19.8 The Runge-Lenz Vector

We studied the hydrogen atom in chapter 11. The Hamiltonian $\hat{H}$ is very simple; it contains a kinetic energy term and a Coulomb potential term $V(r) = -e^2/r$. We calculated the spectrum of this Hamiltonian and found degenerate multiplets. While the energy eigenvalues depend only on a principal quantum number n, for each n there are degenerate multiplets of angular momentum with $\ell = 0, 1, \ldots, n - 1$.

The large amount of degeneracy in this spectrum asks for an explanation. The hydrogen Hamiltonian has in fact some hidden symmetry: there is a conserved quantum Runge-Lenz vector operator. In the following we discuss the *classical* Runge-Lenz vector and its conservation. In the end-of-chapter problems, you will learn about the quantum Runge-Lenz operator. In chapter 20 this knowledge will be used to give a fully algebraic derivation of the hydrogen atom spectrum.

Since the following analysis is classical, the vectors are not operators and carry no hats. Consider the energy function for a particle of momentum **p** moving in a central potential $V(r)$:

$$E = \frac{\mathbf{p}^2}{2m} + V(r). \tag{19.8.1}$$

The force **F** on the particle is given by the negative gradient of the potential:

$$\mathbf{F} = -\nabla V = -V'(r)\frac{\mathbf{r}}{r}, \tag{19.8.2}$$

Here primes denote derivatives with respect to the argument. Newton's equation sets the rate of change of the momentum equal to the force:

$$\frac{d\mathbf{p}}{dt} = -V'(r)\frac{\mathbf{r}}{r}. \tag{19.8.3}$$

Here, $\mathbf{p} = m\dot{\mathbf{r}}$. You should confirm that for motion in a central potential the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is conserved:

$$\frac{d\mathbf{L}}{dt} = 0. \tag{19.8.4}$$

Let us now calculate the time derivative of $\mathbf{p} \times \mathbf{L}$:

$$\begin{aligned} \frac{d}{dt}(\mathbf{p} \times \mathbf{L}) &= \frac{d\mathbf{p}}{dt} \times \mathbf{L} = -\frac{V'(r)}{r}\,\mathbf{r} \times (\mathbf{r} \times \mathbf{p}) \\ &= -\frac{mV'(r)}{r}\,\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}}) = -\frac{mV'(r)}{r}\,[\mathbf{r}(\mathbf{r} \cdot \dot{\mathbf{r}}) - \dot{\mathbf{r}}\,r^2]. \end{aligned} \tag{19.8.5}$$

We now note that

$$\mathbf{r} \cdot \dot{\mathbf{r}} = \frac{1}{2}\frac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) = \frac{1}{2}\frac{d}{dt}r^2 = r\dot{r}. \tag{19.8.6}$$

Using this result, the derivative of $\mathbf{p} \times \mathbf{L}$ becomes

$$\begin{aligned} \frac{d}{dt}(\mathbf{p} \times \mathbf{L}) &= -\frac{mV'(r)}{r}\,[\mathbf{r}\,r\dot{r} - \dot{\mathbf{r}}\,r^2] = mV'(r)r^2\left[\frac{\dot{\mathbf{r}}}{r} - \frac{\mathbf{r}\,\dot{r}}{r^2}\right] \\ &= mV'(r)r^2\,\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right). \end{aligned} \tag{19.8.7}$$

Because of the factor $V'(r)r^2$, the right-hand side fails to be the time derivative of some quantity. But if we focus on potentials for which this factor is a constant, the right-hand side is a time derivative, and we get a conserved quantity. So assume that for some constant γ we have

$$V'(r)\, r^2 \;=\; \gamma. \tag{19.8.8}$$

It then follows that

$$\frac{d}{dt}(\mathbf{p}\times\mathbf{L}) \;=\; m\gamma\,\frac{d}{dt}\Big(\frac{\mathbf{r}}{r}\Big) \quad\Rightarrow\quad \frac{d}{dt}\Big(\mathbf{p}\times\mathbf{L}-m\gamma\,\frac{\mathbf{r}}{r}\Big)=0. \tag{19.8.9}$$

The complicated vector inside the parentheses is constant in time. The condition (19.8.8) on the potential implies that

$$\frac{dV}{dr} \;=\; \frac{\gamma}{r^2} \quad\Rightarrow\quad V(r)=-\frac{\gamma}{r}+c_0. \tag{19.8.10}$$

This is the most general potential for which we get a conserved vector. For $c_0 = 0$ and $\gamma = e^2$, we have the hydrogen atom potential $V(r) = -e^2/r$. For this case the conservation equation reads

$$\frac{d}{dt}\Big(\mathbf{p}\times\mathbf{L}-me^2\,\frac{\mathbf{r}}{r}\Big)=0. \tag{19.8.11}$$

Factoring a constant we obtain the unit-free conserved **Runge-Lenz** vector $\mathbf{R}$ associated with the hydrogen atom classical dynamics:

$$\boxed{\mathbf{R} \;\equiv\; \frac{1}{me^2}\,\mathbf{p}\times\mathbf{L}-\frac{\mathbf{r}}{r}, \qquad \frac{d\mathbf{R}}{dt} \;=\; 0.} \tag{19.8.12}$$

The conservation of the Runge-Lenz vector is a property of inverse-squared central forces. The second term in $\mathbf{R}$ is the inward-directed unit radial vector.

To familiarize ourselves with the Runge-Lenz vector, we first examine its value for a circular orbit, as shown in figure 19.8. With counterclockwise motion, the vector $\mathbf{L}$ points out of the page, and $\mathbf{p} \times \mathbf{L}$ points radially outward. The vector $\mathbf{R}$ is thus a competition between the outward-pointing first term along $\mathbf{p} \times \mathbf{L}$ and the inward-pointing second term along $-\hat{\mathbf{r}}$. If these two terms did not cancel, the result would be a radial vector, outward or inward, but in any case not conserved as it rotates with the particle. This cannot happen, therefore the two terms must cancel. Indeed, for a circular orbit

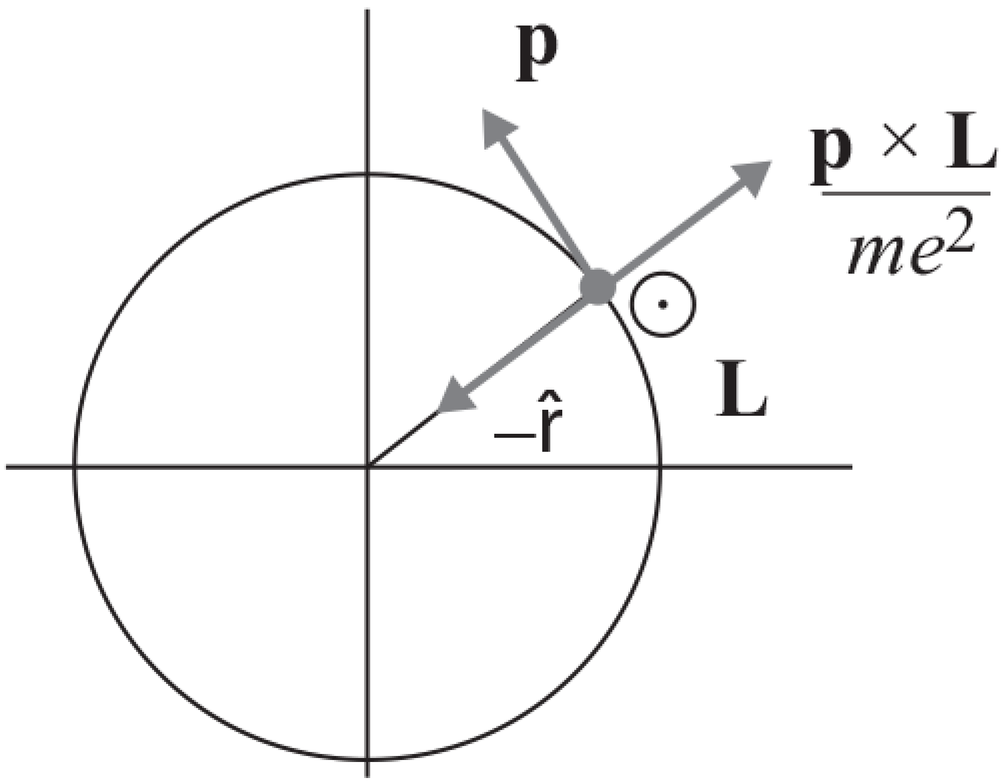

Figure 19.8
The Runge-Lenz vector vanishes for a circular orbit.

$$m\frac{v^2}{r} = \frac{e^2}{r^2} \quad \Rightarrow \quad \frac{mv^2 r}{e^2} = 1 \quad \Rightarrow \quad \frac{(mv)(mvr)}{me^2} = 1 \quad \Rightarrow \quad \frac{pL}{me^2} = 1, \tag{19.8.13}$$

which states that in a circular orbit the first term in **R** is a unit vector. Since it points outward, it cancels with the second term, and the Runge-Lenz vector vanishes for a circular orbit.

We now argue that for an elliptic orbit the Runge-Lenz vector is not zero. Consider figure 19.9, showing a particle in counterclockwise motion around an elliptic orbit. One focus of the ellipse is at the origin $\mathbf{r} = 0$. At all times the conserved **L** points off the page. At the aphelion A, the point farthest away from the focal center, the first term in **R** points outward, and the second term point inward. Thus, if **R** does not vanish it must be a vector along the axis joining the focus and the aphelion, a horizontal vector on the figure. Now consider point B, shown directly above the focus of the orbit. Here, $\mathbf{p}$ is no longer perpendicular to the radial vector, and therefore $\mathbf{p} \times \mathbf{L}$ is no longer radial. As you can see, it points slightly to the left of the vertical. Since $\mathbf{R}$ is conserved, and we know it is horizontal, its pointing to the left allows us to conclude that in an elliptic orbit $\mathbf{R}$ is a nonzero vector pointing *from* the focus *to* the perihelion C, the point of closest approach in the orbit.

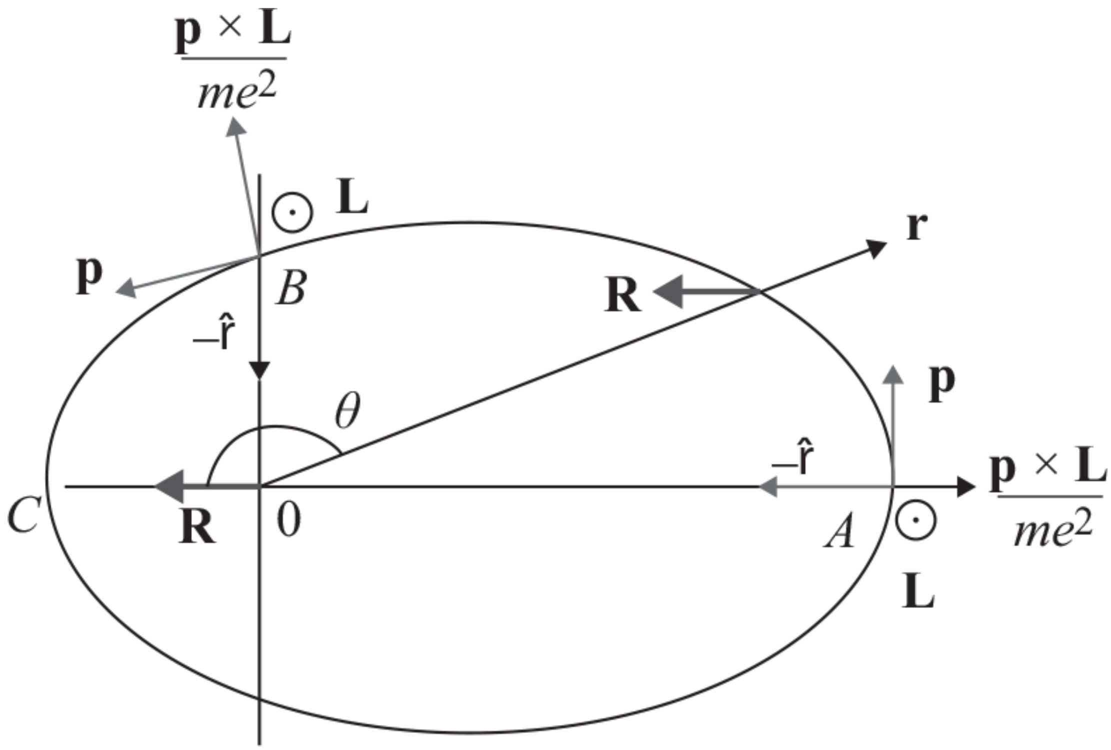

Figure 19.9
In an elliptic orbit, the Runge-Lenz vector is a vector along the major axis of the ellipse and points from the focus to the perihelion C.

Since $\mathbf{R}$ vanishes for circular orbits, the length R of $\mathbf{R}$ must measure the deviation of the orbit from circular. In fact, the magnitude R of the Runge-Lenz vector is precisely the eccentricity of the orbit. To see this we form the dot product of $\mathbf{R}$ with the radial vector $\mathbf{r}$:

$$\mathbf{r} \cdot \mathbf{R} = \frac{1}{me^2}\, \mathbf{r} \cdot (\mathbf{p} \times \mathrm{L}) - r. \tag{19.8.14}$$

Referring to figure 19.9, let θ be the angle measured at the origin, increasing clockwise and defined with $\theta = 0$ the direction to the perihelion.

The angle between $\mathbf{r}$ and $\mathbf{R}$ is then θ and we get

$$rR\cos\theta = \frac{1}{me^2}\mathbf{L}\cdot(\mathbf{r}\times\mathbf{p}) - r = \frac{1}{me^2}L^2 - r. \tag{19.8.15}$$

Collecting terms proportional to r,

$$r(1+R\cos\theta) = \frac{L^2}{me^2} \quad \Rightarrow \quad \boxed{\frac{1}{r} = \frac{me^2}{L^2}(1+R\cos\theta).} \tag{19.8.16}$$

This is one of the standard presentations of an elliptic orbit, and R appears at the place one conventionally has the eccentricity e, thus $e = R$. If $R = 0$, the orbit is circular because r does not depend on θ. The identification of R with e follows from the definition

$$e \equiv \frac{r_{\max} - r_{\min}}{r_{\max} + r_{\min}}. \tag{19.8.17}$$

Here $r_{\min}$ and $r_{\max}$ are, respectively, the minimum and maximum distances to the focus located at the center of force.

Exercise 19.13. *Use equations (19.8.16) and (19.8.17) to confirm that, indeed, $e = R$.*

This analysis thus far has been classical. Quantum mechanically, some things must be changed; happily, not that much! The definition of $\mathbf{R}$ only has to be changed to guarantee that $\hat{R}$ is a Hermitian operator. Hermitization gives

$$\hat{\mathbf{R}} \equiv \frac{1}{2me^2}(\hat{\mathbf{p}}\times\hat{\mathbf{L}} - \hat{\mathbf{L}}\times\hat{\mathbf{p}}) - \frac{\hat{\mathbf{r}}}{r}. \tag{19.8.18}$$

Here $\hat{\mathbf{r}}$ is the position vector operator (not a unit vector).

Exercise 19.14. *Confirm that $\hat{\mathbf{R}}$ defined above is Hermitian and reduces to $\boldsymbol{R}$ when vector operators become classical vectors.*

The quantum mechanical conservation of $\hat{\mathbf{R}}$ is the statement that it commutes with the hydrogen Hamiltonian:

$$[\hat{\mathbf{R}}, \hat{H}] = 0. \tag{19.8.19}$$

The required calculation (problem 19.10) is the quantum analogue of the above classical calculation that showed that the time derivative of **R** is zero. Moreover, the length squared of the operator $\hat{\mathbf{R}}$ is also of interest. The result (problem 19.11) is

$$\hat{\mathbf{R}}^2 = 1 + \frac{2}{me^4}\hat{H}(\hat{\mathbf{L}}^2 + \hbar^2). \tag{19.8.20}$$

These facts above will be used in section 20.8 to show that the symmetries generated by $\hat{\mathbf{R}}$ and the angular momentum $\hat{\mathbf{L}}$ determine completely the spectrum of the hydrogen atom.