18.9 No-Cloning Property

It is often said that one cannot copy or clone a quantum state. While the statement captures the essence of a key quantum property, the precise version of the *no-cloning* property of quantum mechanics is more nuanced. A quantum cloning machine can be designed to clone a few states in a finite-dimensional state space, but a machine that clones arbitrary states cannot be built.

To understand the result properly, one must define what one means by cloning of a quantum state $|\psi\rangle \in V$, with V some vector space. A quantum state is the state of some quantum system—a particle or an atom, for example. In cloning there is no magical device that, for example, takes an electron in some quantum state and creates another electron in that quantum state. We must start with *two* electrons—call them electron one and electron two. Assume electron one is in some state that we want cloned. The cloning machine must copy the state of electron one into electron two without changing the state of electron one. This is analogous to the way a photocopier works: there is a page with information, the original, and there is a blank page. The machine copies the information of the original onto the blank page without altering the original. This is why the no-cloning property is sometimes referred to as the *no-xeroxing* property. Recall that in teleportation we copied a state, but it was done at the cost of destroying the original state.

For an arbitrary spin one-half state $|\psi\rangle = a_+|+\rangle + a_-|-\rangle$ to be cloned, we would require a second spin one-half particle in some "blank" state $|b\rangle$. The blank state is just some arbitrary but fixed state of the spin, perhaps $|+\rangle$ or $|-\rangle$. As we try cloning different states $|\psi\rangle$, the blank state is kept fixed. The cloning machine must act as follows:

$$\text{Cloning machine: } (a_+|+\rangle + a_-|-\rangle) \otimes |b\rangle \;\rightarrow\; e^{i\phi}\,(a_+|+\rangle + a_-|-\rangle) \otimes (a_+|+\rangle + a_-|-\rangle), \tag{18.9.1}$$

with ϕ an arbitrary phase. If perfectly functional, for a fixed blank state $|b\rangle$ the machine must implement this map for all values of a_+ and a_-. The inclusion of an arbitrary phase in the final state will be seen to be immaterial, as it does not help get cloning to work. The above action (18.9.1) with the blank state set equal to $|+\rangle$ is represented in figure 18.4.

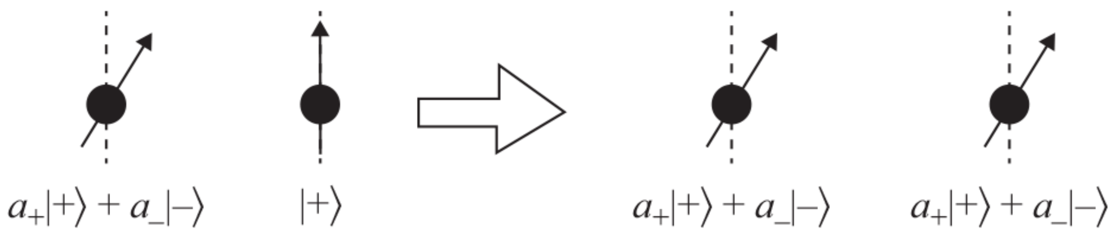

Figure 18.4
Cloning machine, represented by an arrow, copies the state $a_+|+\rangle + a_-|-\rangle$ of the left particle onto the second particle, the blank particle, initially in the state $|+\rangle$.

Since quantum mechanical evolution is unitary, we assume that the cloning machine is just a unitary operator U acting on the tensor product $V \otimes V$. If, instead, the machine involved measurements, the output would not be deterministic, and we do not want this. In general, for a normalized state $|\psi\rangle \in V$ to be copied, and a blank state $|b\rangle \in V$ also normalized, we must have

$$U:\ |\psi\rangle \otimes |b\rangle \ \rightarrow\ e^{i\phi}|\psi\rangle \otimes |\psi\rangle, \quad \langle\psi|\psi\rangle = 1,\ \langle b|b\rangle = 1, \tag{18.9.2}$$

with ϕ a phase the could depend on $|\psi\rangle$ and $|b\rangle$. The no-cloning property is summarized by the statement of the following theorem:

Theorem 18.9.1. *For the arbitrary but fixed, normalized state $|b\rangle \in V$, there is no unitary operator $U \in \mathcal{L}(V \otimes V)$ that implements the map (18.9.2) for arbitrary normalized $|\psi\rangle \in V$.*

Corollary. *In a vector space V of dimension n, the maximal number of states that can be cloned by a unitary U is n. These vectors comprise an orthonormal basis of V.*

Proof. Consider the map (18.9.2) for some fixed state $|\psi_1\rangle$ to be copied:

$$U:\ |\psi_1\rangle|b\rangle \rightarrow e^{i\phi_1}|\psi_1\rangle|\psi_1\rangle. \tag{18.9.3}$$

There is certainly a unitary $U: V \otimes V \rightarrow V \otimes V$ that implements this map. This is clear because the states to the left and to the right of the arrow have the same norm:

$$\begin{aligned} \big\| |\psi_1\rangle|b\rangle \big\|^2 &= \langle\psi_1|\psi_1\rangle\langle b|b\rangle = 1, \\ \big\| e^{i\phi_1}|\psi_1\rangle|\psi_1\rangle \big\|^2 &= \langle\psi_1|\psi_1\rangle\langle\psi_1|\psi_1\rangle = 1. \end{aligned} \tag{18.9.4}$$

Indeed, we can explain this more generally, in a way that helps build the rest of the argument. In an arbitrary space W (corresponding to $V \otimes V$ in our case of interest), a unitary operator U can be easily constructed that maps one chosen unit vector $|e_1\rangle$ to some other unit vector $|f_1\rangle$. For this we use Gram-Schmidt to extend $|e_1\rangle$ to an orthonormal basis $|e_i\rangle$, $i = 1, \ldots,$ dim W, and similarly, we extend $|f_1\rangle$ to an orthonormal basis $|f_i\rangle$, $i = 1, \ldots,$ dim W. The requisite unitary operator U is then $U = \sum_i |f_i\rangle\langle e_i|$. Similarly, given $p \leq$ dim W orthonormal basis vectors $|e_1\rangle, \ldots, |e_p\rangle$ that are mapped, one by one, into p orthonormal basis vectors $|f_1\rangle, \ldots, |f_p\rangle$, there is also a unitary operator that accomplishes this. Again, the operator is constructed by completing the set of vectors into a full orthonormal set.

Applied to the situation in (18.9.3), the above argument shows the existence of a unitary that maps one unit vector in $V \otimes V$, the state to the left of the arrow, to another unit vector in $V \otimes V$, the state to the right of the arrow. Having shown there is a unitary operator U that realizes the map (18.9.3), we ask if the unitary operator can be modified so that it also clones a second state $|\psi_2\rangle$:

$$\begin{aligned} U: \quad & |\psi_1\rangle|b\rangle \to e^{i\phi_1}|\psi_1\rangle|\psi_1\rangle, \\ & |\psi_2\rangle|b\rangle \to e^{i\phi_2}|\psi_2\rangle|\psi_2\rangle. \end{aligned} \tag{18.9.5}$$

The argument sketched above tells us that the unitary U exists if $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal and thus orthonormal vectors in V. This implies that the two $V \otimes V$ states to the left of the arrows are orthonormal and so are the resulting cloned states to the right of the arrows. It is clear now that a unitary U exists that can clone a chosen set of dim V orthonormal states in V. But as we see next, we cannot do better.

For this, consider again the case of two states to be cloned as in (18.9.5), and recall that a unitary operator preserves inner products (see (14.5.21)). If a unitary exists, the inner product of the states to be cloned must equal the inner product of the cloned states:

$$\Big\langle |\psi_1\rangle|b\rangle, |\psi_2\rangle|b\rangle \Big\rangle = \Big\langle e^{i\phi_1}|\psi_1\rangle|\psi_1\rangle, e^{i\phi_2}|\psi_2\rangle|\psi_2\rangle \Big\rangle. \tag{18.9.6}$$

This gives the constraint

$$\langle\psi_1|\psi_2\rangle = e^{i(\phi_2-\phi_1)}(\langle\psi_1|\psi_2\rangle)^2. \tag{18.9.7}$$

One solution of this constraint is $\langle\psi_1|\psi_2\rangle = 0$. We knew this: cloning the two states is possible if the states are orthogonal. The other solution is

$$\langle\psi_1|\psi_2\rangle = e^{i(\phi_1-\phi_2)}. \tag{18.9.8}$$

This condition implies $|\langle\psi_1|\psi_2\rangle|^2 = 1$. But the Schwarz inequality requires $|\langle\psi_1|\psi_2\rangle|^2 \leq \langle\psi_1|\psi_1\rangle\langle\psi_2|\psi_2\rangle = 1$. Since the inequality is saturated, $|\psi_2\rangle$ equals $|\psi_1\rangle$ up to a phase. Then $|\psi_2\rangle$ is not a new state we can clone. This argument implies that if we have a set of clonable states, a *new* state can only be cloned if it is orthogonal to all the clonable states.

Thus, we can begin by picking one state to clone and then add successively orthonormal states that we can clone. Imagine now we have a U that clones dim V orthonormal states, the maximal number of orthonormal states that can be obtained in V. No nontrivial linear combination of these clonable states can be cloned because no such state is orthogonal to all the clonable states. This completes the proof of the no-cloning theorem, as well as that of the corollary.

□

Exercise 18.18. *Alice and Bob are far away from each other but share an entangled pair of spin one-half particles in the singlet state. They aim to communicate information by agreeing that Alice will measure her particle along x if she wins the lottery and along z if she loses. Convince yourself that the strategy will not work unless Bob has a quantum cloning machine.*