16.6 Axioms of Quantum Mechanics

We built the mathematical foundations of quantum mechanics by taking a detailed look at complex vector spaces in chapters 13 and 14. We then turned to the spectral theorem and the spectral representation of operators in chapter 15, learning about complete sets of orthogonal projectors. In this chapter we discussed unitary time evolution and how it implies a Schrödinger equation. Having built, additionally, some experience in quantum systems, it is a good time to scrutinize the axioms of quantum mechanics.

We encountered these axioms before, in section 5.3, but we did not aim to phrase them in all generality. Sometimes these axioms are called "*postulates*" of quantum mechanics. The words *axioms* and *postulates* are largely considered synonymous: they are statements that cannot be proven and are treated as self-evident truths upon which a theory is built. Still, there is a sense in which axioms are used for universal truths, while postulates are used for specific theories or applications. In this spirit we will consider four axioms and two postulates. The axioms apply to any isolated quantum system. A system that is not isolated is called an *open* system, and it interacts with other systems. If all the systems together compose a whole system that is isolated, the axioms apply to the whole system. Open systems will be explored when we discuss density matrices in chapter 22.

Let us now state the axioms, denoted as A1, A2, A3, and A4, valid for any *isolated* quantum system:

A1. States of the system The complete description of a quantum system is given by a *ray* in a Hilbert space $\mathcal{H}$.

Remarks:

- A ray in a vector space is a nonzero vector $|\Psi\rangle$ with the equivalence relation $|\Psi\rangle \simeq c|\Psi\rangle$ for any nonzero $c \in \mathbb{C}$. Any vector in this ray is a representative of the state of the system. The vector $|\Psi\rangle$ is also called a wave function.
- Affirming that the state gives a complete description of the system, the axiom implies that the state describes the *most* that can be known about the system.
- A finite-dimensional Hilbert space is a complex vector space with an inner product $\langle \cdot, \cdot \rangle$ satisfying the axioms listed in section 14.1. When the Hilbert space is infinite-dimensional, the space must also be complete in the norm: all Cauchy sequences of vectors must converge to vectors in the space, as explained above example 14.1. The Hilbert space $\mathcal{H}$ is the state space of the system.
- The state $|\Psi\rangle$ of the system has a representative with unit norm. This representative is a normalized state or wave function.
- However complicated the quantum system and however many particles it contains, just *one* state, one wave function, represents the full quantum state of the system.

A2. Observables Hermitian operators on the state space $\mathcal{H}$ are observables.

Remarks:

- An observable of a system is a property of the system that can be measured. This postulate says that such properties arise from Hermitian operators.
- The spectral theorem (section 15.6) implies that any observable $\hat{A} = \hat{A}^{\dagger}$ can be written as the sum

$$\hat{A} = \sum_k a_k P_k, \tag{16.6.1}$$

where the sum runs over all the *different* eigenvalues a_k of $\hat{A}$, and the P_k are a complete set of orthogonal projectors into the corresponding eigenspaces:

$$P_k^{\dagger} = P_k, \quad P_k P_l = \delta_{kl} P_k, \quad \sum_k P_k = \mathbb{1}. \tag{16.6.2}$$

If an eigenvalue is nondegenerate, the associated projector is rank one. If an eigenvalue has a multiplicity $l > 1$, the associated projector is rank l and projects into an l-dimensional eigenspace.

A3. Measurement Let P_k, with $k = 1, \ldots,$ denote a complete set of orthogonal projectors, and let $\mathcal{H}_k$ denote the subspace P_k projects into. Measurement along this set of projectors is a process in which the state Ψ is projected to $\mathcal{H}_k$ with probability $p(k)$ given by

$$p(k) = \langle \Psi | P_k | \Psi \rangle = \| P_k | \Psi \rangle \|^2. \tag{16.6.3}$$

The normalized state after measurement is

$$\frac{P_k | \Psi \rangle}{\| P_k | \Psi \rangle \|}. \tag{16.6.4}$$

Measuring an observable $\hat{A}$ is measuring along the complete set of orthogonal projectors associated with its spectral decomposition (16.6.1). The probability $p(k)$ for the state to be projected to $\mathcal{H}_k$ is the probability of measuring a_k.

Remarks:

- Measurement of an observable is a nondeterministic physical process. We cannot in general predict the result of the measurement, just the probabilities for the various possible results.
- The measurement axiom does not give any prescription for measuring the state $|\Psi\rangle$ itself. The state is the full description of the system, but it cannot be directly measured. We can only measure observables, and such measurements give us some information about the state.
- The probabilities $p(k)$ add up to one, as they should. Using the completeness of the set of projectors, we indeed find that

$$\sum_k p(k) = \sum_k \langle \Psi | P_k | \Psi \rangle = \langle \Psi | \Big(\sum_k P_k \Big) | \Psi \rangle = \langle \Psi | \Psi \rangle = 1. \qquad (16.6.5)$$

- When measuring $\hat{A}$, if the eigenvalue a_k is nondegenerate with eigenvector $|k\rangle$, then $P_k = |k\rangle\langle k|$, and the probability $p(k)$ is

$$p(k) = \langle \Psi | k \rangle \langle k | \Psi \rangle = |\langle k | \Psi \rangle|^2. \qquad (16.6.6)$$

- When measuring $\hat{A}$, if the eigenvalue a_k is degenerate with multiplicity l, the associated eigenspace is spanned by l orthonormal eigenvectors $|k;1\rangle, \ldots, |k;l\rangle$, and

$$P_k = \sum_{i=1}^{l} |k;i\rangle\langle k;i| \qquad (16.6.7)$$

so that we have

$$p(k) = \langle \Psi | \sum_{i=1}^{l} |k;i\rangle\langle k;i|\Psi\rangle = \sum_{i=1}^{l} |\langle k;i|\Psi\rangle|^2. \qquad (16.6.8)$$

- Measurement along an orthonormal basis $\{|i\rangle\}$ means measuring along the complete set of rank-one orthogonal projectors $P_i = |i\rangle\langle i|$. The probability $p(i)$ of being found in the state $|i\rangle$ arising by projection via P_i is

$$p(i) = \langle \Psi | P_i | \Psi \rangle = |\langle i | \Psi \rangle|^2. \qquad (16.6.9)$$

- When we say we are measuring an orthogonal projector P, we mean treating P as a Hermitian operator, with eigenvalues one and zero.
- The axiom applies (at least formally) to the traditional measurement of the particle position on the real line for a wave function $\psi(x)$. Consider for this purpose the following operator defined for real a and b with $a < b$:

$$P_{a,b} \equiv \int_a^b dx |x\rangle\langle x|. \tag{16.6.10}$$

It is clear that this $P_{a,b}$ is Hermitian, and a quick calculation shows that it satisfies $P_{a,b}P_{a,b} = P_{a,b}$, making it an orthogonal projector. You can also check that any $|x'\rangle$ with $x' \in (a, b)$ is an eigenstate of $P_{a,b}$ with eigenvalue one. The projector $P_{a,b}$ thus projects to the subspace $\mathcal{H}_{ab}$ of states spanned by *all* the kets $|x'\rangle$ with $x' \in (a, b)$.

The projector $P_{a,b}$, together with the projector $\widetilde{p}_{a,b} = \mathbb{1} - P_{a,b}$, forms a complete set of orthogonal projectors. If we measure $|\psi\rangle$ along this set of projectors, the probability $p(a, b)$ of being found in $\mathcal{H}_{ab}$ is

$$\begin{aligned} p(a,b) &= \langle\psi|P_{a,b}|\psi\rangle = \int_a^b dx \langle\psi|x\rangle\langle x|\psi\rangle \\ &= \int_a^b dx\, \psi^*(x)\psi(x) = \int_a^b dx |\psi(x)|^2. \end{aligned} \tag{16.6.11}$$

This is the probability of finding the particle in the range $x \in [a, b]$.

A4. Dynamics Time evolution is unitary: given any state $|\Psi, t_0\rangle$ of the system at time t_0, the state $|\Psi, t_1\rangle$ at time t_1 is obtained by the action of a unitary operator $\mathcal{U}(t_1, t_0)$:

$$|\Psi, t_1\rangle = \mathcal{U}(t_1, t_0)|\Psi, t_0\rangle. \tag{16.6.12}$$

Remarks:

- Time evolution is deterministic: if the state is known exactly at some time it is known exactly at a later time.
- The same operator $\mathcal{U}(t_1, t_0)$ evolves any possible state of the system at time t_0.

- We showed in section 16.2 that unitary time evolution means the state satisfies the Schrödinger equation $i\hbar\partial_t|\Psi\rangle = \hat{H}|\Psi\rangle$ with $\hat{H}$ as the Hamiltonian, a Hermitian operator with units of energy. Thus, axiom A4 implies that any quantum system has a Schrödinger equation that controls the time evolution of the wave function.

It is worth noting how surprising the measurement axiom is. While axiom A4 states that the time evolution of the state is generated by a unitary operator, when we measure, the "evolution" of the state is nondeterministic and happens from the action of one out of several projectors. Projectors, moreover, are not unitary operators: they kill some states, while unitary operators kill none. This means that measurement is not time evolution in the sense of A4. This has mystified many physicists who argue that measurement devices are physical systems governed by quantum laws and wonder why and how unitary time evolution fails to hold. We will discuss this issue further, without resolving it, in section 22.7.

The axiomatic formulation described above follows the *Copenhagen interpretation* of quantum mechanics developed by Bohr, Heisenberg, and others. Perhaps one day it will be improved and replaced with a less mysterious one, but to date, this formulation is consistent with all known facts about quantum mechanics.

In setting up certain quantum mechanical systems, the use of some guiding principles that appear not to follow from the above axioms seem needed. One such principle is relevant to the construction of composite systems, and the other is relevant to systems with identical particles. We will call both of them postulates, as they apply in specific circumstances and lack the generality of the four axioms stated above. Since we have not yet studied composite systems or systems with identical particles, the statements below will not be explained at this point; they are included for completeness. The reader may return to this part after studying the relevant chapters.

P1. Composite system postulate Assume system A has a state space $\mathcal{H}_A$, and system B has a state space $\mathcal{H}_B$. The state space of the composite system AB is the tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$. If system A is prepared in $|\Psi_A\rangle$ and system B is prepared in $|\Psi_B\rangle$, the state of AB is $|\Psi_A\rangle \otimes |\Psi_B\rangle$.

P2. Symmetrization postulate In a system with N identical particles, the states that are physically realized are either totally symmetric under the exchange of the particles, in which case the particles are said to be bosons, or they are totally antisymmetric under the exchange, in which case they are said to be fermions.

The composite system postulate, more than an axiom, seems like a prescription for building or defining a composite system in which we can implement axioms A1 to A4. The material relevant to this postulate is discussed in chapter 18. The symmetrization postulate, stated in chapter 21, is the known way to resolve the problem of *exchange degeneracy*. In fact, this postulate is proven in relativistic quantum field theory under some weak set of assumptions. Moreover, in quantum systems with two spatial dimensions, particles that are neither bosons nor fermions can exist (section 21.4).