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Addition of Angular Momentum

We show that if a space V_1 carries a representation of angular momentum operators $\hat{\mathbf{J}}_1$ and a space V_2 carries a representation of angular momentum operators $\hat{\mathbf{J}}_2$, then the tensor product $V_1 \otimes V_2$ carries a representation of the sum $\hat{\mathbf{J}}_1 + \hat{\mathbf{J}}_2$ of angular momentum operators. We obtain the classic result that the tensor product of two spin one-half vector spaces is the direct sum of a total spin one vector space and a total spin zero vector space. We examine the hyperfine splitting of the ground state of the hydrogen atom and then turn to the spin-orbit interaction, where we discuss complete sets of commuting observables. This leads us to the general study of uncoupled and coupled basis states and the calculation of Clebsch-Gordan coefficients. We derive the selection rule that tells us what representations of the total angular momentum appear in the tensor product $j_1 \otimes j_2$ of two angular momentum representations. We give an algebraic derivation of the spectrum of the hydrogen atom using the conserved orbital angular momentum and Runge-Lenz vector to discover two commuting angular momenta.

20.1 Adding Apples to Oranges?

We are going to be adding angular momenta in a variety of ways. Since angular momenta are operators in quantum mechanics, we are going to be adding angular momentum operators in various ways as well. We may add the *spin* angular momentum $\hat{\mathbf{S}}$ of a particle to its *orbital* angular momentum $\hat{\mathbf{L}}$. Or we may want to add the spin angular momentum $\hat{\mathbf{S}}^{(1)}$ of a particle to the spin angular momentum $\hat{\mathbf{S}}^{(2)}$ of another particle. At first sight we may feel like we are trying to add apples to oranges! For a given particle, spin angular momentum and orbital angular momentum act on different degrees of freedom of the particle. Adding the spins of two different particles also seems unusual if, for example, the particles are far away from each other. Vectors that live at different places are seldom added: you don't typically add the electric field at one point to the electric field at another point because the sum has no obvious interpretation. This is even more severe in general relativity: you *cannot* add vectors that "live" at different points of space-time. To add them you need a procedure to first bring them to a common point. Once they both live at that common point, you can add them.

Despite some differences, however, at an algebraic level all angular momenta are apples (Granny Smith, Red Delicious, McIntosh, Fuji, and so on). Therefore, they can be added, and it is natural to add them. We are not adding apples to oranges; we are adding apples to apples! The physics requires it. For a particle with spin and orbital angular momentum, for example, the sum of these two is the total angular momentum, which is the complete generator of rotations, implementing any rotation on both the spatial and spin degrees of freedom of the particle. Moreover, we will see that in large classes of Hamiltonians, energy eigenstates are eigenstates of a sum of angular momenta. The mathematics allows it: the sum of angular momenta *is* an angular momentum acting in the appropriate *tensor* product. As we will see below, while each angular momentum operator lives on a different vector space, the sum finds a *home* in the tensor product of the vector spaces.

What is an angular momentum? It is a triplet $\hat{J}_i$ of Hermitian operators on some complex vector space V satisfying the commutation relations

$$[\hat{J}_i, \hat{J}_j] = i\hbar\,\epsilon_{ijk}\hat{J}_k. \tag{20.1.1}$$

As we have learned, this is a very powerful statement. When coupled with the requirement that no negative norm-squared states exist, it implies that the vector space V on which these operators act can be decomposed into sums of multiplets, finite-dimensional subspaces that carry irreducible representations of angular momentum.

Let us now assume we have two angular momenta:

$$\begin{aligned} &\text{Hermitian operators } \hat{J}_i^{(1)} \text{ on } V_1 \text{ satisfying } [\hat{J}_i^{(1)}, \hat{J}_j^{(1)}] = i\hbar\,\epsilon_{ijk}\,\hat{J}_k^{(1)}, \\ &\text{Hermitian operators } \hat{J}_i^{(2)} \text{ on } V_2 \text{ satisfying } [\hat{J}_i^{(2)}, \hat{J}_j^{(2)}] = i\hbar\,\epsilon_{ijk}\,\hat{J}_k^{(2)}. \end{aligned} \tag{20.1.2}$$

Our claim is that the "sum" of angular momenta is an angular momentum in the tensor product space:

$$\boxed{\hat{J}_i \equiv \hat{J}_i^{(1)} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{J}_i^{(2)} \text{ satisfy } [\hat{J}_i, \hat{J}_j] = i\hbar\,\epsilon_{ijk}\,\hat{J}_k \text{ acting on } V_1 \otimes V_2.} \tag{20.1.3}$$

Certainly, the sum operator, as defined above, is an operator on $V_1 \otimes V_2$. It is in fact a Hermitian operator on $V_1 \otimes V_2$. We just need to check that the commutator holds:

$$\begin{aligned} [\hat{J}_i, \hat{J}_j] &= [\hat{J}_i^{(1)} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{J}_i^{(2)}\ \hat{J}_j^{(1)} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{J}_j^{(2)}\ ] \\ &= [\hat{J}_i^{(1)} \otimes \mathbb{1},\, \hat{J}_j^{(1)} \otimes \mathbb{1}\,] + [\,\mathbb{1} \otimes \hat{J}_i^{(2)}\ \mathbb{1} \otimes \hat{J}_j^{(2)}\ ], \end{aligned} \tag{20.1.4}$$

since the mixed terms, which represent commutators of the operators in the different spaces, vanish:

$$[\hat{J}_i^{(1)} \otimes \mathbb{1},\, \mathbb{1} \otimes \hat{J}_j^{(2)}\,] = 0, \quad [\mathbb{1} \otimes \hat{J}_i^{(2)},\, \hat{J}_j^{(1)} \otimes \mathbb{1}\,] = 0. \tag{20.1.5}$$

Writing out the commutators, we see that (20.1.4) becomes

$$[\hat{J}_i, \hat{J}_j] = [\hat{J}_i^{(1)}, \hat{J}_j^{(1)}] \otimes \mathbb{1} + \mathbb{1} \otimes [\hat{J}_i^{(2)}, \hat{J}_j^{(2)}\,]. \tag{20.1.6}$$

We can now use the independent algebras of angular momentum to find

$$\begin{aligned} [\hat{J}_i, \hat{J}_j] &= i\hbar\,\epsilon_{ijk}\,\hat{J}_k^{(1)} \otimes \mathbb{1} + i\hbar\,\epsilon_{ijk}\,\mathbb{1} \otimes \hat{J}_k^{(2)} \\ &= i\hbar\,\epsilon_{ijk}\left(\hat{J}_k^{(1)} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{J}_k^{(2)}\right) \\ &= i\hbar\,\epsilon_{ijk}\hat{J}_k, \end{aligned} \tag{20.1.7}$$

which is what we set out to prove.

It is important to note that had we added the two angular momenta with some arbitrary coefficients the sum would not have been an angular momentum. Indeed, suppose we use two *nonzero* constants α and β and write

$$\tilde{J}_i \equiv \alpha\,\hat{J}_j^{(1)} \otimes \mathbb{1} + \beta\,\mathbb{1} \otimes \hat{J}_j^{(2)}. \tag{20.1.8}$$

If the constants are not real, $\tilde{J}_i$ is not Hermitian. The commutator calculation above this time yields

$$[\tilde{J}_i, \tilde{J}_j] = i\hbar\epsilon_{ijk}\left(\alpha^2\hat{J}_k^{(1)} \otimes \mathbb{1} + \beta^2\, \mathbb{1} \otimes \hat{J}_k^{(2)}\right). \tag{20.1.9}$$

We have an algebra of angular momentum if the operator in parentheses is $\tilde{J}_k$. This requires $\alpha^2 = \alpha$ and $\beta^2 = \beta$. Since neither α nor β is zero, the only solution is $\alpha = \beta = 1$. This confirms that (20.1.3) is the *unique* way to add two angular momenta to form a new angular momentum.

Since any vector space where angular momentum operators act can be decomposed into the direct sum of irreducible representations of angular momentum (multiplets), the space $V_1 \otimes V_2$ can be decomposed into sums of irreducible representations of the algebra of *total* angular momentum. This property gives us a powerful tool to understand the spectrum of the Hamiltonian in the physical state space $V_1 \otimes V_2$.