18.7 Quantum Teleportation

Teleportation is impossible in classical physics: there is no basis for dematerializing an object and recreating it somewhere else. In 1993, Bennet, Brassand, Crépeau, Jozsa, Peres, and Wooters discovered that, surprisingly, it *is* possible to teleport a quantum state.

Imagine that Alice is handed a spin one-half particle in some quantum state $|\Psi\rangle$. She does not know what the state is, but of course, it can be written as

$$|\Psi\rangle = \alpha|+\rangle + \beta|-\rangle, \tag{18.7.1}$$

where $\alpha, \beta \in \mathbb{C}$ are constants. We will call particle C the particle imprinted with this state and will write the state of the particle as $|\Psi\rangle_C$:

$$|\Psi\rangle_C = \alpha|+\rangle_C + \beta|-\rangle_C. \tag{18.7.2}$$

Alice's goal is to teleport the state of the particle—called a *quantum bit*, or *qubit* in the language of quantum computation—to Bob, who is far away. Particle C itself does not change position. If Alice and Bob share an entangled pair of spin one-half particles, teleportation will imprint the quantum state $|\Psi\rangle$ on the spin one-half particle available to Bob.

Teleporting is a practical solution to the problem of making the state of particle C available to Bob quickly and efficiently. Alternatively, Alice could perhaps isolate particle C in a safe box and send the box to Bob through the mail. One thing she can't do is clone the state of C and send the copy to Bob. The quantum *no-cloning* principle, to be discussed in section 18.9, prevents Alice from creating a copy of a state that is unknown to her. Indeed, when Alice is handed particle C, she has no way of finding out what α and β are. Measuring the state with some Stern-Gerlach apparatus will not help; the spin will just point up or point down. What has she learned? Almost nothing. Only with many copies of the state would she be able to learn about the values of α and β. Having just one particle, she is unable to measure α and β and send those values to Bob.

A diagram showing the teleportation setup is shown in figure 18.1. The key tool Alice and Bob use is an entangled state of two particles A, B in which Alice has access to particle A, and Bob has access to particle B. One pair AB of entangled particles will allow Alice to teleport the state of particle C. The state of C will be imprinted on particle B. Teleporting quantum states is by now routinely done.

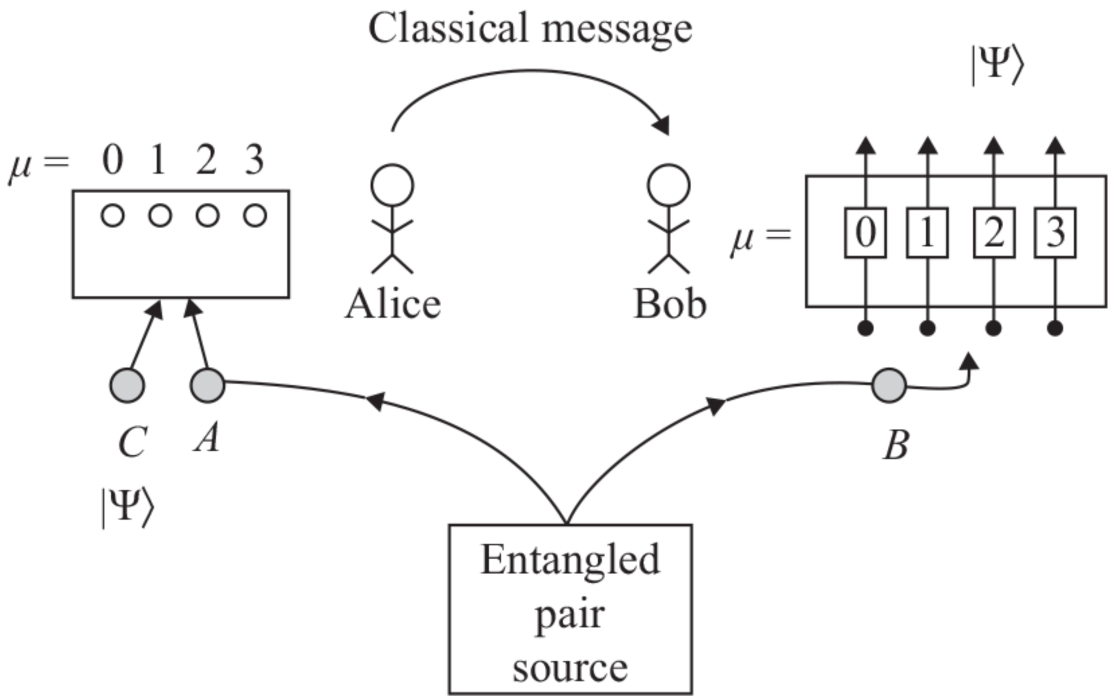

Figure 18.1
Alice has particle C in the state $|\Psi\rangle$, to be teleported, as well as particle A. Particle A is entangled with particle B, which is in Bob's possession. Alice performs a Bell measurement on particles A and C. After she measures, Bob's particle will carry the state $|\Psi\rangle$, up to a simple unitary transformation.

Alice has a console with four lights labeled $\mu = 0, 1, 2, 3$. She will do a Bell measurement on AC, the pair containing particle A of the shared, entangled pair and particle C, whose state is to be teleported. When she does, one of her four lights will blink: if it is the μth light, it is because she ended up with the Bell state $|\Phi_\mu\rangle_{AC}$. Bob, who is in possession of particle B, has a console with four boxes that generate unitary transformations. The first box, labeled $\mu = 0$, does nothing to the state. The ith box (with $i = 1, 2, 3$) applies the operator σ_i. Alice communicates to Bob that the μth light blinked. Then Bob submits particle B to the μth box and out comes, we claim, the state $|\Psi\rangle$ imprinted on particle B.

To prove this, let the entangled, shared pair be the first Bell basis state:

$$|\Phi_0\rangle_{AB} = \frac{1}{\sqrt{2}}\big(|+\rangle_A|+\rangle_B + |-\rangle_A|-\rangle_B\big). \tag{18.7.3}$$

The total state of our three particles A, B, C is therefore

$$
\begin{aligned}
|\Phi_0\rangle_{AB} \otimes |\Psi\rangle_C &= |\Phi_0\rangle_{AB} \otimes (\alpha|+\rangle_C + \beta|-\rangle_C) \\
&= \tfrac{1}{\sqrt{2}}\big(|+\rangle_A|+\rangle_B + |-\rangle_A|-\rangle_B\big) \otimes \big(\alpha|+\rangle_C + \beta|-\rangle_C\big).
\end{aligned} \tag{18.7.4}
$$

Expanding out and reordering the states to have A followed by C and then by B, we have

$$
\begin{aligned}
|\Phi_0\rangle_{AB} \otimes |\Psi\rangle_C = \tfrac{1}{\sqrt{2}}\Big(&\alpha \underbrace{|+\rangle_A|+\rangle_C}\,|+\rangle_B + \beta \underbrace{|+\rangle_A|-\rangle_C}\,|+\rangle_B \\
&+\alpha \underbrace{|-\rangle_A|+\rangle_C}\,|-\rangle_B + \beta \underbrace{|-\rangle_A|-\rangle_C}\,|-\rangle_B\Big).
\end{aligned} \tag{18.7.5}
$$

Note that as long as we label the states, the order in which we write them does not matter. We now write these basis states with braces in the Bell basis using (18.6.9). We find that

$$
\begin{aligned}
|\Phi_0\rangle_{AB} \otimes |\Psi\rangle_C = &\tfrac{1}{2}\Big(|\Phi_0\rangle_{AC} + |\Phi_3\rangle_{AC}\Big)\,\alpha|+\rangle_B + \tfrac{1}{2}\Big(|\Phi_1\rangle_{AC} - i|\Phi_2\rangle_{AC}\Big)\,\beta|+\rangle_B \\
&+ \tfrac{1}{2}\Big(|\Phi_1\rangle_{AC} + i|\Phi_2\rangle_{AC}\Big)\,\alpha|-\rangle_B + \tfrac{1}{2}\Big(|\Phi_0\rangle_{AC} - |\Phi_3\rangle_{AC}\Big)\,\beta|-\rangle_B.
\end{aligned} \tag{18.7.6}
$$

Collecting the Bell states,

$$
\begin{aligned}
|\Phi_0\rangle_{AB} \otimes |\Psi\rangle_C = &\tfrac{1}{2}|\Phi_0\rangle_{AC}\,(\alpha|+\rangle_B + \beta|-\rangle_B) + \tfrac{1}{2}|\Phi_1\rangle_{AC}\,(\alpha|-\rangle_B + \beta|+\rangle_B) \\
&+ \tfrac{1}{2}|\Phi_2\rangle_{AC}\,(i\alpha|-\rangle_B - i\beta|+\rangle_B) + \tfrac{1}{2}|\Phi_3\rangle_{AC}\,(\alpha|+\rangle_B - \beta|-\rangle_B).
\end{aligned} \tag{18.7.7}
$$

We can then see that in this basis, variants of the state $|\Psi\rangle$ are imprinted on particle B:

$$
\begin{aligned}
|\Phi_0\rangle_{AB} \otimes |\Psi\rangle_C = &\tfrac{1}{2}|\Phi_0\rangle_{AC} \otimes |\Psi\rangle_B + \tfrac{1}{2}|\Phi_1\rangle_{AC} \otimes \sigma_1|\Psi\rangle_B \\
&+ \tfrac{1}{2}|\Phi_2\rangle_{AC} \otimes \sigma_2|\Psi\rangle_B + \tfrac{1}{2}|\Phi_3\rangle_{AC} \otimes \sigma_3|\Psi\rangle_B.
\end{aligned} \tag{18.7.8}
$$

Note that all we have done so far is to rewrite the state of the three particles in a convenient form. The above right-hand side allows us to understand what happens when Alice measures the state of AC in the Bell basis. If she finds

- $|\Phi_0\rangle_{AC}$, the B state becomes $|\Psi\rangle_B$,
- $|\Phi_1\rangle_{AC}$, the B state becomes $\sigma_1|\Psi\rangle_B$,
- $|\Phi_2\rangle_{AC}$, the B state becomes $\sigma_2|\Psi\rangle_B$,
- $|\Phi_3\rangle_{AC}$, the B state becomes $\sigma_3|\Psi\rangle_B$.

If Alice gets $|\Phi_0\rangle_{AC}$, then Bob is in possession of the teleported state and has to do nothing. If Alice gets $|\Phi_i\rangle_{AC}$, Bob's particle is in the state $\sigma_i|\Psi\rangle_B$. Bob applies the ith box, which multiplies his state by σ_i, giving him the desired state $|\Psi\rangle_B$. The teleporting is thus complete.

Note that Alice is left with one of the Bell states $|\Phi_\mu\rangle_{AC}$, which has no information whatsoever about the constants α and β that defined the state to be teleported. Thus, the process did not create a copy of the state. The original state was destroyed in the process of teleportation. This is consistent with the no-cloning principle (section 18.9).

All the computational work above led to the key result (18.7.8), which is neatly summarized as the following identity valid for arbitrary states $|\Psi\rangle$:

$$|\Phi_0\rangle_{AB} \otimes |\Psi\rangle_C = \frac{1}{2}\sum_{\mu=0}^{3} |\Phi_\mu\rangle_{AC} \otimes \sigma_\mu |\Psi\rangle_B. \tag{18.7.9}$$

This is an identity for a state of three particles. It expresses the tensor product of an entangled state of the first two particles times a third as a sum of products that involve entangled states of the first and third particle times a state of the second particle.