Formula Sheet

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- Conservation of probability

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}J(x,t) = 0$$

$$\rho(x,t) = |\psi(x,t)|^2 \; ; \qquad J(x,t) = \frac{\hbar}{2im}\left[\psi^*\frac{\partial}{\partial x}\psi - \psi\frac{\partial}{\partial x}\psi^*\right]$$

- Variational principle:

$$E_{gs} \,\le\, \frac{\int dx\,\psi^*(x)H\psi(x)}{\int dx\psi^*(x)\psi(x)} \equiv \langle H\rangle_\psi \quad \text{for all } \psi(x)$$

- Spin-1/2 particle:

$$\text{Stern-Gerlach :} \quad H = -\boldsymbol{\mu}\cdot\mathbf{B}\,, \qquad \boldsymbol{\mu} = g\,\frac{e\hbar}{2m}\,\frac{1}{\hbar}\mathbf{S} = \gamma\mathbf{S}$$

$$\mu_{_B} = \frac{e\hbar}{2m_e}\,, \qquad \boldsymbol{\mu}_e = -2\,\mu_{_B}\frac{\mathbf{S}}{\hbar}\,,$$

$$\text{In the basis } |1\rangle \equiv |z;+\rangle = |+\rangle = \begin{pmatrix}1\\0\end{pmatrix}\;,\;\; |2\rangle \equiv |z;-\rangle = |-\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

$$S_i = \frac{\hbar}{2}\sigma_i \quad \sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\;;\;\sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\;;\;\sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \quad \rightarrow \quad \big[S_i\,,\,S_j\big] = i\hbar\,\epsilon_{ijk}S_k \quad (\epsilon_{123} = +1)$$

$$\sigma_i\sigma_j = \delta_{ij}\mathbf{1} + i\epsilon_{ijk}\sigma_k \quad \rightarrow \quad (\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{b}) = \mathbf{a}\cdot\mathbf{b}\,\mathbf{1} + i\boldsymbol{\sigma}\cdot(\mathbf{a}\times\mathbf{b})$$

$$e^{i\mathbf{M}\theta} = \mathbf{1}\cos\theta + i\,\mathbf{M}\sin\theta\,, \quad \text{if } \mathbf{M}^2 = \mathbf{1}$$

$$\exp\big(i\,\mathbf{a}\cdot\boldsymbol{\sigma}\big) = \mathbf{1}\cos a + i\boldsymbol{\sigma}\cdot\left(\frac{\mathbf{a}}{a}\right)\,\sin a\,, \quad a = |\mathbf{a}|$$

$$\begin{aligned}\exp(i\theta\sigma_3)\,\sigma_1\,\exp(-i\theta\sigma_3) &= \sigma_1\cos(2\theta) - \sigma_2\sin(2\theta)\\ \exp(i\theta\sigma_3)\,\sigma_2\,\exp(-i\theta\sigma_3) &= \sigma_2\cos(2\theta) + \sigma_1\sin(2\theta)\;.\end{aligned}$$

$$S_{\mathbf{n}} = \mathbf{n}\cdot\mathbf{S} = n_xS_x + n_yS_y + n_zS_z = \tfrac{\hbar}{2}\,\mathbf{n}\cdot\boldsymbol{\sigma}\;.$$

$$(n_x, n_y, n_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\,, \qquad S_{\mathbf{n}}\,|\mathbf{n};\pm\rangle = \pm\,\tfrac{\hbar}{2}\,|\mathbf{n};\pm\rangle$$

$$|\mathbf{n};+\rangle \;=\; \cos(\tfrac{1}{2}\theta)|+\rangle \;+\; \sin(\tfrac{1}{2}\theta)\exp(i\phi)|-\rangle$$

$$|\mathbf{n};-\rangle \;=\; -\sin(\tfrac{1}{2}\theta)\exp(-i\phi)|+\rangle \;+\; \cos(\tfrac{1}{2}\theta)|-\rangle$$

$$\big|\langle\mathbf{n}';+|\mathbf{n};+\rangle\big| = \cos(\tfrac{1}{2}\gamma)\,, \quad \gamma \text{ is the angle between } \mathbf{n} \text{ and } \mathbf{n}'$$

$$\langle\mathbf{S}\rangle_{\mathbf{n}} \;=\; \tfrac{\hbar}{2}\,\mathbf{n}\,, \qquad \text{Rotation operator: } R_{\mathbf{n}}(\alpha) \equiv \exp\Big(-\frac{i\alpha S_{\mathbf{n}}}{\hbar}\Big)$$

- Linear algebra

$$\text{Matrix representation of } T \text{ in the basis } (v_1, \ldots, v_n): \quad T v_j = \sum_i T_{ij}\, v_i$$

$$\text{basis change: } u_k = \sum_j A_{jk} v_j \,, \qquad T(\{u\}) = A^{-1} T(\{v\}) A$$

$$\text{Schwarz: } |\langle u, v\rangle| \leq |u|\,|v|$$

$$\text{Adjoint: } \langle u, Tv\rangle = \langle T^\dagger u, v\rangle \,, \quad (T^\dagger)^\dagger = T$$

$$\langle v, Tv\rangle = 0\,, \quad \forall v \in V \quad \to \quad T = 0 \;\; \text{(complex vector space)}$$

- Bras and kets: For an operator Ω and a vector v, we write $|\Omega v\rangle \equiv \Omega|v\rangle$

$$\text{Adjoint: } \langle u|\Omega^\dagger v\rangle = \langle \Omega u|v\rangle$$

$$|\alpha_1 v_1 + \alpha_2 v_2\rangle = \alpha_1|v_1\rangle + \alpha_2|v_2\rangle \quad \longleftrightarrow \quad \langle \alpha_1 v_1 + \alpha_2 v_2| = \alpha_1^*\langle v_1| + \alpha_2^*\langle v_2|$$

- Complete orthonormal basis $|i\rangle$

$$\langle i|j\rangle = \delta_{ij}\,, \qquad \mathbf{1} = \sum_i |i\rangle\langle i|$$

$$\Omega_{ij} = \langle i|\Omega|j\rangle \quad \leftrightarrow \quad \Omega = \sum_{i,j} \Omega_{ij}\; |i\rangle\langle j|$$

$$\langle i|\Omega^\dagger|j\rangle = \langle j|\Omega|i\rangle^*$$

$$\Omega \;\; \text{hermitian: } \Omega^\dagger = \Omega, \qquad U \text{ unitary: } U^\dagger = U^{-1}$$

- Matrix M is normal ($[M, M^\dagger] = 0$) $\longleftrightarrow$ unitarily diagonalizable.

- Position and momentum representations: $\psi(x) = \langle x|\psi\rangle$; $\tilde{\psi}(p) = \langle p|\psi\rangle$;

$$\hat{x}|x\rangle = x|x\rangle\,, \quad \langle x|y\rangle = \delta(x-y)\,, \quad \mathbf{1} = \int dx\, |x\rangle\langle x|\,, \quad \hat{x}^\dagger = \hat{x}$$

$$\hat{p}|p\rangle = p|p\rangle\,, \quad \langle q|p\rangle = \delta(q-p)\,, \quad \mathbf{1} = \int dp\, |p\rangle\langle p|\,, \quad \hat{p}^\dagger = \hat{p}$$

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\Big(\frac{ipx}{\hbar}\Big)\,; \qquad \tilde{\psi}(p) = \int dx \langle p|x\rangle\langle x|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dx \exp\Big(-\frac{ipx}{\hbar}\Big)\psi(x)$$

$$\langle x|\hat{p}^n|\psi\rangle = \Big(\frac{\hbar}{i}\frac{d}{dx}\Big)^n \psi(x)\,; \qquad \langle p|\hat{x}^n|\psi\rangle = \Big(i\hbar\frac{d}{dp}\Big)^n \tilde{\psi}(p)\,; \qquad [\hat{p}, f(\hat{x})] = \frac{\hbar}{i} f'(\hat{x})$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(ikx)dx = \delta(k)$$

- Generalized uncertainty principle

$$\Delta A \;\equiv\; \big|(A - \langle A\rangle \mathbf{1})\Psi\big| \quad \to \quad (\Delta A)^2 = \langle A^2\rangle - \langle A\rangle^2 \geq 0\,.$$

$$\Delta A\,\Delta B \geq \Big|\langle\Psi|\frac{1}{2i}[A,B]|\Psi\rangle\Big|$$

$$\Delta x\,\Delta p \geq \tfrac{\hbar}{2}$$

$$\Delta x = \frac{\Delta}{\sqrt{2}} \;\text{ and }\; \Delta p = \frac{\hbar}{\sqrt{2}\Delta} \;\text{ for }\; \psi \sim \exp\Big(-\frac{1}{2}\frac{x^2}{\Delta^2}\Big)$$

$$\int_{-\infty}^{+\infty} dx \exp\left(-ax^2\right) = \sqrt{\frac{\pi}{a}}$$

$$\text{Time independent operator } Q: \quad \frac{d}{dt}\langle Q\rangle \;=\; \frac{i}{\hbar}\langle\,[H,Q]\rangle$$

$$\Delta H \Delta t \geq \frac{\hbar}{2}\,, \quad \Delta t \equiv \frac{\Delta Q}{\big|\frac{d\langle Q\rangle}{dt}\big|}$$

- Commutator identities

$$[A, BC] = [A,B]C + B[A,C]\,,$$

$$e^A B e^{-A} \;=\; e^{\mathrm{ad}_A} B \;=\; B + [A,B] + \frac{1}{2}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \ldots\,,$$

$$e^A B e^{-A} = B + [A,B]\,, \quad \text{if} \quad [A,[A,B]\,] = 0\,,$$

$$[\,B\,,\,e^A\,] = [\,B\,,A\,]e^A\,, \qquad \text{if} \quad [A,[A,B]\,] = 0$$

$$e^{A+B} \;=\; e^A e^B e^{-\frac{1}{2}[A,B]} \;=\; e^B e^A e^{\frac{1}{2}[A,B]}\;, \quad \text{if } [A,B] \text{ commutes with } A \text{ and with } B$$

- Harmonic Oscillator

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega\,(\hat{N} + \tfrac{1}{2})\,, \quad \hat{N} = \hat{a}^\dagger\hat{a}$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)\;, \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)\;,$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger)\;, \quad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}^\dagger - \hat{a})\,,$$

$$[\hat{x},\hat{p}] = i\hbar\,, \quad [\hat{a},\hat{a}^\dagger] = 1\;, \quad [\hat{N},\hat{a}\,] = -\hat{a}\,, \quad [\hat{N}\,,\hat{a}^\dagger\,] \;=\; \hat{a}^\dagger\,.$$

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle$$

$$\hat{H}|n\rangle = E_n|n\rangle = \hbar\omega\big(n + \tfrac{1}{2}\big)|n\rangle\,, \quad \hat{N}|n\rangle = n|n\rangle\,, \quad \langle m|n\rangle = \delta_{mn}$$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \;, \quad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle \;.$$

$$\psi_0(x) = \langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right) \;.$$

$$x_H(t) = \hat{x}\,\cos\omega t + \frac{\hat{p}}{m\omega}\sin\omega t$$

$$p_H(t) = \hat{p}\,\cos\omega t - m\omega\,\hat{x}\sin\omega t$$

- Coherent states and squeezed states

$$T_{x_0} \;\equiv\; e^{-\frac{i}{\hbar}\hat{p}\,x_0} \,, \quad T_{x_0}|x\rangle = |x+x_0\rangle$$

$$|\widetilde{x}_0\rangle \;\equiv\; T_{x_0}|0\rangle \; = e^{-\frac{i}{\hbar}\hat{p}x_0}|0\rangle \,,$$

$$|\widetilde{x}_0\rangle = e^{-\frac{1}{4}\frac{x_0^2}{d^2}} e^{\frac{x_0}{\sqrt{2}d}a^\dagger}|0\rangle \,, \quad \langle x|\widetilde{x}_0\rangle = \psi_0(x-x_0) \,, \quad d^2 = \frac{\hbar}{m\omega}$$

$$|\alpha\rangle \equiv D(\alpha)|0\rangle = e^{\alpha a^\dagger - \alpha^* a}|0\rangle \,, \quad D(\alpha) \equiv \exp\Big(\alpha a^\dagger - \alpha^* a\Big) \,, \quad \alpha = \frac{\langle\hat{x}\rangle}{\sqrt{2}\,d} \;+\; i\,\frac{\langle\hat{p}\rangle\,d}{\sqrt{2}\,\hbar} \;\in \mathbb{C}$$

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \;, \qquad |\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha\, a^\dagger}|0\rangle \,, \quad |\alpha, t\rangle \;=\; e^{-i\omega t/2}|e^{-i\omega t}\alpha\rangle$$

$$\langle\beta|\alpha\rangle \;=\; e^{-\frac{1}{2}|\alpha-\beta|^2}\, e^{i\,\mathrm{Im}(\beta^*\alpha)}$$

$$|0_\gamma\rangle \;=\; S(\gamma)|0\rangle \,, \qquad S(\gamma) \;=\; \exp\Big(-\frac{\gamma}{2}(a^\dagger a^\dagger - aa)\Big) \,, \quad \gamma \in \mathbb{R}$$

$$|0_\gamma\rangle \;=\; \frac{1}{\sqrt{\cosh\gamma}} \exp\Big(-\frac{1}{2}\,\tanh\gamma\, a^\dagger a^\dagger\Big)\,|0\rangle$$

$$S^\dagger(\gamma)\,aS(\gamma) = \cosh\gamma\, a - \sinh\gamma\, a^\dagger \,, \quad D^\dagger(\alpha)\,aD(\alpha) = a + \alpha$$

$$|\alpha, \gamma\rangle \equiv D(\alpha)S(\gamma)|0\rangle$$

- Time evolution

$$|\Psi, t\rangle = \mathcal{U}(t,0)|\Psi, 0\rangle \,, \quad \mathcal{U} \;\; \text{unitary}$$

$$\mathcal{U}(t,t) = \mathbf{1} \,, \;\; \mathcal{U}(t_2,t_1)\mathcal{U}(t_1,t_0) = \mathcal{U}(t_2,t_0) \,, \;\; \mathcal{U}(t_1,t_2) = \mathcal{U}^\dagger(t_2,t_1)$$

$$i\hbar\frac{d}{dt}|\Psi, t\rangle = \hat{H}(t)|\Psi, t\rangle \quad \leftrightarrow \quad i\hbar\frac{d}{dt}\mathcal{U}(t,t_0) = \hat{H}(t)\mathcal{U}(t,t_0)$$

$$\text{Time independent } \hat{H}\text{:} \quad \mathcal{U}(t,t_0) = \exp\Big[-\frac{i}{\hbar}\,\hat{H}(t-t_0)\Big] = \sum_n e^{-\frac{i}{\hbar}\,E_n(t-t_0)}|n\rangle\langle n|$$

$$\langle A\rangle = \langle \Psi, t|A_S|\Psi, t\rangle = \langle \Psi, 0|A_H(t)|\Psi, 0\rangle \quad \to \quad A_H(t) = \mathcal{U}^\dagger(t,0)A_S\,\mathcal{U}(t,0)$$

$$[A_S,\, B_S\,] = C_S \quad \to \quad [A_H(t)\,,\, B_H(t)\,] = C_H(t)$$

$$i\hbar\frac{d}{dt}\hat{A}_H(t) = [\hat{A}_H(t), \hat{H}_H(t)]\,, \quad \text{for } A_S \text{ time-independent}$$

- Two state systems

$$H = h_0\mathbf{1} + \mathbf{h}\cdot\boldsymbol{\sigma} = h_0\mathbf{1} + h\,\mathbf{n}\cdot\boldsymbol{\sigma}\,, \quad h = |\mathbf{h}|$$

$$\text{Eigenstates: } |\mathbf{n};\pm\rangle\,, \quad E_\pm = h_0 \pm h\,.$$

$$H = -\gamma\,\mathbf{S}\cdot\mathbf{B} \quad \to \quad \text{spin vector } \vec{n} \text{ precesses with Larmor frequency } \boldsymbol{\omega} = -\gamma\mathbf{B}$$

$$\text{NMR magnetic field } \mathbf{B} = B_0\mathbf{z} + B_1\big(\cos\omega t\,\mathbf{x} - \sin\omega t\,\mathbf{y}\big)$$

$$|\Psi(t)\rangle = \exp\Big(\frac{i}{\hbar}\omega t\hat{S}_z\Big)\exp\Big(\frac{i}{\hbar}\gamma\mathbf{B}_{\rm R}\cdot\mathbf{S}\,t\Big)|\Psi(0)\rangle$$

$$\mathbf{B}_{\rm R} = B_0\big(1 - \tfrac{\omega}{\omega_0}\big)\mathbf{z} + B_1\,\mathbf{x}$$

- Tensor products

If $\{e_i\}$ a basis for V and $\{f_j\}$ a basis for W then $\{e_i\otimes f_j\}$ is a basis for $V\otimes W$

If $\Psi = v_*\otimes w_* \in V\otimes W$, Ψ is not entangled.

$$\text{Bell Basis: } |\Phi_0\rangle = \tfrac{1}{\sqrt{2}}(|{+}{+}\rangle + |{-}{-}\rangle)\,, \quad |\Phi_i\rangle \;=\; \mathbf{1}\otimes\sigma_i|\Phi_0\rangle$$

Partial measurement of $\Psi = \sum_i |e_i\rangle\otimes|w_i\rangle$ along first basis $\{|e_i\rangle\}$:

Probability that Ψ is found in $|e_k\rangle = \langle w_k|w_k\rangle$. State after that measurement: $|e_k\rangle\otimes\dfrac{|w_k\rangle}{\sqrt{\langle w_k|w_k\rangle}}$

$$A\otimes B \;=\; \begin{pmatrix} A_{11}B & \dots & A_{1m}B \\ \vdots & \vdots & \vdots \\ A_{m1}B & \dots & A_{mm}B \end{pmatrix}\,,\ A \text{ is } m\times m,\ B \text{ is } n\times n$$

$$\text{basis vectors: } e_1\otimes f_1, \ldots, e_1\otimes f_n\,, \ \ldots\ \ldots\,,\ e_m\otimes f_1\,,\ \ldots\,,\ e_m\otimes f_n$$

- Orbital angular momentum operators

$$\mathbf{a}\cdot\mathbf{b} \;\equiv\; a_i\,b_i\,, \quad (\mathbf{a}\times\mathbf{b})_i \;\equiv\; \epsilon_{ijk}\,a_j\,b_k\,, \quad \mathbf{a}^2 \;\equiv\; \mathbf{a}\cdot\mathbf{a}\,.$$

$$\epsilon_{ijk}\,\epsilon_{ipq} \;=\; \delta_{jp}\delta_{kq} \;-\; \delta_{jq}\delta_{kp}\,, \qquad \epsilon_{ijk}\,\epsilon_{ijq} \;=\; 2\delta_{kq}\,.$$

$$\hat{L}_i = \epsilon_{ijk}\,\hat{x}_j\,\hat{p}_k \quad \Longleftrightarrow \quad \mathbf{L} = \mathbf{r}\times\mathbf{p} \;=\; -\mathbf{p}\times\mathbf{r}$$

Vector $\mathbf{u}$ under rotation: $\quad [\,\hat{L}_i\,,\,\hat{u}_j\,] \;=\; i\hbar\,\epsilon_{ijk}\,\hat{u}_k \quad \Longrightarrow \quad \mathbf{L}\times\mathbf{u} + \mathbf{u}\times\mathbf{L} \;=\; 2i\hbar\,\mathbf{u}$

Scalar S under rotation: $\quad [\,\hat{L}_i\,,\,S\,] \;=\; 0$

$\mathbf{u},\ \mathbf{v}$ vectors under rotations $\quad\to\quad \mathbf{u}\cdot\mathbf{v}$ is a scalar, $\quad\mathbf{u}\times\mathbf{v}$ is a vector

$$\left[\,\hat{L}_i\,,\hat{L}_j\,\right] \;=\; i\hbar\,\epsilon_{ijk}\,\hat{L}_k \quad\Longleftrightarrow\quad \mathbf{L}\times\mathbf{L} \;=\; i\hbar\,\mathbf{L}\,, \qquad [\,\hat{L}_i\,,\,\mathbf{L}^2\,] \;=\; 0\,.$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

$$\hat{L}^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

$$\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}\;; \qquad \hat{L}_\pm = \hbar e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

- Spherical Harmonics

$$Y_{\ell,m}(\theta,\phi) \equiv \langle\theta,\phi|\ell,m\rangle$$

$$Y_{0,0}(\theta,\phi) = \frac{1}{\sqrt{4\pi}}\;; \qquad Y_{1,\pm1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\exp(\pm i\phi)\;; \qquad Y_{1,0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

- Algebra of angular momentum operators **J** (orbital or spin, or sum)

$$[J_i, J_j] = i\hbar\,\epsilon_{ijk}J_k \quad\Longleftrightarrow\quad \mathbf{J}\times\mathbf{J} = i\hbar\,\mathbf{J}\,; \quad\to \qquad [\,\mathbf{J}^2,\,J_i] = 0$$

$$J_\pm = J_x \pm iJ_y\,, \quad (J_\pm)^\dagger = J_\mp \quad J_x = \frac{1}{2}(J_+ + J_-)\,,\; J_y = \frac{1}{2i}(J_+ - J_-)$$

$$[J_z\,, J_\pm] = \pm\hbar\,J_\pm\,,; \qquad [J_+, J_-] = 2\hbar\,J_z \qquad [J^2, J_\pm] = 0$$

$$\mathbf{J}^2 = J_+J_- + J_z^2 - \hbar J_z = J_-J_+ + J_z^2 + \hbar J_z$$

$$\mathbf{J}^2|jm\rangle = \hbar^2\,j(j+1)|jm\rangle\;; \qquad J_z|jm\rangle = \hbar m|jm\rangle\,, \quad m = -j,\ldots,j\,.$$

$$J_\pm|jm\rangle = \hbar\sqrt{j(j+1) - m(m\pm1)}\;|j, m\pm1\rangle$$

- Angular momentum in the two-dimensional oscillator

$$\hat{a}_L = \frac{1}{\sqrt{2}}(\hat{a}_x + i\hat{a}_y)\,,\; \hat{a}_R = \frac{1}{\sqrt{2}}(\hat{a}_x - i\hat{a}_y)\,,\; [\hat{a}_L, \hat{a}_L^\dagger] = [\hat{a}_R\,, \hat{a}_R^\dagger] = 1$$

$$J_+ = \hbar\,\hat{a}_R^\dagger\hat{a}_L\,, \quad J_- = \hbar\,\hat{a}_L^\dagger\hat{a}_R\,, \quad J_z = \tfrac{1}{2}\hbar(\hat{N}_R - \hat{N}_L)$$

$$|j,m\rangle: \qquad j = \tfrac{1}{2}(N_R + N_L)\,, \quad m = \tfrac{1}{2}(N_R - N_L)$$

$$\mathcal{H} = 0\oplus\tfrac{1}{2}\oplus1\oplus\tfrac{3}{2}\oplus\ldots$$

- Radial equation

$$\Big(-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + V(r) + \frac{\hbar^2\,\ell(\ell+1)}{2mr^2}\Big)u_{\nu\ell}(r) = E_{\nu\ell}u_{\nu\ell}(r) \quad \text{(bound states)}$$

$$u_{\nu\ell}(r) \sim r^{\ell+1}\,, \quad \text{as } r \to 0\,.$$

- Hydrogen atom: $H = \frac{\mathbf{p^2}}{2m} - \frac{e^2}{r}$

$$E_n = -\frac{e^2}{2a_0}\frac{1}{n^2}\,, \qquad \psi_{n,\ell,m}(\vec{x}) = \frac{u_{n\ell}(r)}{r}Y_{\ell,m}(\theta,\phi)$$

$$n = 1,2,\dots\,, \qquad \ell = 0,1,\dots,n-1\,, \quad m = -\ell,\dots,\ell$$

$$a_0 = \frac{\hbar^2}{me^2}\,, \qquad \alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137}\,, \qquad \hbar c \simeq 200\,\text{MeV–fm}$$

$$u_{1,0}(r) = \frac{2r}{a_0^{3/2}}\exp(-r/a_0)$$

$$u_{2,0}(r) = \frac{2r}{(2a_0)^{3/2}}\left(1 - \frac{r}{2a_0}\right)\exp(-r/2a_0)$$

$$u_{2,1}(r) = \frac{1}{\sqrt{3}}\frac{1}{(2a_0)^{3/2}}\frac{r^2}{a_0}\exp(-r/2a_0)$$

$$\text{Conserved Runge-Lenz vector:} \quad \mathbf{R} \;\equiv\; \frac{1}{2me^2}(\mathbf{p}\times\mathbf{L} - \mathbf{L}\times\mathbf{p}) - \frac{\mathbf{r}}{r}\,.$$

- Addition of Angular Momentum $\mathbf{J} = \mathbf{J}_1 + \mathbf{J_2}$

Uncoupled basis : $|j_1 j_2; m_1 m_2\rangle$ CSCO : $\{\mathbf{J}_1^2, \mathbf{J}_2^2, J_{1z}, J_{2z}\}$

Coupled basis : $|j_1 j_2; jm\rangle$ CSCO : $\{\mathbf{J}_1^2, \mathbf{J}_2^2, \mathbf{J}^2, J_z\}$

$$j_1 \otimes j_2 = (j_1 + j_2) \oplus (j_1 + j_2 - 1) \oplus \dots \oplus |j_1 - j_2|$$

$$|j_1 j_2; jm\rangle = \sum_{m_1+m_2=m} |j_1 j_2; m_1 m_2\rangle \underbrace{\langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle}_{\text{Clebsch−Gordan coefficient}}$$

$$\mathbf{J}_1\cdot\mathbf{J}_2 = \tfrac{1}{2}(J_{1+}J_{2-} + J_{1-}J_{2+}) + J_{1z}J_{2z} \;=\; \tfrac{1}{2}(\mathbf{J}^2 - \mathbf{J}_1^2 - \mathbf{J}_2^2\,)$$

Combining two spin 1/2 : $\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} = \mathbf{1} \oplus \mathbf{0}$

$$|\,1,1\,\rangle = |\uparrow\uparrow\rangle\,,$$

$$|\,1,0\,\rangle = \tfrac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\,, \qquad |0,0\rangle = \tfrac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

$$|1,-1\rangle = |\downarrow\downarrow\rangle\,.$$