20.2 Adding Two Spin One-Half Angular Momenta

We will now consider the simplest and perhaps the most important case of addition of angular momentum. We assume we have two spin one-half particles, and we focus just on spin degrees of freedom. In such a case, as you know, the two-particle system has a four-dimensional state space, described by basis vectors in which particle one and particle two can be either up or down along z. We want to answer this question: What are the possible values of the total angular momentum, and what are the basis states that realize those values? The question is simply one of finding a *new* basis for the *same* state space.

To set up notation, recall that for a spin one-half particle the state space is $\mathbb{C}^2$, spanned by states $|\frac{1}{2},\frac{1}{2}\rangle$ and $|\frac{1}{2},-\frac{1}{2}\rangle$, satisfying

$$\begin{aligned} \hat{\mathbf{S}}^2|\tfrac{1}{2},m_s\rangle &= \tfrac{3}{4}\hbar^2|\tfrac{1}{2},m_s\rangle, \\ \hat{S}_z|\tfrac{1}{2},m_s\rangle &= \hbar m_s\,|\tfrac{1}{2},m_s\rangle. \end{aligned} \tag{20.2.1}$$

Here, $m_s = \pm\frac{1}{2}$. Now consider a system that features two spin one-half particles. For the first particle, we have the triplet of spin operators $\hat{\mathbf{S}}^{(1)}$ acting on a vector space V_1 spanned by

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle_1,\quad |\tfrac{1}{2},-\tfrac{1}{2}\rangle_1. \tag{20.2.2}$$

For the second particle, we have the triplet spin operators $\hat{\mathbf{S}}^{(2)}$ acting on the vector space V_2 spanned by

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle_2,\quad |\tfrac{1}{2},-\tfrac{1}{2}\rangle_2. \tag{20.2.3}$$

We now form the total spin operators $\hat{S}_i$:

$$\hat{S}_i \equiv \hat{S}_i^{(1)}\otimes \mathbb{1} + \mathbb{1}\otimes \hat{S}_i^{(2)}, \tag{20.2.4}$$

which, for brevity, we write as

$$\hat{S}_i = \hat{S}_i^{(1)} + \hat{S}_i^{(2)}, \tag{20.2.5}$$

with the understanding that each operator on the right-hand side acts on the appropriate factor in the tensor product. The state space for the dynamics of the two particles must contain the tensor product $V_1 \otimes V_2$; more spaces might be needed if the particles have orbital angular momentum or are moving. As we learned before, $V_1 \otimes V_2$ is a four-dimensional complex vector space spanned by the products of states in (20.2.2) and (20.2.3):

$$\begin{aligned}&|\tfrac{1}{2},\tfrac{1}{2}\rangle_1\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle_2,\quad |\tfrac{1}{2},\tfrac{1}{2}\rangle_1\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle_2,\quad |\tfrac{1}{2},-\tfrac{1}{2}\rangle_1\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle_2,\\ &\quad|\tfrac{1}{2},-\tfrac{1}{2}\rangle_1\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle_2.\end{aligned} \tag{20.2.6}$$

Since the total spin operators have a well-defined action on these states and the vector space they span, it must be possible to describe the vector space as a sum of subspaces that carry irreducible representations of the total spin angular momentum. The irreducible representations of an angular momentum $\hat{\mathbf{J}}$ are those characterized by the number j that appears in the eigenvalue of $\hat{\mathbf{J}}^2$. A representation is said to be reducible if it is the direct sum of irreducible representations.

We have four basis states, so the possibilities for multiplets of total spin s are as follows:

1. Four singlets ($s = 0$).
2. Two doublets (($s = \frac{1}{2}$)).
3. One doublet (($s = \frac{1}{2}$)) and two singlets ($s = 0$).

4. One triplet ($s = 1$) and one singlet ($s = 0$).

5. One $s = \frac{3}{2}$ multiplet.

Only the last option is an irreducible representation. All others are reducible. It may be instructive at this point if you pause to make sure no other option exists and then to consider which option you think is the correct one!

The main clue is that the states in the tensor product are eigenstates of $\hat{S}_z$, the total z-component of angular momentum. We see by inspection of (20.2.6) that the possible values of $\hat{S}_z/\hbar$ are $+\,1$, 0, and $-\,1$. Since we have a state with $m = 1$ and no state with higher m, we must have a triplet $s = 1$. Thus, the only option is (4): a triplet and a singlet. This is written as

$$(s=\tfrac{1}{2}) \otimes (s=\tfrac{1}{2}) \;=\; (s=1) \oplus (s=0). \tag{20.2.7}$$

Note that on the left-hand side we have the tensor product of the two state spaces, but on the right-hand side we have the *direct sum* of the representations of total spin angular momentum. This is a fundamental result and is written more briefly as

$$\boxed{\tfrac{1}{2} \otimes \tfrac{1}{2} \;=\; \mathbf{1} \oplus \mathbf{0}.} \tag{20.2.8}$$

We use bold type for the numbers representing j values to make clear that these represent vector spaces. Note that the $\mathbf{0}$ on the right-hand side is neither a zero vector nor a vanishing vector space. It represents the singlet $s = 0$, a vector space with one basis vector, a one-dimensional vector space. Similarly, $\mathbf{1}$ represents the triplet $s = 1$. The equality in (20.2.8) is in fact an equality of vector spaces, and therefore the dimensionality of the vector spaces must agree. This follows from the relation $2 \times 2 = 3 + 1$, where 2×2 is the dimension of the tensor product of two two-dimensional spaces, and $3 + 1$ is the dimension of the direct sum of a three-dimensional space and a one-dimensional space.

Let us understand the decomposition $\mathbf{1} \oplus \mathbf{0}$ explicitly by organizing the basis states according to the eigenvalue m of $\hat{S}_z/\hbar$, the total z-component of angular momentum:

$$
\begin{aligned}
m &= 1: \quad |\tfrac{1}{2},\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle_2, \\
m &= 0: \quad |\tfrac{1}{2},\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle_2, \quad |\tfrac{1}{2},-\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle_2, \\
m &= -1: \quad |\tfrac{1}{2},-\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle_2.
\end{aligned}
\tag{20.2.9}
$$

We get two states with $m = 0$. This is as it should be. One linear combination of these two states must be the $m = 0$ state of the triplet, and another linear combination must be the singlet $s = m = 0$. Those two states are in fact entangled states. Denoting by $|s, m\rangle$ the eigenstates of $\hat{S}^2$ and $\hat{S}_z$ (total spin), we must have a triplet with states

$$
\begin{aligned}
|1,1\rangle &= |\tfrac{1}{2},\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle_2, \\
|1,0\rangle &= \alpha\,|\tfrac{1}{2},\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle_2 + \beta\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle_2, \\
|1,-1\rangle &= |\tfrac{1}{2},-\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle_2,
\end{aligned}
\tag{20.2.10}
$$

for some constants α and β, as well as a singlet

$$
|0,0\rangle = \gamma\,|\tfrac{1}{2},\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle_2 + \delta\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle_2,
\tag{20.2.11}
$$

for some constants γ and δ. We must determine these four constants.

Let us begin by calculating the $|1, 0\rangle$ state in the triplet. To do this, we first recall the general formula

$$
\hat{J}_\pm |j,m\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\,|j, m\pm 1\rangle,
\tag{20.2.12}
$$

which quickly gives us the following preparatory results:

$$
\begin{aligned}
\hat{J}_- |1,1\rangle &= \hbar\sqrt{2}\,|1,0\rangle, \\
\hat{J}_- |\tfrac{1}{2},\tfrac{1}{2}\rangle &= \hbar\sqrt{\tfrac{1}{2}\cdot\tfrac{3}{2} - \tfrac{1}{2}\cdot(-\tfrac{1}{2})}\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \hbar\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle, \\
\hat{J}_+ |\tfrac{1}{2},-\tfrac{1}{2}\rangle &= \hbar\sqrt{\tfrac{1}{2}\cdot\tfrac{3}{2} - (-\tfrac{1}{2})\cdot(\tfrac{1}{2})}\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \hbar\,|\tfrac{1}{2},\tfrac{1}{2}\rangle.
\end{aligned}
\tag{20.2.13}
$$

For our application $\hat{J}$ is total spin $\hat{S}$ so that $\hat{J}_-$ is $\hat{S}_-$. We now apply the lowering operator $\hat{S}_- = \hat{S}_-^{(1)} + \hat{S}_-^{(2)}$ to the top state of the triplet in (20.2.10). We have

$$
\hat{S}_-|1,1\rangle = (\hat{S}_-^{(1)}|\tfrac{1}{2},\tfrac{1}{2}\rangle_1) \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle_2 + |\tfrac{1}{2},\tfrac{1}{2}\rangle_1 \otimes (\hat{S}_-^{(2)}|\tfrac{1}{2},\tfrac{1}{2}\rangle_2).
\tag{20.2.14}
$$

Using the first line of (20.2.13) for the left-hand side and the second line for the right-hand side, we find that

$$\sqrt{2}\hbar\,|1,0\rangle \;=\; \hbar|\tfrac{1}{2},-\tfrac{1}{2}\rangle_1\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle_2 + |\tfrac{1}{2},\tfrac{1}{2}\rangle_1\otimes\hbar\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle_2. \tag{20.2.15}$$

Canceling the common factors of $\hbar$ and switching the order of the terms, we find that the $|1, 0\rangle$ state takes the form

$$|1,0\rangle \;=\; \tfrac{1}{\sqrt{2}}\Big(|\tfrac{1}{2},\tfrac{1}{2}\rangle_1\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle_2 + |\tfrac{1}{2},-\tfrac{1}{2}\rangle_1\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle_2\Big). \tag{20.2.16}$$

Note that the right-hand side, as it should be, is a unit-normalized state. Having found the $m = 0$ state of the $s = 1$ multiplet, there are a number of ways to find the $m = 0$ state of the $s = 0$ singlet. One way is orthogonality: the single $|0, 0\rangle$ must be orthogonal to the $|1, 0\rangle$ state above because these are two states with different eigenvalue s of the Hermitian operator $\hat{S}^2$. Since the overall sign or phase is irrelevant, we can simply take for the singlet the linear combination of $m = 0$ states with a minus sign:

$$|0,0\rangle \;=\; \tfrac{1}{\sqrt{2}}\Big(|\tfrac{1}{2},\tfrac{1}{2}\rangle_1\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle_2 - |\tfrac{1}{2},-\tfrac{1}{2}\rangle_1\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle_2\Big). \tag{20.2.17}$$

You probably remember that we found this state in example 18.3 by searching for a state that is annihilated by the sum of spin angular momentum operators. This is exactly the condition for a singlet.

As an instructive calculation, let us confirm that $\hat{S}^2$ is zero acting on $|0, 0\rangle$. For this it is useful to note that

$$\begin{aligned}\hat{\mathbf{S}}^2 = (\hat{\mathbf{S}}^{(1)}+\hat{\mathbf{S}}^{(2)})^2 \;&=\; (\hat{\mathbf{S}}^{(1)})^2 + (\hat{\mathbf{S}}^{(2)})^2 + 2\hat{\mathbf{S}}^{(1)}\cdot\hat{\mathbf{S}}^{(2)}\\ &= (\hat{\mathbf{S}}^{(1)})^2 + (\hat{\mathbf{S}}^{(2)})^2 + \hat{S}_+^{(1)}\hat{S}_-^{(2)} + \hat{S}_-^{(1)}\hat{S}_+^{(2)} + 2\hat{S}_z^{(1)}\hat{S}_z^{(2)},\end{aligned} \tag{20.2.18}$$

where in the second step we used the general result

$$\hat{\mathbf{J}}^{(1)}\cdot\hat{\mathbf{J}}^{(2)} = \tfrac{1}{2}(\hat{J}_+^{(1)}\hat{J}_-^{(2)} + \hat{J}_-^{(1)}\hat{J}_+^{(2)}) + \hat{J}_z^{(1)}\hat{J}_z^{(2)}, \tag{20.2.19}$$

valid for arbitrary angular momenta. Written in explicit tensor notation, it reads

$$\hat{\mathbf{J}}^{(1)}\cdot\hat{\mathbf{J}}^{(2)} \;\equiv\; \sum_{i=1}^{3}\hat{J}_i^{(1)}\otimes\hat{J}_i^{(2)} = \tfrac{1}{2}(\hat{J}_+^{(1)}\otimes\hat{J}_-^{(2)} + \hat{J}_-^{(1)}\otimes\hat{J}_+^{(2)}) + \hat{J}_z^{(1)}\otimes\hat{J}_z^{(2)}. \tag{20.2.20}$$

Exercise 20.1. *Prove (20.2.20).*

Back to our calculation, all states have $s_1 = s_2 = \frac{1}{2}$, and therefore $(\hat{\mathbf{S}}^{(1)})^2 = (\hat{\mathbf{S}}^{(2)})^2 = \frac{3}{4}\hbar^2$. We thus have

$$\hat{\mathbf{S}}^2\,|0,0\rangle \;=\; \tfrac{3}{2}\hbar^2\,|0,0\rangle + \big(\hat{S}_+^{(1)}\hat{S}_-^{(2)} + \hat{S}_-^{(1)}\hat{S}_+^{(2)} + 2\hat{S}_z^{(1)}\hat{S}_z^{(2)}\big)|0,0\rangle. \tag{20.2.21}$$

It is simple to see that

$$2\hat{S}_z^{(1)}\hat{S}_z^{(2)}\,|0,0\rangle \;=\; 2\,\tfrac{\hbar}{2}\cdot(-\tfrac{\hbar}{2})|0,0\rangle \;=\; -\tfrac{1}{2}\hbar^2\,|0,0\rangle \tag{20.2.22}$$

because the singlet is a superposition of tensor states where each has one state up and one state down. Similarly, recalling that

$$\hat{S}_\pm|\tfrac{1}{2},\mp\tfrac{1}{2}\rangle \;=\; \hbar\,|\tfrac{1}{2},\pm\tfrac{1}{2}\rangle, \tag{20.2.23}$$

we quickly find that

$$\big(\hat{S}_+^{(1)}\hat{S}_-^{(2)} + \hat{S}_-^{(1)}\hat{S}_+^{(2)}\big)|0,0\rangle \;=\; -\hbar^2\,|0,0\rangle, \tag{20.2.24}$$

since each of the operators $\hat{S}_+^{(1)}\hat{S}_-^{(2)}$ and $\hat{S}_-^{(1)}\hat{S}_+^{(2)}$ kills one term in the singlet, and acting on the other term, it gives $\hbar^2$ times the killed one. Check it! Going back to (20.2.21), we get

$$\hat{\mathbf{S}}^2\,|0,0\rangle \;=\; \tfrac{3}{2}\hbar^2\,|0,0\rangle + \big(-\hbar^2 - \tfrac{1}{2}\hbar^2\big)|0,0\rangle \;=\; 0, \tag{20.2.25}$$

as we wanted to show.

Let us summarize our results. The triplet states and singlet states are given by

$$\begin{aligned}
|1,1\rangle \;&=\; |\tfrac{1}{2},\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle_2,\\
|1,0\rangle \;&=\; \tfrac{1}{\sqrt{2}}\Big(|\tfrac{1}{2},\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle_2 + |\tfrac{1}{2},-\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle_2\Big),\\
|1,-1\rangle \;&=\; |\tfrac{1}{2},-\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle_2.\\
|0,0\rangle \;&=\; \tfrac{1}{\sqrt{2}}\Big(|\tfrac{1}{2},\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle_2 - |\tfrac{1}{2},-\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle_2\Big).
\end{aligned} \tag{20.2.26}$$

For briefer notation we replace $|\tfrac{1}{2},\tfrac{1}{2}\rangle \to |\uparrow\rangle$ and $|\tfrac{1}{2},-\tfrac{1}{2}\rangle \to |\downarrow\rangle$:

$$\begin{aligned}
|1,1\rangle \;&=\; |\uparrow\rangle_1 \otimes |\uparrow\rangle_2,\\
|1,0\rangle \;&=\; \tfrac{1}{\sqrt{2}}\Big(|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 + |\downarrow\rangle_1 \otimes |\uparrow\rangle_2\Big),\\
|1,-1\rangle \;&=\; |\downarrow\rangle_1 \otimes |\downarrow\rangle_2.\\
|0,0\rangle \;&=\; \tfrac{1}{\sqrt{2}}\Big(|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 - |\downarrow\rangle_1 \otimes |\uparrow\rangle_2\Big).
\end{aligned} \tag{20.2.27}$$

With the understanding that the first arrow refers to the first particle and the second arrow to the second particle, we can finally write all of this quite briefly. With a dashed line separating the triplet and the singlet, we have

$$\boxed{\begin{array}{rcl} |1,1\rangle &=& |\uparrow\uparrow\rangle, \\[1ex] |1,0\rangle &=& \frac{1}{\sqrt{2}}\big(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\big), \\[1ex] |1,-1\rangle &=& |\downarrow\downarrow\rangle. \\[1ex] \hdashline |0,0\rangle &=& \frac{1}{\sqrt{2}}\big(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\big). \end{array}} \tag{20.2.28}$$

This decomposition of the tensor product of two spin one-half state spaces is needed to calculate the hyperfine splitting in the hydrogen atom (section 20.4). For this system, the relevant spins are those of the proton and the electron.