Combinatorics

# Stars and Bars

Final application

Beyond combinatorics - algebra

Simple derivation of unintuitive result

# Counting Sums

# ways to write 5 as a sum of 3 positive integers, when order matters

# ways to partition 5 items into 3 groups, when order matters

3 + 1 + 1 ☆ ☆ ☆ | ☆ | ☆

2 + 2 + 1 ☆ ☆ | ☆ ☆ | ☆

2 + 1 + 2 ☆ ☆ | ☆ | ☆ ☆

1 + 3 + 1 ☆ | ☆ ☆ ☆ | ☆

1 + 2 + 2 ☆ | ☆ ☆ | ☆ ☆

1 + 1 + 3 ☆ | ☆ | ☆ ☆ ☆

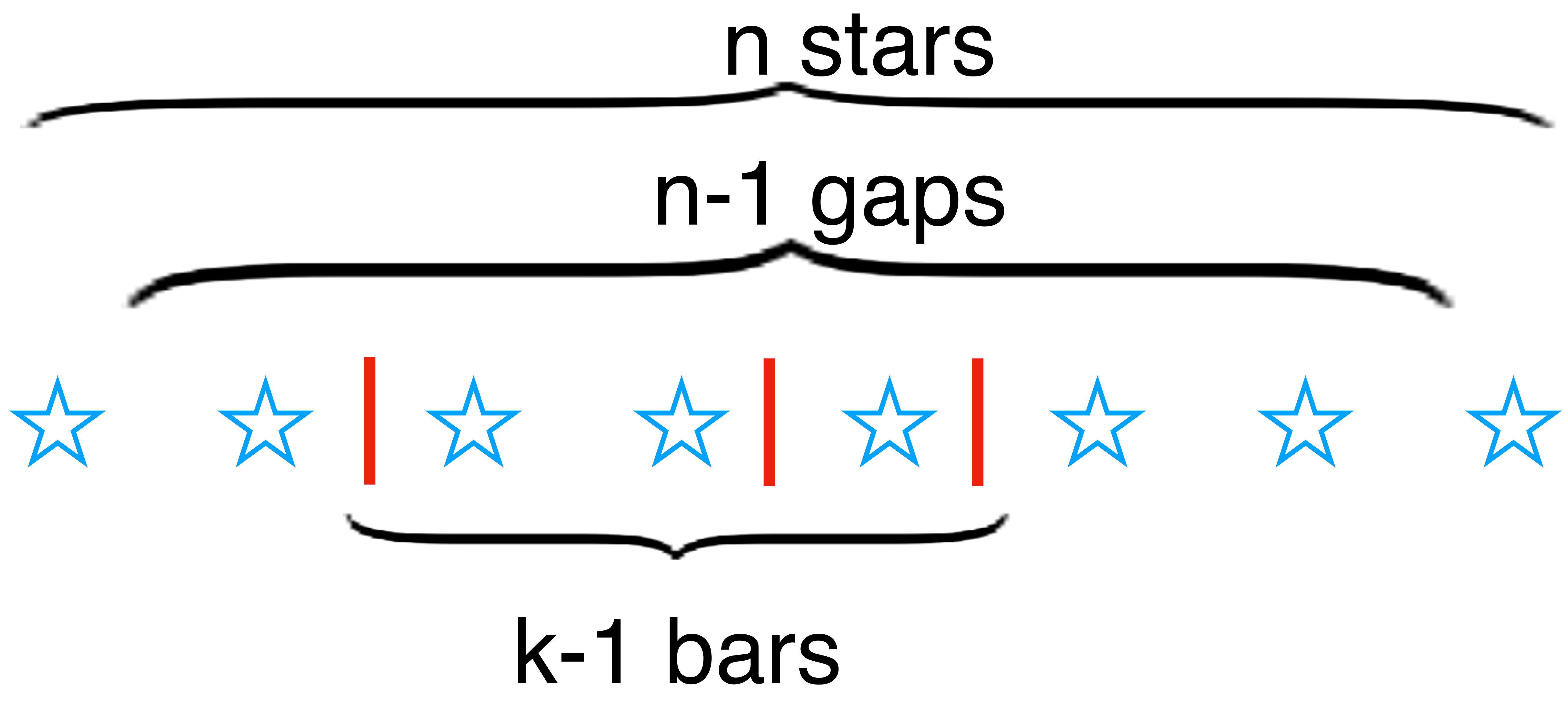
# = 6

| Addition | Partition |
|---|---|
| sum to 5 | 5 stars (items) |
| 3 positive terms | 3 consecutive star intervals |
| 2 +'s separating the numbers | 2 bars separating the intervals |
| | 4 inter-star gaps |
| | Choose 2 of 4 gaps |

$$\binom{4}{2} = 6$$

# k Terms Adding to n

# ways to write n as a sum of k positive integers, when order matters




$$\boxed{\#\text{ sums} = \binom{n-1}{k-1}}$$

# Simple Examples

# ways to write n as sum of k positive integers, when order matters $= \binom{n-1}{k-1}$

| k | sums | $\binom{n-1}{k-1}$ |
|---|---|---|
| 1 | n | $\binom{n-1}{1-1} = 1$ |
| n | 1+1+…+1 | $\binom{n-1}{n-1} = 1$ |
| 2 | 1+(n-1), 2+(n-2), …, (n-1)+1 | $\binom{n-1}{2-1} = n-1$ |
| n-1 | 2+1+..+1, …, 1+..+1+2 | $\binom{n-1}{n-2} = n-1$ |

8 as sum of 4 positive integers $\binom{8-1}{4-1} = \binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$

# Any Sum to n

# ways to write n as a sum of (any # of) positive integers

n=3

| | |
|---|---|
| 3 | ☆ ☆ ☆ |
| 1+2 | ☆ \| ☆ ☆ |
| 2+1 | ☆ ☆ \| ☆ |
| 1+1+1 | ☆ \| ☆ \| ☆ |

# = 4 = $2^2$

general n

$$\overbrace{☆ \quad ☆ \quad ☆ \quad ☆}^{n}$$

bar or no bar

# = $2^{n-1}$


n as sum of k∈[n]: $\binom{n-1}{k-1}$

$$\sum_{k=1}^{n} \binom{n-1}{k-1} = \sum_{i=0}^{n-1} \binom{n-1}{i} = 2^{n-1}$$

# Nonnegative Terms

\# ways to write n as sum of k nonnegative integers

2 as sum of 3 nonnegatives

| | |
|---|---|
| 2+0+0 | ☆ ☆ \| \| |
| 0+2+0 | \| ☆ ☆ \| |
| 0+0+2 | \| \| ☆ ☆ |
| 1+1+0 | ☆ \| ☆ \| |
| 1+0+1 | ☆ \| \| ☆ |
| 0+1+1 | \| ☆ \| ☆ |

\# = 6

As before n ☆ k-1 |

Now ☆ and | can appearing any order

sum ↔ ☆ and |

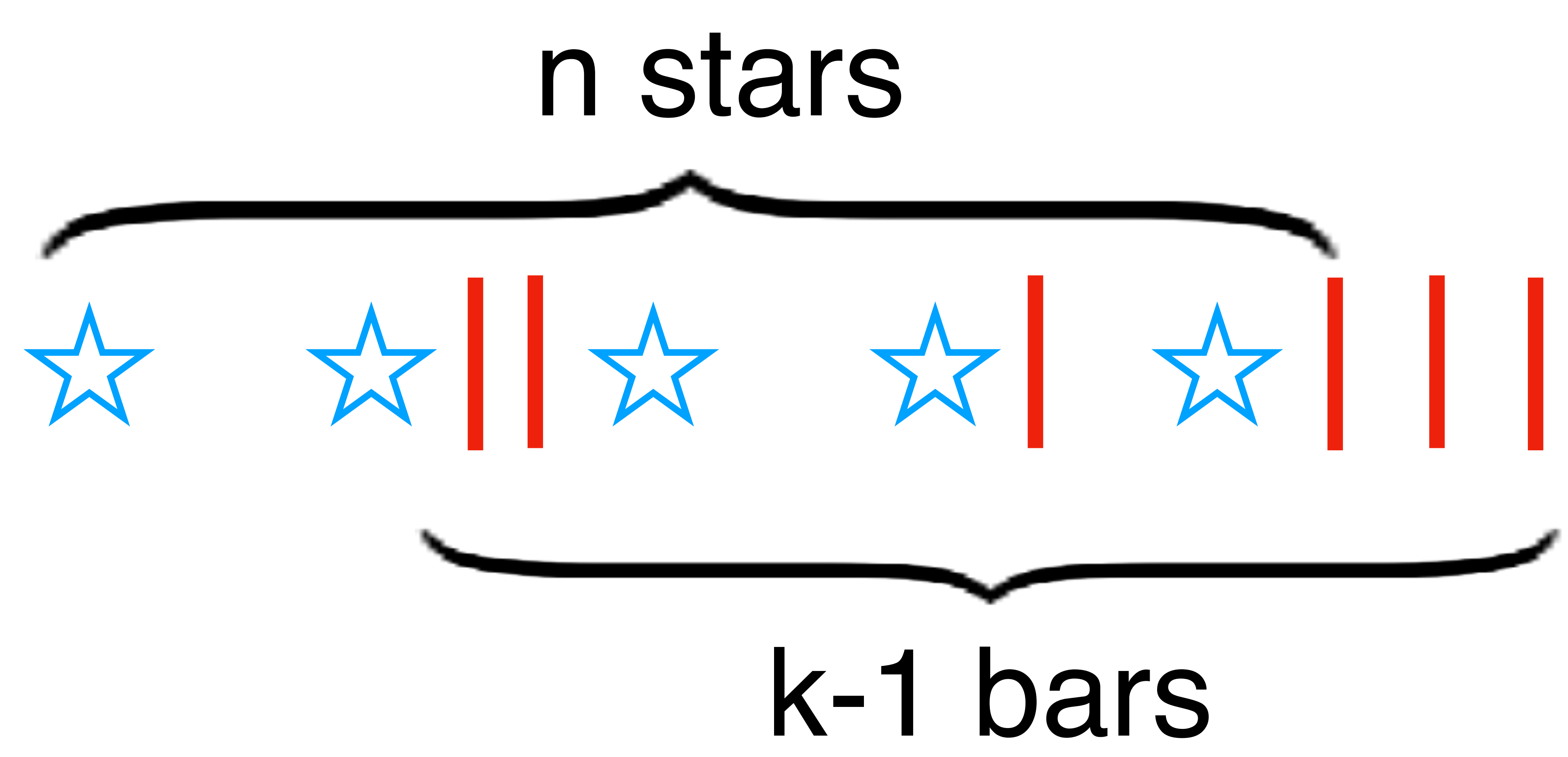
\# sums = # ☆ and |

\# ways to order 2 ☆ and 2 |

$\binom{4}{2}$ = 6

# k Non-negatives Adding to n

# ways to write n as a sum of k non-negative integers, when order matters




# sequences with n stars and k-1 bars

# length n+(k-1) sequences with k-1 bars

$$\boxed{\binom{n+k-1}{k-1}}$$

# 4-Letter Words


# 4-letter words when order matters

$26^4 = 456,976$

# 4-letter words when order does not matter

evil = vile = live = veil = eilv = liev = …

doom = mood = odom = …

Determined by composition:

#a, #b, #c, …, #z

#a + #b + #c + … + #z = 4

26 nonnegative terms

k=26

sum to 4

n=4

$$\binom{4+26-1}{26-1} = \binom{29}{25} = \binom{29}{4}$$

= 23,751

a little ≥ $26^4 / 4!$

# More Applications

Can derive # positive adding to n

# k positive adding to n = # k non-negative adding to n-k

1+2+1+3 = 7    0+1+0+2 = 7-4 = 3

$$\binom{(n-k)+k-1}{k-1} = \binom{n-1}{k-1}$$

Can derive # non-negative adding to ≤ n

# k non-negative adding to ≤ n = # k+1 non-negative adding to n

2+0+3 ≤ 7    2+0+3+2 =7

$$\binom{n+(k+1)-1}{(k+1)-1} = \binom{n+k}{k}$$

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Probability