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15.2 The Uncertainty Principle

The uncertainty principle is an inequality satisfied by the product of the uncertainties of two Hermitian operators that fail to commute. Since the uncertainty of an operator on any given physical state is a number greater than or equal to zero, the product of uncertainties is also a real number greater than or equal to zero. The uncertainty inequality gives us a *lower bound* for the product of uncertainties. When the two operators in question commute, the uncertainty inequality gives no information: it states that the product of uncertainties must be greater than or equal to zero, which we already know.

Let us state the uncertainty inequality. Consider two Hermitian operators $\hat{A}$ and $\hat{B}$ and a physical state Ψ of the quantum system. Let ΔA and ΔB denote the uncertainties of $\hat{A}$ and $\hat{B}$, respectively, in the state Ψ. Then, the uncertainty inequality states that

$$(\Delta A)^2(\Delta B)^2 \geq \left(\langle\Psi|\tfrac{1}{2i}[\hat{A},\hat{B}]|\Psi\rangle\right)^2. \tag{15.2.1}$$

The left-hand side is a real, nonnegative number. For this to be a sensible inequality, the right-hand side must also be a real, nonnegative number. Since the square of a complex number would be either complex or negative, the object within parentheses must be real:

$$\langle\Psi|\tfrac{1}{2i}[\hat{A},\hat{B}]|\Psi\rangle \in \mathbb{R}. \tag{15.2.2}$$

This holds because the operator $\frac{1}{2i}[\hat{A},\hat{B}]$ is in fact Hermitian. To see this, first note that the commutator of two Hermitian operators is *anti*-Hermitian:

$$[\hat{A},\hat{B}]^\dagger = (\hat{A}\hat{B})^\dagger - (\hat{B}\hat{A})^\dagger = \hat{B}^\dagger\hat{A}^\dagger - \hat{A}^\dagger\hat{B}^\dagger = \hat{B}\hat{A} - \hat{A}\hat{B} = -[\hat{A},\hat{B}]. \tag{15.2.3}$$

The presence of the i then makes the operator $\frac{1}{2i}[\hat{A}, \hat{B}]$ Hermitian, as claimed.

Taking square roots, the uncertainty inequality can also be written as

$$\boxed{\Delta A \, \Delta B \geq \left| \langle \Psi | \tfrac{1}{2i}[\hat{A}, \hat{B}] | \Psi \rangle \right|.} \tag{15.2.4}$$

The bars on the right-hand side denote absolute value. The right-hand side is just the norm of the expectation value of $\frac{1}{2i}[\hat{A}, \hat{B}]$ on the state Ψ. In general, all the quantities in the uncertainty inequality can be time dependent, and the inequality must hold for all times. The time dependence arises because Ψ is time dependent, and if it is not an energy eigenstate, expectation values and uncertainties can be time dependent. Of course, if Ψ is an energy eigenstate and $\hat{A}$ and $\hat{B}$ are time-independent operators, all quantities in the uncertainty inequality are time independent.

Before we prove the uncertainty inequality, let's do the canonical example!

Example 15.2. *Position-momentum uncertainty inequality.*
Taking $\hat{A} = \hat{x}$ and $\hat{B} = \hat{p}$ gives the position-momentum uncertainty relation:

$$\Delta x \Delta p \geq \left| \langle \Psi | \tfrac{1}{2i}[\hat{x}, \hat{p}] | \Psi \rangle \right|. \tag{15.2.5}$$

Since $[\hat{x}, \hat{p}\,]/(2i) = \hbar/2$ and Ψ is normalized, we get

$$\boxed{\Delta x \, \Delta p \geq \tfrac{\hbar}{2}.} \tag{15.2.6}$$

This is the most famous application of the uncertainty inequality. It is sometimes called Heisenberg's uncertainty principle.

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We now go over the proof of the uncertainty inequality (15.2.4). We do this not only because you should know how such an important result is derived. The derivation is needed to learn under what conditions, and for what kinds of states, the uncertainty inequality is saturated. Since for an arbitrary state the uncertainties can easily multiply to a value larger than the lower bound in the uncertainty inequality, states that saturate the inequality are in some sense minimum uncertainty states.

Proof of (15.2.4). We define the following two states:

$$\begin{aligned} |f_A\rangle &\equiv (\hat{A} - \langle\hat{A}\rangle\mathbb{1})|\Psi\rangle, \\ |f_B\rangle &\equiv (\hat{B} - \langle\hat{B}\rangle\mathbb{1})|\Psi\rangle. \end{aligned} \tag{15.2.7}$$

Note that by the definition (15.1.2) of uncertainty,

$$\begin{aligned} \langle f_A|f_A\rangle &= (\Delta A)^2, \\ \langle f_B|f_B\rangle &= (\Delta B)^2. \end{aligned} \tag{15.2.8}$$

The Schwarz inequality immediately furnishes an inequality involving the uncertainties:

$$\langle f_A|f_A\rangle\langle f_B|f_B\rangle \geq |\langle f_A|f_B\rangle|^2, \tag{15.2.9}$$

and therefore,

$$(\Delta A)^2(\Delta B)^2 \geq |\langle f_A|f_B\rangle|^2 = (\text{Re}\langle f_A|f_B\rangle)^2 + (\text{Im}\langle f_A|f_B\rangle)^2. \tag{15.2.10}$$

Our task is now to compute each of the two terms on the above right-hand side. For convenience we introduce *checked* operators

$$\begin{aligned} \check{A} &\equiv \hat{A} - \langle\hat{A}\rangle\mathbb{1}, \\ \check{B} &\equiv \hat{B} - \langle\hat{B}\rangle\mathbb{1}. \end{aligned} \tag{15.2.11}$$

We now compute

$$\begin{aligned} \langle f_A|f_B\rangle &= \langle\Psi|\check{A}\check{B}|\Psi\rangle = \langle\Psi|(\hat{A} - \langle\hat{A}\rangle\mathbb{1})(\hat{B} - \langle\hat{B}\rangle\mathbb{1})|\Psi\rangle \\ &= \langle\Psi|\hat{A}\hat{B}|\Psi\rangle - 2\langle\hat{A}\rangle\langle\hat{B}\rangle + \langle\hat{A}\rangle\langle\hat{B}\rangle \end{aligned} \tag{15.2.12}$$

so that, simplifying, we have

$$\begin{aligned} \langle f_A|f_B\rangle &= \langle\Psi|\check{A}\check{B}|\Psi\rangle = \langle\Psi|\hat{A}\hat{B}|\Psi\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle, \\ \langle f_B|f_A\rangle &= \langle\Psi|\check{B}\check{A}|\Psi\rangle = \langle\Psi|\hat{B}\hat{A}|\Psi\rangle - \langle\hat{B}\rangle\langle\hat{A}\rangle, \end{aligned} \tag{15.2.13}$$

where the second equation follows because $|f_A\rangle$ and $|f_B\rangle$ go into each other as we exchange $\hat{A}$ and $\hat{B}$. We now use this to find a nice expression for the imaginary part of $\langle f_A|f_B\rangle$:

$$\text{Im}\langle f_A|f_B\rangle = \frac{1}{2i}(\langle f_A|f_B\rangle - \langle f_B|f_A\rangle) = \frac{1}{2i}\langle\Psi|[\hat{A},\hat{B}]|\Psi\rangle. \tag{15.2.14}$$

For the real part, the expression is not that simple because the product of expectation values does not cancel. It is best to write the real part as the anticommutator of the checked operators:

$$\text{Re}\langle f_A|f_B\rangle = \frac{1}{2}(\langle f_A|f_B\rangle + \langle f_B|f_A\rangle) = \frac{1}{2}\langle\Psi|\{\check{A},\check{B}\}|\Psi\rangle. \tag{15.2.15}$$

Back in (15.2.10) we get

$$(\Delta A)^2(\Delta B)^2 \geq \left(\langle\Psi|\frac{1}{2i}[\hat{A},\hat{B}]|\Psi\rangle\right)^2 + \left(\langle\Psi|\frac{1}{2}\{\check{A},\check{B}\}|\Psi\rangle\right)^2. \tag{15.2.16}$$

This can be viewed as the most complete form of the uncertainty inequality. It turns out, however, that the second term on the right-hand side is seldom simple enough to be of use, and often it can be made equal to zero for certain states. At any rate, the term is positive or zero so it can be dropped while preserving the inequality. This is usually done, giving the familiar forms (15.2.1) and (15.2.4) that we have now established.

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What are the conditions for the uncertainty inequality to be saturated and thus achieve the minimum possible product of uncertainties? As the proof above shows, saturation is achieved under two conditions:

1. The Schwarz inequality is saturated. For this we need $|f_B\rangle = \beta|f_A\rangle$ where $\beta \in \mathbb{C}$.
2. $\text{Re}(\langle f_A|f_B\rangle) = 0$ so that the last term in (15.2.16) vanishes. This means that $\langle f_A|f_B\rangle + \langle f_B|f_A\rangle = 0$.

Using $|f_B\rangle = \beta|f_A\rangle$ in condition (2), we get

$$\langle f_A|f_B\rangle + \langle f_B|f_A\rangle = \beta\langle f_A|f_A\rangle + \beta^*\langle f_A|f_A\rangle = (\beta+\beta^*)\langle f_A|f_A\rangle = 0, \tag{15.2.17}$$

which requires that $\beta + \beta^* = 0$ or equivalently that the real part of β vanishes. It follows that β must be purely imaginary. So $\beta = i\lambda$, with λ real. Therefore, the uncertainty inequality will be saturated if and only if

$$|f_B\rangle = i\lambda|f_A\rangle, \quad \lambda \in \mathbb{R}. \tag{15.2.18}$$

More explicitly, this requires the

$$\boxed{\text{saturation condition:} \quad (\hat{B} - \langle\hat{B}\rangle\,\mathbb{1})|\Psi\rangle = i\lambda\,(\hat{A} - \langle\hat{A}\rangle\mathbb{1})|\Psi\rangle, \quad \lambda \in \mathbb{R}.} \tag{15.2.19}$$

This must be viewed as an equation for Ψ, given any two operators $\hat{A}$ and $\hat{B}$. Moreover, $\langle\hat{A}\rangle$ and $\langle\hat{B}\rangle$ are constants that happen to be Ψ dependent. What is λ, physically? The absolute value of λ is in fact fixed by the equation. Taking the norm of both sides, we get

$$\Delta B = |\lambda|\Delta A \quad \Rightarrow \quad |\lambda| = \frac{\Delta B}{\Delta A}. \tag{15.2.20}$$

The value of λ fixes the uncertainties ΔA and ΔB since the product $\Delta A \Delta B$ is, at saturation, equal to the right-hand side in the inequality and thus fixed once $|\Psi\rangle$ is known.

The saturation condition (15.2.19) can be written as an eigenvalue equation. By moving the $\hat{A}$ operator to the left-hand side and $\langle\hat{B}\rangle$ to the right-hand side, we have

$$(\hat{B} - i\lambda\hat{A})|\Psi\rangle = (\langle\hat{B}\rangle - i\lambda\langle\hat{A}\rangle)|\Psi\rangle. \tag{15.2.21}$$

Since the factor multiplying $|\Psi\rangle$ on the right-hand side is a number, this is indeed an eigenvalue equation. It is a curious one, however, since the operator $\hat{B} - i\lambda\hat{A}$ is *not* Hermitian. The eigenvalues are not real, and they are not expected to be quantized either. The presence of expectation values on the right-hand side may obscure the fact that this *is* a standard eigenvalue problem. Indeed, consider the related equation

$$(\hat{B} - i\lambda\hat{A})|\Psi\rangle = (b - i\lambda a))|\Psi\rangle, \tag{15.2.22}$$

with b and a *real* constants to be determined. This is the familiar eigenvalue problem. Taking expectation value by applying $\langle\Psi|$ to both sides of the equation, we get

$$\langle\hat{B}\rangle - i\lambda\langle\hat{A}\rangle = b - i\lambda a. \tag{15.2.23}$$

Since a, b, $\langle\hat{A}\rangle$, $\langle\hat{B}\rangle$, and λ are real, this equation implies $b = \langle\hat{B}\rangle$ and $a = \langle\hat{A}\rangle$. Therefore, when solving (15.2.21) you can simply take $\langle\hat{A}\rangle$, $\langle\hat{B}\rangle$ to be numbers, and if you get a solution, those numbers will indeed be the expectation values of the operators in your solution!

The classic illustration of this saturation condition is that for the $\hat{x}$, $\hat{p}$ uncertainty inequality $\Delta x \Delta p \geq \hbar/2$. Let us consider this next.

Example 15.3. *Saturating the position-momentum uncertainty inequality.* We are looking here for wave functions $\psi(x)$ for which the uncertainties Δx and Δp multiply exactly to $\hbar/2$, thus saturating the inequality (15.2.6). Such wave functions are sometimes called *minimum uncertainty* states. More precisely, since the product $\Delta x \Delta p$ is fixed, for a fixed momentum uncertainty the states have the minimum position uncertainty, and for a fixed position uncertainty, the states have a minimum momentum uncertainty. The states here, having uncertainties in x and p, are neither position nor momentum eigenstates.

Using equation (15.2.19) with $\hat{A} = \hat{x}$ and $\hat{B} = \hat{p}$, the condition for saturation is

$$(\hat{p} - \langle \hat{p} \rangle \mathbb{1})|\psi\rangle = i\lambda\, (\hat{x} - \langle \hat{x} \rangle \mathbb{1})|\psi\rangle. \tag{15.2.24}$$

As discussed above, in searching for ψ we can treat $\langle \hat{p} \rangle$ and $\langle \hat{x} \rangle$ as constants, which we will call p_0 and x_0, to have more suggestive notation. If we solve the equation, these constants will in fact be, respectively, the $\hat{p}$ and $\hat{x}$ expectation values on ψ. Using the coordinate space description of ψ, the equation reads

$$\left(\frac{\hbar}{i}\frac{d}{dx} - p_0\right)\psi(x) = i\lambda\, (x - x_0)\psi(x). \tag{15.2.25}$$

The wave function $\psi(x)$ is in fact a function of x_0, p_0 and λ. With a little rearrangement, the differential equation becomes

$$\frac{d\psi}{dx} = \left(\frac{ip_0}{\hbar} - \frac{\lambda}{\hbar}(x - x_0)\right)\psi, \tag{15.2.26}$$

and integration gives

$$\ln \psi(x) = \frac{ip_0 x}{\hbar} - \frac{\lambda(x - x_0)^2}{2\hbar} + C', \tag{15.2.27}$$

with C' a constant of integration. Calling $e^{C'} = C$ the wave function takes the form

$$\psi(x) = C \exp\left(-\frac{\lambda(x-x_0)^2}{2\hbar} + \frac{ip_0 x}{\hbar}\right), \quad \lambda \in \mathbb{R}. \tag{15.2.28}$$

Now we see that λ, known to be real, should be positive in order for $\psi(x)$ to be normalizable. The wave function is a Gaussian centered at x_0, with a plane-wave modulating factor $\exp(ip_0x/\hbar)$. This is the minimum uncertainty wave packet. On account of (15.2.20), we now have

$$\lambda = \frac{\Delta p}{\Delta x}. \tag{15.2.29}$$

As $\lambda \to 0$, we have $\Delta p \to 0$, and the wave function becomes a plane wave with momentum p_0. As $\lambda \to \infty$, the wave function localizes at $x = x_0$, and it approaches a position eigenstate. We can write λ in terms of the more physical position uncertainty. Recalling that saturation implies $\Delta x \Delta p = \hbar/2$, we find that

$$\lambda = \frac{\hbar}{2}\frac{1}{(\Delta x)^2}. \tag{15.2.30}$$

We can then rewrite the wave function as follows:

$$\psi(x) = \frac{1}{[2\pi(\Delta x)^2]^{1/4}} \exp\left(-\frac{(x-x_0)^2}{4(\Delta x)^2} + \frac{ip_0 x}{\hbar}\right), \quad x_0 = \langle \hat{x} \rangle, \; p_0 = \langle \hat{p} \rangle. \tag{15.2.31}$$

We have included in this formula the normalization factor. This is our final form for the general Gaussian minimum uncertainty packet. Rewriting equation (15.2.24) with all operators on the left-hand side,

$$(\hat{p} - i\lambda\,\hat{x})|\psi\rangle = (\langle \hat{p} \rangle - i\lambda\langle \hat{x} \rangle)|\psi\rangle, \tag{15.2.32}$$

it becomes clear that the minimum uncertainty packet is in fact an eigenstate of the *non-Hermitian* operator $\hat{p} - i\lambda\,\hat{x}$ with complex eigenvalue $p_0 - i\lambda x_0$. For a fixed $\lambda \geq 0$, eigenstates exist for any complex eigenvalue, since p_0 and x_0 can be fixed to any value. The minimum uncertainty states are some kind of generalized coherent states. Coherent states of the harmonic oscillator, to be studied in chapter 17, are eigenstates of the non-Hermitian annihilation operator $\hat{a} \sim \hat{p} - im\omega\,\hat{x}$, with arbitrary complex eigenvalues! These are minimum uncertainty wave functions with fixed $\lambda = m\omega$, and thus, from (15.2.30), $\Delta x = \sqrt{\frac{\hbar}{2m\omega}}$.