20.7 General Aspects of Addition of Angular Momentum

We have examined the tensor states of two particles with spin one-half and the states of a particle that has both orbital and spin angular momentum and whose state space is a tensor product of $\ell = 1$ orbital states and spin states. In both cases we found it useful to form a basis for the tensor product in which states were eigenstates of the total angular momentum. Now we will consider the general situation and discuss some important regularities.

Consider two state spaces $\hat{\mathcal{H}}_1$ and $\hat{\mathcal{H}}_2$, each of which may contain a number of multiplets of angular momenta $\hat{\mathbf{J}}_1$ and $\hat{\mathbf{J}}_2$, respectively:

$$
\begin{aligned}
\hat{\mathbf{J}}_1: \quad \hat{\mathcal{H}}_1 &= \bigoplus_{j_1} \mathcal{H}_1^{j_1}, \qquad \mathcal{H}_1^{j_1} = \bigoplus_{m_1=-j_1}^{j_1} |j_1, m_1\rangle, \\
\hat{\mathbf{J}}_2: \quad \hat{\mathcal{H}}_2 &= \bigoplus_{j_2} \mathcal{H}_2^{j_2}, \qquad \mathcal{H}_2^{j_2} = \bigoplus_{m_2=-j_2}^{j_2} |j_2, m_2\rangle.
\end{aligned}
\tag{20.7.1}
$$

As indicated above, $\hat{\mathcal{H}}_i$ (with $i = 1, 2$) is a direct sum of multiplets $\mathcal{H}_i^{j_i}$ with angular momentum j_i, each of which contains the familiar $2j_i + 1$ states with different values of m_i. The direct sum symbol is appropriate as the various j_i multiplet vector spaces are put together as orthogonal subspaces inside the big total spaces $\hat{\mathcal{H}}_1$ and $\hat{\mathcal{H}}_2$. The list of j_1 multiplets appearing in $\hat{\mathcal{H}}_1$ and the list of j_2 multiplets appearing in $\hat{\mathcal{H}}_2$ are left arbitrary. The goal is to form the tensor product vector space $\hat{\mathcal{H}}_1 \otimes \hat{\mathcal{H}}_2$ and find a basis of states that are eigenstates of the total angular momentum; more precisely, eigenstates of $\hat{\mathbf{J}}^2$ and $\hat{J}_z$, where

$$\hat{\mathbf{J}} = \hat{\mathbf{J}}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \hat{\mathbf{J}}_2. \tag{20.7.2}$$

The full tensor product is given as a direct sum of the tensor product of the various multiplet subspaces:

$$\hat{\mathcal{H}}_1 \otimes \hat{\mathcal{H}}_2 = \bigoplus_{j_1, j_2} \mathcal{H}_1^{j_1} \otimes \mathcal{H}_2^{j_2}. \tag{20.7.3}$$

It is therefore sufficient for us to understand the structure of the general summand V_{j_1,j_2} in the above relation:

$$V_{j_1,j_2} \equiv \mathcal{H}_1^{j_1} \otimes \mathcal{H}_2^{j_2}, \quad \dim V_{j_1,j_2} = (2j_1+1)(2j_2+1). \tag{20.7.4}$$

We want to understand the general features of V_{j_1,j_2}. For this purpose we note that we have two natural choices of basis states corresponding to alternative CSCOs.

1. The **uncoupled basis states** for V_{j_1,j_2} follow naturally by the tensor product of $\mathcal{H}_1^{(j_1)}$ basis states and $\mathcal{H}_2^{(j_2)}$ basis states:

$$|j_1, j_2, m_1, m_2\rangle \equiv |j_1, m_1\rangle \otimes |j_2, m_2\rangle. \tag{20.7.5}$$

   Here m_1 runs over $2j_1 + 1$ values, and m_2 runs over $2j_2 + 1$ values. The CSCO_u, with subscript u for uncoupled, is given by the following list:

$$\text{CSCO}_u: \quad \{\hat{\mathbf{J}}_1^2, \hat{\mathbf{J}}_2^2, \hat{J}_{1,z}, \hat{J}_{2,z}\}. \tag{20.7.6}$$

   The states $|j_1, j_2, m_1, m_2\rangle$ are eigenstates of the operators in the list with eigenvalues given, respectively, by

$$\{\hbar^2 j_1(j_1+1), \ \hbar^2 j_2(j_2+1), \ \hbar m_1, \ \hbar m_2\}. \tag{20.7.7}$$

2. The **coupled basis states** for V_{j_1,j_2} are eigenstates of the CSCO_c, with subscript c for coupled, that contains the following operators:

$$\text{CSCO}_c: \quad \{\hat{\mathbf{J}}_1^2, \hat{\mathbf{J}}_2^2, \hat{\mathbf{J}}^2, \hat{J}_z\}. \tag{20.7.8}$$

Here $j_1 = j_2 = \frac{1}{2}$. The basis states are

$$|j_1, j_2; j, m\rangle, \tag{20.7.9}$$

and the eigenvalues for the list of operators in the CSCO_c are given by

$$\{\hbar^2 j_1(j_1+1),\ \hbar^2 j_2(j_2+1),\ \hbar^2 j(j+1),\ \hbar m\}. \tag{20.7.10}$$

The states (20.7.9) come in full $\hat{\mathbf{J}}$ multiplets $\mathcal{H}_j$ that contain all allowed m values:

$$\mathcal{H}_j = \bigoplus_{m=-j}^{j} |j_1, j_2; j, m\rangle. \tag{20.7.11}$$

Since we aim to give a basis for the space V_{j_1,j_2}, which has dimension $(2j_1 + 1)(2j_2 + 1)$, we will need a collection of $\hat{\mathbf{J}}$ multiplets $\mathcal{H}_j$ with various values of j:

$$V_{j_1,j_2} = \bigoplus_{j \in S(j_1,j_2)} \mathcal{H}_j. \tag{20.7.12}$$

Here $S(j_1, j_2)$ is a list of values that depend on j_1 and j_2. In combining two spin one-half particles, we learned that for $j_1 = j_2 = \frac{1}{2}$ the list is $j \in \{1, 0\}$. In doing spin-orbit coupling, we learned that for $j_1 = 1, j_2 = \frac{1}{2}$ we have $j \in \{\frac{3}{2}, \frac{1}{2}\}$. What will we get for general j_1 and j_2?

Both the coupled and uncoupled bases are orthonormal bases. One natural question is how to relate the two bases. We did this explicitly for the case of $\frac{1}{2} \otimes \frac{1}{2}$ and for $1 \otimes \frac{1}{2}$, in both cases figuring out how to write the coupled states in terms of the uncoupled ones. The question of relating the two sets of basis vectors is the problem of calculating the **Clebsch-Gordan** (CG) coefficients. To write this out clearly, consider the completeness relation for the uncoupled basis of the space V_{j_1,j_2}. With the sums over m_1 and m_2 running over the usual values,

$$\mathbb{1} = \sum_{m_1,m_2} |j_1 j_2; m_1 m_2\rangle\langle j_1 j_2; m_1 m_2|, \quad \text{acting on} \quad V_{j_1,j_2}. \tag{20.7.13}$$

Apply both sides of this equation to the coupled basis state $|j_1, j_2; j, m\rangle$ to find

$$|j_1, j_2; j, m\rangle = \sum_{m_1,m_2} |j_1 j_2; m_1 m_2\rangle \underbrace{\langle j_1 j_2; m_1 m_2 | j_1, j_2; j, m\rangle}_{\text{Clebsch-Gordan coefficients}}. \tag{20.7.14}$$

The overlaps selected above are the CG coefficients. They are numbers that, if known, determine the relations between the two sets of basis vectors. Schematically, a CG coefficient is an overlap $\langle u_i|c_j\rangle$, where u_i denotes an uncoupled basis element, and c_j denotes a coupled basis element. Since the two bases are orthonormal, the transformation from one to the other is produced by a unitary operator U that can be written as

$$U = \sum_k |c_k\rangle\langle u_k| \tag{20.7.15}$$

and manifestly maps $|u_p\rangle$ to $|c_p\rangle$. The matrix elements of this operator,

$$\langle u_i|U|u_j\rangle = \sum_k \langle u_i|c_k\rangle\langle u_k|u_j\rangle = \sum_k \langle u_i|c_k\rangle\delta_{kj} = \langle u_i|c_j\rangle, \tag{20.7.16}$$

are precisely the CG coefficients. In fact, $\langle u_i|U|u_j\rangle = \langle c_i|U|c_j\rangle$ because U is a basis-changing operator.

There are selection rules for CG coefficients that help their computation:

1. We claim that

$$\boxed{\langle j_1 j_2; m_1 m_2 | j_1, j_2; j, m\rangle = 0, \quad \text{for } m \neq m_1 + m_2.} \tag{20.7.17}$$

   This is easily proven by inserting the $\hat{J}_z$ operator in between the overlap and setting equal the two possible ways of evaluating the matrix element:

$$\langle j_1 j_2; m_1 m_2|\hat{J}_z|j_1, j_2; j, m\rangle = \langle j_1 j_2; m_1 m_2|(\hat{J}_{1,z} + \hat{J}_{2,z})|j_1, j_2; j, m\rangle. \tag{20.7.18}$$

   On the left-hand side, we let the operator act on the ket, and on the right-hand side, we let the operator act on the bra by using its

Hermiticity. We then get

$$\hbar m \langle j_1 j_2; m_1 m_2 | j_1, j_2; j, m\rangle = (\hbar m_1 + \hbar m_2)\langle j_1 j_2; m_1 m_2 | j_1, j_2; j, m\rangle. \tag{20.7.19}$$

Collecting terms, we find that

$$\hbar\big(m - (m_1 + m_2)\big)\langle j_1 j_2; m_1 m_2 | j_1, j_2; j, m\rangle = 0. \tag{20.7.20}$$

This confirms the claim that the CG coefficient must vanish when $m \neq m_1 + m_2$. Note that when we computed coupled states in terms of uncoupled ones we used this rule: a coupled state with $J_z = \hbar m$ could only be related to uncoupled states in which the $\hat{J}_{1,z}$ and $\hat{J}_{2,z}$ eigenvalues added to $\hbar m$; see, for example, the states in (20.5.12).

2. The second selection rule specifies the values of j for which the coupled state can have a nonvanishing overlap with the uncoupled states $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}$. When writing (20.7.12),

$$V_{j_1, j_2} = \mathcal{H}_1^{j_1} \otimes \mathcal{H}_2^{j_2} = \bigoplus_{j \in S(j_1, j_2)} \mathcal{H}_j, \tag{20.7.21}$$

we asked what values of j appear in the list $S(j_1, j_2)$. Only for those values of j does the CG coefficient $\langle j_1 j_2;\ m_1 m_2 |\ j_1,\ j_2;\ j,\ m\rangle$ need not vanish. Additionally, the sum of the dimensions of the $\mathcal{H}_j$'s in the equation above must equal the dimensionality of V_{j_1, j_2}.

We can quickly deduce the largest value of j in the list $S(j_1, j_2)$. The largest value of $\hat{J}_z$ in $\mathcal{H}_1{}^{j_1} \otimes \mathcal{H}_2{}^{j_2}$ is the largest value of $\hat{J}_{1,z} + \hat{J}_{2,z}$. The largest value of $\hat{J}_{1,z}$ is $\hbar j_1$, and the largest value of $\hat{J}_{2,z}$ is $\hbar j_2$. Thus, the largest value of $\hat{J}_z$ is $\hbar(j_1 + j_2)$, implying that the largest value of j in the list is $j_1 + j_2$. As it turns out, the smallest value of j in the list is $|j_1 - j_2|$, and the full list includes all integer values of j in between:

$$S(j_1, j_2) = \{j_1 + j_2,\ j_1 + j_2 - 1,\ \ldots,\ |j_1 - j_2|\,\}. \tag{20.7.22}$$

Note that each value of j appears only once. This result (20.7.21) is usually written by labeling the multiplets with their j value:

$$\boxed{j_1 \otimes j_2 = (j_1 + j_2) \oplus (j_1 + j_2 - 1) \oplus \ldots \oplus |j_1 - j_2|.} \tag{20.7.23}$$

This is an equality between vector spaces showing how to write the vector space on the left-hand side as a direct sum of irreducible, invariant subspaces of $\hat{\mathbf{J}}$. It demonstrates that $j_1 \otimes j_2$ forms a *reducible* representation of the total angular momentum.

There is a basic test of (20.7.23). The dimensionality of $j_1 \otimes j_2$ is $(2j_1 + 1)(2j_2 + 1)$, so the dimensionality of the right-hand side must be the same. Since the j_1 and j_2 labels can be exchanged, we can assume that $j_1 \geq j_2$ without loss of generality. The dimensionality of the right-hand side is the sum of the dimensions of each of the representations. The sum is readily done, and the expected value is obtained:

$$\begin{aligned}
\sum_{j=j_1-j_2}^{j_1+j_2}(2j+1) &= 2\Big(\sum_{j=0}^{j_1+j_2} j - \sum_{j=0}^{j_1-j_2-1} j\Big) + \big(j_1+j_2-(j_1-j_2)+1\big) \\
&= (j_1+j_2)(j_1+j_2+1) - (j_1-j_2-1)(j_1-j_2) + 2j_2 + 1 \\
&= (j_1+j_2)^2 - (j_1-j_2)^2 + 2j_1 + 2j_2 + 1 \\
&= 2j_1(2j_2+1) + 2j_2 + 1 = (2j_1+1)(2j_2+1).
\end{aligned} \tag{20.7.24}$$

It is possible to explain the selection rule for j by considering explicitly all combinations of states in the j_1 and j_2 multiplets—again, assuming that $j_1 \geq j_2$. Let us use two columns to write all the states:

$$\begin{array}{llll}
[1] & |j_1, j_1\rangle & & \\
[2] & |j_1, j_1-1\rangle & & \\
[3] & |j_1, j_1-2\rangle & & \\
\vdots & \vdots & & \\
[2j_1-2j_2+1] & |j_1, 2j_2-j_1\rangle & |j_2, j_2\rangle & [1] \\
 & |j_1, 2j_2-j_1-1\rangle & |j_2, j_2-1\rangle & [2] \\
\vdots & \vdots & \vdots & \vdots \\
[2j_1+1] & |j_1, -j_1\rangle & |j_2, -j_2\rangle & [2j_2+1].
\end{array} \tag{20.7.25}$$

The data is aligned at the bottom: the lowest m states are on the same line. Since $j_1 \geq j_2$, the higher m states of the j_1 multiplet need not have same-line counterparts in the j_2 multiplet. We have also added number labels to the states. They appear in brackets to the left of the j_1 multiplet and to the right of the j_2 multiplet. Those labels allow us to refer quickly to some particular tensor product. Thus, for example,

$$|j_1, j_1 - 2\rangle \otimes |j_2, j_2 - 1\rangle \;\Leftrightarrow\; [3] \times [2]. \qquad (20.7.26)$$

We now walk through the construction of figure 20.2, which shows why the selection rule for j holds. The heavy dots are states, and dots on a horizontal line are states with the same value of $J_z = \hbar m$, the total z-component of angular momentum.

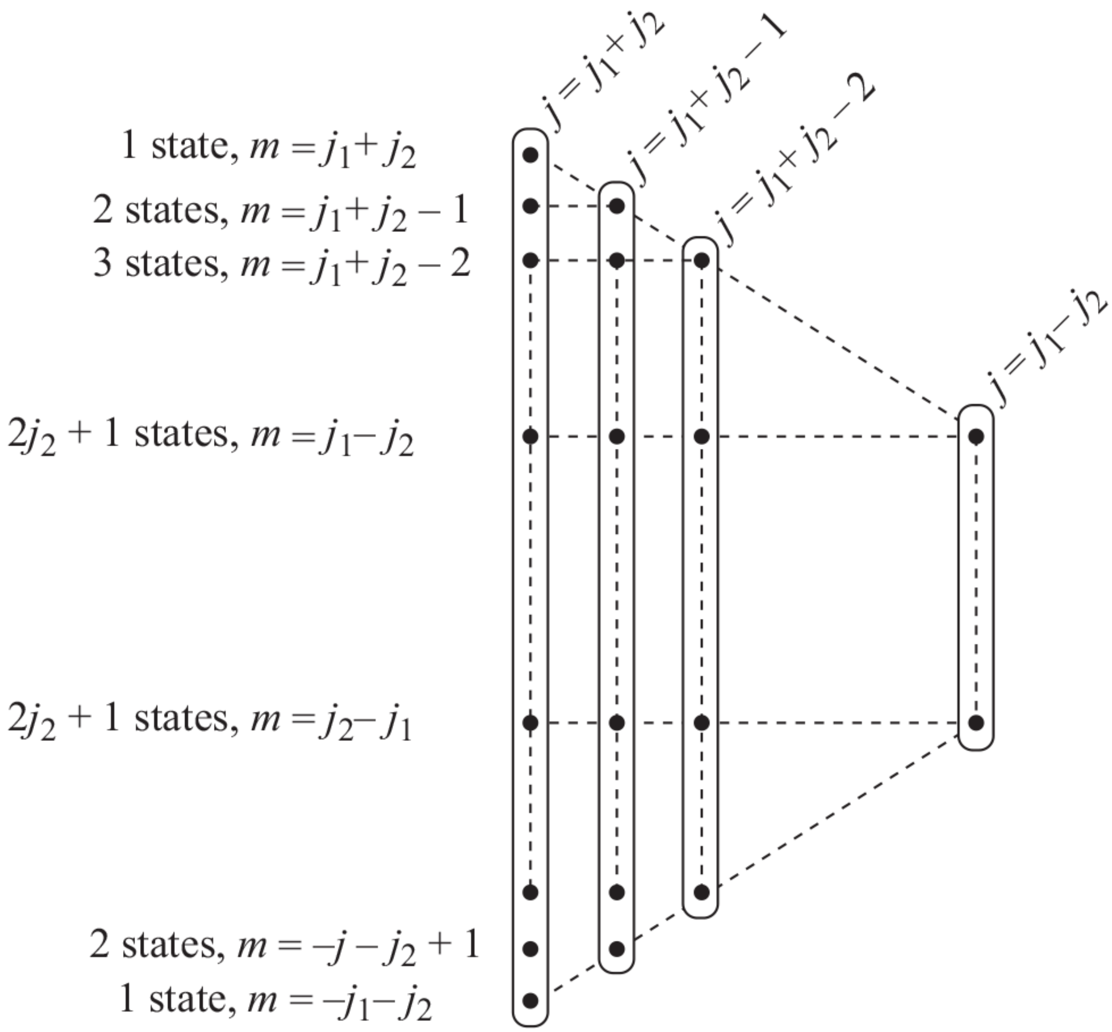

Figure 20.2
Diagram illustrating how $j_1 \otimes j_2$, with $j_1 \geq j_2$, contains total angular momentum representations from $j = j_1 + j_2$ all the way down to $j = j_1 - j_2$. Each heavy dot in the figure is a state. One can view the columns as j multiplets, with j value indicated on the topmost state.

- The first row consists of the single state [1] × [1], with $m = j_1 + j_2$.
- For the second line, we get two states [1] × [2] and [2] × [1], both with $m = j_1 + j_2 - 1$.
- For the third line, we get three states [1] × [3], [2] × [2], and [3] × [1], all with $m = j_1 + j_2 - 2$.
- The number of states on each line will keep growing until we get a maximum of $2j_2 + 1$ states:

$$[1] \times [2j_2 + 1], \; \ldots \; [2j_2 + 1] \times [1], \tag{20.7.27}$$

  all with $m = j_1 - j_2$. At this point we are using all of the states of the j_2 multiplet.

- The number of states on each line now remains the same until we get the set of states

$$[2j_1 - 2j_2 + 1] \times [2j_2 + 1], \; \ldots, \; [2j_1 + 1] \times [1], \tag{20.7.28}$$

  all with $m = j_2 - j_1 = -(j_1 - j_2)$. At this point we are tensoring all the $2j_2 + 1$ matched states at the bottom of the list (20.7.25). Since m values change by one unit as we move from line to line in figure 20.2, there are a total of $j_1 - j_2 - (j_2 - j_1) + 1 = 2(j_1 - j_2) + 1$ lines with the maximum number of states. This implies that the shortest column is the right-most, and it has $2(j_1 - j_2) + 1$ states, with m ranging from $j_1 - j_2$ to $-(j_1 - j_2)$. This corresponds to the $j = j_1 - j_2$ multiplet.

- After this last maximally long horizontal line in figure 20.2, the number of states on each line decreases by one unit as we move from line to line. The last line on the diagram has one state, $[2j_1 + 1] \times [2j_2 + 1]$, with $m = -(j_1 + j_2)$.
- All the columns contain the proper range of m values to form complete j multiplets. The longest column is the first, with $j = j_1 + j_2$. The shortest column was identified before, with $j = j_1 - j_2$. We have columns for all intermediate values of j changing by one unit at a time. This is what we wanted to demonstrate.

This argument shows that there are the right number of states at all values of m to be assembled into the expected multiplets of total angular momentum. In fact, they *have* to assemble in that way, as we explain now. Consider the single state on the first row in the figure, a state with $m = j_1 + j_2$. This m value implies that we have the multiplet $j = j_1 + j_2$. Looking at the second row, we have two states with $m = j_1 + j_2 - 1$. One linear combination must belong to the $j_1 + j_2$ multiplet already identified, but the other linear combination requires a multiplet with $j = j_1 + j_2 - 1$. Looking at the third row, we have three states with $m = j_1 + j_2 - 2$. Two linear combinations belong to the two multiplets already identified, and the third requires a new multiplet with $j = j_1 + j_2 - 2$. Continuing this way, we need a new multiplet with one unit less of j at each step until we get to the first occurrence of the largest row. That row has $2j_2 + 1$ states, all with $m = j_1 - j_2$. We have already identified $2j_2$ multiplets, requiring $2j_2$ linear combinations of the available states. The remaining one linear combination of the states requires one last new multiplet, with $j = j_1 - j_2$. No more multiplets are needed, as the rows no longer grow. We have therefore shown that the expected multiplets appear. Note that if we literally associate the columns in the figure with the multiplets, then each dot will represent some linear combination of the uncoupled basis states listed in the construction.