$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

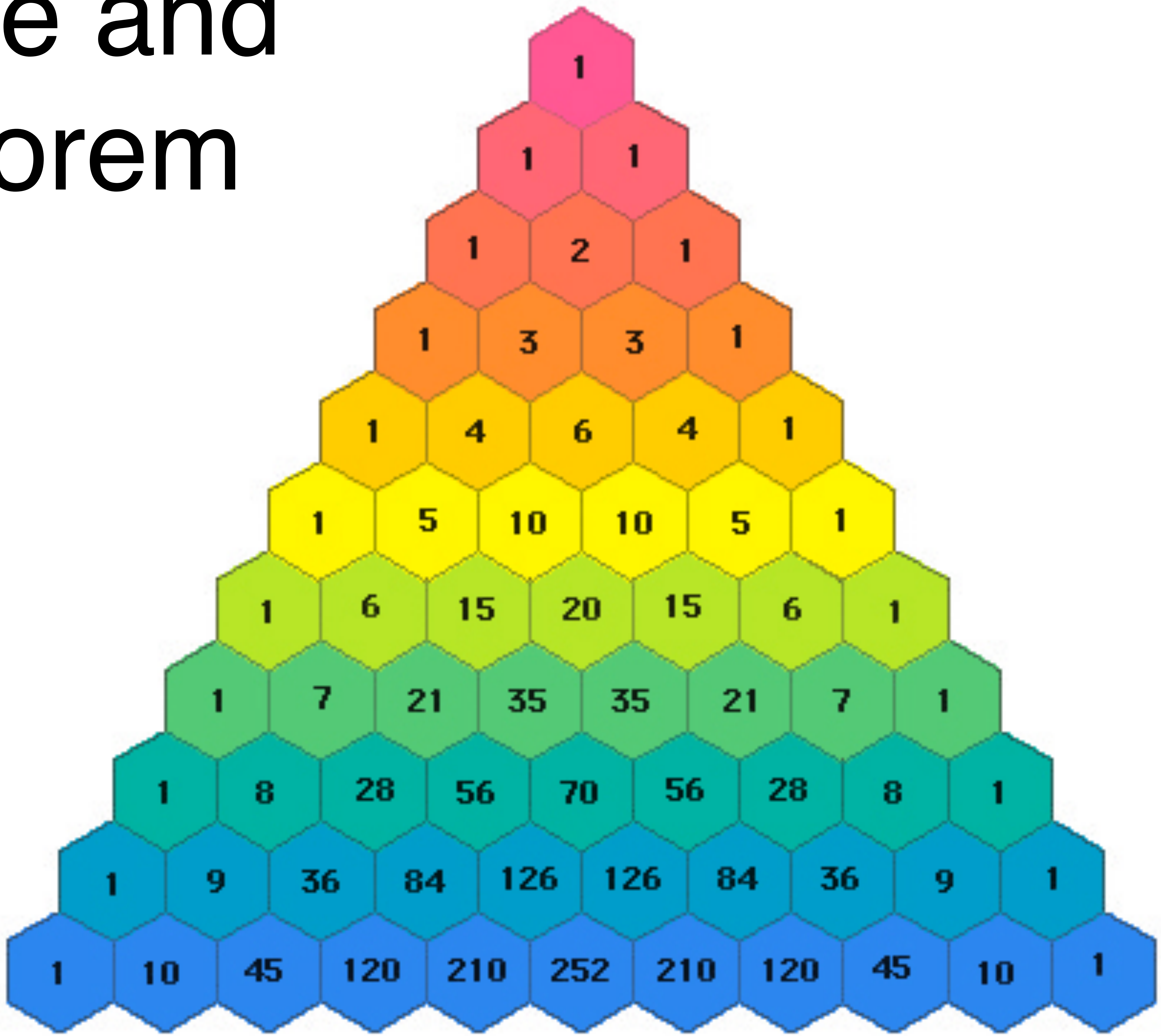
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

Pascal Triangle and Binomial Theorem

Pascal Triangle and Binomial Theorem



Pascal's Identity

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

$$\binom{4}{3} = \binom{3}{3} + \binom{3}{2}$$

$$\begin{array}{ccc}
\begin{array}{c} 0\,1\,1\,1 \\ 1\,0\,1\,1 \\ 1\,1\,0\,1 \\ \underbrace{1\,1\,1\,0}_{\substack{\text{three} \\ \text{1's}}} \end{array}
&
\begin{array}{c} \underbrace{1\,1\,1}_{\substack{\text{three} \\ \text{1's}}}\,\boxed{0} \end{array}
&
\begin{array}{c} \underbrace{\begin{array}{c} 0\,1\,1 \\ 1\,0\,1 \\ 1\,1\,0 \end{array}}_{\substack{\text{two} \\ \text{1's}}} \boxed{\begin{array}{c} 1 \\ 1 \\ 1 \end{array}} \end{array}
\end{array}$$

Pascal's Triangle

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

$$\binom{2}{1} = \binom{1}{1} + \binom{1}{0}$$

$$\binom{3}{1} = \binom{2}{1} + \binom{2}{0}$$

$$\binom{3}{2} = \binom{2}{2} + \binom{2}{1}$$

k

n	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

$$\binom{n}{0} = 1 \qquad \binom{n}{n} = 1$$

Binomial Theorem

$(a + b)^1 = a + b$

$(a + b)^2 = a^2 + 2ab + b^2$

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$$1 = \binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} = 1$$

n \ k	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

$$(a + b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

Binomial Theorem

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i \quad \forall\, a, b \quad \forall\, n \geq 0$$

So important it gives the binomial coefficients their name

Explanation

$$(a+b)^3 = (\underline{a}+b)(\underline{a}+b)(a+\underline{b})$$

Three (a+b) factors

$$= \underbrace{aaa + aab + aba + abb + baa + bab + bba + bbb}$$

Each term: product of a's and b's one selected from each factor

Sum of terms

a's + # b's = 3

$$= a^3 + 3a^2b + 3ab^2 + b^3$$

# of terms with i b's = # ways to select i factors out of the 3 = $\binom{3}{i}$

$$= \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3$$

Generally

$$(a+b)^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3 = \sum_{i=0}^{3}\binom{3}{i}a^{3-i}b^i$$

Similarly

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + ... + \binom{n}{n}b^n = \sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^i$$

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

Gave combinatorial proof

Algebraic proof

$$2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i} 1^{n-i} 1^i = \sum_{i=0}^{n} \binom{n}{i}$$

Polynomial Coefficients

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$$

Coefficient of x^2 in $(1+x)^7$

$$(1+x)^7 = \sum_{i=0}^{7} \binom{7}{i} x^i \qquad \binom{7}{2} = 21$$

Coefficient of x^3 in $(3+2x)^5$

$$(3+2x)^5 = \sum_{i=0}^{5} \binom{5}{i} 3^{5-i} (2x)^i \qquad \binom{5}{3} 3^2 \cdot 2^3 = 720$$

Binomial → Taylor

Taylor expansion

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

$$\left(1 + \frac{x}{n}\right)^n = \sum_{i=0}^{n} \binom{n}{i} \left(\frac{x}{n}\right)^i$$

$$= \sum_{i=0}^{n} \frac{n^{\underline{i}}}{i!} \left(\frac{x}{n}\right)^i$$

$$= \sum_{i=0}^{n} \frac{x^i}{i!} \cdot \frac{n^{\underline{i}}}{n^i}$$

$$n \to \infty$$

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

Binomial Distribution

$$\sum_{i=0}^{n} \binom{n}{i} p^{n-i}(1-p)^i = (p + (1-p))^n = 1^n = 1$$

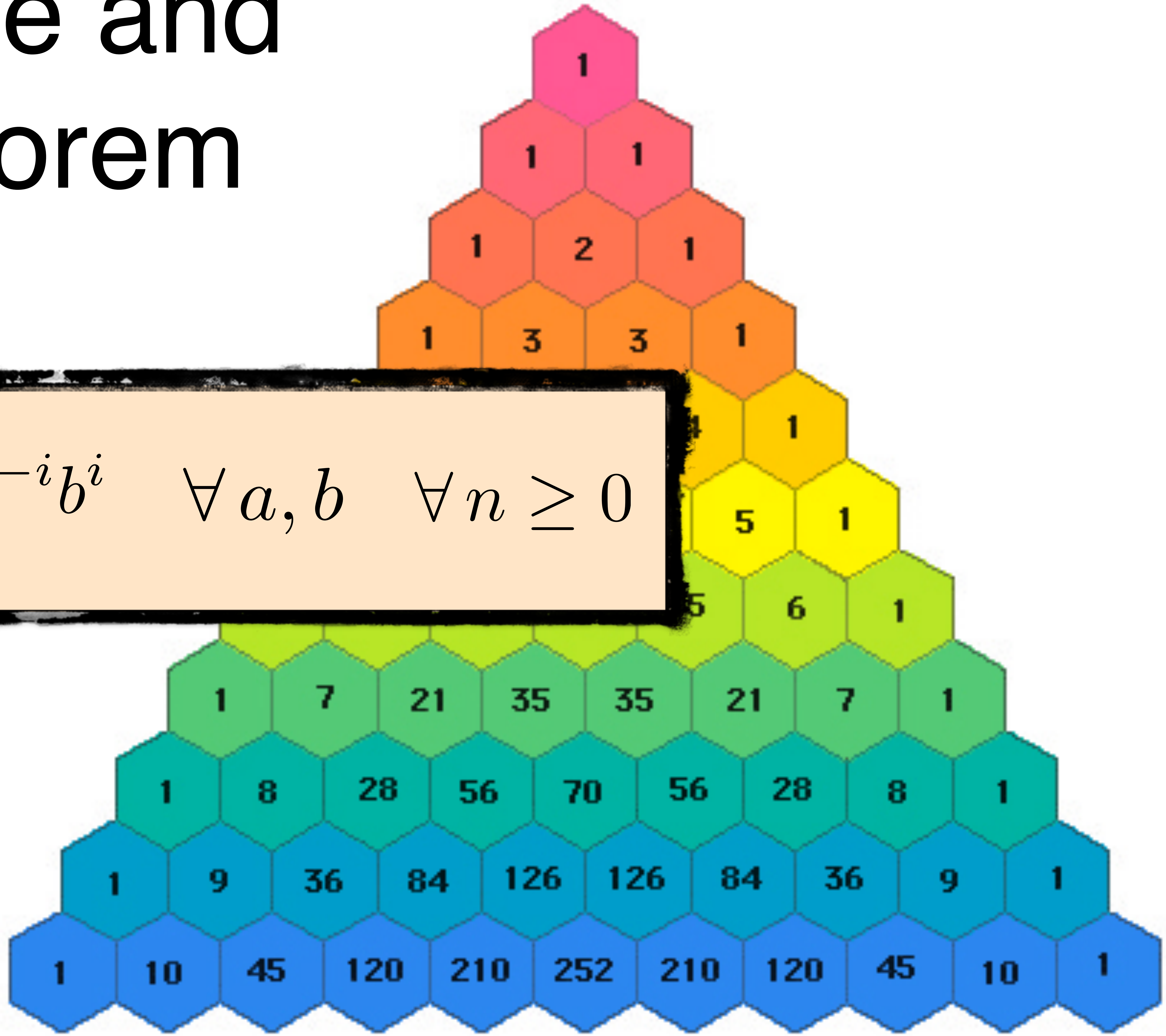
Example

$$\left(\frac{1}{3}\right)^2 + 2\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 = 1$$

More in binomial distribution

Pascal Triangle and Binomial Theorem

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i \quad \forall\, a, b \quad \forall\, n \geq 0$$



So far: Binary Sequences

Next: Larger Alphabets