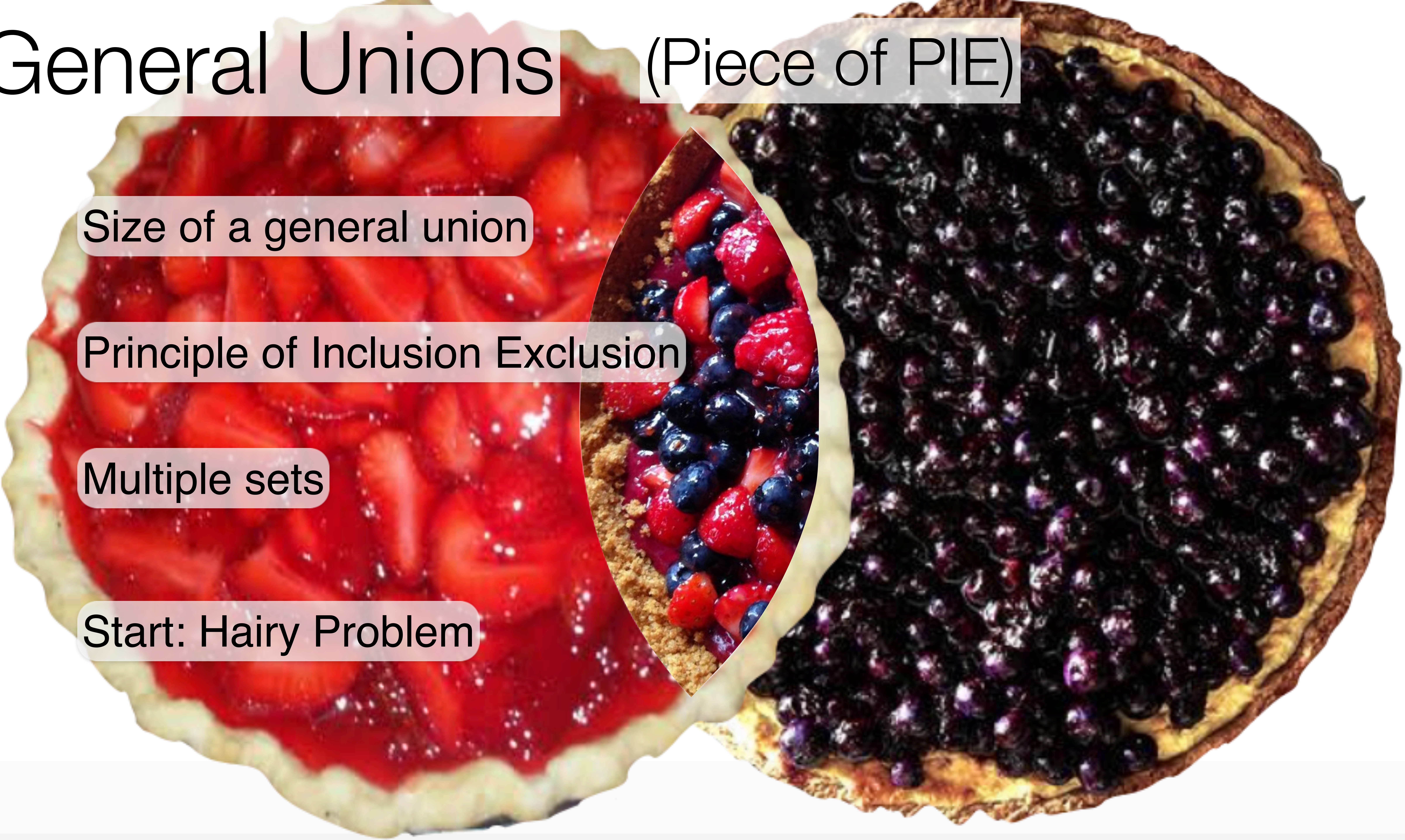
General Unions (Piece of PIE)



Size of a general union

Principle of Inclusion Exclusion

Multiple sets

Start: Hairy Problem

Hairy Problem

A friend claims she can determine the size of any set instantly and exactly

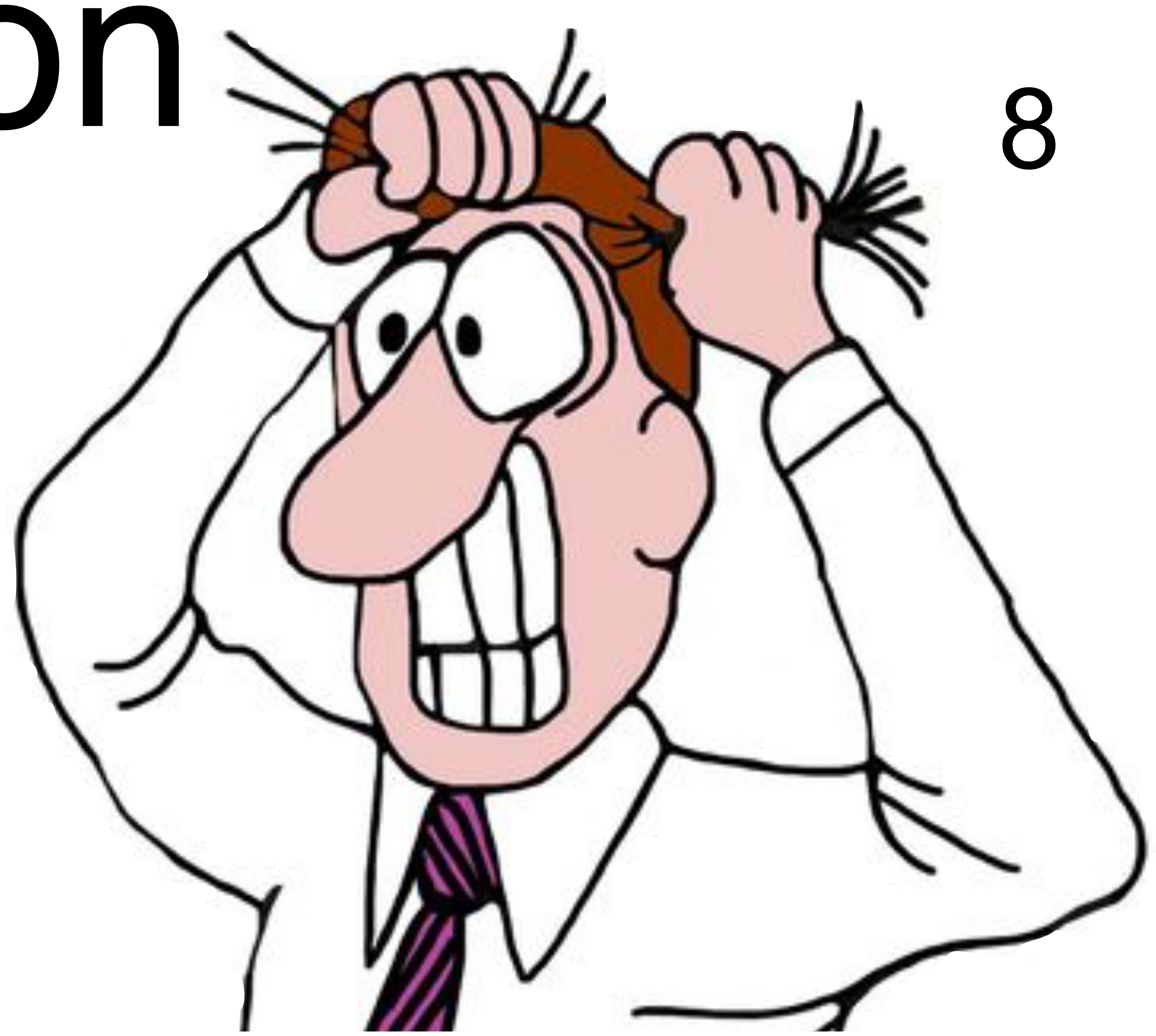
She can determine the exact # of hairs on your head

Can you ask her some questions to be fairly certain if she tells the truth?

Hint

It doesn't just work for humans

Simple Solution



Subtraction Rule

Ask her how many hairs you have

Remove a small number of hairs, say 8

Ask her how many hairs you have now

Difference between her answers should be # hairs removed (8)

Can you ask a single question?

Ask just how many hairs you removed

8 hairs

Zero-knowledge proofs

Prove identity without revealing password

General Unions

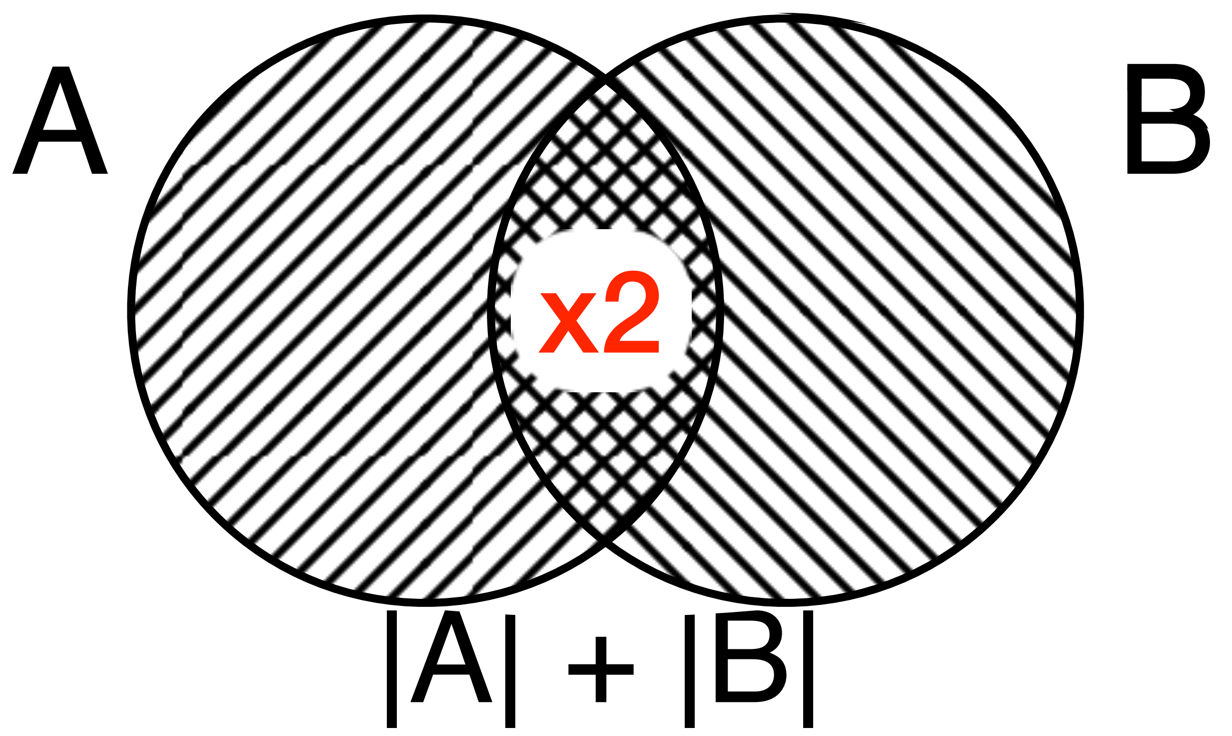
Disjoint A and B: $|A \cup B| = |A| + |B|$

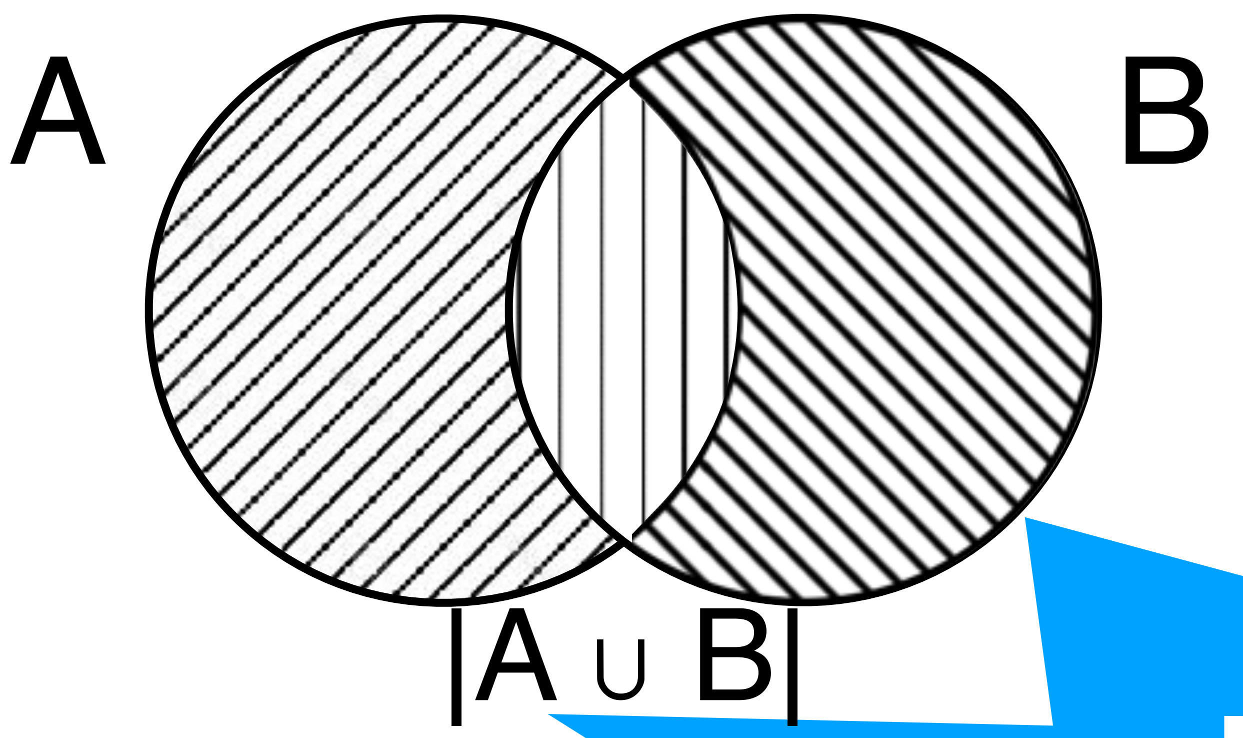
Size of union = sum of sizes

In general: $|A \cup B| \neq |A| + |B|$

$|\{a\} \cup \{a\}| = |\{a\}| = 1 \neq 2 = |\{a\}| + |\{a\}|$

Can we determine $|A \cup B|$ in general?

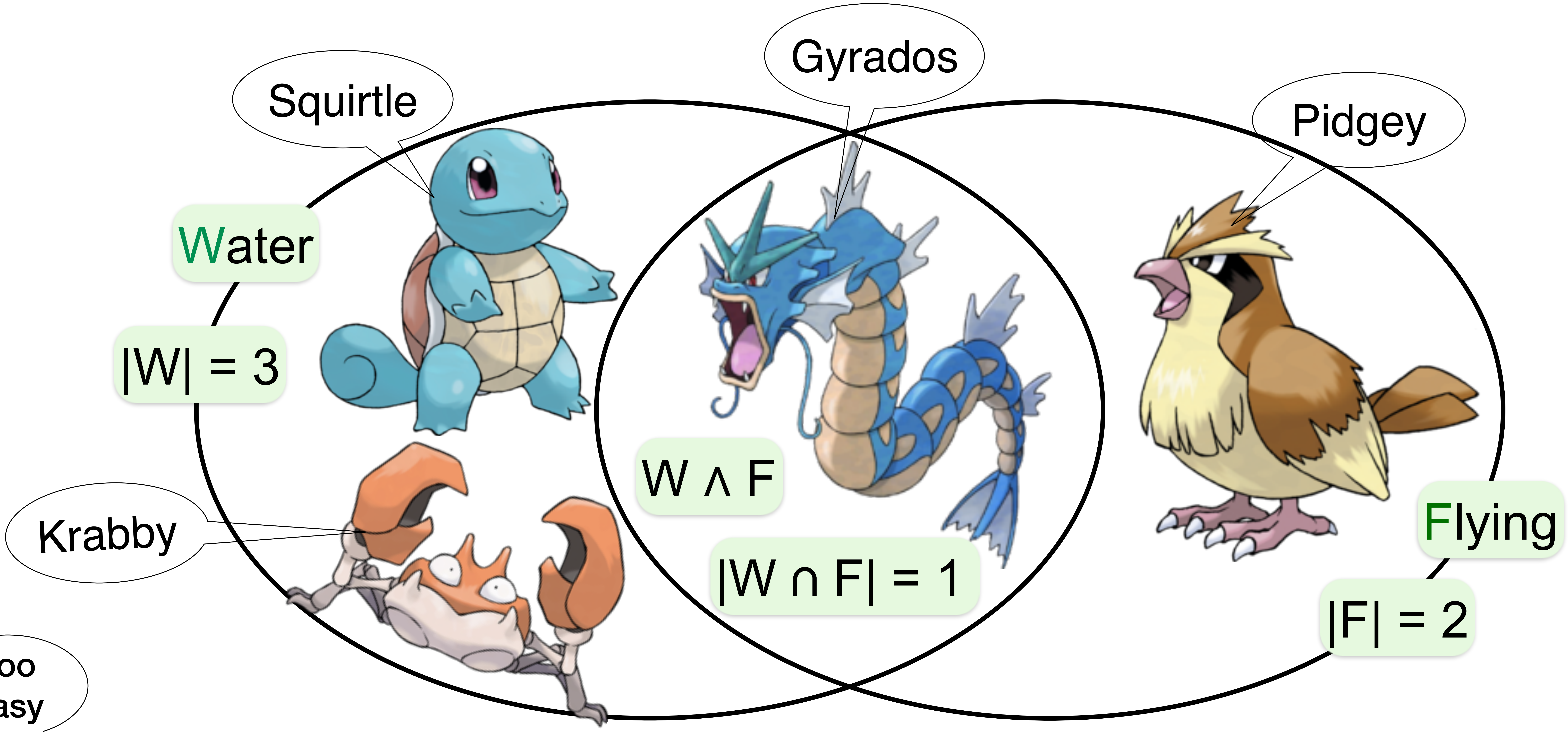




$|A \cup B| = |A| + |B| - |A \cap B|$

Principle of Inclusion-Exclusion (PIE)

Pokémons



Too easy

= 4

$|W \cup F| = |W| + |F| - |W \cap F| = 3+2-1 = 4$ ✅

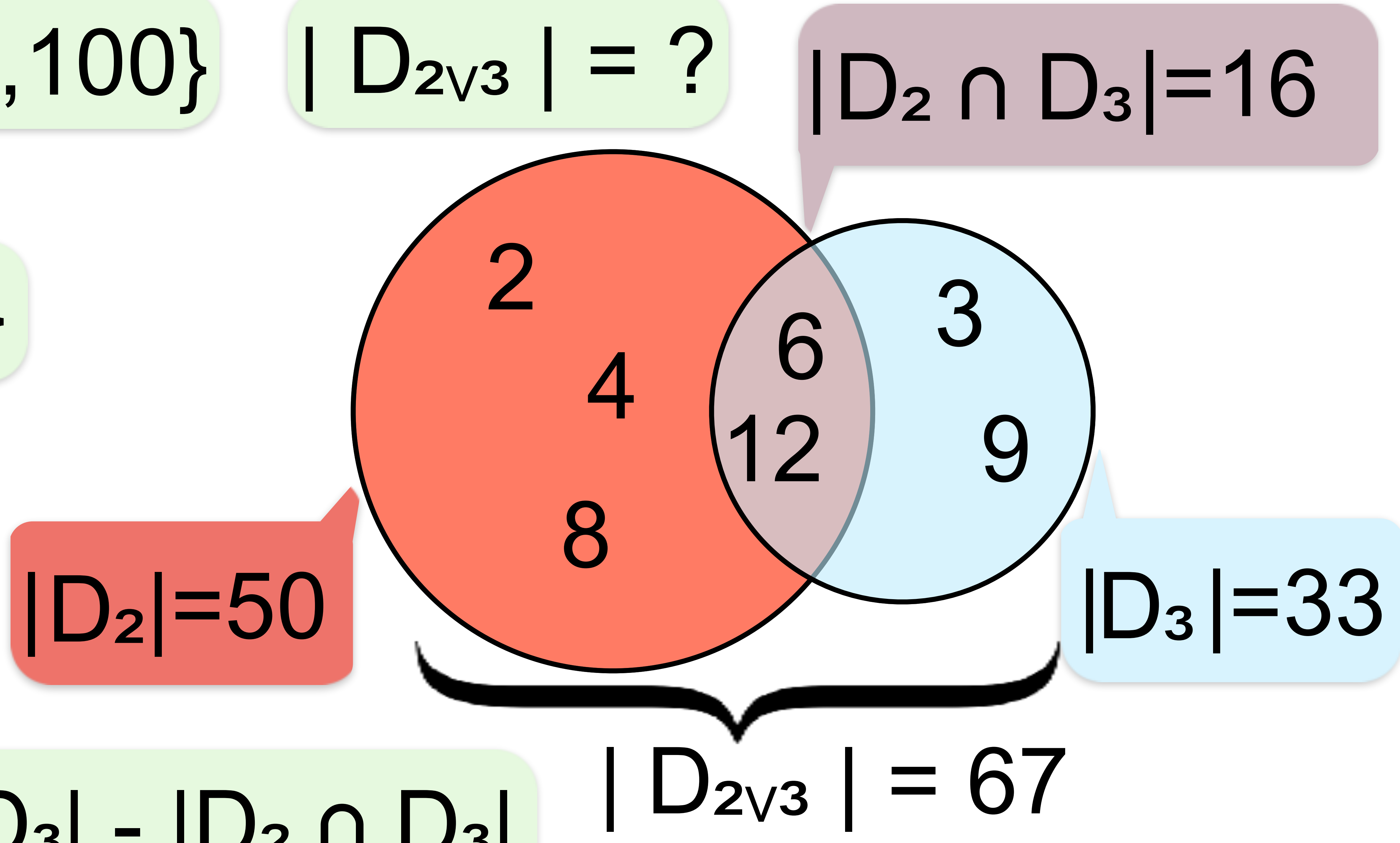
Divisibility by 2 Numbers

$D_{2\vee3}$ = { i∈[100] : 2|i ∨ 3|i } = {2,3,4,6,8,…,100}

| $D_{2\vee3}$ | = ?

D_2 = { i∈[100] : 2|i } = {2, 4, 6,…, 100}

D_3 = { i∈[100] : 3|i } = {3, 6, 9,…, 99}



$D_{2\vee3} = D_2 \cup D_3$

$|D_2 \cup D_3| = |D_2| + |D_3| - |D_2 \cap D_3|$

$D_2 \cap D_3$ = { i∈[100] : 2|i ∧ 3|i } = { i∈[100] : 6|i }

$|D_{2\vee3}| = |D_2| + |D_3| - |D_2 \cap D_3| = 50 + 33 - 16 = 67$

Multiple Sets

Two sets

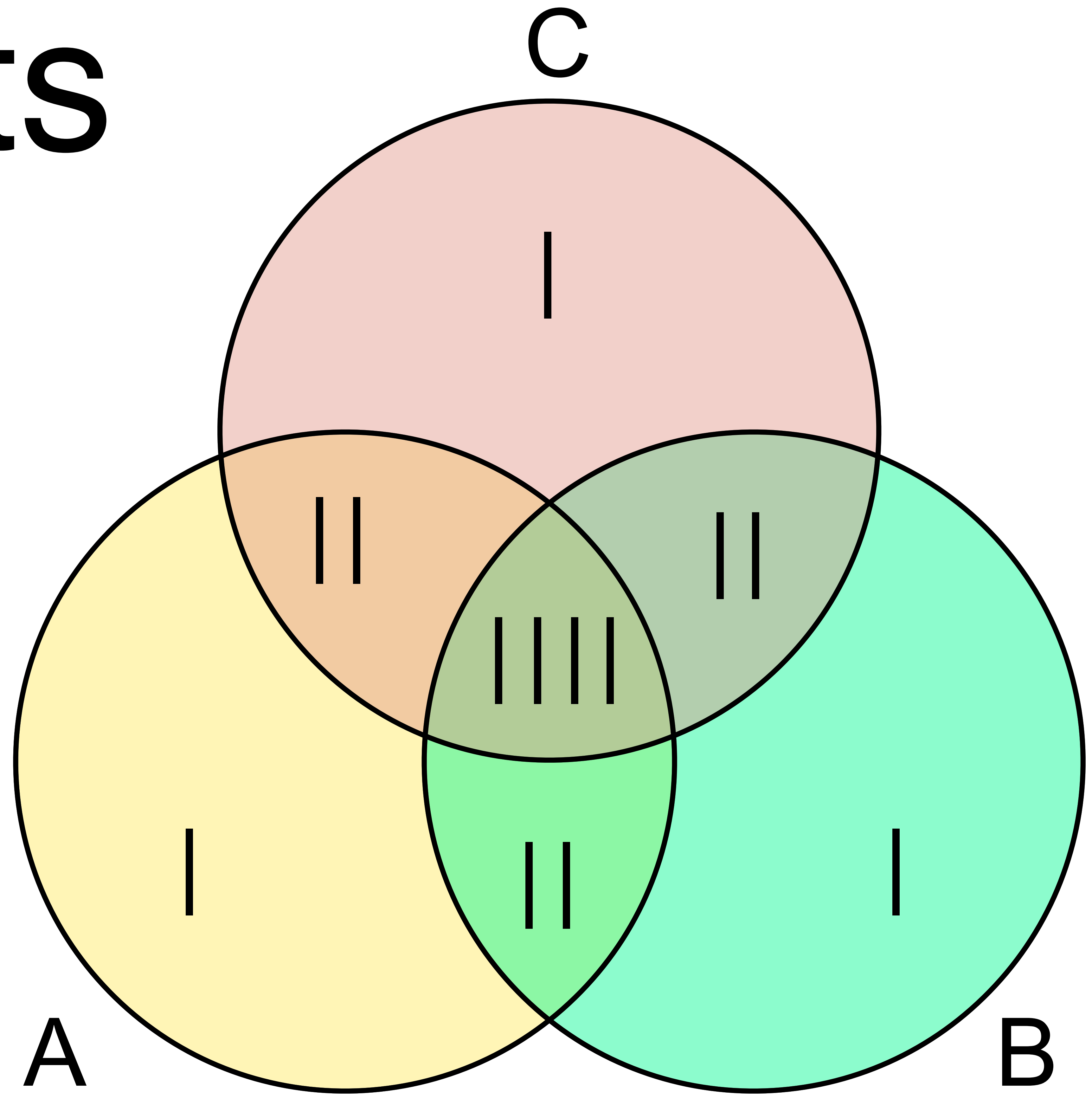
|A ∪ B| = |A| + |B| - |A ∩ B|

Three sets

|A ∪ B ∪ C| = |A| + |B| + |C|
- |A ∩ B| - |A ∩ C| - |B ∩ C|
+ |A ∩ B ∩ C|

n sets

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{1\le i\le n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \cdots + (-1)^{n-1}\left|\bigcap_{i=1}^{n} A_i\right|$$



Polyglots

8 students in class, each "speaks" **C**, **R**, or **Π**thon — $|C \cup R \cup \Pi| = 8$

Each language spoken by 5 students — $|C| = |R| = |\Pi| = 5$

Every language *pair* is spoken by 3 students — $|C \cap R| = |C \cap \Pi| = |R \cap \Pi| = 3$

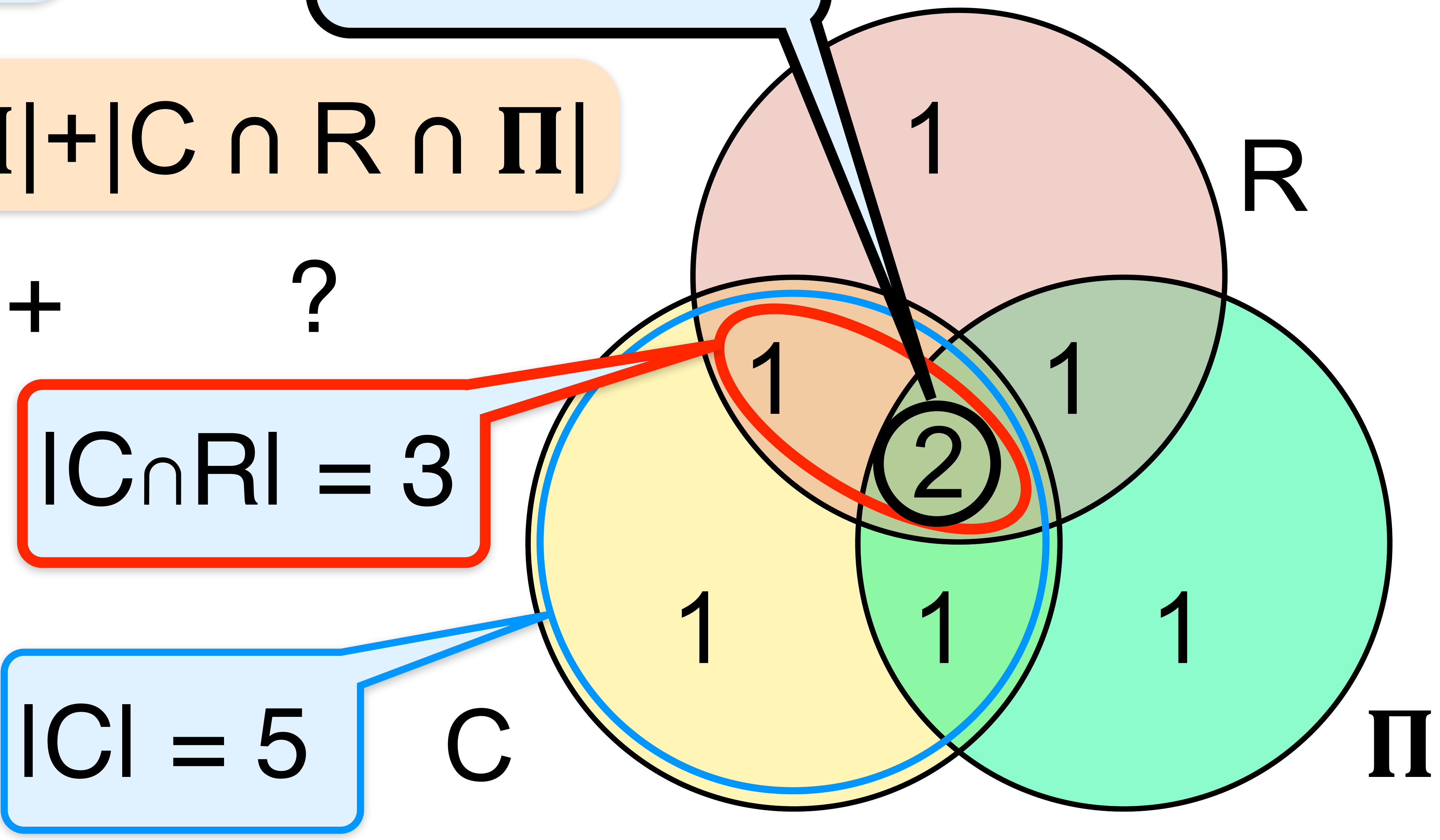
How many students speak all 3 languages? — $|C \cap R \cap \Pi| = ?$

$$|C \cup R \cup \Pi| = |C|+|R|+|F|-|C \cap R|-|C \cap \Pi|-|R \cap \Pi|+|C \cap R \cap \Pi|$$

$$8 = 5 + 5 + 5 - 3 - 3 - 3 + ?$$

$$? = 8 - 5 - 5 - 5 + 3 + 3 + 3$$

$$= 2$$



With that, can say more

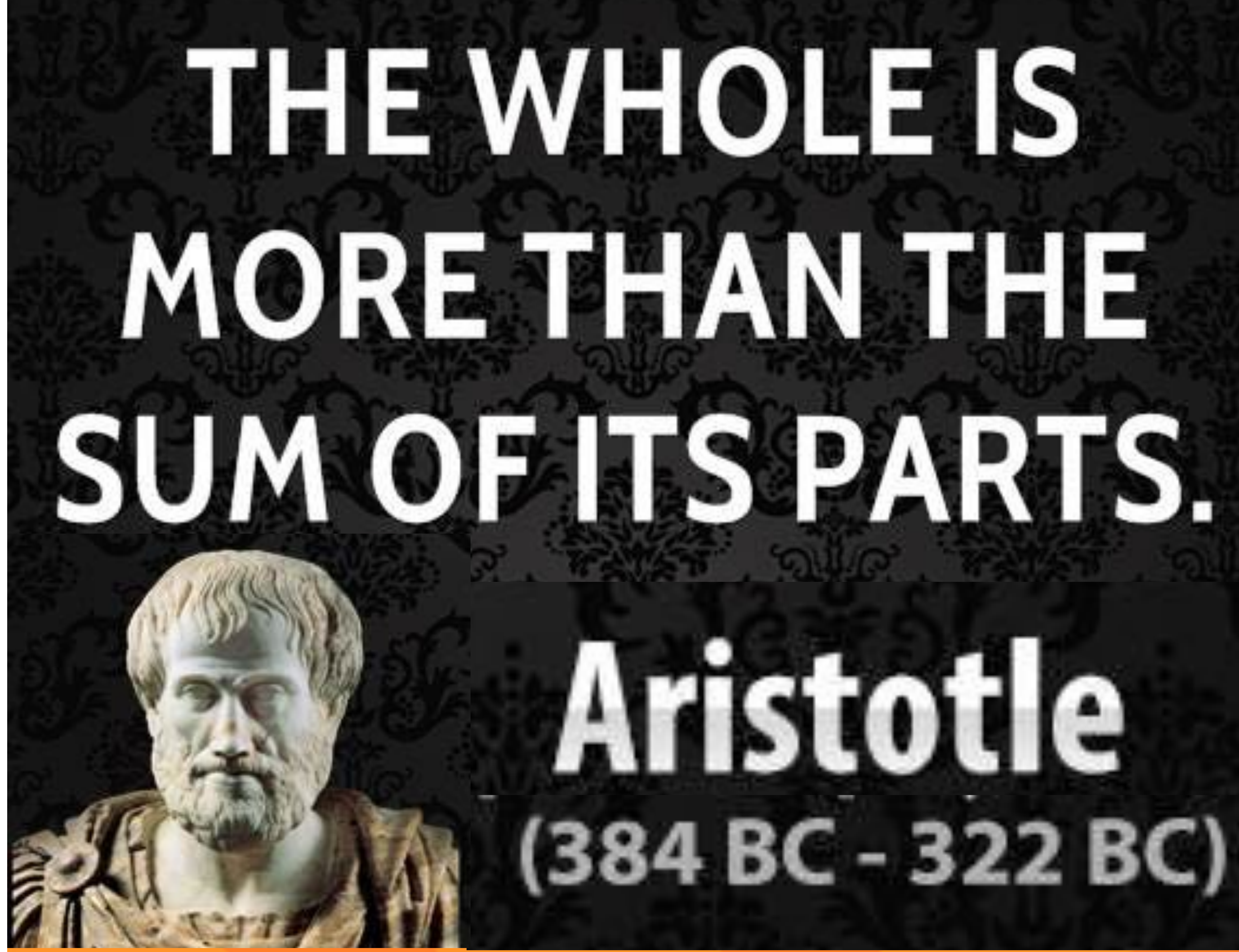
Sanity Checks

Compare  to some expected outcomes

A, B disjoint $|A \cup B| = |A| + |B| - \underbrace{|A \cap B|}_{0} = |A| + |B|$ ✅

Equal sets $|A \cup A| = |A| + |A| - \underbrace{|A \cap A|}_{|A|} = |A|$ ✅

$\max\{|A|, |B|\} \underset{\text{= iff nested}}{\leq} |A \cup B| \underset{\text{= iff disjoint}}{\leq} |A| + |B|$ ✅



Union ≤ sum of its parts

General Unions (Piece of PIE)

Solution to the Hairy Problem

Size of a general union

Principle of Inclusion Exclusion

Multiple sets

Cartesian Products