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13.4 Null Space, Range, and Inverses of Operators

When we encounter a linear operator on a vector space, there are two questions we can ask to determine the most basic properties of the operator: What vectors are mapped to zero by the operator? What vectors in V are obtained from the action of T on V? The first question leads to the concept of null space, the second to the concept of range.

The **null space** or **kernel** of $T \in \mathcal{L}(V)$ is the subset of vectors in V that are mapped to zero by T:

$$\text{null}\, T = \{v \in V;\; Tv = 0\}. \tag{13.4.1}$$

Actually, null T is a *subspace* of V. Indeed, the null space contains the zero vector and is clearly a closed set under addition and scalar multiplication.

A linear operator T: $V \to V$ is said to be **injective** if $Tu = Tv$, with u, $v \in V$, implies $u = v$. An injective map is called a *one-to-one* map because two different elements cannot be mapped to the same one (physicist Sean Carroll has suggested that a better name would be *two-to-two*, as injectivity really means that two different elements are mapped by T to two different elements). In fact, an operator is injective if and only if its null space vanishes:

$$T \text{ injective} \iff \text{null}\, T = 0. \tag{13.4.2}$$

To show this equivalence, we first prove that injectivity implies zero null space. Indeed, if $v \in$ null T then $Tv = T0$ (both sides are zero), and injectivity shows that $v = 0$, proving that null $T = 0$. In the other direction, zero null space means that $T(u-v) = 0$ implies $u-v = 0$ or, equivalently, that $Tu = Tv$ implies $u = v$. This is injectivity.

As mentioned above, it is also of interest to consider the elements of V of the form Tv. We define the **range** of T as the image of V under the map T:

$$\text{range}\, T = \{Tv;\; v \in V\}. \tag{13.4.3}$$

Actually, range T is a *subspace* of V (try proving it!). A linear operator T is said to be **surjective** if range $T = V$. That is, for a surjective T the image of V under T is the complete V.

Example 13.13. *Left- and right-shift operators.*

Recall the action of the left- and right-shift operators on infinite sequences:

$$L(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots), \qquad R(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots). \tag{13.4.4}$$

We can immediately see that null $L = (x_1, 0, 0, \ldots)$. Being different from zero, L is not injective. But L is surjective because any sequence can be obtained by the action of L: $(x_1, x_2, \ldots) = L(x_0, x_1, x_2, \ldots)$ for arbitrary x_0. The null space of R is zero, and thus R is injective. R is not surjective: we cannot get any element whose first entry is nonzero. In summary:

$$\begin{aligned} &L: \quad \text{not injective}, \ \text{surjective}, \\ &R: \quad \text{injective}, \qquad \text{not surjective}. \end{aligned} \tag{13.4.5}$$

We will consider further properties of these operators in example 13.15.

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Let us now consider finite-dimensional vector spaces. Since both the null space and the range of a linear operator $T: V \to V$ are themselves vector spaces, one can calculate their dimensions. The larger the null space of an operator, the more vectors are mapped to zero, and one would expect the range to be reduced accordingly. The smaller the null space, the larger we expect the range to be. This intuition is made precise by the *rank-nullity* theorem. For **any** linear operator on V, the sum of the dimensions of its null space and its range is equal to the dimension of the vector space V:

$$\boxed{\dim(\text{null } T) + \dim(\text{range } T) = \dim V.} \tag{13.4.6}$$

The dimension of the range of T is called the **rank** of T, thus the name rank-nullity theorem. We only sketch the main steps. Let $(e_1, \ldots, e_m)$ with $m = \dim(\text{null } T)$ be a basis for null(T). This basis can be extended to a basis $(e_1, \ldots, e_m, f_1, \ldots, f_n)$ of the full vector space V, where $m + n = \dim V$. The final step consists in showing that the Tf_i form a basis for the range of T. This is done in two steps:

Exercise 13.5. *Show that the vectors $(Tf_1, \ldots, Tf_n)$ (i) span the range of T and (ii) are linearly independent.*

Remark: The rank-nullity theorem in its general form applies to linear maps that relate spaces of different dimensionality. If $T: V \rightarrow W$ is a linear map from a vector space V to a vector space W, then the range of T is a subspace of W, and the null space of T is a subspace of V. Nevertheless, one still has exactly (13.4.6). The result does not involve the dimensionality of the space W.

Example 13.14. *Null space and range of spin operators on* $\mathbb{C}^2$.
Let us consider curious linear combinations $\hat{S}_\pm$ of spin operators of a spin one-half particle:

$$\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y. \tag{13.4.7}$$

While both $\hat{S}_x$ and $\hat{S}_y$ are Hermitian, using i to form a linear combination does not preserve Hermiticity. In fact, $\hat{S}_\pm^\dagger = \hat{S}_\mp$; the operators are Hermitian conjugates of each other. It is instructive to find their matrix representatives

$$\hat{S}_\pm = \frac{\hbar}{2}(\sigma_x \pm i\sigma_y) = \frac{\hbar}{2}\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \pm i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\right] = \frac{\hbar}{2}\begin{pmatrix}0 & 1\pm 1\\ 1\mp 1 & 0\end{pmatrix}. \tag{13.4.8}$$

We thus have the two matrices

$$\hat{S}_+ = \hbar\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \qquad \hat{S}_- = \hbar\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}. \tag{13.4.9}$$

Let us focus on $\hat{S}_+$; the case of $\hat{S}_-$ is completely analogous. Consider the null space of $\hat{S}_+$ first:

$$\hbar\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \quad \rightarrow \quad c_2 = 0 \tag{13.4.10}$$

so that the general vector in the null space is $\begin{pmatrix}c_1\\ 0\end{pmatrix}$ with $c_1 \in \mathbb{C}$. This is the state we call e_1, or $|+\rangle$, a spin state pointing in the positive z-direction. Therefore,

$$\text{null}\,\hat{S}_+ = \text{span}\begin{pmatrix}1\\ 0\end{pmatrix} = \text{span}\,e_1 = \text{span}\,|+\rangle. \tag{13.4.11}$$

Following the notation used to sketch the derivation of the rank-nullity theorem, we have a basis (e_1, f_1) for $\mathbb{C}^2$ with $f_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |-\rangle$. To find the range of $\hat{S}_+$, we let it act on a general vector:

$$\hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \hbar \begin{pmatrix} c_2 \\ 0 \end{pmatrix}, \tag{13.4.12}$$

and we conclude that

$$\text{range } \hat{S}_+ = \text{span } e_1 = \text{span } |+\rangle. \tag{13.4.13}$$

This may seem unusual as the range and null spaces are the same. But all is fine; both are one-dimensional, adding to total dimension two, as required by the rank-nullity theorem. The picture in terms of spin states is simple: acting on $|-\rangle$, the operator $\hat{S}_+$ gives $|+\rangle$, raising the $\hat{S}_z$ eigenvalue of the state. Acting on $|+\rangle$, the operator $\hat{S}_+$ gives zero: the eigenvalue of $\hat{S}_z$ cannot be raised anymore. Clearly, the range of $\hat{S}_+$ and its null space coincide: they are both equal to the span of $|+\rangle$.

Additionally, since $e_1 = \hat{S}_+ f_1$ we actually have range $\hat{S}_+ = \text{span}(\hat{S}_+ f_1)$, in accordance with the sketch of the argument that leads to the rank-nullity theorem. You can quickly confirm that the operator $\hat{S}_-$ lowers the value of $\hat{S}_z$ acting on $|+\rangle$ and kills $|-\rangle$. The range and null space of $\hat{S}_-$ are both the span of $|-\rangle$. The operators $\hat{S}_\pm$ are raising and lowering operators for spin angular momentum. They will be studied further in chapter 19.

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Since linear operators can be multiplied, given an operator we can ask if it has an inverse. It is interesting to consider the question in some detail, as there are some subtleties. To do so we have to discuss left inverses and right inverses.

Let $T \in \mathcal{L}(V)$ be a linear operator. The linear operator S is a **left inverse** for T if

$$ST = \mathbb{1}. \tag{13.4.14}$$

Namely, the product of S and T with S to the left of T is equal to the identity matrix. Analogously, the linear operator S' is a **right inverse** for T if

$$TS' = \mathbb{1}. \tag{13.4.15}$$

Namely, the product of T and S' with S' to the right of T is equal to the identity matrix. If both inverses exist, then they are actually equal. This is easily proven using the above defining relations and associativity of the product:

$$S' = \mathbb{1}S' = (ST)S' = S(TS') = S\mathbb{1} = S. \tag{13.4.16}$$

If both left and right inverses of T exist, then T is said to be **invertible.**

The left and right inverses are relevant for systems of linear equations. Assume we have an operator $T \in \mathcal{L}(V)$, a known vector $c \in V$, and an unknown vector $x \in V$ to be determined from the equation:

$$Tx = c. \tag{13.4.17}$$

Suppose all you have is a left inverse S for T. Then acting with S on the equation gives you $STx = Sc$ and therefore $x = Sc$. Have you solved the equation? Not quite. If you try to check that this is a solution, you fail! Indeed, inserting the value $x = Sc$ on the left-hand side of the equation gives TSc, which may not equal c because S is not known to be a right inverse. All we can say is that $x = Sc$ is the *only possible solution,* given that it follows from the equation but cannot be verified without further analysis. If all you have is a right inverse S', you can now check that $x = S'c$ does solve the equation! This time, however, there is no guarantee that the solution is unique. Indeed, if T has a null space, the solution is clearly not unique since any vector in the null space can be added to the solution to give another solution. Only if both left and right inverses exist are we guaranteed that a unique solution exists!

It is reasonable to ask when a linear operator $T \in \mathcal{L}(V)$ has a left inverse. Think of two pictures of V and T mapping elements from the first picture to elements in the second picture. A left inverse should map each element in the second picture back to the element it came from in the first picture. If T is not injective, two different elements in the first picture are sometimes mapped to the same element in the second picture. The inverse operator can at best map back to one element so it fails to act as an inverse for the other element. This complication is genuine. A left inverse for T exists if and only if T is injective:

$$\boxed{T \text{ has a left inverse} \iff T \text{ is injective.}} \tag{13.4.18}$$

The proof from left to right is easy. Assume $Tv_1 = Tv_2$. Then multiply from the left by the left inverse S, finding $v_1 = v_2$, which proves injectivity. To prove that injectivity implies a left inverse, we begin by considering a basis of V denoted by the collection of vectors $\{v_i\}$. We do not list the vectors because the space V could be infinite-dimensional. Then define

$$Tv_i = w_i \tag{13.4.19}$$

for all v_i's. One can use injectivity to show that the w_i's are linearly independent. Since the map T may not be surjective, the $\{w_i\}$ may not be a basis. They can be completed with a collection $\{y_k\}$ of vectors to form a basis for V. Then we define the action of S by stating how it acts on this basis:

$$\begin{aligned} Sw_i &= v_i, \\ Sy_k &= 0. \end{aligned} \tag{13.4.20}$$

We then verify that

$$ST\Big(\sum_i a_i v_i\Big) = S\Big(\sum_i a_i w_i\Big) = \sum_i a_i v_i, \tag{13.4.21}$$

showing that $ST = \mathbb{1}$ when acting on any element of V. Setting $Sy_k = 0$ is a natural option to define S fully, but the final verification did not make use of that choice.

For the existence of a right inverse S' of T, we need the operator T to be surjective:

$$\boxed{T \text{ has a right inverse} \iff T \text{ is surjective.}} \tag{13.4.22}$$

The necessity of surjectivity is quickly understood: if we have a right inverse, we have $TS'(v) = v$, or, equivalently, $T(S'v) = v$ for all $v \in V$. This says that any $v \in V$ is in the range of T. This is surjectivity of T. A more extended argument is needed to show that surjectivity implies the existence of a right inverse.

Since an operator is invertible if it has both a left and a right inverse, the two boxed results above imply that

$$T \in \mathcal{L}(V): \quad \boxed{T \text{ is invertible } \Longleftrightarrow T \text{ is injective and surjective.}} \tag{13.4.23}$$

This is a completely general result, valid for infinite- and finite-dimensional vector spaces.

Example 13.15. *Inverses for the left- and right-shift operators.*
Recall the properties (13.4.5) of the left- and right-shift operators L and R. Since L is surjective, it must have a right inverse. Since R is injective, it must have a left inverse. The right inverse of L is actually R, and the left inverse of R is actually L. These two facts are encoded by the single equation

$$LR = \mathbb{1}, \tag{13.4.24}$$

which is easily confirmed:

$$LR(x_1, x_2, \ldots,) = L(0, x_1, x_2, \ldots) = (x_1, x_2, \ldots). \tag{13.4.25}$$

Neither L nor R is invertible.

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Example 13.16. *Right inverse for the annihilation operator $\hat{a}$.*
We described in example 13.8 the state space of the harmonic oscillator as a direct sum of one-dimensional spaces U_n spanned by the energy eigenstates φ_n. The operator $\hat{a}$ maps U_n to U_{n-1}, as it is after all the lowering operator. It does so via the relation $\hat{a}\varphi_n = \sqrt{n}\varphi_{n-1}$. It is clear that the operator $\hat{a}$ is not injective: its null space is U_0, the space spanned by the ground state. The operator $\hat{a}$, however, is surjective: the full state space is in the range of $\hat{a}$. This means that $\hat{a}$ has a right inverse S'. This inverse must increase the number of the state by one unit, but, as we will see, it is not the $\hat{a}^\dagger$ operator.

To find the right inverse S' of $\hat{a}$, we write $S'\varphi_n = s_n\varphi_{n+1}$ with s_n a constant to be determined. Then we demand that $\hat{a}S' = \mathbb{1}$ when acting on any U_n:

$$\hat{a}\,S'\varphi_n = s_n\hat{a}\varphi_{n+1} = s_n\sqrt{n+1}\,\varphi_n \quad \rightarrow \quad s_n = \frac{1}{\sqrt{n+1}}. \tag{13.4.26}$$

The right inverse S' of $\hat{a}$ is therefore

$$S'\varphi_n = \frac{1}{\sqrt{n+1}}\,\varphi_{n+1}, \quad \hat{a}\,S' = \mathbb{1}. \tag{13.4.27}$$

Recall that $\hat{a}^\dagger$ satisfies $\hat{a}^\dagger\varphi_n = c_n\varphi_{n+1}$, with $c_n = \sqrt{n+1}$. Therefore, S' is not equal to $\hat{a}^\dagger$. It is not even proportional to $\hat{a}^\dagger$ because s_n/c_n is n dependent. S' acts just like $\hat{a}^\dagger$ on the ground state but differs from the $\hat{a}^\dagger$ action by n-dependent constants on $U_{n>0}$. In fact, $S'\varphi_n = \frac{1}{n+1}\hat{a}^\dagger\varphi_n$.

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Exercise 13.6. *Explain why the creation operator $\hat{a}^\dagger$on the state space of the harmonic oscillator is injective, and determine its left inverse S.*

If the vector space V is finite-dimensional, the results are simpler. Any injective operator is surjective, and any surjective operator is injective. Therefore, any injective operator or any surjective operator is also invertible. The following three properties are therefore completely equivalent for operators on finite-dimensional vector spaces:

$$\dim V = \text{finite:} \quad \boxed{T \text{ is invertible} \iff T \text{ is injective} \iff T \text{ is surjective.}} \tag{13.4.28}$$

Proving these results is accomplished with simple exercises