17.5 Spin Precession in a Magnetic Field

In this section we begin our study of a second, broad class of dynamical systems. This is the class of *two-state* systems. A two-state system does not just have two states! It has two *basis* states; the state space is the twodimensional complex vector space $\mathbb{C}^2$. For such a state space, the Hamiltonian can be viewed as the most general Hermitian 2×2 matrix. When the Hamiltonian is time independent, this Hermitian matrix is characterized by four real numbers.

Two-state systems are useful idealizations when other degrees of freedom can be ignored. A spin one-half particle is a two-state system with regard to spin. Being a particle, however, it may move and thus has position or momentum degrees of freedom that imply a much larger, higher-dimensional state space. Only when we ignore these degrees of freedom—perhaps because the particle is at rest—can we speak of a two-state system.

In this section our two-state system will be a spin one-half particle. This is to prepare for the more complex example of nuclear magnetic resonance to be considered in the following section. The mathematics of two-state systems is always the same, making it possible to visualize any two-state system as a spin system (section 17.7). In the problems at the end of the chapter, we will consider other examples, including the ammonia molecule, which exhibits curious oscillations between two states.

To examine spin precession, let us first recall our earlier discussion of magnetic dipole moments of particles from section 12.2, where we described the Stern-Gerlach experiment. Classically, we had the relation (12.2.9), valid for a spinning charged particle with charge q and mass m uniformly distributed:

$$\boldsymbol{\mu} = \frac{q}{2mc}\mathbf{L}. \tag{17.5.1}$$

Here $\boldsymbol{\mu}$ is the magnetic dipole moment, and $\mathbf{L}$ is its angular momentum. In the quantum world, particles have spin angular momentum operators $\hat{\mathbf{S}}$ and magnetic moment operators $\hat{\boldsymbol{\mu}}$, and the above relation gets modified by the inclusion of a unit-free constant factor g, which differs for each particle:

$$\hat{\boldsymbol{\mu}} = g\,\frac{q}{2mc}\hat{\mathbf{S}} = g\,\frac{q\hbar}{2mc}\frac{\hat{\mathbf{S}}}{\hbar}. \tag{17.5.2}$$

If we consider electrons and protons with masses m_e and m_p, respectively, the Bohr magneton μ_B and the nuclear magneton μ_N are defined as follows:

$$\mu_B = \frac{e\hbar}{2m_e c} = 5.788 \times 10^{-9}\ \frac{\text{eV}}{\text{gauss}}, \quad \mu_N = \frac{e\hbar}{2m_p c} = 3.152 \times 10^{-12}\ \frac{\text{eV}}{\text{gauss}}. \tag{17.5.3}$$

The corresponding expressions in SI units are

$$\mu_B = \frac{e\hbar}{2m_e} = 5.788 \times 10^{-5}\ \frac{\text{eV}}{\text{tesla}}, \quad \mu_N = \frac{e\hbar}{2m_p} = 3.152 \times 10^{-8}\ \frac{\text{eV}}{\text{tesla}}. \tag{17.5.4}$$

Note that the nuclear magneton is about two thousand times smaller than the Bohr magneton. As a result, nuclear magnetic dipole moments are much smaller than the electron dipole moment. For an electron, $g_e = 2$, and since the electron charge is negative, we get

$$\hat{\boldsymbol{\mu}}_e = -2\mu_B \tfrac{1}{\hbar}\,\hat{\mathbf{S}}. \tag{17.5.5}$$

The dipole moment and the angular momentum are antiparallel. For a proton, the experimental result is

$$\hat{\boldsymbol{\mu}}_p = 5.586\,\mu_N \tfrac{1}{\hbar}\,\hat{\mathbf{S}}. \tag{17.5.6}$$

The neutron is neutral, so one may expect no magnetic dipole moment. But the neutron, just like the proton, is not an elementary particle: it is made of gluons, electrically charged valence quarks, and virtual quarks. A dipole moment is thus possible, depending on the way quarks are distributed. Indeed, experimentally,

$$\hat{\boldsymbol{\mu}}_n = -3.826\,\mu_N \tfrac{1}{\hbar}\,\hat{\mathbf{S}}. \tag{17.5.7}$$

Somehow, the negative charge contributes more than the positive charge to the magnetic dipole of the neutron. The compositeness of the proton also accounts for the unusual value of its magnetic dipole moment.

For notational convenience, we introduce the *gyromagnetic ratio* γ that summarizes the relation between the magnetic dipole moment and the spin as follows:

$$\boxed{\hat{\boldsymbol{\mu}} = \gamma\,\hat{\mathbf{S}}.} \tag{17.5.8}$$

The values of γ for the electron, the proton, and the neutron can be read directly from the three previous equations. In general, for any particle or nuclei, equation (17.5.2) gives $\gamma = \frac{gq}{2mc}$, where g, q, and m are, respectively, the g factor, the charge, and the mass of the particle. The $g_e = 2$ factor of the electron has additional quantum corrections that can be evaluated using quantum field theory. These corrections are in agreement with a remarkably precise measurement that gives $g_e = 2.002\,319\,304\,3617(15)$. Also with great accuracy is the Bohr magneton given by $\mu_B = 5.788\,381\,8012(26) \times 10^{-5}$ eV·T^{-1}. Just for fun, we also know that $\hbar = 6.582\,119\,569 \times 10^{-16}$eV·s.

Exercise 17.9. *Show that the SI gyromagnetic ratios γ_e and γ_p for the electron and the proton are*

$$\gamma_e = -1.760\,8596 \times 10^{11}\,\mathrm{s}^{-1}\mathrm{T}^{-1}, \quad \gamma_p = 2.675 \times 10^{8}\,\mathrm{s}^{-1}\mathrm{T}^{-1}. \tag{17.5.9}$$

This gives $|\gamma_e/\gamma_p| \sim 660$.

If we insert the particle in a magnetic field **B**, the Hamiltonian $\hat{H}_S$ for the spin system is

$$\hat{H}_S = -\hat{\boldsymbol{\mu}} \cdot \mathbf{B} = -\gamma\, \mathbf{B} \cdot \hat{\mathbf{S}} = -\gamma\,(B_x \hat{S}_x + B_y \hat{S}_y + B_z \hat{S}_z). \tag{17.5.10}$$

For a magnetic field $\mathbf{B} = B\hat{z}$ along the z-axis, for example, we get

$$\hat{H}_S = -\gamma B \hat{S}_z, \tag{17.5.11}$$

and the associated time-evolution unitary operator is given by

$$\mathcal{U}(t,0) = \exp\Big(-\frac{i\hat{H}_S t}{\hbar}\Big) = \exp\Big(-\frac{i(-\gamma Bt)\hat{S}_z}{\hbar}\Big). \tag{17.5.12}$$

In section 14.7 we discussed in great detail the unitary rotation operator $\hat{R}_{\mathbf{n}}(\alpha)$ defined by a unit vector **n** and an angle α:

$$\hat{R}_{\mathbf{n}}(\alpha) = \exp\Big(-\frac{i\alpha \hat{S}_{\mathbf{n}}}{\hbar}\Big), \text{ with } \hat{S}_{\mathbf{n}} \equiv \mathbf{n} \cdot \hat{\mathbf{S}}. \tag{17.5.13}$$

We showed that, acting on a spin state, the operator rotates it by an angle α about the direction defined by the vector **n**. Comparing (17.5.13) and (17.5.12), we conclude that $\mathcal{U}(t, 0)$ generates a rotation by the angle $(-\gamma Bt)$ about the z-axis. We now confirm this explicitly.

Consider a spin state that at $t = 0$ points along the direction specified by the angles (θ_0, ϕ_0):

$$|\Psi, 0\rangle = \cos\tfrac{\theta_0}{2}|+\rangle + \sin\tfrac{\theta_0}{2}e^{i\phi_0}|-\rangle. \tag{17.5.14}$$

Given the Hamiltonian $\hat{H}_S = -\gamma B\hat{S}_z$ in (17.5.11), we find that

$$\hat{H}_S|\pm\rangle = \mp\tfrac{1}{2}\gamma B\hbar|\pm\rangle. \tag{17.5.15}$$

Therefore, the time-evolved state is given by

$$\begin{aligned} |\Psi, t\rangle &= e^{-i\hat{H}_S t/\hbar}|\Psi, 0\rangle = e^{-i\hat{H}_S t/\hbar}\Big(\cos\tfrac{\theta_0}{2}|+\rangle + \sin\tfrac{\theta_0}{2}e^{i\phi_0}|-\rangle\Big) \\ &= \cos\tfrac{\theta_0}{2}e^{+i\gamma Bt/2}|+\rangle + \sin\tfrac{\theta_0}{2}e^{i\phi_0}e^{-i\gamma Bt/2}|-\rangle. \end{aligned} \tag{17.5.16}$$

To identify the direction of the resulting state, we factor out the phase that multiplies the $|+\rangle$ state:

$$|\Psi, t\rangle = e^{+i\gamma Bt/2}\Big(\cos\tfrac{\theta_0}{2}|+\rangle + \sin\tfrac{\theta_0}{2}e^{i(\phi_0 - \gamma Bt)}|-\rangle\Big). \tag{17.5.17}$$

We now recognize that the spin state points along the direction defined by angles

$$\begin{aligned} \theta(t) &= \theta_0, \\ \phi(t) &= \phi_0 - \gamma Bt. \end{aligned} \tag{17.5.18}$$

Keeping θ constant while changing ϕ indeed corresponds to a rotation about the z-axis and, after time t, the spin has rotated an angle $(-\gamma Bt)$ as claimed above. It rotated with an angular speed of magnitude $\omega = \gamma B$, assuming $\gamma,\ B > 0$. Indeed, with $\gamma,\ B > 0$, the spin direction rotated with angular velocity pointing along the $-z$-direction, so we had $\boldsymbol{\omega} = -\gamma\mathbf{B}$.

We will now show that the above result is general: in a time-independent magnetic field, spin states precess with **Larmor angular frequency** $\boldsymbol{\omega}_L$ given by

$$\boxed{\boldsymbol{\omega}_L = -\gamma\,\mathbf{B}.} \tag{17.5.19}$$

To see this, note that the Hamiltonian of a spin in a magnetic field (17.5.10) becomes

$$\boxed{\hat{H}_S = -\hat{\boldsymbol{\mu}} \cdot \mathbf{B} = -\gamma \mathbf{B} \cdot \hat{\mathbf{S}} = \boldsymbol{\omega}_L \cdot \hat{\mathbf{S}}.} \tag{17.5.20}$$

With the magnetic field assumed time independent, $\boldsymbol{\omega}_L$ is also time independent, and the evolution operator is simply

$$\mathcal{U}(t, 0) = \exp(-i\hat{H}_S t/\hbar) = \exp\left(-i\,\frac{\boldsymbol{\omega}_L \cdot \hat{\mathbf{S}}}{\hbar}\,t\right). \tag{17.5.21}$$

Letting $\mathbf{n}$ denote the direction of $\boldsymbol{\omega}_L$, we write

$$\boldsymbol{\omega}_L = \omega_L\,\mathbf{n}, \quad \mathbf{n} \cdot \mathbf{n} = 1, \quad \omega_L \geq 0. \tag{17.5.22}$$

In this notation, the time-evolution operator becomes

$$\mathcal{U}(t, 0) = \exp\left(-i\,\frac{\omega_L t\,\hat{S}_{\mathbf{n}}}{\hbar}\right) = \hat{R}_{\mathbf{n}}(\omega_L t), \tag{17.5.23}$$

when comparing with (17.5.13). The time-evolution operator □$(t,\ 0)$ rotates the spin states by the angle $\omega_L t$ about the $\mathbf{n}$-axis. In other words, we have shown that

$$\boxed{\text{with } \hat{H}_S = \boldsymbol{\omega}_L \cdot \hat{\mathbf{S}}, \text{ spin states precess with angular velocity } \boldsymbol{\omega}_L.} \tag{17.5.24}$$

The precession of a spin state means the precession of the unit vector $\mathbf{n}(t$ that fixes the direction of the spin state $|\mathbf{n}(t)\rangle$. As the above equation shows, the angular frequency of precession is easily read directly from the Hamiltonian.