s) = \dfrac{s(s+1)}{(s+2)(s^2+2s+5)}$

$$= \frac{s(s+1)}{(s+2)\{(s+1)^2-(2j)^2\}}$$

$$= \frac{s(s+1)}{(s+2)(s+1-2j)(s+1+2j)}$$



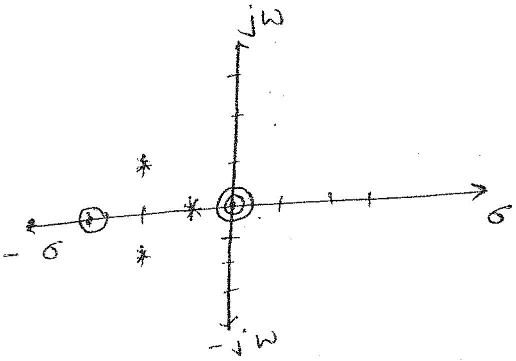
Then, zeros are at, $s = 0$, $s = -1$

poles are at, $s = -2, -1 \pm 2j$

② $N(s) = \dfrac{s^2(s+3)}{(s+1)(s^2+4s+5)}$

Given, $N(s) = \dfrac{s^2(s+3)}{(s+1)(s^2+4s+5)}$

$$= \frac{s^2(s+3)}{(s+1)\{(s+2)^2-j^2\}} = \frac{s^2(s+3)}{(s+1)(s+2\pm j)}$$



Then, zeros are at, $s = 0, 0, -3$

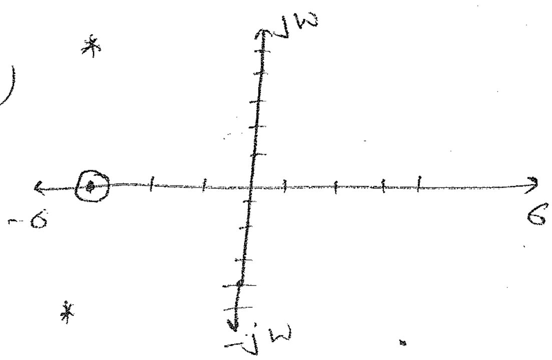
and, poles are at, $s = -1, -2 \pm j$

③ $f(t) = e^{-3t}\cos 5t$

Sol^n: Given, $f(t) = e^{-3t}\cos 5t$

Taking Laplace transform

$$F(s) = \frac{s+3}{(s+3)^2+5^2} = \frac{s+3}{(s+3\pm 5j)}$$



zeros are at, $s = -3$

poles are at, $s = -3 \pm 5j$

Network stability:

Network is said to be stable if its response is finite at any time and system implies that small changes in system input or nitial condition or in system parameter does not results in large change in output.

ondition of stability:

→ If all poles of the transfer function relating the output to input are confirmed on left half of s-plane, the system is stable.

→ If any of the pole that lies on $j\omega$ is repeated or lies on right half of complex s plane, then the system is unstable.

Routh - Hurtiwz criteria for stability (R-H criteria)

It states that "The network or system described by a transfer function for which P(s) is the denominator polynomial, is stable if there are no changes of sign in first column of the array."

R-H array:

Consider a system with characteristics equation,

$$P(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_n = 0$$

Thus, we construct RH array as below:

s^n	a_0	a_2	a_4
s^{n-1}	a_1	a_3	a_5
s^{n-2}	b_1	b_2	b_3
s^{n-3}	c_1	c_2	
s^{n-4}	d_1		
⋮			
s^0			

Where, $b_1 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}$

$b_2 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix}$

$c_1 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix}$

$c_2 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_5 \\ b_1 & b_3 \end{vmatrix}$

$d_1 = -\frac{1}{c_1}\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$

Now, we count the number of sign changes as we go from $a_0, a_1, b_1, c_1, \ldots$ The no. of sign changes of the coefficient of the first column gives the number of the roots of the characteristic equation that are right half of the s-plane.

So, for stable system all the coefficients $a_0, a_1, b_1, c_1, \ldots$ should be of same sign.

<u>Example</u>

1. Check stability of the system with transfer function

$$G(s) = \frac{s^2+s+1}{s^5+15s^4+85s^3+225s^2+274s+120}$$

Soln: R-H array for above transfer function is,

s^5	1	85	274
s^4	15	225	120
s^3	$b_1 = 70$	$b_2 = 266$	×
s^2	$c_1 = 168$	$c_2 = 120$	
s^1	$d_1 = 216$	×	
s^0	$e_1 = 120$		

$$b_1 = -\frac{1}{15}\begin{vmatrix} 1 & 85 \\ 15 & 225 \end{vmatrix} = 70$$

$$b_2 = -\frac{1}{15}\begin{vmatrix} 1 & 274 \\ 15 & 120 \end{vmatrix} = 266$$

$$c_1 = -\frac{1}{70}\begin{vmatrix} 15 & 255 \\ 70 & 266 \end{vmatrix} = 168$$

$$c_2 = -\frac{1}{70}\begin{vmatrix} 15 & 120 \\ 70 & 0 \end{vmatrix} = 120$$

$$d_1 = -\frac{1}{168}\begin{vmatrix} 70 & 266 \\ 168 & 120 \end{vmatrix} = 216$$

$$e_1 = -\frac{1}{216}\begin{vmatrix} 168 & 120 \\ 216 & 0 \end{vmatrix} = 120$$

Since, The first column of R-H array has no sign change i.e. non of the poles lies in the right half of the s-plane. Thus, the system is stable.

2. $s^4 + 3s^3 + 4s^2 + 42s + 40 = 0$

Soln: R-H array of above transfer function is,

s^4	1	4	40
s^3	3	42	0
s^2	$b_1 = -10$	$b_2 = 40$	
s^1	$c_1 = 54$	0	
s^0	40		

$$b_1 = -\frac{1}{3}\begin{vmatrix} 1 & 4 \\ 3 & 42 \end{vmatrix} = -10$$

$$b_2 = -\frac{1}{3}\begin{vmatrix} 1 & 40 \\ 3 & 0 \end{vmatrix} = 40$$

$$c_1 = +\frac{1}{10}\begin{vmatrix} 3 & 42 \\ -10 & 40 \end{vmatrix} = 54$$

Since, the first column of R-H array has sign change i.e. There is a poles in right half of s-plane. Thus, the system is unstable.

Q.3 Find the value of A for which

$$s^3+2s^2+2s+A=0 \text{ is stable.}$$

Soln: R-H array:

s^3	1	2
s^2	2	A
s	$-\frac{1}{2}(A-4)$	
s^0	$\frac{2}{A-4}\left\{\frac{1}{2}A(A-4)\right\}=A$	

The system will be stable if A is greater than zero and $\frac{4-A}{2}>0$

i.e. $4>A$.

But for $A=4$, third row seems to be zero and other element can not be determined. So, in such cases we replace zero by very small number near to zero represented by ϵ.

For $A=4$,

R-H array becomes.

s^3	1	2
s^2	2	4
s	0	
s^0	can not be calculated	

Special case:

When element of 1st row is zero

In such case we replace 0 by small number ϵ. Then we decide the stability depending on values of ϵ.

Now, R-H array becomes:

s^3	1	2
s^2	2	4
s^1	ϵ	
s^0	4	

Thus, for $A = 4$, System is stable if ϵ is greater than zero and system is unstable if ϵ is less than zero.

Q.4 Check the stability for $s^4 + s^3 + 2s^2 + 2s + 3$.

Sol^n: R-H array

$$b_1 = -\frac{1}{1}\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0$$

s^4	1	2	3
s^3	1	2	0
s^2	0	3	
s^1	can not be determined		
s^0	can not be determined		

Since, first column has zero in third row. Fourth row becomes infinity so we can not find stability. In such cases we replace zero by small number ϵ and stability depends on ϵ.

Now, Replacing 0 by ϵ above R-H array becomes

s^4	1	2	3
s^3	1	2	
s^2	ϵ	3	
s^1	$c_1 = 2 - \frac{3}{\epsilon}$	0	
s^0	3		

$$c_1 = -\frac{1}{\epsilon}\begin{vmatrix} 1 & 2 \\ \epsilon & 3 \end{vmatrix} = \frac{2\epsilon - 3}{\epsilon}$$

For ϵ greater than zero, $2 - \frac{3}{\epsilon}$ is negative. So, system is unstable. Also, for ϵ less than zero first element of third row is negative. So, system is again unstable. Hence, the system defined by above equation is unstable.

Q.5 Check the stability for:

Soln: R-H array is,

s^6	1	11	36	36
s^5	5	25	30	0
s^4	$b_1 = 6$	$b_2 = 30$	36	
s^3	$c_1 = 0$	$c_2 = 0$	0	
s^2	can not be determined			
s^1				
s^0				

$$b_1 = -\frac{1}{5}\begin{vmatrix} 1 & 11 \\ 5 & 25 \end{vmatrix} = \frac{55-25}{5} = 6$$

$$b_2 = -\frac{1}{5}\begin{vmatrix} 1 & 36 \\ 5 & 30 \end{vmatrix} = \frac{36\times5-30}{5} = 30$$

$$c_1 = -\frac{1}{6}\begin{vmatrix} 5 & 25 \\ 6 & 30 \end{vmatrix} = \frac{150-150}{6} = 0$$

$$c_2 = -\frac{1}{6}\begin{vmatrix} 5 & 30 \\ 6 & 36 \end{vmatrix} = \frac{180-180}{6} = 0$$

If entire row is zero then we take the polynomial of earlier row and divide the system polynomial by that polynomial to find the quotient then stability is determined by quotient.

$$\frac{1}{6}s^2 + \frac{5}{6}s + 1$$

$$6s^4+30s^2+36 \,\big)\, s^6+5s^5+11s^4+25s^3+36s^2+30s+36 \,\big($$

$$\underline{-s^6 + 5s^4 + 6s^2}$$

$$5s^5+6s^4+25s^3+30s^2+30s+36$$

$$\underline{-5s^5+25s^3+30s}$$

$$6s^4+30s^2+36$$

$$\underline{6s^4+30s^2+36}$$

$$\times$$

Here, quotient is, $\frac{1}{6}s^2+\frac{5}{6}s+1$ then R-H array is

s^2	$\frac{1}{6}$	1
s^1	$\frac{5}{6}$	0
s^0	$b_1 = 1$	

$$b_1 = -\frac{6}{5}\begin{vmatrix} 1/6 & 1 \\ 5/6 & 0 \end{vmatrix} = 1$$

Since the first column of R-H array for quotient has no sign changes. So, the system is stable.

Q.6 Check the stability

$$s^5+s^4+4s^3+24s^2+3s+63=0$$

Soln: R-H array is

s^5	1	4	3
s^4	1	24	63
s^3	-20	-60	0
s^2	21	63	0
s^1	0	0	0
s^0	can not be determined		

$$b_1 = -\frac{1}{1}\begin{vmatrix} 1 & 4 \\ 1 & 24 \end{vmatrix} = -20$$

$$b_2 = -\frac{1}{1}\begin{vmatrix} 1 & 3 \\ 1 & 63 \end{vmatrix} = -60$$

$$c_1 = \frac{1}{20}\begin{vmatrix} 1 & 24 \\ -20 & -60 \end{vmatrix} = 21$$

$$c_2 = \frac{1}{20}\begin{vmatrix} 1 & 63 \\ -20 & 0 \end{vmatrix} = 63$$

$$D_1 = -\frac{1}{21}\begin{vmatrix} -20 & -60 \\ 21 & 63 \end{vmatrix} = 0$$

$$D_2 = -\frac{1}{21}\begin{vmatrix} -20 & 0 \\ 21 & 0 \end{vmatrix} = 0$$

Here, the entire row for s^1 is zero. So we take polynomial just above that row

i.e. $P(s^2) = 21s^2 + 63$

Then, taking derivatives

$P'(s^2) = 2 * 21s = 42s$

Now, R.H. array becomes

s^5	1	4	3
s^4	1	24	63
s^3	-20	-60	0
s^2	21	63	0
s^1	42	0	
s^0	63		

Limitation of R-H criteria:

1. Valid only if characteristic equation is algebric with real coefficients. If any of coefficient is complex or if equation is not algebric RH criteria can not be applied.
2. Gives information about roots only with respect to left half or right half of s-plane i.e. does not gives information about roots on $j\omega$-axis.
3. Can not be applied to discrete time system.

Here, first column of R-H array has two sign changes. So, the system is unstable.

Time domain behaviour of circuit from pole and zero plot:

Consider a transfer function $G_{12}(s)$ that relates the response function $V_2(s)$ with source function $V_1(s)$. Thus, the relation will be as,

$$V_2(s) = G_{12}(s) * V_1(s)$$

Let the transfer function $G_{12}(s)$ has m poles i.e. $P_1, P_2, P_3, \ldots P_m$ and $V_1(s)$ has n poles $P_{m+1}, P_{m+2}, \cdots, P_{m+n}$, then using the partia fraction expansion, the response $V_2(s)$ can be expressed as

$$V_2(s) = \frac{K_1}{s-P_1} + \frac{K_2}{s-P_2} + \cdots + \frac{K_m}{s-P_m} + \frac{K_{m+1}}{s-P_{m+1}} + \cdots + \frac{K_{m+n}}{s-P_{m+n}}$$

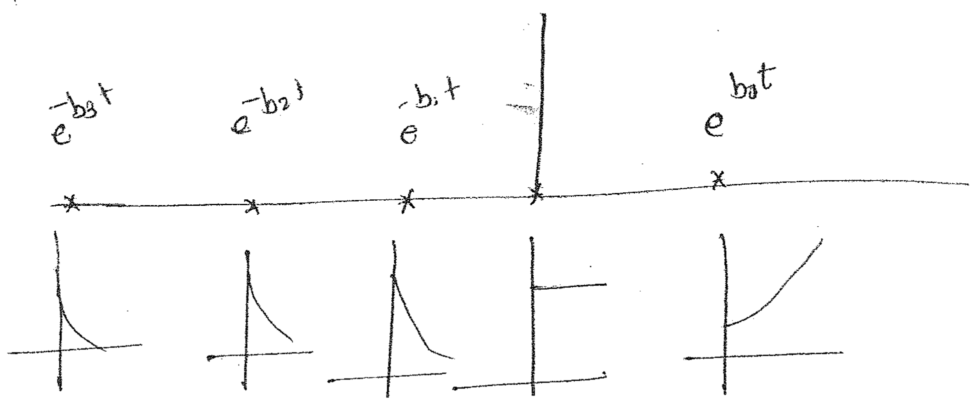
The poles $P_1, P_2, P_3, \ldots, P_m$ contributes to the transient response and the poles $P_{m+1}, P_{m+2}, \ldots, P_{m+n}$ contributes to force response. Taking inverse Laplace transform of the above equation:

$$V_2(t) = K_1 e^{P_1 t} + K_2 e^{P_2 t} + \cdots + K_m e^{P_m t} + K_{m+1} e^{P_{m+1} t} + \cdots + K_{m+n} e^{P_{m+n} t}$$

The poles therefore determines the waveform of the time variation of the response, the o/p voltage. The zeros determines the magnitude of each part of the response i.e. resposible for values $K_1, K_2, \ldots, K_{m+n}$.

In time domain, we see that the real part of each pole appears with an exponential term. If this real part is -ve, the exponential form decays to 0 as time increases.

Hence, for a system all poles of network function should lie on left half of s-plane but zeros are not so restricted.



Complex conjugate pole corresponds to oscillatory nature in time domain.

Time-Domain Response from pole-zero plot:

The time-domain response can be obtained from the pole-zero plot of a network function. Consider a network function given by.

$$H(s) = \frac{N(s)}{D(s)} = K\frac{(s-z_1)(s-z_2)\cdots(s-z_n)}{(s-p_1)(s-p_2)\cdots(s-p_m)}$$

Where, $z_1, z_2, \ldots, z_n$ are zeros and $p_1, p_2, p_3, \ldots, p_m$ are poles of the function $H(s)$.

Assume that poles and zeroes are distinct. Using partial fraction expansion,

$$H(s) = \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \cdots + \frac{k_m}{s-p_m}$$

Where,

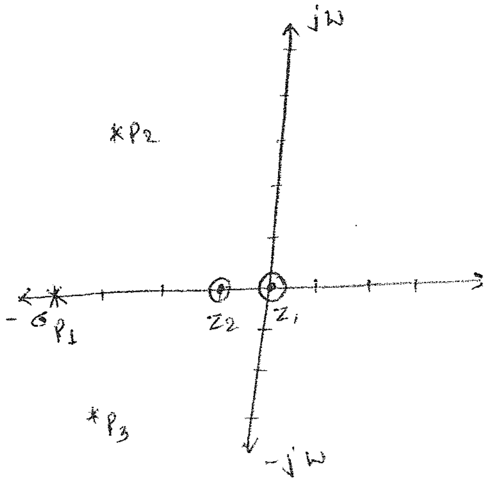
$$k_i = \left[(s-\cancel{z_i})H(s)\right]_{s=p_i} ; \quad i = 1, 2, \ldots, m$$

$$\text{or, } k_i = \frac{K(p_i-z_1)(p_i-z_2)\cdots(p_i-z_n)}{(p_i-p_1)(p_i-p_2)\cdots(p_i-p_m)}$$

Q. If $H(s) = \dfrac{s(s+1)}{(s+4)(s^2+6s+18)}$; find $h(t)$ using the ~~plo~~ pole-zero diagram of the function.

Soln: Here, $H(s) = \dfrac{s(s+1)}{(s+4)(s^2+6s+18)}$

$$= \frac{s(s+1)}{(s+4)\{(s+3)^2+3^2\}} = \frac{s(s+1)}{(s+4)(s+3\pm 3j)}$$



zeroes are at, $s = 0, -1$

poles are at, $s = -4, -3 \pm 3j$

$$H(s) = \frac{k_1}{s+4} + \frac{k_2}{(s+3+3j)} + \frac{k_2}{(s+3-3j)}$$

For pole $P_1 = -4$

$$k_1 = K\frac{(P_1-z_1)(P_1-z_2)}{(P_1-P_2)(P_1-P_3)} = \frac{1(-4-0)(-4+1)}{(-4+3+3j)(-4+3\cancel{+}3j)} = \frac{12}{1^2+3^2}$$

$$= \frac{12}{10} = 1.2$$

For the pole, $P_2 = -3 - 3j$

$$K_2 = \frac{(P_2 - Z_1)(P_2 - Z_2)}{(P_2 - P_1)(P_2 - P_3)} = \frac{(-3-j3-0)(-3-j3+1)}{(-3-j3+4)(-3-j3+3-j3)}$$

$$= \frac{(-3-j3)(-2-j3)}{(1-j3)(-j6)} = \frac{6+9j+6j-9}{-6j-18} = \frac{-3+15j}{-6(j+3)}$$

$$= \frac{1-5j}{6+2j} = \frac{1-5j}{6+2j} \times \frac{6-2j}{6+2j} = \frac{6-2j-30j-10}{36+4}$$

$$= \frac{-4-32j}{40} = \frac{1}{10}(-1-j8)$$

For the pole, $P_3 = -3+3j$

$$K_3 = K_2^* = \frac{1}{10}(-1+j8)$$

$$H(s) = \frac{1.2}{s-4} + \frac{1}{10}(-1-j8)\frac{1}{s+3+3j} + \frac{1}{10}(-1+j8)\frac{1}{s+3-3j}$$

Taking inverse Laplace transform:

$$\therefore h(t) = 1.2e^{-4t} + \frac{1}{10}(-1-j8)e^{(-3-3j)t} + \frac{1}{10}(-1+j8)e^{(-3+3j)t}$$

$$= 1.2e^{-4t} + e^{-3t}\left[-\frac{1}{10}\left(e^{j3t} + e^{-j3t}\right) + \frac{8}{10}j\left(e^{j3t} - e^{-j3t}\right)\right]$$

$$= 1.2e^{-4t} + e^{-3t}\left[-\frac{1}{10}(2\cos 3t) - \frac{8}{10}(2\sin 3t)\right]$$

$$h(t) = 1.2e^{-4t} - \frac{1}{5}e^{-3t}\left[\cos 3t + 8\sin 3t\right]$$

Periodic signals

A signal $f(t)$ is said to be periodic with time period T if $f(t) = f(t+T)$ for all value of t and T is a constant time known as time period.

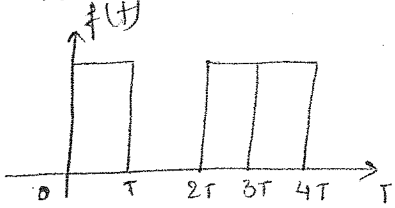


Fig: non - periodic signal

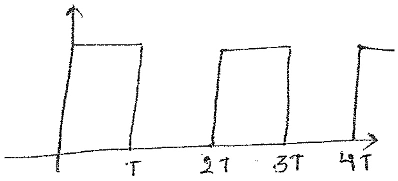


Fig: Periodic signal

Fundamental or Time period.

Minimum positive, non-zero value of T, for which $f(t) = f(t+T)$ is satisfied.

Fundamental frequency $(\omega_0) = \frac{2\pi}{T}$

Even and odd signals:

A Function is said to be even function if it ~~exis~~ satisfies $f(t) = f(-t)$ and a function is said to be odd function if it satisfies $f(t) = -f(-t)$.

<u>Example</u>: t^n for even value of n, $|\sin t|$, $\cos t$ are even function whereas t^n for odd value of n, $\sin t$ are odd function.

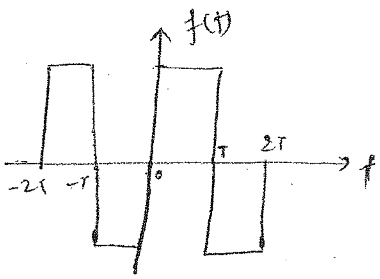


fig: odd function

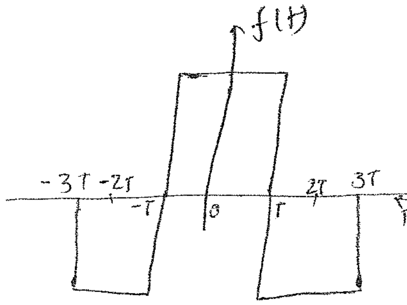


Fig: even function.

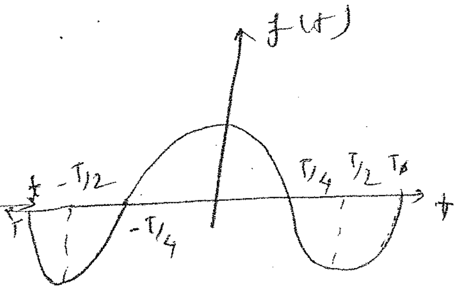


fig: even function

Fourier series

Fourier series is the representation of any periodic signal inter of complex exponential as given by following equation.

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}$$

where, $\omega_0 = \frac{2\pi}{T}$ is fundamental frequency.

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n e^{jn\omega_0 t} + a_n^* e^{-jn\omega_0 t} \right]$$ where a_n is complex fourier

Any periodic signal $f(t)$ can be represented by the infinite sum of sine wave ~~representation~~ components which is called fourier series expansion.

i.e. $$f(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2\pi n t}{T_0} + b_n \sin \frac{2\pi n t}{T_0}\right]$$

$$= a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \omega_0 n t + b_n \sin \omega_0 n t\right] \quad \ldots\ldots ①$$

where, T_0 is the time period of $f(t)$ and $T_0 = \frac{1}{f_0}$ where f_0 is the fundamental frequency. and nf_0 are harmonics of f_0.

a_0, a_n and b_n of equation ① are called trigonometric fourier series coefficients.

Determination of fourier coefficient:

ⓐ For complex exponential functions:

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}$$

where, a_n is given by, $a_n = \frac{1}{T}\int_0^T x(t) e^{-jn\omega_0 t} dt$

for $n=0$, $a_0 = \frac{1}{T}\int_0^T x(t)dt$.

ⓑ For trigonometric function

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

where, $a_0 = \frac{1}{T}\int_0^T x(t)dt$

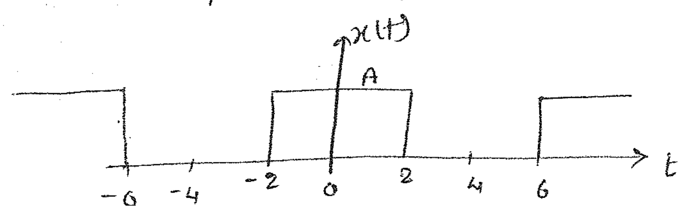
$a_n = \frac{2}{T}\int_0^T x(t) \cos n\omega_0 t\, dt$

$b_n = \frac{2}{T}\int_0^T x(t) \sin n\omega_0 t\, dt$

For even function : $b_n = 0$, $a_n = \frac{4}{T}\int_0^{T/2} x(t) \cos n\omega_0 t\, dt$, $a_0 = \frac{2}{T}\int_0^{T/2} x(t)dt$

For odd function : $a_0 = 0$ and $a_n = 0$, $b_n = \frac{4}{T}\int_0^{T/2} x(t) \sin n\omega_0 t\, dt$

Q.1 Find the exponential fourier series coefficient for the signal given below:



$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

Soln: $T = 8$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{8} = \frac{\pi}{4}$$

Now, $x(t) = \begin{cases} A & \text{for } -2 \text{ to } 2 \\ 0 & \text{for } 2 \text{ to } 6 \end{cases}$

We have, $x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}$

Where, $a_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt$

$$= \frac{1}{8} \int_{-2}^{2} A e^{-jn\omega_0 t} dt$$

$$= \frac{A}{8} \left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0} \right]_{-2}^{2}$$

$$= \frac{A}{8} \left[\frac{e^{-2jn\omega_0} - e^{2jn\omega_0}}{-jn\omega_0} \right]$$

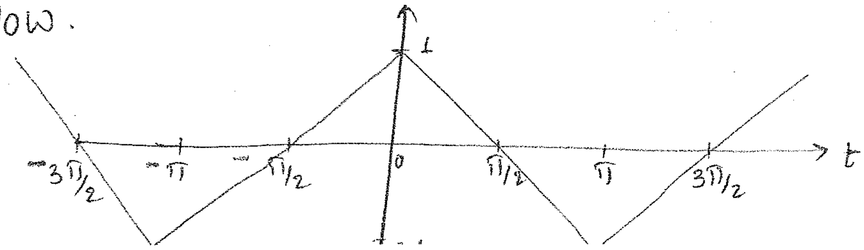
$$= \frac{A}{8} \left[\frac{e^{-jn\frac{\pi}{2}} - e^{jn\frac{\pi}{2}}}{-jn\frac{\pi}{4}} \right]$$

$$= \frac{A}{8} \left[\frac{e^{jn\pi/2} - e^{-jn\pi/2}}{2j} \right] \times \frac{2}{\frac{n\pi}{4}}$$

$$= \frac{A}{2} \frac{\sin n\pi/2}{n\pi/2}$$

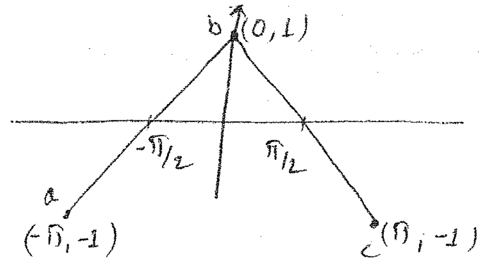
$$= \frac{A}{2} \text{sinc} \frac{n\pi}{2}$$

Q.2. Obtain the trigonometric fourier-series representation for the signal shown below.



Solⁿ: Here,

$$f(t) = \begin{cases} \frac{2}{\pi}t + 1 & \text{for } -\pi \text{ to } 0 \\ -\frac{2}{\pi}t + 1 & \text{for } 0 \text{ to } \pi \end{cases}$$



$T = 2\pi$, $\omega_0 = \frac{2\pi}{T} = 1$

Since function $f(t)$ is an even function. Hence $b_n = 0$

Eqⁿ of ab $\Rightarrow$ $y + 1 = \frac{2}{\pi}(x + \pi)$

$y = \frac{2}{\pi}x + 2 - 1 = \frac{2}{\pi}x + 1$

Eqⁿ of bc $\Rightarrow$ $y - 1 = -\frac{2}{\pi}(x)$

$y = -\frac{2}{\pi}x + 1$

Now,

$$a_0 = \frac{2}{T}\int_0^{T/2} f(t)\,dt$$

$$= \frac{2}{2\pi}\int_0^{\pi}\left(-\frac{2}{\pi}t + 1\right)dt$$

$$= \frac{1}{\pi}\left[-\frac{2}{\pi}\left[\frac{t^2}{2}\right]_0^{\pi} + [t]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{2}{\pi}\times\frac{\pi^2}{2} + \pi\right] = 0$$

$$a_n = \frac{4}{T}\int_0^{T/4} f(t)\cos n\omega_0 t\,dt$$

$$= \frac{4}{2\pi}\int_{-\pi}^{0}\left(\frac{2}{\pi}t + 1\right)\cos n\omega_0 t\,dt$$

$$= \frac{2}{\pi}\left[\int_{-\pi}^{0}\frac{2}{\pi}t\cos n\omega_0 t\,dt + \int_{-\pi}^{0}\cos n\omega_0 t\,dt\right]$$

$$= \frac{2}{\pi}\left[\left\{\frac{\sin n\omega_0 t}{n\omega_0}\right\}_{-\pi}^{0} + \left\{\frac{2}{\pi}\left[\left(\frac{t\sin n\omega_0 t}{n\omega_0}\right)_{-\pi}^{0} - \int_{-\pi}^{0}\frac{\sin n\omega_0 t}{n\omega_0}\,dt\right]\right\}\right]$$

$$= \frac{2}{\pi}\left[\left[\frac{\sin nt}{n}\right]_{-\pi}^{0} + \left[\frac{2}{\pi}\frac{t\sin nt}{n}\right]_{-\pi}^{0} - \frac{2}{n\pi}\left[\frac{\cos nt}{-n}\right]_{-\pi}^{0}\right]$$

$$= \frac{2}{\pi}\left[0 + 0 + \frac{2}{n^2\pi}\left[1 - (-1)^n\right]\right]$$

$$a_n = \frac{4}{n^2\pi^2}\left[1 - (-1)^n\right]$$

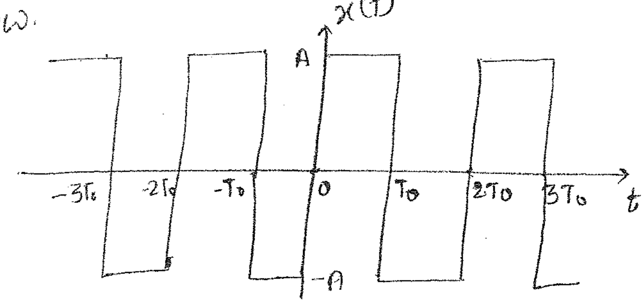
Therefore,

$a_n = 0$; if n is even i.e. 2, 4, 6

$a_n = \frac{8}{n^2\pi^2}$; if n is odd i.e. 1, 3, 5, 7 and $b_n = 0$

$$f(t) = \sum_{n=1} \frac{8}{n^2\pi^2} \cos nt \quad ; n \text{ is odd}$$

Q3. Obtain the trigonometric fourier series representation of the periodic signal as shown in figure below.



Soln:

Here, $x(t) = \begin{cases} A & \text{for } 0 \text{ to } T_0 \\ -A & \text{for } T_0 \text{ to } 2T_0 \end{cases}$

Time period, $T = 2T_0$

$$\omega_0 = \frac{2\pi}{2T_0} = \frac{\pi}{T_0}$$

Since, the function is an odd function, hence $a_0 = 0$, $a_n = 0$.

Now, $$b_n = \frac{4}{T}\int_0^{T/2} x(t)\sin n\omega_0 t\, dt$$

$$= \frac{4}{2T_0}\int_0^{T_0} A \sin n\omega_0 t\, dt$$

$$= \frac{2A}{T_0}\int_0^{T_0} \sin \frac{n\pi t}{T_0}\, dt$$

$$= \frac{2A}{T_0}\left[-\frac{\cos \frac{n\pi t}{T_0}}{\frac{n\pi}{T_0}}\right]_0^{T_0}$$

$$= \frac{2A}{n\pi}\left[-\cos n\pi + 1\right]$$

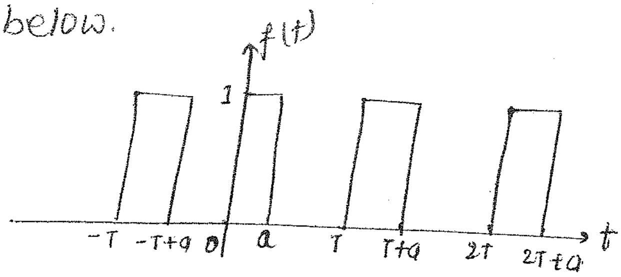
$$= \frac{2A}{n\pi}\left[1 - (-1)^n\right]$$

Therefore, $b_n = 0$; if n is even

$= \frac{4A}{n\pi}$; if n is odd

$$\therefore x(t) = \sum_{n=1}^{\infty} \frac{4A}{n\pi} \sin \frac{n\pi}{T_0} t \quad ; n \text{ is odd.}$$

3.4 Determine the exponential form of fourier series expansion for the periodic waveform shown in figure below.



sol^n: Here, $f(t) = \begin{cases} 1 & \text{for } 0 \text{ to } a \\ 0 & \text{for } a \text{ to } T \end{cases}$

$T = T$

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{T}\int_0^a dt = \frac{a}{T}$$

$$a_n = \frac{1}{T}\int_0^T f(t)\, e^{-jn\omega_0 t}\,dt$$

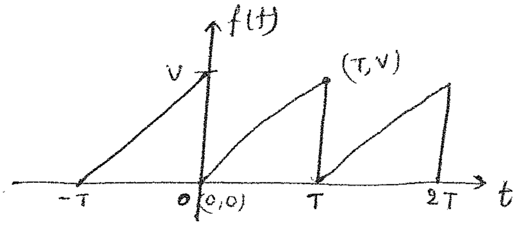
$$= \frac{1}{T}\int_0^T e^{-jn\omega_0 t}\,dt = \frac{1}{T}\left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right]_0^a$$

$$= \frac{1}{T}\left[\frac{e^{-jn\omega_0 a} - 1}{-jn\omega_0}\right]$$

$$= \frac{1 - e^{-jn\omega_0 a}}{jn\omega_0 T}$$

$$\therefore f(t) = \cancel{\frac{a}{T} + \sum_{k=}} \sum_{n=-\infty}^{\infty} \left(\frac{1 - e^{-jn\omega_0 a}}{jn\omega_0 T}\right) e^{jn\omega_0 t}$$

3.5 obtain the trigonometric fourier series of waveform shown below;



sol^n: Here,

$T = T$

$\omega_0 = \frac{2\pi}{T}$

and, $f(t) = \frac{V}{T}t$ for $t = 0$ to T

eq^n $\Rightarrow$ $y = \frac{V}{T}x$

Now, Trigonometric fourier series expansion is given as.

$$f(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t]$$

where, $a_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{T}\int_0^T \frac{V}{T}t\,dt = \frac{V}{T^2}\left[\frac{t^2}{2}\right]_0^T = \frac{V}{2}$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\,dt = \frac{2}{T}\int_0^T \frac{V}{T}t\cos n\omega_0 t\,dt$$

$$= \frac{2V}{T^2}\left\{\left[\frac{t\sin n\omega_0 t}{n\omega_0}\right]_0^T - \int \frac{\sin n\omega_0 t}{n\omega_0}\,dt\right\} = \frac{2V}{T^2}\left\{\frac{T\sin n2\pi}{n\frac{2\pi}{T}} - \frac{1}{n\omega_0}\left[-\frac{\cos n\omega_0}{n\omega_0}\right.\right.$$

$$= \frac{2V}{T^2}\left\{\frac{1}{n^2\omega_0^2}((\cos 2n\pi - 1)\right\}$$

$$= \frac{2V}{T^2}\, \frac{1}{n^2 * \frac{4\pi^2}{T^2}} \left(1 - 1\right)$$

$$= 0$$

and, $b = \frac{2}{T}\int_0^T f(t) \sin n\omega_0 t\, dt$

$$= \frac{2}{T}\int_0^T \frac{V}{T} t \sin \frac{2\pi n t}{T}\, dt$$

$$= \frac{2V}{T^2}\left\{\left[\frac{-t\cos\frac{2\pi n t}{T}}{\frac{2\pi n}{T}}\right]_0^T - \int_0^T \left(-\frac{\cos\frac{2\pi n t}{T}}{\frac{2\pi n}{T}}\right) dt\right\}$$

$$= \frac{2V}{T^2}\left\{-\frac{\cos 2n\pi}{\frac{2n\pi}{T}} + 0 + \frac{1}{\frac{2n\pi}{T}}\left[\frac{\sin\frac{2n\pi t}{T}}{\frac{2n\pi}{T}}\right]_0^T\right\}$$

$$= -\frac{2V}{T} \times \frac{1}{2n\pi} + \frac{2V}{(2n\pi)^2}\left[\cancel{\sin 2n\pi}\right] \rightarrow 0$$

$$= -\frac{V}{n\pi}$$

$$\therefore\ f(t) = \frac{V}{2} + \sum_{n=1}^{\infty}\left[-\frac{V}{n\pi} \sin\frac{2\pi n}{T} t\right].$$

Fourier transform :

Fourier transform is approach to develop the frequency domain representation of a non-periodic signal.

Let, $x(t)$ be non-periodic signal then its fourier transform or fourier integral is given as,

$$x(t) \xleftrightarrow{F.T.} X(j\omega)$$

$$X(j\omega) = F(x(t)) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

and Inverse fourier transform is,

$$x(t) = F^{-1}[X(j\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

Fourier transform is complex function so it has both magnitude and direction

Existence of fourier transform

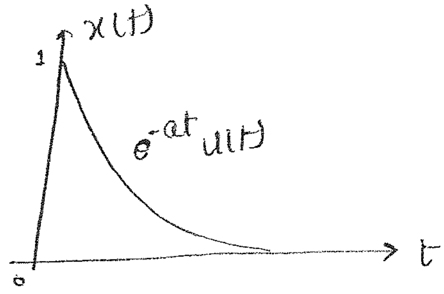
A function $x(t)$ is said to be fourier transformable, if it is satisfied following Dirichlet conditions:

i) The function $x(t)$ is absolutely integrable i.e. $\int_{-\infty}^{\infty} |x(t)|dt < \infty$

ii) The function has finite number of maxima and minima, if any, within a finite interval of time.

iii) The function is single valued and has finite number of discontinuities, if any, within a finite interval of time.

Q1. Obtain the fourier transform of a single-sided exponential function $e^{-at}u(t)$ as shown in figure below. Also draw the spectrum. (where $a>0$).



Soln: Here, $x(t) = e^{-at}u(t)$

Now,

$$X(j\omega) = F[x(t)]$$

$$= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t}dt = \left.\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right|_{0}^{\infty}$$

$$= \frac{1}{a+j\omega}$$

Since, Fourier transform is complex, both amplitude and phase spectrums are necessary i.e.

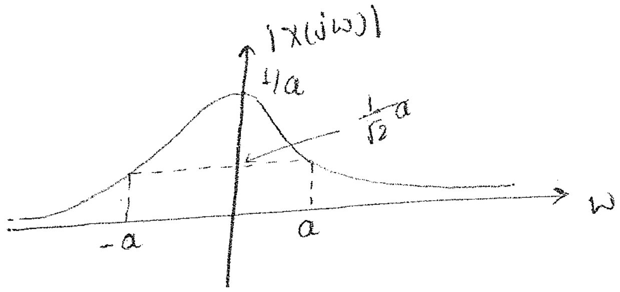
$$X(j\omega) = |X(j\omega)|.e^{j\theta(\omega)}$$

Amplitude spectrum: $|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$

and, phase spectrum, $\theta(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$

ω	$-a$	0	a
$\lvert X(j\omega)\rvert$	$\frac{1}{\sqrt{2}a}$	$\frac{1}{a}$	$\frac{1}{\sqrt{2}}a$

ω	$-a$	0	a
$\theta(\omega)$	$\pi/4$	0	$-\pi/2$



a) Amplitude spectrum

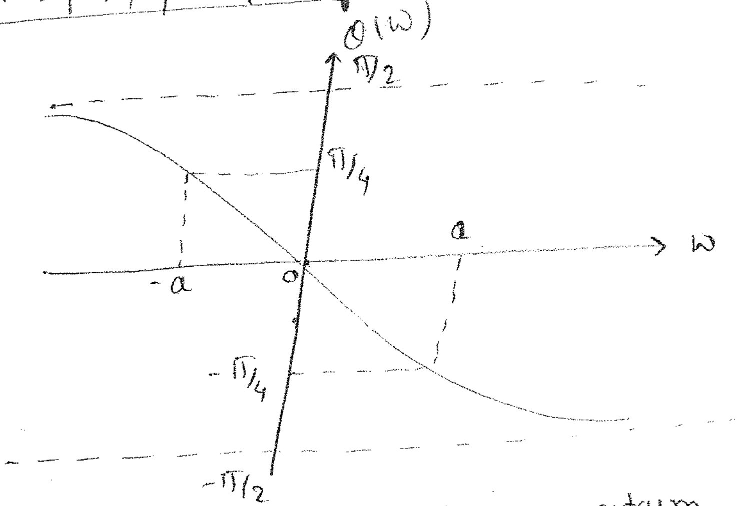
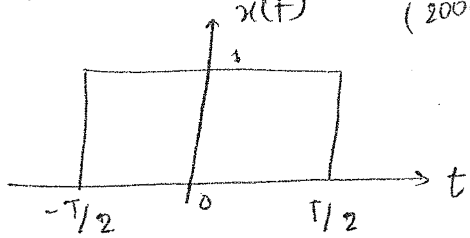


fig: phase spectrum.

Q.2 Find the fourier transform of the gate function shown in figure below. (2009 ~~spring~~ Fall)



Sol^n: Here,

$$x(t) = \begin{cases} 1 \; ; \text{ for } -\frac{T}{2} < t < \frac{T}{2} \\ 0 \; ; \text{ otherwise} \end{cases}$$

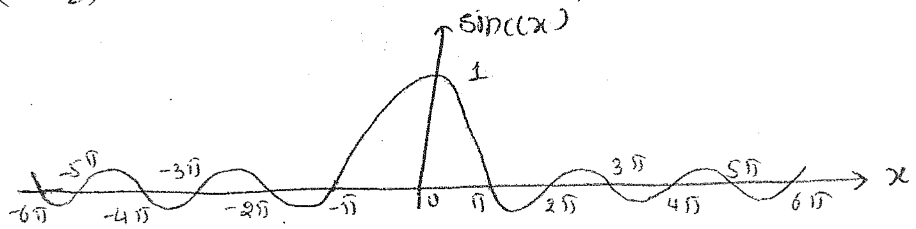
$$X(j\omega) = F[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-T/2}^{T/2} e^{-j\omega t} dt = \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{-T/2}^{T/2}$$

$$= \frac{e^{-j\omega\left(\frac{T}{2}\right)} - e^{-j\omega\left(-\frac{T}{2}\right)}}{-j\omega} = \frac{e^{j\omega\frac{T}{2}} - e^{-j\omega\frac{T}{2}}}{j\omega}$$

$$= \frac{2}{\omega}\left[\frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2j}\right] = \frac{2}{\omega}\sin(\omega T/2)$$

$$= \frac{T \sin(\omega T/2)}{(\omega T/2)} = T \operatorname{sinc}\left(\frac{\omega T}{2}\right)$$

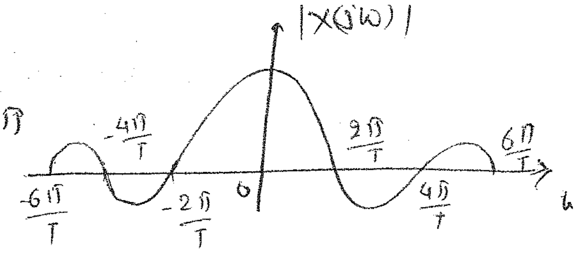


... amplitude spectrum.

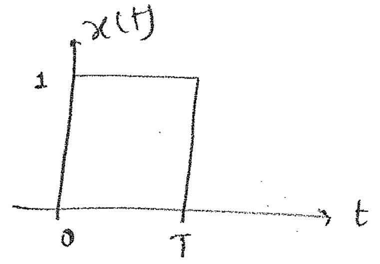
Now, since $\text{sinc}(x) = 0$ when $x = \pm n\pi$

Therefore, $\text{sinc}\left(\frac{\omega T}{2}\right) = 0$ when, $\frac{\omega T}{2} = \pm n\pi$

$\omega = \pm \frac{2n\pi}{T}$.



Q.3. Find the fourier Transform of the rectangular pulse.



Sol^n: Here,

$$x(t) = \begin{cases} 1 ; & \text{for } 0 \text{ to } T \\ 0 & \text{otherwise} \end{cases}$$

$$X(j\omega) = F[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

$$= \int_0^T e^{-j\omega t}\, dt = \left. \frac{e^{-j\omega t}}{-j\omega} \right|_0^T$$

$$= \frac{e^{-j\omega T} - 1}{-j\omega} = \frac{1 - e^{-j\omega T}}{j\omega} = \frac{e^{-j\omega T/2} \cdot e^{j\omega T/2} - e^{-j\omega T/2} \cdot e^{-j\omega T/2}}{j\omega}$$

$$= \frac{2e^{-j\omega T/2}\left[e^{j\omega T/2} - e^{-j\omega T/2}\right]}{\omega \, 2j} = 2e^{-j\omega T/2} \cdot \frac{\sin(\omega T/2)}{\omega T/2} \times (T/2)$$

$$= T e^{-j\omega T/2} \, \text{sinc}\left(\frac{\omega T}{2}\right)$$

Amplitude response/spectrum is same as previous and phase spectrum is,

$$\angle X(j\omega) = -\left(\frac{\omega T}{2}\right)$$

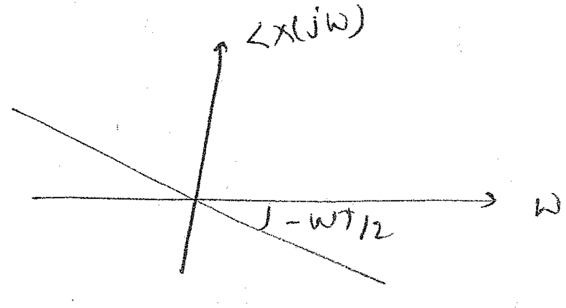


Fig: phase spectrum

It is the study of circuit behaviour according to the change in frequency. Let, $H(s)$ be any network function where, $s = \sigma + j\omega$, since frequency contributed by ω, we consider only imaginary part.

Thus, for $s = j\omega$,

$H(s)$ becomes complex network function $H(j\omega)$. In general form we have,

$$H(s) = \frac{a_0 s^n + a_1 s^{n-1} + \dots + a_n}{b_0 s^m + b_1 s^{m-1} + \dots + b_m}$$

on factorizing,

$$H(s) = \frac{a_0 (s-z_1)(s-z_2) \dots (s-z_n)}{b_0 (s-P_1)(s-P_2) \dots (s-P_m)}$$

For $s = j\omega$, $H(j\omega) = \frac{a_0}{b_0}\left[\frac{(j\omega - z_1)(j\omega - z_2) \dots (j\omega - z_n)}{(j\omega - P_1)(j\omega - P_2) \dots (j\omega - P_m)}\right]$

Thus, frequency response is the curve response for magnitude and phase of transfer function for different values of ω.

Bode plot:

The scientist H.W. Bode suggested a specific method to obtain the frequency response in which logarithmic values are used. It is the graphical representation of transfer function and used to determine stability of transfer function at various frequency.

In generals, Bode plot consists of two plots:

i) Magnitude plot

logarithmic value of magnitude are plotted against logarithmic values of ω.

We have, $H(j\omega) = R(\omega) + j X(\omega)$

$$\phi = \tan^{-1} \frac{X(\omega)}{R(\omega)}, \quad |H(j\omega)| = \sqrt{\{R(\omega)\}^2 + \{X(\omega)\}^2}$$

In dB,

$$|H(j\omega)|_{in\,dB} = 20 \log \sqrt{R^2(\omega) + X^2(\omega)}$$

Therefore, we plot $|H(j\omega)|_{in\,dB}$ vs $\log \omega$

Phase plot

Phase angle in degrees are plotted against logarithmic values of ω.

$$\phi = \tan^{-1}\left[\frac{X(\omega)}{R(\omega)}\right]$$

We plot, ϕ vs. $\log\omega$.

ω,

For Bode plot:

H(s) is expressed as:

$$H(s) = \frac{K(1+sT_a)(1+sT_b)}{s^n(1+sT_1)(1+sT_2)}$$

For $s = j\omega$

$$H(j\omega) = \frac{K(1+j\omega T_a)(1+j\omega T_b)}{(j\omega)^n(1+j\omega T_1)(1+j\omega T_2)}$$

The magnitude $|H(j\omega)|_{in\,dB} = 20\log K + 20\log\sqrt{1+\omega^2T_a^2} + 20\log\sqrt{1+\omega^2T_b^2}$

$$- 20\log\omega^n - 20\log\sqrt{1+\omega^2T_1^2} - 20\log\sqrt{1+\omega^2T_2^2}$$

d, phase, $\phi = \tan^{-1}\left(\frac{0}{K}\right) + \tan^{-1}(\omega T_a) + \tan^{-1}(\omega T_b) - \tan^{-1}\left(\frac{\omega^n}{0}\right) - \tan^{-1}(\omega T_1)$

$$- \tan^{-1}(\omega T_2)$$

$$= \tan^{-1}(\omega T_a) + \tan^{-1}(\omega T_b) - 90° - \tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2)$$

Bode plot for common polynomials

1. Constant, K

$|H(j\omega)|_{in\,dB} = 20\log K$

$\phi = 0°$

2. Integrator, $\frac{1}{s^n}$

$|H(j\omega)|_{in\,dB} = 20\log\left(\frac{1}{\omega^n}\right)$

$= 20\log 1 - 20\log\omega^n$

$= -20\log\omega^n$

$\phi = -90n$

3) Differentiator, s^n

$|H(j\omega)|_{in\,dB} = 20\log(\omega^n)$

$\phi = 90n$

4) $(1+sT)$

$|H(j\omega)|_{in\,dB} = 20\log\sqrt{1+\omega^2T^2}$

$= 20\log\sqrt{1+\left(\frac{\omega}{\omega_m}\right)^2}$ $\left(\because T = \frac{1}{\omega_m}\right)$

for $\omega << \omega_m$

$|H(j\omega)|_{in\,dB} = 20\log 1 = 0$

for $\omega = \omega_m$, $|H(j\omega)|_{in\,dB} = 3.01\,dB$ &

$\phi = \tan^{-1}(1)$

r $\omega >> \omega_m$

	$|H(j\omega)|$ in dB
)ω_m	20dB
0ω_m	40 dB
00ω_m	60 dB.

Similarly, for $\frac{1}{1+ST}$ magnitude decreases at slope of -20dB/dec.

d phase is given by $\phi = -\tan^{-1}\left(\frac{\omega}{\omega_m}\right)$

.	Factor	corner frequency	$H(j\omega)$ in dB	ϕ
.	K	none	$20\log K$	$0°$
.	$S = (j\omega)$	none	$+20$dB/dec line with 0dB at $\omega = 1$	$+90°$
.	$\frac{1}{S} = \frac{1}{j\omega}$	none	-20dB/dec line with 0dB at $\omega = 1$	$-90°$
.	$(1+ST)$	$\omega_m = \frac{1}{T}$	$+20$dB/dec line ~~with~~ ~~≈ 0dB at~~ For $\omega > \omega_m$	$\tan^{-1}(\omega T)$
.	$\frac{1}{1+ST}$	$\omega_m = \frac{1}{T}$	-20dB/dec line ~~with~~ ~~≈ 0dB at~~ For $\omega > \omega_m$	$-\tan^{-1}\left(\frac{\omega}{\omega_m}\right)$
.	$\omega_m^2 + 2\xi\omega_m S + S^2$ $\frac{S^2}{\omega_m^2} + \frac{2\xi S}{\omega_m} + 1$ $\left(\frac{j\omega}{\omega_m}\right)^2 + \frac{2\xi\omega j}{\omega_m} + 1$ $\left\{1 - \left(\frac{\omega}{\omega_m}\right)^2\right\} + j\frac{2\xi\omega}{\omega_m}$	ω_m	$+40$dB/dec line ~~with~~ ~~≈ 0dB at~~ For $\omega > \omega_m$	$\tan^{-1}\frac{2\xi\omega/\omega_m}{1-\left(\frac{\omega}{\omega_m}\right)^2}$
7.	$[\omega_m^2 + 2\xi\omega_m S + S^2]^{-1}$	ω_m	-40dB/dec line ~~with~~ ~~≈ 0dB at~~ For $\omega > \omega_m$	$-\tan^{-1}\frac{2\xi\omega/\omega_m}{1-\left(\frac{\omega}{\omega_m}\right)^2}$

xample:

plot magnitude and phase for

$$G(S) = \frac{20(0.1S+1)(0.002S+1)}{S^2(S+100)}$$

$$= \frac{20(0.1S+1)(0.002S+1)}{100\,S^2(0.01S+1)} = \frac{0.2(0.1S+1)(0.002S+1)}{S^2(0.01S+1)}$$

For $s = j\omega$

$$G(j\omega) = \frac{0.2\,(0.1j\omega+1)(0.002j\omega+1)}{(j\omega)^2\,(0.01j\omega+1)}$$

Table to plot magnitude of G(s)

S.N.	Factor	c.f.	|M| dB
1.	0.2	none	$20\log(0.2) = -14$ dB
2.	$0.1s+1$	10	+20 ~~log~~ dB/dec line ~~with 0dB at~~ For $\omega > 10$ radian
3.	$0.002s+1$	500	+20 dB/dec line ~~with 0dB~~ For ~~at~~ $\omega > 500$ radian
4.	$\frac{1}{s^2}$	none	−40 dB/dec line ~~with 0dB at~~ $\omega = 1$
5.	$\frac{1}{0.01s+1}$	100	− 20 dB/dec line ~~with 0dB~~ For ~~at~~ $\omega > 100$ radian.

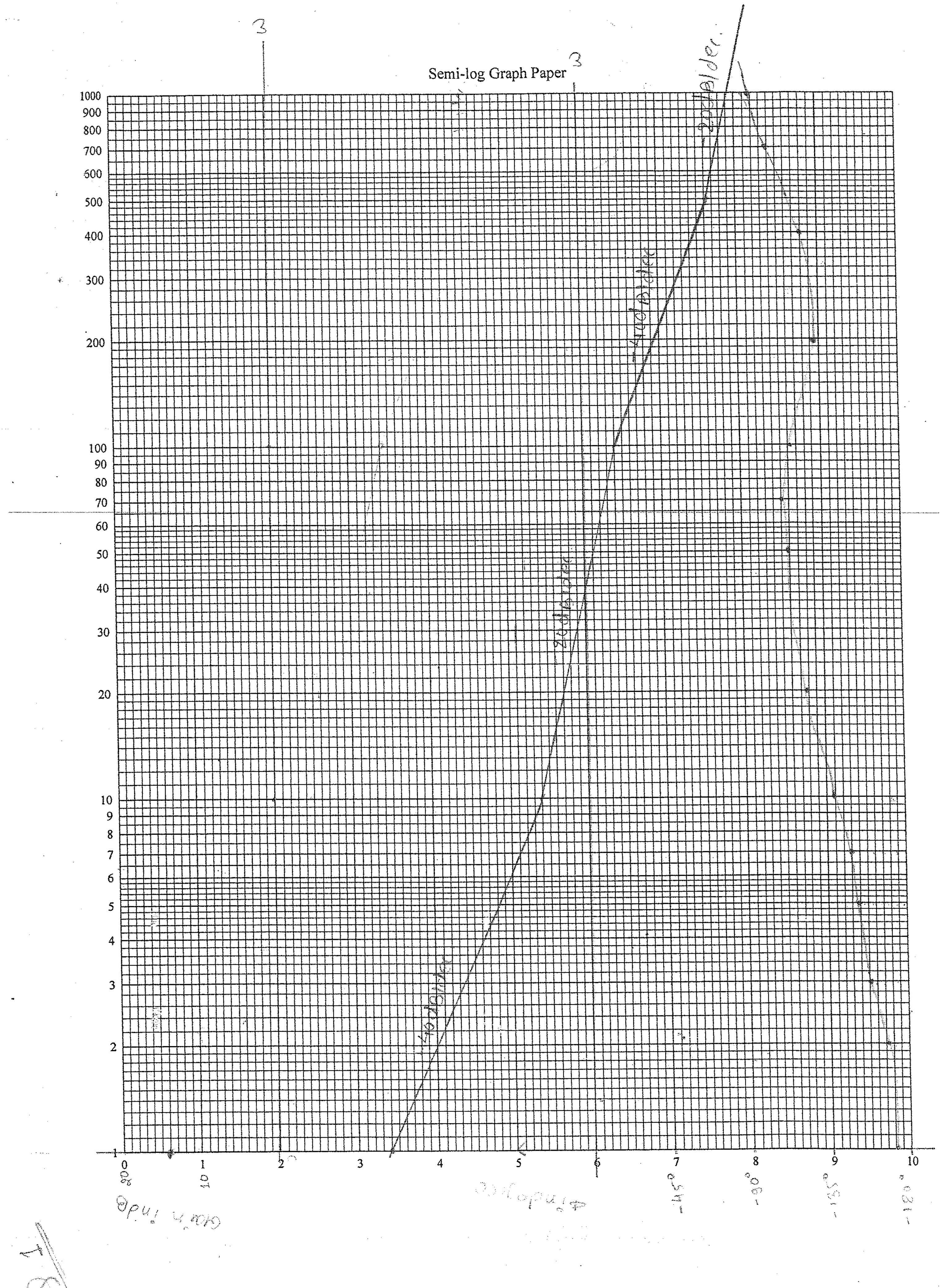
Phase plot

$$\phi = \tan^{-1}(0.1\omega) + \tan^{-1}(0.002\omega) - 180^\circ - \tan^{-1}(0.01\omega)$$

ω	φ
1	−174.7°
2	−169.6°
3	−164.7°
5	−155.7°
7	−148.2°
10	−139.6°
20	−125.6°
50	−122.2°
70	−125.2°
100	−129.4°
200	−134.5°
500	−124.8°
700	−118.2°
1000	−111.4°

Resultant slope table

Range of ω	factor	Resultant slope
starting	pole at origin	−40 dB/dec.
$0 < \omega < 10$	—	−40 dB/dec.
$10 < \omega < 100$	zero at $\omega_{c1} = 10$	−40+20 = −20 dB/dec.
$100 < \omega < 500$	pole at $\omega_{c2} = 100$	−20−20 = −40 dB/dec.
$500 < \omega < \infty$	zero at $\omega_{c3} = 500$	−40+20 = −20 dB/dec.


Semi-log Graph Paper

ample: 2 Draw asymptotic Bode plot for the following function:

$$G(s) = \frac{20(s+2)}{s(s^2+4s+16)}$$

Solⁿ:

$$G(s) = \frac{20(s+2)}{s(s^2+4s+16)}$$

$$= \frac{40(1+0.5s)}{16s\left(\frac{s^2}{16}+\frac{1}{4}s+1\right)}$$

$$= \frac{2.5(1+0.5s)}{s\left(\frac{1}{16}s^2+\frac{1}{4}s+1\right)}$$

For, $s = j\omega$

$$G(j\omega) = \frac{2.5(1+0.5j\omega)}{j\omega\left(\frac{1}{16}(j\omega)^2+\frac{1}{4}j\omega+1\right)} = \frac{2.5(1+0.5j\omega)}{(j\omega)\left\{\left(1-\frac{1}{16}\omega^2\right)+\frac{1}{4}j\omega\right\}}$$

Table to plot magnitude of G(s)

S.N.	Factor	C.F.	$\lvert M\rvert_{dB}$
1.	2.5	none	$20\log(2.5) = 7.95$ dB
~~2.~~	~~$1+0.5s$~~	~~2~~	~~+20 dB/dec line ~~with 0dB at~~ For $\omega > 2$ radian~~
3.	$\frac{1}{s}$	none	-20 dB/dec line with 0dB at $\omega = 1$
4.	$\frac{1}{\frac{1}{16}s^2+\frac{1}{4}s+1}$	4	-40 dB/dec line ~~with 0dB at~~ For $\omega > 100$ radian

Table for phase plot $\phi = \tan^{-1}(0.5\omega) - 90° - \tan^{-1}\left\{\frac{\frac{\omega}{4}}{1-\frac{1}{16}\omega^2}\right\}$

ω	ϕ	ω	ϕ
0.1	$-88.57°$	10	~~92.8°~~ 14.15° = −165.85°
0.2	$-87.16°$	20	~~81.42°~~ −173.95°
0.5	$-83.2°$	50	−177.69°
0.7	$-80.9°$	70	
1	$-78.4°$	100	−178.86°
2	−78.7° 43.9° = −137.1°	200	
5	~~82.9°~~ −155.64°	500	
7	~~80.7°~~	700	
		1000	

Resultant slope table

Range of ω	Factor	Resultant slope
starting	pole at origin	−20 dB/decade
$0 < \omega < 2$	—	−20 dB/dec.
$2 < \omega < 4$	zero ~~pole~~ at $\omega_{c1} = 2$	$-20+20 = 0$ dB/dec.
$4 < \omega < \infty$	pole at $\omega_{c2} = 4$	$0-40 = -40$ dB/dec.

hase crossover frequency

The frequency at which the angle of the system function
s -180° is called phase-crossover frequency. It is denoted by ω_p.

in crossover frequency

The frequency at which the magnitude of the system function
0dB is called gain-crossover frequency. It is denoted by ω_g.

in Margin (GM)

As the system gain changes, the system stability gets
ected. As gain K is increased, the system becomes less and less stable.

The gain margin is the amount of gain in dB that can be
ded to the system before the system becomes unstable.

The gain margin is the reciprocal of the magnitude of the
stem function at the frequency where the phase angle is -180° or
the phase-crossover frequency. Mathematically.

$$GM = \frac{1}{|G(j\omega_p)|}$$

In terms of dB,

$$GM \text{ in dB} = 20\log_{10}\frac{1}{|G(j\omega_p)|} = -20\log_{10}|G(j\omega_p)|$$

The positive gain margin means increase in gain is allowable and
system is stable. On the otherhand, the negative gain margin means
the gain limit has been crossed and system is unstable.

ase margin (PM)

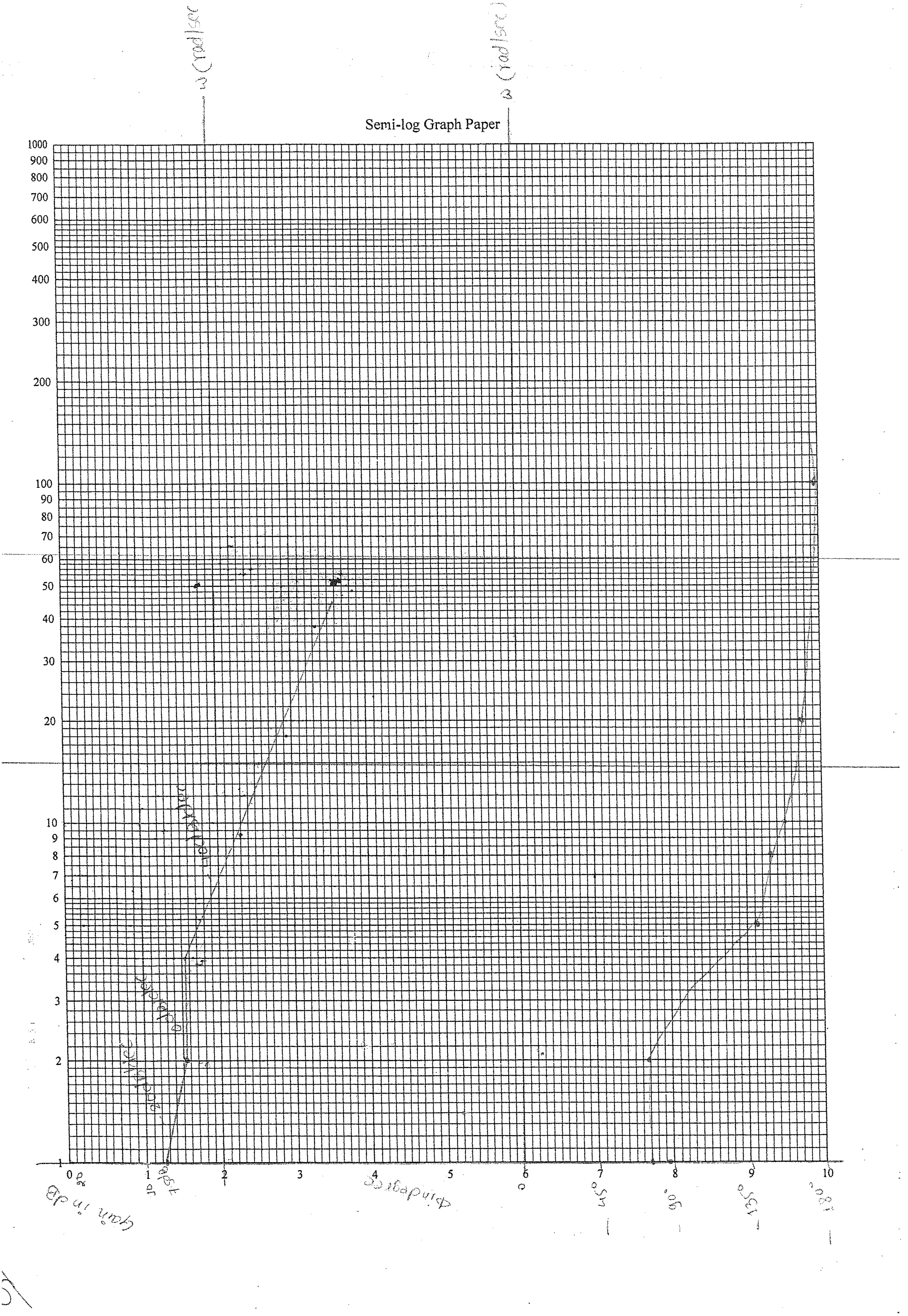
The phase margin is the amount of phase lag in degree that
an be added to the system before the system becomes unstable.

The phase margin is 180° plus the phase angle ϕ at the gain-
rossover frequency. Mathematically,

$$PM = 180^\circ + \phi$$

$$\text{where, } \phi = \angle GM(j\omega_g)$$

The positive phase margin means increase in phase lag is allowable



Semi-log Graph Paper
ω (rad/sec)
ω (rad/sec)
1000
900
800
700
600
500
400
300
200
100
90
80
70
60
50
40
30
20
10
9
8
7
6
5
4
3
2
1
0
1
2
3
4
5
6
7
8
9
10
Gain in dB
φ in degree
45°
90°
135°
180°

ne system is stable. On the otherhand, the negative phase margin eans the phase lag limit has been crossed and system is unstable.

OTE: IF PM is positive, GM must be positive and vice-versa. and similarly if PM is negative, GM must be negative and vice-versa.

Draw the Bode plot for

$$G(S) = \frac{20}{S(S+2)(S+10)}$$

From the Bode plot, also determine (i) phase-crossover frequency (ii) gain-crossover frequency (iii) gain margin (iv) phase margin (v) stability.

olution :

Arrange G(S) in time constant form

$$G(S) = \frac{1}{S\left(1+\frac{S}{2}\right)\left(1+\frac{S}{10}\right)}$$

Table to plot magnitude of G(S)

S.N.	Factor	C.F.	|M| dB
1.	1	none	20log(1) = 0 dB
2.	S	none	−20 dB/dec line with 0 dB at ω = 1
3.	$1+\frac{S}{2}$	2	−20 dB/dec line for ω > 2
4.	$1+\frac{S}{10}$	10	−20 dB/dec line for ω > 10

he resultant slope table

Range of ω	factor	Resultant slope
starting	pole at origin	−20 dB/dec.
$0 < \omega < 2$	—	−20 dB/dec.
$2 < \omega < 10$	pole at $\omega_c = 2$	−20 − 20 = −40 dB/dec.
$10 < \omega < \infty$	pole at $\omega_c = 10$	−40 − 20 = −60 dB/dec.

or phase

$$G(j\omega) = \frac{1}{j\omega\left(1+\frac{j\omega}{2}\right)\left(1+\frac{j\omega}{10}\right)}$$

ω	ϕ
0.1	-93.43°
1	-122.27°
2	-146.3°
0	-213.69°
00	-263.14°
0	-270°

From Bode plot,

phase - crossover frequency, ω_p = 4.7 rad/sec.

gain - crossover frequency, ω_g = 1 rad/sec.

gain - margin, GM = +20dB

phase - margin, PM = +57°

Since, both GM and PM are positive, the given system is stable in nature.

Filters

→ Filters are the combination of electrical components which passes or allows the un-attenuated transmission of electric signals with certain frequency range and stops transmission of electric signals outside the range.

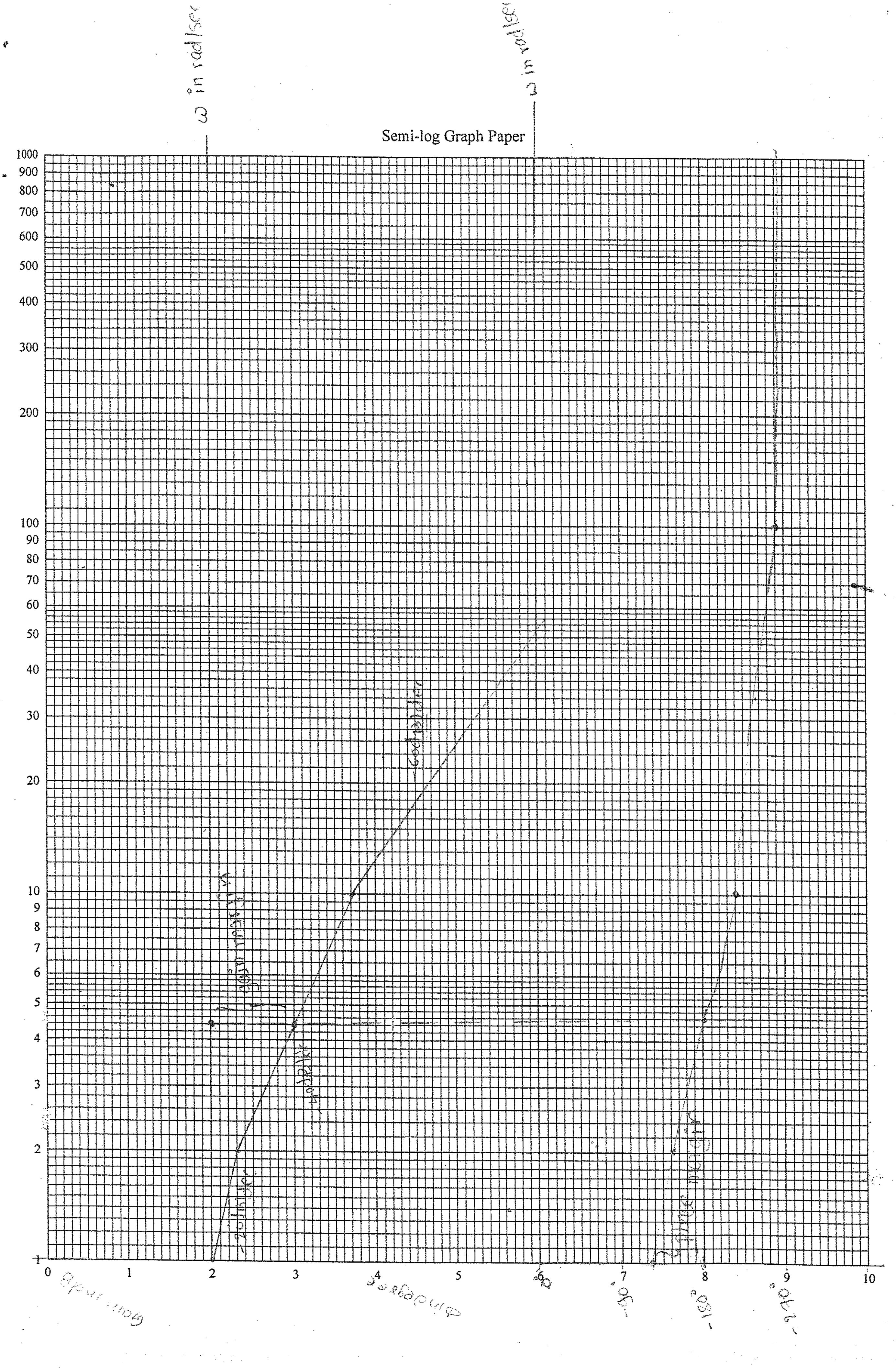
→ Band, in which ideal filters have to pass all frequencies without reduction in magnitude are referred to as pass band.

→ Band, in which ideal filters have to attenuate (or stop) frequencies are referred to as stop band.

→ The frequeny which separates the pass-band and stop-band is defined as the cut-off frequeny of the filter.

Classification of filters:

There are four common types of filters:


Semi-log Graph Paper
ω in rad/sec
ω in rad/sec
Gain in dB
φ in degree
-90°
-180°
-270°

① Low pass filter

These filters reject all frequencies above cut-off frequeny. The attenuation characteristics of an ideal Low pass filter is shown in figure below. Thus, the pass band or transmission band for the L filter is the frequeny range 0 to f_c and the stop band or attenua band is the frequeny range above f_c.

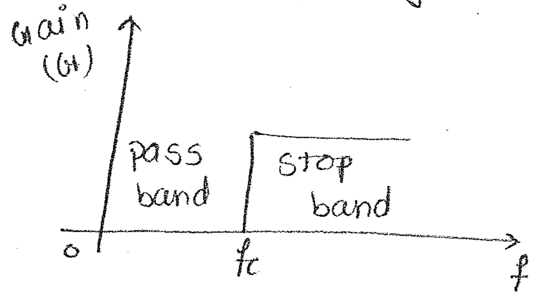


fig: Ideal characteristic of Low pass filter.

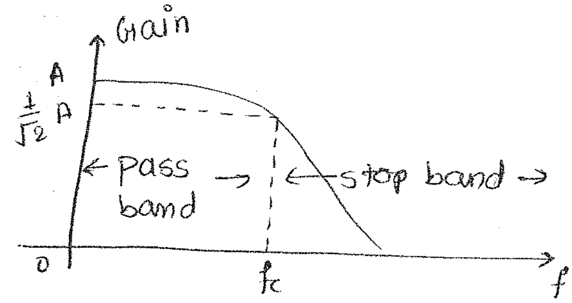


fig: Characteristic of practicle low pass filter.

② High pass filter

These filters reject all frequencies below cut-off frequenc f_c. Thus, the pass band for HP filter is frequeny range above f_c and the stop band or attenuation band is the frequeny range below f_c.

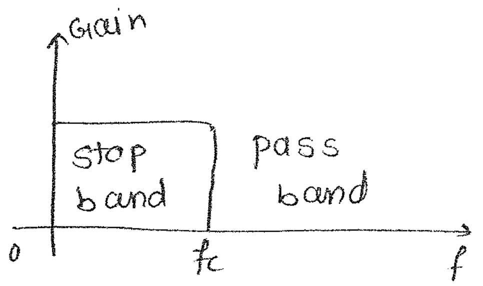


fig: Ideal characteristic of High ~~Low~~ pass filter

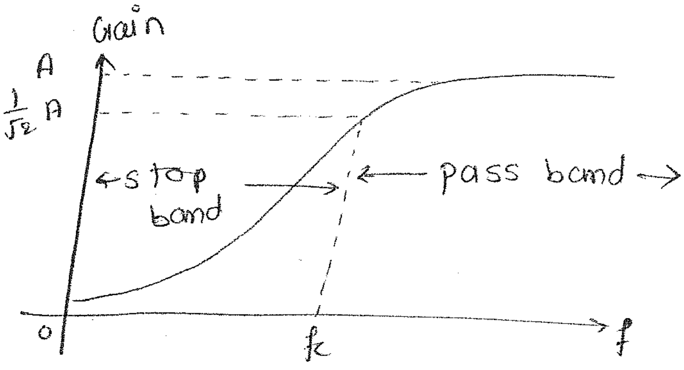


fig: characteristic of High pass filter.

③ Band pass filter :

These filters allow transmission of frequencies between t designated cut off frequencies and reject all other frequencies... Band pass filter has two cut-off frequencies and will have the pa band $f_{c2} - f_{c1}$. f_{c1} is called lower cut-off frequeny while f_{c2} is called higher cut-off frequeny.

The range between higher/upper and lower cut-off frequency is called band width.

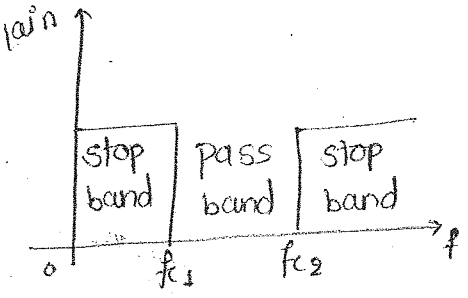


fig: Ideal response of Band pass filter.

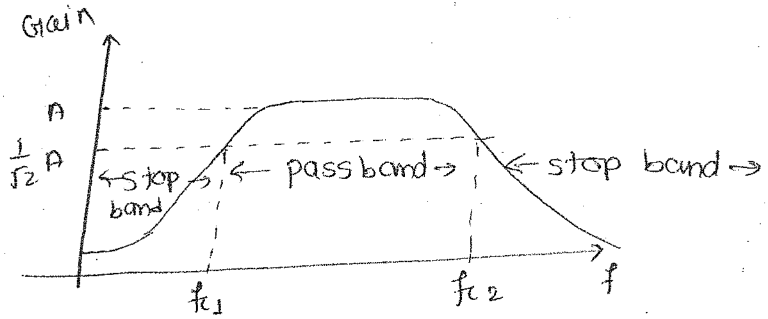
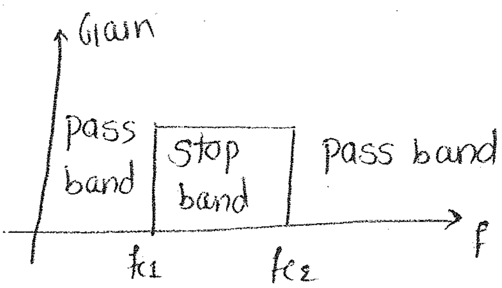
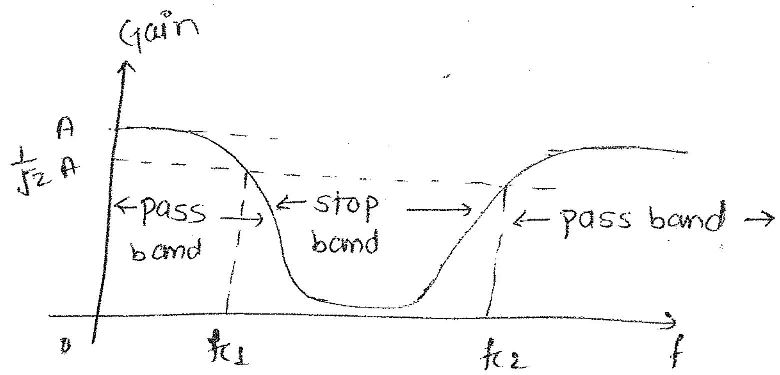


fig: non-ideal response of band pass filter.

④ Band-stop filter:

These filters pass all frequencies lying outside a certain range, while it attenuates all frequencies between the two frequencies f_{c1} and f_{c2}. The characteristic of an ideal and non-ideal band stop filter is shown in below





NOTE:* Roots of $P(s)$ are not permitted at the origin except in case of odd function (polynomial) since one root of odd part $N(s)$ of polynomial is always present at the origin.

'urwitz polynomial

A polynomial $P(s)$ is said to be Hurwitz polynomial if the following nditions are satisfied:

) $P(s)$ is real when s is real

) The roots of $P(s)$ have real parts which are zero or negative.

operties

If the polynomial $P(s)$ can be written as

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$$

hen, All the coefficients a_i must be positive and none of the coefficients ay be zero. except odd and even polynomial.

The continued fraction expansion of the ratio of the odd to even parts $N(s)/M(s)$] or the even to odd parts $[M(s)/N(s)]$ of a Hurwitz polynomial yields l positive quotient terms. In case of odd or even polynomial, ratio of $P(s)$ and '(s) is taken.

1 Check whether the given polynomial $P(s) = s^4 + s^3 + 5s^2 + 3s + 4$ is Hurwitz or not.

r: Here, $P(s) = s^4 + s^3 + 5s^2 + 3s + 4$

condition 1 : ~~Since all coefficients of $P(s)$ are positive and none of the coefficient~~ are zero.

ondition 2 : The even and odd parts of $P(s)$ are

$$M(s) = s^4 + 5s^2 + 4$$

$$N(s) = s^3 + 3s$$

Continued fraction expansion of $\psi(s) = \dfrac{M(s)}{N(s)}$ is given by

$$s^3+3s \,)\, s^4+5s^2+4 \,(\, s$$

$$\underline{s^4 + 3s^2}$$

$$2s^2+4 \,)\, s^3+3s \,(\, \tfrac{1}{2}s$$

$$\underline{s^3 + 2s}$$

$$s \,)\, 2s^2+4 \,(\, 2s$$

$$\underline{2s^2}$$

$$4 \,)\, s \,(\, \tfrac{1}{4}s$$

$$\underline{s}$$

$$\times$$

So that, the continued fraction expansion of $\psi(s)$ is

$$\psi(s) = \frac{M(s)}{N(s)}$$

$$= s + \cfrac{1}{\frac{s}{2} + \cfrac{1}{2s + \cfrac{1}{\frac{s}{4}}}}$$

Since, all the quotient terms of the continued fraction expansion are positive, $p(s)$ is Hurwitz.

2. Check whether the given polynomial $p(s) = s^4 + s^3 + 2s^2 + 4s + 1$ is Hurwitz or not.

Solution: Here, $p(s) = s^4 + s^3 + 2s^2 + 4s + 1$

Condition 1: Since, all the coefficients of $p(s)$ are positive and none of the coefficient are zero.

Condition 2: Even and odd parts of $p(s)$ are

$$M(s) = s^4 + 2s^2 + 1$$

$$N(s) = s^3 + 4s$$

So, continued fraction expansion of $\psi(s) = \frac{M(s)}{N(s)}$ is given as,

$$
\begin{array}{l}
s^3+4s \,)\, s^4+2s^2+1 \,(\, s \\
\qquad\qquad \underline{s^4+4s^2} \\
\qquad\qquad\quad -2s^2+1 \,)\, s^3+4s \,(\, -\frac{1}{2}s \\
\qquad\qquad\qquad\qquad\qquad \underline{s^3-\frac{s}{2}} \\
\qquad\qquad\qquad\qquad\qquad \frac{9}{2}s \,)\, -2s^2+1 \,(\, -\frac{4}{9}s \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{-2s^2} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \,)\, \frac{9}{2}s \,(\, \frac{9}{2}s \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{\frac{9}{2}s} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \times
\end{array}
$$

Since, all the quotient terms of the continued fraction expansion are not positive, $p(s)$ is not Hurwitz.

ositive Real Function

A Function (Polynomial) is said to be positive Real function if it tisfies the following conditions:

ndition 1: ~~the~~ The polynomial should be Hurwitz polynomial. i.e. Function must have no poles on right half of s-plane. This condition can be check through a continued fraction expansion of the odd to even parts or even to odd parts of $D(s)$ in which quotients must be positive.

ndition 2: If poles of given function are on $j\omega$ axis, then residue of poles on $j\omega$ axis ~~are~~ should be positive.

(This condition is tested by making partial fraction expansion of given polynomial.)

ondition 3: $Re[T(j\omega)] \geq 0$ for all ω

Let, $T(s) = \dfrac{N(s)}{D(s)} = \dfrac{M_1(s)+N_1(s)}{M_2(s)+N_2(s)}$

Where, $M_i(s)$ are even polynomial and $N_i(s)$ are odd polynomials.

Rationalizing:

$$T(s) = \frac{M_1(s)+N_1(s)}{M_2(s)+N_2(s)} \times \frac{M_2(s)-N_2(s)}{M_2(s)-N_2(s)} = \frac{M_1M_2 - M_1N_2 + N_1M_2 - N_1N_2}{M_2^2 - N_2^2}$$

$$= \frac{M_1M_2 - N_1N_2}{M_2^2 - N_2^2} + \frac{N_1M_2 - M_1N_2}{M_2^2 - N_2^2}$$

Here, M_1M_2 and N_1N_2 are even function while N_1M_2 and M_1N_2 are odd functions. So, even part of $T(s) = \dfrac{M_1M_2 - N_1N_2}{M_2^2 - N_2^2}$ and odd part of $T(s) = \dfrac{N_1M_2 - M_1N_2}{M_2^2 - N_2^2}$

So, if we let, $s = j\omega$, we see that even parts of the polynomial is real while the odd part of the polynomial is imaginary.

i.e. $Re[T(j\omega)] = Even[T(s)]_{s=j\omega}$

$Im[T(j\omega)] = odd[T(s)]_{s=j\omega}$

So, to test the condition $Re[T(j\omega)] \geq 0$, we determine the real part of $T(s)$ by finding even part of $T(s)$ and put $s=j\omega$

ıen we check whether $Re[T(j\omega)] \geq 0$ for all ω or not.

$\therefore A(\omega^2) = M_1(j\omega) \cdot M_2(j\omega) - N_1(j\omega) N_2(j\omega) \geq 0$ for all ω or not.

operties of PRF :

IF $F(s)$ is PRF then $\frac{1}{F(s)}$ is also PRF.

The sum of two PRF is also PRF but difference may not be

The poles/zeros of PRF can not have positive real parts.

The poles/zeros of PRF occurs either in real or in complex conjugate.

Highest power of numerator and denominator may differ at most by unity.

lowest power of denominator and numerator polynomial may differ at most by unity.

3.1 Test whether the following function is PRF or not

$$Z(s) = \frac{2s^2+5}{s(s^2+1)}$$

soln: For given function, $D(s) = s^3+s$

condition 1 Then, odd polynomial $= s^3+s$ so,

$$D'(s) = 3s^2+1$$

Then, continue fraction is,

$$
\begin{array}{l}
3s^2+1 \,)\, s^3+s \,(\, \frac{1}{3}s \\
\quad\quad\quad\; \underline{-s^3 \mp \frac{1}{3}s} \\
\quad\quad\quad\quad \frac{2s}{3} \,)\, 3s^2+1 \,(\, \frac{9}{2}s \\
\quad\quad\quad\quad\quad\quad\; \underline{3s^2} \\
\quad\quad\quad\quad\quad\quad\quad 1 \,)\, \frac{2s}{3} \,(\, \frac{2s}{3} \\
\quad\quad\quad\quad\quad\quad\quad\quad \underline{-\frac{2s}{3}} \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad \times
\end{array}
$$

Since, all the quotients are positive, $D(s)$ is Hurwitz polynomial.

condition 2

$Z(s)$ has poles at $s = 0$ and $s = \pm j$

Now, to find residues taking partial fraction we get

$$\frac{2s^2+5}{s^3+s} = \frac{2s^2+5}{s(s+1)} = \frac{2s^2+5}{s(s-j)(s+j)} = \frac{A}{s} + \frac{B}{s+j} + \frac{C}{s-j}$$

Now, $A = \left.\frac{2s^2+5}{s(s-j)(s+j)} \times s\right|_{s=0} = \frac{5}{-j^2} = 5$

$B = \left.\frac{2s^2+5}{s(s-j)(s+j)} \times (s+j)\right|_{s=-j} = \frac{3}{-2} = -\frac{3}{2}$

$C = \left.\frac{2s^2+5}{s(s-j)(s+j)} \times (s-j)\right|_{s=j} = -\frac{3}{2}$

Since, residues at poles on imaginary axis i.e. on $s = \pm j$ are egative. So, condition 2 is not satisfied.

$\therefore Z(s)$ is not positive real function (PRF)

Example 2 $Z(s) = \frac{s^2+2s+6}{s(s+3)}$ (~~wrong~~)

Sol^n: For given function, $D(s) = s^2+3s$

onditions Then, even polynomial $= s^2$

~~odd polynomial = 3s~~

Then, continued fraction is, $3s\,)\,s^2\,(\,\frac{1}{3}s$
$\quad s^2$
$\quad \times$

Since, the quotients is positive. D(s) is Hurwitz polynomial.

Condition 2

Z(s) has poles at $s=0$ and $s=-3$

$s=0$ lies on imaginary axis.

For residue test, taking partial fraction

Here, $\frac{s^2+2s+6}{s^2+3s}$ so, $s^2+3s\,)\,s^2+2s+6\,(\,1$
$\quad -s^2-3s$
$\quad -s+6$

$$\frac{s^2+2s+6}{s(s+3)} = 1 + \frac{6-s}{s(s+3)}$$

Now, $\frac{6-s}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}$

$Z(s) = \frac{s^2+2s+6}{s(s^2+3)}$

$D(s) = s^3+3s$

$D'(s) = 3s^2+3$

conditions $3s^2+3\,)\,s^3+3s\,(\,\frac{1}{3}s$
$\quad s^3+s$
$\quad 2s\,)\,3s^2+3\,(\,\frac{3}{2}s$
$\quad 3s^2$
$\quad 3\,)\,2s\,(\,\frac{2}{3}s$
$\quad -2s$
$\quad \times$

Since, the quotients are positive. D(s) is Hurwitz polynomial.

condition 2

Z(s) has poles at $s=0$, $s=\pm\sqrt{3}j$

for residue ~~des~~ test

$$\frac{s^2+2s+6}{s(s^2+3)} = \frac{A}{s} + \frac{B}{s-j\sqrt{3}} + \frac{C}{s+j\sqrt{3}}$$

$A = \left.\frac{s^2+2s+6}{s^3+3s} \times s\right|_{s=0} \Rightarrow 2$, $c = -ve$.

$B = \left.\frac{s^2+2s+6}{s(s^2+3)} \times (s-j\sqrt{3})\right|_{s=j\sqrt{3}}$ $-ve$

$= \frac{-3+2\sqrt{3}j+6}{\ldots} = \frac{3+2\sqrt{3}j}{-6}$

ere, $A = \frac{6-s}{s(s+3)} \times s \Big|_{s=0} = 2$

$B = \frac{6-s}{s(s+3)} \times (s+3) \Big|_{s=-3} = -3$

Since, the residue of pole at $s=0$ is positive. So, condition 2 is satisfied.

ndition 3:

$A(\omega^2) = m_1(j\omega) \cdot m_2(j\omega) - N_1(j\omega) \cdot N_2(j\omega) \geq 0$

For given polynomial,

$M_1(s) = s^2 + 6$ $\quad N_1(s) = 2s$

$M_2(s) = s^2$ $\quad N_2(s) = 3s$

Now, $M_1(s) \cdot M_2(s) - N_1(s) \cdot N_2(s)$

$(s^2+6)\, s^2 - 2s \times 3s$

$= s^4 + 6s^2 - 6s^2 = s^4$

~~$M_1(s) = s^2+6$~~
~~$M_2(s) = 0$~~
~~$N_1(s) = 2s$~~
~~$N_2(s) = s^3+3s$~~
~~$A(s) = 0 - 2s(2s^3+3s)$~~
~~$= -4s^4 + 6s^2$~~
~~$A(\omega^2) = -4(j\omega)^4 + 6(j\omega)^2$~~
~~$= -4\omega^4 + 6\omega^2$~~
~~$A(\omega^2) \ngeq 0$ for all ω so~~
~~$z(s)$ is not PRF~~

$\therefore$ in $j\omega$,

$M_1(j\omega) \cdot M_2(j\omega) - N_1(j\omega) N_2(j\omega) = (j\omega)^4 = \omega^4$

which is positive for all values of ω. so, condition 3 is varified.

Thus, the $z(s)$ is positive real function (PRF).

Example 3 $\quad z(s) = \frac{(s+2)(s+4)}{(s+1)(s+3)}$

Here, $z(s) = \frac{(s+2)(s+4)}{(s+1)(s+3)} = \frac{s^2+6s+8}{s^2+4s+3}$

on: Condition 1:

Since the poles for above function are at $s=-1$, $s=-3$. So, the above function satisfies condition 1 i.e. all poles are at left half side of s-plane.

condition 2:

No roots on $j\omega$ axis. So, condition 2 is not exist.

condition 3:

$A(\omega^2) = M_1(j\omega) \cdot M_2(j\omega) - N_1(j\omega) N_2(j\omega) \geq 0$

or given polynomial,

$M_1(S) = S^2+8 \qquad N_1(S) = 6S$

$M_2(S) = S^2+3 \qquad N_2(S) = 4S$

$\therefore A(\omega^2) = \{(j\omega)^2+8\}\{(j\omega)^2+3\} - (6j\omega)(4j\omega)$

$= (8-\omega^2)(3-\omega^2) + 24\omega^2$

$= 24 - 8\omega^2 - 3\omega^2 + \omega^4 + 24\omega^2$

$= \omega^4 + 13\omega^2 + 24$, This is positive for all value of ω.

so, condition 3 is satisfied.

Hence, the $z(s)$ is Positive Real function (PRF)

1.4.

$$Z(S) = \frac{S^3+5S^2+9S+3}{S^3+4S^2+7S+9}$$

soln: condition 1

Here, $D(S) = S^3+4S^2+7S+9$

odd polynomial $= S^3+7S$

even polynomial $= 4S^2+9$

Then, continued fraction is,

$$\begin{array}{l}
4S^2+9 \,\big)\, S^3+7S \,\big(\, \frac{1}{4}S \\
\quad\quad\quad\;\; -S^3+\frac{9}{4}S \\
\hline
\quad\quad\quad\quad \frac{19}{4}S \,\big)\, 4S^2+9 \,\big(\, \frac{16}{19}S \\
\quad\quad\quad\quad\quad\quad\quad -4S^2 \\
\hline
\quad\quad\quad\quad\quad\quad\quad 9 \,\big)\, \frac{19}{4}S \,\big(\, \frac{19}{36}S \\
\quad\quad\quad\quad\quad\quad\quad\quad\;\; -\frac{19}{4}S \\
\hline
\quad\quad\quad\quad\quad\quad\quad\quad\quad \times
\end{array}$$

Since, all the quotiest are +ve. D(S) is Hurtwitz polynomial.

so, condition 1 is satisfied.

condition 2

By using calculator, poles are at $s = -2.64, -0.68 \pm j1.71$

$\therefore$ Non of the poles lies on imaginary axis. so, condition 2 does not exist.

lition 3 :

$$A(\omega^2) = M_1(j\omega) \cdot M_2(j\omega) - N_1(j\omega) N_2(j\omega) \geq 0$$

$M_1(s) = 5s^2+3$ $\quad$ $N_1(s) = s^3+9s$

$M_2(s) = 4s^2+9$ $\quad$ $N_2(s) = s^3+7s$

Now, $M_1(s) M_2(s) - N_1(s) N_2(s)$

$$(5s^2+3)(4s^2+9) - (s^3+9s)(s^3+7s)$$

$$= 20s^4 + 45s^2 + 12s^2 + 27 - s^6 - 7s^4 - 9s^4 - 63s^2$$

$$= -s^6 + 4s^4 - 6s^2 + 27$$

Now,

$$A(\omega^2) = M_1(j\omega) M_2(j\omega) - N_1(j\omega) N_2(j\omega)$$

$$= -(j\omega)^6 + 4(j\omega)^4 - 6(j\omega)^2 + 27$$

$$= \omega^6 + 4\omega^4 + 6\omega^2 + 27.$$ This is positive for all values of ω.

so, condition 3 is satisfied.

Hence $z(s)$ is PRF.

Synthesis of one port network

Methods of synthesising network are:

① Foster I

- series combination of parallel components.
- use of partial fraction expansion for $z(s)$

② Foster II

- Parallel combination of series components
- use of partial fraction expansion for $Y(s)$

③ Cauer I

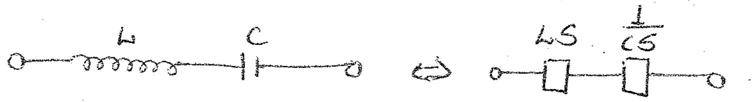
- Continued fraction expansion of $z(s)$ by arranging numerator and denominator in descending order

④ Cauer II

- Continued fraction expansion of $z(s)$ by arranging numerator and denominator in ascending order.

For L-C circuit

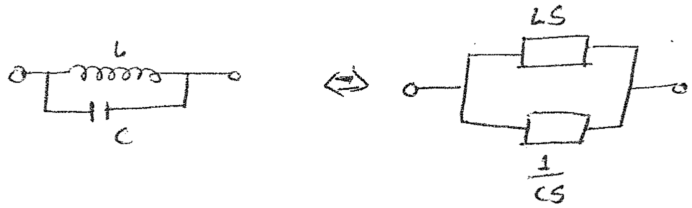
a) L-C series



$$Z(S) = LS + \frac{1}{CS}$$

$$= \frac{LCS^2+1}{SC} = \frac{S^2 + \frac{1}{LC}}{\frac{1}{L}\cdot S} \qquad Y(S) = \frac{\frac{1}{L}\cdot S}{S^2 + \frac{1}{LC}}$$

b) L-C parallel



$$\cancel{Y(S)}\ Z(S) = LS \,\|\, \frac{1}{CS}$$

$$= \frac{LS \times \frac{1}{CS}}{LS + \frac{1}{CS}} = \frac{\frac{L}{C} \times CS}{LCS^2+1} = \frac{LS}{LCS^2+1} = \frac{\frac{1}{C}S}{S^2 + \frac{1}{LC}}$$

$$Y(S) = \frac{S^2 + \frac{1}{LC}}{\frac{1}{C}S}$$

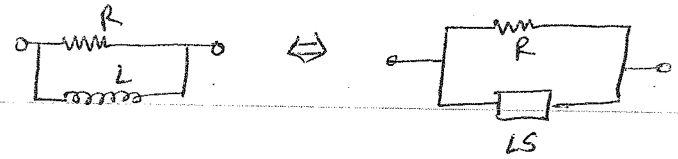
For R-L circuit

i) R-L series

$$Z(S) = R + LS \quad = L\left(S + \frac{R}{L}\right)$$

$$Y(S) = \frac{\frac{1}{L}}{S + \frac{R}{L}}$$

ii) R-L parallel



$$Z(S) = R \,\|\, LS = \frac{RLS}{R+LS}$$

$$= \frac{RS}{S + R/L}$$

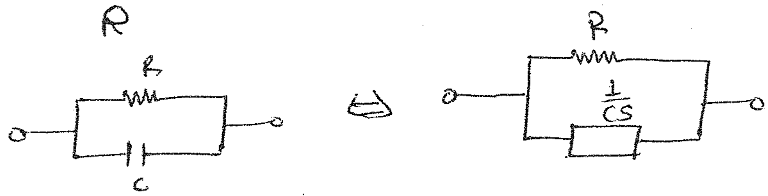
$$Y(S) = \frac{S + R/L}{RS}$$

For R-C circuit

i) R-C series

$$Z(S) = R + \frac{1}{CS} = \frac{RCS+1}{CS} = \frac{S + \frac{1}{RC}}{\frac{1}{R}S}$$

$$Y(S) = \frac{\frac{1}{R}\cdot S}{S + \frac{1}{RC}}$$



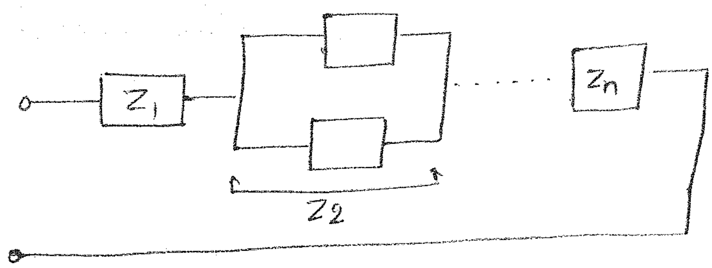
ii) R-C parallel

$$Z(S) = R \,\|\, \frac{1}{CS} = \frac{R \times \frac{1}{CS}}{R + \frac{1}{CS}} = \frac{R}{RCS+1} = \frac{\frac{1}{C}}{S + \frac{1}{RC}}$$

$$Y(S) = S + 1/RC$$

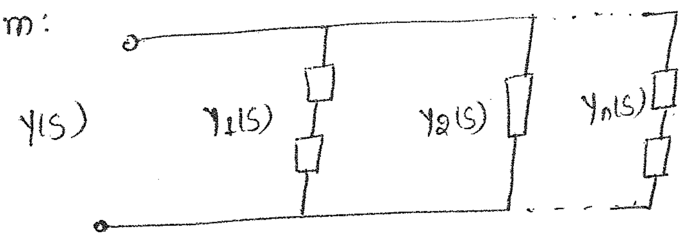
Foster I:

General form:



Foster II:

General form:



Sample

The driving point impedance of one port L-C network is given

y $Z(S) = \dfrac{4(s^2+4)(s^2+25)}{s(s^2+16)}$ obtain Foster I and Foster II form of

quivalent networks.

1ⁿ: We have,

$$Z(S) = \frac{4(s^2+4)(s^2+25)}{s(s^2+16)}$$

For Foster I:

Since order of numerator polynomial is greater than denominator polynomial so we need to divide before partial fraction.

$$s^3+16s \,\big)\, 4s^4+116s^2+400 \,\big(\, 4s$$
$$\underline{4s^4+64s^2}$$
$$52s^2+400$$

i.e. $Z(S) = \dfrac{4(s^2+4)(s^2+25)}{s(s^2+16)} = 4s + \dfrac{52s^2+400}{s(s^2+16)}$

Now, Using partial fraction,

$$Z(S) = 4s + \frac{52s^2+400}{s(s^2+16)} = 4s + \frac{A}{s} + \frac{Bs+C}{s^2+16}$$

Now, $\dfrac{52s^2+400}{s(s^2+16)} = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+16}$

$$52S^2+400 = AS^2+16A+BS^2+CS$$

$$52S^2+400 = (A+B)S^2+CS+16A$$

$16A = 400 \quad \Rightarrow A = 25$

$C = 0 \quad \Rightarrow C = 0$

$A+B = 52 \quad \Rightarrow B = 27$

$$\therefore Z(S) = 4S + \frac{25}{S} + \frac{27S}{S^2+16} = 4S + \frac{1}{\frac{1}{25}S} + \frac{\frac{1}{\frac{1}{27}}S}{S^2+\frac{1}{\frac{27}{16}\cdot\frac{1}{27}}}$$

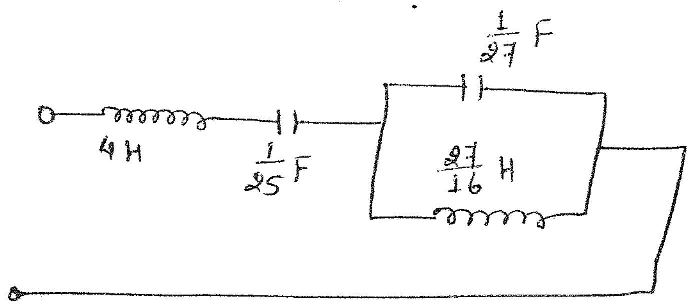


Fig: Foster I

For Foster II :

$$Y(S) = \frac{1}{Z(S)} = \frac{S(S^2+16)}{4(S^2+4)(S^2+25)}$$

Now, using partial fraction

$$Y(S) = \frac{1}{4}\frac{S^3+16S}{(S^2+4)(S^2+25)} = \frac{1}{4}\left[\frac{AS+B}{S^2+4} + \frac{CS+D}{S^2+25}\right]$$

or, $$\frac{S^3+16S}{(S^2+4)(S^2+25)} = \frac{(AS+B)(S^2+25)+(CS+D)(S^2+4)}{(S^2+4)(S^2+25)}$$

or, $$S^3+16S = AS^3+25AS+BS^2+25B+CS^3+4CS+DS^2+4D$$

or, $$S^3+16S = (A+C)S^3+(B+D)S^2+(25A+4C)S+25B+4D$$

Equating coefficient

$A+C = 1 \quad \ldots (1)$

$B+D = 0 \quad \Rightarrow B = -D \quad \ldots (2)$

$25A+4C = 16 \quad \Rightarrow 25A+4(1-A) = 16 \quad \Rightarrow 25A+4-4A = 16 \quad \Rightarrow 21A = 12$

$A = \frac{12}{21}$

$C = 1-A = 1-\frac{12}{21} = \frac{9}{21}$

$25B+4D = 0 \quad \ldots (3)$

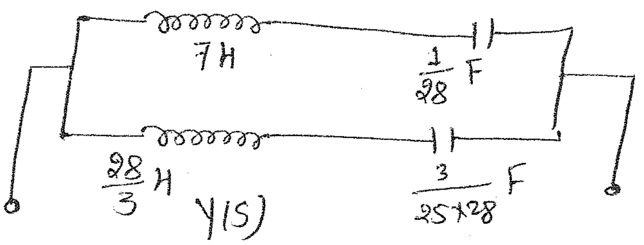
from (2) and (3), we have

$B = 0, \; D = 0$

Thus,

$$Y(S) = \frac{s(s^2+16)}{4(s^2+4)(s^2+25)} = \frac{12s}{4\times 21(s^2+4)} + \frac{9s}{4\times 21(s^2+25)}$$

$$= \frac{\frac{1}{7}s}{s^2+4} + \frac{\frac{3}{28}s}{s^2+25}$$



$$Y_1(S) = \frac{\frac{1}{7}s}{s^2+4} \quad \text{with} \quad \frac{\frac{1}{L}\cdot s}{s^2+\frac{1}{LC}}$$

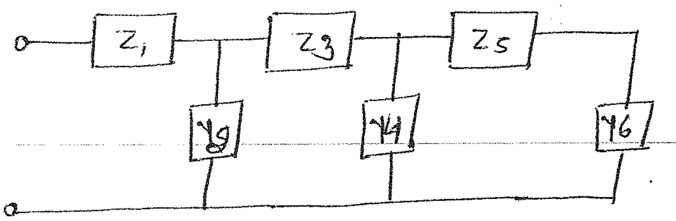
$$Y_2(S) = \frac{\frac{3}{28}s}{s^2+25} \quad \text{with} \quad \frac{\frac{1}{L}\cdot s}{s^2+\frac{1}{LC}}$$

Cauer Method

Cauer method is based on ~~fact that~~ reactance function that may be represented by two different network configuration expressed by continued fraction expansion.

Cauer I :

$$Z_1(S) + \cfrac{1}{Y_2(S) + \cfrac{1}{Z_3(S) + \cfrac{1}{Y_4(S) + \cfrac{1}{Z_5(S) + \dots}}}}$$



① If number of Z(S) is of higher order than denominator divide as usual.

② If number of Z(S) is equal order than denominator divide as usual.

③ If number of Z(S) is lower order than denominator take inverse and divide.

Example 1

Find cauer I form of

$$Z(S) = \frac{12s^4+10s^2+1}{4s^3+2s}$$

Sol^n: Using continued fraction:

$4s^3+2s \overline{)\ 12s^4+10s^2+1\ }(3s = Z_1(s)$; $L = 3H$

$12s^4+6s^2$

$4s^2+1 \overline{)\ 4s^3+2s\ }(s = Y_2(s)$; $C = 1F$

$4s^3+s$

$s \overline{)\ 4s^2+1\ }(4s = Z_3(s)$; $L = 4H$

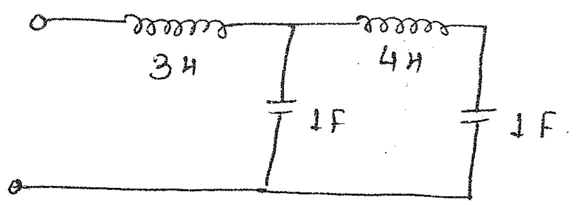
$-4s^2$

$1 \overline{)\ s\ }(s = Y_4(s)$; $C = 1F.$

$-s$

$\times$

Circuit expression is,



__xample 2__

Synthesis the given RC impedance using cauer method.

$$Z(s) = \frac{s^2+4s+3}{s^2+2s}$$

Sol^n^: Given, $Z(s) = \dfrac{s^2+4s+3}{s^2+2s}$

Now, using cauer I method:

$s^2+2s \overline{)\ s^2+4s+3\ }(1 \Rightarrow Z_1(s) \Rightarrow R = 1\Omega$

s^2+2s

$2s+3 \overline{)\ s^2+2s\ }(\frac{1}{2}s \Rightarrow Y_2(s) \Rightarrow C = \frac{1}{2}F$

$-s^2+\frac{3}{2}s$

$\frac{1}{2}s \overline{)\ 2s+3\ }(4 \Rightarrow Z_3(s) \Rightarrow R = 4\Omega$

$-2s$

$3 \overline{)\ \frac{1}{2}s\ }(\frac{s}{6} \Rightarrow Y_4(s) \Rightarrow C = \frac{1}{6}F$

$-\frac{1}{2}s$

$\times$

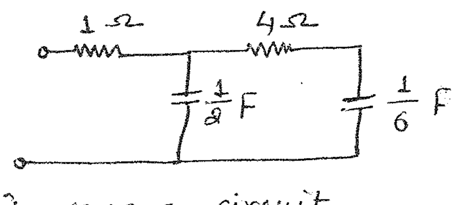


Fig: Cauer I circuit

Cauer II

We take the given polynomial in reverse order and take continued fraction i.e. $Z(S) = \dfrac{3+4S+S^2}{2S+S^2}$

Now, taking long division method

$$2S+S^2 \,\big)\, 3+4S+S^2 \,\big(\, \frac{3}{2S} \Rightarrow Z_1(S) \Rightarrow C_1 = \frac{2}{3}F$$

$$\underline{-3 - \frac{3}{2}S}$$

$$\frac{5}{2}S+S^2 \,\big)\, 2S+S^2 \,\big(\, \frac{4}{5} \Rightarrow Y_2(S) \Rightarrow R_1 = \frac{5}{4}\Omega$$

$$\underline{-2S - \frac{4}{5}S^2}$$

$$\frac{S^2}{5} \,\big)\, \frac{5}{2}S+S^2 \,\big(\, \frac{25}{2S} \Rightarrow Z_3(S) \Rightarrow C_2 = \frac{2}{25}F,$$

$$\underline{-\frac{5}{2}S}$$

$$S^2 \,\big)\, \frac{S^2}{5} \,\big(\, \frac{1}{5} \Rightarrow Y_2(S) \Rightarrow R_2 = 5\,\Omega$$

$$\underline{-\frac{S^2}{5}}$$

$$\times$$

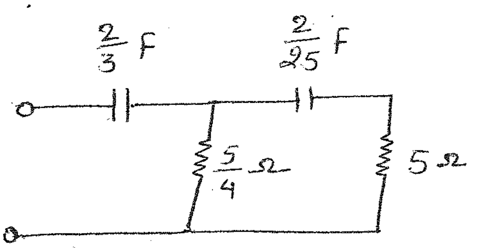
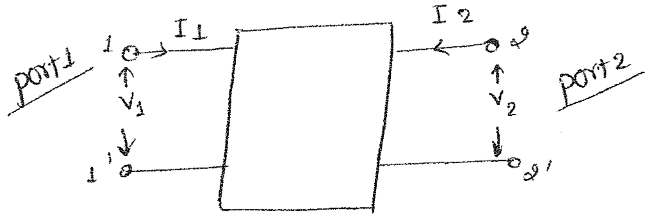


Fig: Cauer II circuit representation.

troduction:

Most often we have seen that the networks with terminals are nected in pairs to other networks. If the current entering one minal of a pair is equal and opposite to the current leaving e other terminal of the pair, then this type of terminal pair is lled as a "port."

I_1 and V_1 : Current and voltage at port 1



I_2 and V_2 : Current and voltage at port 2

Fig: Two port n/w

Out of four variable two are independent variables and emaining two depends on two independent variables and network rameter of that network.

Example

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$
$$V_2 = Z_{21} I_1 + Z_{22} I_2$$
} z-parameter equation

Here,

I_1 and I_2 are independent ~~parameters~~ Variables

V_1 and V_2 are dependent variables

Z_{11}, Z_{12}, Z_{21} and Z_{22} are network parameters (z-parameters)

Two port Network parameters:

There are six combination of network parameters as listed below:

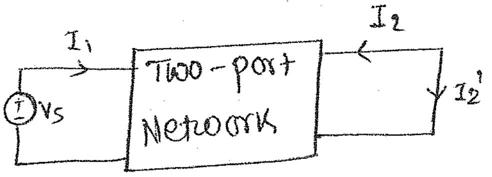
1) z-parameter ⇒ $Z_{11}, Z_{12}, Z_{21}, Z_{22}$ ⇒ Open circuit impedance parameter
2) Y-parameter ⇒ $Y_{11}, Y_{12}, Y_{21}, Y_{22}$ ⇒ Short circuit impedance parameter
3) ABCD parameter ⇒ Transmission parameter
) A'B'C'D' parameter ⇒ Inverse Transmission parameter
) h-Parameter ⇒ hybrid parameter
) g-parameter ⇒ Inverse-hybrid parameter

Relationship of Two port variables:

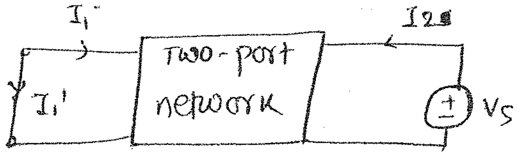
Name	Function: Express	Function: In terms of	Matrix equation
open-circuit impedance [Z]	V_1, V_2	I_1, I_2	$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$
short-circuit admittance [Y]	I_1, I_2	V_1, V_2	$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$
Transmission or chain [T]	V_1, I_1	$V_2, -I_2$	$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$
Inverse transmission [T]	V_2, I_2	$V_1, -I_1$	$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} V_1 \\ -I_1 \end{bmatrix}$
Hybrid (h)	V_1, I_2	I_1, V_2	$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$
Inverse hybrid	I_1, V_2	V_1, I_2	$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$

Reciprocity and symmetry:

A two port network is said to be reciprocal, if the ratio of the excitation to response is invariant to an interchange of the position of the excitation and response in the network. Networks containing resistors, inductors and capacitors are generally reciprocal. Networks that additionally have dependent sources are generally non-reciprocal. Mathematically, $\frac{V_s}{I_2'} = \frac{V_s}{I_1'}$ or $I_2' = I_1'$



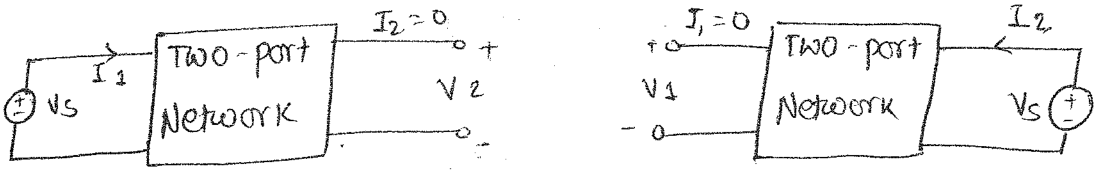
$[V_1 = V_s,\ I_1 = I_1,\ V_2 = 0,\ I_2 = -I_2']$



$[V_2 = V_s,\ I_2 = I_2,\ V_1 = 0,\ I_1 = -I_1']$

fig: Determination of condition for reciprocity.

A two port network is said to be symmetrical if the ports can interchanged without changing the port voltage and currents.
Mathematically, we can say from figure:



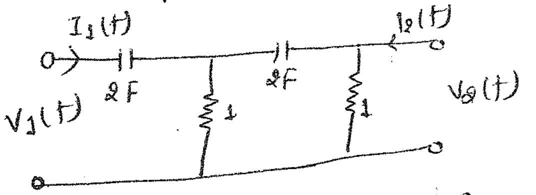
$(V_1 = V_S,\ I_1 = I_1,\ I_2 = 0,\ V_2 = V_2)$ $\quad$ $(V_2 = V_S,\ I_2 = I_2,\ I_1 = 0,\ V_1 = V_1)$

fig. Determination the condition for symmetry.

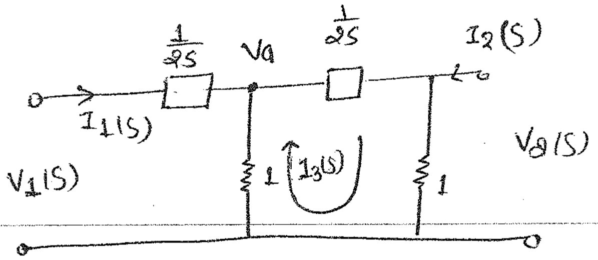
$$\frac{V_S}{I_1}\bigg|_{I_2=0} = \frac{V_S}{I_2}\bigg|_{I_1=0}$$

Example 1.

Obtain z parameter of given two port n/w.



sol^n: The transform circuit is,



~~Apply KCL at Va~~

$$\cancel{\frac{V_1(S) - V_a}{1/2S} = \frac{V_a}{1} + \frac{V_a - V_2(S)}{1/2S}}$$

$$\cancel{(V_1(S) - V_a)\,2S = V_a + (V_a - V_2(S))\,2S}$$

$$V_1(S) = \frac{1}{2S} I_1(S) + I_1(S) - I_3(S) \quad \ldots\ldots (A)$$

$$V_2(S) = I_2(S) + I_3(S) \quad \ldots\ldots (B)$$

$$I_3(S) - I_1(S) + \frac{1}{2S} I_3(S) + I_3(S) + I_2(S) = 0$$

$$I_3(S)\left[2 + \frac{1}{2S}\right] = I_1(S) - I_2(S)$$

$$I_3(S) = \frac{I_1(S) - I_2(S)}{\left(\frac{4S+1}{2S}\right)} = \left(\frac{2S}{4S+1}\right)\left[I_1(S) - I_2(S)\right] \quad \ldots (C)$$

from Ⓐ and Ⓒ

$$V_1(S) = \frac{1}{2S} I_1(S) + I_1(S) - \frac{2S}{4S+1} I_1(S) + \frac{2S}{4S+1} I_2(S)$$

$$V_1(S) = I_1(S) \left[\frac{1}{2S} + 1 - \frac{2S}{4S+1} \right] + \frac{2S}{4S+1} I_2(S)$$

$$V_1(S) = I_1(S) \left[\frac{4S+1+8S^2+2S-4S^2}{8S^2+2S} \right] + \frac{2S}{4S+1} I_2(S)$$

Now,

$$Z_{11} = \left.\frac{V_1(S)}{I_1(S)}\right|_{I_2(S)=0} = \frac{4S^2+6S+1}{2S(4S+1)}$$

$$Z_{12} = \left.\frac{V_1(S)}{I_2(S)}\right|_{I_1(S)=0} = \frac{2S}{4S+1}$$

from Ⓑ and Ⓒ

$$V_2(S) = I_2(S) + \frac{2S}{4S+1} I_1(S) - \frac{2S}{4S+1} I_2(S)$$

$$V_2(S) = \left(1 - \frac{2S}{4S+1}\right) I_2(S) + \frac{2S}{4S+1} I_1(S)$$

$$V_2(S) = \left(\frac{2S+1}{4S+1}\right) I_2(S) + \left(\frac{2S}{4S+1}\right) I_1(S)$$

Now,

$$Z_{21} = \left.\frac{V_2(S)}{I_1(S)}\right|_{I_2(S)=0} = \frac{2S}{4S+1}$$

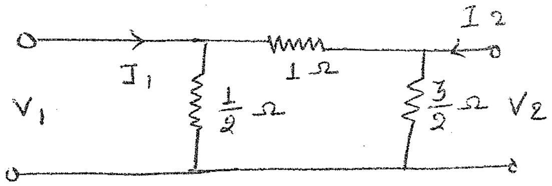
$$Z_{22} = \left.\frac{V_2(S)}{I_2(S)}\right|_{I_1(S)=0} = \frac{2S+1}{4S+1}$$

$$\therefore \quad Z_{11} = \frac{4S^2+6S+1}{2S(4S+1)}, \quad Z_{12} = \frac{2S}{4S+1}$$

$$Z_{21} = \frac{2S}{4S+1} \qquad Z_{22} = \frac{2S+1}{4S+1}$$

xample 2 :

Find ABCD parameter for following resistive n/w.

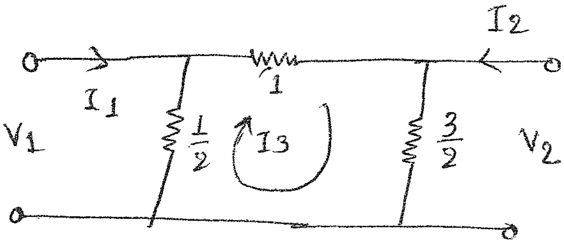


sol^n: The required equation for ABCD parameter are:

$V_1 = AV_2 - BI_2$

$I_1 = CV_2 - DI_2$

Now, transformed circuit is,



Now,

$V_1 = \frac{1}{2}(I_1 - I_3)$

$= \frac{1}{2} I_1 - \frac{1}{2} I_3 \quad \ldots\ldots (1)$

$V_2 = \frac{3}{2}(I_2 + I_3) = \frac{3}{2} I_2 + \frac{3}{2} I_3 \quad \ldots\ldots (2)$

$(I_3 - I_1)\frac{1}{2} + I_3 + (I_2 + I_3)\frac{3}{2} = 0$

or, $I_3\left[\frac{1}{2} + 1 + \frac{3}{2}\right] = \frac{1}{2} I_1 - I_2 \times \frac{3}{2} \Rightarrow I_3\left(\frac{6}{2}\right) = \frac{1}{2} I_1 - \frac{3}{2} I_2$

or, $I_3 = \frac{1}{6} I_1 - \frac{3}{6} I_2 = \frac{1}{6} I_1 - \frac{1}{2} I_2 \quad \ldots (3)$

From eq^n (1) and (3)

$V_1 = \frac{1}{2} I_1 - \frac{1}{2}\left(\frac{1}{6} I_1 - \frac{1}{2} I_2\right)$

$V_1 = \frac{1}{2} I_1 - \frac{1}{12} I_1 + \frac{1}{4} I_2$

$V_1 = \frac{5}{12} I_1 + \frac{1}{4} I_2 \quad \ldots\ldots\ldots\ldots (4)$

From eq^n (2) and (3)

$V_2 = \frac{3}{2} I_2 + \frac{3}{2}\left[\frac{1}{6} I_1 - \frac{1}{2} I_2\right]$

$V_2 = \frac{3}{2} I_2 + \frac{1}{4} I_1 - \frac{3}{4} I_2 = \cancel{\frac{87}{4} I_2 + \frac{15}{8} I_1 \ldots (5)} \quad \frac{3}{4} I_2 + \frac{1}{4} I_1 \ldots ($

Now, from ④ and ⑤

$$\frac{1}{A} = \frac{V_2}{V_1}\bigg|_{I_2=0} = \frac{\frac{1}{4} I_1}{\frac{5}{12} I_1}$$

$$A = \frac{5}{12} \times 4 = \frac{5}{3}$$

Now, when, $V_2 = 0$ eq^n. 5 becomes,

$$I_1 = -3I_2 \quad \ldots\ldots\ (6)$$

from 4,

$$V_1 = \frac{5}{12} \times (-3I_2) + \frac{1}{4} I_2 = -\frac{5}{4} I_2 + \frac{1}{4} I_2 = -I_2$$

$$\therefore -\frac{V_1}{I_2} = B = 1$$

from ⑥

$$\frac{I_1}{I_2} = D = 3$$

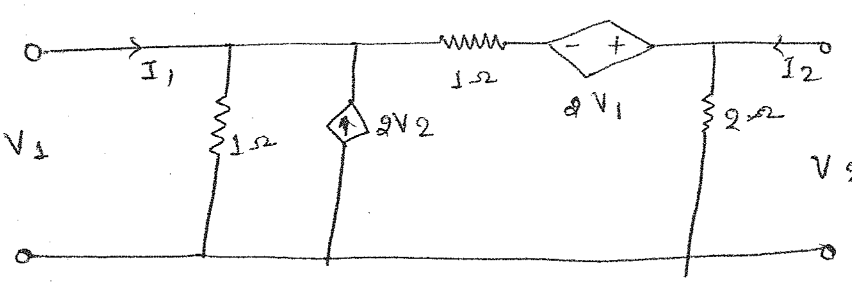
~~from~~ when. $I_2 = 0$ from eq^n ⑤

$$V_2 = \frac{1}{4} I_1$$

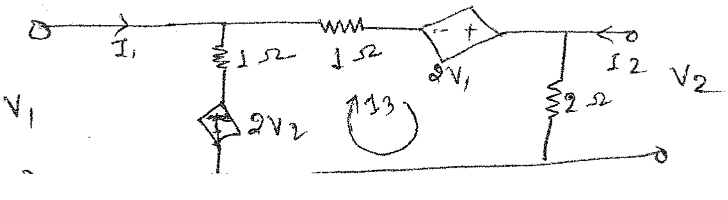
$$\frac{I_1}{V_2} = 4 = C$$

$$\therefore A = \frac{5}{3},\ B = 1,\ C = 4,\ D = 3$$

Example 3 Find Y parameter for the following resistance n/w.



Sol^n:



quation of y parameter are:

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$ i.e. $$I_1 = y_{11}V_1 + y_{12}V_2$$ $$I_2 = y_{21}V_1 + y_{22}V_2$$

From circuit,

$$V_1 = I_1 - I_3 + 2V_2$$

$$\Rightarrow I_3 = I_1 + 2V_2 - V_1 \quad \cdots\cdots (1)$$

nd, $I_3 - 2V_1 + 2(I_3 + I_2) - 2V_2 + I_3 - I_1 = 0$

or, $4I_3 - 2V_1 - 2V_2 + 2I_2 \cancel{=0} - I_1 = 0$

or, $2V_1 + 2V_2 = -I_1 + 2I_2 + 4I_3 \quad \cdots\cdots (2)$

and,

$$V_2 = 2(I_2 + I_3) \quad \cdots\cdots (3)$$

putting value of I_3 from (1) into (2) and (3) we have,

$$2V_1 + 2V_2 = -I_1 + 2I_2 + 4(I_1 + 2V_2 - V_1)$$

or, $2V_1 + 2V_2 = -I_1 + 2I_2 + 4I_1 + 8V_2 - 4V_1$

or, $6V_1 - 6V_2 = 3I_1 + 2I_2 \quad \cdots\cdots (4)$

and, $V_2 = 2I_2 + 2(I_1 + 2V_2 - V_1)$

or, $V_2 = 2I_2 + 2I_1 + 4V_2 - 2V_1$

or, $2V_1 - 3V_2 = 2I_2 + 2I_1 \quad \cdots\cdots (5)$

Now, subtracting (4) from (5), we have

$$-I_1 = -4V_1 + 3V_2$$

$$I_1 = 4V_1 - 3V_2 \quad \cdots\cdots (6)$$

Now, eq^n (5) becomes

$$2V_1 - 3V_2 = 2I_2 + 8V_1 - 6V_2$$

$$I_2 = \frac{3V_2 - 6V_1}{2}$$

$$= -3V_1 + \frac{3}{2}V_2 \quad \cdots\cdots (7)$$

$\therefore$ from eq^n (6) and (7), required y-parameter are $y = \begin{bmatrix} 4 & -3 \\ -3 & \frac{3}{2} \end{bmatrix}$ //

Relationship among parameters:

If we want to express α - parameter interms of parameter, we have to write β - parameter equation then by algebric anipulation we transform it to the form of α - parameter.

xample

Z - parameter in terms of T- Parameter :

Equations of T- parameters are :

$$V_1 = AV_2 - BI_2 \quad \cdots \text{①}$$

$$I_1 = CV_2 - DI_2 \quad \cdots \text{②}$$

From equation, ② $CV_2 = I_1 + DI_2$

$$V_2 = \frac{1}{C} I_1 + \frac{D}{C} I_2 \quad \cdots \text{③}$$

Comparing it with,

$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

We have, $Z_{21} = \frac{1}{C}$, $Z_{22} = \frac{D}{C}$

From equation ① and ③

$$V_1 = \frac{A}{C} I_1 + \frac{AD}{C} I_2 - BI_2 = \frac{A}{C} I_1 + \frac{AD-BC}{C} I_2$$

Comparing it with, $V_1 = Z_{11} I_1 + Z_{12} I_2$

$$\therefore Z_{11} = \frac{A}{C}, \quad Z_{12} = \frac{AD-BC}{C}$$

Y - parameters interms of Z

$$[V] = [Z][I]$$

For, $\frac{[I]}{[V]}$ $[I] = [Z]^{-1} [V]$

$$[Y] = [Z]^{-1}$$

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}^{-1} = \frac{1}{\Delta Z} \begin{bmatrix} Z_{22} & -Z_{12} \\ -Z_{21} & Z_{11} \end{bmatrix}$$

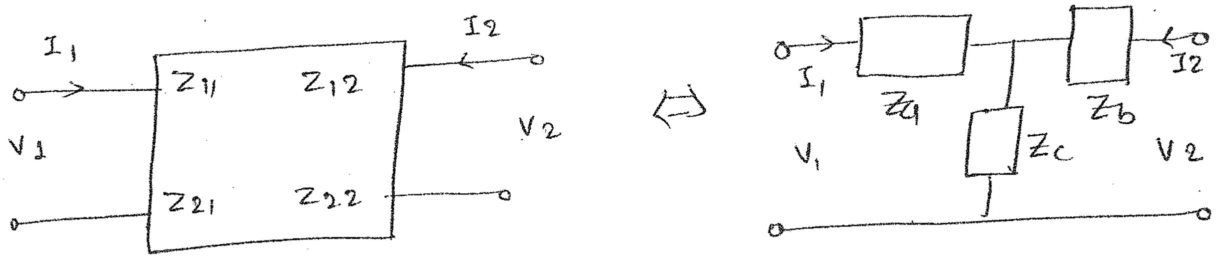
Table for relationship among parameter

Z		Y		T	
Z_{11}	Z_{12}	$\frac{Y_{22}}{\Delta Y}$	$-\frac{Y_{12}}{\Delta Y}$	$\frac{A}{C}$	$\frac{AD-BC}{C}$
Z_{21}	Z_{22}	$-\frac{Y_{21}}{\Delta Y}$	$\frac{Y_{11}}{\Delta Y}$	$\frac{1}{C}$	$\frac{D}{C}$
$\frac{Z_{22}}{\Delta Z}$	$-\frac{Z_{12}}{\Delta Z}$	Y_{11}	Y_{12}	$\frac{D}{B}$	$-\frac{AD-BC}{B}$
$-\frac{Z_{21}}{\Delta Z}$	$\frac{Z_{11}}{\Delta Z}$	Y_{21}	Y_{22}	$-\frac{1}{B}$	$\frac{A}{B}$
$\frac{Z_{11}}{Z_{21}}$	$\frac{\Delta Z}{Z_{21}}$	$-\frac{Y_{22}}{Y_{21}}$	$-\frac{1}{Y_{21}}$	A	B
$\frac{1}{Z_{21}}$	$\frac{Z_{22}}{Z_{21}}$	$-\frac{\Delta Y}{Y_{21}}$	$-\frac{Y_{11}}{Y_{21}}$	C	D

T and π representation of 2-port network:

T- Network

Any two port network can be represented by an equivalent T network as shown in figure below.



Equation for Z-parameter are

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$

$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

For above n/w, for $I_2 = 0$

$$V_1 = (Z_a + Z_c) I_1 \Rightarrow \frac{V_1}{I_1} = Z_a + Z_c = Z_{11}$$

$V_2 = I_1 Z_c \Rightarrow \frac{V_2}{I_1} = Z_c = Z_{21}$

For, $I_1 = 0$

$V_1 = Z_c I_2 \Rightarrow \frac{V_1}{I_2} = Z_c = Z_{12}$

nd, $V_2 = (Z_b + Z_c) I_2 \Rightarrow \frac{V_2}{I_2} = Z_b + Z_c = Z_{22}$

Now,

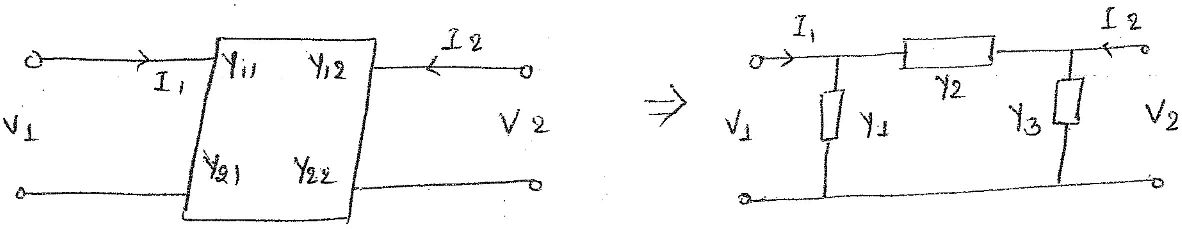
$Z_c = Z_{12} = Z_{21}$

$Z_a + Z_c = Z_{11} \Rightarrow Z_a = Z_{11} - Z_c = Z_{11} - Z_{12}$

$Z_b + Z_c = Z_{22} \Rightarrow Z_b = Z_{22} - Z_c = Z_{22} - Z_{21}$

π- Network

Any two port n/w can be represented by an equivalent π n/w as shown in figure below:

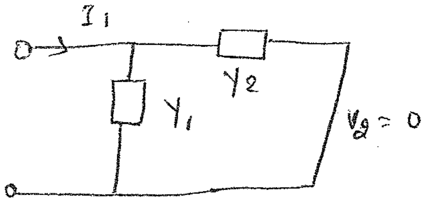


Equation for y- parameter

$I_1 = Y_{11}V_1 + Y_{12}V_2$

$I_2 = Y_{21}V_1 + Y_{22}V_2$

For, above n/w, For $V_2 = 0$



$\Rightarrow I_1 = (Y_1 + Y_2) V_1 \quad \ldots (a)$

$\frac{I_1}{V_1} = Y_1 + Y_2 = Y_{11}$

and, $\frac{I_1 + I_2}{Y_1} + \frac{I_2}{Y_2} = 0$

or, $Y_2 (I_1 + I_2) + I_2 Y_1 = 0$

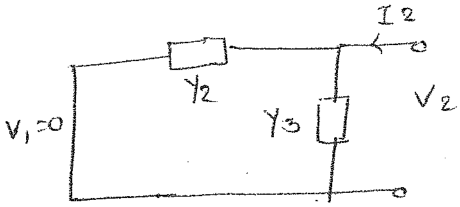
or, $(Y_1 + Y_2) I_2 + Y_2 I_1 = 0$

or, $I_1 = \frac{-I_2 (Y_1 + Y_2)}{Y_2}$

Now, from ⓐ

$$\frac{-I_2(Y_1+Y_2)}{Y_2} = (Y_1+Y_2)V_1$$

$$\frac{I_2}{V_1} = -Y_2 = Y_{21}$$



For $V_1 = 0$

$$I_2 = (Y_2+Y_3)V_2 \quad \ldots ⓑ$$

$$\frac{I_2}{V_2} = Y_2+Y_3 = Y_{22}$$

and, $$\frac{I_1}{Y_2} + \frac{I_1+I_2}{Y_3} = 0$$

$$I_2 = -\frac{I_1(Y_2+Y_3)}{Y_2}$$

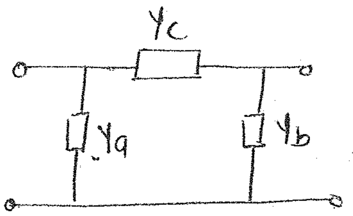
Now, ⓑ becomes

$$-\frac{I_1(Y_2+Y_3)}{Y_2} = (Y_2+Y_3)V_2$$

$$\Rightarrow \frac{I_1}{V_2} = -Y_2 = Y_{12}$$

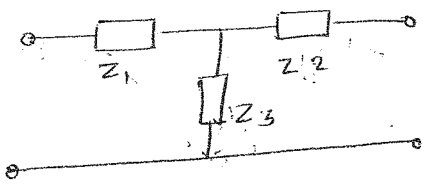
Π to T and T to Π transfer

Π- network



$\Longleftrightarrow$

T-network



Delta to star (Π to T)

$$Z_1 = \frac{\frac{1}{Y_a}\times\frac{1}{Y_c}}{\frac{1}{Y_a}+\frac{1}{Y_b}+\frac{1}{Y_c}} = \frac{\frac{1}{Y_aY_c}}{\frac{Y_aY_b+Y_bY_c+Y_cY_a}{Y_aY_bY_c}} = \frac{Y_b}{Y_aY_b+Y_bY_c+Y_cY_a}$$

$$Z_2 = \frac{Y_a}{Y_aY_b+Y_bY_c+Y_cY_a}$$

$$Y_aY_b + Y_bY_c + Y_cY_a$$

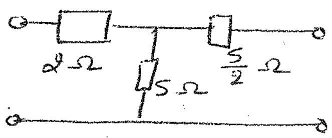
Star to delta (T to π)

$$Y_a = \frac{\frac{1}{Z_1} \times \frac{1}{Z_3}}{\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}} = \frac{\frac{1}{Z_1 Z_3}}{\frac{Z_2Z_3 + Z_1Z_3 + Z_1Z_2}{Z_1Z_2Z_3}} = \frac{Z_2}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}$$

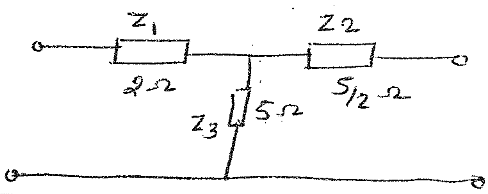
$$Y_b = \frac{Z_1}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}$$

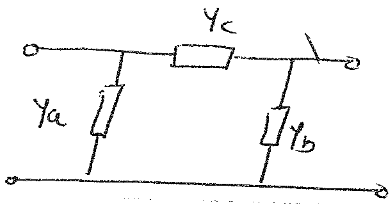
$$Y_c = \frac{Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}$$

Example Find equivalent π n/w for given T n/w.



Sol^n:





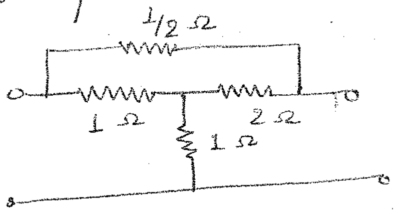
Now,

$$Y_a = \frac{5/2}{2 \times 5/2 + \frac{5}{2} \times 5 + 2 \times 5} = \frac{5/2}{5 + \frac{25}{2} + 10} = \frac{5/2}{\frac{5}{2}(2+5+4)} = \frac{1}{11}\ \mho$$

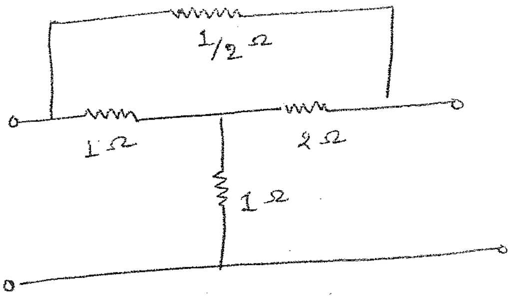
$$Y_b = \frac{Z_1}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1} = \frac{2}{\frac{5}{2}(11)} = \frac{4}{55}\ \mho$$

$$Y_c = \frac{Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1} = \frac{5}{\frac{5}{2} \times 11} = \frac{2}{11}\ \mho$$

Obtain an equivalent T-network



sol^n: Here,



Here, $Y_a = 1\,\Omega$

$Y_b = \frac{1}{2}\,\Omega$

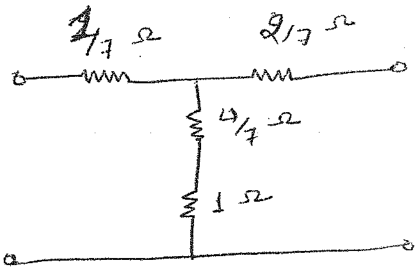
$Y_c = 2\,\Omega$

Then, $Z_2 = \dfrac{Y_a}{Y_aY_b + Y_aY_c + Y_bY_c} = \dfrac{1}{2 + 1 + 1/2} = \dfrac{2}{7}\,\Omega$

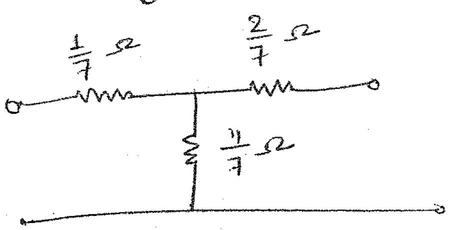
$Z_3 = \dfrac{Y_c}{Y_aY_b + Y_aY_c + Y_bY_c} = \dfrac{2}{\frac{7}{2}} = \dfrac{4}{7}\,\Omega$

$Z_1 = \dfrac{Y_b}{Y_aY_b + Y_aY_c + Y_bY_c} = \dfrac{1/2}{7/2} = \dfrac{1}{7}\,\Omega$

Now,



⇓



Equivalent T-n/w.