8.05x: Quantum Physics II, Exam 1 Formula Sheet

- Gaussian integrals

$$\int_{-\infty}^{\infty} dx e^{-x^2} = \sqrt{\pi}, \qquad \int_{-\infty}^{\infty} dx x^2 e^{-x^2} = \frac{\sqrt{\pi}}{2}, \quad \int_{-\infty}^{\infty} dx x^4 e^{-x^2} = \frac{3\sqrt{\pi}}{4}$$

- Trigonometric functions

$$\begin{array}{ll} \sin x = \left(e^{ix} - e^{-ix}\right)/2i, & \cos x = \left(e^{ix} + e^{-ix}\right)/2 \\ \sinh x = \left(e^{x} - e^{-x}\right)/2, & \cosh x = \left(e^{x} + e^{-x}\right)/2 \end{array}$$

- Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\Psi(x,t) = \hat{H}\Psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right]\Psi(x,t)$$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x)$$

- Conservation of probability

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}J(x,t) = 0$$

$$\rho(x,t) = |\psi(x,t)|^2; \quad J(x,t) = \frac{\hbar}{2im}\left[\psi^* \frac{\partial}{\partial x}\psi - \psi \frac{\partial}{\partial x}\psi^*\right]$$

- Variational principle:

$$E_{gs} \leq \frac{\int dx \psi^*(x) H \psi(x)}{\int dx \psi^*(x)\psi(x)} \equiv \langle H \rangle_\psi \quad \text{for all } \psi(x)$$

- Spin-1/2 particle:

$$\text{Stern-Gerlach}: \quad H = -\boldsymbol{\mu}\cdot\mathbf{B}, \quad \boldsymbol{\mu} = g\frac{e\hbar}{2m}\frac{1}{\hbar}\mathbf{S} = \gamma\mathbf{S}$$
$$\mu_B = \frac{e\hbar}{2m_e}, \quad \boldsymbol{\mu}_e = -2\mu_B\frac{\mathbf{S}}{\hbar},$$

In the basis $|1\rangle \equiv |z;+\rangle = |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |2\rangle \equiv |z;-\rangle = |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$S_i = \frac{\hbar}{2}\sigma_i \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \quad \rightarrow \quad [S_i, S_j] = i\hbar\epsilon_{ijk}S_k \quad (\epsilon_{123} = +1)$$
$$\sigma_i\sigma_j = \delta_{ij}\mathbf{1} + i\epsilon_{ijk}\sigma_k \quad \rightarrow \quad (\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{b}) = \mathbf{a}\cdot\mathbf{b}\mathbf{1} + i\boldsymbol{\sigma}\cdot(\mathbf{a}\times\mathbf{b})$$
$$e^{i\mathbf{M}\theta} = \mathbf{1}\cos\theta + i\mathbf{M}\sin\theta, \quad \text{if } \mathbf{M}^2 = \mathbf{1}$$
$$\exp(i\mathbf{a}\cdot\boldsymbol{\sigma}) = \mathbf{1}\cos a + i\boldsymbol{\sigma}\cdot\left(\frac{\mathbf{a}}{a}\right)\sin a, \quad a = |\mathbf{a}|$$
$$\exp(i\theta\sigma_3)\,\sigma_1\exp(-i\theta\sigma_3) = \sigma_1\cos(2\theta) - \sigma_2\sin(2\theta)$$
$$\exp(i\theta\sigma_3)\,\sigma_2\exp(-i\theta\sigma_3) = \sigma_2\cos(2\theta) + \sigma_1\sin(2\theta).$$
$$S_{\mathbf{n}} = \mathbf{n}\cdot\mathbf{S} = n_xS_x + n_yS_y + n_zS_z = \frac{\hbar}{2}\mathbf{n}\cdot\boldsymbol{\sigma}.$$
$$(n_x, n_y, n_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \quad S_{\mathbf{n}}|\mathbf{n};\pm\rangle = \pm\frac{\hbar}{2}|\mathbf{n};\pm\rangle$$
$$|\mathbf{n};+\rangle = \cos\left(\frac{1}{2}\theta\right)|+\rangle + \sin\left(\frac{1}{2}\theta\right)\exp(i\phi)|-\rangle$$
$$|\mathbf{n};-\rangle = -\sin\left(\frac{1}{2}\theta\right)\exp(-i\phi)|+\rangle + \cos\left(\frac{1}{2}\theta\right)|-\rangle$$
$$|\langle\mathbf{n}';+\mid\mathbf{n};+\rangle| = \cos\left(\frac{1}{2}\gamma\right), \quad \gamma \text{ is the angle between } \mathbf{n} \text{ and } \mathbf{n}'$$
$$\langle\mathbf{S}\rangle_{\mathbf{n}} = \frac{\hbar}{2}\mathbf{n}, \quad \text{Rotation operator: } R_\alpha(\mathbf{n}) \equiv \exp\left(-\frac{i\alpha S_{\mathbf{n}}}{\hbar}\right)$$

- Linear algebra

$$\text{Matrix representation of } T \text{ in the basis } (v_1, \ldots, v_n): Tv_j = \sum_i T_{ij}v_i$$
$$\text{basis change: } u_k = \sum_j A_{jk}v_j, \quad T(\{u\}) = A^{-1}T(\{v\})A$$
$$\text{Schwarz: } |\langle u, v\rangle| \leq |u||v|$$
$$\text{Adjoint: } \langle u, Tv\rangle = \langle T^\dagger u, v\rangle, \quad (T^\dagger)^\dagger = T$$

- Bras and kets: For an operator Ω and a vector v, we write $|\Omega v\rangle \equiv \Omega|v\rangle$

$$\text{Adjoint: } \left\langle u \mid \Omega^{\dagger} v\right\rangle = \langle\Omega u \mid v\rangle$$

$$|\alpha_1 v_1 + \alpha_2 v_2\rangle = \alpha_1 |v_1\rangle + \alpha_2 |v_2\rangle \longleftrightarrow \langle\alpha_1 v_1 + \alpha_2 v_2| = \alpha_1^* \langle v_1| + \alpha_2^* \langle v_2|$$

- Complete orthonormal basis $|i\rangle$

$$\langle i \mid j\rangle = \delta_{ij}, \quad \mathbf{1} = \sum_i |i\rangle\langle i|$$

$$\Omega_{ij} = \langle i|\Omega|j\rangle \leftrightarrow \Omega = \sum_{i,j} \Omega_{ij} |i\rangle\langle j|$$

$$\left\langle i \left|\Omega^{\dagger}\right| j\right\rangle = \langle j|\Omega|i\rangle^*$$

Ω hermitian: $\Omega^{\dagger} = \Omega$, $\quad U$ unitary: $U^{\dagger} = U^{-1}$

- Matrix M is normal $\left(\left[M, M^{\dagger}\right] = 0\right) \longleftrightarrow$ unitarily diagonalizable.

- Position and momentum representations: $\psi(x) = \langle x \mid \psi\rangle$; $\quad \tilde{\psi}(p) = \langle p \mid \psi\rangle$;

$$\hat{x}|x\rangle = x|x\rangle, \quad \langle x \mid y\rangle = \delta(x-y), \quad \mathbf{1} = \int dx|x\rangle\langle x|, \quad \hat{x}^{\dagger} = \hat{x}$$

$$\hat{p}|p\rangle = p|p\rangle, \quad \langle q \mid p\rangle = \delta(q-p), \quad \mathbf{1} = \int dp|p\rangle\langle p|, \quad \hat{p}^{\dagger} = \hat{p}$$

$$\langle x \mid p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ipx}{\hbar}\right); \quad \tilde{\psi}(p) = \int dx \langle p \mid x\rangle\langle x \mid \psi\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dx \exp\left(-\frac{ipx}{\hbar}\right)\psi(x)$$

$$\langle x \left|\hat{p}^n\right| \psi\rangle = \left(\frac{\hbar}{i}\frac{d}{dx}\right)^n \psi(x); \quad \langle p \left|\hat{x}^n\right| \psi\rangle = \left(i\hbar\frac{d}{dp}\right)^n \tilde{\psi}(p); \quad [\hat{p}, f(\hat{x})] = \frac{\hbar}{i} f'(\hat{x})$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(ikx) dx = \delta(k)$$

- Generalized uncertainty principle

$$\Delta A \equiv |(A - \langle A\rangle\mathbf{1})\Psi| \rightarrow \quad (\Delta A)^2 = \left\langle A^2\right\rangle - \langle A\rangle^2 \geq 0.$$

$$\Delta A \Delta B \geq \left|\left\langle\Psi\left|\frac{1}{2i}[A, B]\right|\Psi\right\rangle\right|$$

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

$$\Delta x = \frac{\Delta}{\sqrt{2}} \text{ and } \Delta p = \frac{\hbar}{\sqrt{2}\Delta} \text{ for } \psi \sim \exp\left(-\frac{1}{2}\frac{x^2}{\Delta^2}\right)$$

$$\int_{-\infty}^{+\infty} dx \exp\left(-ax^2\right) = \sqrt{\frac{\pi}{a}}$$

Time independent operator Q : $\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle [H, Q]\rangle$

$$\Delta H \Delta t \geq \frac{\hbar}{2}, \quad \Delta t \equiv \frac{\Delta Q}{\left|\frac{d\langle Q\rangle}{dt}\right|}$$

- Commutator identities

$$\begin{aligned}
[A, BC] &= [A, B]C + B[A, C], \\
[AB, C] &= A[B, C] + [A, C]B, \\
e^{A}Be^{-A} &= e^{\mathrm{ad}_A}\, B = B + [A, B] + \frac{1}{2}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \ldots, \\
e^{A}Be^{-A} &= B + [A, B], \quad \text{if} \quad [A, [A, B]] = 0, \\
\left[B, e^{A}\right] &= [B, A]e^{A}, \quad \text{if} \quad [A, [A, B]] = 0 \\
e^{A+B} &= e^{A}e^{B}e^{-\frac{1}{2}[A,B]} = e^{B}e^{A}e^{\frac{1}{2}[A,B]}, \quad \text{if } [A, B] \text{ commutes with } A \text{ and with } B
\end{aligned}$$

- Gram-Schmidt procedure

Given a basis $\{v_1, \ldots, v_n\}$, an orthonormal basis is given by $\{e_1, \ldots, e_n\}$, where $\tilde{e}_i = v_i - \sum_{j<i} \langle v_i, e_j\rangle\, e_j$ and $e_i = \tilde{e}_i / |\tilde{e}_i|$.

- Infinite square well:

$$V = \begin{cases} 0 & \text{for } 0 < x < L \\ \infty & \text{otherwise} \end{cases}$$

Eigenfunctions and eigenenergies

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \quad E_n = \frac{\hbar^2\pi^2 n^2}{2mL^2}$$

- Harmonic Oscillator

$$\hat{H}=\frac{1}{2m}\hat{p}^2+\frac{1}{2}m\omega^2\hat{x}^2=\hbar\omega\left(\hat{N}+\frac{1}{2}\right),\quad \hat{N}=\hat{a}^\dagger\hat{a}$$
$$\hat{a}=\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}+\frac{i\hat{p}}{m\omega}\right),\quad \hat{a}^\dagger=\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}-\frac{i\hat{p}}{m\omega}\right),$$
$$\hat{x}=\sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}+\hat{a}^\dagger\right),\quad \hat{p}=i\sqrt{\frac{m\omega\hbar}{2}}\left(\hat{a}^\dagger-\hat{a}\right),$$
$$[\hat{x},\hat{p}]=i\hbar,\quad \left[\hat{a},\hat{a}^\dagger\right]=1,\quad [\hat{N},\hat{a}]=-\hat{a},\quad \left[\hat{N},\hat{a}^\dagger\right]=\hat{a}^\dagger.$$
$$\hat{H}|n\rangle=E_n|n\rangle=\hbar\omega\left(n+\frac{1}{2}\right)|n\rangle,\quad \hat{N}|n\rangle=n|n\rangle,\langle m\mid n\rangle=\delta_{mn}$$
$$\hat{a}^\dagger|n\rangle=\sqrt{n!}\left(a^\dagger\right)^n|0\rangle$$
$$\psi_0(x)=\langle x\mid 0\rangle=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right).$$

$$x_H(t)=\hat{x}\cos\omega t+\frac{\hat{p}}{m\omega}\sin\omega t$$
$$p_H(t)=\hat{p}\cos\omega t-m\omega\hat{x}\sin\omega t$$

- Coherent states

$$T_{x_0}\equiv e^{-\frac{i}{\hbar}\hat{p}x_0},\quad T_{x_0}|x\rangle=|x+x_0\rangle$$
$$|\tilde{x}_0\rangle\equiv T_{x_0}|0\rangle=e^{-\frac{i}{\hbar}\hat{p}x_0}|0\rangle,$$
$$|\tilde{x}_0\rangle=e^{-\frac{1}{4}\frac{x_0^2}{d^2}}e^{\frac{x_0}{\sqrt{2}d}a^\dagger}|0\rangle,\quad \langle x\mid\tilde{x}_0\rangle=\psi_0\left(x-x_0\right),\quad d^2=\frac{\hbar}{m\omega}$$
$$|\bar{\alpha}\rangle\equiv D(\alpha)|0\rangle=e^{\alpha a^\dagger-\alpha^* a}|0\rangle,\quad D(\alpha)\equiv\exp\left(\alpha a^\dagger-\alpha^* a\right),\quad \alpha=\frac{\langle\hat{x}\rangle}{\sqrt{2}d}+i\frac{\langle\hat{p}\rangle d}{\sqrt{2}\hbar}\in\mathbb{C}$$
$$|\bar{\alpha}\rangle=e^{\alpha a^\dagger-\alpha^* a}|0\rangle=e^{-\frac{1}{2}|\alpha|^2}e^{\alpha a^\dagger}|0\rangle,\quad \hat{a}|\bar{\alpha}\rangle=\alpha|\bar{\alpha}\rangle,\quad |\bar{\alpha},t\rangle=e^{-i\omega t/2}\left|\overline{e^{-i\omega t}\alpha}\right\rangle$$
$$\langle\bar{\alpha}\mid\bar{\beta}\rangle=\exp\left(-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2+\alpha^*\beta\right)$$
$$|\bar{\alpha}\rangle=e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$
$$1=\int\frac{d^2\alpha}{\pi}|\bar{\alpha}\rangle\langle\bar{\alpha}|$$

- Squeezed states

$$
\begin{gathered}
|0_\gamma\rangle = S(\gamma)|0\rangle, \quad S(\gamma) = \exp\left(-\frac{\gamma}{2}\left(a^\dagger a^\dagger - aa\right)\right), \quad \gamma \in \mathbb{R} \\
|0_\gamma\rangle = \frac{1}{\sqrt{\cosh\gamma}} \exp\left(-\frac{1}{2}\tanh\gamma a^\dagger a^\dagger\right)|0\rangle \\
S^\dagger(\gamma) a S(\gamma) = \cosh\gamma a - \sinh\gamma a^\dagger, \quad D^\dagger(\alpha) a D(\alpha) = a + \alpha \\
|\alpha, \gamma\rangle \equiv D(\alpha) S(\gamma)|0\rangle
\end{gathered}
$$

- Time evolution

$$
\begin{gathered}
|\Psi, t\rangle = U(t, 0)|\Psi, 0\rangle, \quad U \text{ unitary} \\
U(t, t) = 1, \quad U(t_2, t_1) U(t_1, t_0) = U(t_2, t_0), \quad U(t_1, t_2) = U^\dagger(t_2, t_1) \\
i\hbar \frac{d}{dt}|\Psi, t\rangle = \hat{H}(t)|\Psi, t\rangle \quad \leftrightarrow \quad i\hbar \frac{d}{dt} U(t, t_0) = \hat{H}(t) U(t, t_0)
\end{gathered}
$$

Time independent $\hat{H}$: $\quad U(t, t_0) = \exp\left[-\frac{i}{\hbar}\hat{H}(t - t_0)\right] = \sum_n e^{-\frac{i}{\hbar}E_n(t-t_0)}|n\rangle\langle n|$

$$
\begin{gathered}
\langle A\rangle = \langle\Psi, t\,|A_S|\,\Psi, t\rangle = \langle\Psi, 0\,|A_H(t)|\,\Psi, 0\rangle \quad \rightarrow \quad A_H(t) = U^\dagger(t, 0) A_S U(t, 0) \\
[A_S, B_S] = C_S \rightarrow [A_H(t), B_H(t)] = C_H(t) \\
i\hbar \frac{d}{dt}\hat{A}_H(t) = \left[\hat{A}_H(t), \hat{H}_H(t)\right], \text{ for } A_S \text{ time-independent}
\end{gathered}
$$

- Two state systems

$$
H = h_0 \mathbf{1} + \mathbf{h} \cdot \boldsymbol{\sigma} = h_0 \mathbf{1} + h \mathbf{n} \cdot \boldsymbol{\sigma}, \quad h = |\mathbf{h}|
$$

Eigenstates: $|\mathbf{n}; \pm\rangle, \quad E_\pm = h_0 \pm h$.
$H = -\gamma \mathbf{S} \cdot \mathbf{B} \rightarrow$ spin vector $\vec{n}$ precesses with Larmor frequency $\boldsymbol{\omega} = -\gamma \mathbf{B}$