Formula Sheet

- $\hbar c \simeq 197.3$ MeV $\cdot$ fm, $m_e c^2 \simeq 0.511$ MeV, $m_p c^2 = 938$ MeV, $\frac{e^2}{\hbar c} \simeq \frac{1}{137}$
- Relativity: $p = \gamma\, mv\,, \quad E = \gamma mc^2\,, \quad E^2 = p^2c^2 + m^2c^4\,, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}\,, \quad \beta = \frac{v}{c}$
- Photons: $E = h\nu\,, \quad p = \frac{h}{\lambda}\,,$ or $E = \hbar\omega\,, \quad p = \hbar k$
- Wavelengths

$$\text{de Broglie:}\ \ \lambda = \frac{h}{p}\,, \qquad \text{Compton:}\ \ \lambda_C = \frac{h}{mc}\,.$$

- Momentum and position operators

$$p \;=\; \frac{\hbar}{i}\frac{\partial}{\partial x}\,, \quad [x,p\,] \;=\; i\hbar\,, \qquad \mathbf{p} \;=\; \frac{\hbar}{i}\nabla\,, \quad [x_i, p_j\,] = i\hbar\,\delta_{ij}\,, \quad [p_i, f(\mathbf{x})] \;=\; \frac{\hbar}{i}\frac{\partial f}{\partial x_i}$$

- Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t}(\mathbf{x},t) \;=\; \Big(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x},t)\Big)\Psi(\mathbf{x},t)\,,$$

$$\frac{\partial}{\partial t}\rho(\mathbf{x},t) + \nabla\cdot\mathbf{J}(\mathbf{x},t) = 0$$

$$\rho(\mathbf{x},t) = |\Psi(\mathbf{x},t)|^2\ ; \qquad \mathbf{J}(\mathbf{x},t) = \frac{\hbar}{m}\mathrm{Im}\,[\Psi^*\nabla\Psi]$$

- Fourier transforms:

$$\Psi(x) = \frac{1}{\sqrt{2\pi}}\int dk\,\Phi(k)e^{ikx}\,, \quad \Phi(k) = \frac{1}{\sqrt{2\pi}}\int dx\,\Psi(x)e^{-ikx}\,, \quad \int dx\,|\Psi(x)|^2 = \int dk\,|\Phi(k)|^2$$

$$\Psi(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k\,\Phi(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}\,, \ \Phi(\mathbf{k}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3x\,\Psi(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}\,, \ \int d^3x\,|\Psi(\mathbf{x})|^2 = \int d^3k\,|\Phi(\mathbf{k})|^2$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}dx = \delta(k)\,, \qquad \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} e^{i\mathbf{k}\cdot\mathbf{x}}\,d^3x = \delta^{(3)}(\mathbf{k})$$

$$\int_{-\infty}^{+\infty} dx\exp\big(-ax^2+bx\big) = \sqrt{\frac{\pi}{a}}\exp\Big(\frac{b^2}{4a}\Big)\,, \quad \text{when } \mathrm{Re}(a) > 0\,.$$

- Wavepackets

$$v_{group} \;=\; \frac{d\omega}{dk}\,, \quad \Delta k\,\Delta x \simeq 1\,, \quad \text{shape preserving :}\ \ t\,\Delta v \leq \Delta x$$

$$\text{Main contribution from } \int dk\,\Phi(k)e^{i\varphi(k)} \text{ with } \Phi(k) \text{ real and peaked at } k_0 \text{ is from } \frac{d\varphi}{dk}\Big|_{k_0} = 0$$

- Inner product and Hermitian conjugation: $\big(\psi,\varphi\big) = \int dx\,\psi^*(x)\varphi(x)$

$$\big(K\Psi\,,\Phi\big) \;=\; \big(\Psi\,,K^\dagger\Phi\big)\,, \quad \text{If } K^\dagger = K \ \text{ then } K \text{ is Hermitian}$$

- Expectation values

$$\langle Q\rangle(t) \;=\; \int dx\,\Psi^*(x,t)(Q\Psi(x,t))$$

- Time evolution of expectation value. For Q without explicit time dependence

$$i\hbar\frac{d}{dt}\langle Q\rangle \;=\; \big\langle [Q,H]\big\rangle$$

- Commutator identities:

$$[A,BC] = [A,B]C + B[A,C]$$
$$[AB,C] = A[B,C] + [A,C]B$$

- Uncertainty ΔQ of a Hermitian operator Q

$$(\Delta Q)^2 \;=\; \langle Q^2\rangle - \langle Q\rangle^2 \;=\; \big\langle (Q - \langle Q\rangle)^2\big\rangle$$

- Uncertainty principle: $\quad \Delta x\,\Delta p \geq \frac{\hbar}{2}$

$$\Delta x = \Delta \;\text{ and }\; \Delta p = \frac{\hbar}{2\Delta} \;\text{ for }\; \psi \sim \exp\Big(-\frac{1}{4}\frac{x^2}{\Delta^2}\Big)$$

- Stationary state:

$$\Psi(x,t) = \psi(x)e^{-iEt/\hbar}\,, \qquad -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\,\psi(x)$$

- Infinite square well

$$V(x) = \begin{cases} 0\,, & \text{for } \; 0 < x < a, \\ \infty & \text{otherwise} \end{cases}$$

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}\,, \quad E_n = \frac{\hbar^2\pi^2 n^2}{2ma^2}\,, \quad n = 1, 2, \ldots$$

- Finite square well bound states: $E \leq 0$

$$V(x) = \begin{cases} -V_0\,, & \text{for } \; |x| < a, \quad V_0 > 0 \\ \;\;0 & \text{for } \; |x| > a \end{cases}$$

$$\eta^2 \;\equiv (ka)^2 \equiv \; \frac{2m(V_0 - |E|)a^2}{\hbar^2}\,, \quad \xi^2 \;\equiv (\kappa a)^2 \equiv \; \frac{2m|E|a^2}{\hbar^2}\,, \quad z_0^2 \;\equiv\; \frac{2mV_0a^2}{\hbar^2}$$

$$\rightarrow \quad \frac{|E|}{V_0} \;=\; \frac{\xi^2}{z_0^2}\,, \qquad \xi^2 + \eta^2 \;=\; z_0^2$$

$$\text{Even solutions:} \quad \xi \;=\; \eta\tan\eta$$

$$\text{Odd solutions:} \quad \xi \;=\; -\eta\cot\eta$$

- Delta function potential:

$$V = -\frac{\hbar^2}{mL}\,\delta(x), \quad L > 0, \qquad \text{Bound state:} \quad E = -\frac{\hbar^2}{2mL^2}$$

- Harmonic Oscillator

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega\,(\hat{N} + \tfrac{1}{2})\,, \quad \hat{N} = \hat{a}^\dagger\hat{a}$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)\ , \qquad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)\ ,$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger)\ , \qquad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}^\dagger - \hat{a})\,,$$

$$[\hat{x},\hat{p}] = i\hbar\,, \quad [\hat{a},\hat{a}^\dagger] = 1\ , \qquad [\hat{N},\hat{a}\,] = -\hat{a}\,, \quad [\hat{N}\,,\hat{a}^\dagger\,] \ = \ \hat{a}^\dagger\ .$$

$$\hat{a}\varphi_0 \ = \ 0\,, \quad \varphi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\ .$$

$$\varphi_n = \frac{1}{\sqrt{n!}}(a^\dagger)^n\varphi_0$$

$$\hat{H}\,\varphi_n = E_n\,\varphi_n = \hbar\omega\big(n + \tfrac{1}{2}\big)\,\varphi_n\,, \quad \hat{N}\,\varphi_n = n\,\varphi_n\,, \quad (\varphi_m\,,\varphi_n) \ = \ \delta_{mn}$$

$$\hat{a}^\dagger\varphi_n = \sqrt{n+1}\,\varphi_{n+1}\ , \quad \hat{a}\,\varphi_n = \sqrt{n}\,\varphi_{n-1}\ .$$

- Non-normalizable energy eigenstates:

$$\psi(x) \ = \ Ae^{ikx} + Be^{-ikx}\,, \qquad J \ = \ \frac{\hbar k}{m}\big(|A|^2 - |B|^2\big)\,, \qquad E = \frac{\hbar^2k^2}{2m}$$

- Scattering in 1D. $V(x) = \infty$ for $x \le 0$. Solution $\phi(x) = \sin kx$ when $V = 0$.

$$\psi(x) \ = \ e^{i\delta}\,\sin(kx + \delta)\,, \quad x > R \ (R \text{ is the range})$$

Scattered wave: $\psi = \phi + \psi_s$

$$\psi_s = A_s e^{ikx}\,, \quad A_s = e^{i\delta}\sin\delta = \tfrac{1}{2i}(e^{2i\delta} - 1) = \frac{\tan\delta}{1 - i\tan\delta}$$

$$\text{Time delay:} \quad \Delta t \ = \ 2\hbar\,\frac{d\delta}{dE} \quad \to \quad \frac{1}{R}\frac{d\delta}{dk} \ = \ \frac{\Delta t}{\text{free transit time}}$$

$$N_{bound} \ = \ \frac{1}{\pi}\big(\,\delta(0) - \delta(\infty)\big) \quad \text{(Levinson's theorem)}$$

Resonances: Rapid growth in δ, large time delay, large amplitude in the inner region. For a resonance at $k = \alpha$ we have

$$\tan\delta = \frac{\beta}{\alpha - k}\,, \quad \text{pole of } A_s \text{ for } \ \tan\delta = -i, \ \text{at } k = \alpha - i\beta$$

- Orbital angular momentum

$$\hat{L}_x = \hat{y}\,\hat{p}_z - \hat{z}\,\hat{p}_y\,, \quad \hat{L}_y \;=\; \hat{z}\,\hat{p}_x - \hat{x}\,\hat{p}_z\,, \quad \hat{L}_z \;=\; \hat{x}\,\hat{p}_y - \hat{y}\,\hat{p}_x\,.$$

$$[\,\hat{L}_x\,,\,\hat{L}_y\,] \;=\; i\hbar\,\hat{L}_z\,, \quad [\,\hat{L}_y\,,\,\hat{L}_z\,] \;=\; i\hbar\,\hat{L}_x\,, \quad [\,\hat{L}_z\,,\,\hat{L}_x\,] \;=\; i\hbar\,\hat{L}_y\,.$$

$$\hat{L}^2 \;\equiv\; \hat{L}_x\hat{L}_x + \hat{L}_y\hat{L}_y + \hat{L}_z\hat{L}_z\,, \quad [\,\hat{L}^2\,,\hat{L}_i\,] \;=\; 0$$

$$\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2}\,r + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

$$\hat{L}^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

$$\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}\;; \qquad \hat{L}_\pm = \hbar e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

- Spherical Harmonics

$$Y_{\ell,m}(\theta,\phi) \;\equiv\; \mathcal{N}_{\ell,m}\,P_\ell^m(\cos\theta)e^{im\phi}$$

$$\begin{aligned}\hat{L}_z\,Y_{\ell m} &= \hbar m\,Y_{\ell m}\\ \hat{L}^2\,Y_{\ell m} &= \hbar^2\,\ell(\ell+1)\,Y_{\ell m}\end{aligned}$$

$$\int d\Omega\,Y^*_{\ell'm'}(\theta,\phi)\,Y_{\ell m}(\theta,\phi) \;=\; \delta_{\ell',\ell}\delta_{m',m}\,, \qquad \int d\Omega \;=\; \int_0^{2\pi} d\phi\int_{-1}^{1} d(\cos\theta)$$

$$Y_{0,0}(\theta,\phi) = \frac{1}{\sqrt{4\pi}}\;; \qquad Y_{1,\pm1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\exp(\pm i\phi)\;; \qquad Y_{1,0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

- Central potentials: $V(\mathbf{r}) = V(r)$

$$\psi(r,\theta,\phi) \;=\; \frac{u(r)}{r}\,Y_{\ell m}(\theta,\phi)$$

$$\Big(-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + V(r) + \frac{\hbar^2\,\ell(\ell+1)}{2mr^2}\Big)u(r) = Eu(r)$$

$$u(r) \sim r^{\ell+1}\,, \quad \text{as } r\to 0\,.$$

- Hydrogen atom:

$$H \;=\; \frac{\mathbf{p^2}}{2m} - \frac{Ze^2}{r}$$

$$E_n = -\frac{Z^2e^2}{2a_0}\frac{1}{n^2}\,, \qquad a_0 = \frac{\hbar^2}{me^2} \;\simeq\; 0.529\times10^{-10}\mathrm{m}\,, \quad \frac{e^2}{2a_0} \;\simeq\; 13.6\,\mathrm{eV}$$

$$\psi_{n,\ell,m}(\vec{x}) \;=\; A\left(\tfrac{r}{a_0}\right)^{\ell}\Big(\text{Polynomial in } \tfrac{r}{a_0} \text{ of degree } n-(\ell+1)\Big)\, e^{-\frac{Zr}{na_0}}\, Y_{\ell,m}(\theta,\phi)$$

$$n = 1, 2, \ldots\,, \qquad \ell = 0, 1, \ldots, n-1\,, \quad m = -\ell, \ldots, \ell$$

$$\psi_{n,\ell,m}(\vec{x}) = \frac{u_{n\ell}(r)}{r} Y_{\ell,m}(\theta,\phi)$$

$$u_{1,0}(r) = \frac{2r}{a_0^{3/2}} \exp(-r/a_0)$$

$$u_{2,0}(r) = \frac{2r}{(2a_0)^{3/2}}\left(1 - \frac{r}{2a_0}\right)\exp(-r/2a_0)$$

$$u_{2,1}(r) = \frac{1}{\sqrt{3}}\frac{1}{(2a_0)^{3/2}}\frac{r^2}{a_0}\exp(-r/2a_0)$$

Virial theorem: $\langle \hat{T} \rangle = -\frac{1}{2}\langle \hat{V} \rangle$.