22.4 Subsystems and Schmidt Decomposition

Let us consider the physics of a quantum system A that is a part, or a subsystem, of a composite system AB. The composite system is isolated from the rest of the world and is defined on a state space $\mathcal{H}_A \otimes \mathcal{H}_B$, with $\mathcal{H}_A$ and $\mathcal{H}_B$ the state spaces for A and B subsystems, respectively. Typically, one has interactions that couple the two systems A and B, and the systems are entangled. The system AB is a **bipartite** system, which just means it is composed of two parts. In example 22.2 we considered a composite system AB in a pure entangled state and demonstrated that there is no pure state that represents the system A. We need a density matrix. This is truly the only option if AB is not in a pure state.

To fix notation, assume the spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ have dimensions d_A and d_B, respectively, and have orthonormal bases given by

$$\begin{aligned} &\dim \mathcal{H}_A = d_A, \quad (e_1^A, \ldots, e_{d_A}^A) \text{ orthonormal basis,} \\ &\dim \mathcal{H}_B = d_B, \quad (e_1^B, \ldots, e_{d_B}^B) \text{ orthonormal basis.} \end{aligned} \tag{22.4.1}$$

Let there be a density matrix ρ_{AB} for the full AB system. We then ask: What is the relevant density matrix ρ_A that can be used to compute the results of measurements on A? The answer turns out to be quite simple. Since the system B plays no role here, ρ_A, sometimes called the *reduced density matrix*, is obtained by taking the partial trace over $\mathcal{H}_B$ of the full density matrix:

$$\boxed{\rho_A = \text{tr}_B \rho_{AB} = \sum_k \langle e_k^B | \rho_{AB} | e_k^B \rangle \in \mathcal{L}(\mathcal{H}_A).} \tag{22.4.2}$$

This proposal passes a basic consistency check: if ρ_{AB} is a density matrix, so is ρ_A. To see this first note that the trace works out correctly:

$$\text{tr}_A \rho_A = \text{tr}_A \text{tr}_B \rho_{AB} = \text{tr}\, \rho_{AB} = 1, \tag{22.4.3}$$

where tr is the full trace in the tensor product space—see equation (18.4.14). Moreover, ρ_A is a positive semidefinite operator. To prove this we must show that $\langle v_A | \rho_A | v_A \rangle \geq 0$ for any $|v_A\rangle \in \mathcal{H}_A$. This is not complicated:

$$\langle v_A | \rho_A | v_A \rangle = \langle v_A | \sum_k \langle e_k^B | \rho_{AB} | e_k^B \rangle | v_A \rangle = \sum_k \langle v_A | \langle e_k^B | \, \rho_{AB} \, | v_A \rangle | e_k^B \rangle \geq 0, \tag{22.4.4}$$

since every term in the sum is nonnegative because ρ_{AB} is positive semidefinite. Being a positive semidefinite operator of unit trace, ρ_A can represent a density matrix.

The formula (22.4.2) is justified by showing that for an arbitrary operator $\mathcal{O}_A \in \mathcal{L}(\mathcal{H}_A)$ the expectation value obtained using ρ_A equals the expectation value of $\mathcal{O}_A \otimes \mathbb{1}_B$ using the full density matrix ρ_{AB}:

$$\mathrm{tr}_A\left(\rho_A \mathcal{O}_A\right) = \mathrm{tr}\left(\rho_{AB} \mathcal{O}_A \otimes \mathbb{1}_B\right). \tag{22.4.5}$$

To see this let us write a general density matrix ρ_{AB} as the most general operator on $\mathcal{H}_A \otimes \mathcal{H}_B$. Recalling that $\mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B) = \mathcal{L}(\mathcal{H}_A) \otimes \mathcal{L}(\mathcal{H}_B)$, we use basis operators $|e_i^A\rangle\langle e_j^A| \in \mathcal{L}(\mathcal{H}_A)$ and $|e_k^B\rangle\langle e_l^B| \in \mathcal{L}(\mathcal{H}_B)$ to write the most general linear superposition of tensor products:

$$\rho_{AB} = \sum_{i,j,k,l} \rho_{ij,kl}\, |e_i^A\rangle\langle e_j^A| \otimes |e_k^B\rangle\langle e_l^B|. \tag{22.4.6}$$

It follows that

$$\rho_A = \mathrm{tr}_B\, \rho_{AB} = \sum_{i,j,k,l} \rho_{ij,kl}\, |e_i^A\rangle\langle e_j^A| \otimes \langle e_l^B|e_k^B\rangle = \sum_{i,j,k} \rho_{ij,kk}\, |e_i^A\rangle\langle e_j^A|, \tag{22.4.7}$$

and as a result, the left-hand side of (22.4.5) is

$$\mathrm{tr}_A\left(\rho_A\, \mathcal{O}_A\right) = \sum_{i,j,k} \rho_{ij,kk}\, \langle e_j^A|\mathcal{O}_A|e_i^A\rangle. \tag{22.4.8}$$

Similarly, we compute the right-hand side of (22.4.5):

$$\begin{aligned}\mathrm{tr}\left(\rho_{AB}\, \mathcal{O}_A \otimes \mathbb{1}_B\right) &= \mathrm{tr}_A\, \mathrm{tr}_B \sum_{i,j,k,l} \rho_{ij,kl}\, |e_i^A\rangle\langle e_j^A|\mathcal{O}_A \otimes |e_k^B\rangle\langle e_l^B| \\ &= \sum_{i,j,k} \rho_{ij,kk}\, \langle e_j^A|\mathcal{O}_A|e_i^A\rangle,\end{aligned} \tag{22.4.9}$$

making it clear that (22.4.5) holds and thus justifying the claimed formula for the density matrix ρ_A of the subsystem A. Note that nowhere in the proof of (22.4.5) have we used any particular property of the density operator ρ_{AB}. This means that this identity is true for arbitrary operators:

Theorem 22.4.1. *Let $S_{AB} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ be an arbitrary operator, and $S_A = tr_B S_{AB}$. Then for any $\mathcal{O}_A \in \mathcal{L}(\mathcal{H}_A)$ we find that*

$$tr_A\left(S_A \mathcal{O}_A\right) = tr\left(S_{AB} \mathcal{O}_A \otimes \mathbb{1}_B\right). \tag{22.4.10}$$

Example 22.5. *A pure state of two entangled spins and density matrix of a subsystem.*
Consider the pure-state system AB of two spins examined before:

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}\big(|+\rangle_A|-\rangle_B - |-\rangle_A|+\rangle_B\big). \tag{22.4.11}$$

Alice has particle A, and Bob has particle B. We aim to find the density matrix ρ_B for subsystem B, Bob's particle.

The density matrix for the full system is just $\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$, which we can write out conveniently as sums of tensor products of operators:

$$\begin{aligned}\rho_{AB} &= \frac{1}{\sqrt{2}}\big(|+\rangle_A|-\rangle_B - |-\rangle_A|+\rangle_B\big)\frac{1}{\sqrt{2}}\big(\langle+|_A\langle-|_B - \langle-|_A\langle+|_B\big)\\ &= \quad \tfrac{1}{2}\,(|+\rangle\langle+|)_A \otimes (|-\rangle\langle-|)_B\\ &\quad - \tfrac{1}{2}\,(|+\rangle\langle-|)_A \otimes (|-\rangle\langle+|)_B\\ &\quad - \tfrac{1}{2}\,(|-\rangle\langle+|)_A \otimes (|+\rangle\langle-|)_B\\ &\quad + \tfrac{1}{2}\,(|-\rangle\langle-|)_A \otimes (|+\rangle\langle+|)_B.\end{aligned} \tag{22.4.12}$$

We can now take the trace over A to find the density matrix for B. From the four terms displayed on the last right-hand side, the second and third have zero tr_A. The nonvanishing contributions give

$$\rho_B = \text{tr}_A\,\rho_{AB} = \tfrac{1}{2}|-\rangle\langle-| + \tfrac{1}{2}|+\rangle\langle+|. \tag{22.4.13}$$

The state of B is maximally mixed. This was probably expected, as the original pure state $|\psi_{AB}\rangle$ seems as entangled as can be. Another curious fact emerges when we recall that in (22.1.16) we obtained the ensemble that Bob gets when Alice does a measurement of her particle but does not communicate the result. The ensemble is exactly the same as that described by the above ρ_B. Thus, whether or not Alice measures, the state of B is the same, in this case the maximally mixed state. We will understand this remarkable coincidence more generally in the latter part of this section.

□

Exercise 22.5. *Consider the following state $|\hat{\psi}_{AB}\rangle$ of two spin one-half particles A and B:*

$$
\begin{aligned}
|\hat{\psi}_{AB}\rangle &= \tfrac{1}{\sqrt{2}}|+\rangle_A|+\rangle_B + \tfrac{1}{2}|-\rangle_A|+\rangle_B - \tfrac{1}{2}|-\rangle_A|-\rangle_B, \\
&= \tfrac{1}{\sqrt{2}}|+\rangle_A|+\rangle_B + \tfrac{1}{\sqrt{2}}|-\rangle_A|x;-\rangle_B.
\end{aligned}
\tag{22.4.14}
$$

Show that the density matrix ρ_A for particle A is

$$
\rho_A = \tfrac{1}{2}|+\rangle\langle+| + \tfrac{1}{2\sqrt{2}}|+\rangle\langle-| + \tfrac{1}{2\sqrt{2}}|-\rangle\langle+| + \tfrac{1}{2}|-\rangle\langle-|. \tag{22.4.15}
$$

Exercise 22.6. *Diagonalize the above density matrix ρ_A and show that*

$$
\rho_A = \tfrac{1}{2}\big(1 + \tfrac{1}{\sqrt{2}}\big)\,|x;+\rangle\langle x;+| + \tfrac{1}{2}\big(1 - \tfrac{1}{\sqrt{2}}\big)\,|x;-\rangle\langle x;-|. \tag{22.4.16}
$$

Schmidt decomposition The *pure* states $|\psi_{AB}\rangle$ of a bipartite system AB can be written in an insightful way by using as a guide the associated density matrices ρ_A and ρ_B of the subsystems. The result is the *Schmidt decomposition* of the pure state $|\psi_{AB}\rangle$, named in honor of Erhard Schmidt (1876–1959), also known for the Gram-Schmidt procedure that yields orthonormal basis vectors from a set of nonorthonormal ones. The decomposition displays a simple structure—simpler than the general structure allowed from the tensor product:

1. The state $|\psi_{AB}\rangle$ is written in terms of an orthonormal basis $\{|k_A\rangle\}$ of $\mathcal{H}_A$ and an orthonormal basis $\{|k_B\rangle\}$ of $\mathcal{H}_B$ that, respectively, make the reduced density matrices ρ_A and ρ_B diagonal.
2. The decomposition defines an integer r, called the Schmidt index, that characterizes the degree of entanglement of the subsystems A and B.

Suppose we have a bipartite system AB and a pure state $|\Psi_{AB}\rangle$ in which A is entangled with B:

$$
|\Psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B. \tag{22.4.17}
$$

The dimensions and basis states for $\mathcal{H}_A$ and $\mathcal{H}_B$ are as in (22.4.1). Assume that we choose to label the systems so that

$$
d_A \leq d_B. \tag{22.4.18}
$$

A state $|\Psi_{AB}\rangle$ is typically written as an expansion over the obvious basis states $|e_i^A\rangle \otimes |e_j^B\rangle$ of $\mathcal{H}_A \otimes \mathcal{H}_B$:

$$|\Psi_{AB}\rangle = \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} \psi_{ij} |e_i^A\rangle \otimes |e_j^B\rangle. \tag{22.4.19}$$

Here the ψ_{ij} are $d_A \cdot d_B$ expansion coefficients. In the Schmidt decomposition, we will do much better than this. Actually, the above expression can be rewritten as

$$|\Psi_{AB}\rangle = \sum_{i=1}^{d_A} |e_i^A\rangle \otimes |\psi_i^B\rangle, \quad \text{with} \quad |\psi_i^B\rangle = \sum_{j=1}^{d_B} \psi_{ij} |e_j^B\rangle. \tag{22.4.20}$$

While this is nicer, not much can be said about the $|\psi_i^B\rangle$ states; in particular, they need not be orthonormal. In the Schmidt decomposition, the sum analogous to the sum over i runs up to an integer r called the Schmidt index, which can be smaller than d_A. Moreover, the states from $\mathcal{H}_B$ as well as those from $\mathcal{H}_A$ are orthonormal.

To derive the Schmidt decomposition, we consider the state $|\psi_{AB}\rangle$ and the associated density matrices:

$$\rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}|, \quad \rho_A = \text{tr}_B \rho_{AB} = \text{tr}_B \big(|\Psi_{AB}\rangle\langle\Psi_{AB}|\big). \tag{22.4.21}$$

By construction, ρ_A is a Hermitian positive semidefinite $d_A \times d_A$ matrix and can therefore be diagonalized. Let $(p_k, |k_A\rangle\ )$ with $k = 1, \ldots, d_A$ be the eigenvalues and eigenvectors of ρ_A, with the eigenvectors $|k_A\rangle$ chosen to be an orthonormal basis for $\mathcal{H}_A$ and the eigenvalues p_k nonnegative. We see that the density matrix ρ_A has furnished us with a second orthonormal basis of states for $\mathcal{H}_A$. The density matrix ρ_A can then be written as

$$\rho_A = \sum_{k=1}^{d_A} p_k |k_A\rangle\langle k_A|, \quad \sum_{k=1}^{d_A} p_k = 1. \tag{22.4.22}$$

It may happen, for example, that ρ_A is a pure state, in which case the above sum has just one term, and only one p_k is nonzero. In general, the sum defining ρ_A has $r \leq d_A$ terms. Let us assume this and order the list of eigenvectors and eigenvalues so that the first r eigenvalues are nonzero and the rest vanish. Then we will write

$$\rho_A = \sum_{k=1}^{r} p_k |k_A\rangle\langle k_A|, \quad r \leq d_A, \text{ and } p_{k>r} = 0. \tag{22.4.23}$$

Let us now consider $|\psi_{AB}\rangle$. Since the $|k_A\rangle$ span $\mathcal{H}_A$, we can write

$$|\psi_{AB}\rangle = \sum_{k=1}^{d_A} |k_A\rangle \otimes |\psi_k^B\rangle, \tag{22.4.24}$$

with $|\psi_k^B\rangle$ some collection of states in $\mathcal{H}_B$. Note that we have at most d_A terms, not the $d_A \cdot d_B$ terms that would arise if we used the basis states of $\mathcal{H}_B$ to expand the $|\psi_k^B\rangle$ states. Forming the density matrix associated with $|\psi_{AB}\rangle$, we find that

$$\rho_{AB} = \sum_{k,\tilde{k}=1}^{d_A} |k_A\rangle \otimes |\psi_k^B\rangle \ \langle \tilde{k}_A| \otimes \langle \psi_{\tilde{k}}^B|. \tag{22.4.25}$$

Taking the trace over B, we now get

$$\rho_A = \mathrm{tr}_B \rho_{AB} = \sum_{k,\tilde{k}=1}^{d_A} |k_A\rangle\langle \tilde{k}_A| \ \langle \psi_{\tilde{k}}^B|\psi_k^B\rangle. \tag{22.4.26}$$

Compare now with our previous expression for ρ_A in (22.4.23), where no state $|k_A\rangle$ with $k > r$ appears. This means that we should reconsider our ansatz for $|\psi_{AB}\rangle$: no state $|k_A\rangle$ with $k > r$ can appear there either. If they did, there would be some nonvanishing terms in ρ_A that are not included in (22.4.23). We therefore rewrite

$$|\psi_{AB}\rangle = \sum_{k=1}^{r} |k_A\rangle \otimes |\psi_k^B\rangle, \tag{22.4.27}$$

which leads to

$$\rho_{AB} = \sum_{k,\tilde{k}=1}^{r} |k_A\rangle \otimes |\psi_k^B\rangle \langle \tilde{k}_A| \otimes \langle \psi_{\tilde{k}}^B| \ \Rightarrow \ \rho_A = \sum_{k,\tilde{k}=1}^{r} |k_A\rangle\langle \tilde{k}_A| \ \langle \psi_{\tilde{k}}^B|\psi_k^B\rangle. \tag{22.4.28}$$

Once again comparing with ρ_A in (22.4.23), we see there should be no terms with $k \neq \tilde{k}$. Full agreement then requires that

$$\langle \psi_{\tilde{k}}^B|\psi_k^B\rangle = p_k \, \delta_{k\tilde{k}}, \quad k, \tilde{k} = 1, \ldots, r. \tag{22.4.29}$$

In other words, states $|\psi_k^B\rangle$ with different values of k must be orthogonal. It is therefore useful to introduce normalized versions $|k_B\rangle$ of the states $|\psi_k^B\rangle$ as follows:

$$|k_B\rangle \equiv \frac{|\psi_k^B\rangle}{\sqrt{p_k}}, \quad k=1,\ldots,r. \tag{22.4.30}$$

These states satisfy

$$\langle k_B|k'_B\rangle = \delta_{k,k'}, \quad k,k'=1,\ldots,r. \tag{22.4.31}$$

If $r < d_B$, one can define additional orthonormal vectors to have a full basis for $\mathcal{H}_B$. These extra vectors will not feature below.

We have already shown that the pure state $|\psi_{AB}\rangle$ of the bipartite system AB can always be written as a sum of r terms. From (22.4.27) and (22.4.30), we now get the Schmidt decomposition of the pure state $|\psi_{AB}\rangle$:

$$\boxed{|\psi_{AB}\rangle = \sum_{k=1}^{r} \sqrt{p_k}\,|k_A\rangle \otimes |k_B\rangle, \quad r \leq d_A \leq d_B.} \tag{22.4.32}$$

In here,

$$\sum_{k=1}^{r} p_k = 1, \quad p_k > 0, \quad k=1,\ldots,r, \tag{22.4.33}$$

and the states $|k_A\rangle \in \mathcal{H}_A$ and $|k_B\rangle \in \mathcal{H}_B$, with $k = 1, \ldots, r$, form orthonormal sets:

$$\langle k_A|k'_A\rangle = \delta_{k,k'}, \quad \langle k_B|k'_B\rangle = \delta_{k,k'}. \tag{22.4.34}$$

Despite the similar notation, the $|k_A\rangle$ and $|k_B\rangle$ states have nothing to do with each other; they live in different spaces. The Schmidt decomposition (22.4.32) has the properties we anticipated before. It involves the sum of $r \leq d_A$ terms, each a basis state of $\mathcal{H}_A$ multiplied by some state in $\mathcal{H}_B$. Moreover, the $\mathcal{H}_B$ states $|k_B\rangle$ multiplying the $|k_A\rangle$ basis states also form an orthonormal set. Finally, since the construction is inspired by density matrices, the reduced density matrix ρ_A, and in fact $\rho_B = \text{tr}_A \rho_{AB}$ as well, are nicely written in the above language. We already had ρ_A from (22.4.23), and ρ_B follows from a very brief calculation:

$$\boxed{\rho_A = \sum_{k=1}^{r} p_k |k_A\rangle\langle k_A|, \qquad \rho_B = \sum_{k=1}^{r} p_k |k_B\rangle\langle k_B|.} \tag{22.4.35}$$

The $\mathcal{H}_A$ and $\mathcal{H}_B$ basis vectors used in the Schmidt decomposition make the density matrices ρ_A and ρ_B diagonal. Moreover, ρ_A and ρ_B have exactly the same nonzero eigenvalues! This is an important result for any bipartite system in a pure state. Since any pure state of a bipartite system AB has a Schmidt decomposition, the value of r is unambiguously determined. This value is called the **Schmidt number** of the state.

If a state of AB has Schmidt number one, the A and B subsystems are not entangled: the Schmidt decomposition provides a manifest description of the AB state as the tensor product of a state in $\mathcal{H}_A$ and a state in $\mathcal{H}_B$. Moreover, if the Schmidt number r is greater than one, the subsystems are definitely entangled. This is clear because the reduced density matrices ρ_A and ρ_B are mixed (they have $r > 1$ terms), and a state of AB where A and B are not entangled always leads to density matrices ρ_A and ρ_B that represent pure states.

Exercise 22.7. *Prove that for any pure entangled state of AB the purity of ρ_A equals the purity of ρ_B with value:*

$$\zeta(\rho_A) = \zeta(\rho_B) = \sum_{k=1}^{r} p_k^2. \tag{22.4.36}$$

Consider a bipartite system AB where A and B have state spaces of the same dimensionality, and the state of AB is pure. The result of the above exercise shows that when ρ_A is maximally mixed, so is ρ_B. In such a case, we say that A and B are *maximally entangled*.

Example 22.6. *Schmidt decomposition of a state.*
We considered in exercise 22.5 a pure state $|\hat{\psi}_{AB}\rangle$ of a bipartite system AB:

$$|\hat{\psi}_{AB}\rangle = \tfrac{1}{\sqrt{2}}|+\rangle_A|+\rangle_B + \tfrac{1}{2}|-\rangle_A|+\rangle_B - \tfrac{1}{2}|-\rangle_A|-\rangle_B. \tag{22.4.37}$$

We aim to find its Schmidt decomposition. Some of the relevant work was done already. You diagonalized the reduced density matrix ρ_A, finding that

$$\rho_A = \tfrac{1}{2}\big(1 + \tfrac{1}{\sqrt{2}}\big)\,|x;+\rangle\langle x;+| \;+ \tfrac{1}{2}\big(1 - \tfrac{1}{\sqrt{2}}\big)\,|x;-\rangle\langle x;-|. \tag{22.4.38}$$

On account of the relation between ρ_A and the state $|\psi_{AB}\rangle$ it arises from, exemplified in equations (22.4.35) and (22.4.32), we have a simple ansatz for the state $|\hat{\psi}_{AB}\rangle$:

$$|\hat{\psi}_{AB}\rangle = \tfrac{1}{\sqrt{2}}\sqrt{1+\tfrac{1}{\sqrt{2}}}\,|x;+\rangle_A|1_B\rangle + \tfrac{1}{\sqrt{2}}\sqrt{1-\tfrac{1}{\sqrt{2}}}\,|x;-\rangle_A|2_B\rangle, \tag{22.4.39}$$

where the states $|1_B\rangle$ and $|2_B\rangle$ are orthonormal states to be determined. Writing the $|\pm\rangle_A$ states in the expression (22.4.37) for $|\hat{\psi}_{AB}\rangle$ in terms of $|x;\pm\rangle_A$, a short calculation gives

$$\begin{aligned}|\hat{\psi}_{AB}\rangle &= \tfrac{1}{2}|x;+\rangle_A \otimes \Big(\big(1+\tfrac{1}{\sqrt{2}}\big)|+\rangle_B - \tfrac{1}{\sqrt{2}}|-\rangle_B\Big)\\ &\quad + \tfrac{1}{2}|x;-\rangle_A \otimes \Big(\big(1-\tfrac{1}{\sqrt{2}}\big)|+\rangle_B + \tfrac{1}{\sqrt{2}}|-\rangle_B\Big).\end{aligned} \tag{22.4.40}$$

The earlier result (22.4.39) tells us how to rewrite this in a way that orthonormality is manifest. We find that

$$\begin{aligned}|\hat{\psi}_{AB}\rangle &= \tfrac{1}{\sqrt{2}}\sqrt{1+\tfrac{1}{\sqrt{2}}}\,|x;+\rangle_A \otimes \sqrt{1-\tfrac{1}{\sqrt{2}}}\Big(\big(1+\tfrac{1}{\sqrt{2}}\big)|+\rangle_B - \tfrac{1}{\sqrt{2}}|-\rangle_B\Big)\\ &\quad + \tfrac{1}{\sqrt{2}}\sqrt{1-\tfrac{1}{\sqrt{2}}}\,|x;-\rangle_A \otimes \sqrt{1+\tfrac{1}{\sqrt{2}}}\Big(\big(1-\tfrac{1}{\sqrt{2}}\big)|+\rangle_B + \tfrac{1}{\sqrt{2}}|-\rangle_B\Big).\end{aligned} \tag{22.4.41}$$

This is the Schmidt decomposition of $|\hat{\psi}_{AB}\rangle$. You can check that the states to the right of $|x;\pm\rangle_A$ are orthonormal. The Schmidt number is two, and the subsystems A and B are entangled.

□

Measurement along a basis in a subsystem We now want to extend to bipartite systems the result $\tilde{\rho} = \sum_i M_i \rho M_i$ (see (22.2.45)), giving the density matrix $\tilde{\rho}$ after measurement along an orthonormal basis $\{|i\rangle\}$ when the result of the measurement is not known, and the original density matrix is ρ. Here, $M_i = |i\rangle\langle i|$.

Consider therefore a bipartite system AB, and imagine that Alice measures the state of A along a basis $\{|i\rangle_A\}$ associated with projectors $M_i^A = |i\rangle_A\, {}_A\langle i|$, satisfying

$$M_i^{A\dagger} = M_i^A, \quad M_i^A M_i^A = M_i^A, \quad \sum_i M_i^A = \mathbb{1}_A. \tag{22.4.42}$$

Assume that we start with a density matrix ρ_{AB}, and we do not know the result of Alice's measurement. In analogy to the previous result, the density matrix $\tilde{\rho}_{AB}$ after measurement is

$$\tilde{\rho}_{AB} = \sum_i (M_i^A \otimes \mathbb{1}_B)\, \rho_{AB}\, (M_i^A \otimes \mathbb{1}_B). \tag{22.4.43}$$

While clearly very plausible, this claim can be proven explicitly (problem 22.4). With this result we can learn something important about entanglement. We now look for the reduced density matrix $\tilde{\rho}_B$ of B following from $\tilde{\rho}_{AB}$ to see the effect on Bob due to Alice's measurement on A. Can Bob tell that Alice did a measurement? We have already seen in some particular case (example 22.5) that Bob cannot.

We begin our work with the density matrix of B *after* measurement:

$$\tilde{\rho}_B = \mathrm{tr}_A\, \tilde{\rho}_{AB} = \mathrm{tr}_A \sum_i (M_i^A \otimes \mathbb{1}_B)\, \rho_{AB}\, (M_i^A \otimes \mathbb{1}_B). \tag{22.4.44}$$

We wish to compare $\tilde{\rho}_B$ with the density matrix $\rho_B = \mathrm{tr}_A \rho_{AB}$ of B *before* measurement. To analyze this we use a general representation of the original bipartite density matrix in terms of a collection of operators $\mathcal{O}_k^A$ and $\mathcal{O}_k^B$ indexed by some label k:

$$\rho_{AB} = \sum_k \mathcal{O}_k^A \otimes \mathcal{O}_k^B, \quad \mathcal{O}_k^A \in \mathcal{L}(H_A), \quad \mathcal{O}_k^B \in \mathcal{L}(H_B), \quad \forall\, k. \tag{22.4.45}$$

Then we have

$$\begin{aligned} \tilde{\rho}_B &= \mathrm{tr}_A \sum_{i,k} (M_i^A \otimes \mathbb{1}_B)\, \mathcal{O}_k^A \otimes \mathcal{O}_k^B\, (M_i^A \otimes \mathbb{1}_B) \\ &= \mathrm{tr}_A \sum_{i,k} M_i^A \mathcal{O}_k^A M_i^A \otimes \mathcal{O}_k^B = \sum_{i,k} \mathrm{tr}_A (M_i^A \mathcal{O}_k^A M_i^A)\, \mathcal{O}_k^B. \end{aligned} \tag{22.4.46}$$

Recalling the cyclicity of the trace and the projector properties of M_i^A listed above, we have $\mathrm{tr}_A(M_i^A \mathcal{O}_k^A M_i^A) = \mathrm{tr}_A(M_i^A \mathcal{O}_k^A)$. Since the sum of M_i^A's over i gives the identity matrix, we get

$$\tilde{\rho}_B = \sum_k \sum_i \mathrm{tr}_A (M_i^A \mathcal{O}_k^A)\, \mathcal{O}_k^B = \sum_k \mathrm{tr}_A(\mathcal{O}_k^A)\, \mathcal{O}_k^B = \mathrm{tr}_A\, \rho_{AB} = \rho_B. \tag{22.4.47}$$

Alice's measurement, with results unknown to Bob, does *not* change Bob's density matrix. This means that Alice cannot use a measurement to communicate information instantaneously to Bob. Since the particles in entangled pairs can be very far away, this prevents superluminal transfer of information, thus avoiding conflict with special relativity. Note that the result did not depend on using the density matrix of AB. The result holds for an arbitrary operator S_{AB} on AB:

Theorem 22.4.2. No-signaling theorem. *Let $S_{AB} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ be an arbitrary operator and $\tilde{S}_{AB}$ be defined by*

$$\tilde{S}_{AB} = \sum_i (M_i^A \otimes \mathbb{1}_B)\, S_{AB}\, (M_i^A \otimes \mathbb{1}_B), \tag{22.4.48}$$

with M_i^A orthogonal projectors satisfying (22.4.42). Then, $tr_A \tilde{S}_{AB} = tr_A S_{AB}$.

If Alice and Bob share an entangled pair of quantum systems, this theorem prevents Alice from sending a message or a signal to Bob instantaneously by performing measurements. Thus the name *no-signaling theorem*.

We can also imagine that Alice, instead of measuring, applies some Hamiltonian to her system, causing some unitary evolution represented by the operator $\mathcal{U}_A$. In this case the evolved density matrix $\hat{\rho}_{AB}$ of the bipartite system whose initial density matrix is ρ_{AB} takes the form

$$\hat{\rho}_{AB} = (\mathcal{U}_A \otimes \mathbb{1}_B)\, \rho_{AB}\, (\mathcal{U}_A^{\dagger} \otimes \mathbb{1}_B). \tag{22.4.49}$$

It is now simple to show, just as above, that the density matrix for Bob is not affected:

$$\hat{\rho}_B \equiv \text{tr}_A\, \hat{\rho}_{AB} = \rho_B, \tag{22.4.50}$$

where $\rho_B = \text{tr}_A \rho_{AB}$ is the density matrix of B before Alice subjected her particle to unitary evolution. Alice cannot signal Bob by acting on her system with arbitrary unitary evolution.

Exercise 22.8. *Prove that (22.4.50) holds.*