DS:
Shortest Path

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What is Single Source Shortest Path?

- Single source shortest path is about finding a path between a given vertex (called Source) to all other vertices in a graph such that, the total distance between them (source & destination) is minimum.
- Examples
  - Let’s say we have office in 5 different citites and we need to travel from Head office to all other offices.
  - Flight charges between cities are known, what is the cheepest way to reach each office from HQ?

Algorithms: single source shortest path

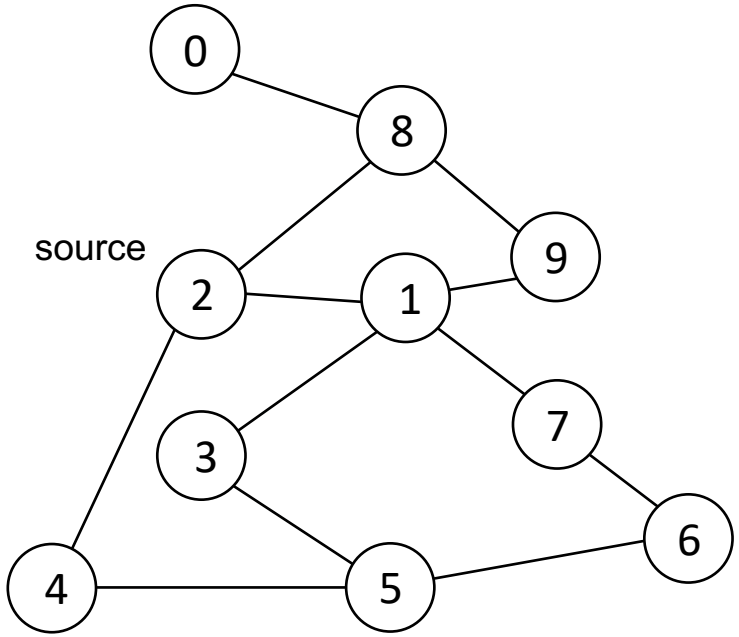
- Breadth first search
- Depth first search
- Dijkstra
- Bellman Ford

What is BFS

- BFS is an algorithm for traversing Graph data structures. It starts at some arbitrary node of a graph and explores the neightboring nodes (which are at current level) first, before moving to the next level neighbors.
- BFS is can be twisted to find Single Source Shortest Path. We need a extra variable called Parent to keep track of path.

Algorithm – Single Source Shortest Path using BFS

```
BFS-for-SSSP (any starting vertex)
        intialize a Queue
        Create a Parent reference in each node
        Enqueue (Source Vertex)
        do until queue is not empty
                currentVertex = Dequeue(Vertex)
                for each adjacent vertices
                        if adjacent vertex is visited
                                don't do anything
                        else
                                enqueue adjacent vertex
                                update their parent as currentVertex
                                mark currentVertex as visited
```



QUEUE

Algorithm – Single Source Shortest Path using BFS

```
BFS-for-SSSP (any starting vertex)
        intialize a Queue                                          O(1)
        Create a Parent reference in each node                     O(1)
        Enqueue (Source Vertex)                                    O(1)
        do until queue is not empty ———————————————————————————— O(V)
                currentVertex = Dequeue(Vertex)                    O(1)
                for each adjacent vertices ——————————————————— O(Adj vertex)   O(Adj vertex)   O(E)
                        if adjacent vertex is visited              O(1)
                                don't do anything                  O(1)
                        else                                       O(1)
                                enqueue adjacent vertex            O(1)
                                update their parent as currentVertex
                                mark currentVertex as visited
```

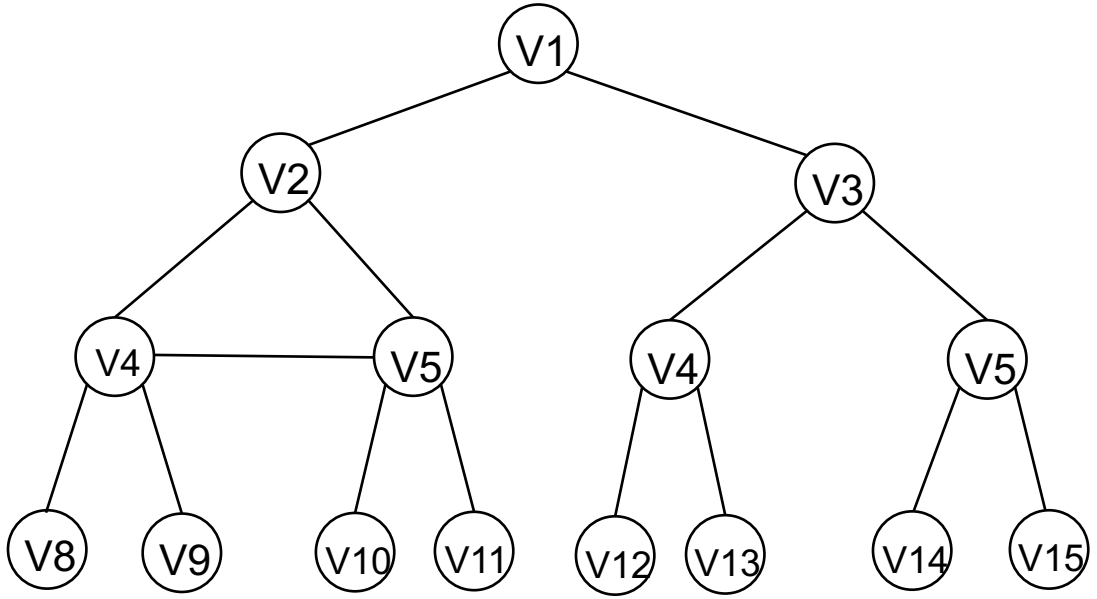
Time complexity: O(E)

Single Source Shortest Path Algorithms

Type of Graphs	BFS
unWeighted-unDirected	V
unWeighted-Directed	V
Positive-Weighted-unDirected	X
Positive-Weighted-Directed	X
Negative-Weighted-unDirected	X
Negative-Weighted-Directed	X

Why Weighted Graph does not work with BFS

- BFS explores a given graph only in breath-way. But there can always be a better route, which is not breadth-way.

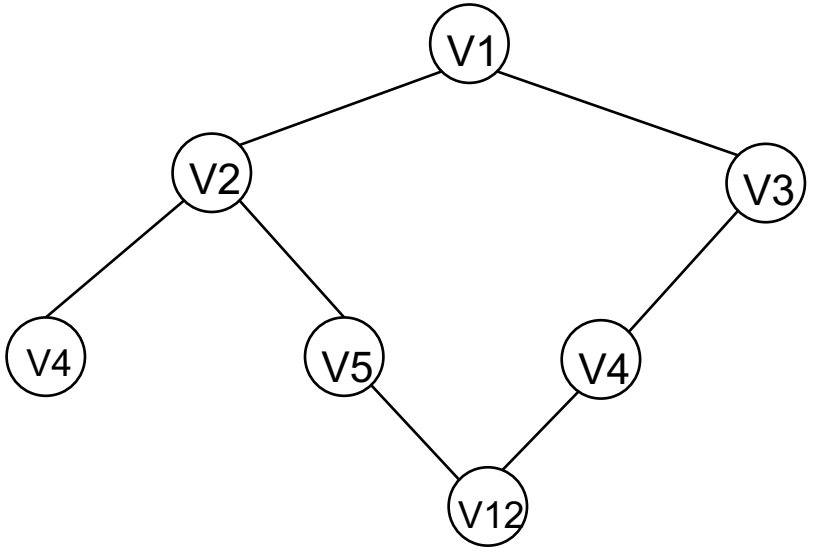


Algorithms: single source shortest path

- Breadth first search
- Depth first search
- Dijkstra
- Bellman Ford

Why DFS does not work for SSSP

- DFS has the tendency to go as far as possible from source. Hence it can never find Shortest Path



Algorithms: single source shortest path

- Breadth first search
- Depth first search
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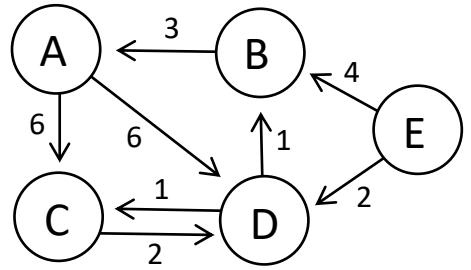
Algorhms - Dijkstra



Dijkstra()

set the distance of all the vertices as infinite and source vertex as 0

Save all the vertices in min-Heap

do until min-Heap is not empty

currentVertex = extract from min-Heap

for each neighbor of currentVertex

if currentVertex's distance + currentEdge < neighbor's distance

update neighbor's distance and parent

Algorhms - Dijkstra

```
Dijkstra()
set the distance of all the vertices as infinite and source vertex as 0      O(V)
save all the vertices in min-Heap                                            O(V)
do until min-Heap is not empty                                               O(V)
        currentVertex = extract from min-Heap                                O(log V)
        for each neighbor of currentVertex                                   O(V)
                if currentVertex's distance + currentEdge < neighbor's distance   O(1)
                        update neighbor's distance and parent                O(1)
```

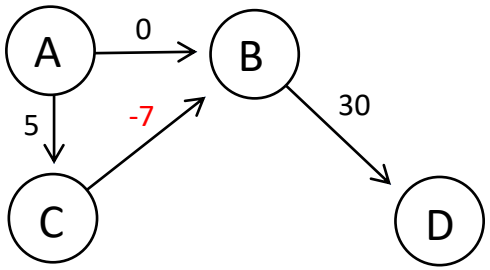
$O(V^2)$

Time complexity: $O(V^2)$

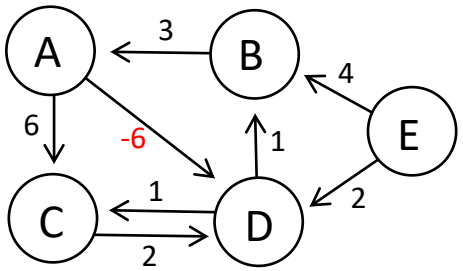
Space complexity: $O(V)$

Single Source Shortest Path Algorithms

Type of Graphs	BFS	Dijkstra
unWeighted-unDirected	V	V
unWeighted-Directed	V	V
Positive-Weighted-unDirected	X	V
Positive-Weighted-Directed	X	V
Negative-Weighted-unDirected	X	X
Negative-Weighted-Directed	X	X
Negative Cycles	X	X



Negative-Weighted



Negative Cycles

Algorithms: single source shortest path

- Breadth first search
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- Dijkstra
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Bellman Ford's Algorithm

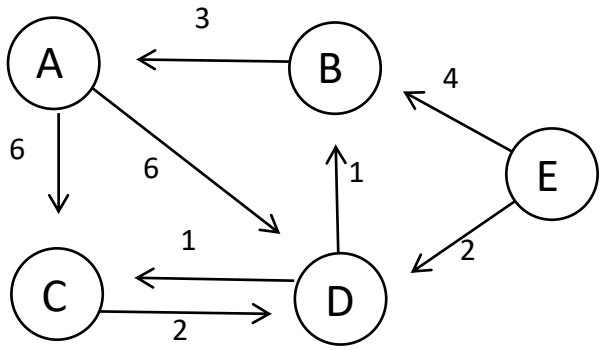
- The Bellman-Ford algorithm is used to find SSSP.
- If a graph contains a "negative cycle" that is reachable from the source, then there is no cheapest path: any path that has a point on the negative cycle can be made cheaper by one more walk around the negative cycle.
- In such a case, the Bellman-Ford algorithm can detect "negative cycles" and report their existence.

Given

Edge	Weight
A>C	6
A>D	6
B>A	3
C>D	2
D>C	1
D>B	1
E>B	4
E>D	2

if (distance of source vertex + current weight between source and destination vertex) < (current distance of destination vertex)
update distance of destination vertex with (the distance of source vertex + current weight between source and destination vertex)

Distance Matrix	
Vertex	Give Distance
A	∞
B	∞
C	∞
D	∞
E	0



Final answer

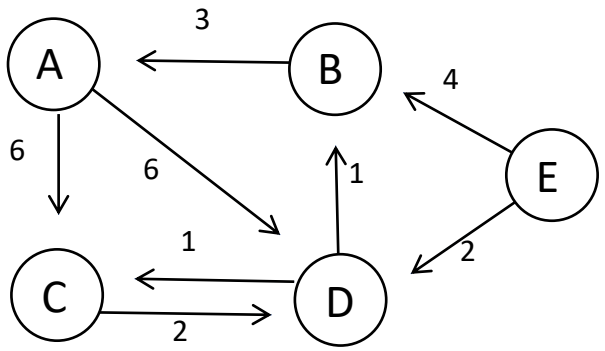
vertex	Distance from E	Path from E
A	6	B>D>E
B	3	D>E
C	3	D>E
D	2	E
E	0	0

Given

Edge	Weight
A>C	6
A>D	6
B>A	3
C>D	2
D>C	1
D>B	1
E>B	4
E>D	2

if (distance of source vertex + current weight between source and destination vertex) < (current distance of destination vertex)
update distance of destination vertex with the distance of source vertex + current weight between source and destination vertex

Distance Matrix			
Vertex	Give Distance	Iteration #1	
		Distanc	Parent
A	∞		
B	∞		
C	∞		
D	∞		
E	0		



Final answer

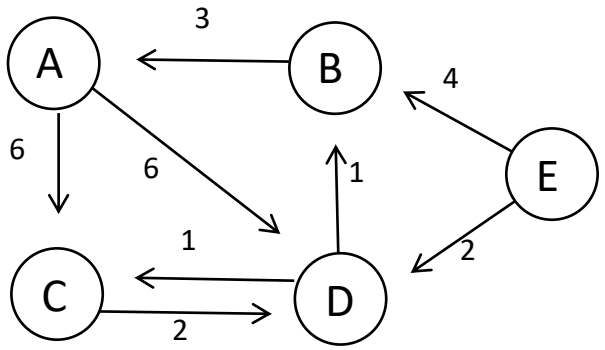
vertex	Distance from E	Path from E
A	6	B>D>E
B	3	D>E
C	3	D>E
D	2	E
E	0	0

Given

Edge	Weight
A>C	6
A>D	6
B>A	3
C>D	2
D>C	1
D>B	1
E>B	4
E>D	2

if (distance of source vertex + current weight between source and destination vertex) < (current distance of destination vertex)
update distance of destination vertex with the distance of source vertex + current weight between source and destination vertex

Distance Matrix					
Vertex	Give Distance	Iteration #1		Iteration #2	
		Distanc	Parent	Distanc	Parent
A	∞	∞	-		
B	∞	4	E		
C	∞	∞	-		
D	∞	2	E		
E	0	0	-		



Final answer

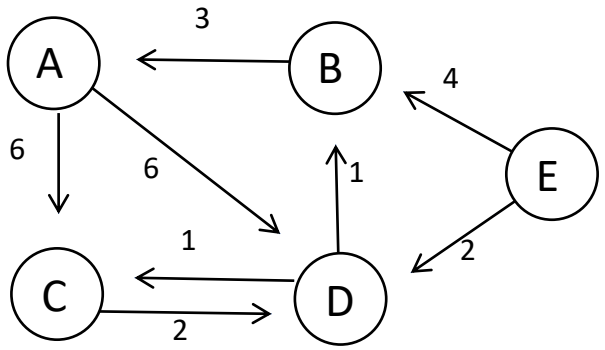
vertex	Distance from E	Path from E
A	6	B>D>E
B	3	D>E
C	3	D>E
D	2	E
E	0	0

Given

Edge	Weight
A>C	6
A>D	6
B>A	3
C>D	2
D>C	1
D>B	1
E>B	4
E>D	2

if (distance of source vertex + current weight between source and destination vertex) < (current distance of destination vertex)
update distance of destination vertex with the distance of source vertex + current weight between source and destination vertex

Distance Matrix					
Vertex	Give Distance	Iteration #1		Iteration #2	
		Distanc	Parent	Distanc	Parent
A	∞	∞	-	4+3=7	B
B	∞	4	E	2+1=3	D
C	∞	∞	-	2+1=3	D
D	∞	2	E	2	E
E	0	0	-	0	-



Final answer

vertex	Distance from E	Path from E
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B	3	D>E
C	3	D>E
D	2	E
E	0	0

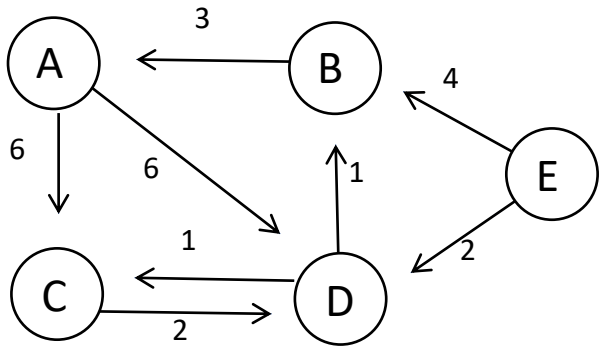
Given

Edge	Weight
A>C	6
A>D	6
B>A	3
C>D	2
D>C	1
D>B	1
E>B	4
E>D	2

if (distance of source vertex + current weight between source and destination vertex) < (current distance of destination vertex)
update distance of destination vertex with the distance of source vertex + current weight between source and destination vertex

Distance Matrix									
Vertex	Give Distance	Iteration #1		Iteration #2		Iteration #3		Iteration #4	
		Distanc	Parent	Distanc	Parent	Distanc	Parent	Distanc	Parent
A	∞	∞	-	4+3=7	B	3+3=6	B	6	B
B	∞	4	E	2+1=3	D	3	D	3	D
C	∞	∞	-	2+1=3	D	3	D	3	D
D	∞	2	E	2	E	2	E	2	E
E	0	0	-	0	-	0	-	0	-

go (v-1) runs



Final answer

vertex	Distance from E	Path from E
A	6	B>D>E
B	3	D>E
C	3	D>E
D	2	E
E	0	0

Bellman Ford Algorithm

```
BellmanFord(G)
    set all the vertex distance to infinite and source as 0
    for 1 to V-1
        for each edge(u,v)
            if d(v) > d(u) + w(u,v)        // if current weight of v is greater
                                           // than current weight of U + current edge
            d(v) = d(u) + w(u,v)
            update parent for 'v'

    for each edge (u,v)
        if d(v) != d(u) + w(u,v)
            the report existence of negative-weight cycle

    print all vertex with distance and parent
```

Bellman Ford Algorithm

```
BellmanFord(G)
    set all the vertex distance to infinite and source as 0          O(V)
    for 1 to V-1                                                     O(E)
        for each edge(u,v)                                           O(1)      O(EV)
            if d(v) > d(u) + w(u,v)
                d(v) = d(u) + w(u,v)                                 O(1)
                update parent for 'v'                                O(1)

    for each edge (u,v)                                              O(E)
        if d(v) != d(u) + w(u,v)                                     O(1)      O(E)
            the report existence of negative-weight cycle            O(1)

    print all vertex with distance and parent                        O(V)
```

O(EV)

Time complexity: O(VE)

Single Source Shortest Path Algorithms

Type of Graphs	BFS	Dijkstra	Bellman Ford
unWeighted-unDirected	V	V	V
unWeighted-Directed	V	V	V
Positive-Weighted-unDirected	X	V	V
Positive-Weighted-Directed	X	V	V
Negative-Weighted-unDirected	X	X	V
Negative-Weighted-Directed	X	X	V
Negative Cycles	X	X	V (can detect)

When to use which algorithms for SSSP

	BFS	Dijkstra	Bellman Ford
Time Complexity	$O(V^2)$	$O(V^2)$	O(V*E)
Implementation	Easy	Moderate	Moderate
Limitation	Does not work for weighted graphs	Does not work for nagetive-weighted graph / nagetive circles	N/A
unWeighted Graph	V	V	V
	Use this as Time Complexity is good and easy to implement	Don't use as implemenation is not as easy as BFS	Don't use as Time Complexity gets bad
Weighted Graph	X	V	V
	Not supported	Use this as Time Complexity is better than Bellman	Don't use as Time Complexity is not good
Negative Circles	X	X	V
	Not supported	Not supported	Use this as BFS and Dijkstra does not supports negative circles