19.2 Properties of Angular Momentum

The orbital angular momentum operators act nicely on the $\hat{x}_i$ and $\hat{p}_i$ operators. The action here is by commutators, and you should verify that

$$\boxed{\begin{aligned} [\,\hat{L}_i\,,\,\hat{x}_j\,] &= i\hbar\,\epsilon_{ijk}\,\hat{x}_k\,, \\ [\,\hat{L}_i\,,\,\hat{p}_j\,] &= i\hbar\,\epsilon_{ijk}\,\hat{p}_k\,. \end{aligned}} \tag{19.2.1}$$

We say that these equations mean that $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$ are vectors under rotations.

Exercise 19.3. *Prove the commutator relations (19.2.1).*

Exercise 19.4. *Use the above relations and (19.1.13) to show that*

$$\hat{\mathbf{p}} \times \hat{\mathbf{L}} = -\,\hat{\mathbf{L}} \times \hat{\mathbf{p}} + 2i\hbar\,\hat{\mathbf{p}}. \tag{19.2.2}$$

Hermitization is the process by which we construct a Hermitian operator starting from a non-Hermitian one. Say Ω is not Hermitian. Its Hermitization Ω_h is defined to be

$$\Omega_h \equiv \tfrac{1}{2}(\Omega + \Omega^\dagger). \tag{19.2.3}$$

Exercise 19.5. *Show that the Hermitization of* $\hat{\mathbf{p}} \times \hat{\mathbf{L}}$ *is*

$$(\hat{\mathbf{p}} \times \hat{\mathbf{L}})_h = \tfrac{1}{2}(\hat{\mathbf{p}} \times \hat{\mathbf{L}} - \hat{\mathbf{L}} \times \hat{\mathbf{p}}) = \hat{\mathbf{p}} \times \hat{\mathbf{L}} - i\hbar\,\hat{\mathbf{p}}. \tag{19.2.4}$$

We have stated that $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$ are vectors under rotations. More generally, we declare that an operator $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$ is a **vector under rotations** if

$$[\,\hat{L}_i\,,\,\hat{u}_j\,] = i\hbar\,\epsilon_{ijk}\,\hat{u}_k\,. \tag{19.2.5}$$

We will also declare that an operator $\hat{Z}$ that commutes with all angular momentum operators is a **scalar under rotations**:

$$[\,\hat{L}_i\,,\,\hat{Z}\,] = 0\,. \tag{19.2.6}$$

Exercise 19.6. *Assume* $\mathbf{n}$ *is a constant vector, and* $\hat{\mathbf{u}}$ *is a vector under rotations. Use the commutator (19.2.5) to show that*

$$[\,\mathbf{n}\cdot\hat{\mathbf{L}},\,\hat{\mathbf{u}}\,] \;=\; -i\hbar\;\mathbf{n}\times\hat{\mathbf{u}}. \tag{19.2.7}$$

We first learned in section 10.1 that angular momentum operators generate rotations, meaning they can be used to build unitary operators that rotate states. These unitary rotation operators also rotate operators, as shown in detail in section 14.7 and summarized in (14.7.25) for arbitrary angular momentum operators. When the unitary rotation operators $\hat{R}_{\mathbf{n}}(\alpha)$ act on a vector operator $\hat{\mathbf{u}}$, the vector is rotated by the action of a rotation matrix $\mathcal{R}_{\mathbf{n}}(\alpha)$, as follows:

$$\hat{R}^{\dagger}_{\mathbf{n}}(\alpha)\,\hat{\mathbf{u}}\,\hat{R}_{\mathbf{n}}(\alpha) = \mathcal{R}_{\mathbf{n}}(\alpha)\,\hat{\mathbf{u}}\,,\quad \hat{R}_{\mathbf{n}}(\alpha) = e^{-i\frac{\alpha}{\hbar}\mathbf{n}\cdot\hat{\mathbf{L}}}. \tag{19.2.8}$$

For more details on the notation, see section 14.7.

Exercise 19.7. *Prove the above formula by going over the steps in theorem 14.7.1 that led to the analogous (14.7.25).*

A scalar operator is simply left invariant by the action of the rotation operators $\hat{R}_{\mathbf{n}}(\alpha)$. We will see later in this chapter that the Hamiltonian for a central potential is a scalar operator.

Given two operators $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ that are vectors under rotations, their dot product is a scalar under rotations, and their cross product is a vector under rotations:

$$\boxed{\begin{aligned} [\,\hat{L}_i,\hat{\mathbf{u}}\cdot\hat{\mathbf{v}}\,] \;&=\; 0,\\ [\,\hat{L}_i,(\hat{\mathbf{u}}\times\hat{\mathbf{v}})_j\,] \;&=\; i\hbar\,\epsilon_{ijk}\,(\hat{\mathbf{u}}\times\hat{\mathbf{v}})_k. \end{aligned}} \tag{19.2.9}$$

Exercise 19.8. *Prove the above equations.*

A number of useful commutator identities follow from (19.2.9). Most importantly, from the second one, taking $\hat{\mathbf{u}} = \hat{\mathbf{r}}$ and $\hat{\mathbf{v}} = \hat{\mathbf{p}}$, we get

$$[\,\hat{L}_i,(\hat{\mathbf{r}}\times\hat{\mathbf{p}})_j\,] \;=\; i\hbar\,\epsilon_{ijk}\,(\hat{\mathbf{r}}\times\hat{\mathbf{p}})_k, \tag{19.2.10}$$

which gives a conceptually clear rederivation of the well-known algebra of angular momentum (10.3.12):

$$\boxed{[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k.} \tag{19.2.11}$$

Note that $\hat{\mathbf{L}}$ itself is a vector under rotations. More explicitly, the above commutators read

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z, \quad [\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x, \quad [\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y. \tag{19.2.12}$$

Note the cyclic nature of these equations: take the first and cycle indices ($x \to y \to z \to x$) once to obtain the second, and cycle again to obtain the third. This is how you can remember these relations by heart! Since the dot product of vector operators is a scalar operator, we have

$$[\hat{L}_i, \hat{\mathbf{r}}^2] = [\hat{L}_i, \hat{\mathbf{p}}^2] = [\hat{L}_i, \hat{\mathbf{r}} \cdot \hat{\mathbf{p}}] = 0, \tag{19.2.13}$$

and, very importantly,

$$[\hat{L}_i, \hat{\mathbf{L}}^2] = 0. \tag{19.2.14}$$

The operator $\hat{\mathbf{L}}^2$ will feature in the complete set of commuting observables for a central potential. An operator, such as $\hat{\mathbf{L}}^2$, that commutes with all $\hat{L}_i$ is called a *Casimir* operator of the algebra of angular momentum. Note that the validity of (19.2.14) just uses the algebra of the $\hat{L}_i$ operators, not their explicit definition in terms of $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$. Since the spin operators also satisfy the algebra of angular momentum, we have $[\hat{S}_i, \hat{\mathbf{S}}^2] = 0$. While $\hat{\mathbf{L}}^2$ commutes with the angular momentum operators, it *fails* to commute with the $\hat{x}_i$ operators and with the $\hat{\mathbf{p}}_i$ operators.

Exercise 19.9. *Use the algebra of $\hat{\mathbf{L}}$ operators to show that*

$$\hat{\mathbf{L}} \times \hat{\mathbf{L}} = i\hbar\hat{\mathbf{L}}. \tag{19.2.15}$$

This is a very elegant way to express the algebra of angular momentum. You can also check that the three components of this vector equation in fact imply the relations (19.2.12). It follows that (19.2.15) is an equivalent statement of the algebra of angular momentum:

$$\boxed{\hat{\mathbf{L}} \times \hat{\mathbf{L}} = i\hbar \hat{\mathbf{L}} \quad \Longleftrightarrow \quad [\hat{L}_i, \hat{L}_j] = i\hbar \epsilon_{ijk} \hat{L}_k.} \tag{19.2.16}$$

More generally, commutation relations of the form

$$[\hat{a}_i, \hat{b}_j] = \epsilon_{ijk} \hat{c}_k \tag{19.2.17}$$

admit a natural rewriting in terms of cross products. From (19.1.13),

$$(\hat{\mathbf{a}} \times \hat{\mathbf{b}})_i + (\hat{\mathbf{b}} \times \hat{\mathbf{a}})_i = \epsilon_{ijk} [\hat{a}_j, \hat{b}_k] = \epsilon_{ijk}\epsilon_{jkp}\hat{c}_p = 2\hat{c}_i. \tag{19.2.18}$$

This means that

$$\boxed{[\hat{a}_i, \hat{b}_j] = \epsilon_{ijk} \hat{c}_k \quad \Longrightarrow \quad \hat{\mathbf{a}} \times \hat{\mathbf{b}} + \hat{\mathbf{b}} \times \hat{\mathbf{a}} = 2\hat{\mathbf{c}}.} \tag{19.2.19}$$

The arrow *does not* work in the reverse direction. One finds that $[\hat{a}_i, \hat{b}_j] = \epsilon_{ijk} \hat{c}_k + \hat{s}_{ij}$ where $\hat{s}_{ij} = \hat{s}_{ji}$ is arbitrary and undetermined. If the arrow could be reversed, then the relation $\hat{\mathbf{a}} \times \hat{\mathbf{b}} + \hat{\mathbf{b}} \times \hat{\mathbf{a}} = \mathbf{0}$ would incorrectly imply that $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ commute. Indeed, while $\hat{\mathbf{r}} \times \hat{\mathbf{p}} + \hat{\mathbf{p}} \times \hat{\mathbf{r}} = 0$ (see (19.1.18)), the operators $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$ don't commute.

Since a vector $\hat{\mathbf{u}}$ under rotations satisfies equation (19.2.5), our result above implies that it also satisfies the vector identity:

$$\hat{\mathbf{u}} \text{ is a vector under rotations} \quad \Longrightarrow \quad \hat{\mathbf{L}} \times \hat{\mathbf{u}} + \hat{\mathbf{u}} \times \hat{\mathbf{L}} = 2i\hbar \hat{\mathbf{u}}. \tag{19.2.20}$$

In spherical coordinates, the angular momentum operators are combinations of angular derivatives. They have no radial dependence, as is expected for operators that generate rotations. We derived such results in section 10.4. For reference, they are repeated here:

$$\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}, \quad \hat{L}_\pm \equiv \hat{L}_x \pm i\hat{L}_y = \pm\hbar e^{\pm i\phi}\Big(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\Big). \tag{19.2.21}$$