Cosmology Programming Assignment 2
Evolution of density with cosmic time

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1 Introduction

In the present assignment, we have explored the time variation of the matter, radiation and dark energy density in a flat universe (k=0) from the time of a few years after big bang till the present age of universe, by plotting and visualizing the same. Since the time evolution of the dark energy is not known, it is assumed as having a constant energy density equal to the present value throughout the history of the universe.

A universe is not entirely composed of radiation or matter alone. Hence, a more general situation is when one has a mixture of both matter and radiation (along with the dark energy). Then there are two separate fluid equations, one for each of the two components. The trick which allows us to write ρ as a function of a(t) (equivalently, as a function of z) still works, so we still have for all times, the density as a function of red-shift (or equivalently, scale parameter) given by[1]

1. $\rho_m \propto (1+z)^3$ (matter density)
2. $\rho_\gamma \propto (1+z)^4$ (radiation density)
3. $\rho_\Lambda =$ constant (dark energy density)

The Friedmann equation now becomes (for a flat universe),

$$H^2 = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}(\rho_m + \rho_\gamma + \rho_\Lambda) \tag{1}$$

and,

$$H_0^2 = \frac{8\pi G}{3}\rho_0 \tag{2}$$

Dividing the above two equations, we get

$$\left(\frac{H}{H_0}\right)^2 = \frac{8\pi G}{3}\Omega = \frac{8\pi G}{3}(\Omega_m + \Omega_\gamma + \Omega_\Lambda) \tag{3}$$

Using the dependence of each density component on the red-shift, we get

$$\left(\frac{H}{H_0}\right)^2 = \frac{8\pi G}{3}\Omega = \frac{8\pi G}{3}(\Omega_{m,0}(1+z)^3 + \Omega_{\gamma,0}(1+z)^4 + \Omega_{\Lambda,0}) \tag{4}$$

This means that the scale factor will have a more complicated behaviour, and so to convert p(a) into p(t) is much harder. It is possible to obtain exact solutions for this situation, but they are very messy. Instead, we consider the simpler situation where one or other of the densities is by far the larger.

In that case, we can say that the Friedmann equation is accurately solved by just including the dominant component. That is, we can use the expansion rates we have already found. For example, suppose radiation is much more important. Then one would have

$$a(t) \propto t^{1/2} \quad ; \quad \rho_\gamma \propto \frac{1}{t^2} \quad ; \quad \rho_m \propto \frac{1}{a^3} \propto \frac{1}{t^{3/2}} \tag{5}$$

Notice that the density in matter falls off more slowly than that in radiation. So the situation of radiation dominating cannot last forever; however small the matter component might be originally it will eventually come to dominate. We can say that domination of the Universe by radiation is an unstable situation.

In the opposite situation, where it is the matter which is dominant. we obtain the solution

$$a(t) \propto t^{2/3} \quad ; \quad \rho_m \propto \frac{1}{t^2} \quad ; \quad \rho_\gamma \propto \frac{1}{a^4} \propto \frac{1}{t^{8/3}} \tag{6}$$

Matter domination is a stable situation. the matter becoming increasingly dominant over the radiation as time goes by.

In the present analysis, we plot the density as a function of time computed by first computing the density values at different red-shift values and then plotting the density as a function of time, where the time corresponding to each red-shift is obtained using the integral relation between the red-shift and time. The other method is to directly compute density as a function of the time using the broken power-law approximation.

The integral relation between the time and red-shift is obtained from the Friedmann equation and is given by

$$(t_1 - t_0) = \frac{-1}{H_0}\int_0^{z_1} \frac{dz}{(1+z)\sqrt{E(z)}} \tag{7}$$

where, t_0 is the present age of the universe, at which the redshift is 0.

2 Results

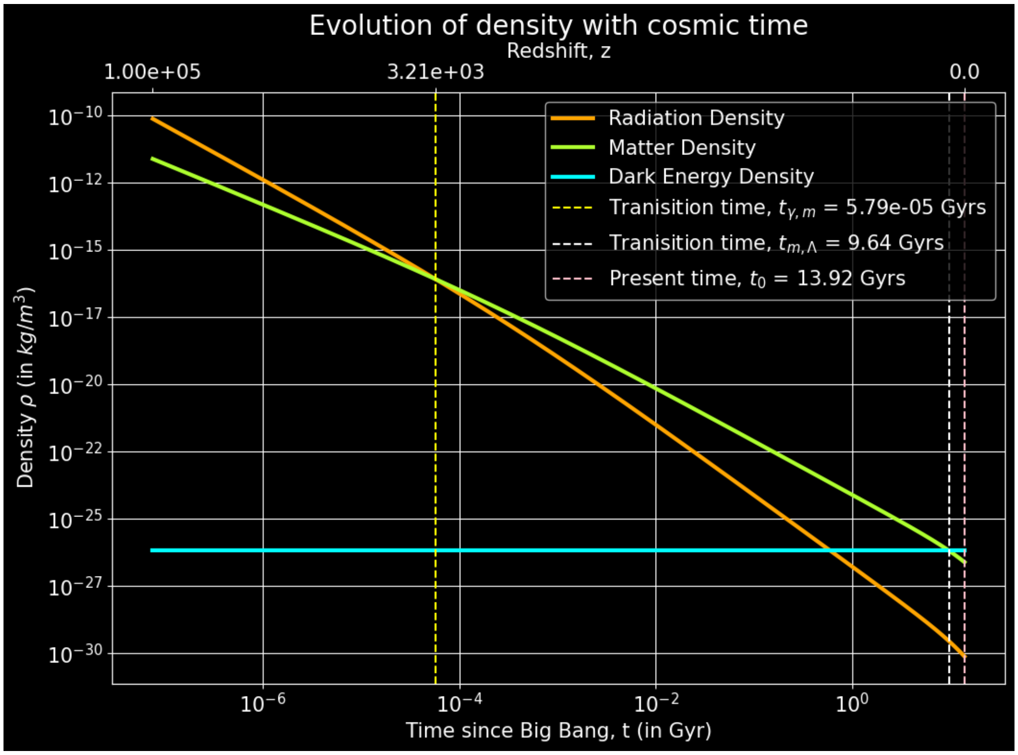

Figure 1: Time evolution of density using density as a function of red-shift

For the present analysis, we make the following approximations and assumptions:

1. The time variation of the scale factor (a(t)) is constant throughout the matter dominated and radiation dominated phases of density evolution.

2. The density evolution with time can be approximated by broken-power-law model with different power-law indices in the two regimes.

The transition red-shift and time are estimated using the following approximate methods:

1. **Method - 1:** The transition red-shift is determined using the density as a function of red-shift. The density values of radiation and matter are equated and the red-shift at which this equality holds is considered as the transition red-shift (z_c).

$$\rho_m(z) = \rho_\gamma(z)$$

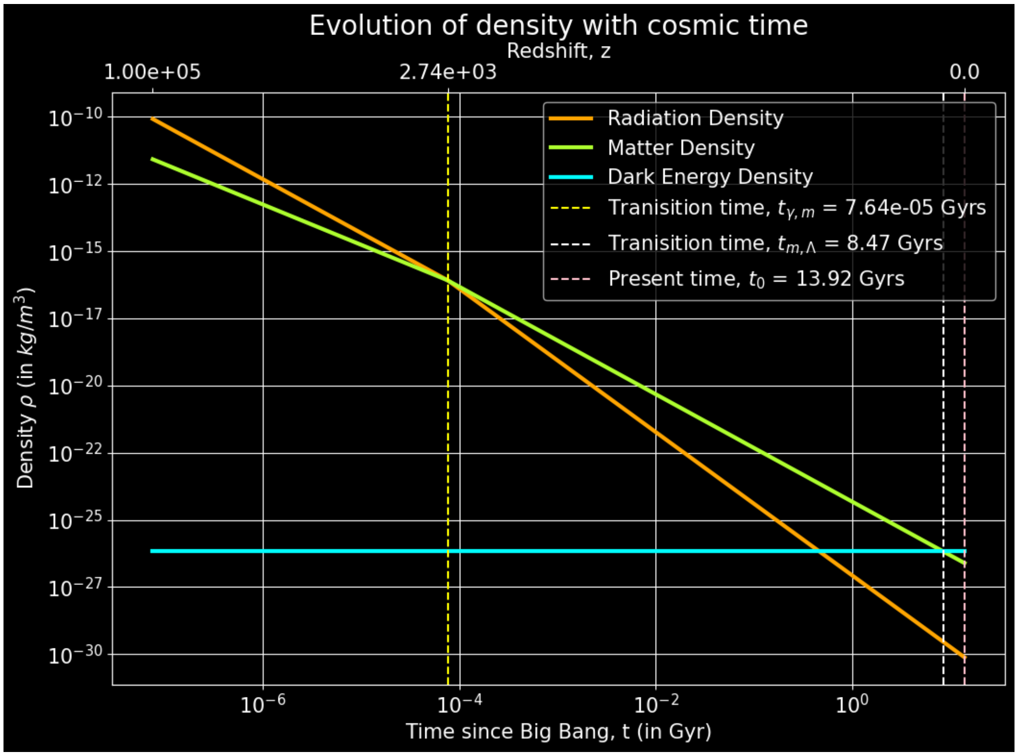

Figure 2: Time evolution of density using density as a function of time, approximated by broken power-law

$$\rho_{m,0}(1+z)^3 = \rho_{\gamma,0}(1+z)^4$$
$$(1+z_c) = \frac{\rho_{m,0}}{\rho_{\gamma,0}}$$

$$z_c = \frac{\rho_{m,0}}{\rho_{\gamma,0}} - 1 \tag{8}$$

2. **Method - 2:** The transition time is determined using the density as a function of time (in matter dominated regime). The density values of radiation and matter are equated and the time at which this equality holds is considered as the transition time (t_c).

$$\rho_m(t) = \rho_\gamma(t)$$
$$\rho_{m,0}\left(\frac{t_0}{t}\right)^2 = \rho_{\gamma,0}\left(\frac{t_0}{t}\right)^{\frac{8}{3}}$$
$$\left(\frac{t_0}{t}\right)^{\frac{2}{3}} = \frac{\rho_{m,0}}{\rho_{\gamma,0}}$$
$$\left(\frac{t_0}{t}\right) = \left(\frac{\rho_{m,0}}{\rho_{\gamma,0}}\right)^{\frac{3}{2}}$$

$$t_c = t_0 \left(\frac{\rho_{\gamma,0}}{\rho_{m,0}} \right)^{\frac{3}{2}} \tag{9}$$

The values of transition red-shift and time computed by the aforementioned methods (approximations) is displayed in the Table 1.

Method	$z_{\gamma,m}$	$t_{\gamma,m}$ (in Yrs)	$z_{m,\Lambda}$	$t_{m,\Lambda}$ (in GYrs)
Equating densities as a function of red-shift	3213.29	57856.65	0.39	9.64
Equating densities as a function of time (post transition)	2738.03	76384.20	0.55	8.47

Table 1: Different values of Transition parameters

During implementation, the plot was obtained using the red-shift values as the independent variable and the time corresponding to each red-shift was obtained by the integral relation between time and red-shift:

$$(t_1 - t_0) = \frac{-1}{H_0} \int_0^{z_1} \frac{dz}{(1+z)\sqrt{E(z)}} \tag{10}$$

3 Question and Answers

Sl. No.	**Question**	**Answer**
1.	Was the universe ever radiation dominated? If so, for what red-shift range?	Yes, the universe was initially radiation dominated for a red-shift ranging from ≈ 3213.29 to ∞.
2.	Did the universe ever flip from a radiation dominated universe to a matter dominated universe? If so, at which redshift, corresponding to what age of the universe?	Yes, the universe did flip from a radiation dominated universe to a matter dominated universe at a red-shift ≈ 3213.29 which corresponds to the age of the universe (at transition) of ≈ 57856.65 years.
3.	At present, what component of the universe dominates the energy density?	At present, the dark energy component of the universe dominates the energy density. 1. $\rho_{\Lambda,0} \approx 6.66 \times 10^{-27}$ 2. $\rho_{m,0} \approx 2.46 \times 10^{-27}$ 3. $\rho_{\gamma,0} \approx 7.66 \times 10^{-31}$ Thus, $\rho_{\Lambda,0} > \rho_{m,0} \gg \rho_{\gamma,0}$.

4.	When did (red-shift and time) the universe become dark energy dominated?	As determined using method-1, the universe became dark energy dominated (from being matter dominated) at a red-shift of 0.39 and time (since the Big Bang) of 9.64 GYr. As determined using method-2, the universe became dark energy dominated at a red-shift of 0.55 and time (since the Big Bang) of 8.47 GYr.

4 Conclusion

Thus, we conclude from the above analysis that the universe was initially radiation dominated and transited to matter dominated phase at a red-shift ≈ 3213.29 ($t_{\gamma,m} \approx 57856.65$ Yrs) and later to dark energy dominated phase (the present phase) at a red-shift ≈ 0.39 ($t_{m,\Lambda} \approx 9.46$ Gyrs). These values which calculated using broken powerlaw approximation to the density functions, we get corresponding values to be ≈ 2738.03 ($t_{\gamma,m} \approx 76384.2$ Yrs) and ≈ 0.55 ($t_{m,\Lambda} \approx 8.47$ Gyrs), respectively.

5 Program

5.1 code.py

```
##############################
#### Importing Libraries ####
##############################

import numpy as np
import matplotlib.pyplot as plt
from astropy import constants as const
from scipy import integrate
from astropy import units as u
plt.style.use('dark_background')

##############################
#### Function Definition ####
##############################

E = lambda z: omega0["rad"]*(1+z)**4 + omega0["m"]*(1+z)**3 + omega0["lambda"
    ]

def rk4(x,y0,f,n=100,*args): # n = number of steps between x0 and x
    y = np.zeros(len(x))
    y[0] = y0
    for j in range(len(y)-1):
        h = (x[j+1]-x[j])
        for i in range(n):
            f0 = f( x[j],y[j],*args)
            f1 = f( x[j]+(0.5*h), y[j] + (0.5*h*f0) ,*args)
            f2 = f( x[j]+(0.5*h), y[j] + (0.5*h*f1) ,*args)
            f3 = f( x[j]+h, y[j] + h*f2 ,*args)
            y[j+1] = y[j] + (h/6.0)*(f0 + 2*f1 + 2*f2 + f3)
    return y[::-1]

```

```python
def z_integrand(z):
    Z = -1.0/(H0*(1+z)*np.sqrt(E(z)))
    return Z

def modified_z_integrand(z):
    Z = -s_to_Gyr/(H0*(1+z)*np.sqrt(E(z)))
    return Z

dzdt = lambda t,z: (-1.0)/z_integrand(z)

#############################
######## Calculation ########
#############################

# given:
pc_to_m = const.pc.value
H0 = 69.7 # km/s/Mpc <- assumed value - not given
print("Assumed value of present Hubble constant = {} km/s/Mpc".format(H0))
H0 = H0*(1e3/(1e6*pc_to_m))
omega0 = {"rad":  8.4e-5, "m": 0.27, "lambda": 0.73}
# assuming a0 = 1
G = const.G.value
rho_c0 = (3*(H0**2))/(8*np.pi*G)
print("Present value of critical density = {} km/m^3".format(rho_c0))
rho0 = dict(zip(list(omega0.keys()),[x*rho_c0 for x in list(omega0.values())
     ]))

s_to_yr = (1*u.second).to(u.yr).value
s_to_Gyr = s_to_yr/1e9
Gyr_to_s = 1.0/s_to_Gyr
t0,dt0 = integrate.quad(z_integrand, 0, np.inf)
t0 *= -(s_to_yr/1e9)
dt0 *= -(s_to_yr/1e9)
print("Present age of universe = {} Gyrs".format(t0))

# finding the boundary in terms of z for matter and radiation dominated
     universes
print("Computing time and redshift at the transition from radiation dominated
      to matter dominated and matter dominated to dark energy dominated
     universe.\n")
print("Computing transition time by first determining the transistion
     redshift and then determining the transition time using integral
     relation between redshift and time.\n")
z_c = (rho0["m"]/rho0["rad"])-1.0
z_d = np.cbrt(rho0["lambda"]/rho0["m"])-1.0
print("Transition redshift (radiation to matter) = {}".format(np.round(z_c,2)
     ))
print("Transition redshift (matter to dark energy) = {}".format(np.round(z_d
     ,2)))
z_arr = np.logspace(-2,5,1000)
z_arr = np.append([0,z_c,z_d],z_arr)
z_arr = np.sort(z_arr)
t_arr = np.array([integrate.quad(modified_z_integrand,0,z)[0] for z in z_arr
     ]) + t0
# determining the time in the history of universe when z = z_c -- by
     integration
t_c,dt_c = integrate.quad(modified_z_integrand,0,z_c)
t_c = t0 + t_c
dt_c = dt0 + dt_c
t_d,dt_d = integrate.quad(modified_z_integrand,0,z_d)
t_d = t0 + t_d
dt_d = dt0 + dt_d
print("Transition time (radiation to matter) = {} Yrs".format(np.round(t_c*1
     e9,2)))
print("Transition time (matter to dark energy) = {} GYrs\n".format(np.round(
     t_d,2)))

```

```python
# finding the boundary in terms of t for matter and radiation dominated universes
print("Computing transition redshift by first determining the transistion time by continuity condition on simplified density function and then determining the transition redshift using time-stepping methods.\n")
t_c_new = t0*(rho0["rad"]/rho0["m"])**1.5
idx = np.where(t_arr > t_c_new)
z_c_new = rk4([t_c_new,t_arr[idx][-1]],z_arr[idx][-1],dzdt,1000)[1]
t_d_new = t0*np.sqrt(rho0["m"]/rho0["lambda"])
idx = np.where(t_arr > t_d_new)
z_d_new = rk4([t_d_new,t_arr[idx][-1]],z_arr[idx][-1],dzdt,1000)[1]
print("Transition redshift (radiation to matter) = {}".format(np.round(z_c_new,2)))
print("Transition redshift (matter to dark energy) = {}".format(np.round(z_d_new,2)))
print("Transition time (radiation to matter) = {} Yrs".format(np.round(t_c_new*1e9,2)))
print("Transition time (matter to dark energy) = {} GYrs\n".format(np.round(t_d_new,2)))

z_str = []
for i in z_arr:
    if(i < 1.0):
        z_str.append("{}".format(np.round(i,2)))
    else:
        z_str.append("{:.2e}".format(i))

f = {"rad": lambda z: rho0["rad"]*(1+z)**4,
"m": lambda z: rho0["m"]*(1+z)**(3.0),
"lambda": lambda z: rho0["lambda"],
}

rho = {}
for i in rho0.keys():
    rho[i] = [f[i](z) for z in z_arr]

f_new = {"rad": lambda t: np.where(t>t_c_new,rho0["rad"]*(t0/t)**(8.0/3.0),(rho0["rad"]*(t0/t_c_new)**(2.0/3.0))*(t0/t)**2),
"m": lambda t: np.where(t>t_c_new,rho0["m"]*(t0/t)**2,(rho0["m"]*np.sqrt(t0/t_c_new))*(t0/t)**(3.0/2.0)),
"lambda": lambda t: rho0["lambda"],
}

rho_new = {}
for i in rho0.keys():
    rho_new[i] = [f_new[i](t) for t in t_arr]

############################
########## Plotting #########
############################

# plotting with the transition redshift determined by first determining the transistion redshift by equating the matter and radiation density and then determining the transition time using integral relation between redshift and time
fs = 15
clrs = ['orange','greenyellow','cyan']
label = ["Radiation Density","Matter Density","Dark Energy Density"]
fig = plt.figure(figsize=(12,8))
ax1 = fig.add_subplot(111)
for i,j in enumerate(rho.keys()):
    ax1.plot(np.log10(t_arr),np.log10(rho[j]),label=label[i],c=clrs[i],lw=3)
ax1.axvline(x=np.log10(t_c), label=r'Tranisition time, $t_{\gamma,m}$' + ' = {:.2e} Gyrs'.format(t_c),c='yellow',ls='--')
ax1.axvline(x=np.log10(t_d), label=r'Tranisition time, $t_{m,\Lambda}$' + ' = {:.2e} Gyrs'.format(t_d),c='white',ls='--')
ax1.axvline(x=np.log10(t_arr[0]), label=r'Present time, $t_0$ = {} Gyrs'.format(np.round(t_arr[0],2)),c='pink',ls='--')
```

```
plt.legend(loc="best",fontsize=fs)
ax1.set_xlabel("Time since Big Bang, t (in Gyr)",fontsize=fs)
ax1.set_ylabel(r"Density $\rho$ (in $kg/m^{-3}$)",fontsize=fs)
ax1.set_title("Evolution of density with cosmic time",fontsize=fs+5)
ax1.tick_params(axis='both', which='major', labelsize=fs)
new_tick_labels = [r"$10^{%s}$" % str(int(lbl)) for lbl in ax1.get_xticks()]
ax1.set_xticklabels(new_tick_labels)
new_tick_labels = [r"$10^{%s}$" % str(int(lbl)) for lbl in ax1.get_yticks()]
ax1.set_yticklabels(new_tick_labels)
ax2 = ax1.twiny()
ax1.grid(True)
new_tick_locations = np.array([t_arr[0],t_c,t_arr[-1]])
new_tick_locations = np.log10(new_tick_locations)
ax2.set_xticks(new_tick_locations)
new_xtick_labels = [z_str[0],"{:.2e}".format(z_c),z_str[-1]]
ax2.set_xlim(ax1.get_xlim())
ax2.set_xticklabels(new_xtick_labels)
ax2.tick_params(axis='both', which='major', labelsize=fs)
ax2.tick_params(axis='both', which='minor', labelsize=fs)
ax2.set_xlabel("Redshift, z",fontsize=fs)
plt.savefig("integral.png",bbox_inches="tight")
plt.close()

# plotting with the transition redshift determined by first determining the
    transistion time by continuity condition on simplified density function
    and then determining the transition redshift using time-stepping methods
fs = 15
clrs = ['orange','greenyellow','cyan']
label = ["Radiation Density","Matter Density","Dark Energy Density"]
fig = plt.figure(figsize=(12,8))
ax1 = fig.add_subplot(111)
for i,j in enumerate(rho.keys()):
    ax1.plot(np.log10(t_arr),np.log10(rho_new[j]),label=label[i],c=clrs[i],lw
    =3)
ax1.axvline(x=np.log10(t_c_new), label=r'Tranisition time, $t_{\gamma,m}$' +
    ' = {:.2e} Gyrs'.format(t_c_new),c='yellow',ls='--')
ax1.axvline(x=np.log10(t_d_new), label=r'Tranisition time, $t_{m,\Lambda}$' +
     ' = {:.2e} Gyrs'.format(t_d_new),c='white',ls='--')
ax1.axvline(x=np.log10(t_arr[0]), label=r'Present time, $t_0$ = {} Gyrs'.
    format(np.round(t_arr[0],2)),c='pink',ls='--')
plt.legend(loc="best",fontsize=fs)
ax1.set_xlabel("Time since Big Bang, t (in Gyr)",fontsize=fs)
ax1.set_ylabel(r"Density $\rho$ (in $kg/m^{-3}$)",fontsize=fs)
ax1.set_title("Evolution of density with cosmic time",fontsize=fs+5)
ax1.tick_params(axis='both', which='major', labelsize=fs)
new_tick_labels = [r"$10^{%s}$" % str(int(lbl)) for lbl in ax1.get_xticks()]
ax1.set_xticklabels(new_tick_labels)
new_tick_labels = [r"$10^{%s}$" % str(int(lbl)) for lbl in ax1.get_yticks()]
ax1.set_yticklabels(new_tick_labels)
ax2 = ax1.twiny()
ax1.grid(True)
new_tick_locations = np.array([t_arr[0],t_c_new,t_arr[-1]])
new_tick_locations = np.log10(new_tick_locations)
ax2.set_xticks(new_tick_locations)
new_xtick_labels = [z_str[0],"{:.2e}".format(z_c_new),z_str[-1]]
ax2.set_xlim(ax1.get_xlim())
ax2.set_xticklabels(new_xtick_labels)
ax2.tick_params(axis='both', which='major', labelsize=fs)
ax2.tick_params(axis='both', which='minor', labelsize=fs)
ax2.set_xlabel("Redshift, z",fontsize=fs)
plt.savefig("simplified.png",bbox_inches="tight")
plt.close()

############################
######## End Of Code ########
############################
```

References

[1] *Andrew R. Liddle, An Introduction to Modern Cosmology*