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Multiparticle States and Tensor Products

We introduce the tensor product $V \otimes W$ of two vector spaces V and W as the state space of two particles, that separately have state spaces V and W, respectively. Operators and inner products on $V \otimes W$ arise from operators and inner products on V and W. Entangled states in $V \otimes W$ are states that can't be factorized into a state in V and a state in W. We construct an entangled state of two spin one-half particles that has zero total angular momentum. We introduce Bell states, which are four entangled states that form a basis for the state space of two spin one-half particles. We explain how quantum teleportation of a spin state works with the help of an entangled pair. We discuss the local realism claims of Einstein, Podolsky, and Rosen (EPR) and the surprising discovery of Bell inequalities, which made the claims of EPR testable and experimentally refuted. Finally, we show it is impossible to construct a machine capable of cloning arbitrary quantum states of any system.

18.1 Introduction to the Tensor Product

In this section we develop the tools needed to describe a system that contains more than one particle. Most of the new ideas appear when we consider systems with two particles. We will assume the particles are distinguishable. For indistinguishable particles quantum mechanics imposes some additional constraints on the allowed set of states. We will study those constraints in chapter 21. The same tools apply to the case of one particle, if such particle has a number of independent degrees of freedom. The material we are about to develop will be needed to understand the addition of angular momenta. In that problem one may be adding the angular momenta of two or more particles in a system or, alternatively, adding the independent angular momenta of a single particle.

Consider two particles. Below we list the state space and the operators associated with each particle:

- Particle 1: Its states are elements of a complex vector space V. In this space we have operators $T_1, T_2, \ldots$.
- Particle 2: Its states are elements of a complex vector space W. In this space we have operators $S_1, S_2, \ldots$.

This list of operators for each particle may include some or many of the operators you are already familiar with: position, momentum, spin, Hamiltonians, projectors, and so on.

Once we have two particles, the two of them *together* form our system. We are after the description of quantum states of this two-particle system. On first thought, we may believe that any state of this system should be described by giving the state $v \in V$ of the first particle and the state $w \in W$ of the second particle. This information could be represented by the ordered list (v, w), where the first item is the state of the first particle and the second item the state of the second particle. This is *a* state of the two-particle system, but it is far from being the general state of the two-particle system. It misses remarkable new possibilities, as we shall soon see.

We thus introduce a new notation. Instead of representing the state of the two-particle system with particle 1 in v and particle 2 in w as (v, w), we will represent it as $v \otimes w$. This element $v \otimes w$ will be viewed as a vector in a new vector space $V \otimes W$ that contains the quantum states of the two-particle system. This $\otimes$ operation is called the **tensor product**. In this case we have two complex vector spaces, and the tensor product $V \otimes W$ is a new complex vector space:

$$v \otimes w \in V \otimes W \quad \text{when} \quad v \in V,\ w \in W. \tag{18.1.1}$$

In the tensor product $v \otimes w$, there is no multiplication to be carried out; we are just placing one vector to the left of $\otimes$ and another to the right of $\otimes$.

We have only described some elements of $V \otimes W$, not quite given its definition yet. We now explain two physically motivated rules that define the tensor product completely.

1. If the vector representing the state of the first particle is scaled by a complex number, this is equivalent to scaling the state of the two particles. The same holds true for the second particle. So we declare

$$\boxed{(av) \otimes w = v \otimes (aw) = a\,(v \otimes w), \qquad a \in \mathbb{C}.} \tag{18.1.2}$$

2. If the state of the first particle is a superposition of two states, the state of the two-particle system is also a superposition. We thus demand distributive properties for the tensor product:

$$\boxed{\begin{aligned}(v_1 + v_2) \otimes w &= v_1 \otimes w + v_2 \otimes w,\\ v \otimes (w_1 + w_2) &= v \otimes w_1 + v \otimes w_2.\end{aligned}} \tag{18.1.3}$$

The tensor product $V \otimes W$ is thus defined to be the vector space whose elements are (complex) linear combinations of elements of the form $v \otimes w$, with $v \in V$, $w \in W$, with the above rules for manipulation. The tensor product $V \otimes W$ is the complex vector space of states of the two-particle system!

Remarks:

1. The vector $0 \in V \otimes W$ is equal to $0 \otimes w$ or $V \otimes 0$. Indeed, with $a = 0$ we have $av = 0$, and the equality of the first and last term in (18.1.2) gives $0 \otimes w = 0(v \otimes w) = 0 \in V \otimes W$, since in any vector space the product of the number zero times any vector is the zero vector.
2. Let v_1, $v_2 \in V$ and w_1, $w_2 \in W$. Consider a vector in $V \otimes W$ built by superposition:

$$\alpha_1(v_1 \otimes w_1) + \alpha_2(v_2 \otimes w_2) \in V \otimes W, \quad \alpha_1, \alpha_2 \in \mathbb{C}. \tag{18.1.4}$$

This is a state of the two-particle system that, with α_1 and α_2 both nonzero, cannot be described by giving the state of the first particle and the state of the second particle. The above superpositions give rise to entangled states. An entangled state of the two particles is one that, roughly, cannot be described by giving the state of the first particle and the state of the second particle. We will make this precise soon.

If $(e_1, \ldots, e_m)$ is a basis of V and $(f_1, \ldots, f_n)$ is a basis of W, then the set of elements $e_i \otimes f_j$ where $i = 1, \ldots, m$ and $j = 1, \ldots, n$ forms a basis for $V \otimes W$:

$$\{ e_i \otimes f_j\,;\ i=1,\ldots,m,\ \ j=1,\ldots,n.\} \text{ is a basis for } \ V \otimes W. \tag{18.1.5}$$

It is simple to see these span $V \otimes W$. First note that for any $v \otimes w$ we have $v=\sum_i v_i e_i$, and $w=\sum_j w_j f_j$ so that

$$v \otimes w \ = \ \big(\sum_i v_i e_i\big) \otimes \big(\sum_j w_j f_j\big) \ = \ \sum_{i,j} v_i w_j\, e_i \otimes f_j. \tag{18.1.6}$$

Since the basis spans elements of the form $v \otimes w$ for all v, w, it will span all linear superpositions of such elements, which is to say, it will span $V \otimes W$. With $n{\cdot}m$ basis vectors, the dimensionality of $V \otimes W$ is equal to the *product* of the dimensionalities of V and W:

$$\dim(V \otimes W) = \dim(V) \times \dim(W). \tag{18.1.7}$$

Dimensions are multiplied, not added, in a tensor product. The most general state in $V \otimes W$ takes the form

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, e_i \otimes f_j, \tag{18.1.8}$$

with $c_{ij} \in \mathbb{C}$ arbitrary numbers.

Example 18.1. *State space of two spin one-half particles.*
Consider two spin one-half particles. For the first particle, we have the state space V_1 with basis states $|+\rangle_1$ and $|-\rangle_1$. For the second particle, we have the state space V_2 with basis states $|+\rangle_2$ and $|-\rangle_2$. The tensor product $V_1 \otimes V_2$ has four basis vectors:

$$|+\rangle_1 \otimes |+\rangle_2; \quad |+\rangle_1 \otimes |-\rangle_2; \quad |-\rangle_1 \otimes |+\rangle_2; \quad |-\rangle_1 \otimes |-\rangle_2. \tag{18.1.9}$$

The most general state of the two-particle system is a linear superposition of the four basis states:

$$|\Psi\rangle = \alpha_1|+\rangle_1 \otimes |+\rangle_2 + \alpha_2|+\rangle_1 \otimes |-\rangle_2 + \alpha_3|-\rangle_1 \otimes |+\rangle_2 + \alpha_4|-\rangle_1 \otimes |-\rangle_2. \tag{18.1.10}$$

Here the α_i with $i = 1, 2, 3, 4$ are complex constants. If we follow the convention that the first ket corresponds to particle 1 and the second ket corresponds to particle 2, we need not write the subscripts, and the notation is simpler. The above state would read

$$|\Psi\rangle = \alpha_1|+\rangle \otimes |+\rangle + \alpha_2|+\rangle \otimes |-\rangle + \alpha_3|-\rangle \otimes |+\rangle + \alpha_4|-\rangle \otimes |-\rangle. \tag{18.1.11}$$

Using the properties of the tensor product, the state can be written in the form

$$|\Psi\rangle = |+\rangle \otimes (\alpha_1|+\rangle + \alpha_2|-\rangle) + |-\rangle \otimes (\alpha_3|+\rangle + \alpha_4|-\rangle). \tag{18.1.12}$$

Expanding out the products, we recover the original form.

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