18.5 EntangledStates

You have learned that $V \otimes W$ includes states

$$\Psi = \sum_i \alpha_i v_i \otimes w_i, \tag{18.5.1}$$

obtained by the linear superposition of simpler states of the form $v_i \otimes w_i$. If handed such a Ψ, you might want to know whether you can write it as a single term $v_* \otimes w_*$ for some $v_* \in V$ and $w_* \in W$:

$$\text{Can one write the state as } \Psi = v_* \otimes w_* \text{ ?} \tag{18.5.2}$$

If no such v_* and w_* exist, we say that Ψ is an **entangled state** of the two particles. If, on the other hand, v_* and w_* exist, then you are able to describe the state of the particles in Ψ independently: particle 1 is in state v_* and particle 2 in state w_*, and we say that Ψ is not an entangled state. Schrödinger called entanglement the essential feature of quantum mechanics.

Entanglement is a basis-independent property. Indeed, if the state can be factorized into $v_* \otimes w_*$ for some basis choice in V and W, it can be factorized for any other basis choice by simply rewriting v_* and w_* in the new basis. If the state cannot be factorized into $v_* \otimes w_*$ for some basis choice in V and U, it cannot be factorized for any other basis choice because factorization with another basis choice would then imply factorization in the original basis choice. The *tensor product* basis vectors can be chosen to not be entangled, as we did for the orthonormal basis $e_i \otimes f_j$ in (18.1.5), or chosen to be entangled, as we will do for the Bell basis relevant to a pair of spin one-half particles.

In the tensor product of two two-dimensional complex vector spaces, it is not hard to decide when a state is entangled. Let V have a basis e_1, e_2 and W have a basis f_1, f_2. Then the most general state in $V \otimes W$ is

$$\Psi_A = A_{11}\, e_1 \otimes f_1 + A_{12}\, e_1 \otimes f_2 + A_{21}\, e_2 \otimes f_1 + A_{22}\, e_2 \otimes f_2, \tag{18.5.3}$$

with coefficients A_{ij} that can be encoded by a matrix A:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \tag{18.5.4}$$

The state is *not* entangled if there exist constants a_1, a_2, b_1, b_2 such that

$$\begin{aligned} &A_{11}\, e_1 \otimes f_1 + A_{12}\, e_1 \otimes f_2 + A_{21}\, e_2 \otimes f_1 + A_{22}\, e_2 \otimes f_2 \\ &= (a_1 e_1 + a_2 e_2) \otimes (b_1 f_1 + b_2 f_2). \end{aligned} \tag{18.5.5}$$

Note that these four unknown constants are not uniquely determined: we can, for example, multiply a_1 and a_2 by some constant $c \neq 0$ and divide b_1 and b_2 by c to obtain a different solution. Indeed $v \otimes w = (cv) \otimes (w/c)$ for any $c \neq 0$. Using the distributive laws for $\otimes$ to expand the right-hand side of (18.5.5) and recalling that $e_i \otimes f_j$ are basis vectors in the tensor product, we see that the equality requires the following four relations:

$$A_{11} = a_1 b_1, \quad A_{12} = a_1 b_2, \quad A_{21} = a_2 b_1, \quad A_{22} = a_2 b_2. \tag{18.5.6}$$

Combining these four expressions gives us a consistency condition:

$$\det A = A_{11} A_{22} - A_{12} A_{21} = a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1 = 0. \tag{18.5.7}$$

In other words, if Ψ_A is *not* entangled the determinant of the matrix A must be zero. We can in fact show that $\det A = 0$ implies that Ψ_A is not entangled. To do this we simply have to present a solution for the equations above under the condition det $A = 0$.

Assume first that $A_{11} = 0$. Then det$A = 0$ implies $A_{12}A_{21} = 0$. If $A_{12} = 0$, then

$$\Psi_A = A_{21} e_2 \otimes f_1 + A_{22} e_2 \otimes f_2 = e_2 \otimes (A_{21} f_1 + A_{22} f_2), \tag{18.5.8}$$

and the state is indeed not entangled. If $A_{21} = 0$, then

$$\Psi_A = A_{12} e_1 \otimes f_2 + A_{22} e_2 \otimes f_2 = (A_{12} e_1 + A_{22} e_2) \otimes f_2, \tag{18.5.9}$$

and again, the state is not entangled. Thus, we can solve all equations when $A_{11} = 0$. Now assuming $A_{11} \neq 0$, we can readily find a factorization that works:

$$\Psi_A = \left(\sqrt{A_{11}} e_1 + \frac{A_{21}}{\sqrt{A_{11}}} e_2\right) \otimes \left(\sqrt{A_{11}} f_1 + \frac{A_{12}}{\sqrt{A_{11}}} f_2\right), \tag{18.5.10}$$

noting that the det$A = 0$ condition means that $A_{22} = A_{12}A_{21}/A_{11}$. We have thus proved that

$$\boxed{\Psi_A = \sum_{i,j=1}^{2} A_{ij}\, e_i \otimes f_j \text{ is entangled if and only if } \det A \neq 0.}$$

Example 18.4. *Entangled state of two spin one-half particles.*

Consider the state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle_1 \otimes |-\rangle_2 - |-\rangle_1 \otimes |+\rangle_2)$ of zero total spin angular momentum (18.3.12). With basis vectors $e_1 = |+\rangle_1$, $e_2 = |-\rangle_1$ and $f_1 = |+\rangle_2$, $f_2 = |-\rangle_2$, the state is

$$|\Psi\rangle = \frac{1}{\sqrt{2}} e_1 \otimes f_2 - \frac{1}{\sqrt{2}} e_2 \otimes f_1 \;\Rightarrow\; A = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \tag{18.5.11}$$

reading the associated A matrix. Since $A = \frac{1}{2} \neq 0$, the state is entangled.

□

Exercise 18.13. *Consider the operator $S \otimes T$ on $\mathcal{L}(V \otimes W)$, with $S \in \mathcal{L}(V)$ and $T \in \mathcal{L}(W)$. Explain why the action of $S \otimes T$ on a nonentangled state leaves it nonentangled.*