18.2 Operators on the Tensor Product Space

How do we construct operators that act in the vector space $V \otimes W$? Let T be an operator in V and S be an operator in W. In other words, $T \in \mathcal{L}(V)$, and $S \in \mathcal{L}(W)$. We can then construct an operator $T \otimes S$ acting on the tensor product:

$$T \otimes S \in \mathcal{L}(V \otimes W). \tag{18.2.1}$$

The operator is defined to act as follows. For any $v \in V$ and $w \in W$,

$$T \otimes S\,(v \otimes w) \equiv Tv \otimes Sw. \tag{18.2.2}$$

This is the only "natural" option: we let T act on the vector it knows how to act on and S act on the vector it knows how to act on. The identity operator in the tensor product is $\mathbb{1} \otimes \mathbb{1}$, the tensor product of the respective identity operators: the $\mathbb{1}$ to the left is the identity operator on V, and the $\mathbb{1}$ to the right is the identity operator on W. A general operator on $V \otimes W$ is a sum $\sum_i T_i \otimes S_i$ with $T_i \in \mathcal{L}(V)$, and $S_i \in \mathcal{L}(W)$. We will elaborate on this idea at the end of this section.

Suppose that we want the operator $T \in \mathcal{L}(V)$ that acts on the first particle to act on the tensor product $V \otimes W$, even though we have not supplied an operator S to act on the W part. The idea is to choose $S = \mathbb{1}$—namely, the identity operator. In this way we "upgrade" the operator T that acts on a single vector space to $T \otimes \mathbb{1}$ that acts on the tensor product:

$$T \in \mathcal{L}(V) \quad \Rightarrow \quad T \otimes \mathbb{1} \in \mathcal{L}(V \otimes W), \qquad T \otimes \mathbb{1}\,(v \otimes w) \equiv Tv \otimes w. \tag{18.2.3}$$

Similarly, an operator S belonging to $\mathcal{L}(W)$ is upgraded to $\mathbb{1} \otimes S$ to act on the tensor product. It is useful to realize that upgraded operators of the first particle *commute* with upgraded operators of the second particle. Indeed,

$$\begin{aligned}(T \otimes \mathbb{1}) \cdot (\mathbb{1} \otimes S)\,(v \otimes w) \;&=\; (T \otimes \mathbb{1})(v \otimes Sw) \;=\; Tv \otimes Sw,\\ (\mathbb{1} \otimes S) \cdot (T \otimes \mathbb{1})\,(v \otimes w) \;&=\; (\mathbb{1} \otimes S)\,(Tv \otimes w) \;=\; Tv \otimes Sw,\end{aligned} \tag{18.2.4}$$

and therefore for any S, T we have

$$[\,T \otimes \mathbb{1},\, \mathbb{1} \otimes S\,] \;=\; 0. \tag{18.2.5}$$

Given a system of two particles, we can construct a simple total Hamiltonian $\hat{H}_T$ describing no interactions by upgrading the single-particle Hamiltonians $\hat{H}_1$ and $\hat{H}_2$ and then adding them:

$$\hat{H}_T \;\equiv\; \hat{H}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \hat{H}_2. \tag{18.2.6}$$

The two terms in $\hat{H}_T$ commute with each other.

Exercise 18.1. *Convince yourself that for an arbitrary operator $\hat{A}$ we have*

$$\exp(\hat{A} \otimes \mathbb{1}) = (\exp \hat{A}) \otimes \mathbb{1}, \quad \textit{and} \quad \exp(\mathbb{1} \otimes \hat{A}) = \mathbb{1} \otimes (\exp \hat{A}). \tag{18.2.7}$$

Exercise 18.2. *Assume $\hat{H}_1$ and $\hat{H}_2$ are time independent. Convince yourself that the time-evolution operator for the two-particle Hamiltonian $\hat{H}_T$ above takes the product form*

$$\exp\Big(-\frac{i\hat{H}_T t}{\hbar}\Big) \;=\; \exp\Big(-\frac{i\hat{H}_1 t}{\hbar}\Big) \otimes \exp\Big(-\frac{i\hat{H}_2 t}{\hbar}\Big). \tag{18.2.8}$$

Example 18.2. *Spin angular momentum of a state of two spin one-half particles.*

Let us now find out how the total angular momentum operator acts on a state of two spin one-half particles. Consider, therefore, a general state $|\Psi\rangle$ of the two particles:

$$|\Psi\rangle = \alpha_1|+\rangle\otimes|+\rangle + \alpha_2|+\rangle\otimes|-\rangle + \alpha_3|-\rangle\otimes|+\rangle + \alpha_4|-\rangle\otimes|-\rangle, \tag{18.2.9}$$

with α_i, $i = 1, \ldots, 4$, complex constants. Recall that in each term on the above right-hand side the first ket corresponds to the first particle, and the second ket corresponds to the second particle. Consider now the *total* z-component of spin angular momentum. Roughly, the total angular momentum in the z-direction would be the sum of the z-components of each individual particle. However, we know better at this point—summing the two angular momenta really means constructing a new operator in the tensor product vector space:

$$\hat{S}_z^{\text{tot}} \equiv \hat{S}_z^{(1)}\otimes\mathbb{1} + \mathbb{1}\otimes\hat{S}_z^{(2)}. \tag{18.2.10}$$

We act with $\hat{S}_z^{\text{tot}}$ on the state $|\Psi\rangle$. The contributions from the two operators on the above right-hand side are

$$\begin{aligned}
(\hat{S}_z^{(1)}\otimes\mathbb{1})|\Psi\rangle &= \alpha_1\hat{S}_z|+\rangle\otimes|+\rangle + \alpha_2\hat{S}_z|+\rangle\otimes|-\rangle + \alpha_3\hat{S}_z|-\rangle\otimes|+\rangle + \alpha_4\hat{S}_z|-\rangle\otimes|-\rangle \\
&= \tfrac{\hbar}{2}\big(\alpha_1|+\rangle\otimes|+\rangle + \alpha_2|+\rangle\otimes|-\rangle - \alpha_3|-\rangle\otimes|+\rangle - \alpha_4|-\rangle\otimes|-\rangle\big), \\
(\mathbb{1}\otimes\hat{S}_z^{(2)})|\Psi\rangle &= \alpha_1|+\rangle\otimes\hat{S}_z|+\rangle + \alpha_2|+\rangle\otimes\hat{S}_z|-\rangle + \alpha_3|-\rangle\otimes\hat{S}_z|+\rangle + \alpha_4|-\rangle\otimes\hat{S}_z|-\rangle \\
&= \tfrac{\hbar}{2}\big(\alpha_1|+\rangle\otimes|+\rangle - \alpha_2|+\rangle\otimes|-\rangle + \alpha_3|-\rangle\otimes|+\rangle - \alpha_4|-\rangle\otimes|-\rangle\big).
\end{aligned}$$

Adding these together, we have

$$\hat{S}_z^{\text{tot}}|\Psi\rangle = \hbar\left(\alpha_1|+\rangle_1\otimes|+\rangle_2 - \alpha_4|-\rangle_1\otimes|-\rangle_2\right). \tag{18.2.11}$$

One can derive this result quickly by noting that since $\hat{S}_z^{(1)}$ is diagonal in the basis for V_1 and $\hat{S}_z^{(2)}$ is diagonal in the basis for V_2, the total $\hat{S}_z^{\text{tot}}$ is diagonal in the tensor product basis. As a result, its eigenvalues on these basis states are the sum of the $\hat{S}_z$ eigenvalues for particle 1 and particle 2. Thus,

$$
\begin{aligned}
\hat{S}_z^{\text{tot}}|+\rangle\otimes|+\rangle &= \left(\tfrac{\hbar}{2}+\tfrac{\hbar}{2}\right)|+\rangle\otimes|+\rangle = \hbar\,|+\rangle\otimes|+\rangle,\\
\hat{S}_z^{\text{tot}}|+\rangle\otimes|-\rangle &= \left(\tfrac{\hbar}{2}-\tfrac{\hbar}{2}\right)|+\rangle\otimes|-\rangle = 0,\\
\hat{S}_z^{\text{tot}}|-\rangle\otimes|+\rangle &= \left(-\tfrac{\hbar}{2}+\tfrac{\hbar}{2}\right)|-\rangle\otimes|+\rangle = 0,\\
\hat{S}_z^{\text{tot}}|-\rangle\otimes|-\rangle &= \left(-\tfrac{\hbar}{2}-\tfrac{\hbar}{2}\right)|-\rangle\otimes|-\rangle = -\hbar\,|-\rangle\otimes|-\rangle.
\end{aligned}
\tag{18.2.12}
$$

The result in (18.2.11) follows quickly from the four relations above. Suppose we are only interested in states in the tensor product that have zero $\hat{S}_z^{\text{tot}}$ or, equivalently, states $|\Psi\rangle$ that satisfy $\hat{S}_z^{\text{tot}}|\Psi\rangle = 0$. This requires that

$$
\alpha_1 = \alpha_4 = 0 \quad \Rightarrow \quad |\Psi\rangle = \alpha_2|+\rangle\otimes|-\rangle + \alpha_3|-\rangle\otimes|+\rangle. \tag{18.2.13}
$$

This is the *most general* state in the tensor product of two spin one-half particles that has zero total spin angular momentum in the z-direction.

□

Example 18.3. *State of two spin one-half particles with zero total spin.* We found in the previous example that the general state with zero total S_z^{tot} is

$$
|\Psi\rangle = \alpha_2|+\rangle\otimes|-\rangle + \alpha_3|-\rangle\otimes|+\rangle, \tag{18.2.14}
$$

with α_2 and α_3 arbitrary complex constants. We now calculate the total x-component $\hat{S}_x^{\text{tot}}$ of spin angular momentum on the above states. For this we recall that $\hat{S}_x|\pm\rangle = \frac{\hbar}{2}|\mp\rangle$, and we write

$$
\hat{S}_x^{\text{tot}} = \hat{S}_x \otimes 1 + 1 \otimes \hat{S}_x. \tag{18.2.15}
$$

The calculation proceeds as follows:

$$
\begin{aligned}
\hat{S}_x^{\text{tot}}|+\rangle\otimes|-\rangle &= \hat{S}_x|+\rangle\otimes|-\rangle + |+\rangle\otimes\hat{S}_x|-\rangle = \tfrac{\hbar}{2}\big(|-\rangle\otimes|-\rangle + |+\rangle\otimes|+\rangle\big),\\
\hat{S}_x^{\text{tot}}|-\rangle\otimes|+\rangle &= \hat{S}_x|-\rangle\otimes|+\rangle + |-\rangle\otimes\hat{S}_x|+\rangle = \tfrac{\hbar}{2}\big(|+\rangle\otimes|+\rangle + |-\rangle\otimes|-\rangle\big).
\end{aligned}
\tag{18.2.16}
$$

Therefore,

$$
\begin{aligned}
\hat{S}_x^{\text{tot}}|\Psi\rangle &= \alpha_2\,\tfrac{\hbar}{2}\big(|-\rangle\otimes|-\rangle + |+\rangle\otimes|+\rangle\big) + \alpha_3\,\tfrac{\hbar}{2}\big(|+\rangle\otimes|+\rangle + |-\rangle\otimes|-\rangle\big)\\
&= \tfrac{\hbar}{2}\,(\alpha_2+\alpha_3)\big(|+\rangle\otimes|+\rangle + |-\rangle\otimes|-\rangle\big).
\end{aligned}
\tag{18.2.17}
$$

If we demand that $\hat{S}_x^{\text{tot}}$ is also zero acting on the state $|\Psi\rangle$ we must have $\alpha_2 = -\alpha_3$. Thus, the following state is the unique state with zero $\hat{S}_x^{\text{tot}}$ and $\hat{S}_z^{\text{tot}}$:

$$|\Psi\rangle = \alpha\Big(|+\rangle\otimes|-\rangle - |-\rangle\otimes|+\rangle\Big). \tag{18.2.18}$$

As it turns out, this state also has zero $\hat{S}_y^{\text{tot}}$ (exercise below). Since all three operators $\hat{S}_z^{\text{tot}}, \hat{S}_x^{\text{tot}}$, and $\hat{S}_y^{\text{tot}}$ annihilate $|\Psi\rangle$, the state has zero total spin angular momentum. It is a state of two spin one-half particles in which the spin angular momenta add up to zero.

□

Exercise 18.3. *Verify that the state (18.2.18) satisfies* $\hat{S}_y^{\text{tot}}|\Psi\rangle = 0$.

Let us discuss further the structure of linear operators on the tensor product $V \otimes W$. The space of those linear operators is called $\mathcal{L}(V \otimes W)$. We claimed that the most general element in this space was constructed as the sum $\sum_i T_i \otimes S_i$ with $T_i \in \mathcal{L}(V)$ and $S_i \in \mathcal{L}(W)$. In fact, the precise statement is that

$$\mathcal{L}(V\otimes W) = \mathcal{L}(V)\otimes\mathcal{L}(W). \tag{18.2.19}$$

We can explain why this holds using basis states. Suppose we have basis states $|e_i^V\rangle$ with $i = 1, \ldots, m$ for the space V and basis states $|e_a^W\rangle$ with $a = 1, \ldots, n$ for the space W. Let us now list basis vectors for the other relevant spaces:

$$\begin{aligned} &\text{basis vectors for } \mathcal{L}(V) = \{\,|e_i^V\rangle\langle e_j^V|,\ i,j = 1,\ldots,m\},\\ &\text{basis vectors for } \mathcal{L}(W) = \{\,|e_a^W\rangle\langle e_b^W|, a, b = 1,\ldots,n\}, \end{aligned} \tag{18.2.20}$$

since the general operator in a vector space is a linear superposition of basis ket-bra operators. For the tensor product we then have

$$\text{basis vectors for } \mathcal{L}(V)\otimes\mathcal{L}(W) = \left\{ |e_i^V\rangle\langle e_j^V| \otimes |e_a^W\rangle\langle e_b^W|,\ \begin{matrix} i,j = 1,\ldots,m \\ a,b = 1,\ldots,n \end{matrix} \right\}, \tag{18.2.21}$$

since here a basis vector is a basis vector in the first factor $\mathcal{L}(V)$ tensored with a basis vector in the second factor $\mathcal{L}(W)$. Finally, we also have

$$\text{basis vectors for } \mathcal{L}(V\otimes W) = \left\{ |e_i^V\rangle\otimes|e_a^W\rangle\,\langle e_j^V|\otimes\langle e_b^W|,\ \begin{matrix} i,j = 1,\ldots,m \\ a,b = 1,\ldots,n \end{matrix} \right\}, \tag{18.2.22}$$

constructed as ket-bra operators of $V \otimes W$. We can now discuss the claimed relation (18.2.19). We assert that the basis vectors of $\mathcal{L}(V \otimes W)$

and the basis vectors of $\mathcal{L}(V) \otimes \mathcal{L}(W)$ in fact represent the same operator. This follows from the way these basis vectors are defined to act. Letting similarly labeled basis vectors act on $|v\rangle \otimes |w\rangle$, we have

$$\text{from } \mathcal{L}(V \otimes W): \ |e_i^V\rangle \otimes |e_a^W\rangle\, \langle e_j^V| \otimes \langle e_b^W| \ \ |v\rangle \otimes |w\rangle = |e_i^V\rangle \otimes |e_a^W\rangle\, \langle e_j^V|v\rangle \langle e_b^W|w\rangle,$$

$$\text{from } \mathcal{L}(V) \otimes \mathcal{L}(W): \ |e_i^V\rangle\, \langle e_j^V| \ \otimes \ |e_a^W\rangle\, \langle e_b^W| \ \ |v\rangle \otimes |w\rangle = |e_i^V\rangle\, \langle e_j^V|v\rangle \otimes |e_a^W\rangle \langle e_b^W|w\rangle.$$

We can see that the results in both lines are the same. This shows that the spaces of linear operators on the left-hand side and the right-hand side of (18.2.19) agree.

It is interesting to consider the case when the spaces V and W are the same space V, and we have a "swap" operator $S \in \mathcal{L}(V \otimes V)$ that acts as follows:

$$S(v \otimes \tilde{v}) = \tilde{v} \otimes v, \quad \text{for all } v, \tilde{v} \in V. \tag{18.2.23}$$

It may seem puzzling that S can be constructed from sums of products of operators that act separately on the two vector spaces. But, in fact, one can easily build this operator. With basis vectors $|e_i\rangle$, $i = 1, \ldots, n$ for V, the swap operator $S \in \mathcal{L}(V \otimes V)$ is given by

$$S = \sum_{i,j=1}^{n} |e_i\rangle \langle e_j| \otimes |e_j\rangle \langle e_i|. \tag{18.2.24}$$

Exercise 18.4. *Show that this operator satisfies the requisite action (18.2.23).*