14.10 Nondenumerable Basis States

In this section we describe the use of bras and kets for the position and momentum states of a particle moving on the real line $x \in \mathbb{R}$. Let us begin with position. We will introduce position states $|x\rangle$ where the label x in the ket is the value of the position. Roughly, $|x\rangle$ represents the state of the system where the particle is at position x. The full state space requires position states $|x\rangle$ for all values of x. Physically, we consider all of these states to be linearly independent: the state of a particle at some point x_0 can't be built by the superposition of states where the particle is elsewhere. Since x is a continuous variable, the basis states form a nondenumerable infinite set:

$$\text{basis states: } |x\rangle, \quad \forall x \in \mathbb{R}. \tag{14.10.1}$$

Since we have an infinite number of basis vectors, this state space is an infinite-dimensional complex vector space. This should not surprise you. The states of a particle on the real line can be represented by wave functions, and the set of possible wave functions form an infinite-dimensional complex vector space.

Note here that the label in the ket is not a vector; it is the position on a line. If we did not have the decoration provided by the ket, it would be hard to recognize that the object is a state in an infinite-dimensional complex vector space. Therefore, the following should be noted:

$$\begin{aligned} |ax\rangle &\neq a|x\rangle, && \text{for any real } a \neq 1, \\ |-x\rangle &\neq -|x\rangle, && \text{unless } x = 0, \\ |x_1 + x_2\rangle &\neq |x_1\rangle + |x_2\rangle. \end{aligned} \tag{14.10.2}$$

All these equations would hold if the labels inside the kets were vectors. In the first line, roughly, the left-hand side is a state with a particle at ax, while the right-hand side is a state with a particle at x. Analogous remarks hold for the other lines. Note also that $|0\rangle$ represents a particle at $x = 0$, not the zero vector on the state space, for which we would probably have to use the symbol 0.

For the quantum mechanics of a particle moving in three spatial dimensions, we would have position states $|\mathbf{x}\rangle$. Here the label is a vector in a three-dimensional real vector space, while the ket is a vector in the infinite-dimensional complex vector space of the theory. Again, the decoration enclosing the vector label plays a crucial role: it reminds us that the state lives in an infinite-dimensional complex vector space.

Let us go back to our position basis states for the one-dimensional problem. The inner product must be defined, so we will take

$$\langle x|y\rangle \equiv \delta(x-y). \tag{14.10.3}$$

It follows that position states with different positions are orthogonal to each other. The norm of a position state is infinite: $\langle x|x\rangle = \delta(0) = \infty$, so these are not allowed states of particles. We visualize the state $|x\rangle$ as the state of a particle perfectly localized at x, but this is an idealization. We can easily construct normalizable states using the superpositions of position states. We also have a completeness relation:

$$\mathbb{1} = \int_{-\infty}^{\infty} dx\, |x\rangle\langle x|. \tag{14.10.4}$$

This is consistent with our inner product above. Letting the above equation act on $|y\rangle$, we find an equality:

$$|y\rangle = \int dx\, |x\rangle\langle x|y\rangle = \int dx\, |x\rangle\, \delta(x-y) \ = \ |y\rangle. \tag{14.10.5}$$

All integrals are now assumed to run from $-\infty$ to $+\ \infty$. The position operator $\hat{x}$ is defined by its action on the position states. Not surprisingly, we define

$$\hat{x}\,|x\rangle \equiv x\,|x\rangle, \tag{14.10.6}$$

thus declaring that $|x\rangle$ are $\hat{x}$ eigenstates with eigenvalue equal to the position x. We can also show that $\hat{x}$ is a Hermitian operator by checking that $\hat{x}^{\dagger}$ and $\hat{x}$ have the same matrix elements:

$$\langle x_1|\hat{x}^{\dagger}|x_2\rangle = \langle x_2|\hat{x}|x_1\rangle^{*} = [x_1\delta(x_1-x_2)]^{*} = x_2\delta(x_1-x_2) = \langle x_1|\hat{x}|x_2\rangle, \tag{14.10.7}$$

using the reality of x_1 and $\delta(x_1 - x_2)$ and the symmetry of the delta function to change x_1 into x_2. We thus conclude that $\hat{x}^{\dagger} = \hat{x}$. As a result, the bra associated with (14.10.6) is

$$\langle x|\hat{x} = x\langle x|. \tag{14.10.8}$$

The wave function associated with a state is formed by taking the inner product of a position state with the given state. Given the state $|\psi\rangle$ of a particle, we define the associated position state wave function $\psi(x)$ by

$$\psi(x) \equiv \langle x|\psi\rangle \ \in \mathbb{C}. \tag{14.10.9}$$

This is sensible: $\langle x|\psi\rangle$ is a number that depends on the value of x and is thus a function of x. We can now do a number of basic computations. First, we write any state as a superposition of position eigenstates by inserting $\mathbb{1}$, as in the completeness relation:

$$|\psi\rangle = \mathbb{1}|\psi\rangle = \int dx\, |x\rangle \langle x|\psi\rangle = \int dx\, |x\rangle\, \psi(x). \tag{14.10.10}$$

As expected, $\psi(x)$ is the component of $|\psi\rangle$ along the state $|x\rangle$. The overlap of states can also be written in position space:

$$\langle\phi|\psi\rangle = \langle\phi|\mathbb{1}|\psi\rangle = \int dx\, \langle\phi|x\rangle \langle x|\psi\rangle = \int dx\, \phi^*(x)\psi(x). \tag{14.10.11}$$

Matrix elements involving $\hat{x}$ are also easily evaluated:

$$\begin{aligned} \langle\phi|\hat{x}|\psi\rangle &= \langle\phi|\hat{x}\mathbb{1}|\psi\rangle = \int dx\, \langle\phi|\hat{x}|x\rangle \langle x|\psi\rangle \\ &= \int dx\, \langle\phi|x\rangle\, x\, \langle x|\psi\rangle = \int dx\, \phi^*(x)\, x\, \psi(x). \end{aligned} \tag{14.10.12}$$

We now introduce momentum states $|p\rangle$ that are eigenstates of the momentum operator $\hat{p}$, in complete analogy to the position states:

$$\begin{aligned} &\text{Basis states: } |p\rangle, \quad \forall p \in \mathbb{R}. \\ \langle p'|p\rangle &= \delta(p - p'), \\ \mathbb{1} &= \int dp\, |p\rangle\langle p|, \\ \hat{p}\,|p\rangle &= p\,|p\rangle. \end{aligned} \tag{14.10.13}$$

Just as for position space, we also find that

$$\hat{p}^\dagger = \hat{p}, \quad \text{and} \qquad \langle p|\hat{p} = p\langle p|. \tag{14.10.14}$$

In order to relate the two bases, we need the value of the overlap $\langle x|p\rangle$. Since $|p\rangle$ is the state of a particle with momentum p, we must interpret $\langle x|p\rangle$ as the wave function for a particle with momentum p:

$$\langle x|p\rangle = \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}, \tag{14.10.15}$$

where the normalization was adjusted to be compatible with the completeness relations. Indeed, consider the $\langle p'|p\rangle$ overlap and use completeness in x to evaluate it:

$$\begin{aligned}\langle p'|p\rangle &= \int dx \langle p'|x\rangle\langle x|p\rangle = \frac{1}{2\pi\hbar}\int dx e^{i(p-p')x/\hbar} \\ &= \frac{1}{2\pi}\int du\, e^{i(p-p')u} = \delta(p-p'),\end{aligned} \tag{14.10.16}$$

where we let $u = x/\hbar$ and used the integral representation of the delta function obtained from Fourier's theorem in (4.4.5).

We can now ask: What is $\langle p|\psi\rangle$? The answer is quickly obtained by computation:

$$\langle p|\psi\rangle = \int dx \langle p|x\rangle\langle x|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int dx e^{-ipx/\hbar}\psi(x) = \tilde{\psi}(p), \tag{14.10.17}$$

which is the Fourier transform of $\psi(x)$, as defined in (4.4.12). Thus, the *Fourier transform of* $\psi(x)$ *is the wave function in the momentum representation*.

It is often necessary to evaluate $\langle x|\hat{p}\,|\psi\rangle$. This is, by definition, the wave function of the state $\hat{p}\,|\psi\rangle$. We would expect it to equal the familiar action of the momentum operator on the wave function for $|\psi\rangle$. There is no need to speculate, because we can calculate this matrix element with the rules defined so far. We do so by inserting a complete set of momentum states:

$$\langle x|\hat{p}\,|\psi\rangle = \int dp\,\langle x|p\rangle\langle p|\hat{p}\,|\psi\rangle = \int dp\,(p\langle x|p\rangle)\langle p|\psi\rangle. \tag{14.10.18}$$

Now we notice that

$$p\langle x|p\rangle = \frac{\hbar}{i}\frac{d}{dx}\langle x|p\rangle, \tag{14.10.19}$$

and therefore,

$$\langle x|\hat{p}\,|\psi\rangle = \int dp\left(\frac{\hbar}{i}\frac{d}{dx}\langle x|p\rangle\right)\langle p|\psi\rangle. \tag{14.10.20}$$

The derivative can be moved out of the integral since no other part of the integrand depends on x:

$$\langle x|\,\hat{p}\,|\psi\rangle = \frac{\hbar}{i}\frac{d}{dx}\int dp\,\langle x|p\rangle\langle p|\psi\rangle. \tag{14.10.21}$$

The completeness sum is now trivial and can be discarded to obtain, as anticipated,

$$\boxed{\langle x|\,\hat{p}\,|\psi\rangle = \frac{\hbar}{i}\frac{d}{dx}\langle x|\psi\rangle = \frac{\hbar}{i}\frac{d}{dx}\psi(x).} \tag{14.10.22}$$

Exercise 14.10. *Show that*

$$\langle x|\,\hat{p}^n\,|\psi\rangle = \Big(\frac{\hbar}{i}\frac{d}{dx}\Big)^n\psi(x). \tag{14.10.23}$$

Exercise 14.11. *Show that*

$$\langle p|\,\hat{x}\,|\psi\rangle = i\hbar\frac{d}{dp}\tilde{\psi}(p). \tag{14.10.24}$$

Example 14.15. *Ket version of the Schrödinger equation.*
Given a state $|\psi\rangle$, we defined the wave function $\psi(x) = \langle x|\psi\rangle$. For the time-dependent state $|\Psi, t\rangle$, we define the Schrödinger wave function $\Psi(x, t)$ similarly:

$$\Psi(x,t) \equiv \langle x|\Psi,t\rangle. \tag{14.10.25}$$

In here, the time dependence simply goes along for the ride. Consider the familiar form of the Schrödinger equation for a particle of mass m moving in a one-dimensional potential $V(x, t)$:

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = \Big(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\Big)\Psi(x,t). \tag{14.10.26}$$

Since the bra $\langle x|$ is time independent, (14.10.25) implies that the left-hand side of the above equation can be written as follows:

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = \langle x|i\hbar\frac{\partial}{\partial t}|\Psi,t\rangle. \tag{14.10.27}$$

Similarly, using (14.10.23) for $n = 2$, we find that

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) = \langle x|\,\frac{\hat{p}^2}{2m}|\Psi,t\rangle. \tag{14.10.28}$$

Finally,

$$V(x,t)\Psi(x,t) = \langle x|V(\hat{x},t)|\Psi,t\rangle. \tag{14.10.29}$$

It follows from the last three equations that the Schrödinger equation (14.10.26) can be rewritten as

$$\langle x|\, i\hbar\frac{\partial}{\partial t}|\Psi,t\rangle = \langle x|\Big(\frac{\hat{p}^2}{2m} + V(\hat{x},t)\Big)|\Psi,t\rangle. \tag{14.10.30}$$

Since this holds for arbitrary bra $\langle x|$, it follows that we have the ket equality:

$$i\hbar\frac{\partial}{\partial t}|\Psi,t\rangle = \Big(\frac{\hat{p}^2}{2m} + V(\hat{x},t)\Big)|\Psi,t\rangle. \tag{14.10.31}$$

Identifying the operator on the right-hand side as the Hamiltonian $\hat{H}$, we get

$$i\hbar\frac{\partial}{\partial t}|\Psi,t\rangle = \hat{H}\,|\Psi,t\rangle, \tag{14.10.32}$$

which is the "ket" version of the Schrödinger equation.

□

Example 14.16. *Harmonic oscillator in bra-ket notation.*
The harmonic oscillator is a quantum system with a countable set of basis states: the energy eigenstates $\varphi_n(x)$ with $n = 0, 1, \ldots$. Since these wave functions have x dependence, it is natural to define kets $|n\rangle$ representing energy eigenstates and identify the wave functions as overlaps with the position states:

$$\varphi_n(x) = \langle x|n\rangle. \tag{14.10.33}$$

In particular, the ground state is now called $|0\rangle$, and we have

$$\varphi_0(x) = \langle x|0\rangle. \tag{14.10.34}$$

Do not confuse the oscillator ground state $|0\rangle$ with the zero vector or with a state of zero energy! The wave function $\varphi_0(x)$ arises from the condition $\hat{a}|0\rangle = 0$. To get a differential equation for φ_0, we act on $\hat{a}|0\rangle = 0$ with the position bra $\langle x|$:

$$\langle x|\hat{a}|0\rangle = 0 \quad \Rightarrow \quad \langle x|\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)|0\rangle = 0. \tag{14.10.35}$$

Using the identity (14.10.22) to turn $\hat{p}$ into a differential operator, we find that

$$\left(x + \frac{i}{m\omega}\frac{\hbar}{i}\frac{d}{dx}\right)\varphi_0(x) = 0 \Rightarrow \left(\frac{d}{dx} + \frac{x}{L_0^2}\right)\varphi_0 = 0, \tag{14.10.36}$$

where L_0 is the familiar oscillator size. The solution is indeed $\varphi_0(x) = N_0 \exp(-\frac{x^2}{2L_0^2})$, with $N_0^2 = \frac{1}{\sqrt{\pi}}\frac{1}{L_0}$ for unit normalization. In the new notation, the formula $\varphi_n = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n\varphi_0$, derived in section 9.4, takes the form

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle. \tag{14.10.37}$$

These states are eigenstates of the number operator with eigenvalue n: $\hat{N}|n\rangle = n|n\rangle$. Recall also that the Hamiltonian is $\hat{H} = \hbar\omega(\hat{N} + \frac{1}{2})$. The action of creation and annihilation operators on the energy eigenstates was determined in (9.4.26). In the new notation,

$$\begin{aligned} \hat{a}^\dagger|n\rangle &= \sqrt{n+1}\,|n+1\rangle \\ \hat{a}\,|n\rangle &= \sqrt{n}\;|n-1\rangle. \end{aligned} \tag{14.10.38}$$

The matrix elements of an operator $\Box$ are rewritten as $\langle\varphi_m, \Box\varphi_n\rangle = \langle m|\Box|n\rangle$. □

Problems

Problem 14.1. *Schwarz inequality and triangle inequality.*

1. For real vector spaces, the dot product satisfies the Schwarz inequality $(\mathbf{a}\cdot\mathbf{b})^2 \le (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b})$. Prove this inequality as follows. Consider the vector $\mathbf{a} - \lambda\mathbf{b}$, with λ a real constant. Note that

$$f(\lambda) \equiv (\mathbf{a} - \lambda\mathbf{b})\cdot(\mathbf{a} - \lambda\mathbf{b}) \ge 0 \tag{14.10.39}$$

for all λ, and therefore the minimum over λ is still nonnegative: $\min_\lambda f(\lambda) \ge 0$. When is the Schwarz inequality saturated?

2. For a complex vector space, the Schwarz inequality reads $|\langle a, b\rangle| \leq \|a\|\|b\|$, with the norm defined by $\|a\|^2 = \langle a, a\rangle$. Prove this inequality using the vector $v(\lambda) \equiv a - \lambda b$, with λ a *complex* constant and noting that

$$f(\lambda) \equiv \langle v(\lambda), v(\lambda)\rangle \geq 0, \tag{14.10.40}$$

for all λ so that the minimum of $f(\lambda)$ over λ is nonnegative. When is the Schwarz inequality saturated? [Hint: To minimize over a complex variable (such as λ), one must vary the real and imaginary parts. Equivalently, show that you can treat λ and λ^* as if they were independent variables in the sense of partial derivatives. Confirm that since $f(\lambda)$ is real, the stationary condition for λ is equivalent to the stationary condition for λ^*.]

3. For a complex vector space, one has the *triangle inequality*

$$\|a+b\| \leq \|a\| + \|b\| . \tag{14.10.41}$$

Prove this inequality starting from the expansion of $\|a+b\|^2$. You will have to use the property $|\mathrm{Re}(z)| \leq |z|$, which holds for any complex number z, as well as the Schwarz inequality. Show that the triangle inequality is saturated if and only if $a = cb$ for c, a *real* positive constant.

Problem 14.2. *Overlap of two spin one-half states.*

Consider a spin state $|\mathbf{n}\rangle$ where $\mathbf{n}$ is the unit vector defined by the polar and azimuthal angles θ and ϕ and the spin state $|\mathbf{n}'\rangle$ where $\mathbf{n}'$ is the unit vector defined by the polar and azimuthal angles θ' and ϕ'. Let γ denote the angle between the vectors $\mathbf{n}$ and $\mathbf{n}'$: $\mathbf{n}\cdot\mathbf{n}' = \cos\gamma$. Show by direct computation that the overlap of the associated spin states is controlled by *half* the angle between the unit vectors:

$$|\langle\mathbf{n}'|\mathbf{n}\rangle|^2 = \cos^2\tfrac{\gamma}{2} = \tfrac{1}{2}(1+\mathbf{n}\cdot\mathbf{n}'). \tag{14.10.42}$$

Problem 14.3. *Orthogonal projections and approximations (Axler).*

Consider a vector space V with an inner product and a subspace U of V. The question is: Given a vector $v \in V$ that is not in U, what is the vector in

U that best approximates v? As we also have a norm, we can ask a more precise question: What is the vector $u \in U$ for which $|v - u|$ is smallest? The answer is nice and simple: the vector u is given by $P_U v$, the orthogonal projection of v to U!

1. Prove the above claim by showing that for any $u \in U$ one has

$$|v - u| \ \geq |v - P_U v|. \tag{14.10.43}$$

As an application consider the infinite-dimensional vector space of real functions in the interval $x \in [-1, 1]$. The inner product of two functions f and g on this interval is taken to be

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) dx. \tag{14.10.44}$$

Take U to be the six-dimensional subspace of functions spanned by $\{1, x, x^2, x^3, x^4, x^5\}$. (For this problem please use an algebraic manipulator that does integrals.)

2. Use the Gram-Schmidt algorithm to find an orthonormal basis $(e_1, \ldots, e_6)$ for U.
3. Consider approximating the functions $\sin \pi x$ and $\cos \pi x$ with the best possible representatives from U. Calculate exactly these two representatives and write them as polynomials in x with coefficients that depend on powers of π and other constants. Also write the polynomials using numerical coefficients with six significant digits.
4. Do a plot for each of the functions ($\sin \pi x$ and $\cos \pi x$) where you show the original function, its best approximation in U calculated above, and the approximation in U that corresponds to the truncated Taylor expansion about $x = 0$.

The polynomials you found in (2) are in fact proportional to the Legendre polynomials, which are orthogonal but normalized differently.

Problem 14.4. *Elaborations on a theorem.*

Theorem 14.5.1 states that for complex vector spaces, the condition $\langle v, Tv \rangle = 0$ for all $v \in V$ implies that $T = 0$. The result does not hold for real vector spaces. To distinguish the two cases, we consider separate conditions:

$$\text{Real case: } \langle u, Su \rangle = 0, \text{ for all } u, \quad \text{complex case: } \langle v, Tv \rangle = 0, \text{ for all } v. \tag{14.10.45}$$

We first examine the case of dimension two to see why the theorem is true and why it fails for real vector spaces. Then we extend to higher dimensions.

1. Let S be represented by a real 2×2 matrix S_{ij} and u by two real components u_i, with $i, j = 1, 2$. Similarly, let T be represented by a 2×2 matrix T_{ij} with complex entries and v by two complex components v_i, with $i, j = 1, 2$. Write out the quadratic forms and then apply the conditions under which they vanish for all u and v, respectively. Show that $T_{ij} = 0$. For what kind of matrices S does the vanishing of $\langle u, Su \rangle$ imply the vanishing of S?
2. Extend your argument to arbitrary size matrices, showing that $T_{ij} = 0$ and stating for what kind of matrices S the theorem holds in the real case.
3. Consider a complex vector space and an arbitrary linear operator. A basis can be shown to exist for which the matrix representing the operator has an upper-triangular form (the elements below the diagonal vanish). In light of the above analysis, explain why the same does not hold for arbitrary linear operators on real vector spaces.

Problem 14.5. *Another characterization of orthogonal projectors.*

Consider a vector space V and a linear operator P that satisfies the equation $P^2 = P$. Theorem 14.5.4 demonstrates that this implies $V = \text{null } P \oplus \text{range } P$. This is not enough, however, to show that P is an orthogonal projector. Show that orthogonality is guaranteed if

$$|Pv| \le |v| \quad \text{for all } v \in V. \tag{14.10.46}$$

You may find it useful to prove first the following characterization of orthogonal vectors: Let $u, v \in V$. Then $\langle u, v \rangle = 0$ if and only if $|u| \le |u + av|$ for any constant a.

Problem 14.6. *Rotation matrix for vectors.*

Given a vector $\mathbf{u}$ in $\mathbb{R}^3$, the rotation matrix $\mathcal{R}_{\mathbf{n}}(\alpha)$ acting on the vector is supposed to give us the result of rotating $\mathbf{u}$ by an angle α about an axis oriented along $\mathbf{n}$. We want to show that, as indicated in (14.7.6),

$$\mathcal{R}_{\mathbf{n}}(\alpha)\,\mathbf{u} = (1-\cos\alpha)\,(\mathbf{n}\cdot\mathbf{u})\,\mathbf{n} + (\cos\alpha)\,\mathbf{u} + (\sin\alpha)\,(\mathbf{n}\times\mathbf{u}). \tag{1}$$

We can denote the rotated vector by $\mathbf{u}(\alpha) = \mathcal{R}_{\mathbf{n}}(\alpha)\ \mathbf{u}$, with $\mathbf{u} = \mathbf{u}(0)$. As discussed, arising by rotation means that it satisfies the differential equation

$$\frac{d\mathbf{u}(\alpha)}{d\alpha} = \mathbf{n}\times\mathbf{u}(\alpha). \tag{2}$$

1. Verify the correctness of (1) by showing that it satisfies the differential equation (2).
2. Construct the solution (1) directly by inspection of the rotation geometry. For this write the $\alpha = 0$ vector $\mathbf{u}$ as follows:

   $$\mathbf{u} = (\mathbf{u}\cdot\mathbf{n})\mathbf{n} + \mathbf{u}_\perp,\quad \mathbf{u}_\perp = \mathbf{u} - (\mathbf{u}\cdot\mathbf{n})\,\mathbf{n}.$$

   Note that as α starts to differ from zero it is $\mathbf{u}_\perp$ that begins changing in time by rotating in a plane spanned by $\mathbf{u}_\perp$ and $\mathbf{n}\times\mathbf{u}_\perp$.
3. Use index notation to describe the rotation as $u_i(\alpha) = \mathcal{R}_{\mathbf{n}}(\alpha)_{ij}u_j$ with

   $$\mathcal{R}_{\mathbf{n}}(\alpha)_{ij} = (1-\cos\alpha)n_i n_j + \cos\alpha\,\delta_{ij} + \sin\alpha\,\epsilon_{ikj}n_k.$$

   Use this to prove the rotational invariance (14.7.8) of the dot product.
4. Confirm explicitly the composition rule (14.7.7) for rotations.

Problem 14.7. *Sum rules and the quantum virial theorem.*

Consider the Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$ for a one-dimensional quantum system. Assume $\hat{H}$ has a discrete set of eigenfunctions: $\hat{H}|a\rangle = E_a|a\rangle$, with a running over some set of values.

1. Prove the Thomas-Reiche-Kuhn sum rule: $\sum_{a'}\left|\langle a|\hat{x}|a'\rangle\right|^2(E_{a'}-E_a) = \frac{\hbar^2}{2m}$. (Hint: consider $[[\hat{x}, \hat{H}], \hat{x}]$.)
2. Show that $\langle a|\hat{p}|a'\rangle = \frac{im}{\hbar}(E_a - E_{a'})\langle a|\hat{x}|a'\rangle$. (Hint: consider $[\hat{H}, \hat{x}]$.) Use this result to prove the energy-weighted sum rule:

   $$\sum_{a'}\left|\langle a|\hat{x}|a'\rangle\right|^2(E_a - E_{a'})^2 = \frac{\hbar^2}{m^2}\langle a|\hat{p}^2|a\rangle.$$

3. Consider the commutator $[\hat{x}\hat{p}\,,\,\hat{H}]$ to show that

$$2\langle a|\frac{\hat{p}^2}{2m}|a\rangle = \langle a|\hat{x}\partial_{\hat{x}}V(\hat{x})|a\rangle.$$

This is, in fact, the quantum-mechanical virial theorem, usually stated as $2\langle\hat{T}\rangle = \langle x\frac{dV}{dx}\rangle$, where $\hat{T}$ denotes kinetic energy, and the expectation values are for a stationary state. Write the resulting relation between expectation values of the kinetic and potential energy when $V(x) = \alpha x^n$.

Problem 14.8. *Exercises on the one-dimensional harmonic oscillator.*

1. Show that a state of the oscillator with a negative number has a negative norm squared. This means such a state is inconsistent.
2. We showed that the ground energy eigenstate $|0\rangle$ is the unique state annihilated by the lowering operator $\hat{a}$.
   - Show algebraically that the excited states of the oscillator are nondegenerate by showing that a degeneracy would imply a degeneracy of the ground state.
   - Show that the existence of a state with a positive but fractional number implies the existence of states of negative norm squared.
3. At $t = 0$ a particle in the harmonic oscillator is in the superposition $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. Find the time-dependent expectation values $\langle\hat{x}\rangle(t)$ and $\langle\hat{p}\rangle(t)$.
4. Consider a normalized state $|\lambda\rangle$ in the harmonic oscillator satisfying

$$\hat{a}|\lambda\rangle = \lambda\,|\lambda\rangle, \quad \lambda \in \mathbb{C}, \tag{1}$$

   where $\hat{a}$ is the annihilation operator, and λ is a complex constant. Note that the states above are coherent states, as discussed in example 13.21, with $\psi_\lambda(x) = \langle x|\lambda\rangle$. Calculate both the expectation value $\langle\hat{H}\rangle$ of the harmonic oscillator Hamiltonian and the energy uncertainty ΔH in the $|\lambda\rangle$ state. Equation (1) is the only property of the states needed for these computations.

Problem 14.9. *Parity operator and oscillator states.*

Let P denote a parity operator defined by its action on position eigenstates:

$$P : |x\rangle \to |-x\rangle, \quad \text{for all } x \in \mathbb{R}.$$

1. Given a state $|\psi\rangle$ with position space wave function $\psi(x)$, what is the wave function associated with the state $P|\psi\rangle$? What does P give when acting on the ground state of the harmonic oscillator?
2. Show that $P\,\hat{x} = -\hat{x}P$ and $P\,\hat{p} = -\hat{p}\,P$, where $\hat{x}$ and $\hat{p}$ are the position and momentum operators, respectively. Conclude that $P\,\hat{a}^\dagger = -\hat{a}^\dagger P$, where $\hat{a}^\dagger$ is the harmonic oscillator creation operator.
3. Show that the energy eigenstate $|n\rangle$ of the harmonic oscillator satisfies $P|n\rangle = (-1)^n|n\rangle$. What does this imply for the associated wave function $\varphi_n(x) = \langle x|n\rangle$?