17.3 General Coherent States

Consider again the coherent state $|x_0\rangle c$, written in terms of creation and annihilation operators. As we had in (17.2.22),

$$|x_0\rangle_c = e^{\alpha\,(\hat{a}^\dagger - \hat{a})}|0\rangle, \quad \text{with} \;\; \alpha \; = \; \frac{1}{\sqrt{2}}\frac{x_0}{L_0}. \tag{17.3.1}$$

An obvious generalization is to let α be a complex number: $\alpha \in \mathbb{C}$. This must be done with care since the operator in the exponential (17.3.1) must be anti-Hermitian, making the exponential unitary. We must therefore replace $\alpha\ (\hat{a}^\dagger - \hat{a})$ with $\alpha\hat{a}^\dagger - \alpha^*\hat{a}$, which is anti-Hermitian when α is complex. We thus define

$$\boxed{|\alpha\rangle \;\equiv\; D(\alpha)|0\rangle \;\equiv\; \exp\big(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\big)|0\rangle, \quad \text{with} \;\; \alpha \in \mathbb{C}.} \tag{17.3.2}$$

The coherent state, now denoted by $|\alpha\rangle$, is obtained by acting on the vacuum with the unitary *displacement* operator

$$D(\alpha) \equiv \exp\big(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\big). \tag{17.3.3}$$

Since $D(\alpha)$ is unitary, it is clear that $\langle\alpha|\alpha\rangle = 1$. In the present notation, the coherent state is recognized by using a Greek letter as the label of the ket.

The action of the annihilation operator on the states $|\alpha\rangle$ is quite interesting:

$$\begin{aligned}\hat{a}|\alpha\rangle \;&=\; \hat{a}\, e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}|0\rangle \;=\; [\hat{a},\, e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}]|0\rangle \\ &=\; [\hat{a},\, \alpha\hat{a}^\dagger - \alpha^*\hat{a}]e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}|0\rangle \;=\; \alpha e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}|0\rangle,\end{aligned} \tag{17.3.4}$$

where the commutator was evaluated using (13.7.24). We conclude that

$$\boxed{\hat{a}\,|\alpha\rangle \;=\; \alpha\,|\alpha\rangle.} \tag{17.3.5}$$

This result is a bit shocking: we have found eigenstates of the *non-Hermitian* operator $\hat{a}$. Because $\hat{a}$ is not Hermitian, our theorems about eigenstates and eigenvectors of Hermitian operators do not apply. For example, the eigenvalues need not be real ($\alpha \in \mathbb{C}$), two eigenvectors with different eigenvalues need not be orthogonal (they are not!), and the set of eigenvectors need not form a complete basis (coherent states actually give an overcomplete basis!). We actually determined explicitly the eigenstates of the annihilation operator in example 13.21.

Exercise 17.5. *Ordering the exponential in the state* $|\alpha\rangle$ *in (17.3.2), show that*

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha \hat{a}^\dagger} |0\rangle. \tag{17.3.6}$$

Exercise 17.6. *Show that*

$$\langle\beta|\alpha\rangle = \exp\Big(-\tfrac{1}{2}(|\alpha|^2 + |\beta|^2) + \beta^*\alpha\Big). \tag{17.3.7}$$

Hint: You may find it helpful to evaluate $e^{\beta^*\hat{a}+\alpha\hat{a}^\dagger}$ *in two different ways using (17.2.23).*

Exercise 17.7. *The above formula for the overlap* $\langle\beta|\alpha\rangle$ *does not make* $\langle\alpha|\alpha\rangle = 1$ *manifest. Show that the above can be rewritten as*

$$\langle\beta|\alpha\rangle = e^{-\frac{1}{2}|\alpha-\beta|^2}\, e^{i\,\mathrm{Im}(\beta^*\alpha)}. \tag{17.3.8}$$

When $\beta = \alpha$*, both exponents vanish manifestly, and thus the overlap is clearly equal to one. Note that the result implies that* $|\langle\beta|\alpha\rangle|^2 = e^{|\alpha-\beta|^2}$.

To find the physical interpretation of the complex number α, we first note that when real, as in (17.3.1), α encodes the initial position x_0 of the coherent state. More precisely, it encodes the expectation value of $\hat{x}$ in the state at $t = 0$. For complex α, its real part is still related to the initial position:

$$\langle\alpha|\hat{x}|\alpha\rangle = \frac{L_0}{\sqrt{2}}\langle\alpha|(\hat{a}+\hat{a}^\dagger)|\alpha\rangle = \frac{L_0}{\sqrt{2}}(\alpha+\alpha^*) = L_0\sqrt{2}\,\mathrm{Re}(\alpha), \tag{17.3.9}$$

where we used (17.3.5), both on bras and on kets. We have thus learned that

$$\mathrm{Re}(\alpha) = \frac{1}{\sqrt{2}}\frac{\langle\hat{x}\rangle}{L_0}. \tag{17.3.10}$$

It is natural to conjecture that the imaginary part of α is related to the momentum expectation value on the initial state. So we explore

$$\langle\alpha|\hat{p}|\alpha\rangle = \frac{i}{\sqrt{2}}\frac{\hbar}{L_0}\langle\alpha|(\hat{a}^\dagger - \hat{a})|\alpha\rangle = -\frac{i}{\sqrt{2}}\frac{\hbar}{L_0}(\alpha - \alpha^*) = \sqrt{2}\frac{\hbar}{L_0}\,\mathrm{Im}(\alpha) \tag{17.3.11}$$

and learn that

$$\mathrm{Im}(\alpha) = \frac{1}{\sqrt{2}}\frac{L_0}{\hbar}\langle\hat{p}\rangle. \tag{17.3.12}$$

The identification of α in terms of expectation values of $\hat{x}$ and $\hat{p}$ is now complete:

$$\boxed{\alpha = \frac{1}{\sqrt{2}}\Big(\frac{\langle\hat{x}\rangle}{L_0} + i\,\frac{L_0\langle\hat{p}\rangle}{\hbar}\Big).} \tag{17.3.13}$$

Exercise 17.8. *Show that the anti-Hermitian operator in the exponent of $D(\alpha)$ can be written as follows:*

$$\alpha\,\hat{a}^\dagger - \alpha^*\hat{a} = -\tfrac{i}{\hbar}\big(\hat{p}\,\langle x\rangle - \langle\hat{p}\rangle\,\hat{x}\big). \tag{17.3.14}$$

Assuming α is defined as in (17.3.13), this result allows us to rewrite the general coherent state (17.3.2) as follows:

$$\boxed{|\alpha\rangle = \exp\Big(-\frac{i\hat{p}\,\langle\hat{x}\rangle}{\hbar} + \frac{i\langle\hat{p}\rangle\,\hat{x}}{\hbar}\Big)|0\rangle.} \tag{17.3.15}$$

In order to find the time dependence of the general coherent state $|\alpha\rangle$, we use the time-evolution operator:

$$|\alpha, t\rangle \equiv e^{-\frac{i\hat{H}t}{\hbar}}|\alpha\rangle = \Big(e^{-i\frac{Ht}{\hbar}}\exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})\,e^{i\frac{Ht}{\hbar}}\Big)e^{-i\frac{Ht}{\hbar}}|0\rangle. \tag{17.3.16}$$

To proceed, we can use the Heisenberg picture. For a time-independent Hamiltonian (as that of the simple harmonic oscillator) and a Schrödinger operator $\square$, we have $\square_H(t) = e^{iHt/\hbar}\square\, e^{-iHt/\hbar}$. With the opposite signs for the exponentials, we get $e^{-iHt/\hbar}\square\, e^{iHt/\hbar} = \square_H(-t)$. Such a relation is also valid for any function of an operator: $e^{-iHt/\hbar}F(\square)e^{iHt/\hbar} = F(\square_H(-t))$, as you can convince yourself whenever $F(x)$ has a convergent Taylor expansion in powers of x. It then follows that back in (17.3.16) we have:

$$|\alpha, t\rangle = \exp\Big(\alpha\hat{a}^\dagger(-t) - \alpha^*\hat{a}(-t)\Big)e^{-i\omega t/2}|0\rangle. \tag{17.3.17}$$

Recalling from (16.5.31) that $\hat{a}(t) = e^{-i\omega t}\,\hat{a}$ and $\hat{a}^\dagger(t) = e^{i\omega t}\,\hat{a}^\dagger$, we find that

$$|\alpha, t\rangle = e^{-i\omega t/2} \exp\left(\alpha e^{-i\omega t}\hat{a}^\dagger - \alpha^* e^{i\omega t}\hat{a}\right)|0\rangle. \tag{17.3.18}$$

Looking at the exponential, we see that it is in fact the displacement operator with α replaced by $\alpha e^{-i\omega t}$. As a result, we have shown that

$$\boxed{|\alpha, t\rangle = e^{-i\omega t/2}|e^{-i\omega t}\alpha\rangle.} \tag{17.3.19}$$

This is how a coherent state $|\alpha\rangle$ evolves in time: up to an irrelevant phase, the state remains a coherent state with a time-varying parameter $e^{-i\omega t}\alpha$. In the complex α plane, the state is represented by a vector that rotates clockwise with angular velocity ω. The α plane can be viewed as having a real axis that gives $\langle \hat{x} \rangle$, up to a proportionality constant, and an imaginary axis that gives $\langle \hat{p} \rangle$, up to a proportionality constant. The evolution of any state is represented by a circle. This is illustrated in figure 17.2.

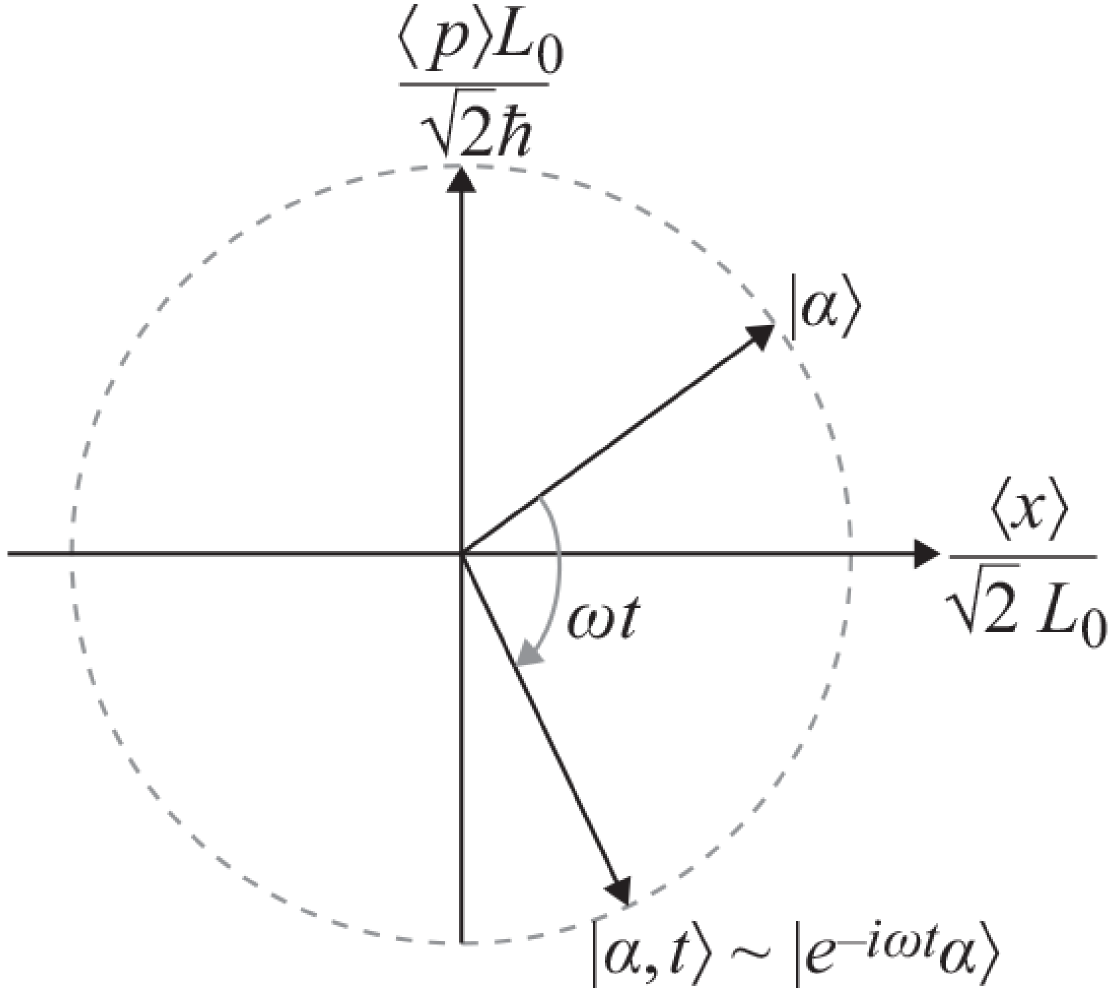

Figure17.2

Time evolution of the coherent state $|\alpha\rangle$. The real and imaginary parts of α determine, respectively, the expectation values $\langle \hat{x}\rangle$ and $\langle \hat{p}\ \rangle$. As time goes by, the α parameter of the coherent state rotates clockwise with angular velocity ω.