1) No, it is not possible to make it inconsistent because it is known that the dimension of Nullspace of the Coffecient matrix A of the linear system is $\dim(Nul(A))$ is same as the number of free Variables, so $\dim(Nul(A)) = 2$. The non homogenous system has 6 linear equations with 8 unknowns, so the matrix is 6×8 matrix. And the $\dim(Col A)$ is $8 - 2 = 6$, $\dim(Col(A)) = 6$, and the system has a solution if and only if $b \in Col(A)$, for every b the system has soln, so it is not possible to make it inconsistent if we change some constants

2) standard basis in $P_3(t)$ is $\{1, t, t^2, t^3\}$

$P_0 = 1 \longrightarrow 1(1) + (0)t + (0)t^2 + (0)t^3 = 1\ 0\ 0\ 0$

$P_1 = 1 - t \longrightarrow 1(1) + (-1)t + (0)t^2 + (0)t^3 = 1\ -1\ 0\ 0$

$P_2 = 2 - 4t + t^2 \longrightarrow 1(2) + (-4)t + (1)t^2 + (0)t^3 = 2\ -4\ 1\ 0$

$P_3 = 6 - 18t + 9t^2 - t^3 \rightarrow 1(6) + (-18)t + 9(t^2) - 1(t^3 = 6\ -18\ 9\ -1$

writing above polynomials in matrix form

$$\begin{array}{cccc} P_0 & P_1 & P_2 & P_3 \end{array}$$
$$\begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

this matrix shows that the Rank is 4, and it has a pivot point in every Column which make it linearly Independent

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② b) $c = \{P_0, P_1, P_2, P_3\}$

$P(t) = 7 - 8t + 3t^2$

Let $P(t) = Q_0 P_0 + Q_1 P_1 + Q_2 P_2 + Q_3 P_3$

$\rightarrow 7 - 8t + 3t^2 = Q_0(1) + Q_1(1-t) + Q_2(2-4t+t^2) + Q_3(6-18t+9t^2 - t^3)$

$\rightarrow 7 - 8t + 3t^2 = (Q_0 + Q_1 + 2Q_2 + 6Q_3) + t(-Q_1 - 4Q_2 - 18Q_3) + t^2(Q_2 + 9Q_3 + t^3(-Q_3)$

$7 = Q_0 + Q_1 + 2Q_2 + 6Q_3 \quad (I)$

$-8 = -Q_1 - 4Q_2 - 18Q_3 \quad (II)$

$3 = Q_2 + 9Q_3 \quad (III)$

$0 = -Q_3 \quad (IV)$

from (IV), $Q_3 = 0$

from (III), $Q_2 = 3 - 9Q_3 = 3 - 0 = \underline{\underline{3}}$

from (II), $Q_1 = -4Q_2 - 18Q_3 + 8 = -4(3) - 18(0) + 8 = \underline{\underline{-4}}$

from (I): $Q_0 = 7 - Q_1 - 2Q_2 - 6Q_3 = 7 - (-4) - 2(3) - 6(0)$

$= \underline{\underline{5}}$

$\therefore 7 - 8 + 3t^2 = \underline{\underline{5P_0 - 4P_1 + 3P_2 + 0P_3}}$

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3) a) $C = \{P_0, P_1, P_2, P_3\}$ standard basis $\{1, t, t^2, t^3\}$

we have already found the matrix in 4(a)

$$P = \begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Now let $t^3 = x_0 P_0 + x_1 P_1 + x_2 P_2 + x_3 P_3$

$t^3 = (x_0 + x_1 + 2x_2 + 6x_3) + t(-x_1 - 4x_2 - 18x_3)$
$+ t^2(x_2 + 9x_3) + t^3(-x_3)$

$\rightarrow x_0 + x_1 + 2x_2 + 6x_3 = 0$ — (i)

$-x_1 - 4x_2 - 18x_3 = 0$ — (ii)

$x_2 + 9x_3 = 0$ — (iii)

$-x_3 = 1$ — (iv)

from (iv) $= x_3 = -1$

from (iii) $= x_2 = -9x_3 = -9(-1) = 9$

from (ii) $= x_1 = -4(x_2) - 18x_3 = -4(9) - 18(-1) = -18$

from (i) $= x_0 = -x_1 - 2x_2 - 6x_3 = 18 - 18 + 6 = 6$

b) $\therefore t^3 = 6P_0 - 18P_1 + 9P_2 - 1P_3$

4) Diagonalize the matrix $A = \begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix}$

$\Rightarrow PDP^{-1} = A$

$\Rightarrow$ finding eigenvalues,

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & -3 & 3 \\ 0 & -5-\lambda & 6 \\ 0 & -3 & 4-\lambda \end{bmatrix}\right)$$

$$\Rightarrow (1-\lambda)\left((-5-\lambda)(4-\lambda) - (-18)\right)$$

$$\Rightarrow (1-\lambda)\left((-5-\lambda)(4-\lambda) + 18\right)$$

$$\Rightarrow (1-\lambda)(-20 + 5\lambda - 4\lambda + \lambda^2 + 18)$$

$$\Rightarrow (1-\lambda)(-20 + \lambda + \lambda^2 + 18)$$

$$\Rightarrow (1-\lambda)(\lambda^2 + \lambda - 2)$$

$$\Rightarrow (1-\lambda)(\lambda - 1)(\lambda + 2)$$

$$\Rightarrow \lambda = 1, -2$$

When $\lambda = 1$ $\begin{bmatrix} (1-1) & -3 & 3 \\ 0 & (-5-1) & 6 \\ 0 & -3 & 4-(1) \end{bmatrix} = \begin{bmatrix} 0 & -3 & 3 \\ 0 & -6 & 6 \\ 0 & -3 & 3 \end{bmatrix} \sim$

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_1 = x_3$, $x_2 = 2x_3$, $x_3 = x_3$ $\quad = x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ $\quad - P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

$$D = P^{-1}AP = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} //$$

5) a) the image of $p(t) = 18 - 7t + 5t^2$, $T(p(t)) = (t+1)p(t-2)$

$\Rightarrow (t+1)(18 - 7(t-2) + 5(t-2)^2)$

$\Rightarrow (t+1)(18 - 7t + 14 + 5(t^2 - 4t + 4))$

$\Rightarrow (t+1)(18 - 7t + 14 + 5t^2 - 20t + 20)$

$\Rightarrow (t+1)(52 - 27t + 5t^2)$

$\Rightarrow 52t - 27t^2 + 5t^3 + 52 + 13t + 5t^2$

$\Rightarrow 5t^3 + 18t^2 + 65t + 52$

b) Let $P_1(t), P_2(t) \in P_2$ be arbitrary and c be a scalar in the field $\mathbb{R}$

Then $T(cP_1(t) + P_2(t)) = (t+1)(cP_1(t-2) + P_2(t-2))$

$= c(t+1)P_1(t-2) + (t+1)P_2(t-2)$

$cT(P_1(t)) + T(P_2(t)) = cT(P_1(t-2)) + T(P_2(t-2))$

This shows that T is a linear transformation

c) $T(1) = (t+1) \cdot 1 = t+1 = 0t^3 + 0 \cdot t^2 + 1 \cdot t + 1 \cdot 1$

$T(t) = (t+1) \cdot t = t^2 + t = 0 \cdot t^3 + 1 \cdot t^2 + 1 \cdot t + 0 \cdot 1$

$T(t^2) = (t+1)t^2 = t^3 + t = 1 \cdot t^3 + 0 \cdot t^2 + 1 \cdot t + 0 \cdot 1$

$T(t^3) = (t+1)t^3 = t^4 + t^3 = 1 \cdot t^4 + 1 \cdot t^3 + 0 \cdot t^2 + 0 \cdot t + 0 \cdot 0$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

d) $T(P(t)) = T(18-7t+5t^2)$, $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Coordinate Vector to basis $\{1, t, t^2\}$ for $p(t) = 18-7t+5t^2$ is $[18\ -7\ 5]^T = \begin{bmatrix} 18 \\ -7 \\ 5 \end{bmatrix}$ $(\because P(t) = 18\cdot 1 + (-7)t + 5t^2)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 18 \\ -7 \\ 5 \end{bmatrix} = \begin{bmatrix} 18 \\ 16 \\ -7 \\ 5 \end{bmatrix}$$

$T(P(t)) = 18\cdot 1 + 16\cdot t - 7(t^2) + 5(t)^3$

$\therefore T(P(t)) = T(18-7t+5t^2) = 18+16t-7t^2+5t^3$

6) a) $T(x^2e^x) = \frac{d}{dx}(x^2e^x) = x^2e^x = x^2e^x + 2xe^x$

$T(xe^x) = d/dx\,(xe^x) = xe^x + e^x$

$T(e^x) = d/dx\,(e^x) = e^x$

so, the matrix of $[T]_B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

b) $L(x^2e^x) = \frac{d^2}{dx^2}(x^2e^x) = x^2e^x + 2xe^x + 2xe^x + 2e^x = x^2e^x + 4xe^x + 2e^x$

$L(xe^x) = d^2/dx^2\,(xe^x) = xe^x + 2e^x$

so, the matrix of $[L]_B = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

c) Let $f \in V$ then $L(f) = \frac{d^2f}{dx} = \frac{d}{dx}\left(\frac{df}{dx}\right) = T(T(f)) = T^2(f)$

$$[T]_B^2 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 2+2+0 & 0+2+0 \\ 0+0+0 & 0+1+0 & 0+1+1 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = [L]_B$$

$[L]_B = [T]_B^2$

7) a) Change of Coordinates matrix from B to C

$\Rightarrow$ from trigonometry $\sin^2 x = \frac{1 - \cos^2 x}{2}$

$= \left(-\frac{1}{2}\right) \cos 2x + \left(\frac{1}{2}\right) 1$

So, Coordinates of $\sin^2 x$ are $-\frac{1}{2}, \frac{1}{2}$

$\Rightarrow$ from trigonometry $\cos^2 x = \frac{1 + \cos 2x}{2}$

$= \left(\frac{1}{2}\right) \cos 2x + \left(\frac{1}{2}\right) 1$

So, Coordinates of $\cos^2 x$ are $\frac{1}{2}, \frac{1}{2}$

$\Rightarrow$ The Change of Coordinates matrix from B to C is $\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

b) Change of Coordinates matrix from C to B

$\Rightarrow$ from trigonometry $\cos 2x = \cos^2 x - \sin^2 x$

$= (-1) \sin^2 x + (1) \cos^2 x$

So, Coordinates of $\sin^2 x$ are $-1, 1$

$\Rightarrow$ from trigonometry $1 = \cos^2 x + \sin^2 x$

$= (1) \sin^2 x + (1) \cos^2 x$

So, Coordinates of 1 are 1, 1

$\Rightarrow$ The change of Coordinates matrix from C to B is $\begin{bmatrix} -1, & 1 \\ 1, & 1 \end{bmatrix}$

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8) a) Given basis $B = \{1, x, x^2\}$

$T(p(x)) = p(2x-1)$

$T(1) = 1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2$

$T(x) = 2x-1 = (1\cdot -1) + 2\cdot x + 0\cdot x^2$

$T(x^2) = (2x-1)^2 = 4x^2 - 4x + 1 = 1\cdot 1 + (-4\cdot x) + (4\cdot x^2)$

So, the matrix representation is $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix}$

b) c.p of $[T]$ is $C_x[T] = (\lambda - 1)(\lambda - 2)(\lambda - 4)$

eigen Values are $\lambda = 1, 2, 4$

For $\lambda = 1$, $x = [x_1, x_2, x_3]^T$

$(T - \lambda I)x = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow$ $-x_2 + x_3 = 0$

$x_2 - 4x_3 = 0$, $x_2 = 0$

$3x_3 = 0$, $x_3 = 0$

x_1 is a free Variable, assume $x_1 = 1$

So $x = (x_1, x_2, x_3) = (\underline{1, 0, 0})$ is the eigenvector Corresponding to 1

For $\lambda = 2$, $x = [x_1, x_2, x_3]^T$

$(T - \lambda I)x = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 0 & -4 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow$ $-x_1 - x_2 + x_3 = 0$

$-4x_3 = 0$

$2x_3 = 0$, $x_3 = 0$

$\Rightarrow x_1 = -x_2$

assume $= x_1 = 1$

$x_2 = -1$

$x = [x_1, x_2, x_3] = (\underline{1, -1, 0})$ is the eigenvectors Corresponding to 2

(8) For $\lambda = 4$ $x = [x_1, x_2, x_3]^T$

(b)

$$\begin{bmatrix} -3 & -1 & 1 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

$-3x_1 - x_2 + x_3 = 0$

$-2x_2 - 4x_3 = 0$

$x_3 = x_3$

$-3x_1 + 2 + 1 = 0$

$x_1 = 1$

$-2x_2 - 4 = 0$

$-2x_2 = 4$

$x_2 = -2$

$x_3 = 1$

So $[x_1, x_2, x_3] = (1, -2, 1)$ is the eigen Vector Corresponding to 4

Eigen Vector of $\lambda_1 = (1, 0, 0)$

$= 1$ (in terms of P_2)

Eigen Vector of $\lambda_2 = (1, -1, 0)$

$= 1 - x$ (in terms of P_2)

Eigen Vector of $\lambda_3 = (1, -2, 1)$

$= 1 - 2x + x^2$

(c) Eigen Vectors for $\lambda = 1$ = $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Eigen Vectors for $\lambda = 2$ = $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

Eigen Vectors for $\lambda = 4$ = $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Now $x + 1 = 1 \cdot 1 + 1 \cdot x + 0 \cdot x^2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$[x]_C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{matrix} x_1 = 2 \\ x_2 = -1 \\ x_3 = 0 \end{matrix}$$

$\therefore$ Coordinate Vector of $x+1$ is $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ //

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d) $T(a+bx+cx^2) = a + b(2x-1) + c(2x-1)^2$

(4)

$T(1) = 1$

$T(1-x) = 1-(2x-1) = 2-2x$

$T(1-2x+x^2) = 1 - 2(2x-1) + (2x-1)^2$

$= 1 - 4x + 2 + 4x^2 - 4x + 1$

$= 4x^2 - 8x + 4$

$1 = (1)(1) + 0(1-x) + 0(1-2x+x^2)$

$2-2x = 0(1) + 2(1-x) + 0(1-2x+x^2)$

$4x^2-8x+4 = 0(1) + 0(1-x) + 4(1-2x+x^2)$

$$[T]_C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$[T^4]_C = \begin{bmatrix} 1^4 & 0 & 0 \\ 0 & 2^4 & 0 \\ 0 & 0 & 4^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 256 \end{bmatrix}$$

i) $T^4(P(x)) = a + 16bx + 256cx^2$

$T^4(a+bx+cx^2) = a + 16bx + 256cx^2$

so, $T^4(x+1) = 1 + 16x$

$C = \{1, 1-x, 1-2x+x^2\}$

$1+16x = 18/2\,(1) - 16/2\,(1-x) + 0(1-2x+x^2)$

Coordinate Vector of $1+16x$ to C is $\begin{bmatrix} 9 \\ -8 \\ 0 \end{bmatrix}$

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(7)

(ii) Let $t_3 =$

$= x_0 P_0 + x_1 P_1 + x_2 P_2$

$$\begin{bmatrix} 1 & \cdot 1 & 1 \\ 0 & 16 & 0 \\ 0 & 0 & 256 \end{bmatrix}$$

$t^3 = (x_0 - x_1 + x_2) + t(16x_2) + t^2(256x_3)$

$x_0 - x_1 + x_2 = 0$

$16x_2 = 0$

$256x_2 = 0, \; x_3 = 0$

$$= \begin{bmatrix} 1 \\ 16 \\ 256 \end{bmatrix}$$

iii) $T(a + bx + cx^2) = a + b(2x-1) + c(2x-1)^2$

$T^4 = (1+x) = T \cdot T \cdot T(1+x)$

$T(1+x) = 1 + (2x-1) = 2x$

$T \cdot T(1+x) = T(2x) = 2(2x-1) = 4x - 2$

$T \cdot T \cdot T(1+x) = T(4x-2) = -2 + 4(4x-2)$

$= -2 + 16x - 8 = -10 + 16x$

$T \cdot T \cdot T(1+x) = T^4(1+x) = T(-10+16x)$

$= -10 + 16(2x-1)$

$= -10 + 32x - 16$

$= -26 + 32x$

$T^4(1+x) = -\underline{26+32x}$

9) $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \xrightarrow[R_4 - 2R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -2 \\ 0 & 1 & 1 \\ 0 & -3 & -1 \end{bmatrix} \xrightarrow{-1/3 R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2/3 \\ 0 & 1 & 1 \\ 0 & -3 & -1 \end{bmatrix}$

$\xrightarrow[R_4 + 3R_2]{R_3 - R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2/3 \\ 0 & 0 & 1/3 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{3 \times R_3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2/3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_4 - R_3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2/3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow[R_1 - R_3]{R_2 - 2/3 R_3} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ Basis for Column Space is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

By Gram-schmidt process

Let $v_1 = (1,2,0,2)$, $v_2 = (2,1,1,1)$, $v_3 = (1,0,1,1)$

$u_1 = v_1 = (1,2,0,2)$

$\|u_1\| = \sqrt{1+4+0+4} = \sqrt{9} = 3$

$e_1 = \dfrac{u_1}{\|u_1\|} = \left(\tfrac{1}{3}, \tfrac{2}{3}, 0, \tfrac{2}{3}\right)$

$u_2 = v_2 - \dfrac{u_1 \cdot v_2}{\|u_1\|^2} u_1 = (2,1,1,1) - \tfrac{6}{9}\left(\tfrac{1}{3}, \tfrac{2}{3}, 0, \tfrac{2}{3}\right)$

$= \left(\tfrac{4}{3}, -\tfrac{1}{3}, 1, -\tfrac{1}{3}\right)$

(9) $\|u_2\| = \sqrt{16/9 + 1/9 + 1 + 1/9} = \sqrt{27/9} = \sqrt{3}$

$$e_2 = \frac{u_2}{\|u_2\|} = \left(\frac{4}{3\sqrt{3}}, \frac{-1}{3\sqrt{3}}, 1/3\sqrt{3}, \frac{-1}{3\sqrt{3}}\right)$$

$$\Rightarrow u_3 = v_3 - \frac{u_1 \cdot v_3}{\|u_1\|^2} u_1 - \frac{u_2 \cdot v_3}{\|u_2\|^2} u_2$$

$$= (1,0,1,1) - 3/9\,(1,2,0,2) - 2/3\,(4/3, -1/3, 1, -1/3)$$

$$= (-2/9, -4/9, 1/3, 5/9)$$

$$\|u_3\| = \sqrt{4/81 + \frac{16}{81} + 1/9 + \frac{25}{81}} = \sqrt{2/3}$$

$$e_3 = \frac{u_3}{\|u_3\|} = \left(\frac{-\sqrt{2}}{3\sqrt{3}}, \frac{-2\sqrt{2}}{3\sqrt{3}}, 1/\sqrt{6}, \frac{5}{3\sqrt{6}}\right)$$

$\therefore$ orthonormal basis for Column Space of A is

$$\begin{bmatrix} 1/3 \\ 2/3 \\ 0 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 4/3\sqrt{3} \\ -1/3\sqrt{3} \\ 1/3\sqrt{3} \\ -1/3\sqrt{3} \end{bmatrix}, \begin{bmatrix} -2/3\sqrt{3} \\ -2\sqrt{2}/3\sqrt{3} \\ 1/\sqrt{6} \\ 5/3\sqrt{6} \end{bmatrix}$$

10) $W = \text{span}\{V_1, V_2, V_3\}$ where $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, V_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, V_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$

$b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ on W.

$$\Rightarrow \text{Proj}_W \vec{b} = \hat{b} = \left(\frac{\vec{b}\cdot\vec{V_1}}{\vec{V_1}\cdot\vec{V_1}}\right)\vec{V_1} + \left(\frac{\vec{b}\cdot\vec{V_2}}{\vec{V_2}\cdot\vec{V_2}}\right)\vec{V_2} + \left(\frac{\vec{b}\cdot\vec{V_3}}{\vec{V_3}\cdot\vec{V_3}}\right)\vec{V_3}$$

$$\Rightarrow \hat{b} = \left(\frac{1\cdot 1 + 2\cdot 1 + 3\cdot 1 + 4\cdot 1}{1^2+1^2+1^2+1^2}\right)\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \left(\frac{(1\cdot -1)+(2\cdot 1)+(3\cdot -1)+(4\cdot 1)}{(-1)^2+(1)^2+(-1)^2+(1)^2}\right)\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$+ \left(\frac{1\cdot 1 + (2\cdot -1) + (3\cdot 1) + (4\cdot 1)}{(1)^2+(-1)^2+(1)^2+(1)^2}\right)\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \left(\frac{10}{4}\right)\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \left(\frac{2}{4}\right)\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \left(\frac{6}{4}\right)\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7/2 \\ 3/2 \\ 7/2 \\ 9/2 \end{bmatrix} = \hat{b}$$

1 ii) Suppose that a $n \times n$ matrix A is similar to the $n \times n$ matrix B That means, there exists a $n \times n$ invertible matrix P such that $A = P^{-1}BP$ Then

i) $\det A = \det(P^{-1}BP)$

$= \det P^{-1} \cdot \det B \cdot \det P$

$= 1/\det P \cdot \det B \cdot \det P$

$= \det B$

Therefore $\det A \neq 0$ if and only if $\det B \neq 0$

A $n \times n$ matrix A is invertible if and only if $\det A \neq 0$

Therefore A is invertible if and only B is invertible

ii) A $n \times n$ matrix A is invertible if and only if $\det A \neq 0$

from eqn 1

$A^{-1} = (P^{-1}BP)^{-1}$

$= PB^{-1}(P^{-1})$ Using $(AB)^{-1} = B^{-1}A^{-1}$

$= KB^{-1}K^{-1}$ where $K = P^{-1}$

Therefore A^{-1} is similar to B^{-1}

12) Since A and B are similar matrices, there exist an invertible matrix P such that $A = PBP^{-1}$

Characteristics polynomial of A

$= \det(A - \lambda I)$

$= \det(PBP^{-1} - \lambda I)$

$= \det(PBP^{-1} - P\lambda I P^{-1})$

$= \det(P(B - \lambda I)P^{-1})$

$= \det(P) \det(B - \lambda I) \det(P^{-1})$

$(\because \det(P^{-1}) = 1/\det P)$

$= \det(B - \lambda I)$

Characteristics polynomial of B

$\rightarrow$ Characteristics polynomial of A and B are the same.