Repeated Experiments Replacement

Composite experiments

Repetition

With and without replacement

Part I

Order matters, or not

Part II

Composite Experiments

Experiments often consist of several parts

Student major year GPA

Ad product audience cost

Still can be viewed as a single experiment

Outcomes more complex

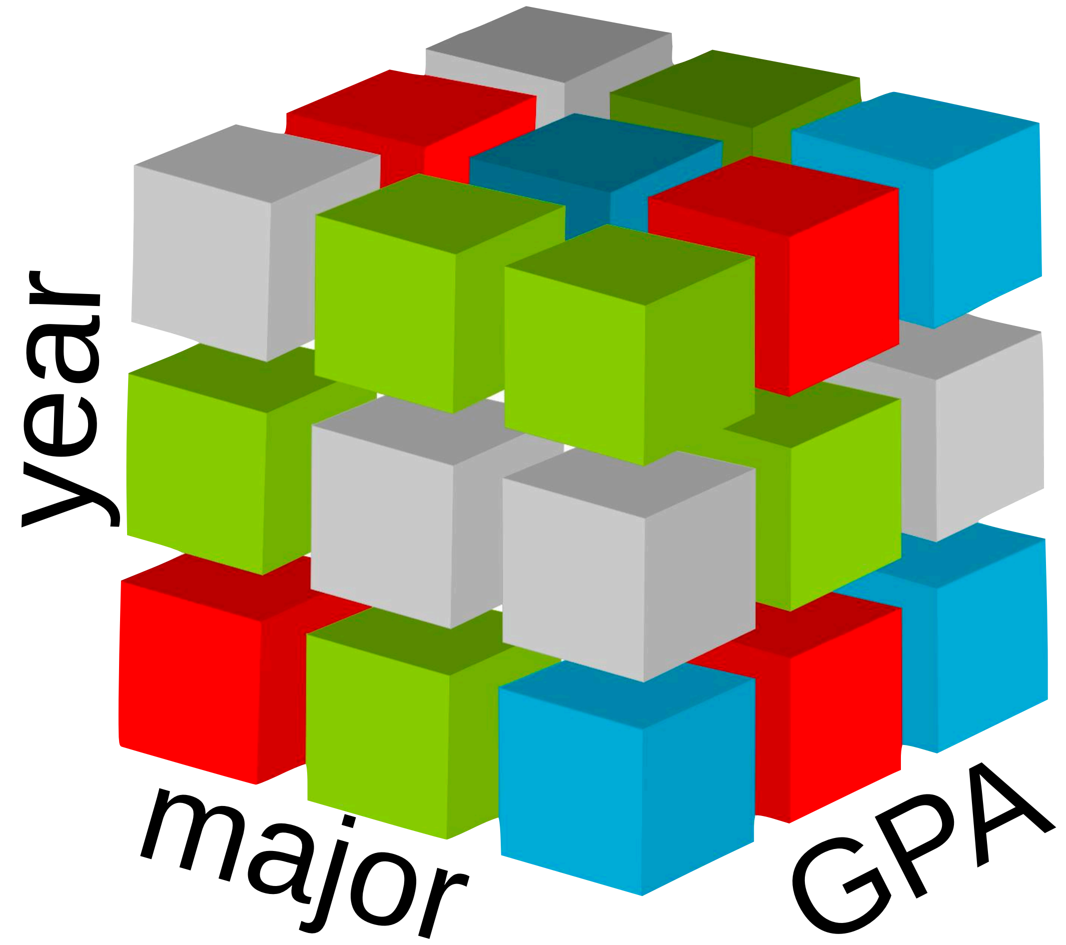
3-tuple

(CS, senior, 3.8)

(book, teenage, $9.99)

Sample space

Cartesian product



Goal

Understand simple

Analyze complex

Start

Independent Repetitions

Repetition All experiments of same type

Daily temperatures

Daily stock prices

Coin flips

Die rolls

Card draws

Independent Different components unrelated

First coin heads

Second coin 50% heads / tails ✅

Second coin more likely heads ❌

Two Coins

Fair

Independent flips

Outcomes

Probabilities

		coin 2	½	½
			h	t
coin 1	½	h	¼	¼
	½	t	¼	¼

$\Omega = \{ hh, ht, th, tt \} = \{h,t\}^2$

Cartesian power

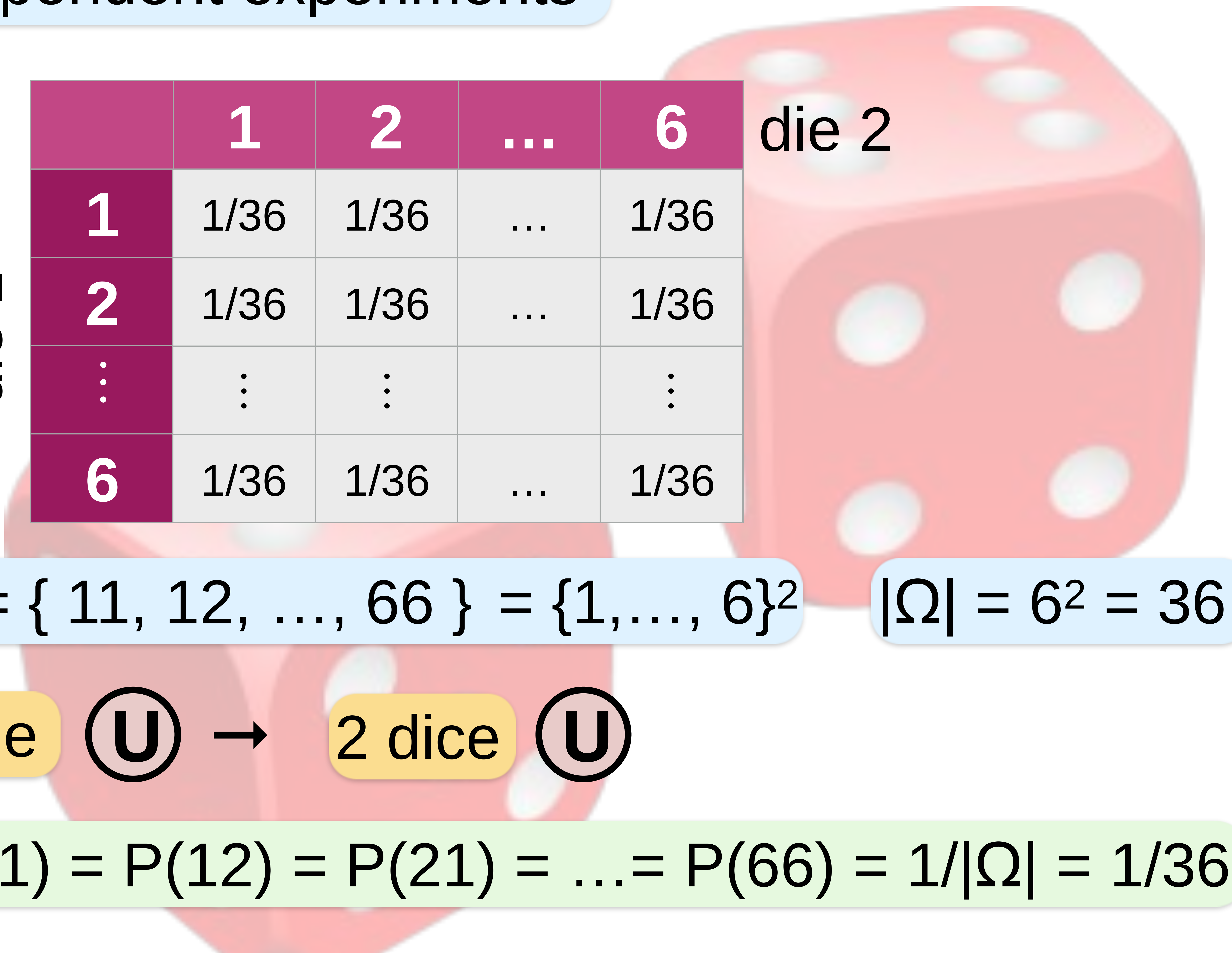
$|\Omega| = 2^2 = 4$

1 coin U → 2 coins U

$P(hh) = P(ht) = P(th) = P(tt) = 1/|\Omega| = 1/4$

Two Dice

Fair   Independent experiments



	1	2	…	6	die 2
1	1/36	1/36	…	1/36	
2	1/36	1/36	…	1/36	
⋮	⋮	⋮		⋮	
6	1/36	1/36	…	1/36	

die 1

$\Omega = \{\ 11, 12, \ldots, 66\ \} = \{1,\ldots, 6\}^2$ $|\Omega| = 6^2 = 36$

1 die U → 2 dice U

$P(11) = P(12) = P(21) = \ldots = P(66) = 1/|\Omega| = 1/36$

Events

$$P(E) = P(X \in E) = \sum_{x \in E} P(x)$$

U → P(E) = |E| / |Ω|

2 coins

$|\Omega| = 2^2 = 4$

U P(Different outcomes) = P({ht,th}) = 2 / |Ω| = 2/4 = 1/2

P(At least one h) = P({ ht,th,hh }) = 3 / |Ω| = 3/4

3 coins

$|\Omega| = 2^3 = 8$

U P(Alternating) = P({ hth, tht }) = 2 / 8 = 1/4

P(odd # h's) = P({ htt, tht, tth, hhh}) = 4 / 8 = 1/2

Sampling

Many sources of randomness

Coin

Die

…

Often sample (select) physical objects

Patients in a study

Customers at a restaurant

Products for quality control

Visitors to web pages

Cards from a deck

Balls from an urn

Two sampling types

With

Without

Replacement

Replacement

Sequentially select physical objects

With replacement

Replace (reuse) selected element

Outcomes can repeat

Experiments often independent

Like coins dice

Without replacement

Do not replace (reuse) selected element

Outcomes cannot repeat

Experiments dependent

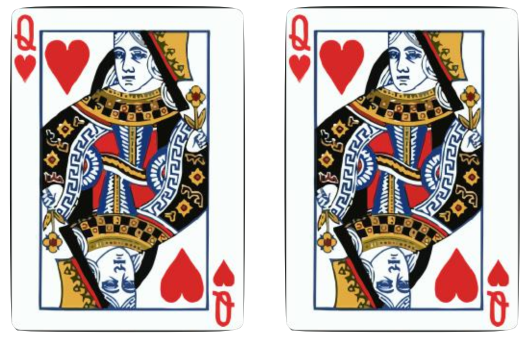
Difference largest for small Ω

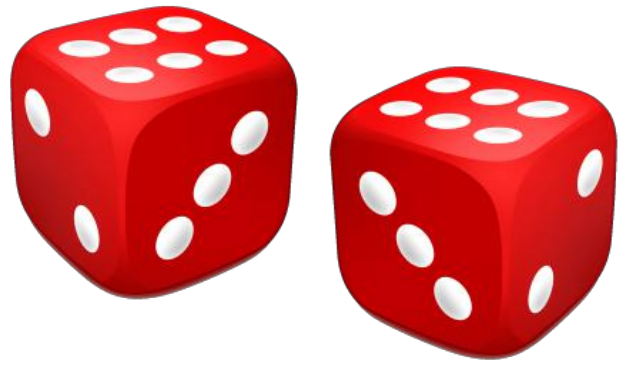
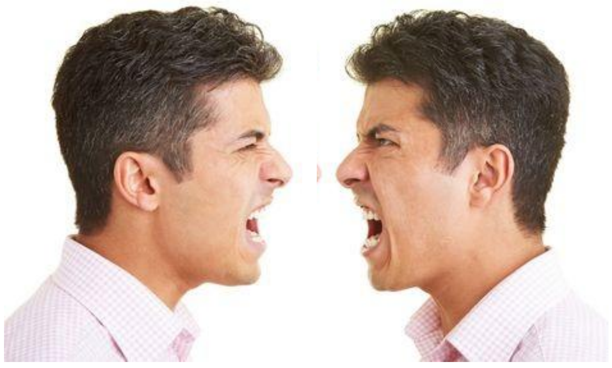
With / Without Differences

Sampling (selection) with without replacement

repeat as if from scratch repeat with smaller set

Same element can cannot be selected again

✅  coin cards  ❌

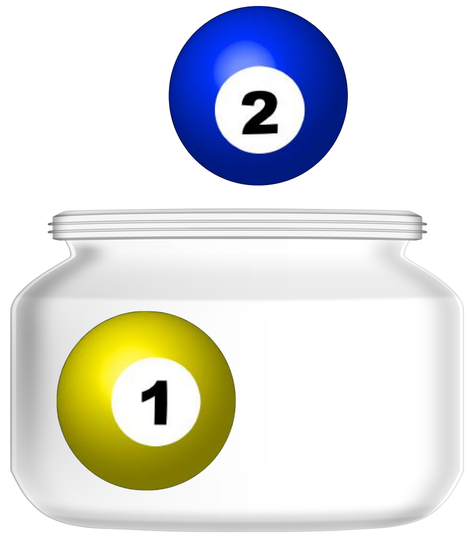
✅  die people  ❌

Examples

Balls in a Jar… with Replacement

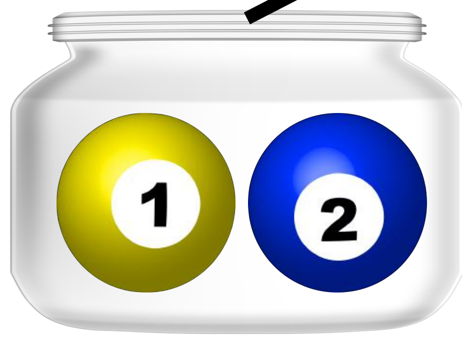
1 and 2 ball in a jar

Pick 1 ball

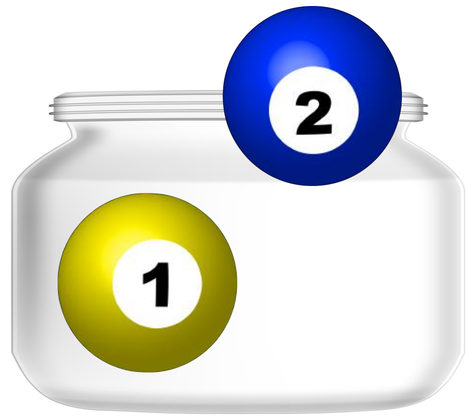


Return

Replace it

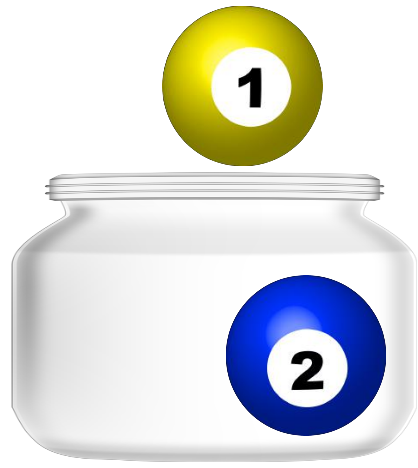


might repeat



→ 2 2

Pick again



→ 2 1

Selection with replacement

Second selection - from original set

Probabilities

2nd ball

1st ball		½ 1	½ 2
½ 1		¼	¼
½ 2		¼	¼

$\Omega = \{11, 12, 21, 22\}$

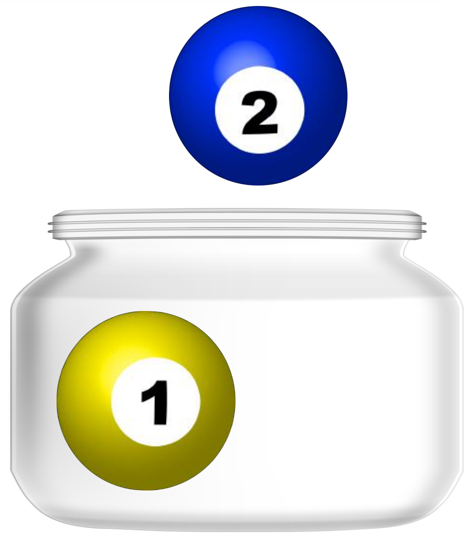
$|\Omega| = 4$

U $P(11) = \ldots = P(22) = ¼$

Balls in a Jar… without Replacement

1 and 2 ball in a jar

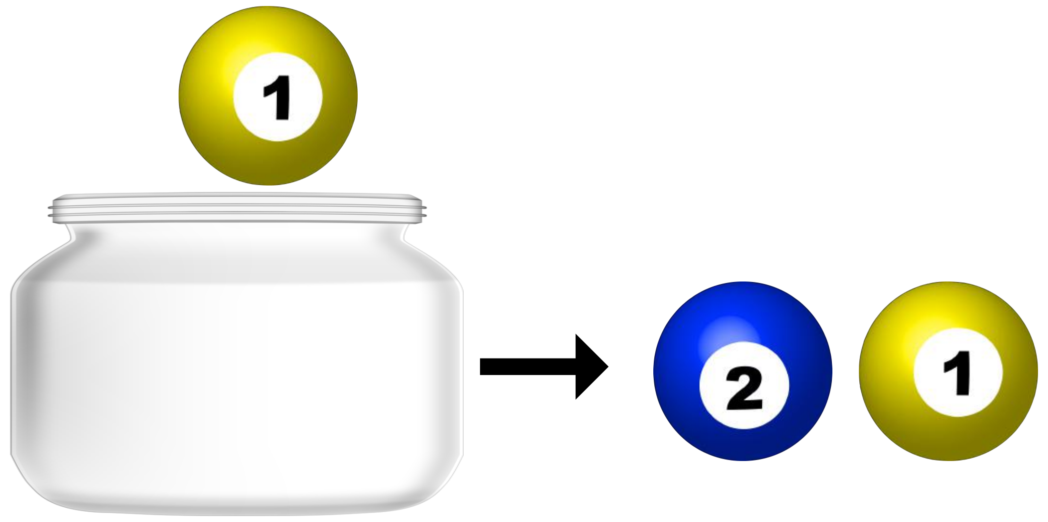
Pick one ball



Return

Do not replace

Pick second ball



Selection without replacement

Second selection - from a subset

Probabilities

		2nd ball: 1	2nd ball: 2
1st ball	½ 1	0	½
	½ 2	½	0

Ω = {1 2, 2 1}

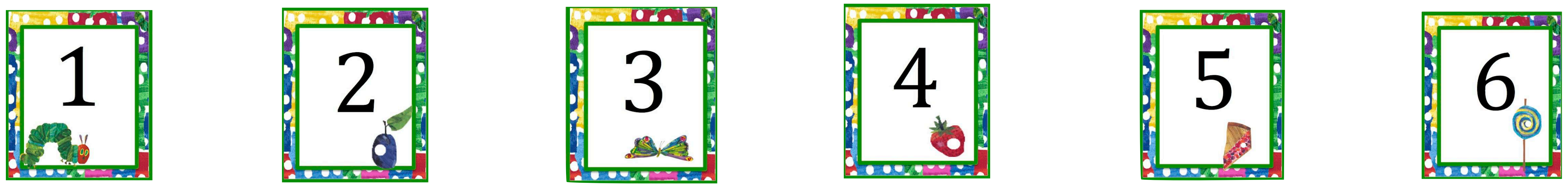
|Ω| = 2

U

P(1 2) = P(2 1) = ½

Drawing Cards

Six cards



2-permutations of {1,..,6}

Draw one

Without replacement, draw a second

Outcomes $\Omega = \{12,\ldots, 16, 21,\ldots, 26,\ldots, 65\} = (6)^{\underline{2}}$ $|\Omega| = 6^{\underline{2}} = 6\cdot 5 = 30$

Probabilities i=j Cannot happen

i≠j $P(i,j) = \frac{1}{6}\cdot\frac{1}{5} = 1/30$

$P(i,j) = 1/|\Omega| = 1/30$

U

card 2

card 1	1	2	…	6
1	0	1/30	…	1/30
2	1/30	0		1/30
⋮	⋮			
6	1/30	1/30	…	0

Replacement Summary

2 selections

	Original	With replacement	Without replacement
Description		Outcomes can repeat	Outcomes cannot repeat
Sample space	Ω	$\{ (x,y) : x,y\in\Omega \} = \Omega^2$	$\{ (x,y): x,y\in\Omega,\ x\neq y \} = \Omega^{\underline{2}}$
| Sample space |	$\|\Omega\|$	$\|\Omega\|^2$	$\|\Omega\|^{\underline{2}} = \|\Omega\|\cdot(\|\Omega\|-1)$

Probabilities

Uniformity	Unfiform	Uniform	Uniform
P (element)	$1 / \|\Omega\|$	$1 / \|\Omega\|^2$	$1 / [\ \|\Omega\|\cdot(\|\Omega\|-1)\ ]$

Repeated Experiments Replacement

Compound experiments

Repetition

Uniform → Uniform

With replacement $|\Omega| = |\Omega|^2$

Without replacement $|\Omega| = |\Omega| \cdot (|\Omega|-1)$

Part I

Order matters, or not

Part II

Repeated Experiments - Order Matters

Part I:

- Compound experiments
- Repetition: Uniform → Uniform
  - With replacement: $|\Omega| = |\Omega_1|^2$
  - Without replacement: $|\Omega| = |\Omega_1| \cdot (|\Omega_1|-1)$

Part II:

- Order matters, or not

Order Does Not Matter

So far | order matters | Card 5 → 3 | Stock 10 → 50 | 50 → 10

Sometimes | does not matter | Elections | Dem | Rep

When order does not matter | Shopping

Tuple of outcomes → Set of outcomes

(2,5) (5,2) → {2,5} | Event { (2,5), (5,2) }

(4,4) → {4,4}

← Order matters

→ Order doesn't matter

With & w/o replacement

With Replacement

2 cards ∈ {1,..,6} with replacement

Equivalently, 2 dice

Order ← Matters $P(1,1) = P(1,2) = \ldots = P(6,5) = P(6,6) = 1/36$

→ Doesn't matter $P(\{1,2\}) = P(\{(1,2), (2,1)\}) = P(1,2) + P(2,1) = 2/36$

$P(\{1,1\}) = P(1,1) = 1/36$

Not Uniform!

	1	2	…	6
1	1/36	1/36	…	1/36
2	1/36	1/36	…	1/36
⋮	⋮	⋮		⋮
6	1/36	1/36	…	1/36

Order matters

Sanity Check

$\sum$ Probabilities $= 1$

$$\binom{6}{2} \cdot \frac{2}{36} + \binom{6}{1} \cdot \frac{1}{36} = \frac{5}{6} + \frac{1}{6} = 1$$ ✅

Without Replacement

2 cards ∊ {1,..,6} without replacement

	1	2	...	6
1		1/30	1/30	1/30
2	1/30		1/30	1/30
⋮	1/30	1/30		1/30
6	1/30	1/30	1/30	

Order ← Matters $i \neq j$ $P(i,j) = \frac{1}{6} \cdot \frac{1}{5} = 1/30$

→ Doesn't matter $P(\{1,2\}) = P(\{(1,2), (2,1)\}) = P(1,2) + P(2,1) = 2/30$

{1,1} cannot happen

Uniform!

Sanity Check Probabilities sum to 1 $\binom{6}{2} \cdot \frac{2}{30} = 1$ ✅

Alternative Calculation

2 cards $\in \{1,..,6\}$ sequentially without replacement

$P(\{1,2\}) = P(1,2) + P(2,1) = 2/30$

	1	2	…	6
1		1/30	1/30	1/30
2	1/30		1/30	1/30
⋮	1/30	1/30		1/30
6	1/30	1/30	1/30	

Alternatively Select both cards simultaneously

$\Omega = \{ \{1,2\}, \{1,3\}, \dots \{5,6\} \} = \binom{[6]}{2}$

$|\Omega| = \binom{6}{2} = 15$ U

$P(\{1,2\}) = 1/15$ ✅

Sequential
Simultaneous
Same

Poker Hand Probabilities

Deck 52 cards **Hand** 5 cards

Ω = { possible hands } = {  ,  ,  , …}

≈ 50·50·10·100 = 2.5 M

$$|\Omega| = \binom{52}{5} = \frac{52!}{5!\cdot 47!} = \frac{52\cdot 51\cdot \overset{10}{\cancel{50}}\cdot 49\cdot \overset{2}{\cancel{48}}}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = 2,598,960 \approx \text{ 2.6 million}$$

All hands equally likely **Equiprobable**

U

P() ≈ 1/ 2.6 million

or any other hand More to come

Repeated Experiments - Order Matters

Repetition	Uniform → Uniform	Part I
With replacement	$\|\Omega\| = \|\Omega_1\|^2$	
Without replacement	$\|\Omega\| = \|\Omega_1\| \cdot (\|\Omega_1\|-1)$	

Order doesn't matter	With replacement	non-uniform	$P(\{1,2\}) = 2 \cdot P(\{1,1\})$	Part II
	Without replacement	uniform	$\{1,1\} \notin \Omega$