14.8 From Inner Products to Bra-kets

Paul Dirac invented an alternative notation for inner products that leads to the concepts of *bras* and *kets*. Dirac's notation is sometimes more efficient than the conventional mathematical notation we have developed. It is also widely used.

In this and the following sections, we discuss Dirac's notation and spend some time rewriting some of our results using bras and kets. This will afford you the opportunity to appreciate the identities by looking at them in a different notation. Operators can also be written in terms of bras and kets, using their matrix representation. We are providing here a detailed explanation for some of the rules we followed when we began writing spin states as kets in chapter 12.

A classic application of the bra-ket notation is to systems with a nondenumerable basis of states, such as position states $|x\rangle$ and momentum states $|p\rangle$ of a particle moving in one dimension. We will consider these in section 14.10.

The construction begins by writing the inner product differently. The first step in the Dirac notation is to define the so called bra-ket pair $\langle u|v\rangle$ of two vectors u and v using inner products:

$$\langle u|\,v\rangle \equiv \langle\, u, v\,\rangle. \tag{14.8.1}$$

It is as if the inner-product comma is replaced by a vertical bar! Since things look a bit different in this notation, let us rewrite a few of the properties of inner products in bra-ket notation.

We now say, for example, that $\langle v|v\rangle \geq 0$ for all v, while $\langle v|v\rangle = 0$ if and only if $v = 0$. The conjugate exchange symmetry becomes $\langle u|v\rangle = \langle v|u\rangle^*$. Additivity and homogeneity on the second entry is written as

$$\langle u|c_1 v_1 + c_2 v_2\rangle \;=\; c_1\langle u|v_1\rangle + c_2\langle u|v_2\rangle, \quad c_1, c_2 \in \mathbb{C}, \tag{14.8.2}$$

while conjugate homogeneity (14.1.15) on the first entry is summarized by

$$\langle c_1 u_1 + c_2 u_2|v\rangle \;=\; c_1^*\,\langle u_1|v\rangle + c_2^*\,\langle u_2|v\rangle. \tag{14.8.3}$$

Two vectors u and v are orthogonal if $\langle u|v\rangle = 0$, and the norm $\|v\|$ of a vector is $\|v\|^2 = \langle v|v\rangle$. The Schwarz inequality for any pair of vectors u and v, reads $|\langle u|v\rangle| \leq \|u\|\,\|v\|$.

A set of basis vectors $\{e_i\}$ with $i = 1, \ldots, n$ is said to be orthonormal if

$$\langle e_i|e_j\rangle \;=\; \delta_{ij}. \tag{14.8.4}$$

An arbitrary vector can be written as a linear superposition of basis states:

$$v \;=\; \sum_i \alpha_i\, e_i. \tag{14.8.5}$$

We then see that the coefficients are determined by the inner product

$$\langle e_k|v\rangle \;=\; \Big\langle e_k\Big|\sum_i \alpha_i e_i\Big\rangle \;=\; \sum_i \alpha_i \langle e_k|e_i\rangle \;=\; \alpha_k. \tag{14.8.6}$$

We can therefore write, just as we did in (14.2.6),

$$v \;=\; \sum_i e_i\,\langle e_i|v\rangle. \tag{14.8.7}$$

The next step is to isolate bras and kets from the bra-ket. To do this we reinterpret the bra-ket form of the inner product. We want to "split" the bra-ket into two ingredients, a bra and a ket:

$$\langle u|v\rangle \;\Rightarrow\; \langle u| \;\; |v\rangle. \tag{14.8.8}$$

Here the symbol $|v\rangle$ is called a **ket**, and the symbol $\langle u|$ is called a **bra**. The bra-ket is recovered when the space between the bra and the ket collapses.

We will view the ket $|v\rangle$ as another way to write the vector v. There is a bit of redundancy in this notation that may be confusing: both $v \in V$ and $|v\rangle \in V$. Both are vectors in V, but sometimes the ket $|v\rangle$ is called a *state* in V. The enclosing symbol $|\;\rangle$ is a decoration added to the vector v without changing its meaning, perhaps like the familiar arrows added above a symbol to denote a vector. In this case the label in the ket is a vector, and the ket itself is that vector!

When the label of the ket is a vector, the bra-ket notation is a direct rewriting of the mathematical notation. Sometimes, however, the label of the ket is not a vector. The label could be the value of some quantity that characterizes the state. In such cases the notation affords some extra flexibility. We used such labeling, for example, when we wrote $|+\rangle$ and $|-\rangle$ for the spin states that point along the positive z-direction and along the negative z-direction, respectively. We will encounter similar situations in this chapter.

$$\boxed{\begin{array}{c}\text{Sometimes the label inside a ket is the vector itself;}\\ \text{other times it is an object that characterizes the vector.}\end{array}} \tag{14.8.9}$$

Let T be an operator in a vector space V. We wrote Tv as the vector obtained by the action of T on the vector v. Now the same action would be written as $T|v\rangle$. With kets labeled by vectors, we can simply identify

$$|Tv\rangle \equiv T|v\rangle. \tag{14.8.10}$$

When kets are labeled by vectors, operators go in or out of the ket without change. If the ket labels are not vectors, the above identification is not possible. Imagine a nondegenerate system where we label the states by their energies, as in $|E_i\rangle$, where E_i is the value of the energy for the ith state. Acting with the momentum operator $\hat{p}$ on the state is denoted as $\hat{p}$ $|E_i\rangle$. It would be confusing, however, to rewrite this as $|\hat{p}\, E_i\rangle$ since E_i is not a vector. It is an energy, and $\hat{p}$ does not act on energies.

Bras are rather different from kets, although we also label them by vectors. Bras are linear functionals on the vector space V. We defined linear functionals in section 14.4: they are linear maps ϕ from V to the numbers: $\phi(v) \in \mathbb{F}$. The set of *all* linear functionals on V is in fact a new vector space over $\mathbb{F}$, the vector space V^* **dual to** V. The vector space structure of V^* follows from the natural definitions of sums of linear functionals and the multiplication of linear functionals by numbers:

1. For ϕ_1, $\phi_2 \in V^*$, we define the sum $\phi_1 + \phi_2 \in V^*$ by

$$(\phi_1 + \phi_2)v \equiv \phi_1(v) + \phi_2(v). \tag{14.8.11}$$

2. For $\phi \in V^*$ and $a \in \mathbb{F}$, we define $a\phi \in V^*$ by

$$(a\phi)(v) = a\phi(v). \tag{14.8.12}$$

We proved before that for any linear functional $\phi \in V^*$ there is a unique vector $u \in V$ such that $\phi(v) = \langle u, v\rangle$. We can make this more explicit by labeling the linear functional by u and thus writing

$$\phi_u(v) = \langle u, v\rangle. \tag{14.8.13}$$

Since the elements of V^* are characterized uniquely by vectors, the vector space V^* has the same dimensionality as V.

A bra is also labeled by a vector. The bra $\langle u|$ can also be viewed as a linear functional because it has a natural action on vectors. The bra $\langle u|$, acting on the vector $|v\rangle$, is defined to give the bra-ket number $\langle u|v\rangle$:

$$\langle u| \,:\, |v\rangle \;\rightarrow\; \langle u|v\rangle. \tag{14.8.14}$$

Compare this with

$$\phi_u \,:\, v \;\rightarrow\; \langle u, v\rangle. \tag{14.8.15}$$

Since $\langle u, v\rangle = \langle u|v\rangle$, the last two equations mean that we can identify

$$\boxed{\phi_u \iff \langle u|.} \tag{14.8.16}$$

This identification will allow us to work out how to manipulate bras.

Once we choose a basis, a vector can be represented by a column vector, as discussed in (13.5.6). If kets are viewed as column vectors, then *bras should be viewed as row vectors*. In this way a bra to the left of a ket in the bra-ket makes sense: matrix multiplication of a row vector times a column vector gives a number. Indeed, for vectors

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \tag{14.8.17}$$

the canonical inner product gives

$$\langle u|v\rangle \;=\; u_1^* v_1 + \cdots + u_n^* v_n. \tag{14.8.18}$$

If we think of this as having a bra and a ket,

$$\langle u| = (u_1^*, \ldots, u_n^*), \quad |v\rangle = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \tag{14.8.19}$$

then matrix multiplication gives us the desired bra-ket:

$$\langle u|v\rangle \;=\; (u_1^*, \ldots, u_n^*) \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \;=\; u_1^* v_1 + \cdots + u_n^* v_n. \tag{14.8.20}$$

The row representative of the bra $\langle u|$ was obtained by the transposition and complex conjugation of the column vector representative of $|u\rangle$.

The key properties needed to manipulate bras follow from the properties of linear functionals and identification (14.8.16). For our linear functionals, you can quickly verify that for $u_1, u_2 \in V$ and $a \in \mathbb{F}$,

$$\begin{aligned} \phi_{u_1+u_2} \;&=\; \phi_{u_1} + \phi_{u_2}, \\ \phi_{au} \;&=\; a^*\, \phi_u. \end{aligned} \tag{14.8.21}$$

With the noted identification with bras, these become

$$\boxed{\begin{aligned}\langle u_1 + u_2| &= \langle u_1| + \langle u_2|,\\ \langle au| &= a^* \langle u|.\end{aligned}} \tag{14.8.22}$$

If $\phi_u = \phi_{u'}$, then $u = u'$. Thus, we conclude that $\langle u| = \langle u'|$ implies $u = u'$.

A rule to pass from general kets to general bras is useful. We can obtain such a rule by considering the ket

$$|v\rangle = |\alpha_1 u_1 + \alpha_2 u_2\rangle = \alpha_1 |u_1\rangle + \alpha_2 |u_2\rangle. \tag{14.8.23}$$

Then

$$\langle v| = \langle \alpha_1 u_1 + \alpha_2 u_2| = \alpha_1^* \langle u_1| + \alpha_2^* \langle u_2|, \tag{14.8.24}$$

using the relations in (14.8.22). We have thus shown that the rule to pass from kets to bras, and *vice versa*, is

$$\boxed{|v\rangle = \alpha_1 |u_1\rangle + \alpha_2 |u_2\rangle \iff \langle v| = \alpha_1^* \langle u_1| + \alpha_2^* \langle u_2|.} \tag{14.8.25}$$

As we mentioned earlier, we sometimes write kets with labels other than vectors. Let us reconsider the basis vectors $\{e_i\}$ discussed in (14.8.4). The ket $|e_i\rangle$ is simply called $|i\rangle$, and the orthonormal condition reads

$$\langle i | j \rangle = \delta_{ij}. \tag{14.8.26}$$

The expansion (14.8.5) of a vector now reads

$$|v\rangle = \sum_i |i\rangle \alpha_i. \tag{14.8.27}$$

As in (14.8.6), the expansion coefficients are $\alpha_i = \langle i \mid v \rangle$ so that

$$\boxed{|v\rangle = \sum_i |i\rangle \langle i \mid v\rangle.} \tag{14.8.28}$$

We placed the numerical component $\langle i \mid v \rangle$ to the right of the ket $|\, i\rangle$. This is useful because we will soon discover that the sum $\sum_i |\, i\rangle\langle\, i|$ has a special meaning.