Cosmology Programming Assignment 3
Angular Diameter and Luminosity Distances with Redshift

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1 Introduction

The concept of distance measurement in Cosmology is quite abstract, given that the universe is constantly expanding. The primary distance that one defines is a **co-moving distance**, which is invariant under change in the scale factor of the universe (expansion of the universe). Next, is to define a **proper distance**, in which the scale used for distance measurement is fixed, and hence, the distance measurement varies with the epoch of observation in an non-static (expanding or contracting) universe. The other two commonly used distances are - **Luminosity distance** and **Angular diameter distance**

1.1 Luminosity Distance

The luminosity distance is a way of expressing the amount of light received from a distant object. Let us suppose we observe an object with a certain flux. The luminosity distance is the distance that the object appears to have, assuming the inverse square law for the reduction of light intensity with distance holds. The luminosity distance is not the physical distance to the object, because in the real Universe the inverse square law does not hold. It is broken both because the geometry of the Universe need not be flat, and the universe is expanding. For generality, while in the following discussion it is presumed the object is observed at the present epoch.

We begin with definitions as follows. The luminosity L of an object is defined as the energy emitted per unit solid angle per second; since the total solid angle is 4π steradians, this equals the total power output divided by 4π. The radiation flux density S received by us is defined as the energy received per unit area per second. Then

$$d_{lum}^2 = \frac{L}{S} \tag{1}$$

because L/ S is the unit area per unit solid angle.

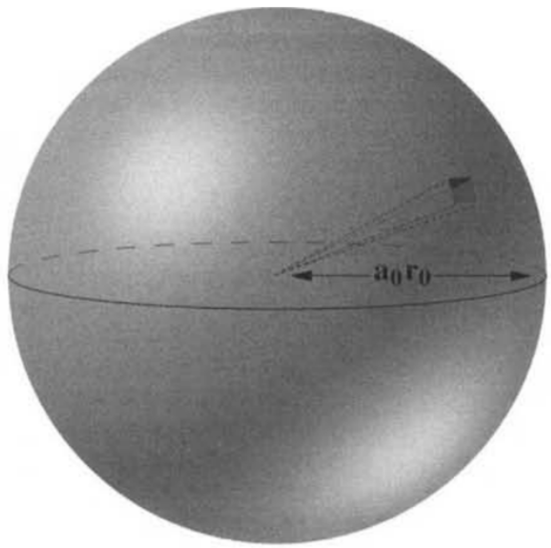

Figure 1: We receive light a distance $a_o r_o$ from the source. The surface area of the sphere at that distance is $4\pi a_o^2 r_o^2$, and so our detector of unit area intercepts a fraction $1/4\pi a_o^2 r_o^2$ of the total light output 4π L.

This is best visualized by placing the radiating object at the centre of a sphere, co-moving radius r_o, with us holding our detector at the surface of the sphere, as shown in Figure 1. The physical radius of the sphere is $a_o r_o$, and so its total surface area is $4\pi a_o^2 r_o^2$. In this representation, the effect of the geometry is in the determination of r_o; it doesn't appear explicitly in the area.

If we were in a static space the radiation flux received would simply be $S = \frac{L}{a_o^2 r_o^2}$, but we have to allow for the expansion of the Universe and how that affects the photons as they propagate from the source to the observer. There arc actually two effects:

- The individual photons lose energy $\propto$ (1 + z), so have less energy when they arrive.
- The photons arrive less frequently $\propto (1 + z)$.

Combining the two, the received flux is

$$S = \frac{L}{a_o^2 r_o^2 (1+z)^2} \tag{2}$$

and hence, the luminosity distance is given by

$$d_{lum} = a_o r_o (1+z) \tag{3}$$

Distant objects appear to be further away than they really are because of the effect of redshift reducing their apparent luminosity. For example, consider a flat spatial geometry k = 0. Then for a radial ray ds = a(t)dr and so the physical distance to a source is given by integrating this at fixed time

$$d_{phy} = a_o r_o \tag{4}$$

For nearby objects z « 1 and so $d_{1um} \simeq d_{phys}$. i.e. the objects really are just as far away as they look. But more distant objects appear further away ($d_{1um} > d_{phys}$) than they really are.

1.2 Angular Distance Diameter

1be angular diameter diameter is a measure or how large objects appear to be. As with the luminosity distance. it is defined as the distance that an object of known physical extent appears to be at, under the assumption of Euclidean geometry. If we lake the object to lie perpendicular t0 the line of sight and to have physical extent, "l". the angular diameter is therefore,

$$d_{diam} = \frac{1}{sin\theta} \simeq \frac{1}{\theta} \tag{5}$$

where the small-angle approximation used in the final expression is valid in almost any astronomical context.

To find an expression for this, it is most convenient this time to place ourselves at the origin, and the object at radial coordinate r_o. We need to use the metric at the time the light was emitted, t_e, and we align our 'rod' in the θ direction of the metric. The physical size "l" is measured using ds, now entirely in the θ direction, as

$$l = ds = r_o a(t_e) d\theta \tag{6}$$

The light rays from each end of the rod propagate radially towards us, and so this angular extent is preserved even if the Universe is expanding. The angular size we perceive is

$$d\theta = \frac{l}{r_o a(t_e)} = \frac{l(1+z)}{a_o r_o} \tag{7}$$

where the redshift term accounts for the evolution of the scale factor between emission and the present. Accordingly

$$d_{diam} = \frac{a_o r_o}{(1+z)} = \frac{d_{lum}}{(1+z)^2} \tag{8}$$

The angular diameter and luminosity distances therefore have similar forms, but have a different dependence on redshift.

2 Results

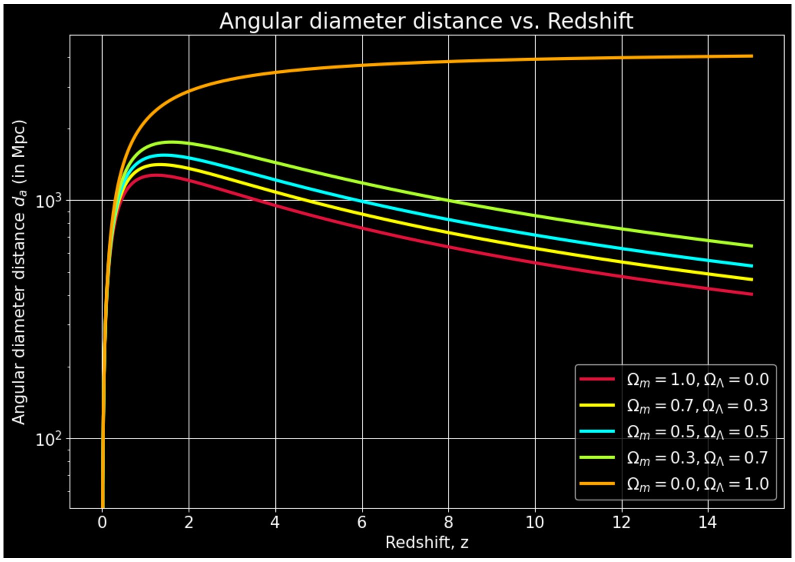

Figure 2: Semilogy plot of angular diameter distance as a function of redshift

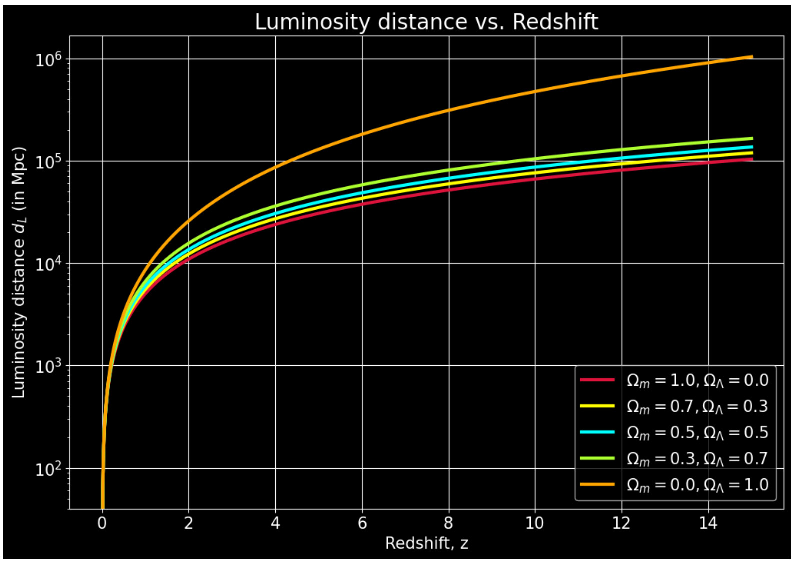

Figure 3: Semi-log plot of luminosity distance as a function of redshift

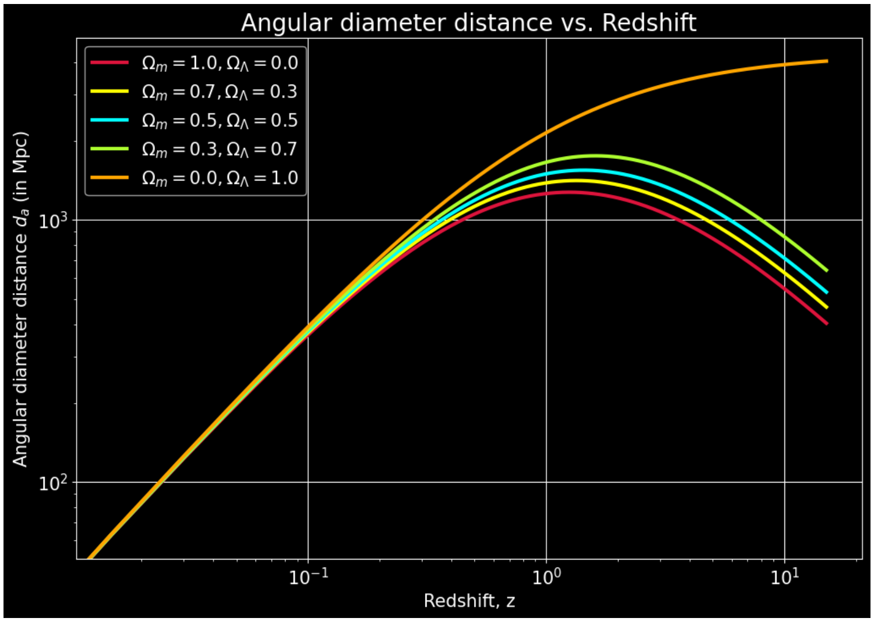

Figure 4: Log-Log plot of angular diameter distance as a function of redshift

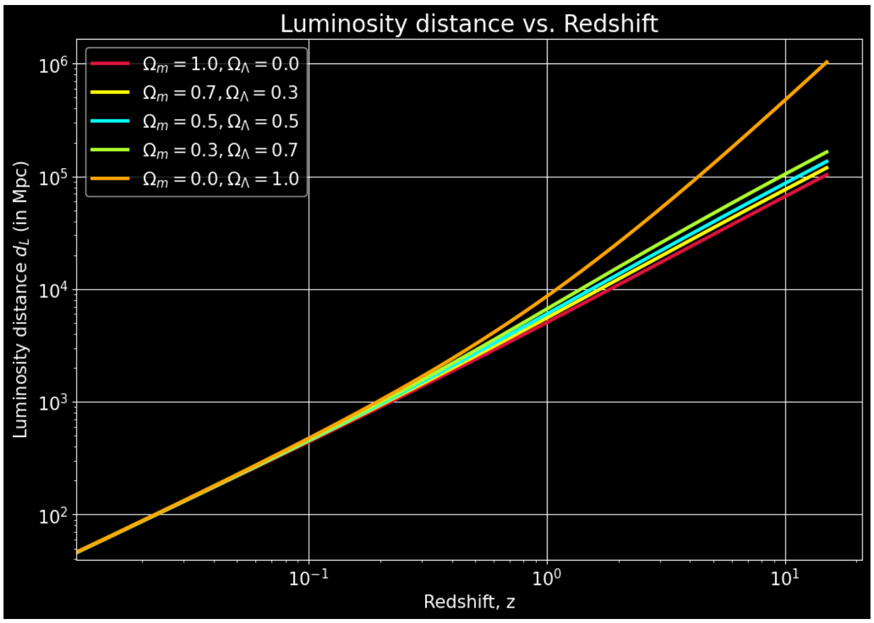

Figure 5: Log-Log plot of luminosity distance as a function of redshift

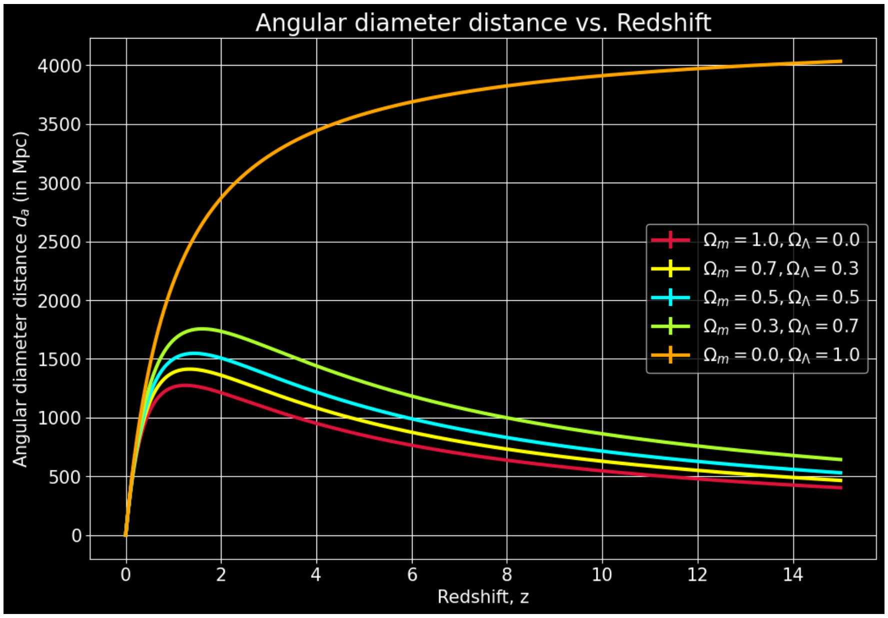

Figure 6: Plot of angular diameter distance (with error-bars) as a function of redshift

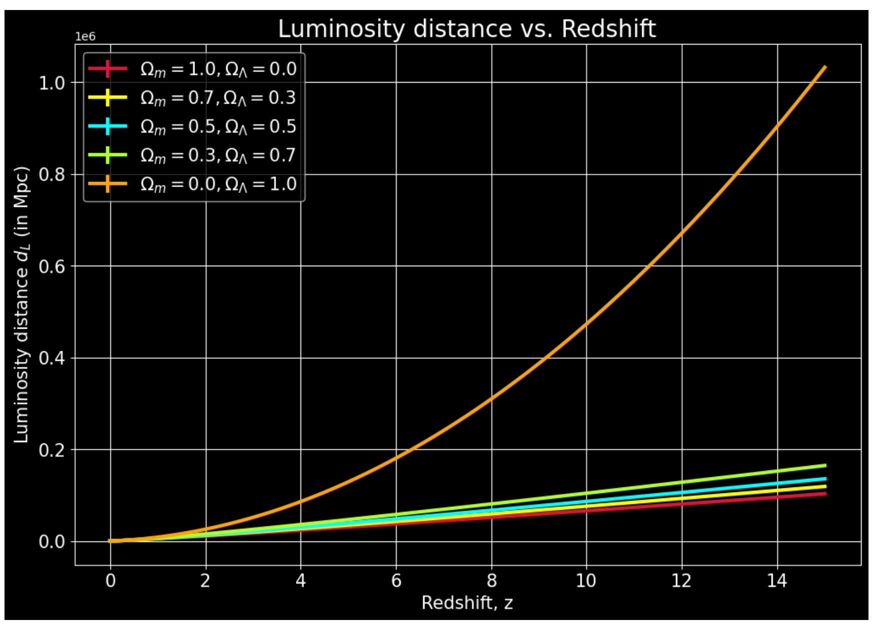

Figure 7: Plot of luminosity distance (with error-bars) as a function of redshift

3 Observations and Inferences

Sl. No.	Observations	Inferences
1.	We observe from the plot of angular angular diameter distance as a function of redshift, that the curve attains a maximum (for each universe under consideration) and then declines. That is, the inverse mapping between angular diameter distance and redshift is a one-to-many mapping.	Even for distant objects $a_o r_o$ remains finite, but the light becomes infinitely redshifted. Hence $d_{diam} \to 0$ as $z \to \infty$, meaning that distant objects appear to be nearby! Once objects are far enough away, moving them further actually makes their angular extent larger (though they do get fainter as according to the luminosity distance). This is because the diameter distance refers to objects of fixed physical size l, so the earlier we are considering, the larger a co-moving size they have.
2.	We observe that for a universe completely composed of dark energy only, the angular diameter distance is a monotonically increasing function of redshift and tends to a value $= \frac{c}{H_0}$.	This can be thought of as the limiting distance one can observe in a universe completely composed of dark energy. The limiting speed of any object in the universe is the speed of light in vacuum (c) and the characteristic age of the universe is the Hubble time ($t_H = \frac{1}{H_0}$), hence, the limiting distance is the distance travelled by the photons in Hubble time, which is $ct_H = \frac{c}{H_0}$.
3.	We observe that for all universes considered, the luminosity distance is a monotonically increasing function of redshift.	Distant objects appear to be further away than they really are, because of the effect of redshift reducing their apparent luminosity.

4 Conclusion

Thus, in the present analysis, we have explored the variation of the two important distance estimates in Cosmology - angular diameter distance and luminosity distance. Based on the plots generated, various observations are made and inferences drawn. Some of the key findings are that for a universe with non-vanishing matter density parameter, one can obtain same angular diameter distance estimate for two different values of redshifts (except for the case of the universe completely composed of dark energy). Whereas, the luminosity distance is found to be monotonically increasing function of the redshift.

5 Program

```
#############################
#### Importing Libraries ####
#############################

import numpy as np
import matplotlib.pyplot as plt
from astropy import constants as const
from scipy import integrate
plt.style.use('dark_background')

#############################
#### Function Definition ####
#############################

E = lambda z,omega_m,omega_lam: omega_m*(1+z)**3 + omega_lam

def Integrand(z,omega_m,omega_lam):
    return 1.0/(H0*np.sqrt(E(z,omega_m,omega_lam)))

def get_Da(z,omega_m,omega_lam,unit="Mpc"):
    integrand = lambda z: Integrand(z,omega_m=omega_m,omega_lam=omega_lam)
    if(unit == "m"):
        d,dd = integrate.quad(integrand,0,z)
        d *= (c/(1+z))
        dd *= (c/(1+z))
        return [d,dd]
    if(unit == "Mpc"):
        d,dd = integrate.quad(integrand,0,z)
        d *= ((c*m_to_Mpc)/(1+z))
        dd *= ((c*m_to_Mpc)/(1+z))
        return [d,dd]

def plot(x,y,dy=None,typ="semilogy",title="da"):
    fs = 15
    clrs = ['crimson','yellow','cyan','greenyellow','orange']
    fig = plt.figure(figsize=(12,8))
    ax = fig.add_subplot(111)
    if(typ=="semilogy"):
        for ctr,(i,j) in enumerate(zip(omega_m,omega_lambda)):
            ax.semilogy(x,y[ctr],label=r"$\Omega_m = {}, \Omega_\Lambda = {}$".format(i,j),c=clrs[ctr],lw=3)
    if(typ=="loglog"):
        for ctr,(i,j) in enumerate(zip(omega_m,omega_lambda)):
            ax.loglog(x,y[ctr],label=r"$\Omega_m = {}, \Omega_\Lambda = {}$".format(i,j),c=clrs[ctr],lw=3)
    if(typ=="errorbar"):
        for ctr,(i,j) in enumerate(zip(omega_m,omega_lambda)):
            ax.errorbar(x,y[ctr],dy[ctr],label=r"$\Omega_m = {}, \Omega_\Lambda = {}$".format(i,j),c=clrs[ctr],lw=3)

    plt.legend(loc="best",fontsize=fs)
    ax.set_xlabel("Redshift, z",fontsize=fs)
    if(title=="da"):
        ax.set_ylabel(r"Angular diameter distance $d_a$ (in Mpc)",fontsize=fs)
        ax.set_title("Angular diameter distance vs. Redshift",fontsize=fs+5)
    if(title=="dl"):
        ax.set_ylabel(r"Luminosity distance $d_L$ (in Mpc)",fontsize=fs)
        ax.set_title("Luminosity distance vs. Redshift",fontsize=fs+5)
    ax.tick_params(axis='both', which='major', labelsize=fs)
    ax.grid(True)
    plt.savefig("{}_{}.png".format(title,typ))
    plt.close()
```

```
##############################
######## Calculation ########
##############################

# given:
c = const.c.value
m_to_pc = 1.0/const.pc.value
m_to_Mpc = (1e-6)/const.pc.value
pc_to_m = const.pc.value
H0 = 69.7 # km/s/Mpc
H0 = H0*(1e3/(1e6*pc_to_m))
omega_m = np.array([1.0,0.7,0.5,0.3,0.0])
omega_lambda = np.copy(omega_m[::-1])

# angular diameter distance
z_arr = np.linspace(0,15,1000)
Da_arr = np.zeros((5,len(z_arr)))
dDa_arr = np.zeros((5,len(z_arr)))
for ctr,(i,j) in enumerate(zip(omega_m,omega_lambda)):
    Da = np.array([get_Da(z,i,j) for z in z_arr])
    Da_arr[ctr] = Da.T[0]
    dDa_arr[ctr] = Da.T[1]

# luminosty distance
Dl_arr = np.zeros((5,len(z_arr)))
dDl_arr = np.zeros((5,len(z_arr)))
for ctr,(i,j) in enumerate(zip(omega_m,omega_lambda)):
    Dl_arr[ctr] = ((1+z_arr)**2)*np.copy(Da_arr[ctr])
    dDl_arr[ctr] = ((1+z_arr)**2)*np.copy(dDa_arr[ctr])

##############################
########## Plotting #########
##############################

plot(x=z_arr,y=Da_arr,typ="semilogy",title="da")
plot(x=z_arr,y=Da_arr,typ="loglog",title="da")
plot(x=z_arr,y=Da_arr,dy=dDa_arr,typ="errorbar",title="da")
plot(x=z_arr,y=Dl_arr,typ="semilogy",title="dl")
plot(x=z_arr,y=Dl_arr,typ="loglog",title="dl")
plot(x=z_arr,y=Dl_arr,dy=dDl_arr,typ="errorbar",title="dl")

##############################
######## End Of Code ########
##############################
```

References

[1] *Andrew R. Liddle, An Introduction to Modern Cosmology*