16.5 Heisenberg Equations of Motion

We can calculate the Heisenberg operator associated with a Schrödinger one using the definition (16.4.2). Alternatively, Heisenberg operators satisfy a differential equation: the Heisenberg equation of motion. This equation looks very much like the equations of motion of classical dynamical variables—so much so that people trying to invent quantum theories sometimes begin with the equations of motion of some classical system and postulate the existence of Heisenberg operators that satisfy similar equations. In that case they must also find a Heisenberg Hamiltonian and show that the equations of motion indeed arise in the quantum theory.

To determine the equation of motion of Heisenberg operators, we will simply take time derivatives of the definition (16.4.2). For this purpose we recall (16.3.1), which we copy here using the subscript S for the Hamiltonian:

$$i\hbar \frac{\partial \mathcal{U}(t,t_0)}{\partial t} = \hat{H}_S(t)\mathcal{U}(t,t_0). \tag{16.5.1}$$

Taking the adjoint of this equation, we find that

$$i\hbar \frac{\partial \mathcal{U}^\dagger(t,t_0)}{\partial t} = -\mathcal{U}^\dagger(t,t_0)\hat{H}_S(t). \tag{16.5.2}$$

We can now calculate. Using (16.4.2) we find

$$\begin{aligned} i\hbar\frac{d}{dt}\hat{A}_H(t) &= \Big(i\hbar\frac{\partial \mathcal{U}^\dagger}{\partial t}(t,0)\Big)\hat{A}_S(t)\mathcal{U}(t,0) + \mathcal{U}^\dagger(t,0)\hat{A}_S(t)\Big(i\hbar\frac{\partial \mathcal{U}}{\partial t}(t,0)\Big) \\ &\quad + \mathcal{U}^\dagger(t,0)\, i\hbar\frac{\partial \hat{A}_S(t)}{\partial t}\,\mathcal{U}(t,0). \end{aligned} \tag{16.5.3}$$

Using (16.5.1) and (16.5.2), we have

$$\begin{aligned} i\hbar\frac{d}{dt}\hat{A}_H(t) &= -\mathcal{U}^\dagger(t,0)\hat{H}_S(t)\hat{A}_S(t)\mathcal{U}(t,0) + \mathcal{U}^\dagger(t,0)\hat{A}_S(t)\hat{H}_S(t)\mathcal{U}(t,0) \\ &\quad + \mathcal{U}^\dagger(t,0)\, i\hbar\frac{\partial \hat{A}_S(t)}{\partial t}\,\mathcal{U}(t,0). \end{aligned} \tag{16.5.4}$$

We now use (16.4.5) for the top line and recognize that in the bottom line we have the Heisenberg operator associated with the time derivative of $\hat{A}_S$:

$$\begin{aligned} i\hbar\frac{d}{dt}\hat{A}_H(t) &= -\hat{H}_H(t)\hat{A}_H(t) + \hat{A}_H(t)H_H(t) + i\hbar\Big(\frac{\partial \hat{A}_S(t)}{\partial t}\Big)_H \\ &= [\hat{A}_H(t), \hat{H}_H(t)] + i\hbar\Big(\frac{\partial \hat{A}_S(t)}{\partial t}\Big)_H. \end{aligned} \tag{16.5.5}$$

Properly understood, in the last term, the operations of taking a time derivative and then going into the Heisenberg picture commute. To make the point clearly, note that in general $\hat{A}_S$ is built from some time-independent operators that we can denote collectively by $\hat{\mathcal{O}}$ and, separately, has some explicit time dependence. To reflect this, the operator is written as $\hat{A}_S(\hat{\mathcal{O}}; t)$. It then follows that

$$
\begin{aligned}
\left(\frac{\partial \hat{A}_S}{\partial t}\right)_H &= \left(\lim_{\epsilon\to 0}\frac{1}{\epsilon}\big(\hat{A}_S(\hat{\mathcal{O}};t+\epsilon)-\hat{A}_S(\hat{\mathcal{O}};t)\big)\right)_H \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\big(\hat{A}_S(\hat{\mathcal{O}};t+\epsilon)-\hat{A}_S(\hat{\mathcal{O}};t)\big)_H \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\big(\hat{A}_S(\hat{\mathcal{O}}_H(t);t+\epsilon)-\hat{A}_S(\hat{\mathcal{O}}_H(t);t)\big) \\
&= \frac{\partial}{\partial t}\hat{A}_S(\hat{\mathcal{O}}_H(t);t) = \frac{\partial \hat{A}_H}{\partial t},
\end{aligned}
\tag{16.5.6}
$$

where we note that $\hat{A}_H = \hat{A}_S(\hat{\mathcal{O}}_H(t);\ t)$, since making $\hat{A}_S$ into a Heisenberg operator just means turning $\hat{\mathcal{O}}$ into a Heisenberg operator. We emphasize that the partial time derivative does *not* act on the Heisenberg operator $\hat{\mathcal{O}}_H(t)$. With this understanding, we write (16.5.5) in its final form:

$$
\boxed{i\hbar\,\frac{d\hat{A}_H(t)}{dt} = [\hat{A}_H(t),\,\hat{H}_H(t)] + i\hbar\,\frac{\partial \hat{A}_H(t)}{\partial t}.}
\tag{16.5.7}
$$

A few comments are in order.

1. Schrödinger operators without time dependence: if the operator $\hat{A}_S$ has no explicit time dependence, the last term in (16.5.7) vanishes, and we have the simpler

$$
i\hbar\,\frac{d\hat{A}_H(t)}{dt} = [\hat{A}_H(t),\,\hat{H}_H(t)].
\tag{16.5.8}
$$

2. Time dependence of expectation values: Let $\hat{A}_S$ be a Schrödinger operator without time dependence. Let us now take the time derivative of the expectation value relation in (16.4.16):

$$
\begin{aligned}
i\hbar\frac{d}{dt}\langle\Psi,t|\hat{A}_S|\Psi,t\rangle &= i\hbar\frac{d}{dt}\langle\Psi,0|\,\hat{A}_H(t)\,|\Psi,0\rangle = \langle\Psi,0|\;i\hbar\,\frac{d\hat{A}_H(t)}{dt}\;|\Psi,0\rangle \\
&= \langle\Psi,0|\;[\hat{A}_H(t),\,\hat{H}_H(t)]\,|\Psi,0\rangle.
\end{aligned}
\tag{16.5.9}
$$

We write this as

$$
\boxed{i\hbar\frac{d}{dt}\,\langle\hat{A}_H(t)\rangle = \big\langle[\hat{A}_H(t),\,\hat{H}_H(t)]\big\rangle.}
\tag{16.5.10}
$$

Notice that this equation takes exactly the same form in the Schrödinger picture (recall the comments below (16.4.17)):

$$\boxed{i\hbar \frac{d}{dt} \langle \hat{A}_S \rangle = \langle [\hat{A}_S, \hat{H}_S] \rangle.} \tag{16.5.11}$$

3. A time-independent operator $\hat{A}_S$ is said to be **conserved** if it commutes with the Hamiltonian:

$$\text{conserved operator } \hat{A}_S: \quad [\hat{A}_S, \hat{H}_S] = 0. \tag{16.5.12}$$

It follows that $[\hat{A}_H(t), \hat{H}_H(t)] = 0$, and using (16.5.8), we find that

$$\frac{d\hat{A}_H(t)}{dt} = 0. \tag{16.5.13}$$

The Heisenberg operator is constant. Thus, the expectation value of the operator is also constant. This is consistent with (6) in the previous section: $\hat{A}_H$ is in fact equal to $\hat{A}_S$!

Example 16.1. *Heisenberg operators for the harmonic oscillator.*
Our main goal here is to obtain the explicit form of the Heisenberg operators $\hat{x}_H(t)$ and $\hat{p}_H(t)$ for the simple harmonic oscillator. For this we will have to solve their Heisenberg equations of motion.

The Schrödinger Hamiltonian $\hat{H}_S$ for the oscillator takes the form

$$\hat{H}_S = \frac{\hat{p}^2}{2m} + \tfrac{1}{2} m\omega^2 \hat{x}^2. \tag{16.5.14}$$

Using (16.4.10), the associated Heisenberg Hamiltonian $\hat{H}_H(t)$ is obtained by replacing $\hat{x}$ and $\hat{p}$ above by their Heisenberg counterparts:

$$\hat{H}_H(t) = \frac{\hat{p}_H^2(t)}{2m} + \tfrac{1}{2} m\omega^2 \hat{x}_H^2(t). \tag{16.5.15}$$

The Heisenberg equation of motion for $\hat{x}_H(t)$ is

$$\begin{aligned} \frac{d}{dt} \hat{x}_H(t) &= \frac{1}{i\hbar} [\hat{x}_H(t), \hat{H}_H(t)] = \frac{1}{i\hbar} \Big[\hat{x}_H(t), \frac{\hat{p}_H^2(t)}{2m} \Big] \\ &= \frac{1}{i\hbar}\, 2 \frac{\hat{p}_H(t)}{2m} \Big[\hat{x}_H(t), \hat{p}_H(t) \Big] = \frac{1}{i\hbar} \frac{\hat{p}_H(t)}{m}\, i\hbar = \frac{\hat{p}_H(t)}{m} \end{aligned} \tag{16.5.16}$$

so that our first equation is

$$\frac{d}{dt}\,\hat{x}_H(t) \;=\; \frac{\hat{p}_H(t)}{m}. \tag{16.5.17}$$

For the momentum operator, we get

$$\begin{aligned}\frac{d}{dt}\,\hat{p}_H(t) \;&=\; \frac{1}{i\hbar}[\hat{p}_H(t), \hat{H}_H(t)] \;=\; \frac{1}{i\hbar}\Big[\hat{p}_H(t),\ \frac{1}{2}m\omega^2\, x_H^2(t)\Big]\\ &=\; \frac{1}{i\hbar}\,\frac{1}{2}m\omega^2 \cdot 2(-i\hbar)\hat{x}_H(t) \;=\; -m\omega^2\,\hat{x}_H(t),\end{aligned} \tag{16.5.18}$$

so our second equation is

$$\frac{d}{dt}\,\hat{p}_H(t) = -m\omega^2\,\hat{x}_H(t). \tag{16.5.19}$$

Taking another time derivative of our first equation and using the second one, we get

$$\frac{d^2}{dt^2}\,\hat{x}_H(t) = -\omega^2\,\hat{x}_H(t). \tag{16.5.20}$$

The solution of this differential equation takes the form

$$\hat{x}_H(t) \;=\; \hat{A}\,\cos\omega t + \hat{B}\,\sin\omega t, \tag{16.5.21}$$

where $\hat{A}$ and $\hat{B}$ are time-independent operators to be determined by initial conditions. Using (16.5.17), we can calculate the associated momentum operator:

$$\hat{p}_H(t) \;=\; m\frac{d}{dt}\hat{x}_H(t) \;=\; -m\omega\hat{A}\sin\omega t + m\omega\hat{B}\cos\omega t. \tag{16.5.22}$$

At zero time the Heisenberg operators must equal the Schrödinger ones, so

$$\hat{x}_H(0) = \hat{A} = \hat{x}, \quad \hat{p}_H(0) = m\omega\hat{B} = \hat{p}. \tag{16.5.23}$$

We thus find that

$$\hat{A} \;= \hat{x}, \quad \hat{B} \;=\; \frac{1}{m\omega}\,\hat{p}. \tag{16.5.24}$$

Finally, back in (16.5.21) and (16.5.22) we have the full solution for the simple harmonic oscillator Heisenberg operators:

$$\boxed{\begin{aligned}\hat{x}_H(t) &= \hat{x}\,\cos\omega t + \frac{1}{m\omega}\,\hat{p}\,\sin\omega t,\\ \hat{p}_H(t) &= \hat{p}\cos\omega t - m\omega\,\hat{x}\,\sin\omega t.\end{aligned}} \tag{16.5.25}$$

Let us confirm that the Heisenberg Hamiltonian is time independent and in fact equal to the Schrödinger Hamiltonian. This must hold because the Schrödinger Hamiltonian is time independent. Starting with (16.5.15) and using (16.5.25), we show that

$$\begin{aligned}\hat{H}_H(t) &= \frac{\hat{p}_H^2(t)}{2m} + \frac{1}{2}m\omega^2\hat{x}_H^2(t)\\ &= \frac{1}{2m}\big(\hat{p}\cos\omega t - m\omega\,\hat{x}\,\sin\omega t\big)^2 + \frac{1}{2}m\omega^2\Big(\hat{x}\,\cos\omega t + \frac{1}{m\omega}\,\hat{p}\,\sin\omega t\Big)^2\\ &= \frac{\cos^2\omega t}{2m}\hat{p}^2 + \frac{m^2\omega^2\sin^2\omega t}{2m}\,\hat{x}^2 - \frac{\omega}{2}\sin\omega t\cos\omega t(\hat{p}\hat{x}+\hat{x}\hat{p})\\ &\quad + \frac{\sin^2\omega t}{2m}\hat{p}^2 + \frac{m\omega^2\cos^2\omega t}{2}\,\hat{x}^2 + \frac{\omega}{2}\cos\omega t\sin\omega t\,(\hat{x}\hat{p}+\hat{p}\hat{x})\\ &= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2,\end{aligned} \tag{16.5.26}$$

confirming that $\hat{H}_H(t) = \hat{H}_S$.

□

Example 16.2. *Heisenberg creation and annihilation operators.*
Let us determine the Heisenberg operators corresponding to the simple harmonic oscillator creation and annihilation operators. For simplicity, the Heisenberg operator associated with $\hat{a}$ will be denoted by $\hat{a}(t)$. Since the harmonic oscillator Hamiltonian is time independent, $\square = e^{-i\hat{H}t/\hbar}$ and we have

$$\hat{a}(t) \equiv e^{i\hat{H}t/\hbar}\,\hat{a}\,e^{-i\hat{H}\,t/\hbar} = e^{i\omega\,t\hat{N}}\,a\,e^{-i\omega t\hat{N}}, \tag{16.5.27}$$

where we wrote $\hat{H} = \hbar\omega(\hat{N}+\frac{1}{2})$ and noted that the additive constant has no effect on the result. A simple way to evaluate $\hat{a}(t)$ goes through a differential equation. We take the time derivative of the above to find

$$\frac{d}{dt}\,\hat{a}(t) = i\omega\,e^{i\omega\,t\hat{N}}\,[\hat{N},\hat{a}\,]e^{-i\omega t\hat{N}} = -i\omega\,e^{i\omega\,t\hat{N}}\,\hat{a}\,e^{-i\omega t\hat{N}}. \tag{16.5.28}$$

We recognize on the final right-hand side the operator $\hat{a}(t)$, so we have obtained the differential equation

$$\frac{d}{dt}\hat{a}(t) = -i\omega\,\hat{a}(t). \tag{16.5.29}$$

Since $\hat{a}(t = 0) = \hat{a}$, the solution is

$$\hat{a}(t) = e^{-i\omega t}\,\hat{a}. \tag{16.5.30}$$

Together with the adjoint of this formula, we have:

$$\boxed{\begin{aligned} \hat{a}(t) &= e^{-i\omega t}\,\hat{a}, \\ \hat{a}^\dagger(t) &= e^{i\omega t}\;\hat{a}^\dagger. \end{aligned}} \tag{16.5.31}$$

Clearly, $[\hat{a}(t), \hat{a}^\dagger(t)] = 1$, as expected.

□