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15.8 Complete Set of Commuting Observables

We have discussed the problem of finding the eigenstates and eigenvalues of a Hermitian operator S. In a quantum system, S is a quantum mechanical observable. The eigenstates of S are states of the system in which the observable S can be measured with certainty. The result of the measurement is the eigenvalue associated with the eigenstate. Moreover, the eigenstates of S form a basis for the state space.

If the Hermitian operator S has a nondegenerate spectrum, all eigenvalues are different, and we have a rather nice situation in which each eigenstate can be uniquely labeled with the corresponding eigenvalue of S. The physical value of the observable distinguishes the various eigenstates. In this case the operator S provides a *complete set of commuting observables* or a CSCO. The set here has just one observable, the operator S.

The situation is more nontrivial if the Hermitian operator S has a degenerate spectrum. This means that V has an S-invariant subspace of dimension $d > 1$ spanned by orthonormal eigenstates $(u_1, \ldots, u_d)$, all of which have a common S eigenvalue. This time, the eigenvalue of S does not allow us to uniquely label the basis eigenstates of the invariant subspace. Physically, this is an unsatisfactory situation as we have different basis states—the various u_i's—that we cannot tell apart by the measurement of S. This time S does not provide a CSCO.

We are thus physically motivated to find another Hermitian operator T that is compatible with S. Two Hermitian operators are said to be **compatible observables** if they commute since then we can find a basis of V comprised by simultaneous eigenvectors of the operators. These states can be labeled by two observables—namely, the two eigenvalues. If we are lucky, the basis eigenstates in each of the S-invariant subspaces of dimension higher than one can be organized into T eigenstates of different eigenvalues. In this case T breaks the spectral degeneracy of S, and using T eigenvalues, as well as S eigenvalues, we can uniquely label a basis of orthonormal states of V. In this case we say that S and T form a CSCO.

We are now ready for a definition of a complete set of commuting observables. Consider a set of commuting observables—namely, a set $\{S_1, \ldots, S_k\}$ of Hermitian operators, all of which commute with each other. The operators act on a complex vector space V that represents the physical state-space of some quantum system. By the corollary in the previous section, we can find an orthonormal basis of vectors in V such that each vector is an eigenstate of every operator in the set. Let each eigenstate in the basis be labeled by the ordered list of eigenvalues of the S_i operators, with $i = 1, \ldots, k$. The set $\{S_1, \ldots, S_k\}$ is said to be a CSCO if no two eigenstates have the same labels.

This idea can be expressed with equations by writing the spectral decomposition for each of the commuting observables $S_1, \ldots, S_k$. Assume $\dim V = N$, and in writing the spectral decompositions, we use the rank-one projectors $P_1, \ldots, P_N$. Each projector is written as $P_i = |u_i\rangle\langle u_i|$, with $|u_i\rangle$, $i = 1, \ldots, N$, the common basis of eigenvectors. We then have

$$\begin{array}{ccccc} S_1 = & \lambda_1^{(1)} P_1 + & \cdots & + \lambda_N^{(1)} P_N, \\ \vdots & \vdots & \vdots & \vdots \\ S_k = & \lambda_1^{(k)} P_1 + & \cdots & + \lambda_N^{(k)} P_N. \end{array} \tag{15.8.1}$$

In this presentation the operators S_i all manifestly commute. In each row a number of eigenvalues are repeated, as each operator has a degenerate spectrum (if one operator has no degeneracies, it could serve as a complete set just by itself!). We have a complete set of commuting observables if the "columns" of eigenvalues associated to a single projector are all different. As ordered sets, we must have

$$\{\lambda_i^{(1)}, \ldots, \lambda_i^{(k)}\} \neq \{\lambda_j^{(1)}, \ldots, \lambda_j^{(k)}\}, \quad \text{for } i \neq j. \tag{15.8.2}$$

This is in fact the condition that each eigenstate u_i is uniquely determined by the list of eigenvalues provided by the complete set of observables.

It is a physically motivated assumption that for any quantum system there is a complete set of commuting observables, for otherwise there is no physical way to distinguish the various states that span the vector space. It would mean having states that are manifestly different but have no single observable property in which they differ! In any physical problem, we are urged to find commuting observables, and we must add observables until all degeneracies are resolved. A CSCO need not be unique. Once we have a CSCO, adding another observable causes no harm, although it is not necessary. Also, if the pair (S_1, S_2) form a CSCO, so will the pair $(S_1 + S_2,\ S_1 - S_2)$, for example. It is often useful to have CSCOs with the smallest possible number of operators.

The first operator that is usually included in a CSCO is the Hamiltonian $\hat{H}$. For bound state problems in one dimension, energy eigenstates are nondegenerate, and the energy can be used to uniquely label the $\hat{H}$ eigenstates. A simple example is the infinite square well. Another example is the one-dimensional harmonic oscillator. In such cases $\hat{H}$ forms the CSCO. If we have, however, a two-dimensional isotropic harmonic oscillator in the (x, y) plane, the Hamiltonian has degeneracies. At the first excited level, we can have the first excited state of the x harmonic oscillator or, at the same energy, the first excited state of the y harmonic oscillator. We thus need another observable that can be used to distinguish these states. There are several options (problem 15.12).

Example 15.5. *CSCO for bound states of hydrogen.*
The hydrogen atom bound states, obtained in section 11.3, are represented by wave functions $\psi_{n\ell m}(r, \theta, \phi)$ taking the form

$$\psi_{n\ell m}(r,\theta,\phi) = \frac{u_{n\ell}(r)}{r}\, Y_{\ell m}(\theta,\phi). \tag{15.8.3}$$

The labels on the wave function suffice to distinguish all states and tell us about observables. Here $n \geq 1$ is the principal quantum number, and it, alone, determines the value of the energy E_n of the state. With Hamiltonian $\hat{H}$ we have

$$\hat{H}\psi_{n\ell m} = E_n \psi_{n\ell m}, \quad E_n = -\frac{e^2}{2a_0}\frac{1}{n^2}. \tag{15.8.4}$$

We include $\hat{H}$ in the set of commuting observables. Since for each n we have $\ell = 0, \ldots, n-1$ and for each value of ℓ we have $(2\ell + 1)$ values of m, there is plenty of degeneracy and clearly the label n does not suffice. The other two labels, ℓ and m, arise because the spherical harmonics are eigenstates of $\hat{\mathbf{L}}^2$ and $\hat{L}_z$, as we showed in section 10.5:

$$
\begin{aligned}
\hat{\mathbf{L}}^2\, Y_{\ell m} &= \hbar^2\, \ell(\ell+1)\, Y_{\ell m}, \\
\hat{L}_z\, Y_{\ell m} &= \hbar m\, Y_{\ell m}.
\end{aligned}
\tag{15.8.5}
$$

The angular momentum operators only involve angular derivatives and are independent of r. As a result, they ignore the radial dependence in the wave functions $\psi_{n\ell m}$, and we have

$$
\begin{aligned}
\hat{\mathbf{L}}^2\, \psi_{n\ell m} &= \hbar^2\, \ell(\ell+1)\, \psi_{n\ell m}, \\
\hat{L}_z\, \psi_{n\ell m} &= \hbar m\, \psi_{n\ell m}.
\end{aligned}
\tag{15.8.6}
$$

The label ℓ encodes the eigenvalue of $\hat{\mathbf{L}}^2$, and the label m encodes the eigenvalue of $\hat{L}_z$. We thus choose the set of observables to be

$$
\{\hat{H},\, \hat{\mathbf{L}}^2,\, \hat{L}_z\}. \tag{15.8.7}
$$

We have seen that their labels uniquely specify the states. The only question that remains is if all commutators vanish. We checked in section 10.4 that $\hat{\mathbf{L}}^2$ and $\hat{L}_z$ commute. Both of these operators, we claim, commute with the Hamiltonian, which, as we showed in (10.2.11), takes the form

$$
\hat{H} = -\frac{\hbar^2}{2m}\,\frac{1}{r}\,\frac{\partial^2}{\partial r^2}\, r + \frac{1}{2mr^2}\,\hat{\mathbf{L}}^2 + V(r). \tag{15.8.8}
$$

The $[H, \hat{\mathbf{L}}^2] = 0$ claim is clear: $\hat{\mathbf{L}}^2$ commutes with itself and ignores all radial dependence—it is a purely angular operator. The $[H, \hat{L}_z] = 0$ claim is also clear: $\hat{L}_z$ also ignores all radial dependence and commutes with $\hat{\mathbf{L}}^2$. This confirms that the set in (15.8.7) is indeed a CSCO for the bound state spectrum of hydrogen.

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