Q.1) Find the optimal control input and corresponding state trajectory for the dynamical system with 2 states given below. The states are denoted by $X(t) = [x_1(t),\, x_2(t)]^T$, and the control input is given by $u(t)$.

$$\frac{dX(t)}{dt} = \dot{X}(t) = f(X(t)) + g(X(t))u(t) \tag{1}$$

$$f(X(t)) = \begin{pmatrix} -x_1(t) + x_2(t) \\ -\frac{1}{2}x_1(t) - \frac{1}{2}x_2(t) + \frac{1}{2}x_2(t)\, sin^2(x_1(t)) \end{pmatrix} \tag{2}$$

$$g(X(t)) = \begin{pmatrix} 0 \\ sin(x_1(t)) \end{pmatrix} \tag{3}$$

The cost to be minimized is given below.

$$J(X(t)) = \frac{1}{2}\int_t^{\infty} \left(x_1^2(\tau) + x_2^2(\tau) + u^2(\tau)\right) d\tau \tag{4}$$

Initial conditions: $X(0) = [1, 2]^T$.
NB: Here consider $f(X(t))$ is unknown

Q.2) Find the optimal control input and corresponding state trajectory for the dynamical system with 2 states given below. The states are denoted by $X(k) = [x_1(k),\, x_2(k)]^T$, and the control input is given by $u(k)$.

$$X(k) = AX(k) + Bu(k) \tag{5}$$

The cost to be minimized is given below.

$$J = \frac{1}{2}\sum_{k=0}^{k=120} \left(X(k)^T Q X(k) + Ru^2(k)\right) \tag{6}$$

$X(0) = [5,\, -1]^T$

$$A = \begin{pmatrix} 0.9974 & 0.0539 \\ -0.1078 & 1.1591 \end{pmatrix} \tag{7}$$

$$B = \begin{pmatrix} 0.0013 \\ 0.0539 \end{pmatrix} \tag{8}$$

$$Q = \begin{pmatrix} 0.25 & 0.0 \\ 0.0 & 0.50 \end{pmatrix} \tag{9}$$

$$R = 0.15 \tag{10}$$

Case 1: A is known
Case 2: A is unknown

Q.3) Find the optimal control input and corresponding state trajectory for the dynamical system with 3 states given below. The states are denoted by $X(t) = [x_1(t),\, x_2(t),\, x_3(t)]^T$, and the control input is given by $u(t)$.

$$\dot{X}(t) = \frac{dX(t)}{dt} = AX(t) + Bu(t) \tag{11}$$

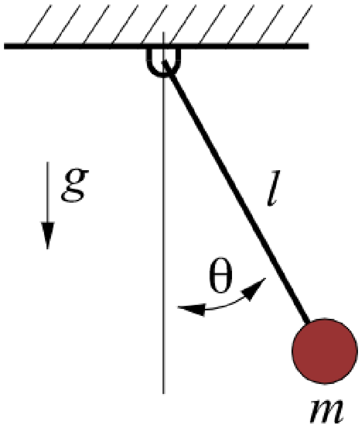

Figure 1: Pendulum

$$A = \begin{pmatrix} -1.01887 & 0.90506 & -0.00215 \\ 0.82225 & -1.07741 & -0.1755 \\ 0 & 0 & -20.2 \end{pmatrix} \tag{12}$$

$$B = \begin{pmatrix} 0 \\ 0 \\ 20.2 \end{pmatrix} \tag{13}$$

The cost to be minimized is given below.

$$J = \frac{1}{2}\int_0^{\infty} (X(t)^T Q X(t) + Ru^2(t))\, dt \tag{14}$$

$$Q = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 20 \end{pmatrix} \tag{15}$$

$$R = 50 \tag{16}$$

$X(0) = [5,\ -2,\ 0]^T$.

Case 1: A is known
Case 2: A is unknown

Q.4) The model equations of the pendulum shown in Figure 1 is given below.

$$ml^2 \frac{d^2\theta(t)}{dt^2} + b\frac{d\theta(t)}{dt} + mglsin(\theta(t)) = u(t) \tag{17}$$

Here m is the mass in kg, l is the length in m, g is the acceleration due to gravity, b is the coefficient of air friction, $\theta(t)$ is the angle from vertical and $u(t)$ is the control input torque. Find the optimal control input that takes the pendulum from $\theta(0) = 0$ radians to $\theta(t_f) = \pi$ radians? (Here, the final time t_f is a free variable.)
Formulate an appropriate cost function to be minimized?
Case 1: Find the optimal control for m=0.5 kg, l=1 m, b=0.1
Case 2: Find the optimal control when m=0.5 kg, l=1 m, and b is unknown.

Q.5) The dynamics of the cart-pole system shown in Figure 2 is given below. Here M and m_p is the mass (kg) of the cart and pole respectively. The linear displacement (in m) of the cart

$$\ddot{\theta} = \frac{-m_p L \sin\theta \cos\theta \dot{\theta}^2 + (M + m_p)\, g \sin\theta + \cos\theta F_x}{\left(M + m_p\left(1 - \cos^2\theta\right)\right) L}$$

$$\ddot{x} = \frac{-m_p L \sin\theta \dot{\theta}^2 + m_p g \sin\theta \cos\theta + F_x}{M + m_p\left(1 - \cos^2\theta\right)}$$

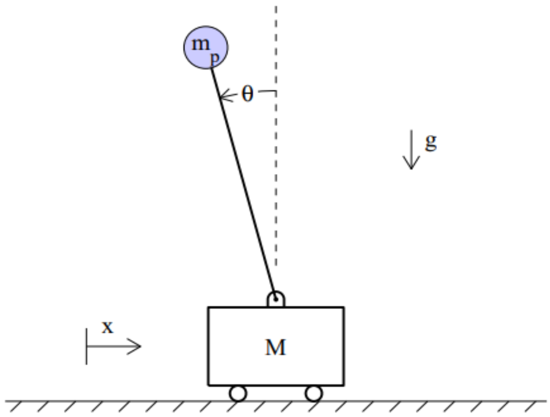

Figure 2: Cart-Pole

is denoted by x, g is the acceleration due to gravity, θ is the angular displacement (radians) of the pole of length L (in m) and F_x is the input force applied to the cart (in N).
Find the optimal control input (F_x) that takes the pole from the initial angular position $\theta(0) = \frac{\pi}{6}$ to the desired angular position $\theta(t_f) = 0$? (Here, the final time t_f is a free variable.)
M= 20 Kg, m_p= 0.5 kg, $L = 0.5\ m$,

Q.6) Find the optimal control input and corresponding state trajectory for the dynamical system with 2 states given below. The states are denoted by $X(k) = [x_1(k),\ x_2(k)]^T$, and the control input is given by $u(k)$.

$$X(k) = AX(k) + Bu(k) \tag{18}$$

The cost to be minimized is given below.

$$J = \frac{1}{2}\sum_{k=0}^{k=200}\left(X(k)^T Q X(k) + Ru^2(k)\right) \tag{19}$$

$X(0) = [1.9,\ -1.5]^T$

$$A = \begin{pmatrix} 0.9974 & 0.0539 \\ -0.1078 & 1.1591 \end{pmatrix} \tag{20}$$

$$B = \begin{pmatrix} 0.0013 \\ 0.0539 \end{pmatrix} \tag{21}$$

$$Q = \begin{pmatrix} 0.25 & 0.0 \\ 0.0 & 0.50 \end{pmatrix} \tag{22}$$

$$R = 0.15 \tag{23}$$

Case 1: A is known and $|x_1(k)| \leq 2$, $|x_2(k)| \leq 2$, $|u(k)| \leq 1.0$.
Case 2: A is unknown and $|x_1(k)| \leq 2$, $|x_2(k)| \leq 2$, $|u(k)| \leq 1.0$.