15.3 Energy-Time Uncertainty

A more subtle form of the uncertainty relation deals with energy and time. The inequality is sometimes stated vaguely in the form $\Delta E \Delta t \gtrsim \hbar$. In here, there is no problem in defining ΔE precisely; after all, we have the Hamiltonian operator, and its uncertainty ΔH is a perfect candidate for the energy uncertainty. The problem is time. Time is not an operator in quantum mechanics; it is a parameter, a real number used to describe the way systems change. Unless we define an uncertainty Δt precisely, we cannot hope for a well-defined uncertainty relation.

We describe a familiar setting in order to illustrate the spirit of the inequality $\Delta E \Delta t \gtrsim \hbar$. Consider a photon that is detected at some point in space, as a passing oscillatory wave of exact duration Δt. Without any quantum mechanical considerations, we can ask the observer for the value of the angular frequency ω of the pulse. In order to answer our question, the observer will attempt to count the number N of complete oscillations of the waveform that went through. Of course, this number N is given by Δt divided by the period $2\pi/\omega$ of the wave:

$$N = \frac{\omega}{2\pi} \Delta t. \tag{15.3.1}$$

The observer, however, will typically fail to count full waves because as the pulse gets started from zero and later dies off, the waveform may cease to follow the sinusoidal pattern. Thus, we expect an uncertainty $\Delta N \gtrsim 1$. Given the above relation, with Δt known exactly, this implies an uncertainty $\Delta \omega$ in the value of the angular frequency:

$$\Delta\omega\, \Delta t \gtrsim 2\pi. \tag{15.3.2}$$

This is all still classical; the above identity is routinely used by electrical engineers. It represents a limit on the ability to ascertain the frequency of a wave that is observed for a limited amount of time. It becomes quantum mechanical if we speak of a single photon whose energy is $E = \hbar\omega$. Then $\Delta E = \hbar \Delta\omega$ so that multiplying the above inequality by $\hbar$ we get

$$\Delta E\, \Delta t \gtrsim h. \tag{15.3.3}$$

In this uncertainty inequality, Δt is the duration of the pulse. It is a reasonable relation, but the presence of $\gtrsim$ betrays its lack of precision.

Russian physicists Leonid Mandelstam and Igor Tamm found a way out shortly after the formulation of the uncertainty principle. Consider a time-independent Hermitian operator $\hat{Q}$ that measures "Q-ness." We can find a precise energy-Q uncertainty inequality by applying the uncertainty inequality to the Hamiltonian $\hat{H}$ and $\hat{Q}$, finding

$$\Delta H\, \Delta Q \geq \left| \langle \Psi | \frac{1}{2i} [\hat{H}, \hat{Q}] | \Psi \rangle \right|. \tag{15.3.4}$$

The right-hand side is interesting because the expectation value of the commutator $[\hat{H}, \hat{Q}]$ controls the time dependence of the expectation value of $\hat{Q}$. Since $\hat{Q}$ is time independent, the time dependence of the expectation value originates from the time dependence of the states. The relevant equation was derived in section 5.2 and reads

$$\frac{\hbar}{i} \frac{d}{dt} \langle \hat{Q} \rangle = \left\langle [\hat{H}, \hat{Q}] \right\rangle \quad \text{for time-independent } \hat{Q}. \tag{15.3.5}$$

This equation reminds us that an operator that commutes with $\hat{H}$ is conserved: its *expectation value* is time independent. With this result, the inequality (15.3.4) can be simplified:

$$\Delta H \Delta Q \geq \left| \left\langle \frac{1}{2i} [\hat{H}, \hat{Q}] \right\rangle \right| = \left| \frac{1}{2i} \frac{\hbar}{i} \frac{d\langle \hat{Q} \rangle}{dt} \right| = \frac{\hbar}{2} \left| \frac{d\langle \hat{Q} \rangle}{dt} \right|. \tag{15.3.6}$$

Therefore, we find that

$$\boxed{\Delta H\, \Delta Q \geq \frac{\hbar}{2} \left| \frac{d\langle \hat{Q} \rangle}{dt} \right|, \quad \text{for time-independent } \hat{Q}.} \tag{15.3.7}$$

This is a precise uncertainty inequality. It also suggests a definition of a time Δt_Q:

$$\Delta t_Q \equiv \frac{\Delta Q}{\left| \frac{d\langle \hat{Q} \rangle}{dt} \right|}. \tag{15.3.8}$$

This quantity has units of time. It is the time it would take $\langle \hat{Q} \rangle$ to change by ΔQ if both ΔQ and the velocity $\frac{d\langle \hat{Q} \rangle}{dt}$ were time independent. When $\langle \hat{Q} \rangle$ and ΔQ are roughly of the same size, we can view Δt_Q as the time for "appreciable" change in $\langle \hat{Q} \rangle$. In terms of Δt_Q, the uncertainty inequality reads

$$\boxed{\Delta H \Delta t_Q \geq \frac{\hbar}{2}.} \tag{15.3.9}$$

This is a precise inequality because Δt_Q has a precise definition.

The uncertainty relation involves ΔH. It is natural to ask if this quantity is time dependent. We will now show that the uncertainty ΔH, evaluated for any state of a quantum system, is time independent if the Hamiltonian is also time independent. Indeed, if $\hat{H}$ is time independent, we can use $\hat{H}$ and $\hat{H}^2$ for $\hat{Q}$ in (15.3.5) so that

$$\begin{aligned} \frac{d}{dt}\langle \hat{H} \rangle &= \frac{i}{\hbar}\big\langle [\hat{H}, \hat{H}] \big\rangle = 0, \\ \frac{d}{dt}\langle \hat{H}^2 \rangle &= \frac{i}{\hbar}\big\langle [\hat{H}, \hat{H}^2] \big\rangle = 0. \end{aligned} \tag{15.3.10}$$

It then follows that the energy uncertainty is conserved:

$$\frac{d}{dt}(\Delta H)^2 = \frac{d}{dt}\big(\langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2\big) = 0, \tag{15.3.11}$$

showing that ΔH is a constant. So we have shown that

$$\boxed{\text{if } \hat{H} \text{ is time independent, the uncertainty } \Delta H \text{ is constant in time.}} \tag{15.3.12}$$

The conservation of energy uncertainty can be used to understand some aspects of atomic decays. Consider the hyperfine transition in the hydrogen atom. Since both the proton and the electron are spin one-half particles, the ground state of hydrogen is fourfold degenerate, corresponding to the four possible combinations of spins (up-up, up-down, down-up, down-down). The magnetic interaction between the spins actually breaks this degeneracy and produces the so-called hyperfine splitting. This is a very tiny split: 5.88×10^{-6}eV, much smaller than the

13.6 eV binding energy of the ground state. For a hyperfine atomic transition, the emitted photon carries the energy difference: $E_\gamma = 5.88 \times 10^{-6}$eV, resulting in a wavelength of 21.1 cm and a frequency of ν = 1420.405751786(30) MHz. The eleven significant digits of this frequency attest to the sharpness of the emission line.

The issue of uncertainty arises because the excited state of the hyperfine splitting has a lifetime τ_H for decay to the ground state and emission of a photon. This lifetime is extremely long—in fact, $\tau_H \sim 11$ million years ($= 3.4 \times 10^{14}$ s (second), recalling that a year is about $\pi \times 10^7$s, accurate to better than 1%). This lifetime can be viewed as the time it takes some observable of the electron-proton system to change significantly (its total spin angular momentum, perhaps), so by the uncertainty principle, it must be related to some energy uncertainty $\Delta E \sim \hbar/\tau_H \simeq 2 \times 10^{-30}$ eV of the original excited state of the hydrogen atom. Once the decay takes place, the atom goes to the fully stable ground state, without any possible energy uncertainty. By the conservation of energy uncertainty, the photon must carry the uncertainty ΔE. But $\Delta E/E_\gamma \sim 3 \times 10^{-25}$, an absolutely infinitesimal effect on the photon, resulting in no broadening of the 21 cm line! That's one reason it is so useful in astronomy. For decays with much shorter lifetimes, there can be an observable broadening of an emission line due to the energy-time uncertainty principle.

The energy-Q uncertainty relation (15.3.7) can be used to derive an upper bound for the rate of change of the overlap $|\langle\Psi(0)|\Psi(t)\rangle|$ between a state and its initial value at time equal zero (problem 15.10). Sometimes a state can evolve in such a way that the above overlap vanishes for some value of time, the state turning orthogonal to the time-equal-zero state. The time $\Delta t_\perp$ that it takes to do so is in fact bounded below:

$$\Delta H \, \Delta t_\perp \geq \frac{h}{4}. \tag{15.3.13}$$

Note the h, not $\hbar$, to the right of the inequality. Moreover, the constant ΔH can be evaluated on the initial state. A state cannot evolve into an orthogonal state arbitrarily fast; there is a lower bound for the time it takes to do so, and this bound is inversely proportional to the energy uncertainty. Indeed, if the energy uncertainty is zero, the state is in fact an energy eigenstate and evolves only by a phase: $|\Psi(t)\rangle = \exp(-iEt/\hbar)|\Psi(0)\rangle$, with E the energy. In that case the overlap $|\langle\Psi(0)|\Psi(t)\rangle|$ remains one for all times $t > 0$. The bound may play a role in limiting the maximum possible speed of a quantum computer.