① **Finite Range:**

$$P(x) = \frac{1}{n} \quad \forall \; x \in \{1,2,3,\dots,n-1,n\} = \mathcal{X}$$

P.M.F conditions,

$$P(x = v_i) = \frac{1}{n} > 0 \quad \forall x \in \mathcal{X}$$

$$\sum_{i=1}^{n} P(v_i) = \sum_{i=1}^{n} \frac{1}{n} = \frac{1}{n}\left(\sum_{i=1}^{n} 1\right) = \frac{n}{n} = 1$$

Infinite Range:

$$P(x) = \frac{1}{2^x} \quad \forall \; x \in \mathbb{N}$$

P.M.F conditions,

$$P(x) = \frac{1}{2^x} > 0 \quad \forall \; x \in \mathbb{N}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} P(v_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2^i}$$

$$= \underbrace{\left[\frac{1}{2} + \frac{1}{2^2} + \cdots\cdots\right]}_{G.P}$$

$$= \frac{1/2}{1 - 1/2}$$

$$= 1$$

② $$U(a,b) = \begin{cases} \dfrac{1}{(b-a)} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

$$\mu = \int_a^b x P(x)\,dx = \int_a^b x\left[\frac{1}{b-a}\right] dx$$

$$= \frac{1}{2(b-a)}\left[b^2 - a^2\right] = (b+a)/2$$

</br>

$$\sigma^2 = \int_a^b (x-\mu)^2 P(x)\,dx$$

$$= \frac{1}{(b-a)} \int_a^b (x-\mu)^2\,dx$$

$$= \frac{1}{(b-a)} \left.\frac{(x-\mu)^3}{3}\right]_a^b$$

$$= \frac{(b-\mu)^3 - (a-\mu)^3}{3(b-a)}$$

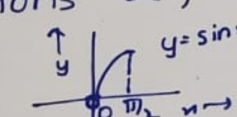
$$= \frac{\left(b - \left(\frac{b+a}{2}\right)\right)^3 - \left(a - \left(\frac{b+a}{2}\right)\right)^3}{3(b-a)}$$

$$= \frac{3\left(\frac{b-a}{2}\right)^3}{3(b-a)} = \frac{(b-a)^2}{12}$$

$$\boxed{\sigma^2 = \frac{(b-a)^2}{12}}$$

③ The two density functions be,

$$\boxed{P_1(x) = \sin x}, \quad x \in \left(0, \frac{\pi}{2}\right]$$



μ, σ

$$\boxed{P_2(x) = N(\mu,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}}$$

Normal Density function.

</br>

let us calculate μ, σ for $P_1(x) = \sin x$

$$\mu = \int_{0^+}^{\pi/2} (\sin x)\, x \, dx$$

$$= -\int_{0^+}^{\pi/2} x \, d(\cos x) \qquad \boxed{\int u\,dv = uv - \int v\,du}$$

$$= -\left[x\cos x \Big]_0^{\pi/2} - \int_0^{\pi/2} \cos x \, dx \right]$$

$$= \int_0^{\pi/2} \cos x \, dx - x\cos x \Big]_0^{\pi/2}$$

$$= 1 - 0 = 1$$

$$\boxed{\mu = \int_0^{\pi/2} x \sin x \, dx = 1} = \mathcal{E}(x)$$

$$\sigma^2 = \mathcal{E}(x^2) - (\mathcal{E}(x))^2$$

$$= \mathcal{E}(x^2) - 1$$

$$= \int_{0^+}^{\pi/2} x^2 \sin x \, dx - 1$$

$$= -\int_0^{\pi/2} x^2 d(\cos x) - 1$$

$$= -\left[x^2 \cos x \Big]_0^{\pi/2} - \int_0^{\pi/2} \cos x \,(2x) dx \right] - 1$$

$$= 2\int_0^{\pi/2} x \cos x \, dx - 1$$

$$= 2\left[\left[\int_0^{\pi/2} \left(\frac{\pi}{2} - x\right) \sin x \, dx\right] - 1\right]$$

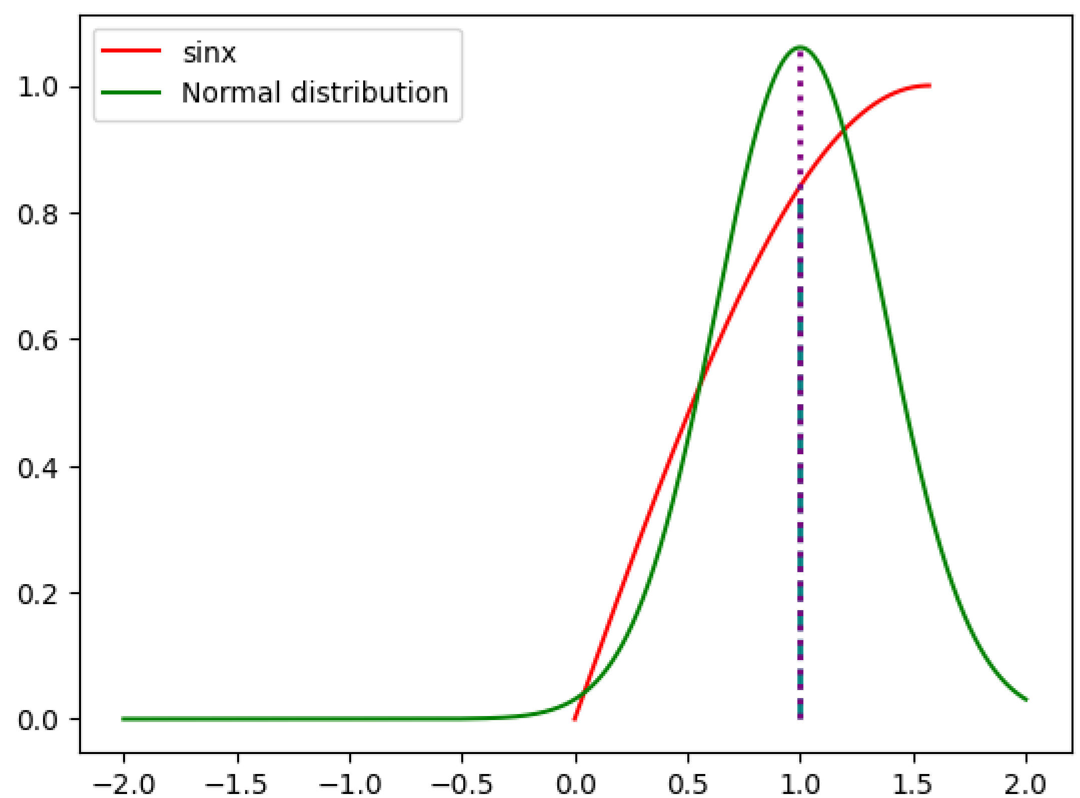
$$= 2\left[\frac{\pi}{2} \int_0^{\pi/2} \sin x \, dx - \mathcal{E}(x)\right] - 1 = \pi - 3$$

$$\boxed{\sigma^2 = \pi - 3}$$

</br>

In [1]:
```python
from cProfile import label
import matplotlib.pyplot as plt
import numpy as np
import math

plt.style.use('default')
pi = math.pi
```

```
c = 1/math.sqrt(2 * pi * (pi - 3))

x = np.linspace(0.00000001, pi/2, 100000)
y = np.linspace(-2, 2, 100000)


def f(x): return math.sin(x)


vect_f = np.vectorize(f)


def g(y): return c * ((math.e)**(-(y - 1)*(y - 1)/(2 * (pi - 3))))


vect_g = np.vectorize(g)


plt.plot(x, vect_f(x), 'r', label='sinx')
plt.plot(y, vect_g(y), 'g', label='Normal distribution')

plt.vlines(x=[1], ymin=[0], ymax=[f(1)], colors='teal', ls='--',
           lw=2)
plt.vlines(x=[1], ymin=[0], ymax=[g(1)], colors='purple', ls=':',
           lw=2)

plt.legend()

plt.savefig('3.png')
```



$$\boxed{P_2(x) = N(1, \sqrt{\pi-3}) = \frac{1}{\sqrt{2\pi(\pi-3)}}\, e^{-\frac{(x-1)^2}{2(\pi-3)}}}$$

(4) $\mathrm{Var}(x) \equiv \sigma^2 = \mathcal{E}\left[(x-\mu)^2\right]$

$$\mathcal{E}(x) \equiv \mu = \sum_{x \in \mathcal{X}} x\, p(x) = \sum_{i=1}^{n} v_i\, p(v_i)$$

$$\sigma^2 = \mathcal{E}\left[(x-\mu)^2\right] = \sum_{i=1}^{n} (v_i-\mu)^2 p(v_i)$$

$$= \left(\underbrace{\sum_{i=1}^{n} (v_i^2 p(v_i))} + \mu^2 \sum_{i=1}^{n} p(v_i) - 2\mu \sum_{i=1}^{n} v_i p(v_i)\right)$$

$$= \mathcal{E}(x^2) + \mu^2(1) - 2\mu(\mu)$$

$$= \mathcal{E}(x^2) - \mu^2$$

$$= \mathcal{E}(x^2) - (\mathcal{E}(x))^2$$

$$\boxed{\sigma^2 = \mathcal{E}(x^2) - (\mathcal{E}(x))^2}$$

(5) $N(\mu, \sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$$\mu' = \int_{-\infty}^{\infty} (N(\mu,\sigma)) \cdot x\, dx$$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right) \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} x \cdot dx$$

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$$\boxed{c = \frac{1}{\sqrt{2\pi}\,\sigma}}$$

$$\mu' = c \int_{-\infty}^{\infty} \left(e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right) \cdot x \cdot dx$$

$$\boxed{x \longrightarrow t+\mu}$$

$$\mu' = c \int_{-\infty}^{\infty} e^{-\left(\frac{t^2}{2\sigma^2}\right)} \cdot (t+\mu)\, d(x-\mu).$$

$$= c\left[\mu \int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}} dt + \underbrace{\int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}} \cdot t\, dt}_{\to 0}\right]$$

$$= \boxed{c\mu \int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}} dt} \text{ --- } (1)$$

$$= 2c\mu \int_{0}^{\infty} e^{-\frac{t^2}{2\sigma^2}} dt$$

$$= \frac{2\mu}{\sqrt{2\pi}\,\sigma} \int_{0}^{\infty} e^{-\left(\frac{t}{\sqrt{2}\sigma}\right)^2} dt$$

$$= \frac{2\mu}{\sqrt{\pi}} \int_{0}^{\infty} e^{-\left(\frac{t}{\sqrt{2}\sigma}\right)^2} d\left(\frac{t}{\sqrt{2}\sigma}\right)$$

$$\boxed{\frac{t}{\sqrt{2}\sigma} \longrightarrow x}$$

$$= \frac{2\mu}{\sqrt{\pi}} \int_{0}^{\infty} e^{-x^2} dx = \frac{2\mu}{\sqrt{\pi}}\left(\frac{\sqrt{\pi}}{2}\right) = \mu$$

Using gaussian Integral Trick

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}} \qquad \boxed{\int_{0}^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}}$$

$$\therefore \boxed{\mu' = \mu} \checkmark$$

</br>

$$(\sigma')^2 = E(x^2) - (E(x))^2$$

$$(\sigma')^2 = \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdot x^2\, dx - \mu^2\right]$$

$$E(x^2) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} x^2\, dx$$

$$\boxed{x \to t+\mu}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t)^2}{2\sigma^2}} (t+\mu)^2\, d(x-\mu)$$

$$\boxed{c = \frac{1}{\sqrt{2\pi}\sigma}}$$

$$= c\int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}} (t^2+\mu^2+2\mu t)\, dt$$

$$= c\left(\int_{-\infty}^{\infty} t^2 e^{-\frac{t^2}{2\sigma^2}} dt + \mu^2 \int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}} dt + \overbrace{2\mu \int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}} t\, dt}^{0}\right)$$

$$= c\int_{-\infty}^{\infty} t^2 e^{-\frac{t^2}{2\sigma^2}} dt + c\mu^2 \int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}} dt$$

From ①, $\boxed{c\mu \int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}} dt = \mu}$

</br>

$$E(x^2) = c\int_{-\infty}^{\infty} t^2 \cdot e^{-\frac{t^2}{2\sigma^2}} dt + \mu \cdot (\mu)$$

$$= c\int_{-\infty}^{\infty} t^2 e^{-\frac{t^2}{2\sigma^2}} dt + \mu^2 = K + \mu^2$$

$$K = c\int_{-\infty}^{\infty} t^2 e^{-\frac{t^2}{2\sigma^2}} dt$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} t^2 \cdot e^{-\frac{t^2}{2\sigma^2}} dt$$

$$= \frac{1}{(\sqrt{2}\sigma)\sqrt{\pi}} \int_{-\infty}^{\infty} t^2 e^{-\frac{t^2}{2\sigma^2}} dt$$

$$\boxed{\frac{t}{\sqrt{2}\sigma} \to x}$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (\sqrt{2}\sigma x)^2 e^{-x^2} \cdot (\sqrt{2}\sigma)\, dx$$

$$= \frac{1}{\sqrt{\cancel{2}\pi}\cancel{\sigma}} \int_{-\infty}^{\infty} (2\sigma^2 x^2) \cdot e^{-x^2} (\cancel{\sqrt{2}\sigma})\, dx$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (2\sigma^2)(x^2 e^{-x^2})\, dx$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} (x^2 e^{-x^2})\, dx = \sigma^2$$

$$\boxed{\int_{-\infty}^{\infty} (x^2 e^{-x^2})\, dx = \frac{\sqrt{\pi}}{2}}$$ using Gaussian Integral Trick

</br>

$$E(x^2) = \sigma^2 + \mu^2$$

$$(\sigma')^2 = (\sigma^2 + \mu^2) - (E(x))^2$$

$$= (\sigma^2 + \mu^2) - \mu^2$$

$$= \sigma^2$$

$$\therefore \boxed{\sigma' = \sigma}$$

- $\therefore \mu, \sigma$ are indeed parameters of $N(\mu, \sigma)$.

</br>

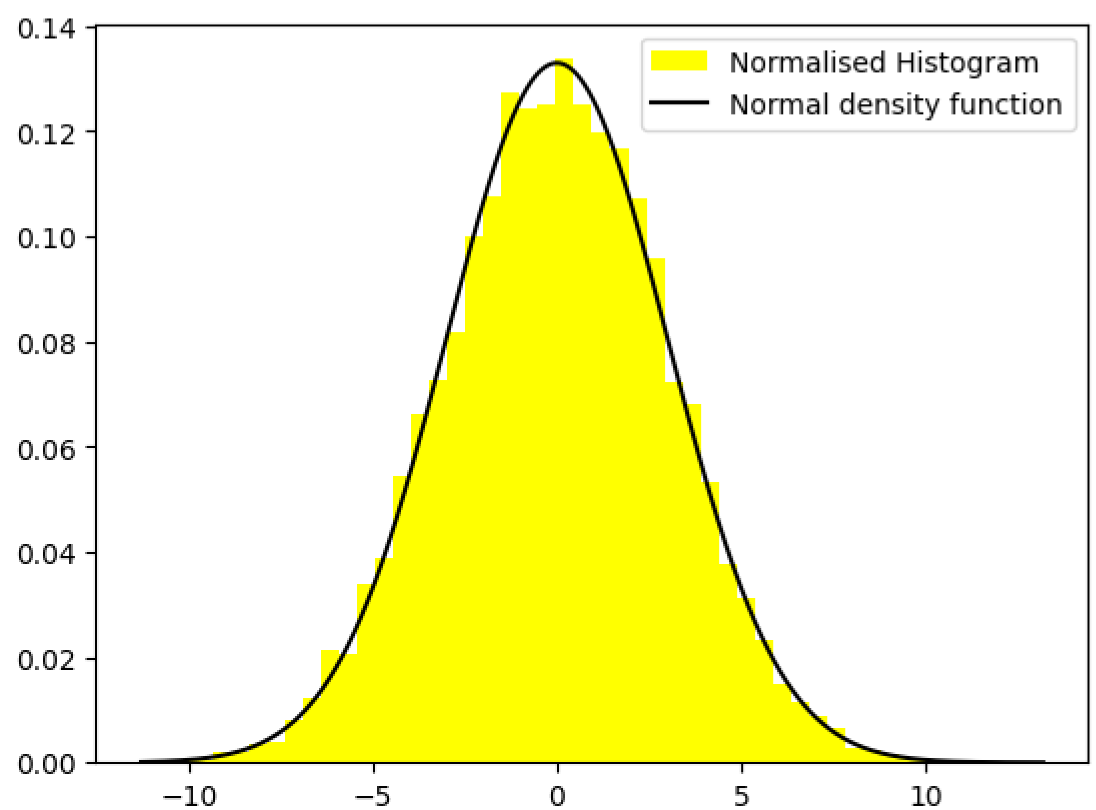
Qusestion-6

In [2]:
```
from cProfile import label
import matplotlib.pyplot as plt
from matplotlib import colors
from matplotlib.ticker import Formatter
import numpy as np
# import math
from scipy.stats import norm, rayleigh, expon
# from sklearn.preprocessing import scale
```

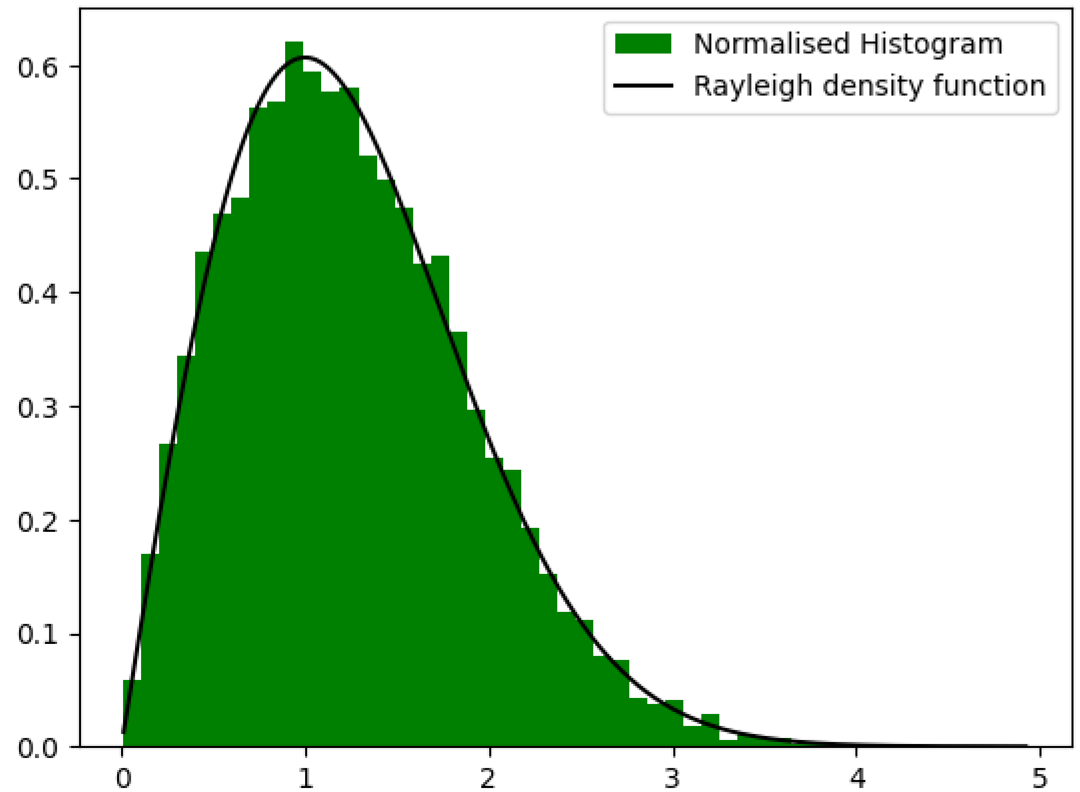
In [3]:
```python
plt.style.use('default')

# y = np.random(0, 1, 10000)
y = np.random.rand(10000)


# mapping of random numbers

norm_x = norm.ppf(y, loc=0, scale=3)
rayleigh_x = rayleigh.ppf(y, loc=0, scale=1 * 1)
expon_x = expon.ppf(y, loc=0, scale=1 / 1.5)

x1 = np.linspace(min(norm_x), max(norm_x), len(norm_x))
x2 = np.linspace(min(rayleigh_x), max(rayleigh_x), len(rayleigh_x))
x3 = np.linspace(min(expon_x), max(expon_x), len(expon_x))

norm = norm.pdf(x1, loc=0, scale=3)
rayleigh = rayleigh.pdf(x2, loc=0, scale=1 * 1)
expon = expon.pdf(x3,loc=0, scale=1 / 1.5)




print(norm_x, rayleigh_x, expon_x)
```

```
[ 0.6250538  -4.89816193  1.76722461 ... -6.01160259  0.93408464
 -1.18995897] [1.32176018 0.32442128 1.60029626 ... 0.21354196 1.39534121 0.92126003] [0.58234999 0.03508305 0.85364937 ... 0.01520006 0.64899237 0.28290668]
```

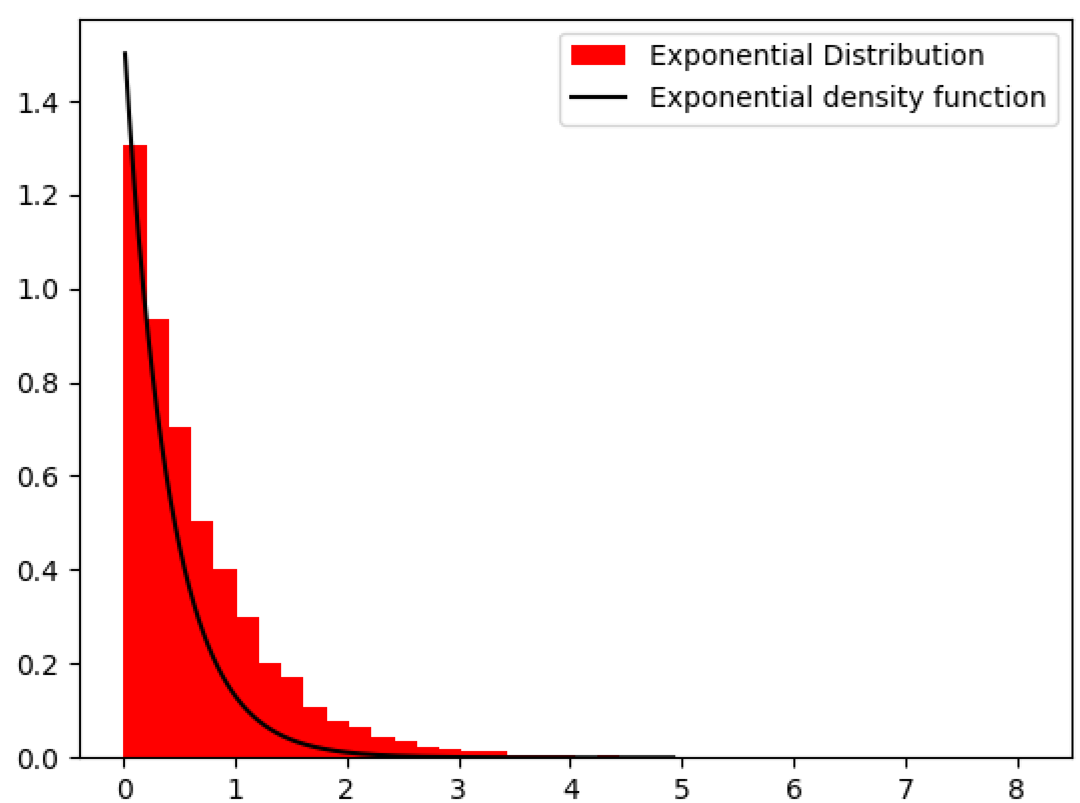
In [4]:
```python
from turtle import color


plt.hist(norm_x, density=True, bins= 50,color="yellow", label="Normalised Histogram")

plt.plot(x1, norm, color = "black", label = "Normal density function")
plt.plot()
plt.legend()
plt.savefig("Norm.png")
```



In [5]:
```
plt.hist(rayleigh_x, density=True, bins=50,
         color="green", label="Normalised Histogram")
plt.plot(x2, rayleigh, color = "black", label = "Rayleigh density function")
plt.legend()
plt.savefig("Rayleigh.png")
```



In [6]:
```
plt.hist(expon_x, density=True, bins=40,
         color="red", label="Exponential Distribution")
plt.plot(x2, expon, color = "black", label = "Exponential density function")
plt.legend()
plt.savefig("exp.png")
```

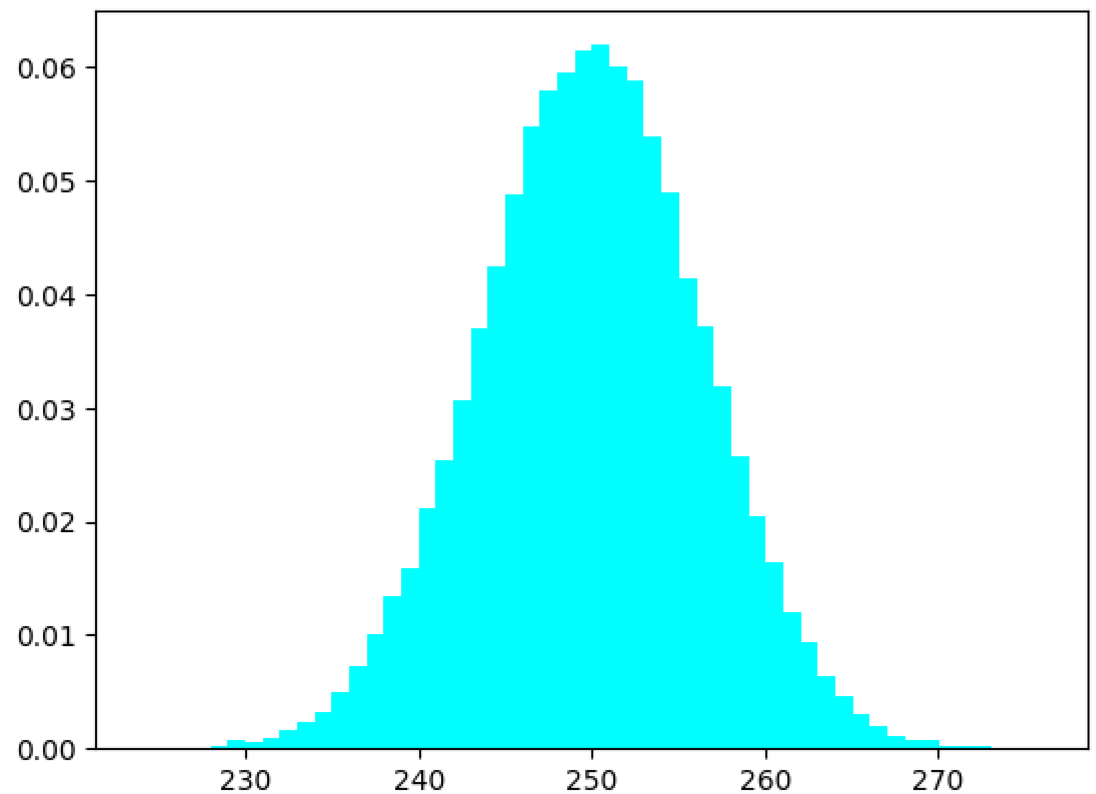


> Given a random variable y that follows the pdf U [0, 1], the random variable $x = C^{-1}$ (y) will follow a PDF with corresponding CDF as C(). With the inverse of CDF we got the values of x and plotted the corresponding pdf(x) above which traces the histograms.

Question-7

In [7]:
```
plt.style.use('default')


def generate_random_numbers():
    y = np.random.rand(500)
    return np.sum(y)

random_numbers = [generate_random_numbers() for i in range(50000)]

# print(random_numbers)
```

In [8]:
```
plt.hist(random_numbers, density=True, bins=int(abs(max(random_numbers)) - abs(min(random_numbers))),
         color="cyan")
plt.savefig("7.png")
```



> Peak ok the histogram is at in the range (250 - δ, 250 + δ) because, the mean of the distribution of the 500 random numbers generated in the generate_random_numbers() function has a mean (0.5) so the expected value should be 500 * (0.5) = 250. This distribution looks like a normal distribution with mean 250.