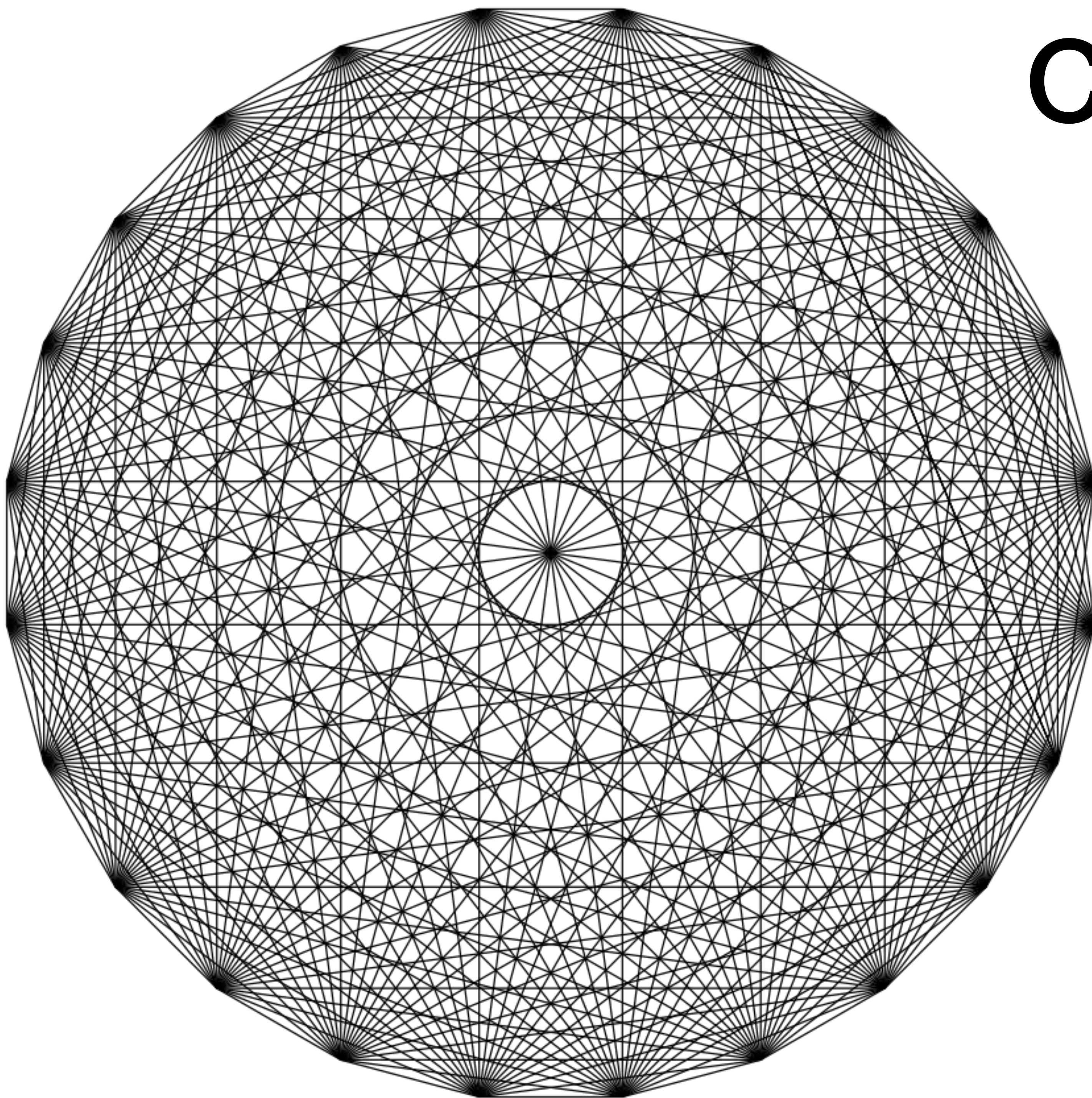


# Combinations



# k-subsets

A k-element set is called a **k-set**, a k-element subset is a **k-subset**

$\binom{[n]}{k}$  collection of k-subsets of  $[n] = \{1, 2, \dots, n\}$

$\binom{[3]}{1}$   $\{ \{1\}, \{2\}, \{3\} \}$

$\binom{[3]}{2}$   $\{ \{1, 2\}, \{1, 3\}, \{2, 3\} \}$

$\binom{[4]}{2}$   $\{ \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\} \}$



# Sequences with k 1's

$\binom{[n]}{k}$  collection of k-subsets of  $[n] = \{1, 2, \dots, n\}$

1-1 correspondence to n-bit sequences with k 1's

	Subsets	Binary Sequences
$\binom{[3]}{1}$	$\{1\}, \{2\}, \{3\}$	100, 010, 001
$\binom{[3]}{2}$	$\{1, 2\}, \{1, 3\}, \{2, 3\}$	110, 101, 011
$\binom{[4]}{2}$	$\{1, 2\}, \{1, 3\}, \dots, \{3, 4\}$	1100, 1010, ..., 0011

# Two Interpretations

$\binom{[n]}{k}$       k-subsets of an n-set      n-bit sequences with k 1's

$\binom{[3]}{2}$        $\{\{1,2\}, \{1,3\}, \{2,3\}\}$       110, 101, 011

1 - 1 correspondence

Same number of elements

Mostly count sequences

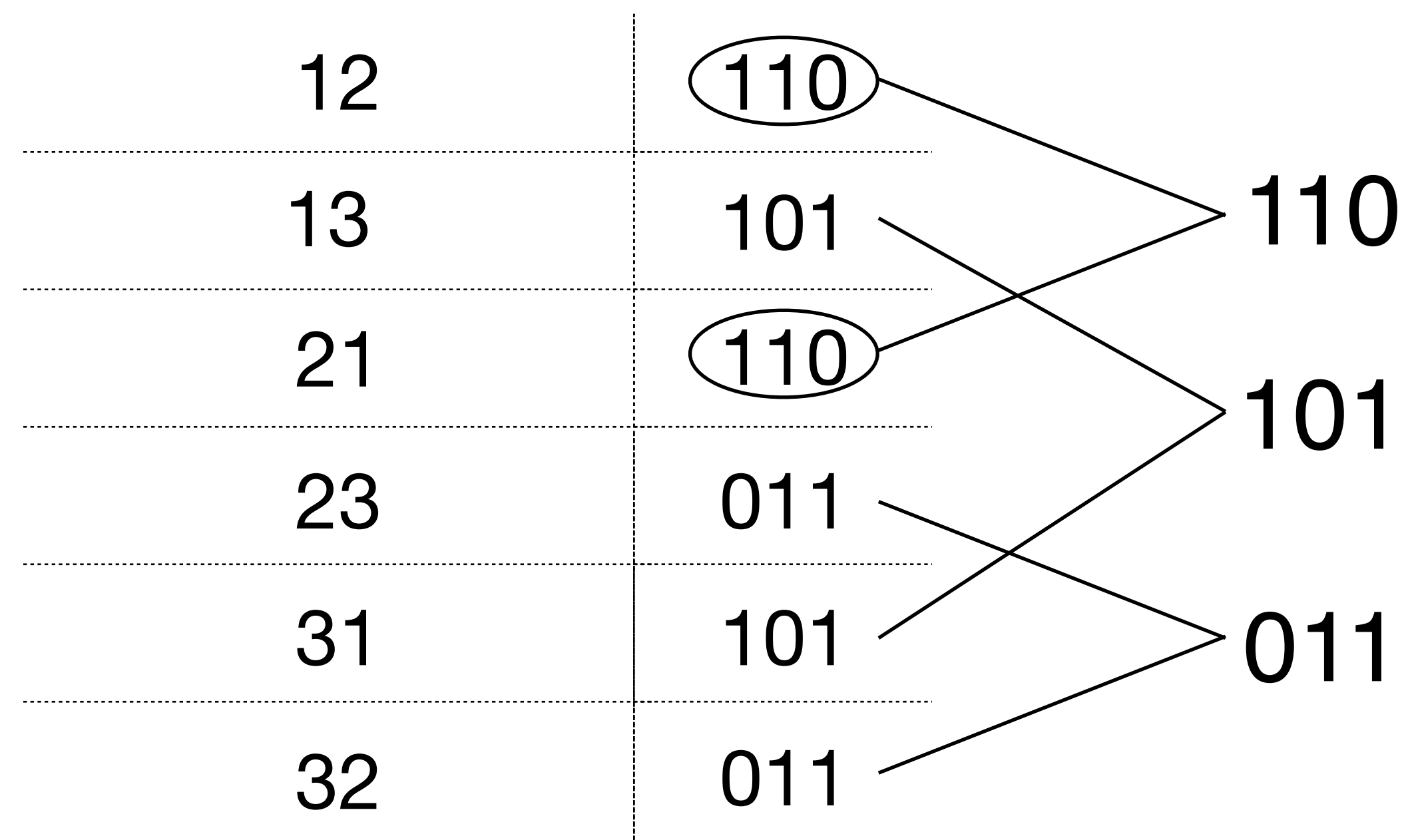
Same applies to subsets

# Number of n-Bit Sequences with k 1's

$$\binom{n}{k} \triangleq \left| \binom{[n]}{k} \right| = \# \text{ n-bit sequences with k 1's} \quad \boxed{\text{binomial coefficient}}$$

$$\binom{3}{2} = \left| \binom{[3]}{2} \right| = |\{110, 101, 011\}| = 3$$

Locations of 1's: Ordered Pairs from  $\{1,2,3\}$        $\# = 3^2 = 6$



$$\binom{3}{2} = \frac{3^2}{2} = \frac{6}{2} = 3$$

# Calculating the Binomial Coefficients

$$\binom{n}{k} \triangleq \left| \binom{[n]}{k} \right| = ?$$

number      set

Each location  $\in [n]$

Specify locations of the  $k$  1's in order

123, 531, 213, ...

# ordered locations       $n^k$

Every binary sequence with  $k$  1's corresponds to  $k!$  ordered locations

10101  $\leftrightarrow$  1,3,5    1,5,3    3,1,5    3,5,1    5,1,3    5,3,1

$$k! \binom{n}{k} = n^k$$

$$\binom{n}{k} = \frac{n^k}{k!} = \frac{n!}{k!(n-k)!}$$

$$\left| \binom{[n]}{k} \right| = \binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{for small } n, k$$

$$\binom{[3]}{1} = \left\{ \begin{array}{c} 001 \\ 010 \\ 100 \end{array} \right\}$$

$$\binom{3}{1} = \frac{3!}{1!2!} = 3$$

Choose location of 1

$$\binom{[3]}{2} = \left\{ \begin{array}{c} 011 \\ 101 \\ 110 \end{array} \right\}$$

$$\binom{3}{2} = \frac{3!}{2!1!} = 3$$

Choose location of 0

$$\binom{[4]}{2} = \left\{ \begin{array}{c} 0011 \\ 0101 \\ 0110 \\ 1001 \\ 1010 \\ 1100 \end{array} \right\}$$

$$\binom{4}{2} = \frac{4!}{2!2!} = 6$$

Choose location of 1's  
1st location: 4 choices  
2nd location: 3 choices  
Each chosen twice

# Simple $\binom{n}{k}$

$$\binom{n}{0} = \frac{n!}{0! n!} = 1$$

All-zero sequence

$$\binom{n}{n} = \frac{n!}{n! 0!} = 1$$

All-one sequence

$$\binom{n}{1} = \frac{n!}{1! (n-1)!} = n$$

Choose location of single 1

$$\binom{n}{2} = \frac{n!}{2! (n-2)!} = \frac{n(n-1)}{2}$$

1st location:  $n$  ways  
2nd location:  $n-1$  ways  
Each sequence chosen twice



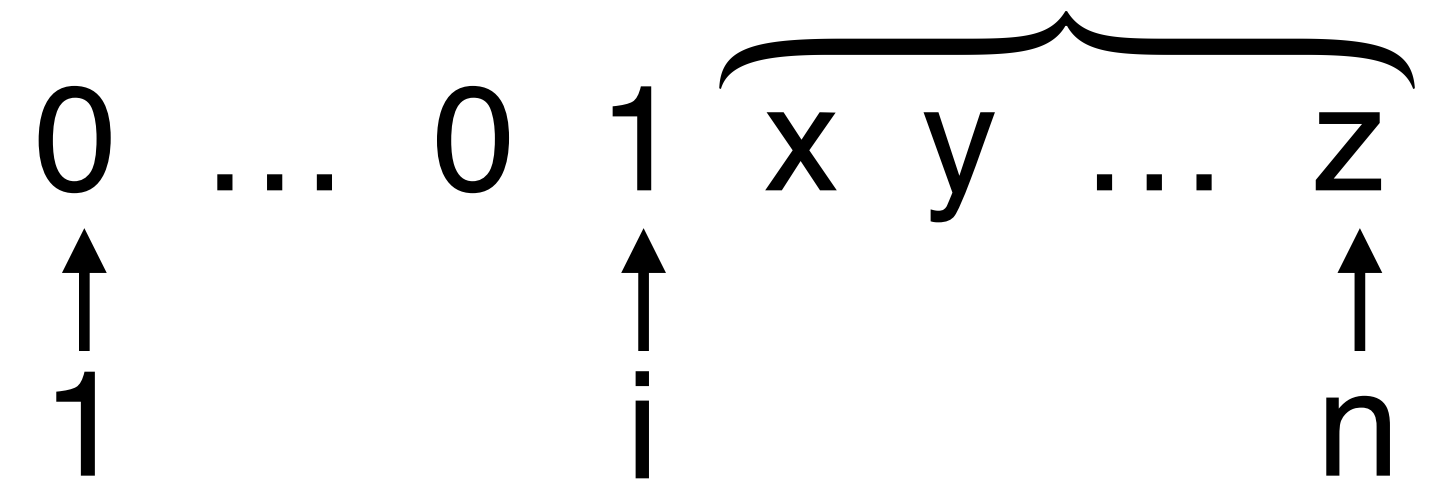
# Alternative Explanation of $\binom{n}{2} = \frac{n(n-1)}{2}$

$$\binom{[n]}{2} = \{\text{n-bit strings with two 1's}\}$$

$$A_i = \{x^n: \text{first 1 at location } i\} \quad (1 \leq i \leq n-1)$$

n-i locations w/  
exactly one 1

$$|A_i| = n - i \quad A_i\text{'s disjoint}$$



$$\binom{[n]}{2} = A_1 \uplus A_2 \uplus \dots \uplus A_{n-1}$$

$$\binom{n}{2} = |\binom{[n]}{2}| = |A_1| + \dots + |A_{n-1}| = (n-1) + \dots + 1 = \frac{n(n-1)}{2}$$

# Binomial Coefficients by Hand

Simply calculate binomial coefficients

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!}$$

$$\binom{7}{2} = \frac{\cancel{7 \cdot 6}^3}{\cancel{2 \cdot 1}} = 7 \cdot 3 = 21$$

$$\binom{n}{k} = \binom{n}{n-k}, \text{ if } k > \frac{n}{2} \text{ calculate } \binom{n}{n-k}$$

$$\binom{7}{5} = \binom{7}{2} = 21$$

Cancel terms

$$\binom{12}{9} = \binom{12}{3} = \frac{\cancel{12 \cdot 11 \cdot 10}^2}{\cancel{3 \cdot 2 \cdot 1}} = 220$$

Can calculate fairly large binomial coefficients by hand

# Permutations or Combinations?

Permutations

Order matters

Combinations

Order doesn't matter

Type of lock you used to secure your bike?

Combinations





# Combinations

$\binom{[n]}{k}$  Collection of k-element subsets of [n]  $\binom{[3]}{2} = \{ \{1,2\}, \{1,3\}, \{2,3\} \}$

$\binom{n}{k} \triangleq \left| \binom{[n]}{k} \right|$  Number of k-element subsets of [n]  $\binom{3}{2} = \left| \binom{[3]}{2} \right| = 3$

n choose k

Binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$\binom{n}{k} = \#$  n-bit sequences with k 1's