16.4 Heisenberg Picture

The idea here is to confine the dynamical evolution to the operators. We will "fold" the time dependence of the states into the operators. Since the objects we usually calculate are time dependent expectation values of operators, this approach turns out to be quite effective. The result is a new "picture" of quantum mechanics, the *Heisenberg picture* in which the operators carry the time dependence and the states do not evolve.

We will define time-dependent Heisenberg operators starting from Schrödinger operators. Schrödinger operators are in fact the operators we have been using all along, such as $\hat{x}$, $\hat{p}$, $\hat{L}_i$, $\hat{S}_i$, and others, as well as operators constructed from them, possibly including time. Schrödinger operators come in two types: time independent, like $\hat{x}$ or $\hat{p}$, and time dependent, like Hamiltonians with time-dependent potentials. For each Schrödinger operator, we will associate a Heisenberg operator.

Let us consider a Schrödinger operator $\hat{A}_S$, with the subscript S for Schrödinger. This operator may or may not have time dependence. We now examine a matrix element of $\hat{A}_S$ in between time-dependent states $|\alpha, t\rangle$ and $|\beta, t\rangle$ and use the time-evolution operator to convert the states to time zero:

$$\langle\alpha, t|\hat{A}_S|\beta, t\rangle = \langle\alpha, 0|\mathcal{U}^\dagger(t, 0)\,\hat{A}_S\,\mathcal{U}(t, 0)\,|\beta, 0\rangle. \tag{16.4.1}$$

We simply define the Heisenberg operator $\hat{A}_H(t)$ associated with $\hat{A}_S$ as the object in between the time equal zero states:

$$\boxed{\hat{A}_H(t) \equiv \mathcal{U}^\dagger(t,0)\,\hat{A}_S\,\mathcal{U}(t,0).} \tag{16.4.2}$$

The Heisenberg operator is obtained from the Schrödinger operator by a similarity transformation generated by the time-evolution operator □(*t,* 0).

Let us consider a number of important consequences of this definition.

1. At $t = 0$, the Heisenberg operator becomes equal to the Schrödinger operator:

$$\hat{A}_H(0) = \hat{A}_S. \tag{16.4.3}$$

The Heisenberg operator associated with the identity operator is the identity operator:

$$\mathbb{1}_H = \mathcal{U}^\dagger(t,0)\,\mathbb{1}\,\mathcal{U}(t,0) = \mathbb{1}. \tag{16.4.4}$$

2. The Heisenberg operator associated with the product of Schrödinger operators is equal to the product of the corresponding Heisenberg operators:

$$\hat{C}_S = \hat{A}_S\hat{B}_S \quad \Rightarrow \quad \hat{C}_H(t) = \hat{A}_H(t)\hat{B}_H(t). \tag{16.4.5}$$

Indeed,

$$\begin{aligned}\hat{C}_H(t) &= \mathcal{U}^\dagger(t,0)\,\hat{C}_S\mathcal{U}(t,0) = \mathcal{U}^\dagger(t,0)\,\hat{A}_S\hat{B}_S\mathcal{U}(t,0)\\ &= \hat{\mathcal{U}}^\dagger(t,0)\,\hat{A}_S\mathcal{U}(t,0)\mathcal{U}^\dagger(t,0)\,\hat{B}_S\mathcal{U}(t,0) = \hat{A}_H(t)\hat{B}_H(t).\end{aligned} \tag{16.4.6}$$

3. It also follows from (16.4.5) that if we have a commutator of Schrödinger operators the corresponding Heisenberg operators satisfy the same commutation relations:

$$[\hat{A}_S, \hat{B}_S] = \hat{C}_S \quad \Rightarrow \quad [\hat{A}_H(t), \hat{B}_H(t)] = \hat{C}_H(t). \tag{16.4.7}$$

Since $\mathbb{1}_H = \mathbb{1}$, equation (16.4.7) implies that, for example,

$$[\hat{x}, \hat{p}] = i\hbar\mathbb{1} \quad \Rightarrow \quad [\hat{x}_H(t), \hat{p}_H(t)] = i\hbar\mathbb{1}. \tag{16.4.8}$$

4. Schrödinger and Heisenberg Hamiltonians: Consider a Schrödinger Hamiltonian H_S that depends on some Schrödinger momenta and

position operators $\hat{p}$ and $\hat{x}$:

$$\hat{H}_S(\hat{x}, \hat{p}\,;\, t). \tag{16.4.9}$$

Since the $\hat{x}$ and $\hat{p}$ operators in $\hat{H}_S$ appear in products, property (2) implies that the associated Heisenberg Hamiltonian $\hat{H}_H$ takes the same form but with $\hat{x}$ and $\hat{p}$ replaced by their Heisenberg counterparts:

$$\hat{H}_H(t) = \hat{H}_S(\hat{x}_H(t), \hat{p}_H(t)\,;\, t). \tag{16.4.10}$$

5. Equality of Hamiltonians: When $[\hat{H}_S(t), \hat{H}_S(t')] = 0$, for all t, t', the Heisenberg Hamiltonian is in fact equal to the Schrödinger Hamiltonian. To see this, recall that for this type of Hamiltonian the time evolution operator is

$$\mathcal{U}(t, 0) = \exp\Big[-\frac{i}{\hbar}\int_0^t dt'\hat{H}_S(t')\Big]. \tag{16.4.11}$$

Moreover, by definition,

$$\hat{H}_H(t) = \mathcal{U}^\dagger(t, 0)\,\hat{H}_S(t)\,\mathcal{U}(t, 0). \tag{16.4.12}$$

Since the $\hat{H}_S$ commute at different times, $\hat{H}_S(t)$ commutes both with $\mathcal{U}(t, 0)$ and $\mathcal{U}^\dagger(t, 0)$. Therefore, the $\hat{H}_S(t)$ in (16.4.12) can be moved, say, to the right, giving us

$$\boxed{\hat{H}_H(t) = \hat{H}_S(t), \quad \text{when } [\hat{H}_S(t), \hat{H}_S(t')] = 0.} \tag{16.4.13}$$

Clearly, this equality holds for time-independent Schrödinger Hamiltonians. The meaning of this equality becomes clearer when we use (16.4.10) and (16.4.9) to write

$$\hat{H}_S(\hat{x}_H(t), \hat{p}_H(t)\,;\, t) = \hat{H}_S(\hat{x}, \hat{p}\,;\, t). \tag{16.4.14}$$

Operationally, this means that if we take $\hat{x}_H(t)$ and $\hat{p}_H(t)$ and plug them into the Schrödinger Hamiltonian (left-hand side), the result is as if we had simply plugged $\hat{x}$ and $\hat{p}$. We will confirm this for the case of the simple harmonic oscillator.

6. Equality of operators: If a Schrödinger operator $\hat{A}_S$ commutes with the Hamiltonian $\hat{H}_S(t)$ for all times, then $\hat{A}_S$ commutes with □$(t, 0)$ since this operator (even in the most complicated of cases) is built using $\hat{H}_S(t)$. It follows that $\hat{A}_H(t) = \hat{A}_S$; the Heisenberg operator is equal to the Schrödinger operator. In summary,

$$[A_S, \hat{H}_S(t)] = 0, \quad \forall\, t \quad \Rightarrow \quad \hat{A}_H(t) = \hat{A}_S. \tag{16.4.15}$$

7. Expectation values: Consider (16.4.1) and let $|\alpha, t\rangle = |\beta, t\rangle = |\Psi, t\rangle$. The matrix element now becomes an expectation value and we have

$$\boxed{\langle \Psi, t|\hat{A}_S|\Psi, t\rangle \;=\; \langle \Psi, 0|\, \hat{A}_H(t)\, |\Psi, 0\rangle.} \tag{16.4.16}$$

With a little abuse of notation, we simply write this equation as

$$\boxed{\langle \hat{A}_S\rangle \;=\; \langle \hat{A}_H(t)\rangle.} \tag{16.4.17}$$

When writing such an equation, you should realize that on the left-hand side you compute the expectation value using the time-dependent state, while on the right-hand side you compute the expectation value using the state at time equal zero.