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13.2 Subspaces, Direct Sums, and Dimensionality

To better understand a vector space, one can try to figure out its possible subspaces. A **subspace** of a vector space V is a subset of V that is also a vector space. To verify that a subset U of V is a subspace, you must check that U contains the vector 0 and that U is closed under addition and scalar multiplication. All other properties required by the axioms for U to be a vector space are automatically satisfied because U is contained in V (think about this!).

Example 13.6. *Subspaces of* $\mathbb{R}_2$.
Let $V = \mathbb{R}^2$ so that elements of V are pairs (v_1, v_2) with $v_1, v_2 \in \mathbb{R}$. Now introduce the subsets W_r defined by a real number r:

$$W_r \equiv \{(v_1, v_2) \big| \; 3v_1 + 4v_2 = r, \text{ with } r \in \mathbb{R}\}. \tag{13.2.1}$$

When is W_r a subspace of $\mathbb{R}$? Since we need the zero vector (0, 0) to be contained, this requires $3 \cdot 0 + 4 \cdot 0 = r$ or $r = 0$. Indeed, one can readily verify that W_0 is closed under addition and scalar multiplication and is therefore a subspace of V.

It is possible to visualize all nontrivial subspaces of $\mathbb{R}^2$. These are the lines that go through the origin. Each line is a vector space: it contains the zero vector (the origin), and all vectors defined by points on the line can be added or multiplied to find vectors on the same line.

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Exercise 13.1. *Let U_1 and U_2 be two subspaces of V. Is $U_1 \cap U_2$ a subspace of V?*

To understand a complicated vector space, it is useful to consider subspaces that together build up the space. Let $U_1, \ldots, U_m$ be a collection of subspaces of V. We say that the space V is the **direct sum of the subspaces** $U_1, \ldots, U_m$ and we write

$$V = U_1 \oplus \cdots \oplus U_m \tag{13.2.2}$$

if any vector in V can be written *uniquely* as the sum

$$u_1 + \cdots + u_m, \quad \text{where} \quad u_i \in U_i. \tag{13.2.3}$$

This can be viewed as a decomposition of any vector into a sum of vectors, one in each of the subspaces. Part of the intuition here is that while the set of all subspaces fills the whole space, the various subspaces cannot overlap. More precisely, their only common element is zero: $U_i \cap U_j = \{0\}$ for $i \neq j$. If this is violated, the decomposition of vectors in V would not be unique. Indeed, for if some vector $v \in U_i \cap U_j$ ($i \neq j$) then also $-v \in U_i \cap U_j$ (why?), and therefore letting $u_i \to u_i + v$ and $u_j \to u_j - v$ would leave the total sum unchanged, making the decomposition nonunique. The condition of zero mutual overlaps is necessary for the uniqueness of the decomposition, but it is not in general sufficient. It suffices, however, when we have two summands: to show that $V = U \oplus W$, one must prove that any vector can be written as $u + w$ with $u \in U$ and $w \in W$ and that $U \cap W = 0$. In general, uniqueness of the sum in (13.2.3) follows if the only way to write 0 as a sum $u_1 + \cdots + u_m$ with $u_i \in U_i$ is by taking all u_i's equal to zero. Direct sum decompositions appear rather naturally when we consider the addition of angular momentum.

Given a vector space, we can produce lists of vectors. A **list** $(v_1, \ldots, v_n)$ of vectors in V contains, by definition, a *finite* number of vectors. The number of vectors in a list is the length of the list. The **span** of a list of vectors $(v_1, \ldots, v_n)$ in V, denoted as $\text{span}(v_1, \ldots, v_n)$, is the set of all linear combinations of these vectors:

$$a_1 v_1 + \cdots + a_n v_n, \qquad a_i \in \mathbb{F}. \tag{13.2.4}$$

A vector space V is spanned by a list $(v_1, \ldots, v_n)$ if $V = \text{span}(v_1, \ldots, v_n)$.

Now comes a very natural definition: A vector space V is said to be **finite-dimensional** if it is spanned by some list of vectors in V. If V is not finite-dimensional, it is **infinite-dimensional**. In such a case, no list of vectors from V can span V. Note that by definition, any finite-dimensional vector space has a spanning list.

Let us explain why the vector space of all polynomials $p(z)$ in example 13.1, item 4, is an infinite-dimensional vector space. Indeed, consider any list of polynomials. Since a list is always of finite length, there is a polynomial of maximum degree in the list. Thus, polynomials of higher degree are not in the span of the list. Since no list can span the space, it is infinite-dimensional.

For example 13.1, item 1, consider the list of vectors $(e_1, \ldots, e_N)$ with

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots \quad e_N = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}. \tag{13.2.5}$$

This list spans the space: the general vector displayed in (13.1.1) is $a_1 e_1 + \cdots + a_N e_N$. This vector space is therefore finite-dimensional.

To make further progress, we need the concept of linear independence. A list of vectors $(v_1, \ldots, v_n)$ with $v_i \in V$ is said to be **linearly independent** if the equation

$$a_1 v_1 + \cdots + a_n v_n = 0 \tag{13.2.6}$$

only has the solution $a_1 = \cdots = a_n = 0$. One can prove a key result: *the length of any linearly independent list is less than or equal to the length of any spanning list*. This is reasonable, as we discuss now. Spanning lists can be enlarged as much as desired because adding vectors to a spanning list still gives a spanning list. They cannot be reduced arbitrarily, however, because at some point the remaining vectors will fail to span. For linearly independent lists, the situation is exactly reversed: they can be easily shortened because dropping vectors will not disturb the linear independence but cannot be enlarged arbitrarily because at some point the new vectors can be expressed in terms of those already in the list. As it happens, in a finite vector space the length of the longest list of linearly independent vectors is the same as the length of the shortest list of spanning vectors. This leads to the concept of dimensionality, as we will see below.

We can now explain what a basis for a vector space is. A **basis** of V is a list of vectors in V that both spans V and is linearly independent. It is not hard to prove that any finite-dimensional vector space has a basis. While bases are not unique, all bases of a finite-dimensional vector space have the same length. The **dimension** of a finite-dimensional vector space is equal to the length of any list of basis vectors. If V is a space of dimension n, we write $\dim V = n$. It is also true that for a finite-dimensional vector space a list of vectors of length $\dim V$ is a basis if it is a linearly independent list or if it is a spanning list.

The list $(e_1, \ldots, e_N)$ in (13.2.5) is not only a spanning list but a linearly independent list (prove it!). Thus, the dimensionality of the space is N.

Exercise 13.2. *Explain why the vector space in example 13.1, item 2, has dimension $M \cdot N$.*

The vector space $\mathbb{F}^\infty$ of infinite sequences in example 13.1, item 5, is infinite-dimensional, as we now justify. Assume $\mathbb{F}^\infty$ is finite-dimensional, in which case it has a spanning list of some length n. Define s_k as the element in $\mathbb{F}^\infty$ with a one in the kth position and zero elsewhere. The list $(s_1, \ldots, s_m)$ is clearly a linearly independent list of length m, with m arbitrary. Choosing $m > n$, we have a linearly independent list longer than a spanning list. This is a contradiction, and therefore $\mathbb{F}^\infty$ cannot be finite-dimensional. Recall this is the state space of the square well. The space of complex functions on the interval $[0, L]$ (example 13.1, item 5) is also infinite-dimensional.

Equipped with the concept of dimensionality, there is a simple way to see if we have a direct sum decomposition of a vector space. In fact, we have $V = U_1 \oplus \cdots \oplus U_m$ if any vector in V can be written as $u_1 + \cdots + u_m$, with $u_i \in U_i$ and if $\dim U_1 + \cdots + \dim U_m = \dim V$. The proof of this result is not complicated.

Example 13.7. *The real vector space of 2 × 2 Hermitian matrices.*
Consider example 13.1, item 3, and focus on the case of the vector space of 2 × 2 Hermitian matrices. Recall that the most general Hermitian 2 × 2 matrix takes the form

$$\begin{pmatrix} a_0 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & a_0 - a_3 \end{pmatrix}, \qquad a_0, a_1, a_2, a_3 \in \mathbb{R}. \tag{13.2.7}$$

Now consider the following list of four "vectors," $(\mathbb{1}, \sigma_1, \sigma_2, \sigma_3)$, with σ_i the Pauli matrices (12.1.20) and $\mathbb{1}$ the 2 × 2 identity matrix. All entries in this list are Hermitian matrices, so this is a list of vectors in the space. Moreover, the list spans the space since the general Hermitian matrix shown above is $a_0\mathbb{1} + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3$. The list is linearly independent since

$$a_0\mathbb{1} + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 = 0 \quad \Rightarrow \quad \begin{pmatrix} a_0 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & a_0 - a_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{13.2.8}$$

and you can quickly see that this implies that a_0, a_1, a_2, and a_3 are all zero. So the list is a basis, and the space in question is a four-dimensional real vector space.

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Example 13.8. *State space $\mathcal{H}$ of the simple harmonic oscillator.*
The energy eigenstates of the harmonic oscillator can be used to give a direct sum representation of the state space $\mathcal{H}$ for the harmonic oscillator. Let U_n be the one-dimensional subspace that is the span of the energy eigenstate φ_n of the oscillator (see chapter 9). The state φ_n is an eigenstate of the number operator $\hat{N}$ with eigenvalue n:

$$U_n \equiv \{\alpha\varphi_n,\ \alpha \in \mathbb{C},\ \hat{N}\varphi_n = n\varphi_n\}. \tag{13.2.9}$$

The space U_n is an $\hat{N}$-invariant subspace of $\mathcal{H}$. Since any state of the oscillator can be written uniquely as a sum of energy eigenstates, we have the direct sum decomposition:

$$\mathcal{H} = \bigoplus_{n=1}^{\infty} U_n = U_0 \oplus U_1 \oplus U_2 \oplus \cdots . \tag{13.2.10}$$

The space $\mathcal{H}$, of course, is infinite-dimensional, the direct sum of an infinite countable set of one-dimensional subspaces. A little care is needed here to describe $\mathcal{H}$ precisely. While it is clear that any harmonic oscillator state can be written uniquely as a sum of energy eigenstates, not all sums of energy eigenstates correspond to physical states of the harmonic oscillator. Finite linear combinations always do; these are the Fock states considered in example 13.4. Some but not all infinite linear combinations also give physical states. Physical states must be normalizable.

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Example 13.9. *State space of hydrogen atom bound states.*
The bound states of the hydrogen atom span an important subspace $\mathcal{H}$ of this quantum system. The bound state spectrum was determined in section 11.3, with energy levels indexed by the principal quantum number $n = 1, 2, \ldots$. Calling $\mathcal{H}_n$ the vector subspace spanned by the degenerate energy eigenstates at principal quantum number n, we see that

$$\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots . \tag{13.2.11}$$

Ignoring the spin of the electron, $\dim \mathcal{H}_n = n^2$. We can refine the description by giving a direct sum decomposition of $\mathcal{H}_n$. In fact, for any fixed n the orbital angular momentum runs from $\ell = 0$ to $\ell = n - 1$, a total of n angular momentum multiplets. We write this as

$$\mathcal{H}_n = \bigoplus_{\ell=0}^{n-1} \mathcal{H}_{n,\ell} = \mathcal{H}_{n,0} \oplus \cdots \oplus \mathcal{H}_{n,n-1}. \tag{13.2.12}$$

The space $\mathcal{H}_{n,\ell}$ is simply a vector space of states with angular momentum ℓ and principal quantum number n. It has dimension $2\ell + 1$, and on account of example 13.3, $\mathcal{H}_{n,\ell} = \mathbb{C}^{2\ell+1}$. The hydrogen atom spectrum is special in that it has a large amount of degeneracies: multiplets with different values of ℓ but the same value of n are degenerate.

For a general central potential, the only degeneracies are those within ℓ multiplets. The spectrum of bound states can *always* be organized by angular momentum, and we find that

$$\mathcal{H} = \bigoplus_{\ell=0}^{\infty} \hat{\mathcal{H}}_\ell. \tag{13.2.13}$$

Each $\hat{\mathcal{H}}_\ell$ is a collection of many, perhaps infinitely many, multiplets of angular momentum ℓ. For the hydrogen atom, for example, $\hat{\mathcal{H}}_{\ell=0}$ contains states with $n = 1, 2, \ldots$. More generally, in hydrogen $\hat{\mathcal{H}}_\ell$ has states with principal quantum number $n > \ell$, for all such values of n.

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Example 13.10. *Review of index manipulations.*
Let us review and summarize the basic elements of index manipulation. For any vector $\mathbf{a} = (a_1, a_2, a_3)$, we denote the components with an index so that we have components a_i with index i running over the set $i = 1, 2, 3$. With another vector $\mathbf{b} = (b_1, b_2, b_3)$, the dot product is written as

$$\mathbf{a}\cdot\mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{i=1}^{3} a_i b_i. \tag{13.2.14}$$

It is a useful convention that repeated indices are summed over the values they run over. Thus, in the above, since we have the repeated i, we simply write

$$\mathbf{a}\cdot\mathbf{b} = a_i\, b_i. \tag{13.2.15}$$

A repeated index is sometimes called a *dummy* index, and the particular letter we use for it is immaterial: $a_i b_i = a_k b_k$, for example. In general, there should be no more than two indices with the same label in any expression. A useful symbol is the Kronecker delta δ_{ij}, symmetric in i and j and defined as

$$\delta_{ij} = \begin{cases} 0, \text{ if } i \neq j, \\ 1, \text{ if } i = j. \end{cases} \tag{13.2.16}$$

Note that by the summation convention $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$. Moreover, one often has to simplify $\delta_{ij}B_j$. In fact,

$$\delta_{ij}B_j = B_i. \tag{13.2.17}$$

This holds because as we sum over j, the Kronecker delta vanishes unless j is equal to i, in which case it equals one.

The other important object is the three-index Levi-Civita symbol ϵ_{ijk}. We encountered this object in describing the commutator of angular momentum operators: $[\hat{L}_i, \hat{L}_j] = i\hbar\, \epsilon_{ijk}\, \hat{L}_k$. Since each index can run over three values, in principle this object has $3 \times 3 \times 3 = 27$ values to be specified. But the ϵ symbol is defined as being totally antisymmetric, meaning antisymmetric under the exchange of any pair of indices: $\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}$. This implies that in order for ϵ_{ijk} to be nonvanishing no two indices can have the same value, and therefore i, j, k must be some permutation of 1, 2, 3. We declare that

$$\epsilon_{123} = 1, \tag{13.2.18}$$

and this determines all other cases, such as, for example, $\epsilon_{312} = -\epsilon_{132} = +\epsilon_{123} = +1$. The ϵ symbol is often used to write cross products. As you know, the cross product has components

$$\mathbf{a} \times \mathbf{b} = \left(a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \right), \tag{13.2.19}$$

where the three objects in parentheses are the three components of $(\mathbf{a}\times\mathbf{b})$. We claim that

$$(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk}\, a_jb_k. \tag{13.2.20}$$

We check one case (you do the others!). Take the first component:

$$(\mathbf{a} \times \mathbf{b})_1 = \epsilon_{1jk}a_jb_k = \epsilon_{123}a_2b_3 + \epsilon_{132}a_3b_2 = a_2b_3 - a_3b_2. \tag{13.2.21}$$

The product of two epsilon symbols with one common index satisfies a useful identity:

$$\epsilon_{ijk}\epsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}. \tag{13.2.22}$$

As a simple consistency check, you should verify that the right-hand side, just like the left-hand side, is antisymmetric under the exchange of j and k as well as under the exchange of p and q. A consequence of this identity is a formula for the product of two symbols with two summed indices. For this we set $p = j$, finding that

$$\epsilon_{ijk}\epsilon_{ijq} = \delta_{jj}\delta_{kq} - \delta_{jq}\delta_{kj} = 3\delta_{kq} - \delta_{kq} \quad \Rightarrow \quad \epsilon_{ijk}\epsilon_{ijq} = 2\delta_{kq}. \tag{13.2.23}$$

The classic application of the double epsilon identity is to the simplification of the double cross product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. To do this we calculate its ith component:

$$\begin{aligned}(\mathbf{a} \times (\mathbf{b} \times \mathbf{c}))_i &= \epsilon_{ijk} a_j (\mathbf{b} \times \mathbf{c})_k \\ &= \epsilon_{ijk} a_j \, \epsilon_{kpq} b_p c_q \\ &= \epsilon_{kij}\epsilon_{kpq} \, a_j \, b_p \, c_q \\ &= (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}) \, a_j \, b_p \, c_q \\ &= a_j b_i c_j - a_j b_j c_i = b_i(\mathbf{a} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{b}) c_i.\end{aligned} \tag{13.2.24}$$

From this it follows that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}. \tag{13.2.25}$$

Most identities of vector algebra can be derived using the above methods.

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Exercise13.3. *Write* $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$ *in terms of dot products only.*