18.8 EPR and Bell Inequalities

In this section we begin by studying some properties of the singlet state of two particles of spin one-half. We then turn to the claims of Einstein, Podolsky, and Rosen (EPR) concerning entangled states in quantum mechanics. Finally, we discuss the so-called Bell inequalities that would follow if EPR were right. Of course, quantum mechanics violates these inequalities, which experiment indeed shows. EPR were wrong.

We have been talking about the singlet state of two spin one-half particles. This state emerges, for example, in particle decays. The neutral η_0 meson, of rest mass 547 MeV, sometimes decays into a muon and an antimuon of opposite charge:

$$\eta_0 \to \mu^+ + \mu^-. \tag{18.8.1}$$

The meson is a spinless particle that being at rest has zero orbital angular momentum. As a result, it has zero total angular momentum. As it decays, the final state of the two muons must have zero total angular momentum as well. Most often, the state of the two muons has zero orbital angular momentum. In such a situation, conservation of angular momentum requires zero total spin angular momentum. The muon antimuon pair, flying away from each other with zero orbital angular momentum, are in a singlet state. This state takes the form

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\big(|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2\big). \tag{18.8.2}$$

As an angular momentum singlet, this state is rotational invariant (see also problem 18.1). The state is is fact the same for whatever direction **n** we use to define a basis of spin states:

$$|\Psi\rangle \;=\; \frac{1}{\sqrt{2}}\big(|\mathbf{n};+\rangle_1|\mathbf{n};-\rangle_2 - |\mathbf{n};-\rangle_1|\mathbf{n};+\rangle_2\big). \tag{18.8.3}$$

We now ask: In this singlet what is the probability $P(\mathbf{a}, \mathbf{b})$ that the first particle is in the state $|\mathbf{a}; +\rangle$, and the second particle is in the state $|\mathbf{b}; +\rangle$, with **a** and **b** two arbitrarily chosen unit vectors? To help ourselves, we write the singlet state using the first vector:

$$|\Psi\rangle \;=\; \frac{1}{\sqrt{2}}\big(|\mathbf{a};+\rangle_1|\mathbf{a};-\rangle_2 - |\mathbf{a};-\rangle_1|\mathbf{a};+\rangle_2\big). \tag{18.8.4}$$

By definition, the probability we want is

$$P(\mathbf{a},\mathbf{b}) \;=\; \Big|{}_1\langle\mathbf{a};+|{}_2\langle\mathbf{b};+|\Psi\rangle\Big|^2. \tag{18.8.5}$$

Only the first term in (18.8.4) contributes and we get

$$P(\mathbf{a},\mathbf{b}) \;=\; \tfrac{1}{2}\big|\langle\mathbf{b};+|\mathbf{a};-\rangle\big|^2. \tag{18.8.6}$$

We recall that the overlap squared between two spin states is given by the cosine squared of half the angle in between them (example 14.2). Using figure 18.2, we see that the angle between **b** and $-\mathbf{a}$ is $\pi - \theta_{ab}$, where θ_{ab} is the angle between **b** and **a**. Therefore,

$$P(\mathbf{a},\mathbf{b}) \;=\; \tfrac{1}{2}\cos^2\!\big(\tfrac{1}{2}(\pi - \theta_{ab})\big). \tag{18.8.7}$$

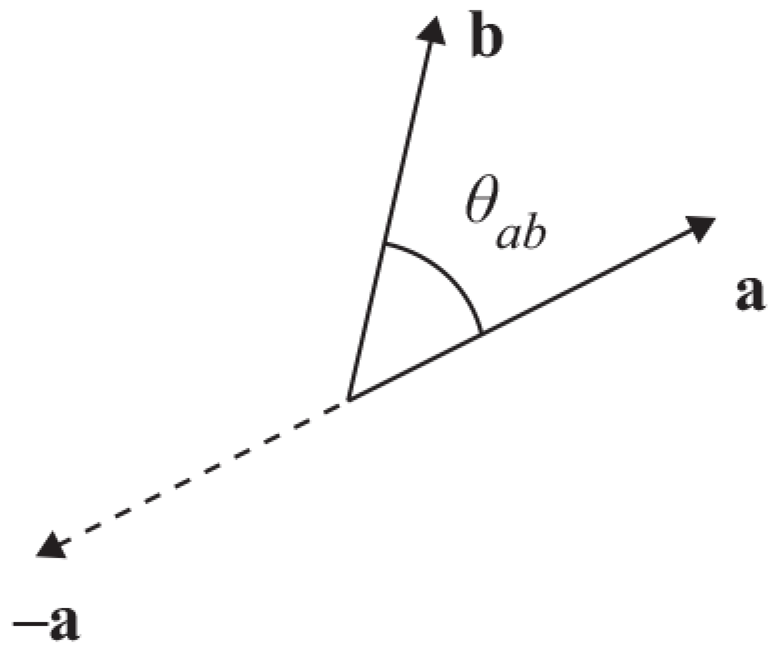

Figure 18.2
Directions associated with the vectors **a** and **b**.

Our final result is therefore

$$\boxed{P(\mathbf{a},\mathbf{b}) \;=\; \tfrac{1}{2}\sin^2\!\left(\tfrac{1}{2}\theta_{ab}\right).} \tag{18.8.8}$$

As a simple consistency check, if $\mathbf{b} = -\mathbf{a}$, then $\theta_{ab} = \pi$, and $P(\mathbf{a}, -\mathbf{a}) = 1/2$, which is what we expect. If we measure using orthogonal vectors, like the unit vectors $\hat{\mathrm{x}}$ and $\hat{\mathbf{z}}$, we get

$$P(\hat{\mathbf{z}},\hat{\mathbf{x}}) = \tfrac{1}{2}\sin^2 45^\circ = \tfrac{1}{2}\cdot\tfrac{1}{2} = \tfrac{1}{4}. \tag{18.8.9}$$

This is all we will need to know about singlet states.

The statements by EPR dealt with entangled states and formulated the notion of **local realism**. This is understood as two properties of measurement:

1. The result of a measurement at one point cannot depend on whatever action takes place at a faraway point at the same time.
2. The result of a measurement on a particle in an entangled pair corresponds to some element of reality. If the measurement of an

observable gives a value, that value was a definite property of the state before measurement.

Both properties seem eminently reasonable at first thought, but both are violated in quantum mechanics. The violation of the first is disturbing, given our intuition that simultaneous, spatially separated events can't affect each other. But there is something nonlocal about quantum mechanics. The violation of the second is something we have already seen repeatedly in nonentangled contexts. Measurement involves collapse of the wave function; the result is not preordained and does not correspond to an unequivocal property of the system.

Following the logic of EPR, the so-called entangled singlet pairs are just pairs of particles that have definite spins. Moreover, in this logic the results of quantum mechanical measurements are reproduced if our large ensemble of pairs has the following distribution of states:

- In 50% of pairs, particle 1 has spin along $\hat{z}$, and particle 2 has spin along $-\hat{z}$.
- In 50% of pairs, particle 1 has spin along $-\hat{z}$, and particle 2 has spin along $\hat{z}$.

This would explain the perfect correlations between the spins of the two particles and is consistent, for example, with $P(\hat{z}, -\hat{z}) = 1/2$, which we obtained quantum mechanically.

The challenge for the EPR proposal is to keep reproducing the results of more complicated measurements. Suppose each of the two observers can measure spin along two possible axes: the x- and z-axes. They measure in any of these two directions. EPR logic would posit that in any entangled pair each particle has a definite state of spin in these two directions. For example, a particle of type $[\hat{z}, -\hat{x}]$ is one that if measured along z always gives $\hbar/2$, while if measured along x gives $-\hbar/2$. This of course is not possible in quantum mechanics: if we want to guarantee that measurement along z gives $\hbar/2$, the state of the particle must be $|+\rangle$, in which case the result of measuring along x cannot be predicted. EPR logic implies that there are states of particles in which noncommuting variables have fixed, predictable values. In this setup the observed quantum mechanical results are matched if our ensemble of pairs has the following properties:

- 25% of pairs have particle 1 in $[\hat{z}, \hat{x}]$ and particle 2 in $[-\hat{z}, -\hat{x}]$,
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- 25% of pairs have particle 1 in $[-\hat{z}, -\hat{x}]$ and particle 2 in $[\hat{z}, \hat{x}]$.

First, note some perfect correlations in this EPR-like setup: particles 1 and 2 have opposite spins in each possible direction, so knowing the spin of one in a particular direction tells us the spin of the second. This is, of course, needed to match the properties of the quantum mechanical singlets: if both measure in the same direction, they must get opposite values of the spin. We can ask: What is the probability $P(\hat{z}, -\hat{z})$ that particle 1 is along $\hat{z}$ and particle 2 along $-\hat{z}$? The first two cases above apply, and thus this probability is 1/2, consistent with quantum mechanics. We can also ask for $P(\hat{z}, \hat{x})$. This time only the second case applies, giving us a probability of 1/4, as we obtained earlier in (18.8.9). The quantum mechanical answers indeed arise for all questions.

The insight of Bell was to realize that when we can measure in *three* directions the quantum mechanical answers *cannot* be reproduced by suitable ensembles with pairs of particles having their own definite spin states. For this, he showed that such ensembles imply inequalities—the Bell inequalities—that are violated in quantum mechanics. Indeed, suppose each observer can measure along any one of the three vectors **a, b, c**. Each particle is just measured once, along one of these directions. Let us assume that we have a large number N of pairs that, following the EPR logic, contain particles with well-defined spins on these three directions. A particle of type [**a**, −**b**, **c**], for example, would give $\hbar/2$, $-\hbar/2$, and $\hbar/2$ if measured along **a**, **b**, or **c**, respectively. Again, this kind of state does not exist in quantum mechanics. Now, EPR logic would try to give a distribution of pairs of different types that would match quantum mechanical results. We will now show that any distribution will disagree with quantum mechanics. To do this, we keep the values of the populations general:

Populations	**Particle 1**	**Particle 2**
N_1	$[\ \mathbf{a}, \mathbf{b}, \mathbf{c}]$	$[-\mathbf{a}, -\mathbf{b}, -\mathbf{c}]$
N_2	$[\ \mathbf{a}, \mathbf{b}, -\mathbf{c}]$	$[-\mathbf{a}, -\mathbf{b}, \mathbf{c}]$
N_3	$[\ \mathbf{a}, -\mathbf{b}, \mathbf{c}]$	$[-\mathbf{a}, \mathbf{b}, -\mathbf{c}]$
N_4	$[\ \mathbf{a}, -\mathbf{b}, -\mathbf{c}]$	$[-\mathbf{a}, \mathbf{b}, \mathbf{c}]$
N_5	$[-\mathbf{a}, \mathbf{b}, \mathbf{c}]$	$[\ \mathbf{a}, -\mathbf{b}, -\mathbf{c}]$
N_6	$[-\mathbf{a}, \mathbf{b}, -\mathbf{c}]$	$[\ \mathbf{a}, -\mathbf{b}, \mathbf{c}]$
N_7	$[-\mathbf{a}, -\mathbf{b}, \mathbf{c}]$	$[\ \mathbf{a}, \mathbf{b}, -\mathbf{c}]$
N_8	$[-\mathbf{a}, -\mathbf{b}, -\mathbf{c}]$	$[\ \mathbf{a}, \mathbf{b}, \mathbf{c}]$

As required, all spins are properly correlated in particles 1 and 2. Moreover, $N = \sum_{i=1}^{8} N_i$. To find the probability $P(\mathbf{a}, \mathbf{b})$, for example, we look for the lines in the table that have particle 1 in **a** as well as particle 2 in **b**. The populations that satisfy this are N_3 and N_4, and their sum divided by the total number N of pairs is the desired probability. In this way we record the following probabilities:

$$P(\mathbf{a},\mathbf{b}) = \frac{N_3 + N_4}{N}, \quad P(\mathbf{a},\mathbf{c}) = \frac{N_2 + N_4}{N}, \quad P(\mathbf{c},\mathbf{b}) = \frac{N_3 + N_7}{N}. \tag{18.8.10}$$

Consider now the trivially correct inequality:

$$N_3 + N_4 \ \le\ N_2 + N_4 + \ N_3 + N_7 \tag{18.8.11}$$

which on account of (18.8.10) implies the **Bell inequality**:

$$\boxed{P(\mathbf{a},\mathbf{b}) \ \le\ P(\mathbf{a},\mathbf{c}) + P(\mathbf{c},\mathbf{b}).} \tag{18.8.12}$$

If true quantum mechanically, given (18.8.8) we would have

$$\tfrac{1}{2}\sin^2 \tfrac{1}{2}\theta_{ab} \ \le\ \tfrac{1}{2}\sin^2 \tfrac{1}{2}\theta_{ac} \ + \tfrac{1}{2}\sin^2 \tfrac{1}{2}\theta_{cb}. \tag{18.8.13}$$

But this is violated for many choices of angles. Take, for example, the planar configuration in figure 18.3:

$$\theta_{ab} = 2\theta, \quad \theta_{ac} = \theta_{cb} = \theta. \tag{18.8.14}$$

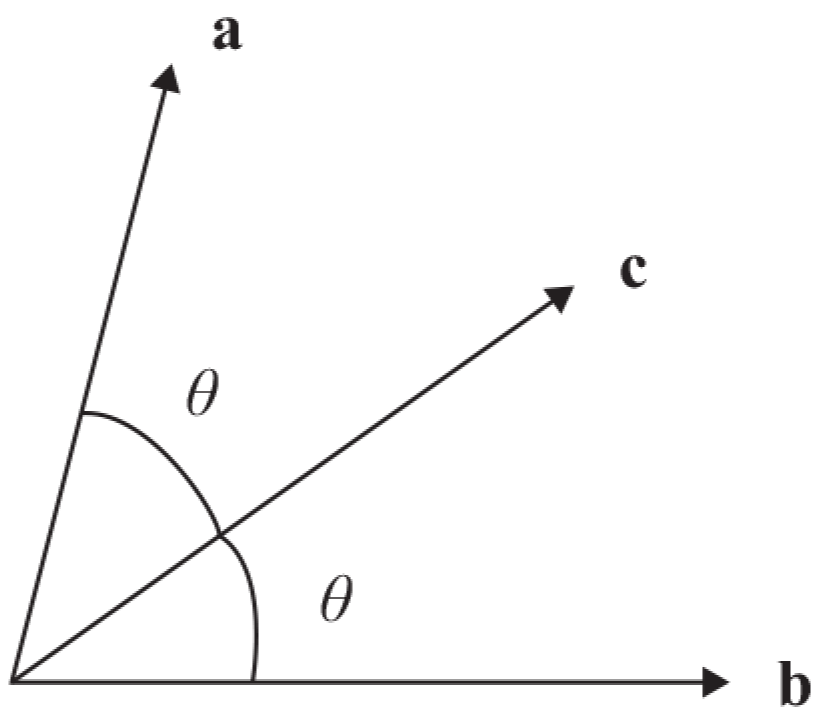

Figure 18.3
A planar configuration for the unit vectors **a**, **b**, and **c**. For $\theta < \frac{\pi}{2}$ the Bell inequality is violated quantum mechanically.

For this situation, the inequality becomes

$$\tfrac{1}{2}\sin^2\theta \le \sin^2 \tfrac{1}{2}\theta. \tag{18.8.15}$$

This fails for sufficiently small θ: $\frac{1}{2}\theta^2 \le \frac{\theta^2}{4}$ is just plain wrong. In fact, the Bell inequality is violated in quantum mechanics for any $\theta < \frac{\pi}{2}$. Experimental results have confirmed that Bell inequalities are violated, and thus the original claim of local realism by EPR is incorrect.

The arguments of Bell have been extended in several directions, and various examples have provided extra insight into the remarkable properties of entangled quantum states. Some of these directions are considered in the problems. Let us look at some of the ideas:

1. We have spoken about correlations. Let's be more precise about this term. In the spin singlet state, the spins of the two particles are said to be perfectly anticorrelated *when* measured in the same direction. Measuring the spin of one along **n** and reading the result tells us the

spin of the other particle along **n**: it will be the opposite. There are also *correlation functions*. In a composite system AB with parts A and B and in the state $|\Psi_{AB}\rangle$, the expectation value of $\mathcal{O}_A \otimes \mathcal{O}_B$,

$$\langle \mathcal{O}_A \otimes \mathcal{O}_B \rangle = \langle \Psi_{AB} | \mathcal{O}_A \otimes \mathcal{O}_B | \Psi_{AB} \rangle, \tag{18.8.16}$$

with $\mathcal{O}_A \in \mathcal{L}(\mathcal{H}_A)$ and $\mathcal{O}_B \in \mathcal{L}(\mathcal{H}_B)$ is called the correlation function of $\mathcal{O}_A$ and $\mathcal{O}_B$. If the state of the system is a product state $|\Psi_{AB}\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$, then

$$\begin{aligned}
\langle \mathcal{O}_A \otimes \mathcal{O}_B \rangle &= \langle \Psi_A | \otimes \langle \Psi_B | \mathcal{O}_A \otimes \mathcal{O}_B | \Psi_A \rangle \otimes | \Psi_B \rangle \\
&= \langle \Psi_A | \mathcal{O}_A | \Psi_A \rangle \, \langle \Psi_B | \mathcal{O}_B | \Psi_B \rangle \\
&= \langle \mathcal{O}_A \rangle \langle \mathcal{O}_B \rangle,
\end{aligned} \tag{18.8.17}$$

showing that the correlator factorizes into the product of expectation values when the state of the full system is not entangled. If the state is entangled, the correlation generically fails to factorize. For example, in the spin singlet state of two spin one-half particles and for unit vectors **a** and **b** you will show (problem 18.4) that the product of spin operators has the correlation

$$\langle \mathbf{a} \cdot \boldsymbol{\sigma}^{(1)} \otimes \mathbf{b} \cdot \boldsymbol{\sigma}^{(2)} \rangle = -\cos\theta_{ab}, \tag{18.8.18}$$

where θ_{ab} is the angle between the two unit vectors. This right-hand side does not factorize into the product of a function that depends on **a** and a function that depends on **b**, showing the entanglement of the state. The EPR analogs of correlators of this type lead to a version of Bell inequalities (problem 18.4) that is indeed violated by the quantum correlators.

2. A Bell-type inequality obtained by Clauser, Horn, Shimony, and Holt (CHSH) is explored in problem 18.5. They consider an entangled system AB and two operators $\hat{A}_1$, $\hat{A}_2$ in system A as well as two operators $\hat{B}_1$, $\hat{B}_2$ in system B. All these operators have eigenvalues ± 1. As it turns out, on the entangled state the expectation value of the operator $\hat{Q}$ defined as

$$\hat{Q} \equiv \hat{A}_1 \otimes \hat{B}_1 - \hat{A}_1 \otimes \hat{B}_2 + \hat{A}_2 \otimes \hat{B}_1 + \hat{A}_2 \otimes \hat{B}_2 \tag{18.8.19}$$

exceeds the value it would take if the $\hat{A}$ and $\hat{B}$ operators were replaced by random variables with deterministic values of ± 1.

3. Bell inequalities can be derived from simple classes of *local hidden variable* theories. A hidden variable is a hypothetical quantity whose values are not accessible to the experimentalist. Let us denote by the generic label λ a set of hidden variables. These hidden variables are presumed to come with a probability distribution. When EPR talk about a particle measured by Alice of type $[\hat{z}, \hat{x}]$, it means that Alice measures + 1 for σ_z and for σ_x. In a theory of hidden variables, one would have two functions $A(z, \lambda)$ and $A(x, \lambda)$, both taking values of ± 1. The first one, $A(z, \lambda)$, is Alice's measured value of spin along z when the hidden variable takes values λ. Similarly, $A(x, \lambda)$ is Alice's measured value of spin along x when the hidden variable takes values λ. For a particle of type $[\hat{z}, \hat{x}]$, one must have λ such that both these functions give + 1. Bob has similar functions, and for each entangled pair, the value of λ is the same as Alice's, reflecting the locality of the hidden variable theory. The functions determine uniquely the measured values once we know λ. The analog of quantum expectation values is the averaging over the values of the hidden variables, weighted by the probability distribution. Interestingly, independent of the probability distribution one can derive inequalities that are violated by quantum probabilities. The Bell inequality considered in problem 18.4 can be derived from a hidden variable theory.

4. Bell inequalities, as discussed above, involve probabilities or expectation values of operators. To determine these quantities we require repeated measurements. Some tests of quantum mechanics involve deterministic quantities—namely, the measurement of operators that happen to take definite values on the entangled state. The hidden variable theory analysis is done with a definite value of the hidden variable, and its prediction plainly disagrees with the quantum prediction. A particularly simple and elegant entangled state of three spin one-half particles illustrating this possibility (problem 18.6) was discussed by Greenberg, Horne, and Zeilinger (GHZ):

$$|\Phi\rangle = \frac{1}{\sqrt{2}}\big(\,|+\rangle|+\rangle|+\rangle\, - \,|-\rangle|-\rangle|-\rangle\,\big). \qquad (18.8.20)$$

One considers operators $\hat{O}_i$, with $i = 1, 2, 3$, each acting on the state space of the three particles. The state $|\Phi\rangle$ is an eigenstate of all three $\hat{O}_i$'s, with eigenvalues all equal to $+1$. The operators are such that $\hat{O}_1\hat{O}_2\hat{O}_3 = -\hat{O}$, with $\hat{O}$ an operator for which $|\Phi\rangle$ must be an eigenstate of eigenvalue -1. In the hidden variable theory, whenever the $\hat{O}_i$ analogs have value $+1$ so does the analog of $\hat{O}$, giving us a discrepancy with quantum mechanics.

5. If Alice and Bob are far away and each has a particle from an entangled pair, they cannot use this pair to send information to each other. The correlations in the pair allow for no signaling. While we have focused on nonrelativistic quantum mechanics, signaling would be a problem in relativistic quantum mechanics, for it could allow information to be sent faster than light. We will discuss no signaling in detail in section 22.4 (theorem 22.4.2). You can see in problem 18.7 the failure of signaling with a particular strategy. Sharing entangled pairs, however, can somehow help Alice and Bob, who are far apart and cannot communicate with each other, do better in a game they play against Charlie (problems 18.8 and 18.9).

Exercise 18.17. *Alice, Bob, and Charlie are in possession of particles, 1, 2, and 3, respectively, in an entangled state*

$$|\Phi\rangle = \frac{1}{\sqrt{2}}\big(|+\rangle|+\rangle|+\rangle - |-\rangle|-\rangle|-\rangle\big). \tag{18.8.21}$$

Assume Alice measures the spin of particle 1 along the z-direction. Describe the possible states of particles 2 and 3 after her measurement. Are particles 2 and 3 entangled? Assume instead that Alice measures the spin of particle 1 along the x-direction. Describe the possible states of particles 2 and 3 after measurement. Are particles 2 and 3 entangled?