Exercise 18.14. *Consider the operator $S \otimes T$ on $\mathcal{L}(V \otimes W)$, with $S \in \mathcal{L}(V)$ and $T \in \mathcal{L}(W)$. Does the action of $S \otimes T$ on an entangled state give an entangled state? If yes, prove it. If no, give an example.*

18.6 Bell Basis States

Bell states are a set of four *entangled, orthonormal basis vectors* in the state space of two spin one-half particles. To describe this basis consider the tensor product $V_1 \otimes V_2$, with V_1 and V_2 both the two-dimensional complex vector space appropriate to spin one-half particles. For brevity of notation, we will leave out the 1 and 2 subscripts on the states as well as the $\otimes$ in between the states. It is always understood that in $V_1 \otimes V_2$ the state in V_1 appears to the left of the state of V_2. Consider now the state

$$|\Phi_0\rangle \equiv \frac{1}{\sqrt{2}}\big(|+\rangle|+\rangle + |-\rangle|-\rangle\big). \tag{18.6.1}$$

This is clearly an entangled state: its associated matrix is diagonal with equal entries of $1/\sqrt{2}$ and thus a nonzero determinant. Moreover, this state is unit normalized:

$$\langle\Phi_0|\Phi_0\rangle = 1. \tag{18.6.2}$$

We use this state as the first of our basis vectors for $V_1 \otimes V_2$. Since this tensor product is four-dimensional, we need three more entangled basis states. Here they are

$$|\Phi_i\rangle \equiv (\mathbb{1} \otimes \sigma_i)|\Phi_0\rangle, \; i = 1, 2, 3. \tag{18.6.3}$$

It is clear that these states are entangled. If $|\Phi_i\rangle$ were not entangled, it would follow that $(\mathbb{1} \otimes \sigma_i)|\Phi_i\rangle$ (i not summed) is also not entangled (exercise 18.13). But using $\sigma_i^2 = \mathbb{1}$, we see that this last state is in fact $|\Phi_0\rangle$, which is entangled. This contradiction shows that $|\Phi_i\rangle$ must be entangled. Since the operator $\mathbb{1} \otimes \sigma_i$ is unitary, it follows from the definition that all $|\Phi_i\rangle$ are unit normalized.

Let us look at the form of $|\Phi_1\rangle$:

$$|\Phi_1\rangle = (\mathbb{1} \otimes \sigma_1)\frac{1}{\sqrt{2}}\big(|+\rangle|+\rangle + |-\rangle|-\rangle\big) = \frac{1}{\sqrt{2}}\big(|+\rangle|-\rangle + |-\rangle|+\rangle\big). \tag{18.6.4}$$

By analogous calculations we obtain the full list of **Bell states**:

$$
\begin{aligned}
|\Phi_0\rangle &= \mathbb{1}\otimes\mathbb{1}\,|\Phi_0\rangle = \tfrac{1}{\sqrt{2}}\big(|+\rangle|+\rangle+|-\rangle|-\rangle\big),\\
|\Phi_1\rangle &= \mathbb{1}\otimes\sigma_1|\Phi_0\rangle = \tfrac{1}{\sqrt{2}}\big(|+\rangle|-\rangle+|-\rangle|+\rangle\big),\\
|\Phi_2\rangle &= \mathbb{1}\otimes\sigma_2|\Phi_0\rangle = \tfrac{i}{\sqrt{2}}\big(|+\rangle|-\rangle-|-\rangle|+\rangle\big),\\
|\Phi_3\rangle &= \mathbb{1}\otimes\sigma_3|\Phi_0\rangle = \tfrac{1}{\sqrt{2}}\big(|+\rangle|+\rangle-|-\rangle|-\rangle\big).
\end{aligned}
\tag{18.6.5}
$$

Note that $|\Phi_2\rangle$ is the spin singlet state (18.3.12). We can confirm by inspection that Φ_0 is orthogonal to the other three: $\langle\Phi_0|\Phi_i\rangle = 0$. It is not much work either to see that the basis is in fact orthonormal. But the calculation is kind of fun. Since $(S\otimes T)^\dagger = S^\dagger\otimes T^\dagger$ and $\sigma_i^\dagger = \sigma_i$, we find that

$$
\langle\Phi_i| = \langle\Phi_0|(\mathbb{1}\otimes\sigma_i). \tag{18.6.6}
$$

We can then compute

$$
\begin{aligned}
\langle\Phi_i|\Phi_j\rangle &= \langle\Phi_0|(\mathbb{1}\otimes\sigma_i)(\mathbb{1}\otimes\sigma_j)|\Phi_0\rangle\\
&= \langle\Phi_0|\mathbb{1}\otimes\sigma_i\sigma_j|\Phi_0\rangle\\
&= \langle\Phi_0|\mathbb{1}\otimes\big(\mathbb{1}\delta_{ij}+i\epsilon_{ijk}\sigma_k\big)|\Phi_0\rangle\\
&= \delta_{ij}\langle\Phi_0|\mathbb{1}\otimes\mathbb{1}|\Phi_0\rangle+i\epsilon_{ijk}\langle\Phi_0|\mathbb{1}\otimes\sigma_k|\Phi_0\rangle\\
&= \delta_{ij}\langle\Phi_0|\Phi_0\rangle+i\epsilon_{ijk}\langle\Phi_0|\Phi_k\rangle = \delta_{ij}.
\end{aligned}
\tag{18.6.7}
$$

Indeed, we have an orthonormal basis of entangled states.

We can solve for the nonentangled basis states in terms of the Bell states. We quickly find from (18.6.5) that

$$
\begin{aligned}
|+\rangle|+\rangle &= \tfrac{1}{\sqrt{2}}\big(|\Phi_0\rangle+|\Phi_3\rangle\big),\\
|-\rangle|-\rangle &= \tfrac{1}{\sqrt{2}}\big(|\Phi_0\rangle-|\Phi_3\rangle\big),\\
|+\rangle|-\rangle &= \tfrac{1}{\sqrt{2}}\big(|\Phi_1\rangle-i|\Phi_2\rangle\big),\\
|-\rangle|+\rangle &= \tfrac{1}{\sqrt{2}}\big(|\Phi_1\rangle+i|\Phi_2\rangle\big).
\end{aligned}
\tag{18.6.8}
$$

Introducing labels A and B for the two spaces in a tensor product $V_A\otimes V_B$, we can rewrite the above equations as

$$
\begin{aligned}
|+\rangle_A|+\rangle_B &= \tfrac{1}{\sqrt{2}}\big(|\Phi_0\rangle_{AB}+|\Phi_3\rangle_{AB}\big),\\
|-\rangle_A|-\rangle_B &= \tfrac{1}{\sqrt{2}}\big(|\Phi_0\rangle_{AB}-|\Phi_3\rangle_{AB}\big),\\
|+\rangle_A|-\rangle_B &= \tfrac{1}{\sqrt{2}}\big(|\Phi_1\rangle_{AB}-i|\Phi_2\rangle_{AB}\big),\\
|-\rangle_A|+\rangle_B &= \tfrac{1}{\sqrt{2}}\big(|\Phi_1\rangle_{AB}+i|\Phi_2\rangle_{AB}\big),
\end{aligned}
\tag{18.6.9}
$$

where $|\Phi_i\rangle_{AB}$ are the Bell states we defined above, with the first state in V_A and the second state in V_B.

Let us now discuss measurements that can be done on an entangled pair of particles. Recall that given an orthonormal basis $|e_1\rangle, \ldots, |e_n\rangle$ we can measure a state $|\Psi\rangle$ along this basis (see axiom A3 and equation (16.6.9)). We have the probability $p(i) = |\langle e_i|\Psi\rangle|^2$ of being found in the state $|i\rangle$. After measurement, the state will be in one of the basis states $|e_i\rangle$.

For a state of two spin one-half particles A, B, we may choose the four Bell states as our orthonormal basis for measurement. If so, after measurement the state will be in one of the Bell states $|\Phi_i\rangle_{AB}$, with probability $|{}_{AB}\langle\Phi_i|\Psi\rangle|^2$.

If Alice and Bob each has one of the particles in an entangled pair, more sophisticated measurements are possible. We examine those now.

Partial measurement Suppose we have a general entangled state $|\Psi\rangle \in V \otimes W$ of two particles. Alice has access to the first particle and decides to measure along the basis $|e_1\rangle, \ldots, |e_n\rangle$ of V. This is analyzed with measurement axiom A3, using a complete set of mutually orthogonal projectors M_i:

$$
M_i \equiv |e_i\rangle\langle e_i| \otimes \mathbb{1}, \quad i = 1, \ldots, n. \tag{18.6.10}
$$

The projectors act trivially on the state space of the second particle and act on the state space of the first particle as expected. Clearly, $M_i^\dagger = M_i$, $M_iM_j = M_i\delta_{ij}$, and $\sum_{i=1}^n M_i = \mathbb{1}\otimes\mathbb{1}$, which is the identity in the tensor product. To simplify the writing of probabilities, consider the measurement of a general state $|\Psi\rangle$ written with the help of the basis vectors $|e_i\rangle$ as

$$
|\Psi\rangle = \sum_i |e_i\rangle \otimes |w_i\rangle. \tag{18.6.11}
$$

Here, the $|w_i\rangle \in W$ are some calculable vectors that in general are neither normalized nor orthogonal. Such a writing of $|\Psi\rangle$ is always possible. From axiom A3, the probability $p(i)$ that Alice will find the first particle to be in the state $|i\rangle$ is

$$p(i) = \langle\Psi|M_i|\Psi\rangle = \sum_{p,q} \langle e_p| \otimes \langle w_p| \left(|e_i\rangle\langle e_i| \otimes \mathbb{1}\right) |e_q\rangle \otimes |w_q\rangle = \sum_{p,q} \langle e_p|e_i\rangle\langle e_i|e_q\rangle \langle w_p|w_q\rangle.$$

Using the orthonormality of the basis,

$$p(i) = \langle w_i|w_i\rangle. \tag{18.6.12}$$

If Alice finds her particle in $|e_i\rangle$, the state of the system after measurement is $M_i|\Psi\rangle$, suitably normalized:

$$M_i|\Psi\rangle = |e_i\rangle\langle e_i| \otimes \mathbb{1} \sum_p |e_p\rangle \otimes |w_p\rangle = |e_i\rangle \otimes |w_i\rangle. \tag{18.6.13}$$

Normalizing, we see that after the measurement the state of the system will be

$$|e_i\rangle \otimes \frac{|w_i\rangle}{\sqrt{\langle w_i|w_i\rangle}}, \quad \text{for some value of } i. \tag{18.6.14}$$

Exercise 18.15. *Show that one also has* $p(i) = \|\langle e_i|\Psi\rangle\|^2$*, an expression formally analogous to the familiar rule for measuring along a basis. Note that the norm is needed because* $\langle e_i|\Psi\rangle \in W$.

Exercise 18.16. *Alice and Bob measure the entangled state* $|\Psi\rangle$ *along the basis states* $|e_i\rangle \otimes |f_j\rangle$ *of* $V \otimes W$*. Show that the probability of finding the state in* $|e_i\rangle \otimes |f_j\rangle$ *is* $p(i, j) = |\langle e_i| \otimes \langle f_j|\Psi\rangle|^2$*. Show, additionally, that* $p(i) = \sum_j p(i, j)$.

As a simple illustration of partial measurement, consider the entangled spin single state:

$$|\Psi\rangle \ = \ \tfrac{1}{\sqrt{2}}\left(|+\rangle_1 \otimes |-\rangle_2 \ - \ |-\rangle_1 \otimes |+\rangle_2\right). \tag{18.6.15}$$

If we are to measure the first particle along the $|+\rangle$, $|-\rangle$ basis, we rewrite the state in the form (18.6.11):

$$|\Psi\rangle \ = \ |+\rangle_1 \otimes \tfrac{1}{\sqrt{2}}|-\rangle_2 \ + \ |-\rangle_1 \otimes \left(-\tfrac{1}{\sqrt{2}}|+\rangle_2\right). \tag{18.6.16}$$

On account of (18.6.12), the probabilities $p(+)$ and $p(-)$ are then given by

$$\begin{aligned} p(+) &= \tfrac{1}{\sqrt{2}}\tfrac{1}{\sqrt{2}}\langle -|-\rangle = \tfrac{1}{2}\,; \quad \text{state after measurement: } |+\rangle_1 \otimes |-\rangle_2, \\ p(-) &= \left(-\tfrac{1}{\sqrt{2}}\right)^2 \langle +|+\rangle = \tfrac{1}{2}\,; \quad \text{state after measurement: } |-\rangle_1 \otimes |+\rangle_2, \end{aligned} \tag{18.6.17}$$

the states after measurement given in the form (18.6.14). After the measurement of the first particle, a measurement of the second particle will show that its spin is always opposite to the spin of the first particle.

As a more nontrivial example, consider a state of three particles A, B, C. Such a state lives in $V_A \otimes V_B \otimes V_C$. To analyze what happens if Alice decides to do a Bell measurement of the pair AB, the state Ψ of the system must be written in the form

$$|\Psi\rangle = |\Phi_0\rangle_{AB} \otimes |u_0\rangle_C + \sum_{i=1}^{3} |\Phi_i\rangle_{AB} \otimes |u_i\rangle_C. \tag{18.6.18}$$

In general, the states $|u_\mu\rangle$ with $\mu = 0, 1, 2, 3$ are neither normalized nor orthogonal to each other. After measurement, the state of the particles AB will be one of the Bell states $|\Phi_\mu\rangle_{AB}$. The probability $p_{AB}(\Phi_\mu)$ that the AB particles are in the state $|\Phi_\mu\rangle$ is

$$p_{AB}(\Phi_\mu) = \langle u_\mu | u_\mu \rangle. \tag{18.6.19}$$

Moreover, the state after measurement is

$$|\Phi_\mu\rangle_{AB} \otimes \frac{|u_\mu\rangle_C}{\sqrt{\langle u_\mu | u_\mu\rangle}}, \quad \text{for some } \mu \in \{0, 1, 2, 3\}. \tag{18.6.20}$$