Combinatorics Permutations

Permutations

Ordering of elements

permutations of n-set

n factorial

n!

Anagrams

Word reorderings

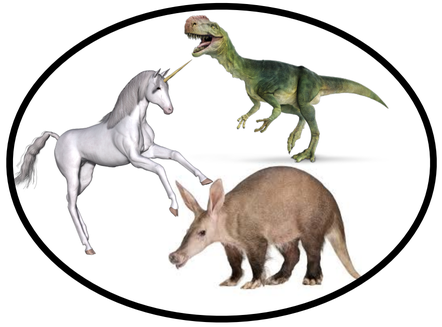
Circular orders

Musical Chairs, Bryant Park, NYC, Summer 2013

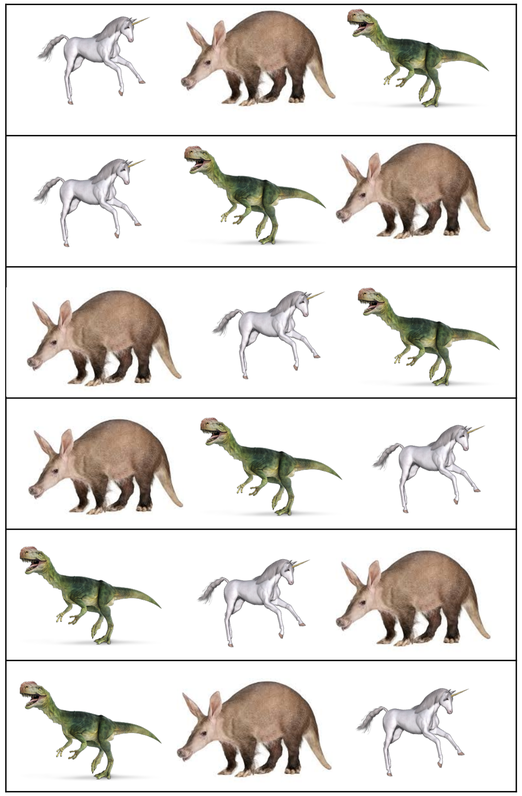
Permutations

A permutation of a set is an ordering of its elements

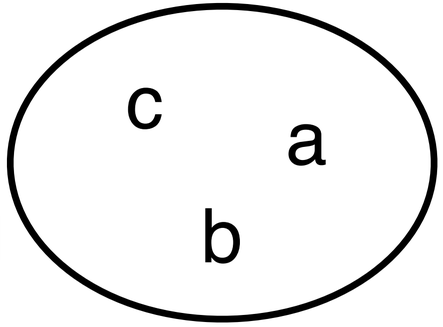
Set



Permutations



#	
1	abc
2	acb
3	bac
4	bca
5	cab
6	cba

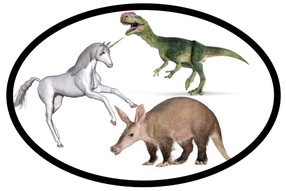
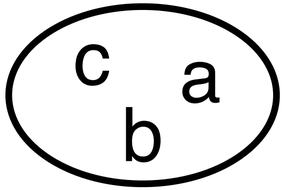


permutations of a set

determined by just set size

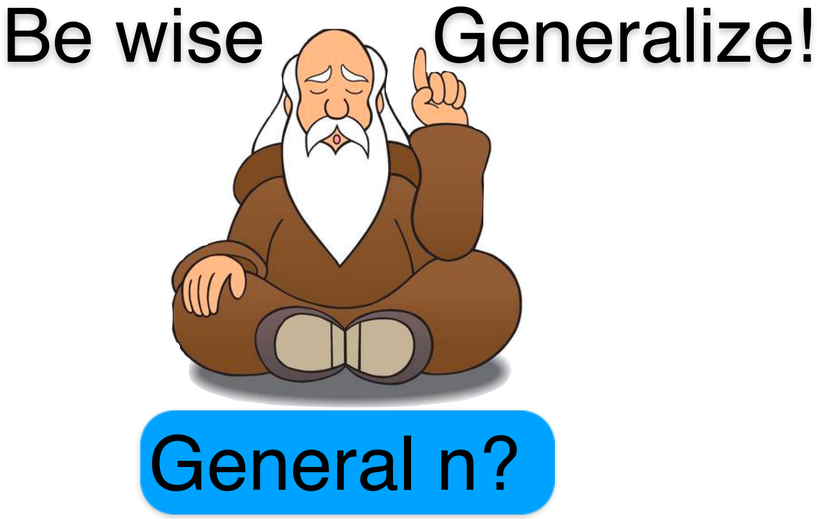
permutations of an n-set?

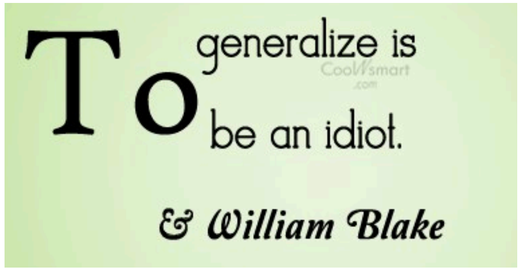
set of size n

 or  ?

Permutations of Small Sets

n	1	2	3
Set	{a}	{a,b}	{a,b,c}
Permutations	a	a b b a	a b c a c b b a c b c a c a b c b a
#	1	2	6



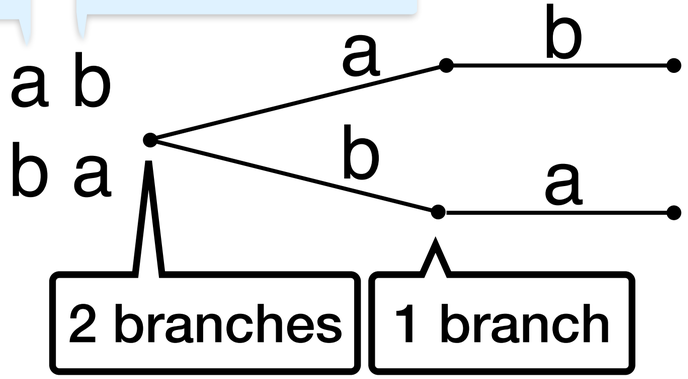


Permutations of an n-set

2 objects

2 choices

1 choice



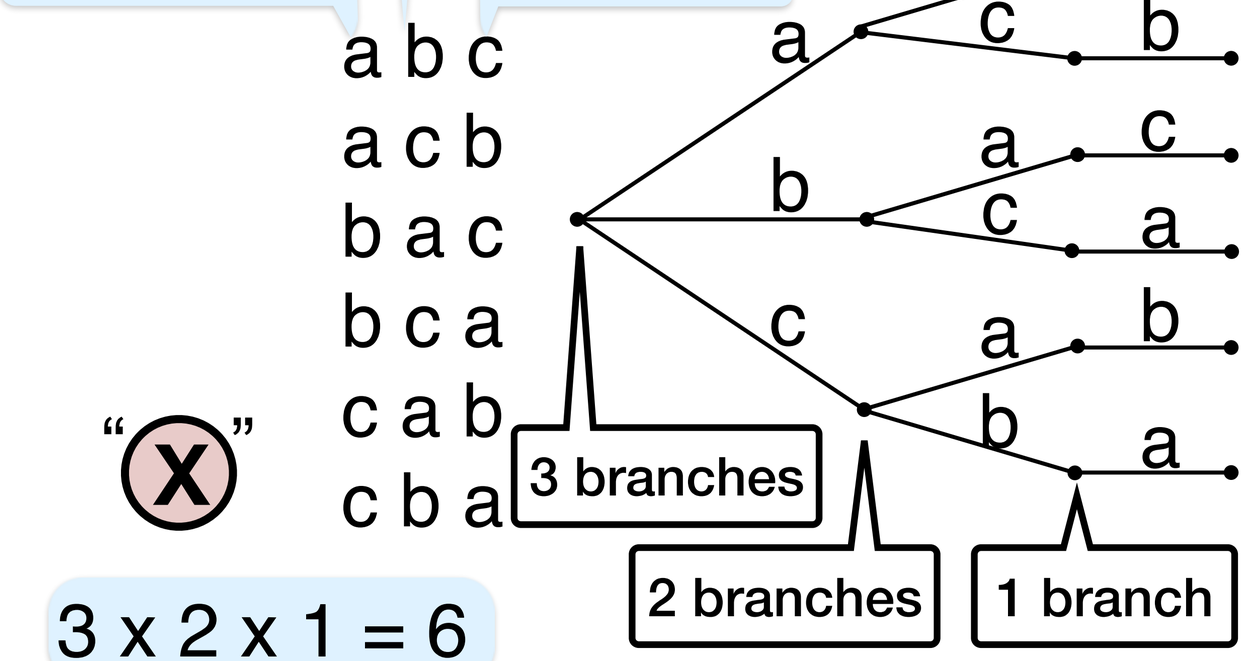
2 x 1 = 2

3 objects

2 choices

3 choices

1 choice



“X”

3 x 2 x 1 = 6

permutations of an n-set

n × (n-1) × … × 2 × 1 ≜ n!

n factorial

0 Factorial

For $n \geq 1$ $n!$ = # permutations of an n-set = $n \times (n-1) \times \ldots \times 2 \times 1$

$0! = ?$ 0? 1?

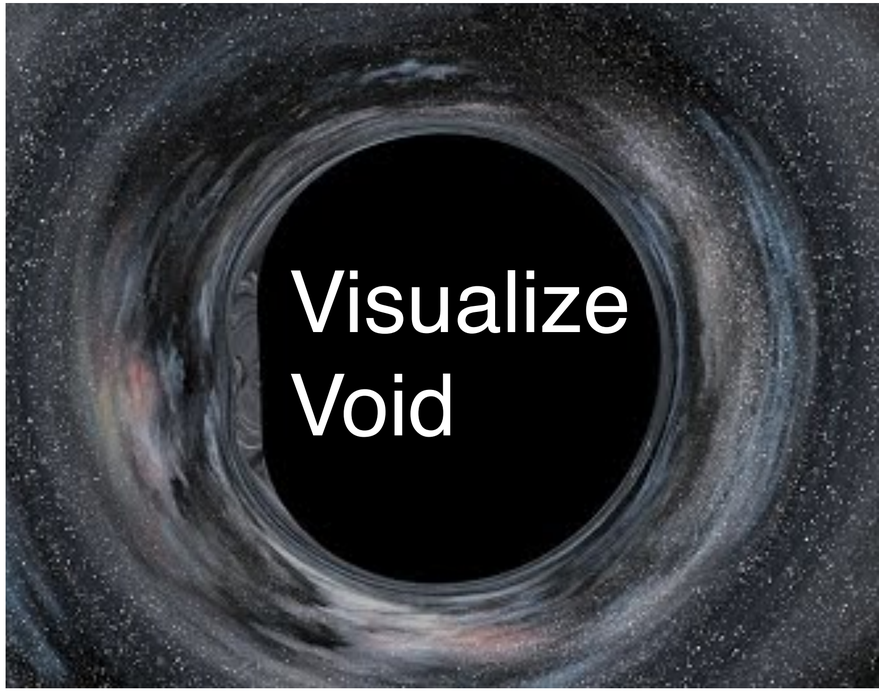
How many ways can you permute 0 objects?

n	Set	Permutations	n!
2	{a, b}	ab, ba	2
1	{a}	a	1
0	{}		?

Invisible

(Permutations)	n!
(ab), (ba)	2
(a)	1
( )	1

Surround permutations by ()



$0! = 1$ Exact same reason as $2^0=1$

Beautiful but Backwards

So far showed

permutations

$n! = n \times (n-1) \times \ldots \times 2 \times 1$

Product of consecutive integers

What nicer after 2^n ?

Why did we get numbers in reverse order?

Is there a natural proof for

$n! = 1 \times 2 \times 3 \times \ldots \times n$

Alternative Factorial View

n! n x (n-1) x … x 2 x 1 → 1 x 2 x 3 x … x n

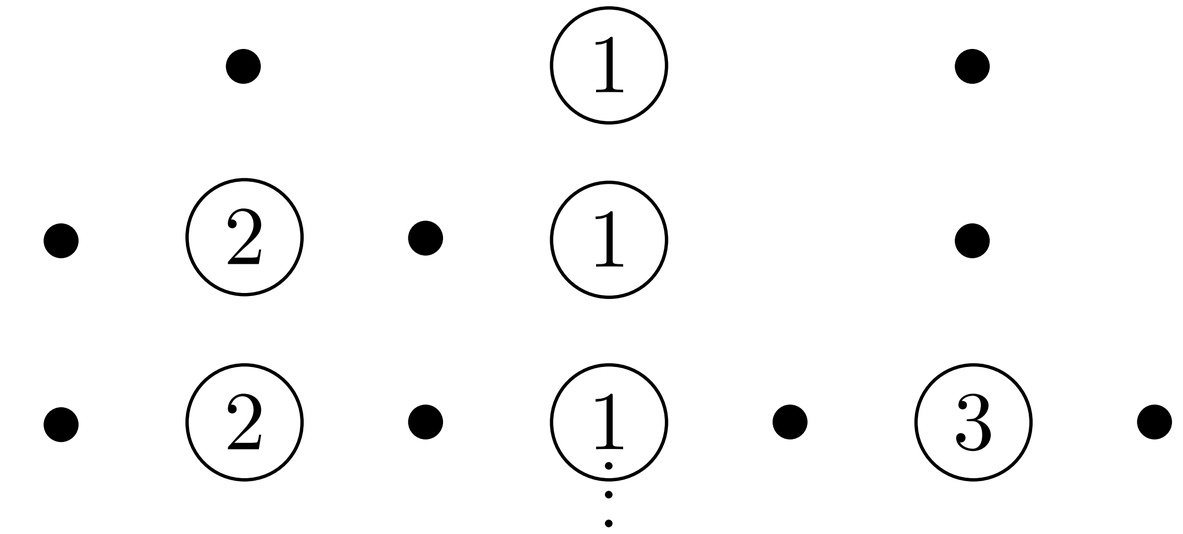
Write elts Left to right → Smallest to largest

One position for the 1st (smallest) element

Two for 2nd

Three for 3rd

Four for 4th



n! = 1 x 2 x 3 x … x n

(n-1)! → n!

n! simply related to (n-1)!

$$n! = \underbrace{1 \cdot 2 \cdot \ldots \cdot (n-1)}_{(n-1)!} \cdot n$$

$$= (n-1)! \cdot n$$

n! = (n-1)! · n ∀ n ≥ ~~2~~ 1

From definition of 0!=1
1! = 1·1 = 1·0!
Hence also works for n=1

n	Product	n!	
0	1	1	x 1
1	1	1	x 2
2	1 × 2	2	x 3
3	1 × 2 × 3	6	x 4
4	1 × 2 × 3 × 4	24	x 5
5	1 × 2 × 3 × 4 × 5	120	x 6
6	1 × 2 × 3 × 4 × 5 × 6	720	

Can determine n! recursively

Examples & Applications

Basic Permutations

orders to visit 3 cities

LA, SD, SF

LA SD SF
....
SF SD LA

$3! = 3 \times 2 \times 1 = 6$

ways to rank 4 students

Don, Hil, Jeb, Liz

Don > Hil > Jeb > Liz
....
Liz > Jeb > Hil > Don

$4! = 4 \times 3 \times 2 \times 1 = 24$

Anagrams

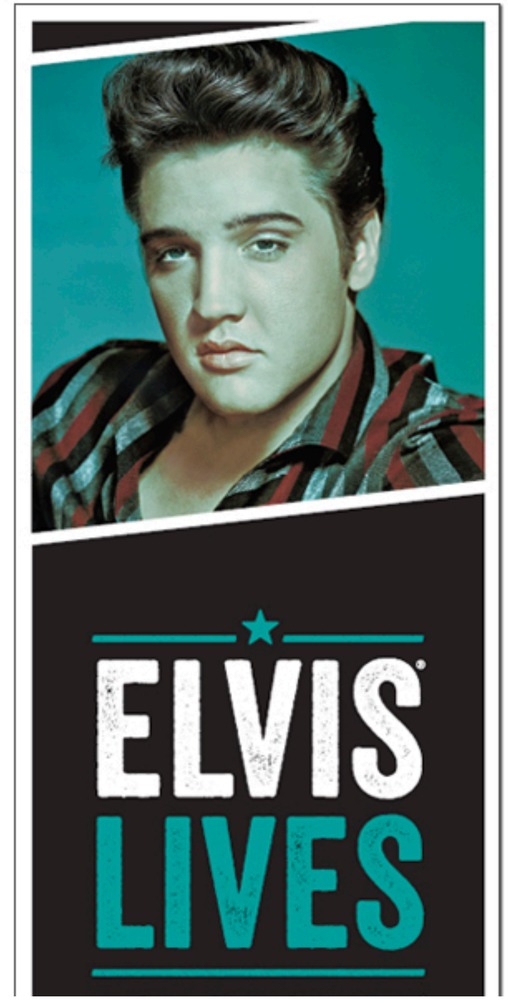
Definition of *Anagram* (Entry 1 of 2)

1 : a word or phrase made by transposing the letters of another word or phrase

// The word "secure" is an *anagram* of "rescue."

LISTEN

SILENT



Ignore whether the word has a meaning

For now, all letters distinct

Later, letters repeat

Anagrams of PEARS

PEARS
SPEAR
....
EAPRS

All permutations of {P,E,A,R,S}

5! = 5 x 4 x 3 x 2 x 1 = 120

Constrained Anagrams of PEARS

A,R stay adjacent in order

PARSE
.....
SEPAR

Permutations of P E AR S 4! = 4 x 3 x 2 x 1 = 24

A,R are adjacent in either order

SPARE
.....
RAESP

2 orders 24 anagrams each "X" 2 x 24 = 48

A,R are not adjacent

AESPR
.....
SRPAE

− 5! - 48 = 120 - 48 = 72

More Constrained Permutations

ways 3 distinct boys and 2 distinct girls can stand in a row

Unconstrained

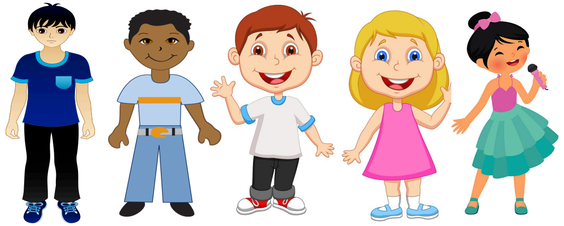
(3+2)! = 5! = 120

Alternating boys and girls

must be b,g,b,g,b

3! x 2! = 6 x 2 = 12 ⊗

Boys together and girls together



3b,2g or 2g,3b

2 x 3! x 2! = 24

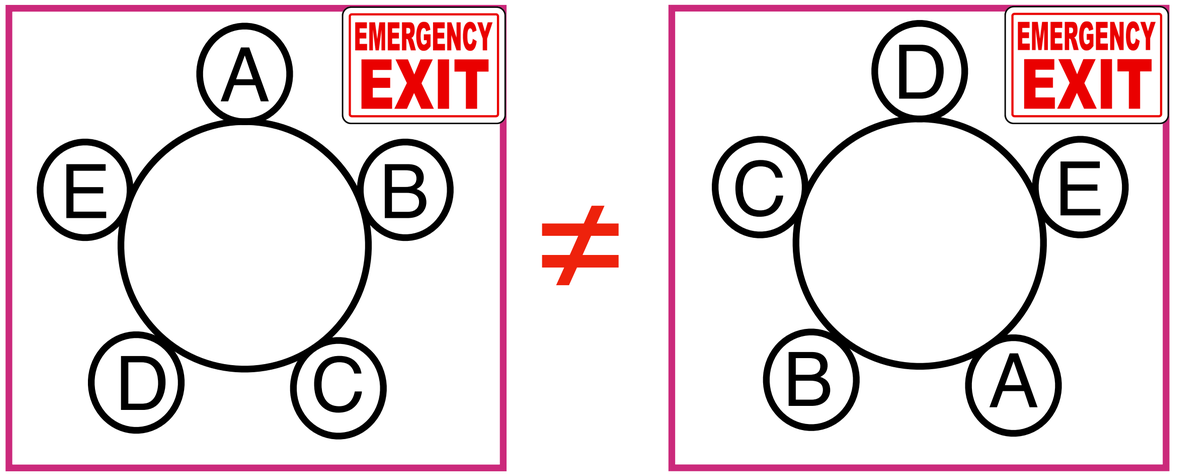
Unconstrained, but orientation (left to right) doesn't matter

5! / 2 = 60

Circular Arrangements

ways 5 people can sit at a round table = ?

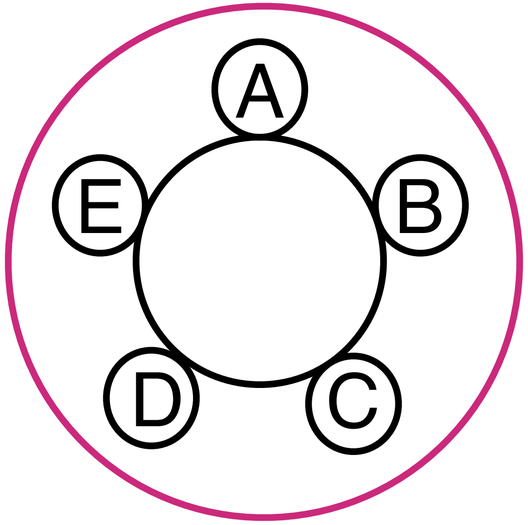
Rotations matter



≠

5! = 120

Rotations don't matter



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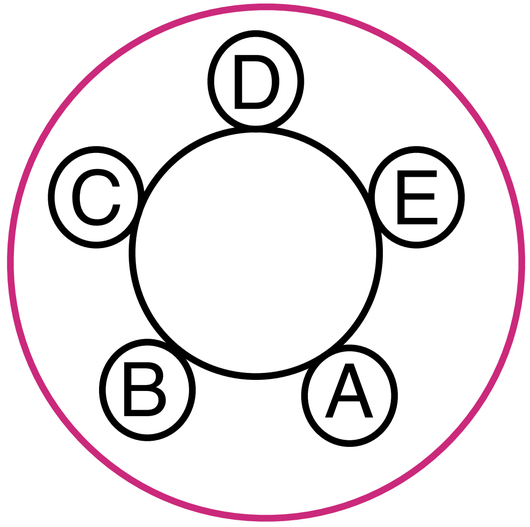
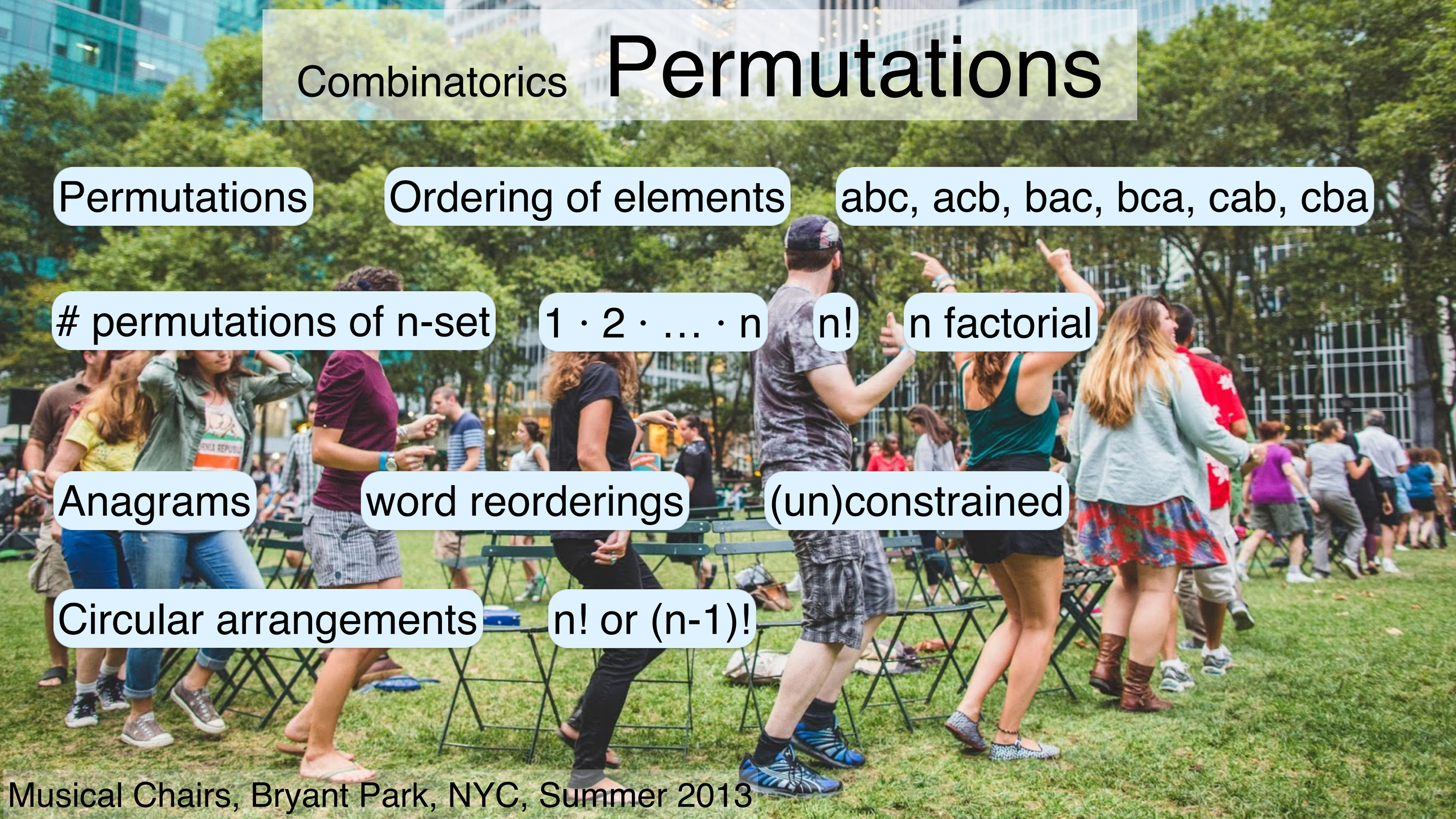


Alternatively, start with A and arrange 4 others clockwise

5! / 5 = 4!
= 24

Combinatorics Permutations



Permutations

Ordering of elements

abc, acb, bac, bca, cab, cba

permutations of n-set

$1 \cdot 2 \cdot \ldots \cdot n$

$n!$

n factorial

Anagrams

word reorderings

(un)constrained

Circular arrangements

n! or (n-1)!

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