## 14.7 Rotation Operators for Spin States

We have seen examples of orbital angular momentum operators generating particular rotations (section 10.1). The operators generating rotations are unitary operators. We now wish to construct general rotation operators. For this purpose we will focus here on spin angular momentum. As we will explain, the main results apply for general angular momentum, such as orbital angular momentum.

We will show how unitary operators constructed from the spin operators $\hat{\mathbf{S}}$ act on the spin operators themselves, and from this, we will learn how they rotate spin states. A few relations are useful in our analysis, and we discuss them now. The first one gives the expectation value of $\hat{\mathbf{S}}$ in a spin state $|\mathbf{n}\rangle$. We claim that

$$\langle\mathbf{n}|\hat{\mathbf{S}}|\mathbf{n}\rangle = \frac{\hbar}{2}\,\mathbf{n}. \tag{14.7.1}$$

This is, in fact, a way to measure the direction $\mathbf{n}$ of a spin state: the components of $\mathbf{n}$ are proportional to the expectation values of the components of the spin operator. This is quickly checked explicitly, recalling that with $\mathbf{n} = (n_x, n_y, n_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ we have $|\mathbf{n}\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}e^{i\phi}|-\rangle$. Thus, for example,

$$\begin{aligned}\langle\mathbf{n}|\hat{S}_x|\mathbf{n}\rangle &= \frac{\hbar}{2}\Big(\cos\tfrac{\theta}{2}\langle+| + \sin\tfrac{\theta}{2}e^{-i\phi}\langle-|\Big)\sigma_x\Big(\cos\tfrac{\theta}{2}|+\rangle + \sin\tfrac{\theta}{2}e^{i\phi}|-\rangle\Big)\\ &= \frac{\hbar}{2}\Big(\cos\tfrac{\theta}{2}\langle+| + \sin\tfrac{\theta}{2}e^{-i\phi}\langle-|\Big)\Big(\cos\tfrac{\theta}{2}|-\rangle + \sin\tfrac{\theta}{2}e^{i\phi}|+\rangle\Big)\\ &= \frac{\hbar}{2}\cos\tfrac{\theta}{2}\sin\tfrac{\theta}{2}(e^{i\phi}+e^{-i\phi}) = \frac{\hbar}{2}\sin\theta\cos\phi = \frac{\hbar}{2}n_x.\end{aligned} \tag{14.7.2}$$

**Exercise 14.5.** *Check that* $\langle\mathbf{n}|\hat{S}_y|\mathbf{n}\rangle = \frac{\hbar}{2}n_y$, *and* $\langle\mathbf{n}|\hat{S}_z|\mathbf{n}\rangle = \frac{\hbar}{2}n_z$, *thus completing the verification of (14.7.1).*

The second property we need is just a simple commutator. For an arbitrary vector $\mathbf{a}$, we have

$$[\mathbf{a}\cdot\hat{\mathbf{S}}, \hat{\mathbf{S}}] = -i\hbar\,\mathbf{a}\times\hat{\mathbf{S}}. \tag{14.7.3}$$

This is quickly checked by explicit computation:

**Exercise 14.6.** *Verify that* $[\mathbf{a}\cdot\hat{\mathbf{S}}, \hat{S}_k] = -i\hbar\,(\mathbf{a}\times\hat{\mathbf{S}})_k$, *thus proving (14.7.3).*

The third and final set of needed results involves rotations. It is known in mechanics that a vector $\mathbf{v}$ in three-dimensional space rotating around the directed axis $\mathbf{n}$ with angular velocity $\omega$ satisfies the differential equation

$$\frac{d\mathbf{v}}{dt} = \boldsymbol{\omega}\times\mathbf{v}, \quad \text{with } \boldsymbol{\omega}\equiv\omega\mathbf{n}. \tag{14.7.4}$$

At any instant of time, the vector $\mathbf{v}(t)$ is obtained from the time equal zero vector $\mathbf{v}(0)$ by a rotation of angle $\omega t$ about the axis $\mathbf{n}$. This rotation can be viewed as a rotation matrix $\mathcal{R}_{\mathbf{n}}(\omega t)$ acting on the time equal zero vector:

$$\mathbf{v}(t) = \mathcal{R}_{\mathbf{n}}(\omega t)\,\mathbf{v}(0). \tag{14.7.5}$$

The subscript in the rotation matrix gives the unit vector specifying the directed axis of rotation, and the argument is the angle of rotation. Interestingly (problem 14.6), the explicit time evolution of the vector can be written neatly because the rotation matrix takes a simple form. For a rotation angle $\alpha$ about the directed axis $\mathbf{n}$, we find that the action on an arbitrary vector $\mathbf{u}$ gives

$$\mathcal{R}_{\mathbf{n}}(\alpha)\,\mathbf{u} = (1-\cos\alpha)\,(\mathbf{n}\cdot\mathbf{u})\,\mathbf{n} + (\cos\alpha)\,\mathbf{u} + (\sin\alpha)\,(\mathbf{n}\times\mathbf{u}). \tag{14.7.6}$$

Note that $\mathcal{R}_{\mathbf{n}}(0)\mathbf{u} = \mathbf{u}$, as expected: the zero angle rotation is the identity matrix. The rotation matrix is a periodic function of $\alpha$ with period $2\pi$, also as expected. Acting on a vector, $\mathcal{R}_{\mathbf{n}}(\alpha)$ rotates it by an angle $\alpha$ about the axis defined by $\mathbf{n}$. A couple of extra properties of rotations are worth discussing (their verification is in problem 14.6). If we perform rotations about the *same* axis successively, the composition is simply a rotation about that axis with the total angle given by the sum of the angles:

$$\mathcal{R}_{\mathbf{n}}(\beta)\,\mathcal{R}_{\mathbf{n}}(\alpha) = \mathcal{R}_{\mathbf{n}}(\alpha+\beta). \tag{14.7.7}$$

This shows that $\mathcal{R}_{\mathbf{n}}(-\alpha)$ is the inverse of $\mathcal{R}_{\mathbf{n}}(\alpha)$, as is also clear from the definition. Another property is interesting. In $\mathbb{R}^3$ the dot product is invariant under rotations. This means that the inner product of two vectors is unchanged when both vectors are subject to the *same rotation*. Thus, for vectors $\mathbf{u}$ and $\mathbf{v}$ we have

$$\mathbf{u}\cdot\mathbf{v} = \big(\mathcal{R}_{\mathbf{n}}(\alpha)\,\mathbf{u}\big)\cdot\big(\mathcal{R}_{\mathbf{n}}(\alpha)\,\mathbf{v}\big). \tag{14.7.8}$$

We can now begin the detailed analysis of spin rotations. For this we define the *unitary* operator $\hat{R}_{\mathbf{n}}(\alpha)$:

$$\hat{R}_{\mathbf{n}}(\alpha) = e^{-\frac{i}{\hbar}\alpha\hat{S}_{\mathbf{n}}} = e^{-\frac{i\alpha}{\hbar}\mathbf{n}\cdot\hat{\mathbf{S}}} = e^{-\frac{i\alpha}{2}\mathbf{n}\cdot\boldsymbol{\sigma}}. \tag{14.7.9}$$

We now claim that the action of $\hat{R}_{\mathbf{n}}(\alpha)$ on the vector operator, or operator triplet $\hat{\mathbf{S}}$, rotates the components as if they were the components of an ordinary vector in three dimensions and that its action on a spin state $|\mathbf{n}'\rangle$ gives a spin state pointing at the direction defined by the rotation of $\mathbf{n}'$. This is the content of the following theorem:

**Theorem 14.7.1.** *With* $\hat{R}_{\mathbf{n}}(\alpha) = \exp(-\frac{i}{\hbar}\alpha\hat{S}_{\mathbf{n}})$*, we find that*

$$\hat{R}^{\dagger}_{\mathbf{n}}(\alpha)\,\hat{\mathbf{S}}\,\hat{R}_{\mathbf{n}}(\alpha) \;=\; \mathcal{R}_{\mathbf{n}}(\alpha)\,\hat{\mathbf{S}}, \tag{14.7.10}$$

*as well as*

$$\hat{R}_{\mathbf{n}}(\alpha)|\mathbf{n}'\rangle = |\mathbf{n}''\rangle, \quad \textit{with} \;\; \mathbf{n}'' = \mathcal{R}_{\mathbf{n}}(\alpha)\,\mathbf{n}', \tag{14.7.11}$$

*with the spin states defined up to phases.*

*Comments:* Note that the first equation above, (14.7.10), relates triplets: on the left-hand side the unitary operator acts on each component of $\hat{\mathbf{S}}$ by similarity, and on the right-hand side $\mathcal{R}_{\mathbf{n}}(\alpha)\,\hat{\mathbf{S}}$ is defined as in (14.7.6). We emphasize that $\mathcal{R}_{\mathbf{n}}(\alpha)$ is defined to act on vectors, or triplets, while $\hat{R}_{\mathbf{n}}(\alpha)$ is a unitary operator on spin space $\mathbb{C}^2$. The second equation, (14.7.11), tells us how $\hat{R}_{\mathbf{n}}(\alpha)$ rotates spin states.

*Proof.* We define the vector function $\mathbf{G}(\alpha)$ as the quantity we wish to evaluate:

$$\mathbf{G}(\alpha) \equiv \hat{R}^{\dagger}_{\mathbf{n}}(\alpha)\,\hat{\mathbf{S}}\,\hat{R}_{\mathbf{n}}(\alpha) = e^{\frac{i\alpha}{\hbar}\mathbf{n}\cdot\hat{\mathbf{S}}}\,\hat{\mathbf{S}}\,e^{-\frac{i\alpha}{\hbar}\mathbf{n}\cdot\hat{\mathbf{S}}}. \tag{14.7.12}$$

Note that $\mathbf{G}(0) = \hat{\mathbf{S}}$. Now differentiate with respect to $\alpha$ to obtain

$$\frac{d\mathbf{G}(\alpha)}{d\alpha} = \frac{i}{\hbar}\,e^{\frac{i\alpha}{\hbar}\mathbf{n}\cdot\hat{\mathbf{S}}}\,[\mathbf{n}\cdot\hat{\mathbf{S}},\,\hat{\mathbf{S}}]\,e^{-\frac{i\alpha}{\hbar}\mathbf{n}\cdot\hat{\mathbf{S}}}. \tag{14.7.13}$$

Using the commutator identity (14.7.3), we now get

$$\frac{d\mathbf{G}(\alpha)}{d\alpha} = e^{\frac{i\alpha}{\hbar}\mathbf{n}\cdot\hat{\mathbf{S}}}\,\mathbf{n}\times\hat{\mathbf{S}}\,e^{-\frac{i\alpha}{\hbar}\mathbf{n}\cdot\hat{\mathbf{S}}} = \mathbf{n}\times\left(e^{\frac{i\alpha}{\hbar}\mathbf{n}\cdot\hat{\mathbf{S}}}\,\hat{\mathbf{S}}\,e^{-\frac{i\alpha}{\hbar}\mathbf{n}\cdot\hat{\mathbf{S}}}\right), \tag{14.7.14}$$

showing that we have the differential equation:

$$\frac{d\mathbf{G}(\alpha)}{d\alpha} = \mathbf{n}\times\mathbf{G}(\alpha). \tag{14.7.15}$$

We recognize this as the equation (14.7.4) for a rotating operator, with the role of time played by $\alpha$ and with angular speed $\omega = 1$. The solution (14.7.5) applies; after all, the equation is a linear matrix equation, and it makes no difference whether the unknowns are operators or commuting objects. We therefore find that

$$\mathbf{G}(\alpha) = \mathcal{R}_{\mathbf{n}}(\alpha)\,\mathbf{G}(0) = \mathcal{R}_{\mathbf{n}}(\alpha)\,\hat{\mathbf{S}}. \tag{14.7.16}$$

This completes the proof of (14.7.10). To prove the second claim, giving the specification of $\mathbf{n}''$, we evaluate the expectation value of $\hat{\mathbf{S}}$ on the state $|\mathbf{n}''\rangle$:

$$\langle \mathbf{n}''|\hat{\mathbf{S}}\,|\mathbf{n}''\rangle = \langle \mathbf{n}'|\hat{R}_{\mathbf{n}}^{\dagger}(\alpha)\,\hat{\mathbf{S}}\,\hat{R}_{\mathbf{n}}(\alpha)|\mathbf{n}'\rangle = \langle \mathbf{n}'|\mathcal{R}_{\mathbf{n}}(\alpha)\hat{\mathbf{S}}\,|\mathbf{n}'\rangle, \tag{14.7.17}$$

where the rotation matrix is acting on $\hat{\mathbf{S}}$. Equivalently, it is acting on the vector defined by the expectation value:

$$\langle \mathbf{n}''|\hat{\mathbf{S}}\,|\mathbf{n}''\rangle = \mathcal{R}_{\mathbf{n}}(\alpha)\langle \mathbf{n}'|\hat{\mathbf{S}}\,|\mathbf{n}'\rangle. \tag{14.7.18}$$

Recalling the expectation value in (14.7.1), we get the relation we wished to prove:

$$\tfrac{\hbar}{2}\,\mathbf{n}'' = \tfrac{\hbar}{2}\,\mathcal{R}_{\mathbf{n}}(\alpha)\mathbf{n}' \quad \Rightarrow \quad \mathbf{n}'' = \mathcal{R}_{\mathbf{n}}(\alpha)\mathbf{n}'. \tag{14.7.19}$$

The operator $\hat{R}_{\mathbf{n}}(\alpha)$ rotates a spin state by rotating the vector that defines it with the associated matrix operator $\mathcal{R}_{\mathbf{n}}(\alpha)$.

□

It is useful to elucidate how the property $\hat{R}_{\mathbf{n}}^{\dagger}(\alpha)\,\hat{\mathbf{S}}\,\hat{R}_{\mathbf{n}}(\alpha) = \mathcal{R}_{\mathbf{n}}(\alpha)\,\hat{\mathbf{S}}$ leads to the action of the rotation operators on a spin operator $\hat{S}_{\mathbf{n}'}$. For this we take the dot product of this equation with the unit vector **n'**:

$$\hat{R}_{\mathbf{n}}^{\dagger}(\alpha)\,\mathbf{n}'\cdot\hat{\mathbf{S}}\,\hat{R}_{\mathbf{n}}(\alpha) = \mathbf{n}'\cdot\mathcal{R}_{\mathbf{n}}(\alpha)\,\hat{\mathbf{S}}. \tag{14.7.20}$$

On the left-hand side, the dot product gives the spin operator $\hat{S}_{\mathbf{n}'}$. On the right-hand side, we use the invariance (14.7.8) of the dot product to act with $\mathcal{R}_{\mathbf{n}}(-\alpha)$ on both vectors:

$$\hat{R}_{\mathbf{n}}^{\dagger}(\alpha)\,\hat{S}_{\mathbf{n}'}\,\hat{R}_{\mathbf{n}}(\alpha) = (\mathcal{R}_{\mathbf{n}}(-\alpha)\mathbf{n}')\cdot\hat{\mathbf{S}}. \tag{14.7.21}$$

The right-hand side is now a spin operator, and we find that

$$\hat{R}_{\mathbf{n}}^{\dagger}(\alpha)\,\hat{S}_{\mathbf{n}'}\,\hat{R}_{\mathbf{n}}(\alpha) = \hat{S}_{\tilde{\mathbf{n}}'} \quad \text{with} \quad \tilde{\mathbf{n}}' = \mathcal{R}_{\mathbf{n}}(-\alpha)\mathbf{n}'. \tag{14.7.22}$$

The vector **n'** that defines the spin operator becomes $\tilde{\mathbf{n}}'$ using the inverse of the rotation that transformed the states in (14.7.11). Letting $\alpha \to -\alpha$ in the above relation and recalling that the rotation operators are unitary, we find that

$$\boxed{\hat{R}_{\mathbf{n}}(\alpha)\,\hat{S}_{\mathbf{n}'}\,\hat{R}_{\mathbf{n}}^{\dagger}(\alpha) = \hat{S}_{\mathbf{n}''} \quad \text{with} \quad \mathbf{n}'' = \mathcal{R}_{\mathbf{n}}(\alpha)\mathbf{n}'.} \tag{14.7.23}$$

You can quickly see that an inner product $\langle \mathbf{n}_1 | \hat{S}_{\mathbf{n}_2} | \mathbf{n}_3 \rangle$ is invariant under the simultaneous transformations:

$$|\mathbf{n}_1\rangle \rightarrow \hat{R}_{\mathbf{n}}(\alpha)|\mathbf{n}_1\rangle, \quad |\mathbf{n}_3\rangle \rightarrow \hat{R}_{\mathbf{n}}(\alpha)|\mathbf{n}_3\rangle, \quad \hat{S}_{\mathbf{n}_2} \rightarrow \hat{R}_{\mathbf{n}}(\alpha)\,\hat{S}_{\mathbf{n}_2}\,\hat{R}^{\dagger}_{\mathbf{n}}(\alpha). \tag{14.7.24}$$

This invariance is the statement that the theory has rotational symmetry: nothing changes when states and operators are simultaneously rotated.

If you go over the above derivations, the key result, $\hat{R}^{\dagger}_{\mathbf{n}}(\alpha)\,\hat{\mathbf{S}}\,\hat{R}_{\mathbf{n}}(\alpha) = \mathcal{R}_{\mathbf{n}}(\alpha)\,\hat{\mathbf{S}}$, was obtained using *only* the commutator algebra of the spin angular momentum operators. Since orbital angular momentum operators obey the same commutator algebra, this result also holds for orbital angular momentum. More generally, letting $\hat{\mathbf{J}} = (\hat{J}_1, \hat{J}_2, \hat{J}_3)$ be angular momentum operators $[\hat{J}_i, \hat{J}_j] = i\hbar\epsilon_{ijk}\hat{J}_k$, we have

$$\boxed{\hat{R}^{\dagger}_{\mathbf{n}}(\alpha)\,\hat{\mathbf{J}}\,\hat{R}_{\mathbf{n}}(\alpha) = \mathcal{R}_{\mathbf{n}}(\alpha)\,\hat{\mathbf{J}}, \quad \hat{R}_{\mathbf{n}}(\alpha) = e^{-i\frac{\alpha}{\hbar}\mathbf{n}\cdot\hat{\mathbf{J}}}.} \tag{14.7.25}$$

The rotation matrix $\mathcal{R}_{\mathbf{n}}(\alpha)$ is the same one we used above.