17 Dynamics of Quantum Systems

To illustrate the methods used to determine the dynamics of quantum systems, we focus on two broad classes of examples. In the first class, we consider coherent states of the harmonic oscillator. In their simplest form, they arise by translation of the ground state. We use the Heisenberg picture to show that, as opposed to energy eigenstates, coherent states have classical behavior. Looking at the dynamics of electromagnetic fields in a cavity, we are led to a harmonic oscillator description of photon states as well as field operators for electric and magnetic fields. For the second class, we consider two-state systems, quantum systems with two basis states. The general Hamiltonian of such a system is a 2 × 2 Hermitian matrix. We discuss Larmor precession of spin one-half states in a magnetic field and nuclear magnetic resonance, in which one follows the evolution of spin states in a magnetic field with a large longitudinal component and a rotating radio-frequency component. Finally, we discuss the factorization, or supersymmetric method, that gives an algebraic solution for the spectrum of one-dimensional potentials.

17.1 Basics of Coherent States

Coherent states are quantum states that exhibit classical behavior. We will introduce them here and explore their properties. In preparation for this, we first examine translation operators. We took a brief look at translation operators in section 10.1.

Let us construct a unitary **translation** operator T_{x_0} that acting on states moves them, or translates them, by a distance x_0, where x_0 is a constant with units of length. We then claim as the

$$\boxed{\text{translation operator:} \quad T_{x_0} \equiv e^{-\frac{i}{\hbar}\hat{p}\, x_0}.} \tag{17.1.1}$$

This operator is unitary because it is the exponential of an anti-Hermitian operator (see example 14.11). The multiplication of two such operators is simple:

$$T_{x_0} T_{y_0} = e^{-\frac{i}{\hbar}\hat{p}\, x_0} e^{-\frac{i}{\hbar}\hat{p}\, y_0} = e^{-\frac{i}{\hbar}\hat{p}\,(x_0+y_0)}, \tag{17.1.2}$$

since the exponents commute, and $e^A e^B = e^{A+B}$ if $[A, B] = 0$. As a result,

$$T_{x_0} T_{y_0} = T_{x_0+y_0}. \tag{17.1.3}$$

The translation operators form a group: the product of two translations is a translation, there is a unit element $T_0 = \mathbb{1}$, corresponding to $x_0 = 0$, and each element T_{x_0} has an inverse T_{-x_0}. The group multiplication rule is commutative. It follows from the explicit definition of the translation operator that

$$(T_{x_0})^\dagger = e^{\frac{i}{\hbar}\hat{p}\, x_0} = e^{-\frac{i}{\hbar}\hat{p}\,(-x_0)} = T_{-x_0} = (T_{x_0})^{-1}, \tag{17.1.4}$$

confirming that the operator is unitary. In the following we write $(T_{x_0})^\dagger$ simply as $T_{x_0}{}^\dagger$. We say that T_{x_0} translates by x_0 because its action on the operator $\hat{x}$ is as follows:

$$T_{x_0}^\dagger\, \hat{x}\, T_{x_0} = e^{\frac{i}{\hbar}\hat{p}\, x_0}\, \hat{x}\, e^{-\frac{i}{\hbar}\hat{p}\, x_0} = \hat{x} + \frac{i}{\hbar}[\hat{p}, \hat{x}]x_0 = \hat{x} + x_0, \tag{17.1.5}$$

where we used $e^A B e^{-A} = B + [A, B]$, valid when $[A, B]$ commutes with A. Further motivation for the identification of the T operators as translations is obtained by considering a normalized state $|\psi\rangle$ and the expectation value of $\hat{x}$ on this state:

$$\langle \hat{x} \rangle_\psi = \langle \psi |\, \hat{x}\, | \psi \rangle. \tag{17.1.6}$$

Now we ask: What is the expectation value of $\hat{x}$ on the state $T_{x_0}|\psi\rangle$? We find

$$\langle \hat{x} \rangle_{T_{x_0}\psi} = \langle \psi | T_{x_0}^\dagger\, \hat{x}\, T_{x_0} | \psi \rangle = \langle \psi | (\hat{x} + x_0) | \psi \rangle = \langle \hat{x} \rangle_\psi + x_0. \tag{17.1.7}$$

The expectation value of $\hat{x}$ on the displaced state is indeed equal to the expectation value of $\hat{x}$ in the original state plus x_0, confirming that *we should view $T_{x_0}|\psi\rangle$ as the state $|\psi\rangle$ displaced a distance x_0.*

We claim that when acting on position states the translation operator T_{x_0} does what we would expect; it displaces the state by x_0:

$$T_{x_0}|x_1\rangle = |x_1 + x_0\rangle. \tag{17.1.8}$$

We can prove this by acting on the above left-hand side with an arbitrary momentum bra $\langle p|$:

$$\langle p|T_{x_0}|x_1\rangle = \langle p|e^{-\frac{i}{\hbar}\hat{p}x_0}|x_1\rangle = e^{-\frac{i}{\hbar}px_0}\langle p|x_1\rangle. \tag{17.1.9}$$

Recalling the value $\langle x|p\rangle = e^{ipx}/\sqrt{2\pi\hbar}$ of the overlap (14.10.15), we get

$$\langle p|T_{x_0}|x_1\rangle = e^{-\frac{i}{\hbar}px_0}\frac{e^{-\frac{i}{\hbar}px_1}}{\sqrt{2\pi\hbar}} = \frac{e^{-\frac{i}{\hbar}p(x_1+x_0)}}{\sqrt{2\pi\hbar}} = \langle p|x_1 + x_0\rangle, \tag{17.1.10}$$

proving the desired result, given that $\langle p|$ is arbitrary. It also follows from unitarity and (17.1.8) that

$$T^{\dagger}_{x_0}|x_1\rangle = T_{-x_0}|x_1\rangle = |x_1 - x_0\rangle. \tag{17.1.11}$$

Passing to bras, the previous result gives

$$\langle x_1|T_{x_0} = \langle x_1 - x_0|. \tag{17.1.12}$$

We can also discuss the action of the translation operator in terms of arbitrary states $|\psi\rangle$ and their wave functions $\psi(x) = \langle x|\psi\rangle$. Then the "translated" state $T_{x_0}|\psi\rangle$ has a wave function

$$\langle x|T_{x_0}|\psi\rangle = \langle x - x_0|\psi\rangle = \psi(x - x_0). \tag{17.1.13}$$

Indeed, $\psi(x-x_0)$ is the function $\psi(x)$ translated by the distance $+\,x_0$. For example, the value that $\psi(x)$ takes at $x = 0$ is taken by the function $\psi(x - x_0)$ at $x = x_0$.

We are finally ready to define a coherent state $|x_0\rangle c$ of the simple harmonic oscillator. The state is labeled by x_0, and the c subscript on the ket is there to remind you that it is a coherent state, *not* a position state. Here we define the

$$\boxed{\text{coherent state:} \quad |x_0\rangle_c \equiv T_{x_0}|0\rangle = e^{-\frac{i}{\hbar}\hat{p}x_0}|0\rangle,} \tag{17.1.14}$$

where $|0\rangle$ denotes the ground state of the oscillator. The coherent state is simply the translation of the ground state by a distance x_0. This state has no time dependence displayed, so it may be thought of as the state of the system at $t = 0$. As t increases, the state will evolve according to the Schrödinger equation, and we will later consider this evolution. Note that the coherent state is normalized:

$$_c\langle x_0|x_0\rangle_c = \langle 0|T_{x_0}^\dagger T_{x_0}|0\rangle = \langle 0|0\rangle = 1. \tag{17.1.15}$$

This had to be: it is the action of the unitary operator T_{x_0} on the normalized ground state.

The wave function ψ_{x_0} associated to the coherent state is easily obtained:

$$\psi_{x_0}(x) \equiv \langle x|x_0\rangle_c = \langle x|T_{x_0}|0\rangle = \langle x - x_0|0\rangle = \varphi_0(x - x_0), \tag{17.1.16}$$

where we used (17.1.12), and $\langle x|0\rangle = \varphi_0(x)$ is the ground state wave function. As expected the wave function for the coherent state is just the ground state wave function displaced a distance x_0 to the right. This is illustrated in figure 17.1.

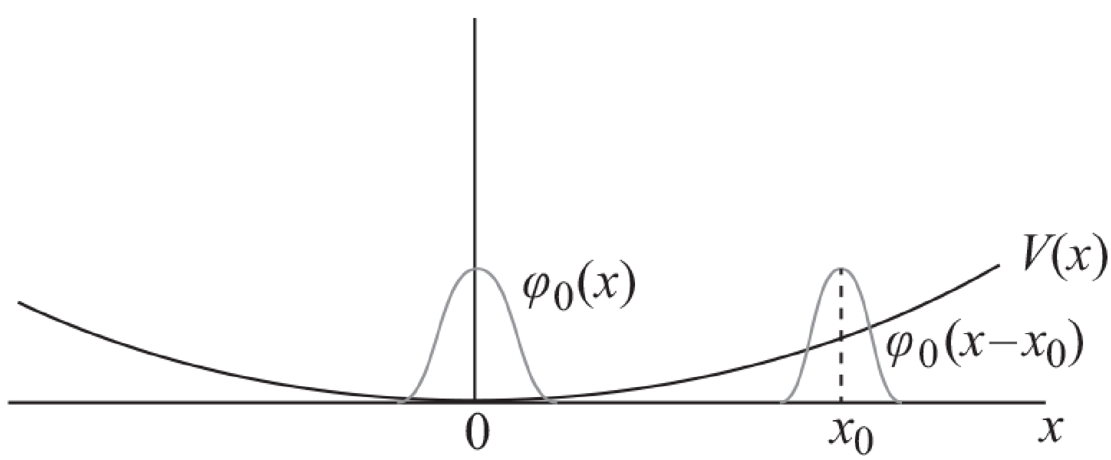

Figure 17.1
The ground state wave function $\varphi_0(x)$ displaced to the right a distance x_0 is the wave function $\varphi_0(x - x_0)$. The corresponding state, denoted as $|x_0\rangle c$, is the simplest example of a coherent state.

Let us now do a few sample calculations to better understand these states.

1. We first find the expectation value of $\hat{x}$ in a coherent state. This is quickly done:

$$_c\langle x_0|\hat{x}\,|x_0\rangle_c \;=\; \langle 0|T^\dagger_{x_0}\,\hat{x}\,T_{x_0}|0\rangle \;=\; \langle 0|(\hat{x}+x_0)|0\rangle, \tag{17.1.17}$$

   where we used the action (17.1.5) of the translation operator on $\hat{x}$. Recalling now that $\langle 0|\hat{x}|0\rangle = 0$, we get

$$_c\langle x_0|\hat{x}\,|x_0\rangle_c \;=\; x_0. \tag{17.1.18}$$

   Not that surprising! The state is centered at x_0.

2. Now we calculate the expectation value of $\hat{p}$ in a coherent state. Since $\hat{p}$ commutes with T_{x_0}, we find that

$$_c\langle x_0|\hat{p}\,|x_0\rangle_c \;=\; \langle 0|T^\dagger_{x_0}\,\hat{p}\,T_{x_0}|0\rangle \;=\; \langle 0|\,\hat{p}\,T^\dagger_{x_0}\,T_{x_0}|0\rangle \;=\; \langle 0|\,\hat{p}\,|0\rangle \;=\; 0, \tag{17.1.19}$$

   recalling that the expectation value of $\hat{p}$ in any harmonic oscillator energy eigenstate vanishes. The coherent state we built has no momentum. We will later consider more general coherent states that have momentum.

3. Finally, we calculate the expectation value of the energy in a coherent state. Note that the coherent state is not an energy eigenstate. It is also neither a position eigenstate nor a momentum eigenstate! With $\hat{H}$ the Hamiltonian, we have

$$_c\langle x_0|\hat{H}|x_0\rangle_c \;=\; \langle 0|T^\dagger_{x_0}\hat{H}T_{x_0}|0\rangle. \tag{17.1.20}$$

   We now compute

$$\begin{aligned} T^\dagger_{x_0}\hat{H}T_{x_0} \;&=\; T^\dagger_{x_0}\Big(\frac{\hat{p}^2}{2m}+\tfrac{1}{2}m\omega^2\hat{x}^2\Big)T_{x_0} \;=\; \frac{\hat{p}^2}{2m}+\tfrac{1}{2}m\omega^2(\hat{x}+x_0)^2 \\ &=\; \hat{H}+m\omega^2x_0\hat{x}+\tfrac{1}{2}m\omega^2x_0^2. \end{aligned} \tag{17.1.21}$$

   Back in (17.1.20),

$$_c\langle x_0|\hat{H}|x_0\rangle_c \;=\; \langle 0|\hat{H}|0\rangle+m\omega^2x_0\langle 0|\hat{x}|0\rangle+\tfrac{1}{2}m\omega^2x_0^2. \tag{17.1.22}$$

Recalling that the ground state energy is $\hbar\omega/2$ and that in the ground state $\hat{x}$ has no expectation value, we finally get

$$ {}_c\langle x_0|\hat{H}|x_0\rangle_c = \tfrac{1}{2}\hbar\omega + \tfrac{1}{2}m\omega^2 x_0^2. \tag{17.1.23}$$

This is reasonable: the total energy is the zero-point energy plus the potential energy of a particle at x_0. The coherent state $|x_0\rangle c$ is the quantum version of a point particle on a spring held stretched at $x = x_0$.