18.3 Inner Products for Tensor Spaces

We now consider the definition of an *inner product* in $V \otimes W$. This product can be defined naturally if we have inner products on V and on W.
In this case, with vectors $v, \tilde{v} \in V$ and $w, \tilde{w} \in W$, we simply let

$$\langle v \otimes w,\ \tilde{v} \otimes \tilde{w} \rangle \equiv \langle v,\ \tilde{v} \rangle \langle w, \tilde{w} \rangle\,, \tag{18.3.1}$$

where the inner products on the right-hand side are those in V and in W. To find the inner product of general vectors in $V \otimes W$, we must declare that with vectors $X, Y, Z \in V \otimes W$ the following distributive properties hold:

$$
\begin{aligned}
\langle X+Y,\, Z\rangle &= \langle X,\, Z\rangle + \langle Y,\, Z\rangle, \\
\langle X,\, Y+Z\rangle &= \langle X, Y\rangle + \langle X, Z\rangle.
\end{aligned}
\tag{18.3.2}
$$

Note that (18.3.1) implies that for a complex constant a,

$$
\begin{aligned}
&\big\langle v\otimes w,\ a(\tilde{v}\otimes\tilde{w})\big\rangle = \big\langle v\otimes w,\ (a\tilde{v})\otimes\tilde{w}\big\rangle = \langle v, a\tilde{v}\rangle\langle w,\tilde{w}\rangle = a\big\langle v\otimes w,\ \tilde{v}\otimes\tilde{w}\big\rangle, \\
&\big\langle a(v\otimes w),\ \tilde{v}\otimes\tilde{w}\big\rangle = \big\langle (av)\otimes w,\ \tilde{v}\otimes\tilde{w}\big\rangle = \langle av,\tilde{v}\rangle\,\langle w,\tilde{w}\rangle = a^*\big\langle v\otimes w,\ \tilde{v}\otimes\tilde{w}\big\rangle,
\end{aligned}
\tag{18.3.3}
$$

using the properties of the inner product in V. The distributive properties then show that, in general, the inner product in $V \otimes W$ satisfies the expected

$$
\begin{aligned}
\langle X, aY\rangle &= a\langle X, Y\rangle, \\
\langle aX, Y\rangle &= a^*\langle X, Y\rangle.
\end{aligned}
\tag{18.3.4}
$$

It is useful to describe the inner product in a basis $\{e_i \otimes f_j;\ i = 1, \ldots, m,\ j = 1, \ldots, n\}$ for the tensor product, with $\{e_i;\ i = 1, \ldots, m\}$ and $\{f_j;\ j = 1, \ldots, m\}$ *orthonormal* bases for V and W. Using (18.3.1), we immediately find

$$
\big\langle e_i\otimes f_j,\ e_p\otimes f_q\big\rangle = \langle e_i, e_p\rangle\langle f_j, f_q\rangle = \delta_{ip}\delta_{jq}.
\tag{18.3.5}
$$

This makes the tensor product basis vectors $e_i \otimes f_j$ orthonormal. The verification that the inner product on $V \otimes W$ satisfies the remaining axioms of an inner product (section 14.1) is addressed in the exercises below. For both exercises, assume that $X, Y \in V \otimes W$, and write the most general such vectors as $X = \sum_{ij} x_{ij}\, e_i \otimes f_j$, and $Y = \sum_{ij} y_{ij}\, e_i \otimes f_j$. Then proceed using (18.3.2), (18.3.4), and (18.3.5).

Exercise 18.5. *Show that* $\langle X, X\rangle \geq 0$, *and* $\langle X, X\rangle = 0$ *if and only if* $X = 0$.

Exercise 18.6. *Show that* $\langle X, Y\rangle = \langle Y, X\rangle^*$.

It is often convenient to use bra-ket notation for inner products in the tensor product. Kets and bras in the tensor product are often written as follows:

$$
\begin{aligned}
|v\otimes w\rangle &= |v\rangle_1 \otimes |w\rangle_2, \\
\langle v\otimes w| &= {}_1\langle v| \otimes {}_2\langle w|.
\end{aligned}
\tag{18.3.6}
$$

Notice that for both bras and kets we write the state of particle 1 to the left of the state of particle 2. We then write (18.3.1) as

$$\langle v\otimes w|\tilde{v}\otimes\tilde{w}\rangle \;=\; \left({}_1\langle v|\otimes{}_2\langle w|\right)\left(|\tilde{v}\rangle_1\otimes|\tilde{w}\rangle_2\right)=\langle v|\tilde{v}\rangle\,\langle w|\tilde{w}\rangle. \tag{18.3.7}$$

Going back to example 18.3, we can now normalize the state we built with two spin one-half particles and zero total spin angular momentum. Our four basis vectors $|+\rangle_1\otimes|+\rangle_2$, $|+\rangle_1\otimes|-\rangle_2$, $|-\rangle_1\otimes|+\rangle_2$, and $|-\rangle_1\otimes|-\rangle_2$ are orthonormal. We had the unnormalized state in (18.2.18) given by

$$|\Psi\rangle=\alpha\left(|+\rangle_1\otimes|-\rangle_2\;-\;|-\rangle_1\otimes|+\rangle_2\right). \tag{18.3.8}$$

The associated bra is then

$$\langle\Psi|=\alpha^*\left({}_1\langle+|\otimes{}_2\langle-|\;-\;{}_1\langle-|\otimes{}_2\langle+|\right). \tag{18.3.9}$$

We then have

$$\begin{aligned}\langle\Psi|\Psi\rangle \;&=\; \alpha\alpha^*\left({}_1\langle+|\otimes{}_2\langle-|\;-\;{}_1\langle-|\otimes{}_2\langle+|\right)\left(|+\rangle_1\otimes|-\rangle_2\;-\;|-\rangle_1\otimes|+\rangle_2\right)\\ &=\;\alpha\alpha^*\left({}_1\langle+|\otimes{}_2\langle-||+\rangle_1\otimes|-\rangle_2\;+\;{}_1\langle-|\otimes{}_2\langle+|\,|-\rangle_1\otimes|+\rangle_2\right)\end{aligned} \tag{18.3.10}$$

since only terms where the spin states are the same for the first particle and for the second particle survive. We thus have, for normalization,

$$\langle\Psi|\Psi\rangle \;=\; |\alpha|^2(1+1)=2|\alpha|^2=1 \quad\Rightarrow\quad \alpha=\frac{1}{\sqrt{2}}. \tag{18.3.11}$$

The normalized state with zero total spin angular momentum is then

$$\boxed{|\Psi\rangle=\frac{1}{\sqrt{2}}\left(|+\rangle_1\otimes|-\rangle_2-|-\rangle_1\otimes|+\rangle_2\right).} \tag{18.3.12}$$

This is a rather interesting state, sometimes called the spin singlet state. We will see that it is an entangled state of the two particles. Finally, consistent with having zero total spin, it is a rotationally invariant state (problem 18.1).

Exercise18.7. *Show that $(S\otimes T)^\dagger=S^\dagger\otimes T^\dagger$. For this it suffices to show that $\langle(S\otimes T)u,\,v\rangle=\langle u,\,(S^\dagger\otimes T^\dagger)v\rangle$, where u, v are basis vectors of the type $e_i\otimes f_j$.*