16

Pictures of Quantum Mechanics

We explain the postulate of unitary time evolution and use it to derive the Schrödinger equation. In the Schrödinger picture of quantum mechanics, physical states evolve in time, and we have a set of fundamental time-independent operators. We introduce the Heisenberg picture of quantum mechanics in which we define time-dependent Heisenberg operators associated to Schrödinger operators. In this picture the dynamics of Schrödinger states is captured by the time-dependent operators, and Heisenberg states are taken to be time independent. We then turn to the axioms of quantum mechanics, defining the states of the system, observables, measurement, and dynamics. Two additional postulates explain how to build composite systems and how to treat identical particles.

16.1 Schrödinger Picture and Unitary Time Evolution

The state space of quantum mechanics—the Hilbert space $\mathcal{H}$ of states—is best thought of as a space with time-independent basis vectors. There is no role for time in the definition of the state space $\mathcal{H}$. In the Schrödinger "picture" of the dynamics, the state that represents a quantum system depends on time. Time is viewed as a parameter: at different times the state of the system is represented by different states in the Hilbert space. We write the state vector as

$$|\Psi, t\rangle, \tag{16.1.1}$$

and its components along the basis vectors of $\mathcal{H}$ are time dependent. If we call those basis vectors $|u_i\rangle$, we write

$$|\Psi, t\rangle = \sum_i |u_i\rangle c_i(t), \tag{16.1.2}$$

where the $c_i(t)$ are some functions of time. Assuming the state is normalized, which is almost always convenient, we can imagine $|\Psi, t\rangle$ as a unit vector whose tip, as a function of time, sweeps a trajectory in $\mathcal{H}$. We will discuss the postulate of unitary time evolution in this section and then show that the Schrödinger equation follows from this postulate.

We declare that for any quantum system there is a *unitary* operator $\mathcal{U}(t, t_0)$ such that for *any* state $|\Psi, t_0\rangle$ of the system at time t_0 the state at time t is

$$\boxed{|\Psi, t\rangle = \mathcal{U}(t, t_0)|\Psi, t_0\rangle, \quad \forall\, t, t_0.} \tag{16.1.3}$$

It must be emphasized that the operator $\mathcal{U}$ generates time evolution for *any* possible state at time t_0—it does *not* depend on the chosen state at time t_0. A physical system has a single operator $\mathcal{U}$ that generates the time evolution of all possible states. The above equation is valid for all times t, so t can be greater than, equal to, or less than t_0. As defined, the operator $\mathcal{U}$ is unique: if there is another operator $\mathcal{U}'$ that generates exactly the same evolution, then $(\mathcal{U}-\mathcal{U}')|\Psi, t_0\rangle = 0$, and since the state $|\Psi, t_0\rangle$ is arbitrary, the operator $\mathcal{U}-\mathcal{U}'$ vanishes, showing that $\mathcal{U} = \mathcal{U}'$.

The unitary property of $\mathcal{U}$ means that its Hermitian conjugate is its inverse:

$$(\mathcal{U}(t, t_0))^\dagger \mathcal{U}(t, t_0) = \mathbb{1}. \tag{16.1.4}$$

In order to avoid extra parentheses, we will write

$$\mathcal{U}^\dagger(t, t_0) \equiv (\mathcal{U}(t, t_0))^\dagger \tag{16.1.5}$$

so the unitarity property reads

$$\mathcal{U}^\dagger(t, t_0)\mathcal{U}(t, t_0) = \mathbb{1}. \tag{16.1.6}$$

The unitarity of □ also implies that the norm of the state is conserved by time evolution:

$$\langle \Psi, t \,|\Psi, t\rangle \;=\; \langle \Psi, t_0 \,|\mathcal{U}^\dagger(t, t_0)\mathcal{U}(t, t_0)|\Psi, t_0\rangle \;=\; \langle \Psi, t_0|\, \Psi, t_0\rangle. \tag{16.1.7}$$

This is illustrated in figure 16.1.

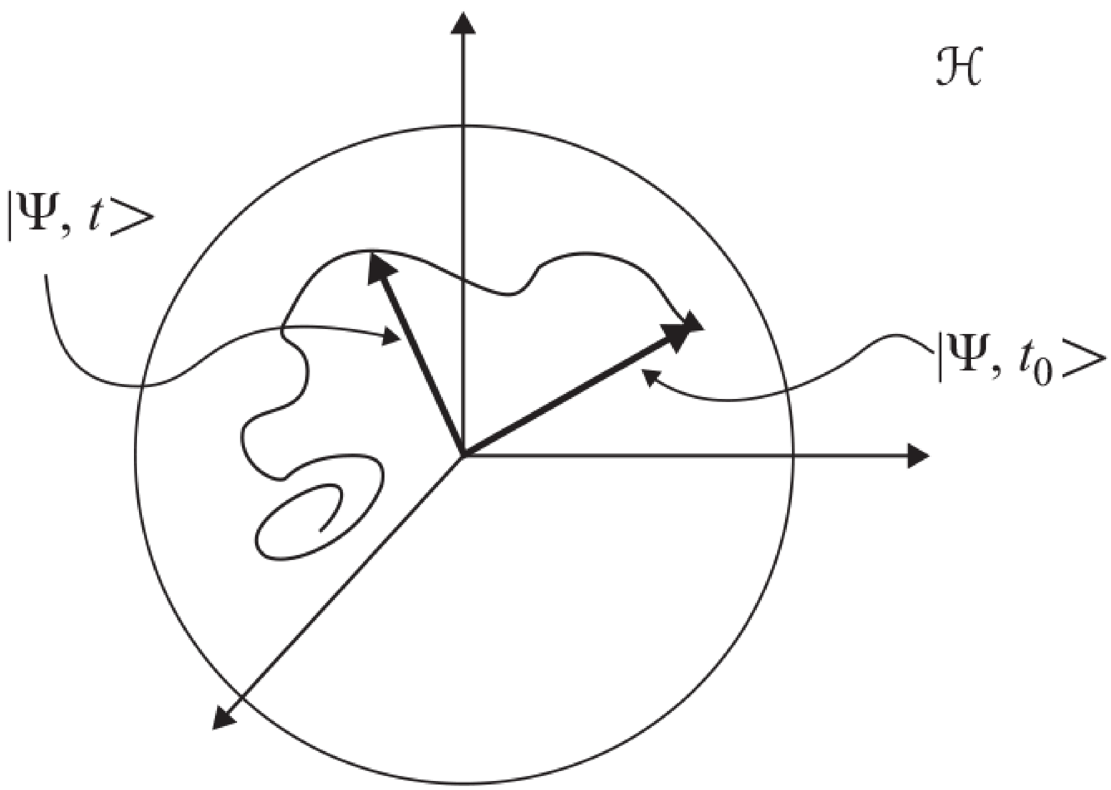

Figure 16.1
The initial state $|\Psi, t_0\rangle$ can be viewed as a vector in the complex vector space $\mathcal{H}$. As time goes by the vector moves, evolving by unitary transformations, so that its norm is preserved.

Remarks:

1. For time $t = t_0$, equation (16.1.3) gives no time evolution:

$$|\Psi, t_0\rangle \;=\; \mathcal{U}(t_0, t_0)|\Psi, t_0\rangle. \tag{16.1.8}$$

Since this equality holds for *any* possible state at $t = t_0$, the unitary evolution operator with equal time arguments must be the identity operator:

$$\mathcal{U}(t_0, t_0) \;=\; \mathbb{1}, \;\; \forall t_0. \tag{16.1.9}$$

2. Composition: Consider the evolution from t_0 to t_2 as a two-step procedure, from t_0 to t_1 first, followed by evolution from t_1 to t_2:

$$|\Psi, t_2\rangle = \mathcal{U}(t_2, t_1)|\Psi, t_1\rangle = \mathcal{U}(t_2, t_1)\mathcal{U}(t_1, t_0)|\Psi, t_0\rangle. \tag{16.1.10}$$

This equation and $|\Psi, t_2\rangle = \mathcal{U}(t_2, t_0)|\Psi, t_0\rangle$ imply that $\mathcal{U}$ composes as follows:

$$\mathcal{U}(t_2, t_0) = \mathcal{U}(t_2, t_1)\mathcal{U}(t_1, t_0). \tag{16.1.11}$$

3. Inverses: Consider the above composition law (16.1.11), and set $t_2 = t_0$ and $t_1 = t$. Then using (16.1.9), we get

$$\mathbb{1} = \mathcal{U}(t_0, t)\mathcal{U}(t, t_0). \tag{16.1.12}$$

We then have

$$\mathcal{U}(t_0, t) = (\mathcal{U}(t, t_0))^{-1} = (\mathcal{U}(t, t_0))^{\dagger}, \tag{16.1.13}$$

where the first relation follows from (16.1.12) and the second by unitarity. Again, declining to use parentheses that are not really needed, we write

$$\boxed{\mathcal{U}(t_0, t) = \mathcal{U}^{-1}(t, t_0) = \mathcal{U}^{\dagger}(t, t_0).} \tag{16.1.14}$$

Simply said, inverses or the Hermitian conjugation of $\mathcal{U}$ reverses the order of the time arguments.