17.4 Photon States

As an application of the harmonic oscillator and coherent states, we will now consider electromagnetic oscillations in a cavity. As it turns out, their quantum description is through a harmonic oscillator whose states are photon states. Moreover, the coherent states of this oscillator turn out to represent approximate classical states of the electromagnetic field.

The energy E in a classical electromagnetic field is obtained by adding the contributions of the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$:

$$E = \int d^3x \, \frac{1}{8\pi} \Big[\mathbf{E}^2(\mathbf{x},t) + \mathbf{B}^2(\mathbf{x},t) \Big]. \tag{17.4.1}$$

Now consider a rectangular cavity of volume V with a single mode of the electromagnetic field—namely, a single frequency ω, with corresponding wave number $k = \omega/c$ and a single polarization state. The electromagnetic fields form a standing wave in which the electric and magnetic fields are out of phase. They can take the form

$$E_x(z,t) = \sqrt{\frac{8\pi}{V}} \, \omega \, q(t) \sin kz, \quad B_y(z,t) = \sqrt{\frac{8\pi}{V}} \, p(t) \cos kz. \tag{17.4.2}$$

Here, $q(t)$ and $p(t)$ are classical time-dependent functions. As we will see below, in the quantum theory they become Heisenberg operators $\hat{q}(t)$ and $\hat{p}(t)$ satisfying $[\hat{q}(t), \hat{p}(t)] = i\hbar$.

The energy (17.4.1) associated with the above fields is quickly calculated, recalling that with periodic boundary conditions on the fields, the average of $(\sin kz)^2$ or $(\cos kz)^2$ over the volume V is 1/2. We then find that

$$E = \tfrac{1}{2}\big(p^2(t) + \omega^2 q^2(t)\big). \tag{17.4.3}$$

There is some funny business here with units. The variables $q(t)$ and $p(t)$ do not have their familiar units, as you can see from the expression for the energy. We are missing a quantity with units of mass that divides the p^2 contribution and multiplies the q^2 contribution. Here p has units of $\sqrt{E}$, and q has units of $T\sqrt{E}$. Still, the product of q and p has units of $\hbar$, which is useful. Since photons are massless particles, there is no quantity with units of mass that we can use. Note that the dynamical variable $q(t)$ is not a position; it is essentially the electric field. The dynamical variable $p(t)$ is not a momentum; it is essentially the magnetic field.

The quantum theory of this electromagnetic field uses the structure implied by the classical results above. From the energy above, we *postulate* a Hamiltonian $\hat{H}$ of the form

$$\hat{H} = \tfrac{1}{2}\big(\hat{p}^2 + \omega^2 \hat{q}^2\big), \tag{17.4.4}$$

with Schrödinger operators $\hat{q}$ and $\hat{p}$ that satisfy $[\hat{q}, \hat{p}] = i\hbar$ and associated Heisenberg operators $\hat{q}(t)$ and $\hat{p}(t)$ with the same commutator. As soon as we declare that the classical variables $q(t)$ and $p(t)$ are to become operators, the electric and magnetic fields in (17.4.2) become *field operators*, space and time-dependent operators. This oscillator is our familiar oscillator but with m set equal to one, which is allowed given the unusual units of $\hat{q}$ and $\hat{p}$. With the familiar (9.3.11) and $m = 1$, we have

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} + i\hat{p}) \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} - i\hat{p}), \qquad [\hat{a}, \hat{a}^\dagger] = 1. \tag{17.4.5}$$

It follows that

$$\hbar\omega\,\hat{a}^\dagger\hat{a} = \tfrac{1}{2}(\omega\hat{q} - i\hat{p})(\omega\hat{q} + i\hat{p}) = \tfrac{1}{2}(\hat{p}^2 + \omega^2\hat{q}^2 + i\omega[\hat{q}, \hat{p}]) = \tfrac{1}{2}(\hat{p}^2 + \omega^2\hat{q}^2 - \hbar\omega), \tag{17.4.6}$$

and comparing with (17.4.4), we can rewrite the Hamiltonian in terms of $\hat{a}$ and $\hat{a}^\dagger$:

$$H = \hbar\omega\Big(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}\Big) = \hbar\omega(\hat{N} + \tfrac{1}{2}). \tag{17.4.7}$$

This was the expected answer: this formula does not depend on m, and our setting $m = 1$ had no import. At this point we got photons: We *interpret* the state $|n\rangle$ of the above harmonic oscillator as the state with *n photons*. This state has energy $\hbar\omega(n + \frac{1}{2})$, which is, up to the zero-point energy $\hbar\omega/2$, the energy of n photons of energy $\hbar\omega$ each. A photon is the basic quantum of the electromagnetic field.

For more intuition we consider the electric field operator. For this we first note that

$$\hat{q} = \sqrt{\frac{\hbar}{2\omega}}\,(\hat{a} + \hat{a}^\dagger), \tag{17.4.8}$$

and the corresponding Heisenberg operator is, using (16.5.31),

$$\hat{q}(t) = \sqrt{\frac{\hbar}{2\omega}}\,(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}). \tag{17.4.9}$$

In quantum field theory—which is what we are doing here—the electric field is a Hermitian operator. Its form is obtained by substituting (17.4.9) into (17.4.2):

$$\boxed{\hat{E}_x(z,t) = \mathcal{E}_0 \left(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\right) \sin kz, \quad \mathcal{E}_0 = \sqrt{\frac{4\pi\hbar\omega}{V}}.} \tag{17.4.10}$$

This is a *field operator*: an operator that depends on position, in this case z, as well as on time. The coordinates x, y, or z are *not* operators in this analysis. The constant $\mathcal{E}_0$ is sometimes called the electric field of a photon.

A classical electric field can be identified as the expectation value of the electric field operator in the given photon state. We immediately see that in the n photon state $|n\rangle$ the expectation value of $\hat{E}_x$ vanishes! Indeed,

$$\langle \hat{E}_x(z,t)\rangle = \mathcal{E}_0 \left(\langle n|\hat{a}|n\rangle e^{-i\omega t} + \langle n|\hat{a}^\dagger|n\rangle e^{i\omega t}\right) \sin kz = 0, \tag{17.4.11}$$

since the matrix elements of $\hat{a}$ and $\hat{a}^\dagger$ vanish. Energy eigenstates of the photon field do not correspond to classical electromagnetic fields. Now consider the expectation value of the field in a coherent state $|\alpha\rangle$, with $\alpha \in \mathbb{C}$. This time, we get

$$\langle \hat{E}_x(z,t)\rangle = \mathcal{E}_0 \left(\langle \alpha|\hat{a}|\alpha\rangle e^{-i\omega t} + \langle \alpha|\hat{a}^\dagger|\alpha\rangle e^{i\omega t}\right) \sin kz. \tag{17.4.12}$$

Recalling that $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$,

$$\langle \hat{E}_x(z,t)\rangle = \mathcal{E}_0 \left(\alpha\, e^{-i\omega t} + \alpha^*\, e^{i\omega t}\right) \sin kz. \tag{17.4.13}$$

This is a standing wave. To make this clear, we write $\alpha = |\alpha|e^{i\theta}$ and find that

$$\langle \hat{E}_x(z,t)\rangle = 2\mathcal{E}_0\, \mathrm{Re}(\alpha e^{-i\omega t}) \sin kz = 2\mathcal{E}_0\, |\alpha|\, \cos(\omega t - \theta) \sin kz. \tag{17.4.14}$$

Coherent photon states with large $|\alpha|$ give rise to classical electric fields! In the state $|\alpha\rangle$, the expectation value of the number operator $\hat{N}$ is $|\alpha|^2$. Thus, the above electric field is the classical field associated with a quantum state with about $|\alpha|^2$ photons.