17.2 Heisenberg Picture for Coherent States

We will later discuss the explicit time evolution of coherent states. In the meantime we can study the time evolution of expectation values quite efficiently using the Heisenberg picture since we already calculated in (16.5.25) the time-dependent Heisenberg operators $\hat{x}_H(t)$ and $\hat{p}_H(t)$.

If at time equal zero we have the coherent state $|x_0\rangle c$, at time t we write the time-evolved state as $|x_0,\ t\rangle c$. We now ask what the (time-dependent) expectation value of $\hat{x}$ is on this state:

$$\langle\hat{x}\rangle_{x_0}(t) = {}_c\langle x_0, t|\,\hat{x}\,|x_0, t\rangle_c = {}_c\langle x_0|\,\hat{x}_H(t)\,|x_0\rangle_c, \tag{17.2.1}$$

the last equality being the definition of the Heisenberg operator itself. Using the explicit form of $\hat{x}_H(t)$, we get

$$\langle\hat{x}\rangle_{x_0}(t) = {}_c\langle x_0|\left(\hat{x}\,\cos\omega t + \frac{1}{m\omega}\,\hat{p}\,\sin\omega t\right)|x_0\rangle_c. \tag{17.2.2}$$

Finally, using (17.1.18) and (17.1.19), we find that

$$\langle\hat{x}\rangle_{x_0}(t) = x_0\cos\omega t. \tag{17.2.3}$$

The expectation value of $\hat{x}$ is performing oscillatory motion! This confirms the classical interpretation of the coherent state. For the momentum the calculation is quite similar:

$$\langle\hat{p}\rangle_{x_0}(t) = {}_c\langle x_0|\,\hat{p}_H(t)\,|x_0\rangle_c = {}_c\langle x_0|\left(\hat{p}\cos\omega t - m\omega\,\hat{x}\,\sin\omega t\right)|x_0\rangle_c, \tag{17.2.4}$$

resulting in

$$\langle\hat{p}\rangle_{x_0}(t) = -m\omega\,x_0\,\sin\omega t, \tag{17.2.5}$$

which is the expected result, as it is equal to $m\frac{d}{dt}\langle\hat{x}\rangle_{x_0}(t)$.

We know that the harmonic oscillator ground state is a minimum uncertainty state. We will now discuss the extension of this fact to coherent states. We begin by calculating the uncertainties Δx and Δp in a coherent state at $t = 0$. We will see that, just like the ground state, the coherent state minimizes the product of uncertainties. Then we will calculate uncertainties of the coherent state as a function of time!

Let us begin with the position uncertainty Δx. We find that

$$ {}_c\langle x_0|\hat{x}^2|x_0\rangle_c \;=\; \langle 0|T_{x_0}^\dagger \hat{x}^2 T_{x_0}|0\rangle \;=\; \langle 0|(\hat{x}+x_0)^2|0\rangle \;=\; \langle 0|\hat{x}^2|0\rangle + x_0^2. \tag{17.2.6} $$

The first term on the right-hand side was calculated in (9.4.31) and results in

$$ {}_c\langle x_0|\hat{x}^2|x_0\rangle_c \;=\; \tfrac{1}{2}L_0^2 + x_0^2. \tag{17.2.7} $$

Since $c\langle x_0|\hat{x}|x_0\rangle c = x_0$, the position uncertainty is given by

$$ (\Delta x)^2 = \tfrac{1}{2}L_0^2, \quad \text{on the state } |x_0\rangle_c. \tag{17.2.8} $$

For the momentum uncertainty, the computation is quite analogous:

$$ {}_c\langle x_0|\hat{p}^2|x_0\rangle_c \;=\; \langle 0|T_{x_0}^\dagger \hat{p}^2 T_{x_0}|0\rangle \;=\; \langle 0|\hat{p}^2|0\rangle \;=\; \tfrac{1}{2}\left(\tfrac{\hbar}{L_0}\right)^2, \tag{17.2.9} $$

where the last equality follows from (9.4.33). Recalling that $c\langle x_0|\hat{p}\;|x_0\rangle c$ = 0, we have

$$ (\Delta p)^2 = \tfrac{1}{2}\left(\tfrac{\hbar}{L_0}\right)^2, \quad \text{on the state } |x_0\rangle_c. \tag{17.2.10} $$

As a result,

$$ \Delta x \Delta p = \tfrac{\hbar}{2}, \quad \text{on the state } |x_0\rangle_c. \tag{17.2.11} $$

The coherent state has minimum $\Delta x \Delta p$ at time equal zero, as befits a state that is just a displaced ground state.

To find the time-dependent uncertainties, our expectation values must be evaluated on the time-dependent state $|x_0,\ t\rangle c$. We begin with the position operator:

$$
\begin{aligned}
(\Delta x)^2(t) &= {}_c\langle x_0, t|\hat{x}^2|x_0, t\rangle_c - {}_c\langle x_0, t|\hat{x}|x_0, t\rangle_c^2 \\
&= {}_c\langle x_0|\hat{x}_H^2(t)|x_0\rangle_c - {}_c\langle x_0|\hat{x}_H(t)|x_0\rangle_c^2 \\
&= {}_c\langle x_0|\hat{x}_H^2(t)|x_0\rangle_c - x_0^2\cos^2\omega t,
\end{aligned}
\tag{17.2.12}
$$

using (17.2.3). The computation of the first term, $I = c\langle x_0|\hat{x}_H{}^2(t)|x_0\rangle c$, takes a few steps:

$$
\begin{aligned}
I &= {}_c\langle x_0|\Big(\hat{x}\,\cos\omega t + \frac{1}{m\omega}\,\hat{p}\,\sin\omega t\Big)^2|x_0\rangle_c \\
&= {}_c\langle x_0|\hat{x}^2|x_0\rangle_c\cos^2\omega t + {}_c\langle x_0|\hat{p}^2|x_0\rangle_c\Big(\frac{\sin\omega t}{m\omega}\Big)^2 + \frac{\cos\omega t\sin\omega t}{m\omega}\,{}_c\langle x_0|\big(\hat{x}\hat{p}+\hat{p}\hat{x}\big)|x_0\rangle_c \\
&= \Big(\tfrac{1}{2}L_0^2 + x_0^2\Big)\cos^2\omega t + \frac{m\hbar\omega}{2}\Big(\frac{\sin\omega t}{m\omega}\Big)^2 + \frac{\cos\omega t\sin\omega t}{m\omega}\,{}_c\langle x_0|\big(\hat{x}\hat{p}+\hat{p}\hat{x}\big)|x_0\rangle_c.
\end{aligned}
$$

The last expectation value vanishes, as you should prove:

Exercise 17.1. *Show that* $c\langle x_0|(.\hat{x}\hat{p} + \hat{p}\,\hat{x}).|x_0\rangle c = \langle 0|(.\hat{x}\hat{p} + \hat{p}\,\hat{x}).|0\rangle = 0$. *(The vanishing of* $\langle 0|(.\hat{x}\hat{p}+ \hat{p}\,\hat{x}).|0\rangle$ *was also discussed in problem 9.5.)*

Returning to our computation, the expectation value I of $x_H{}^2(t)$ in the coherent state is

$$
I = \Big(\tfrac{1}{2}L_0^2 + x_0^2\Big)\cos^2\omega t + \frac{m\hbar\omega}{2}\Big(\frac{\sin\omega t}{m\omega}\Big)^2 = \tfrac{1}{2}L_0^2 + x_0^2\cos^2\omega t. \tag{17.2.13}
$$

Therefore, finally, going back to (17.2.12) we get

$$
(\Delta x)^2(t) = \tfrac{1}{2}L_0^2. \tag{17.2.14}
$$

The uncertainty Δx does not change in time as the state evolves! This suggests, but does not yet prove, that the state does not change shape as it moves. Not changing shape means that at different times $|\psi(x, t)|^2$ and $|\psi(x, t')|^2$ differ only by an overall displacement in x. To settle the issue of shape change, it is useful to calculate the time-dependent uncertainty in the momentum:

$$
\begin{aligned}
(\Delta p)^2(t) &= {}_c\langle x_0, t|\hat{p}^2|x_0, t\rangle_c - {}_c\langle x_0, t|\hat{p}|x_0, t\rangle_c^2 \\
&= {}_c\langle x_0|\hat{p}_H^2(t)|x_0\rangle_c - {}_c\langle x_0|\hat{p}_H(t)|x_0\rangle_c^2 \\
&= {}_c\langle x_0|\hat{p}_H^2(t)|x_0\rangle_c - m^2\omega^2\,x_0^2\sin^2\omega t,
\end{aligned}
\tag{17.2.15}
$$

where we used (17.2.5). The rest of the computation is recommended:

Exercise 17.2. *Show that*

$$ {}_c\langle x_0|\hat{p}_H^2(t)|x_0\rangle_c = \tfrac{1}{2}\left(\tfrac{\hbar}{L_0}\right)^2 + m^2\omega^2 x_0^2 \sin^2 \omega t. \tag{17.2.16}$$

This result then implies that

$$(\Delta p)^2(t) = \tfrac{1}{2}\left(\tfrac{\hbar}{L_0}\right)^2. \tag{17.2.17}$$

This, together with (17.2.14), gives

$$\Delta x(t)\Delta p(t) = \tfrac{\hbar}{2}, \quad \text{on the state } |x_0, t\rangle_c. \tag{17.2.18}$$

The coherent state remains a minimum uncertainty packet for all times. Since only Gaussians have such minimum uncertainty (example 15.3), the state remains a Gaussian for all times. Since Δx is constant, the Gaussian does not change shape, thus the name *coherent state*. The state does not spread out in time, changing shape; it just moves "coherently."

Note that

$$\Delta x(t) = \frac{L_0}{\sqrt{2}}, \text{ and } \Delta p(t) = \frac{1}{\sqrt{2}}\frac{\hbar}{L_0}. \tag{17.2.19}$$

The quantum oscillator size L_0 is very small for a macroscopic oscillator. A coherent state with a large $x_0 \gg L_0$ is classical in the sense that the typical excursion x_0 is much larger than the position uncertainty $\sim L0$. Similarly, the typical momentum $m\omega x_0 \sim \frac{\hbar}{L_0}\frac{x_0}{L_0}$ is much larger than the momentum uncertainty, by just the same factor $\sim x_0/L0$.

Exercise 17.3. *Prove that on the coherent state* $|x_0\rangle c$ *we have*

$$\frac{\Delta p(t)}{\sqrt{\overline{\langle \hat{p}^2\rangle(t)}}} = \frac{\Delta x(t)}{\sqrt{\overline{\langle \hat{x}^2\rangle(t)}}} = \frac{1}{\sqrt{1+\frac{x_0^2}{L_0^2}}}, \tag{17.2.20}$$

where the overlines on the expectation values denote time average.

Coherent states in the energy basis We can get an interesting expression for the coherent state $|x_0\rangle c$ by rewriting the momentum operator in terms of creation and annihilation operators. From (9.3.12) we have that

$$\hat{p} = \frac{i}{\sqrt{2}} \frac{\hbar}{L_0} (\hat{a}^\dagger - \hat{a}). \tag{17.2.21}$$

It follows that the coherent state (17.1.14) is given by

$$|x_0\rangle_c = \exp\Big(-\frac{i}{\hbar}\,\hat{p}\,x_0\Big)|0\rangle = \exp\Big(\frac{1}{\sqrt{2}}\frac{x_0}{L_0}(\hat{a}^\dagger - \hat{a})\Big)|0\rangle. \tag{17.2.22}$$

Since $\hat{a}|0\rangle = 0$, the above formula admits simplification: we should be able to get rid of all the $\hat{a}$'s! We could do this if we could split the exponential into two exponentials, one with the $\hat{a}^\dagger$'s to the *left* of another one with the $\hat{a}$'s. The exponential with the $\hat{a}$'s would stand near the vacuum and give no contribution, as we will see below. For this purpose we recall the commutator identity (13.7.31):

$$e^{A+B} = e^A\, e^B\, e^{-\frac{1}{2}[A,B]}, \quad \text{if } [A,B] \text{ commutes with } A \text{ and with } B. \tag{17.2.23}$$

Consider the exponential in (17.2.22), and write it as e^{A+B}, identifying A and B as follows:

$$A = \frac{1}{\sqrt{2}}\frac{x_0}{L_0}\,\hat{a}^\dagger, \quad B = -\frac{1}{\sqrt{2}}\frac{x_0}{L_0}\,\hat{a}. \tag{17.2.24}$$

Then $[A, B] = x_0^2/(2L_0^2)$ and we find

$$\exp\Big(\frac{x_0}{\sqrt{2}L_0}\,\hat{a}^\dagger - \frac{x_0}{\sqrt{2}L_0}\,\hat{a}\Big) = \exp\Big(\frac{x_0}{\sqrt{2}L_0}\,\hat{a}^\dagger\Big)\exp\Big(-\frac{x_0}{\sqrt{2}L_0}\,\hat{a}\Big)\exp\Big(-\frac{1}{4}\frac{x_0^2}{L_0^2}\Big). \tag{17.2.25}$$

Since the last exponential is just a number and the second exponential acts like the identity on the vacuum, the coherent state in (17.2.22) becomes

$$\boxed{|x_0\rangle_c = \exp\Big(-\frac{i\hat{p}\,x_0}{\hbar}\Big)|0\rangle = \exp\Big(-\frac{1}{4}\frac{x_0^2}{L_0^2}\Big)\exp\Big(\frac{x_0}{\sqrt{2}L_0}\,\hat{a}^\dagger\Big)|0\rangle.} \tag{17.2.26}$$

While this form is poised to produce an expansion in energy eigenstates, the unit normalization of the state is no longer manifest. Expanding the exponential with creation operators, we get

$$|x_0\rangle_c = \sum_{n=0}^{\infty} \exp\Big(-\frac{1}{4}\frac{x_0^2}{L_0^2}\Big) \cdot \frac{1}{n!}\Big(\frac{x_0}{\sqrt{2}L_0}\Big)^n (\hat{a}^\dagger)^n|0\rangle$$
$$= \sum_{n=0}^{\infty} \exp\Big(-\frac{1}{4}\frac{x_0^2}{L_0^2}\Big) \cdot \frac{1}{\sqrt{n!}}\Big(\frac{x_0}{\sqrt{2}L_0}\Big)^n |n\rangle. \tag{17.2.27}$$

We thus have the desired expansion of the coherent state as a linear superposition of all energy eigenstates:

$$\boxed{|x_0\rangle_c = \sum_{n=0}^{\infty} c_n|n\rangle, \quad \text{with } c_n = \exp\Big(-\frac{1}{4}\frac{x_0^2}{L_0^2}\Big) \cdot \frac{1}{\sqrt{n!}}\Big(\frac{x_0}{\sqrt{2}L_0}\Big)^n.} \tag{17.2.28}$$

Since the probability P_n of finding the energy $E_n = \hbar\omega(n+\frac{1}{2})$ is equal to $|c_n|^2 = c_n^2$, we have

$$P_n = c_n^2 = \exp\Big(-\frac{x_0^2}{2L_0^2}\Big) \cdot \frac{1}{n!}\Big(\frac{x_0^2}{2L_0^2}\Big)^n. \tag{17.2.29}$$

Introducing a dimensionless λ, the probability can be written neatly as follows:

$$\boxed{P_n = \frac{\lambda^n}{n!}e^{-\lambda}, \quad \text{with } \lambda \equiv \frac{x_0^2}{2L_0^2}.} \tag{17.2.30}$$

The values of P_n must define a probability distribution for all integers $n \geq 0$, parameterized by λ. This is in fact the familiar *Poisson distribution*. It is straightforward to verify that, as required,

$$\sum_{n=0}^{\infty} P_n = e^{-\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!} = e^{-\lambda}e^{\lambda} = 1. \tag{17.2.31}$$

The physical interpretation of λ can be obtained by computing the expectation value of n in this probability distribution:

$$\langle n\rangle \equiv \sum_{n=0}^{\infty} nP_n = e^{-\lambda}\sum_{n=0}^{\infty} n\frac{\lambda^n}{n!} = e^{-\lambda}\sum_{n=0}^{\infty}\lambda\frac{d}{d\lambda}\frac{\lambda^n}{n!} = e^{-\lambda}\lambda\frac{d}{d\lambda}e^{\lambda} = \lambda. \tag{17.2.32}$$

Therefore, λ is equal to the expected value $\langle n\rangle$—that is, the expected value of the number operator $\hat{N}$ on the coherent state.

Exercise 17.4. *Show that*

$$\langle n^2 \rangle \equiv \sum_{n=0}^{\infty} n^2 P_n = \lambda^2 + \lambda. \tag{17.2.33}$$

It now follows that

$$(\Delta n)^2 = \langle n^2 \rangle - \langle n \rangle^2 = \lambda \quad \Rightarrow \quad \Delta n = \sqrt{\lambda}. \tag{17.2.34}$$

In terms of energy, we have $E = \hbar\omega(n + \frac{1}{2})$, and therefore

$$\langle E \rangle \;=\; \hbar\omega\big(\langle n \rangle + \tfrac{1}{2}\big) \;=\; \hbar\omega\big(\lambda + \tfrac{1}{2}\big). \tag{17.2.35}$$

Moreover, $E = \hbar\omega(n + \frac{1}{2})$ also implies that $\Delta E = \hbar\omega\Delta n$; the shift by $\hbar/2$ is immaterial to the uncertainty, and the scaling by $\hbar\omega$ just goes through. Therefore,

$$\Delta E \;=\; \hbar\omega\sqrt{\lambda} \;=\; \hbar\omega\,\frac{x_0}{\sqrt{2}L_0}. \tag{17.2.36}$$

Note now that for large λ the average energy is much larger than its uncertainty:

$$\frac{\langle E \rangle}{\Delta E} \;=\; \sqrt{\lambda} + \frac{1}{2\sqrt{\lambda}} \;\simeq\; \sqrt{\lambda}. \tag{17.2.37}$$

All in all, for large λ,

$$\boxed{\sqrt{\lambda} \;=\; \frac{\Delta E}{\hbar\omega} \;\simeq\; \frac{\langle E \rangle}{\Delta E}.} \tag{17.2.38}$$

We see that the uncertainty ΔE is much larger than the separation between energy levels: it is big enough to contain about $\sqrt{\lambda}$ levels. At the same time, ΔE is much smaller than the expected value $\langle E \rangle$ of the energy, the latter a factor $\sqrt{\lambda}$ larger than the former. Of course, large $\sqrt{\lambda}$ means $x_0/L0 \gg 1$.