20.5 Computation of $\mathbf{1}\otimes\frac{1}{2}$

We already worked out how the tensor product of two $\frac{1}{2}$ multiplets breaks into irreducible representations of the total angular momentum. We found that the result is encapsulated by the relation $\frac{1}{2}\otimes\frac{1}{2}=\mathbf{1}\oplus\mathbf{0}$.

This example is perhaps too simple to illustrate the general ways in which we approach the problem of adding two angular momenta. In this section we will consider two multiplets: a multiplet $j_1 = 1$ of an angular momentum $\hat{\mathbf{J}}_1$ and a multiplet $j_2=\frac{1}{2}$ of an angular momentum $\hat{\mathbf{J}}_2$. We are going to tensor these two vector spaces to form $\mathbf{1}\otimes\frac{1}{2}$ and find out how the tensor product decomposes in multiplets of the total angular momentum $\hat{\mathbf{J}}\equiv\hat{\mathbf{J}}_1+\hat{\mathbf{J}}_2$. This calculation is instructive and, moreover, will be useful in our analysis of spin-orbit coupling in the following section.

Since the $j_1 = 1$ multiplet is three-dimensional and the $j_2=\frac{1}{2}$ multiplet is two-dimensional, the tensor product has six basis states:

$$|j_1,m_1\rangle\otimes|j_2,m_2\rangle,\quad j_1=1,\quad j_2=\tfrac{1}{2}, \tag{20.5.1}$$

which exist for all combinations of $m_1 = -1,\ 0,\ 1$, and $m_2=\pm\frac{1}{2}$. We have chosen to write the j_1 multiplet states to the left of the j_2 multiplet states. The states above form what is called the *uncoupled basis* because the products are eigenstates of both $(\hat{\mathbf{J}}_1^2,\hat{J}_{1,z})$ and $(\hat{\mathbf{J}}_2^2,\hat{J}_{2,z})$, the uncoupled angular momenta. Our goal in this section is to determine the *coupled* states, the eigenstates of the total angular momentum operators $\hat{\mathbf{J}}^2$ and $\hat{J}_z$.

We note that even the uncoupled states are eigenstates of $\hat{J}_z=\hat{J}_{1,z}+\hat{J}_{2,z}$:

We note that even the uncoupled states are eigenstates of $\hat{J}_z=\hat{J}_{1,z}+\hat{J}_{2,z}$:

$$\begin{aligned}\hat{J}_z|j_1,m_1\rangle\otimes|j_2,m_2\rangle &= \hat{J}_{1,z}|j_1,m_1\rangle\otimes|j_2,m_2\rangle+|j_1,m_1\rangle\otimes\hat{J}_{2,z}|j_2,m_2\rangle\\ &= \hbar m_1\,|j_1,m_1\rangle\otimes|j_2,m_2\rangle+\hbar m_2|j_1,m_1\rangle\otimes|j_2,m_2\rangle\\ &=\hbar(m_1+m_2)|j_1,m_1\rangle\otimes|j_2,m_2\rangle.\end{aligned} \tag{20.5.2}$$

Thus, the $\hat{J}_z$ eigenvalue of an uncoupled state is obtained by adding the values of m_1 and m_2 and multiplying by $\hbar$. Our first step in the analysis is to organize the six states by the $\hat{J}_z$ eigenvalue. We quickly find the following table:

$$\begin{aligned}
\tfrac{1}{\hbar}J_z &= \tfrac{3}{2}: \qquad |1,1\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle,\\
\tfrac{1}{\hbar}J_z &= \tfrac{1}{2}: \qquad |1,0\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle \quad |1,1\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle,\\
\tfrac{1}{\hbar}J_z &= -\tfrac{1}{2}: \qquad |1,0\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle \quad |1,-1\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle,\\
\tfrac{1}{\hbar}J_z &= -\tfrac{3}{2}: \qquad |1,-1\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle.
\end{aligned} \tag{20.5.3}$$

We want to discover the $\hat{\mathbf{J}}^2$ eigenstates. A couple of states are quickly recognized. The top state on this list has $J_z = \frac{3}{2}\hbar$, and since there are no states with higher J_z, it must be the top state of a $j = \frac{3}{2}$ multiplet:

$$|j=\tfrac{3}{2}, m=\tfrac{3}{2}\rangle = |1,1\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle. \tag{20.5.4}$$

Notice that $\hat{J}_+ = \hat{J}_{1,+} + \hat{J}_{2,+}$ kills the state, as it should: $\hat{J}_{1,+}$ kills the first factor, and $\hat{J}_{2,+}$ kills the second factor. The bottom state of this $j = \frac{3}{2}$ multiplet is also recognized; it is the bottom state in the table above:

$$|j=\tfrac{3}{2}, m=-\tfrac{3}{2}\rangle = |1,-1\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle. \tag{20.5.5}$$

Since we must get complete multiplets, we must get the four states in $j = \frac{3}{2}$. The state with $J_z = \frac{1}{2}\hbar$ must arise from a linear combination of states on the second line of the table, and the state with $J_z = -\frac{1}{2}\hbar$ must arise from a linear combination of states on the third line of the table. We are then left with two basis states, one with $J_z = \frac{1}{2}\hbar$ and one with $J_z = -\frac{1}{2}\hbar$. These *must* assemble into an additional $j = \frac{1}{2}$ multiplet! Therefore, we write

$$\mathbf{1}\otimes\tfrac{1}{2} = \tfrac{3}{2}\oplus\tfrac{1}{2}. \tag{20.5.6}$$

The left-hand side indicates that we have tensored the vector spaces of $\ell = 1$ and $s = \frac{1}{2}$. On the right-hand side, we have the direct sum of $j = \frac{3}{2}$ and $j = \frac{1}{2}$. In terms of numbers of basis states, the above relation holds because 3 × 2 = 4 + 2. To find the $|j=\frac{3}{2}, m=\frac{1}{2}\rangle$ state, we will apply $\hat{J}_-$ to equation (20.5.4). Recalling that

$$\hat{J}_\pm|j,m\rangle = \hbar\sqrt{j(j+1)-m(m\pm 1)}\,|j,m\pm 1\rangle, \tag{20.5.7}$$

we see that the left-hand side gives

$$\hat{J}_-|j=\tfrac{3}{2}, m=\tfrac{3}{2}\rangle = \hbar\sqrt{\tfrac{3}{2}\cdot\tfrac{5}{2}-\tfrac{3}{2}\cdot\tfrac{1}{2}}\,|j=\tfrac{3}{2}, m=\tfrac{1}{2}\rangle = \hbar\sqrt{3}\,|j=\tfrac{3}{2}, m=\tfrac{1}{2}\rangle. \tag{20.5.8}$$

On the right-hand side, we see that

$$\begin{aligned}\hat{J}_-|1,1\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle &= (\hat{J}_{1,-}\otimes\mathbb{1}+\mathbb{1}\otimes\hat{J}_{2,-})|1,1\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle\\ &= \hat{J}_-|1,1\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle+|1,1\rangle\otimes\hat{J}_-|\tfrac{1}{2},\tfrac{1}{2}\rangle\\ &= \hbar\sqrt{2}\,|1,0\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle+\hbar\,|1,1\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle.\end{aligned} \tag{20.5.9}$$

Equating the results of the two last calculations, we find the desired state:

$$|j=\tfrac{3}{2},m=\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|1,0\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle+\sqrt{\tfrac{1}{3}}\,|1,1\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle. \tag{20.5.10}$$

The state came out normalized, as it should! As expected, this state is a linear combination of the two states with $J_z=\frac{1}{2}\hbar$. The state $|j=\frac{1}{2},m=\frac{1}{2}\rangle$ must be an orthogonal linear combination of the *same* two states because eigenstates of Hermitian operators with different eigenvalues are orthogonal. This allows us to easily write

$$|j=\tfrac{1}{2},m=\tfrac{1}{2}\rangle = -\sqrt{\tfrac{1}{3}}\,|1,0\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle+\sqrt{\tfrac{2}{3}}\,|1,1\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle. \tag{20.5.11}$$

The overall sign of this state is arbitrary and a matter of convention. The above state could also have been calculated as follows.

Exercise 20.2. *Consider the ansatz $|j=\frac{1}{2},m=\frac{1}{2}\rangle=\alpha\,|1,0\rangle\otimes|\frac{1}{2},\frac{1}{2}\rangle+\beta\,|1,1\rangle\otimes|\frac{1}{2},-\frac{1}{2}\rangle$ and determine the coefficients α and β, up to normalization, by the condition that the state is annihilated by $\hat{J}_+$.*

We have by now identified four out of the six coupled states (three states in the $j=\frac{3}{2}$ multiplet and one in the $j=\frac{1}{2}$ multiplet). The remaining two are quickly obtained. Writing the $|j, m\rangle$ states without the explicit j = … and m = …, the full $j=\frac{3}{2}$ and $j=\frac{1}{2}$ multiplets are given below.

$$
\boxed{
\begin{aligned}
&\mathbf{j}=\tfrac{3}{2}: \\
&|\tfrac{3}{2},\tfrac{3}{2}\rangle = |1,1\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle \\
&|\tfrac{3}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|1,0\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle+\sqrt{\tfrac{1}{3}}\,|1,1\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle \\
&|\tfrac{3}{2},-\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|1,0\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle+\sqrt{\tfrac{1}{3}}\,|1,-1\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle \\
&|\tfrac{3}{2},-\tfrac{3}{2}\rangle = |1,-1\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle \\
&-\;- \\
&\mathbf{j}=\tfrac{1}{2}: \\
&|\tfrac{1}{2},\tfrac{1}{2}\rangle = -\sqrt{\tfrac{1}{3}}\,|1,0\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle+\sqrt{\tfrac{2}{3}}\,|1,1\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle \\
&|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\,|1,0\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle-\sqrt{\tfrac{2}{3}}\,|1,-1\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle.
\end{aligned}
}
\tag{20.5.12}
$$

Let us comment on the states we did not derive. The state $|\frac{3}{2},-\frac{1}{2}\rangle$ can be found by applying $\hat{J}_-$ to $|\frac{3}{2},\frac{1}{2}\rangle$ or by applying $\hat{J}_+$ to $|\frac{3}{2},-\frac{3}{2}\rangle$, both of which were determined earlier. Similarly, the state $|\frac{1}{2},-\frac{1}{2}\rangle$ can be found by applying $\hat{J}_-$ to $|\frac{1}{2},\frac{1}{2}\rangle$ or by demanding orthogonality to $|\frac{3}{2},-\frac{1}{2}\rangle$. You can test your knowledge by doing these short computations.

Exercise 20.3. *Are the states in (20.5.12) eigenstates of* $\hat{\mathbf{J}}_1^2$*? Are they eigenstates of* $\hat{\mathbf{J}}_2^2$*?*