8.05x: Quantum Physics II Formula Sheet Exam 2

Department of Physics, Massachusetts Institute of Technology

- Gaussian integrals ($\alpha > 0$)

$$\int_{-\infty}^{\infty} \mathrm{d}x e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}, \quad \int_{-\infty}^{\infty} \mathrm{d}x x^2 e^{-\alpha x^2} = \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}}, \quad \int_{-\infty}^{\infty} \mathrm{d}x x^4 e^{-\alpha x^2} = \frac{3}{4}\sqrt{\frac{\pi}{\alpha^5}}$$

- Trigonometric functions

$$\sin x = \left(e^{ix} - e^{-ix}\right)/2i, \quad \cos x = \left(e^{ix} + e^{-ix}\right)/2$$
$$\sinh x = \left(e^{x} - e^{-x}\right)/2, \quad \cosh x = \left(e^{x} + e^{-x}\right)/2$$

- Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \hat{H}\Psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right]\Psi(x,t)$$
$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x)$$

- Conservation of probability

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}J(x,t) = 0$$
$$\rho(x,t) = |\psi(x,t)|^2; \quad J(x,t) = \frac{\hbar}{2im}\left[\psi^*\frac{\partial}{\partial x}\psi - \psi\frac{\partial}{\partial x}\psi^*\right]$$

- Variational principle:

$$E_{gs} \leq \frac{\int dx \psi^*(x) H \psi(x)}{\int dx \psi^*(x)\psi(x)} \equiv \langle H \rangle_\psi \quad \text{for all } \psi(x)$$

- Spin-1/2 particle:

$$\text{Stern-Gerlach}: \quad H = -\boldsymbol{\mu}\cdot\mathbf{B}, \quad \boldsymbol{\mu} = g\frac{e\hbar}{2m}\frac{1}{\hbar}\mathbf{S} = \gamma\mathbf{S}$$
$$\mu_B = \frac{e\hbar}{2m_e}, \quad \boldsymbol{\mu}_e = -2\mu_B\frac{\mathbf{S}}{\hbar}$$

In the basis $|1\rangle \equiv |z;+\rangle = |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |2\rangle \equiv |z;-\rangle = |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$S_i = \frac{\hbar}{2}\sigma_i \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \to [S_i, S_j] = i\hbar\epsilon_{ijk}S_k \quad (\epsilon_{123} = +1)$$

$$\sigma_i\sigma_j = \delta_{ij}\mathbf{1} + i\epsilon_{ijk}\sigma_k \quad \to \quad (\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{b}) = \mathbf{a}\cdot\mathbf{b}\mathbf{1} + i\boldsymbol{\sigma}\cdot(\mathbf{a}\times\mathbf{b})$$

$$e^{i\mathbf{M}\theta} = \mathbf{1}\cos\theta + i\mathbf{M}\sin\theta, \quad \text{if } \mathbf{M}^2 = \mathbf{1}$$

$$\exp(i\mathbf{a}\cdot\boldsymbol{\sigma}) = \mathbf{1}\cos a + i\boldsymbol{\sigma}\cdot\left(\frac{\mathbf{a}}{a}\right)\sin a, \quad a = |\mathbf{a}|$$

$$\exp(i\theta\sigma_3)\,\sigma_1\exp(-i\theta\sigma_3) = \sigma_1\cos(2\theta) - \sigma_2\sin(2\theta)$$

$$\exp(i\theta\sigma_3)\,\sigma_2\exp(-i\theta\sigma_3) = \sigma_2\cos(2\theta) + \sigma_1\sin(2\theta)$$

$$S_{\mathbf{n}} = \mathbf{n}\cdot\mathbf{S} = n_xS_x + n_yS_y + n_zS_z = \frac{\hbar}{2}\mathbf{n}\cdot\boldsymbol{\sigma}$$

$$(n_x, n_y, n_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \quad S_{\mathbf{n}}|\mathbf{n};\pm\rangle = \pm\frac{\hbar}{2}|\mathbf{n};\pm\rangle$$

$$|\mathbf{n};+\rangle = \cos\left(\frac{1}{2}\theta\right)|+\rangle + \sin\left(\frac{1}{2}\theta\right)\exp(i\phi)|-\rangle$$

$$|\mathbf{n};-\rangle = -\sin\left(\frac{1}{2}\theta\right)\exp(-i\phi)|+\rangle + \cos\left(\frac{1}{2}\theta\right)|-\rangle$$

$$|\langle\mathbf{n}';+\mid\mathbf{n};+\rangle| = \cos\left(\frac{1}{2}\gamma\right), \quad \gamma \text{ is the angle between } \mathbf{n} \text{ and } \mathbf{n}'$$

$$\langle\mathbf{S}\rangle_{\mathbf{n}} = \frac{\hbar}{2}\mathbf{n}, \quad \text{Rotation operator: } R_\alpha(\mathbf{n}) \equiv \exp\left(-\frac{i\alpha S_{\mathbf{n}}}{\hbar}\right)$$

- Linear algebra

$$\text{Matrix representation of } T \text{ in the basis } (v_1, \ldots, v_n): Tv_j = \sum_i T_{ij}v_i$$

$$\text{basis change: } u_k = \sum_j A_{jk}v_j, \quad T(\{u\}) = A^{-1}T(\{v\})A$$

$$\text{Schwarz: } |\langle u, v\rangle| \leq |u||v|$$

$$\text{Adjoint: } \langle u, Tv\rangle = \langle T^\dagger u, v\rangle, \quad (T^\dagger)^\dagger = T$$

- Bras and kets: For an operator Ω and a vector v, we write $|\Omega v\rangle \equiv \Omega|v\rangle$

$$\text{Adjoint: } \langle u \mid \Omega^\dagger v\rangle = \langle\Omega u \mid v\rangle$$

$$|\alpha_1v_1 + \alpha_2v_2\rangle = \alpha_1|v_1\rangle + \alpha_2|v_2\rangle \longleftrightarrow \langle\alpha_1v_1 + \alpha_2v_2| = \alpha_1^*\langle v_1| + \alpha_2^*\langle v_2|$$

- Complete orthonormal basis $|i\rangle$

$$\langle i \mid j\rangle=\delta_{ij}, \quad \mathbf{1}=\sum_i |i\rangle\langle i|$$

$$\Omega_{ij}=\langle i|\Omega|j\rangle \leftrightarrow \Omega=\sum_{i,j} \Omega_{ij}|i\rangle\langle j|$$

$$\left\langle i\left|\Omega^{\dagger}\right| j\right\rangle=\langle j|\Omega| i\rangle^{*}$$

Ω hermitian: $\Omega^{\dagger}=\Omega, \quad U$ unitary: $U^{\dagger}=U^{-1}$

- Matrix M is normal $\left(\left[M, M^{\dagger}\right]=0\right) \longleftrightarrow$ unitarily diagonalizable.
- Position and momentum representations: $\psi(x)=\langle x \mid \psi\rangle; \quad \tilde{\psi}(p)=\langle p \mid \psi\rangle;$

$$\hat{x}|x\rangle=x|x\rangle, \quad\langle x \mid y\rangle=\delta(x-y), \quad \mathbf{1}=\int dx|x\rangle\langle x|, \quad \hat{x}^{\dagger}=\hat{x}$$

$$\hat{p}|p\rangle=p|p\rangle, \quad\langle q \mid p\rangle=\delta(q-p), \quad \mathbf{1}=\int dp|p\rangle\langle p|, \quad \hat{p}^{\dagger}=\hat{p}$$

$$\langle x \mid p\rangle=\frac{1}{\sqrt{2 \pi \hbar}} \exp \left(\frac{i p x}{\hbar}\right); \quad \tilde{\psi}(p)=\int dx\langle p \mid x\rangle\langle x \mid \psi\rangle=\frac{1}{\sqrt{2 \pi \hbar}} \int dx \exp \left(-\frac{i p x}{\hbar}\right) \psi(x)$$

$$\langle x\left|\hat{p}^{n}\right| \psi\rangle=\left(\frac{\hbar}{i} \frac{d}{dx}\right)^{n} \psi(x); \quad\langle p\left|\hat{x}^{n}\right| \psi\rangle=\left(i \hbar \frac{d}{dp}\right)^{n} \tilde{\psi}(p); \quad[\hat{p}, f(\hat{x})]=\frac{\hbar}{i} f^{\prime}(\hat{x})$$

$$\frac{1}{2 \pi} \int_{-\infty}^{\infty} \exp (i k x) dx=\delta(k)$$

- Generalized uncertainty principle

$$\Delta A \equiv|(A-\langle A\rangle \mathbf{1}) \Psi| \quad(\Delta A)^{2}=\left\langle A^{2}\right\rangle-\langle A\rangle^{2} \geq 0.$$

$$\frac{\Delta A \Delta B \geq|\langle\Psi| 1}{2 i[A, B]|\Psi\rangle \mid}$$

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

$$\Delta x=\frac{\Delta}{\sqrt{2}} \text{ and } \Delta p=\frac{\hbar}{\sqrt{2} \Delta} \text{ for } \psi \sim \exp \left(-\frac{1}{2} \frac{x^{2}}{\Delta^{2}}\right)$$

$$\int_{-\infty}^{+\infty} dx \exp \left(-a x^{2}\right)=\sqrt{\frac{\pi}{a}}$$

Time independent operator $Q: \quad \frac{d}{dt}\langle Q\rangle=\frac{i}{\hbar}\langle[H, Q]\rangle$

$$\Delta H \Delta t \geq \frac{\hbar}{2}, \quad \Delta t \equiv \frac{\Delta Q}{\left|\frac{d\langle Q\rangle}{dt}\right|}$$

- Commutator identities

$$
\begin{aligned}
[A, BC] &= [A, B]C + B[A, C], \\
[AB, C] &= A[B, C] + [A, C]B, \\
e^{A} B e^{-A} &= e^{\mathrm{ad}_A} B = B + [A, B] + \frac{1}{2}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \dots, \\
e^{A} B e^{-A} &= B + [A, B], \quad \text{if } [A, [A, B]] = 0, \\
\left[B, e^{A}\right] &= [B, A] e^{A}, \quad \text{if } [A, [A, B]] = 0 \\
e^{A+B} &= e^{A} e^{B} e^{-\frac{1}{2}[A,B]} = e^{B} e^{A} e^{\frac{1}{2}[A,B]}, \quad \text{if } [A, B] \text{ commutes with } A \text{ and with } B
\end{aligned}
$$

- Gram-Schmidt procedure

  Given a basis $\{v_1, \dots, v_n\}$, an orthonormal basis is given by $\{e_1, \dots, e_n\}$, where $\tilde{e}_i = v_i - \sum_{j<i} \langle v_i, e_j \rangle e_j$ and $e_i = \tilde{e}_i / |\tilde{e}_i|$.

- Infinite square well:

$$
V = \begin{cases} 0 & \text{for } 0 < x < L \\ \infty & \text{otherwise} \end{cases}
$$

  Eigenfunctions and eigenenergies

$$
\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \quad E_n = \frac{\hbar^2 \pi^2 n^2}{2mL^2}
$$

- Harmonic Oscillator

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega\left(\hat{N} + \frac{1}{2}\right), \quad \hat{N} = \hat{a}^\dagger\hat{a}$$
$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right), \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right),$$
$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right), \quad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}\left(\hat{a}^\dagger - \hat{a}\right),$$
$$[\hat{x}, \hat{p}] = i\hbar, \quad \left[\hat{a}, \hat{a}^\dagger\right] = 1, \quad [\hat{N}, \hat{a}] = -\hat{a}, \quad \left[\hat{N}, \hat{a}^\dagger\right] = \hat{a}^\dagger.$$
$$\hat{H}|n\rangle = E_n|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle, \quad \hat{N}|n\rangle = n|n\rangle, \quad \langle m \mid n\rangle = \delta_{mn}$$
$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle.$$
$$\psi_0(x) = \langle x \mid 0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right).$$

$$x_H(t) = \hat{x}\cos\omega t + \frac{\hat{p}}{m\omega}\sin\omega t$$
$$p_H(t) = \hat{p}\cos\omega t - m\omega\hat{x}\sin\omega t$$

- Coherent states

$$T_{x_0} \equiv e^{-\frac{i}{\hbar}\hat{p}x_0}, \quad T_{x_0}|x\rangle = |x + x_0\rangle$$
$$|\tilde{x}_0\rangle \equiv T_{x_0}|0\rangle = e^{-\frac{i}{\hbar}\hat{p}x_0}|0\rangle$$
$$|\tilde{x}_0\rangle = e^{-\frac{1}{4}\frac{x_0^2}{d^2}}e^{\frac{x_0}{\sqrt{2}d}a^\dagger}|0\rangle, \quad \langle x \mid \tilde{x}_0\rangle = \psi_0\left(x - x_0\right), \quad d^2 = \frac{\hbar}{m\omega}$$
$$|\bar{\alpha}\rangle \equiv D(\alpha)|0\rangle = e^{\alpha a^\dagger - \alpha^* a}|0\rangle, \quad D(\alpha) \equiv \exp\left(\alpha a^\dagger - \alpha^* a\right), \quad \alpha = \frac{\langle\hat{x}\rangle}{\sqrt{2}d} + i\frac{\langle\hat{p}\rangle d}{\sqrt{2}\hbar} \in \mathbb{C}$$
$$|\bar{\alpha}\rangle = e^{\alpha a^\dagger - \alpha^* a}|0\rangle = e^{-\frac{1}{2}|\alpha|^2}e^{\alpha a^\dagger}|0\rangle, \quad \hat{a}|\bar{\alpha}\rangle = \alpha|\bar{\alpha}\rangle, \quad |\bar{\alpha}, t\rangle = e^{-i\omega t/2}\left|\overline{e^{-i\omega t}\alpha}\right\rangle$$
$$\langle\bar{\alpha} \mid \bar{\beta}\rangle = \exp\left(-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha^*\beta\right)$$
$$|\bar{\alpha}\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$
$$1 = \int\frac{d^2\alpha}{\pi}|\bar{\alpha}\rangle\langle\bar{\alpha}|$$

- Squeezed states

$$|0_\gamma\rangle = S(\gamma)|0\rangle, \quad S(\gamma) = \exp\left(-\frac{\gamma}{2}\left(a^\dagger a^\dagger - aa\right)\right), \quad \gamma \in \mathbb{R}$$
$$|0_\gamma\rangle = \frac{1}{\sqrt{\cosh\gamma}}\exp\left(-\frac{1}{2}\tanh\gamma a^\dagger a^\dagger\right)|0\rangle$$
$$S^\dagger(\gamma)aS(\gamma) = \cosh\gamma a - \sinh\gamma a^\dagger, \quad D^\dagger(\alpha)aD(\alpha) = a + \alpha$$
$$|\alpha,\gamma\rangle \equiv D(\alpha)S(\gamma)|0\rangle$$

- Time evolution

$$|\Psi,t\rangle = U(t,0)|\Psi,0\rangle, \quad U \text{ unitary}$$
$$U(t,t) = \mathbf{1}, \quad U(t_2,t_1)\,U(t_1,t_0) = U(t_2,t_0), \quad U(t_1,t_2) = U^\dagger(t_2,t_1)$$
$$i\hbar\frac{d}{dt}|\Psi,t\rangle = \hat{H}(t)|\Psi,t\rangle \quad \leftrightarrow \quad i\hbar\frac{d}{dt}U(t,t_0) = \hat{H}(t)U(t,t_0)$$

Time independent $\hat{H}$: $\quad U(t,t_0) = \exp\left[-\frac{i}{\hbar}\hat{H}(t-t_0)\right] = \sum_n e^{-\frac{i}{\hbar}E_n(t-t_0)}|n\rangle\langle n|$

$$\langle A\rangle = \langle\Psi,t\,|A_S|\,\Psi,t\rangle = \langle\Psi,0\,|A_H(t)|\,\Psi,0\rangle \quad \rightarrow \quad A_H(t) = U^\dagger(t,0)A_SU(t,0)$$
$$[A_S,B_S] = C_S \quad \rightarrow \quad [A_H(t),B_H(t)] = C_H(t)$$
$$i\hbar\frac{d}{dt}\hat{A}_H(t) = \left[\hat{A}_H(t),\hat{H}_H(t)\right], \text{ for } A_S \text{ time-independent}$$

- Two state systems

$$H = h_0\mathbf{1} + \mathbf{h}\cdot\boldsymbol{\sigma} = h_0\mathbf{1} + h\mathbf{n}\cdot\boldsymbol{\sigma}, \quad h = |\mathbf{h}|$$
$$\text{Eigenstates: } |\mathbf{n};\pm\rangle, \quad E_\pm = h_0 \pm h.$$

$H = -\gamma\mathbf{S}\cdot\mathbf{B} \rightarrow$ spin vector $\vec{n}$ precesses with Larmor frequency $\boldsymbol{\omega} = -\gamma\mathbf{B}$

- Orbital angular momentum operators

$$\mathbf{a}\cdot\mathbf{b} \equiv a_ib_i, \quad (\mathbf{a}\times\mathbf{b})_i \equiv \epsilon_{ijk}a_jb_k, \quad \mathbf{a}^2 \equiv \mathbf{a}\cdot\mathbf{a}.$$
$$\epsilon_{ijk}\epsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}, \quad \epsilon_{ijk}\epsilon_{ijq} = 2\delta_{kq}$$
$$\hat{L}_i = \epsilon_{ijk}\hat{x}_j\hat{p}_k \Longleftrightarrow \mathbf{L} = \mathbf{r}\times\mathbf{p} = -\mathbf{p}\times\mathbf{r}$$

Vector $\mathbf{u}$ under rotation: $\left[\hat{L}_i,\hat{u}_j\right] = i\hbar\epsilon_{ijk}\hat{u}_k \quad \Longrightarrow \quad \mathbf{L}\times\mathbf{u}+\mathbf{u}\times\mathbf{L} = 2i\hbar\mathbf{u}$

Scalar S under rotation: $\left[\hat{L}_i,S\right] = 0$

$\mathbf{u},\mathbf{v}$ vectors under rotations $\rightarrow \mathbf{u}\cdot\mathbf{v}$ is a scalar, $\mathbf{u}\times\mathbf{v}$ is a vector

$$\left[\hat{L}_i, \hat{L}_j\right] = i\hbar\epsilon_{ijk}\hat{L}_k \Longleftrightarrow \mathbf{L}\times\mathbf{L} = i\hbar\mathbf{L}, \quad \left[\hat{L}_i, \mathbf{L}^2\right] = 0.$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

$$\hat{L}^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

$$\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}; \quad \hat{L}_\pm = \hbar e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

- Spherical Harmonics

$$Y_{\ell,m}(\theta,\phi) \equiv \langle\theta,\phi \mid \ell, m\rangle$$

$$Y_{0,0}(\theta,\phi) = \frac{1}{\sqrt{4\pi}}; \quad Y_{1,\pm 1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\exp(\pm i\phi); \quad Y_{1,0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

- Algebra of angular momentum operators $\mathbf{J}$ (orbital or spin, or sum)

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k \Longleftrightarrow \mathbf{J}\times\mathbf{J} = i\hbar\mathbf{J}; \quad \rightarrow \quad \left[\mathbf{J}^2, J_i\right] = 0$$

$$J_\pm = J_x \pm iJ_y, \quad (J_\pm)^\dagger = J_\mp \quad J_x = \frac{1}{2}\left(J_+ + J_-\right), J_y = \frac{1}{2i}\left(J_+ - J_-\right)$$

$$[J_z, J_\pm] = \pm\hbar J_\pm, ; \quad [J_+, J_-] = 2\hbar J_z \quad \left[J^2, J_\pm\right] = 0$$

$$\mathbf{J}^2 = J_+J_- + J_z^2 - \hbar J_z = J_-J_+ + J_z^2 + \hbar J_z$$

$$\mathbf{J}^2|jm\rangle = \hbar^2 j(j+1)|jm\rangle; \quad J_z|jm\rangle = \hbar m|jm\rangle, \quad m = -j, \ldots, j.$$

$$J_\pm|jm\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}|j, m\pm 1\rangle$$

- Angular momentum in the two-dimensional oscillator

$$\hat{a}_L = \frac{1}{\sqrt{2}}\left(\hat{a}_x + i\hat{a}_y\right), \quad \hat{a}_R = \frac{1}{\sqrt{2}}\left(\hat{a}_x - i\hat{a}_y\right), \quad \left[\hat{a}_L, \hat{a}_L^\dagger\right] = \left[\hat{a}_R, \hat{a}_R^\dagger\right] = 1$$

$$J_+ = \hbar\hat{a}_R^\dagger\hat{a}_L, \quad J_- = \hbar\hat{a}_L^\dagger\hat{a}_R, \quad J_z = \frac{1}{2}\hbar\left(\hat{N}_R - \hat{N}_L\right)$$

$$|j,m\rangle: \quad j = \frac{1}{2}\left(N_R + N_L\right), \quad m = \frac{1}{2}\left(N_R - N_L\right)$$

$$\mathcal{H} = 0 \oplus \frac{1}{2} \oplus 1 \oplus \frac{3}{2} \oplus \ldots$$

- Radial equation

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + V(r) + \frac{\hbar^2\ell(\ell+1)}{2mr^2}\right)u_{\nu\ell}(r) = E_{\nu\ell}u_{\nu\ell}(r) \quad \text{(bound states)}$$

$$u_{\nu\ell}(r) \sim r^{\ell+1}, \quad \text{as } r \to 0.$$

- Hydrogen atom

$$E_n = -\frac{e^2}{2a_0}\frac{1}{n^2}, \quad \psi_{n,\ell,m}(\vec{x}) = \frac{u_{n\ell}(r)}{r}Y_{\ell,m}(\theta,\phi)$$
$$n = 1,2,\ldots, \quad \ell = 0,1,\ldots,n-1, \quad m = -\ell,\ldots,\ell$$
$$a_0 = \frac{\hbar^2}{me^2}, \quad \alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137}, \quad \hbar c \simeq 200\mathrm{MeV}-\mathrm{fm}$$
$$u_{1,0}(r) = \frac{2r}{a_0^{3/2}}\exp\left(-r/a_0\right)$$
$$u_{2,0}(r) = \frac{2r}{(2a_0)^{3/2}}\left(1-\frac{r}{2a_0}\right)\exp\left(-r/2a_0\right)$$
$$u_{2,1}(r) = \frac{1}{\sqrt{3}}\frac{1}{(2a_0)^{3/2}}\frac{r^2}{a_0}\exp\left(-r/2a_0\right)$$

- Addition of Angular Momentum $\mathbf{J} = \mathbf{J}_1 + \mathbf{J_2}$
  Uncoupled basis : $|j_1 j_2; m_1 m_2\rangle$ CSCO : $\left\{\mathbf{J}_1^2, \mathbf{J}_2^2, J_{1z}, J_{2z}\right\}$
  Coupled basis : $\quad |j_1 j_2; jm\rangle \quad$ CSCO : $\left\{\mathbf{J}_1^2, \mathbf{J}_2^2, \mathbf{J}^2, J_z\right\}$

$$j_1 \otimes j_2 = (j_1+j_2) \oplus (j_1+j_2-1) \oplus \ldots \oplus |j_1-j_2|$$
$$|j_1 j_2; jm\rangle = \sum_{m_1+m_2=m} |j_1 j_2; m_1 m_2\rangle \underbrace{\langle j_1 j_2; m_1 m_2 \mid j_1 j_2; jm\rangle}_{\text{Clebsch-Gordan coefficient}}$$
$$\mathbf{J}_1\cdot\mathbf{J}_2 = \frac{1}{2}\left(J_{1+}J_{2-} + J_{1-}J_{2+}\right) + J_{1z}J_{2z}$$
$$\mathbf{J}_1\cdot\mathbf{J}_2 = \frac{1}{2}\left(\mathbf{J}^2 - \mathbf{J}_1^2 - \mathbf{J}_2^2\right)$$

  Combining two spin 1/2 : $\quad \frac{1}{2}\otimes\frac{1}{2} = 1 \oplus 0$

$$\begin{aligned} |1,1\rangle &= |\uparrow\uparrow\rangle \\ |1,0\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \quad |0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \\ |1,-1\rangle &= |\downarrow\downarrow\rangle. \end{aligned}$$

- density matrix: $\quad E = \left\{(p_1, |\psi_1\rangle), \ldots, (p_n, |\psi_n\rangle)\right\}, \quad p_1,\ldots,p_n > 0, p_1 + \cdots + p_n = 1$

$$\rho_E \equiv \sum_{a=1}^{n} p_a |\psi_a\rangle\langle\psi_a|, \quad \langle\hat{Q}\rangle_E = \mathrm{tr}\left(\hat{Q}\rho_E\right)$$

  General ρ is positive semidefinite, and $\mathrm{tr}\,\rho = 1$. Pure state $\leftrightarrow \mathrm{tr}\,\rho^2 = 1$.
  spin one-half density matrix: $\quad \rho = \frac{1}{2}(\mathbb{1} + \mathbf{a}\cdot\boldsymbol{\sigma}), \quad |\mathbf{a}| \leq 1.$

time evolution: $i\hbar\frac{\partial\rho}{\partial t} = [\hat{H}, \rho]$.

Schmidt decomposition: $|\psi_{AB}\rangle = \sum_{k=1}^{r} \sqrt{p_k}\, |k_A\rangle \otimes |k_B\rangle\,, \quad r \leq d_A \leq d_B,$

$$\rho_A = \sum_{k=1}^{r} p_k \, |k_A\rangle \left\langle k_A \right|, \quad \rho_B = \sum_{k=1}^{r} p_k \left| k_B \right\rangle \langle k_B|\,, \quad \langle k_A \mid k'_A\rangle = \delta_{k,k'}, \quad \langle k_B \mid k'_B\rangle = \delta_{k,k'}.$$

Lindblad equation: $\frac{\partial\rho}{\partial t} = \frac{1}{i\hbar}[H, \rho] + \sum_k \left(L_k \rho L_k^\dagger - \frac{1}{2} \left\{ L_k^\dagger L_k, \rho \right\} \right)$.