19.7 The Three-Dimensional Isotropic Oscillator

For a particle of mass m in a three-dimensional isotropic harmonic oscillator, the Hamiltonian $\hat{H}$ defining the system takes the form

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{\hat{p}_y^2}{2m} + \frac{\hat{p}_z^2}{2m} + \tfrac{1}{2}m\omega^2(x^2 + y^2 + z^2). \tag{19.7.1}$$

Isotropy means that the system does not single out any special direction in space. The isotropy of the above Hamiltonian is a consequence of a single frequency ω appearing in the potential term: the frequencies for the independent x, y, and z oscillations are identical. This Hamiltonian is the sum of three commuting, one-dimensional Hamiltonians,

$$\hat{H} = \hat{H}_x + \hat{H}_y + \hat{H}_z, \tag{19.7.2}$$

where

$$\hat{H}_x = \frac{\hat{p}_x^2}{2m} + \tfrac{1}{2}m\omega^2x^2, \quad \hat{H}_y = \frac{\hat{p}_y^2}{2m} + \tfrac{1}{2}m\omega^2y^2 \quad \hat{H}_z = \frac{\hat{p}_z^2}{2m} + \tfrac{1}{2}m\omega^2z^2. \tag{19.7.3}$$

The Hamiltonian $\hat{H}$ does not describe three particles oscillating in different directions; it describes just one particle. Interestingly, as we will see below, tensor products are relevant here, the state space being the tensor product of three one-dimensional oscillator state spaces. The isotropy of the oscillator is manifest when we note that $x^2 + y^2 + z^2 = r^2$ and rewrite the Hamiltonian as follows:

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + \tfrac{1}{2}m\omega^2r^2. \tag{19.7.4}$$

Both terms are manifestly rotational invariant. The potential term here is a central potential:

$$V(r) = \tfrac{1}{2}m\omega^2r^2. \tag{19.7.5}$$

Therefore, the spectrum of this system consists of multiplets of angular momentum—in fact, an infinite number of them. As we will see, however, the spectrum has degeneracies: multiplets with different values of ℓ but the same energy. These degeneracies are the result of a *hidden symmetry*, a symmetry that is far from obvious in the Hamiltonian. The quantum three-dimensional oscillator is a lot more symmetric than the infinite spherical well, which had no degeneracies whatsoever.

To describe the three-dimensional oscillator, we can use the creation and annihilation operators $\hat{a}_x^\dagger, \hat{a}_y^\dagger, \hat{a}_z^\dagger$ and $\hat{a}_x$, $\hat{a}_y$, $\hat{a}_z$ associated with one-dimensional oscillations in the x-, y-, and z-directions. The Hamiltonian then takes the form

$$\hat{H} = \hbar\omega\big(\hat{N}_x + \hat{N}_y + \hat{N}_z + \tfrac{3}{2}\big) \;=\; \hbar\omega\big(\hat{N} + \tfrac{3}{2}\big), \tag{19.7.6}$$

where we define the total number operator $\hat{N}$ as $\hat{N} \equiv \hat{N}_x + \hat{N}_y + \hat{N}_z$. Here, $\hat{N}_x = \hat{a}_x^\dagger \hat{a}_x$, $\hat{N}_y = \hat{a}_y^\dagger \hat{a}_y$, and $\hat{a}_x^\dagger, \hat{a}_y^\dagger, \hat{a}_z^\dagger$ are the number operators associated with the three directions x, y, and z, respectively.

Tensor products are relevant to the three-dimensional oscillator. We have used tensor products to describe multiparticle states. Tensor products, however, are also relevant to single particles if they have degrees of freedom that live in different spaces or more than one set of attributes, each of which is described by states in some vector space. For the three-dimensional oscillator, the Hamiltonian is the sum of commuting Hamiltonians of one-dimensional oscillators for the x, y, and z directions. Thus, the general states are obtained by tensoring the state spaces $\mathcal{H}_x$, $\mathcal{H}_y$, and $\mathcal{H}_z$ of the three independent oscillators. It is a single particle oscillating, but the description of what it is doing entails saying what it is doing in each of the independent directions. Thus, for the three-dimensional state space $\mathcal{H}_{3D}$ we write

$$\mathcal{H}_{3D} \;=\; \mathcal{H}_x \otimes \mathcal{H}_y \otimes \mathcal{H}_z. \tag{19.7.7}$$

In this notation the total Hamiltonian in (19.7.2) is written as

$$\hat{H} \;=\; \hat{H}_x \otimes \mathbb{1} \otimes \mathbb{1} \;+\; \mathbb{1} \otimes \hat{H}_y \otimes \mathbb{1} \;+\; \mathbb{1} \otimes \mathbb{1} \otimes \hat{H}_z. \tag{19.7.8}$$

The vacuum state $|0\rangle$ of the three-dimensional oscillator can be viewed as

$$|0\rangle \;\equiv\; |0\rangle_x \otimes |0\rangle_y \otimes |0\rangle_z. \tag{19.7.9}$$

The associated ground state wave function φ is

$$\varphi(x, y, z) = \langle x| \otimes \langle y| \otimes \langle z| \, |0\rangle = \langle x|0\rangle_x \langle y|0\rangle_y \langle z|0\rangle_z = \varphi_0(x)\varphi_0(y)\varphi_0(z), \tag{19.7.10}$$

where φ_0 is the ground state wave function of the one-dimensional oscillator. Recalling the form of the (nonnormalized) basis states for $\mathcal{H}_x$, $\mathcal{H}_y$, and $\mathcal{H}_z$,

$$\begin{aligned}
&\mathcal{H}_x: \ (\hat{a}_x^\dagger)^{n_x}|0\rangle_x, \ n_x = 0, 1, \ldots, \\
&\mathcal{H}_y: \ (\hat{a}_y^\dagger)^{n_y}|0\rangle_y, \ n_y = 0, 1, \ldots, \\
&\mathcal{H}_z: \ (\hat{a}_z^\dagger)^{n_z}|0\rangle_z, \ n_z = 0, 1, \ldots,
\end{aligned} \tag{19.7.11}$$

the basis states for the three-dimensional oscillator state space are

$$(\hat{a}_x^\dagger)^{n_x}|0\rangle_x \otimes (\hat{a}_y^\dagger)^{n_y}|0\rangle_y \otimes (\hat{a}_z^\dagger)^{n_z}|0\rangle_z, \quad n_x, n_y, n_z \in \{0, 1, \ldots\}. \tag{19.7.12}$$

This is what we would expect intuitively: we simply pile arbitrary numbers of $\hat{a}_x^\dagger, \hat{a}_y^\dagger$, and $\hat{a}_z^\dagger$ operators on the vacuum. For brevity, we write such basis states as

$$(\hat{a}_x^\dagger)^{n_x}(\hat{a}_y^\dagger)^{n_y}(\hat{a}_z^\dagger)^{n_z}|0\rangle. \tag{19.7.13}$$

Each of these states has a wave function that is the product of x-, y-, and z-dependent wave functions. Once we form superpositions of such states, the total wave function can no longer be factored into x-, y-, and z dependent wave functions. The x, y, and z-dependences become "entangled." In that sense these are the analogues of entangled states of three particles.

We are ready to construct the states of the three-dimensional isotropic oscillator. The key property is that the states must organize themselves into representations of angular momentum, the multiplets we explored in section 19.3. Since angular momentum operators commute with the Hamiltonian, angular momentum multiplets represent degenerate states. As we will see, the multiplets that appear at each level can be deduced by simple reasoning.

We already built the ground state, which is a single state with $\hat{N}$ eigenvalue $N = 0$. All other states have higher energies, so this state must be, by itself, a representation of angular momentum. It can only be the singlet $\ell = 0$. Thus, we have, from (19.7.6),

$$N=0,\ E=\tfrac{3}{2}\hbar\omega,\quad |0\rangle \longleftrightarrow \ell=0. \tag{19.7.14}$$

The states with $N = 1$ have $E=\frac{5}{2}\hbar\omega$ and are

$$\hat{a}_x^\dagger|0\rangle,\ \hat{a}_y^\dagger|0\rangle,\ \hat{a}_z^\dagger|0\rangle. \tag{19.7.15}$$

These three degenerate states fit precisely into an $\ell = 1$ multiplet (a triplet). There is, in fact, no other possibility. Any higher-ℓ multiplet has too many states. Moreover, we cannot have three singlets because this would mean a degeneracy in the bound state spectrum of the $\ell = 0$ radial Schrödinger equation, which is impossible. The $\ell = 0$ ground state and the $\ell = 1$ triplet at the first excited level are indicated in figure 19.7.

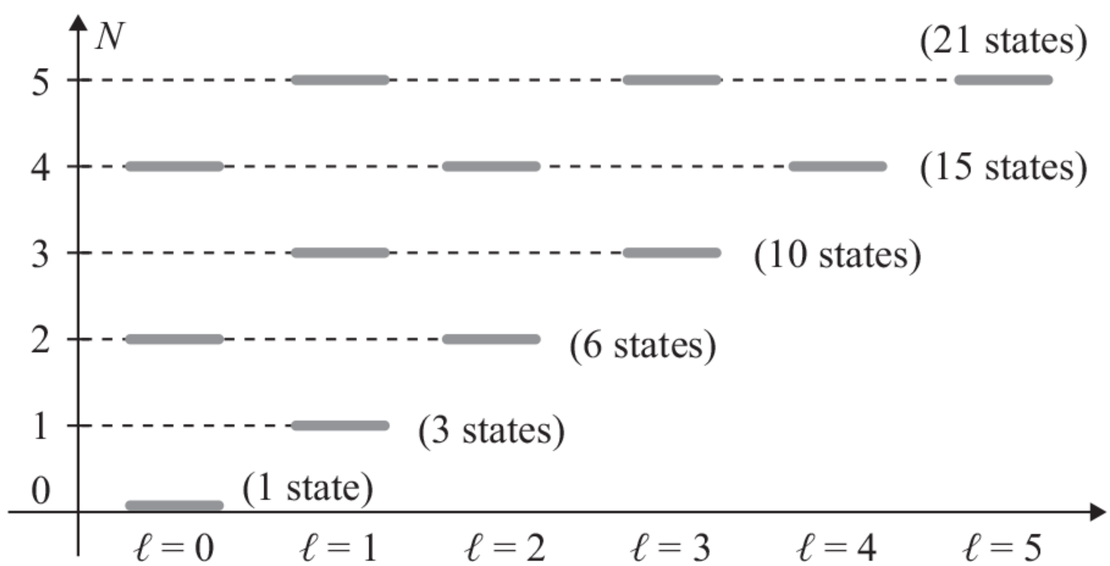

Figure 19.7

Spectral diagram for angular momentum multiplets in the three-dimensional isotropic harmonic oscillator. The energy is $E=\hbar\omega(N+\frac{3}{2})$, with N the total number eigenvalue. At level N the maximal ℓ also equals N.

Let us proceed now with the states at $N = 2$ or $E=\frac{7}{2}\hbar\omega$. There are six states:

$$\hat{a}_x^\dagger\hat{a}_x^\dagger|0\rangle,\ \hat{a}_y^\dagger\hat{a}_y^\dagger|0\rangle,\ \hat{a}_z^\dagger\hat{a}_z^\dagger|0\rangle,\ \hat{a}_x^\dagger\hat{a}_y^\dagger|0\rangle,\ \hat{a}_x^\dagger\hat{a}_z^\dagger|0\rangle,\ \hat{a}_y^\dagger\hat{a}_z^\dagger|0\rangle. \tag{19.7.16}$$

To help ourselves in trying to find the angular momentum multiplets, recall that the number of states # for a given ℓ is $2\ell + 1$:

ℓ	0	1	2	3	4	5	6	7
#	1	3	5	7	9	11	13	15

We cannot use the triplet twice: this would imply a degeneracy in the spectrum of the $\ell = 1$ radial equation. In general, at any fixed energy level any ℓ multiplet that appears can only appear once. Therefore, the only way to get six states is having five from $\ell = 2$ and one from $\ell = 0$:

$$\text{Six } N=2 \text{ states: } (\ell=2) \oplus (\ell=0). \tag{19.7.17}$$

We use the direct sum, not the tensor product. The six states define a six-dimensional vector space spanned by five vectors in $\ell = 2$ and one vector in $\ell = 0$.

Let us continue to determine the pattern. At $N = 3$, with $E = \frac{9}{2}\hbar\omega$, we actually have ten states (count them!). It would seem now that there are two options for multiplets:

$$(\ell=3) \oplus (\ell=1) \text{ or } (\ell=4) \oplus (\ell=0). \tag{19.7.18}$$

We can see that the second option is problematic. If true, an $\ell = 3$ multiplet, which has not appeared yet, would not arise at this level. If it appeared eventually, it would do so at a higher energy, and we would have the lowest $\ell = 3$ multiplet with higher energy than the lowest $\ell = 4$ multiplet. This is not possible, as discussed at the end of section 19.4. You may think perhaps that $\ell = 3$ multiplets never appear, avoiding the inconsistency, but this is not true. At any rate we will provide below a rigorous argument that confirms the first option is the one realized. Therefore, the ten degenerate states at $N = 3$ consist of the following multiplets:

$$\text{Ten } N=3 \text{ states: } (\ell=3) \oplus (\ell=1). \tag{19.7.19}$$

Let us do one more level. At $N = 4$ we find fifteen states. Instead of writing them out, let us count them without listing them. In fact, we can easily do the general case of arbitrary integer $N \geq 1$. The states we are looking for are of the form

$$(\hat{a}_x^\dagger)^{n_x}(\hat{a}_y^\dagger)^{n_y}(\hat{a}_z^\dagger)^{n_z}|0\rangle, \quad \text{with} \quad n_x+n_y+n_z=N. \tag{19.7.20}$$

The number of different solutions of $n_x + n_y + n_z = N$, with $n_x, n_y, n_z \geq 0$, is the number of degenerate states at total number N. To visualize this think of $n_x + n_y + n_z = N$ as the equation for a plane in three-dimensional space with axes n_x, n_y, n_z. Since no integer can be negative, we are looking for points with integer coordinates in the region of the plane that lies on the positive octant, as shown in figure 19.6. Starting at one of the three corners, say, $(n_x, n_y, n_z) = (N, 0, 0)$, we have one point, then moving toward the origin we encounter two points, then three, and so on until we find $N+1$ points on the (n_y, n_z) plane. Thus, the number deg(N) of degenerate states with number N is

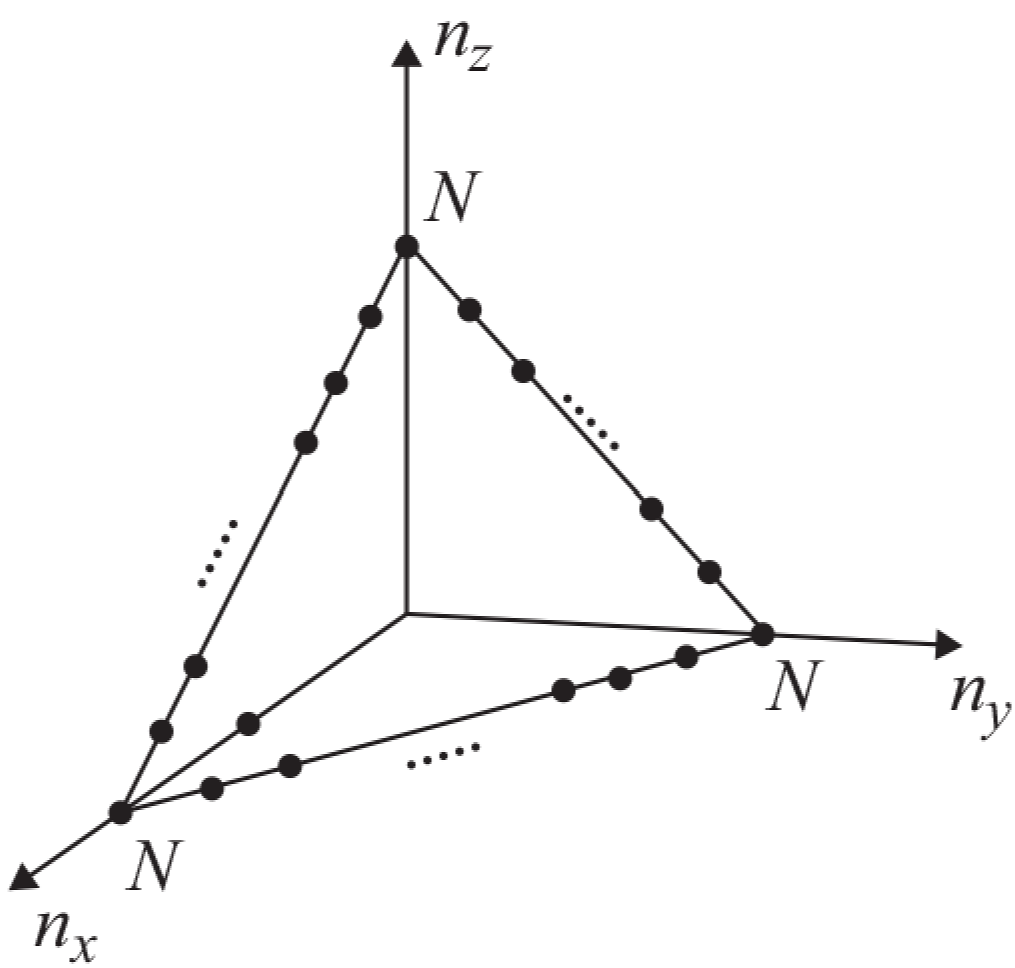

Figure 19.6
Counting the number of degenerate states with total number N in the isotropic harmonic oscillator.

$$\deg(N) = 1 + 2 + \cdots + (N+1) \ = \ \frac{(N+1)(N+2)}{2}. \tag{19.7.21}$$

Back to the $N = 4$ level, deg(4) = 15. We rule out a single $\ell = 7$ multiplet since the multiplets $\ell = 4, 5, 6$ have not appeared yet. No two of those multiplets can appear simultaneously either, for it would imply that the ground states of two potentials with different ℓ coincide. Therefore, the only one that can appear is the lowest-ℓ multiplet, the $\ell = 4$ multiplet, with nine states. The remaining six states must then appear as $\ell = 2$ plus $\ell = 0$. we then have

$$\text{Fifteen } N = 4 \text{ states: } (\ell = 4) \oplus (\ell = 2) \oplus (\ell = 0). \tag{19.7.22}$$

We can see that ℓ jumps by steps of two, starting from the maximal ℓ. This is in fact the rule. It is quickly confirmed that the 21 states with $N = 5$ (deg(5) = 21), would arise from $(\ell = 5) \oplus (\ell = 3) \oplus (\ell = 1)$. All this is shown in figure 19.7.

Some of the structure of angular momentum multiplets can be seen more explicitly by trading the $\hat{a}_x$ and $\hat{a}_y$ operators for complex linear combinations $\hat{a}_L$ and $\hat{a}_R$:

$$\hat{a}_L \ = \ \frac{1}{\sqrt{2}}(\hat{a}_x + i\hat{a}_y), \ \ \hat{a}_R \ = \ \frac{1}{\sqrt{2}}(\hat{a}_x - i\hat{a}_y). \tag{19.7.23}$$

The $\hat{a}_L^\dagger$ and $\hat{a}_R^\dagger$ operators are obtained by taking the Hermitian conjugate of those above. L and R operators commute with each other while, as expected, $[\hat{a}_L, \hat{a}_L^\dagger] = [\hat{a}_R, \hat{a}_R^\dagger] = 1$. With number operators $\hat{N}_R = \hat{a}_R^\dagger \hat{a}_R$ and $\hat{N}_L = \hat{a}_L^\dagger \hat{a}_L$, the Hamiltonian reads

$$\hat{H} = \hbar\omega(\hat{N}_R + \hat{N}_L + \hat{N}_z + \tfrac{3}{2}). \tag{19.7.24}$$

More importantly, the z-component $\hat{L}_z$ of angular momentum takes the simple form

$$\hat{L}_z = \hbar(\hat{N}_R - \hat{N}_L). \tag{19.7.25}$$

The above claims are confirmed in problem 19.7. Note that $\hat{a}_z$ carries no z-component of angular momentum. States are built acting with arbitrary numbers of $\hat{a}_L{}^\dagger$, $\hat{a}_R{}^\dagger$, and $\hat{a}_z{}^\dagger$ operators on the vacuum. The $N = 1$ states are then presented as

$$\hat{a}_R^\dagger|0\rangle,\ \hat{a}_z^\dagger|0\rangle,\ \hat{a}_L^\dagger|0\rangle. \tag{19.7.26}$$

We see that the first state has $\hat{L}_z$ eigenvalue $L_z = \hbar$, the second $L_z = 0$, and the third $L_z = -\hbar$, exactly the three expected values of the $\ell = 1$ multiplet previously identified. For number $N = 2$, the state with highest $\hat{L}_z$ eigenvalue is $(\hat{a}_R^\dagger)^2|0\rangle$, which has $L_z = 2\hbar$. This shows that the highest ℓ multiplet is $\ell = 2$. For arbitrary positive integer number N, the state with highest $\hat{L}_z$ eigenvalue is $(\hat{a}_R^\dagger)^N|0\rangle$, which has $L_z = \hbar N$. This shows we must have an $\ell = N$ multiplet. This is in fact what we got before. We can also understand why the top multiplet $\ell = N$ is accompanied by an $\ell = N - 2$ multiplet and no $\ell = N - 1$ multiplet. Consider the above state with maximal $L_z/\hbar$ equal to N and then the states with one and two units less of $L_z/\hbar$:

$$\begin{aligned}
L_z/\hbar = N: &\qquad (\hat{a}_R^\dagger)^N|0\rangle,\\
L_z/\hbar = N-1: &\qquad (\hat{a}_R^\dagger)^{N-1}\hat{a}_z^\dagger|0\rangle,\\
L_z/\hbar = N-2: &\qquad (\hat{a}_R^\dagger)^{N-2}(\hat{a}_z^\dagger)^2|0\rangle,\ \ (\hat{a}_R^\dagger)^{N-1}\,\hat{a}_L^\dagger|0\rangle.
\end{aligned} \tag{19.7.27}$$

While there is only one state with one unit less of $L_z/\hbar$, there are two states with two units less. One linear combination of these two states must belong to the $\ell = N$ multiplet, but the other linear combination must be the top state of an $\ell = N - 2$ multiplet! This is the reason for the jump of two units.

For arbitrary N we can see why the total number of states deg(N) in (19.7.21) can be reproduced by ℓ multiplets skipping by two:

$$\begin{aligned}
N\text{ odd}:\ \ \deg(N) &= \underbrace{1+2}_{\ell=1}+\underbrace{3+4}_{\ell=3}+\underbrace{5+6}_{\ell=5}+\underbrace{7+8}_{\ell=7}+\cdots+\underbrace{N+(N+1)}_{\ell=N},\\
N\text{ even}:\ \ \deg(N) &= \underbrace{1}_{\ell=0}+\underbrace{2+3}_{\ell=2}+\underbrace{4+5}_{\ell=4}+\underbrace{6+7}_{\ell=6}+\cdots+\underbrace{N+(N+1)}_{\ell=N}.
\end{aligned} \tag{19.7.28}$$

The degeneracy of the spectrum is “explained” if we identify operators that commute with the Hamiltonian and connect the various ℓ multiplets that appear for a fixed number N. Such operators are symmetries of the theory. Consider, for example,

$$K \equiv \hat{a}_R^\dagger\hat{a}_L. \tag{19.7.29}$$

It is simple to check that K commutes with the Hamiltonian. With more work one can show that acting on the top state of the $\ell = N - 2$ multiplet, K gives the top state of the $\ell = N$ multiplet (problem 19.7). This means that these two multiplets are degenerate. A more systematic analysis of the degeneracy would begin by finding all the independent operators that commute with the Hamiltonian. We will not do this here.

The two-dimensional isotropic oscillator also has degeneracies that ask for an explanation. In this case one can show that there are three operators $\hat{J}_i$ that commute with the Hamiltonian and in fact form an algebra of angular momentum (problem 19.8). This is *not* the algebra of orbital angular momentum because there are no such operators for motion in a plane; while there is an $\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$, there is no $\hat{L}_x$ or $\hat{L}_y$ because we have no $(\hat{z},\ \hat{p}_z)$ operators. The angular momentum operators $\hat{J}_i$ here generate a *hidden* symmetry, and degenerate states must fall into multiplets of $\hat{\mathbf{J}}$. It turns out that at each energy level of the two-dimensional isotropic oscillator there is a single multiplet of $\hat{\mathbf{J}}$. In fact the full spectrum is precisely the list of *all* possible multiplets $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. This is clearly a remarkable system.