Cartesian Powers

- sequences
- subsets
- functions
- exponential growth

+, -, X, ...

Analogies between number and set operations

Numbers	**Sets**
Addition	Disjoint union
Subtraction	Complement
Multiplication	Cartesian product
Exponents	?

Cartesian Powers of a Set

Cartesian product of a set with itself is a Cartesian power

$A^2 = A \times A$

Cartesian square

$$A^n \overset{\text{def}}{=} \underbrace{A \times A \times \ldots \times A}_{n}$$

n’th Cartesian power

ⓧ

$$|A^n| = |A \times A \times \ldots \times A| = |A| \times |A| \times \ldots \times |A| = |A|^n$$

Practical and theoretical applications

California License Plates

Till 1904 no registration

1905-1912 various registration formats one-time $2 fee

1913

≤6 digits $10^6 = 1$ million If all OK

1956

$26^3 \times 10^3 \approx 17.6$ m

Sam?

1969

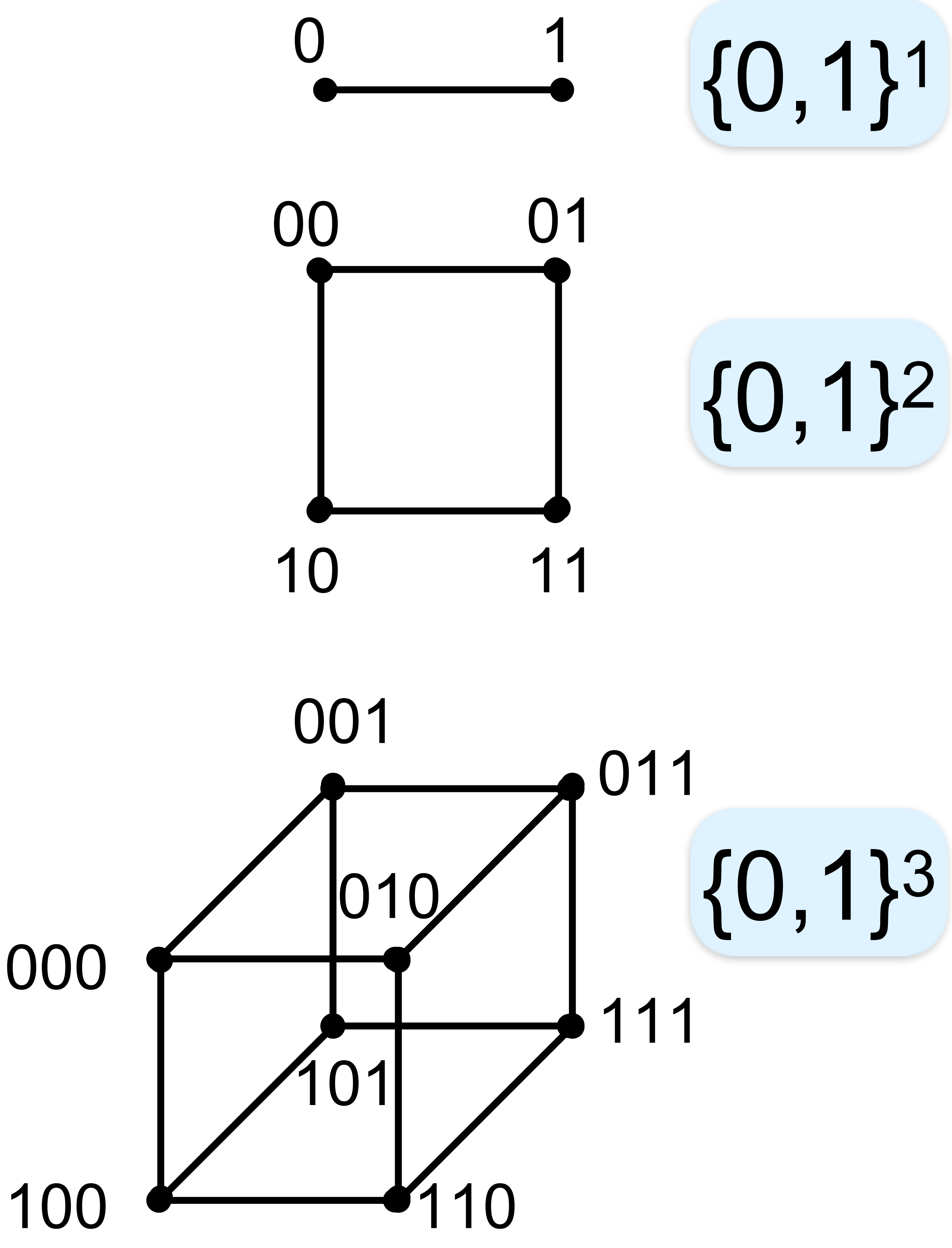
$26^3 \times 10^4 \approx 176$ m

Binary Strings

$\{0,1\}^n$ = { length-n binary strings }

n-bit strings

n	Set	Strings	Size
0	$\{0,1\}^0$	Λ	1
1	$\{0,1\}^1$	0, 1	2
2	$\{0,1\}^2$	00, 01, 10, 11	4
3	$\{0,1\}^3$	000, 001, 011, 010, 100, 110, 101, 111	8
…	…	…	…
n	$\{0,1\}^n$	0…0, …, 1…1	2^n



$| \{0,1\}^n | = |\{0,1\}|^n = 2^n$

Subsets

The power set of S, denoted $\mathbb{P}(S)$, is the collection of all subsets of S

$\mathbb{P}(\{a,b\}) = \{ \{\}, \{a\}, \{b\}, \{a,b\} \}$

$|\mathbb{P}(S)| = ?$

Subsets of S

Binary strings of length |S|

1-1 correspondence between $\mathbb{P}(S)$ and $\{0,1\}^{|S|}$

$\mathbb{P}(\{a,b\})$ and $\{0,1\}^2$

$\mathbb{P}(\{a,b\})$	a	b	$\{0,1\}^2$
{}	✗	✗	00
{b}	✗	✓	01
{a}	✓	✗	10
{a,b}	✓	✓	11

$$\boxed{|\mathbb{P}(S)| = |\{0,1\}^{|S|}| = 2^{|S|}}$$

The size of the power set is the power of the set size

Functions

A function from A to B maps every element $a \in A$ to an element $f(a) \in B$

Define a function f: specify f(a) for every $a \in A$

f from {1,2,3} to {p, u} specify f(1), f(2), f(3) f(1)=p, f(2)=u, f(3)=p

f: 3-tuple (f(1), f(2), f(3)) (p, u, p)

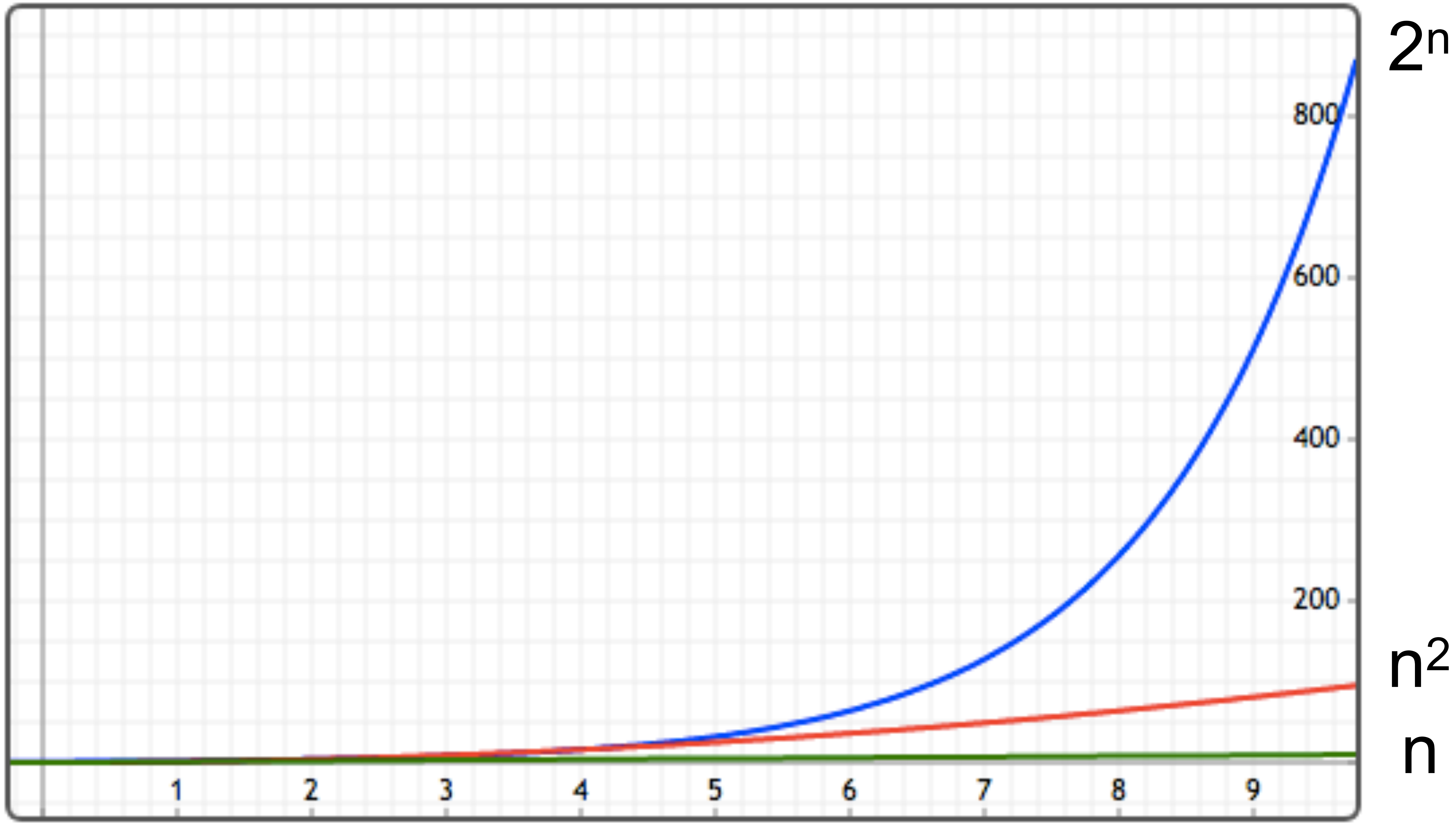
{ functions from {1,2,3} to {p,u} } {p,u} x {p,u} x {p,u}

\# functions from {1,2,3} to {p,u} $= 2 \times 2 \times 2 = 2^3 = |\{p,u\}|^{|\{1,2,3\}|}$

{ functions from A to B } $\underbrace{B \times B \times \ldots \times B}_{|A|} = B^{|A|}$

\# functions from A to B $= |B^{|A|}| = |B|^{|A|}$ Ⓧ

Exponential Growth



Cartesian Powers

Cartesian Powers & Exponentials

Cartesian power

Again use **product** function in **itertools** library

```
import itertools
print(set(itertools.product({2, 5, 9}, repeat = 2)))
{(5,9),(5,5),(2,9),(9,2),(9,9),(2,2),(9,5),(2,5),(5,2)}
```

Exponent

`**`

```
print(3**2)
9
```

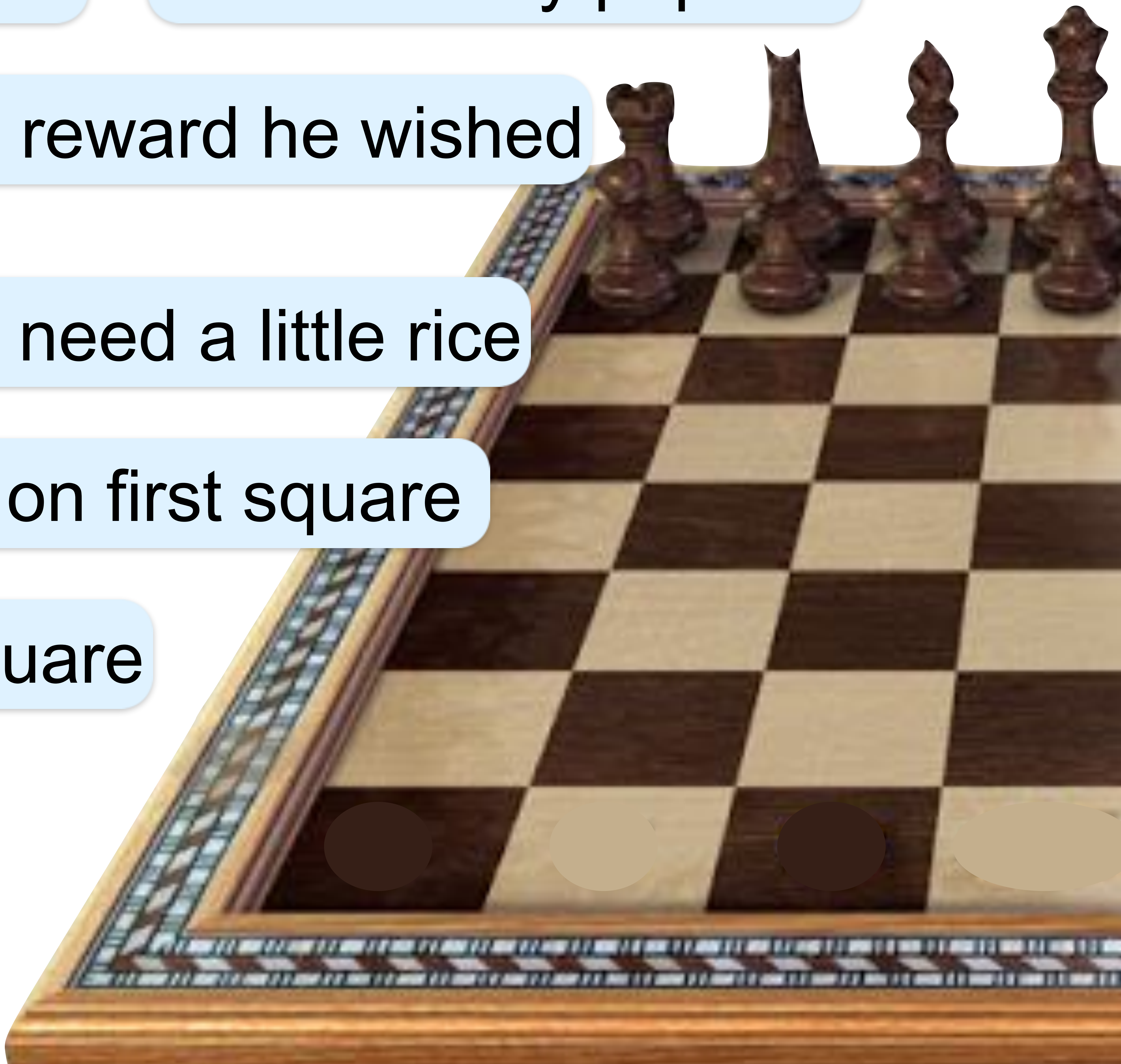
Notebook

Compute exponentials

Compare to other growth rates

Chess-Rice Legend

(Indian & Persian versions)



Chess

Invented by poor yet clever peasant

Became very popular

King liked it

Offered peasant any reward he wished

Peasant

Poor and humble farmer

Just need a little rice

Kindly place a single rice grain on first square

Double on each subsequent square

King

Such a modest request

Granted!

Chess-Rice Legend (ctd.)

King Placed one (2^0) grain on first square

Two (2^1) on second

Four (2^2) on third..

All Americans' worth $\$90 \cdot 10^{12}$

All humans' worth $\$600 \cdot 10^{12}$

World richest person worth $\$90 \cdot 10^{9}$

64th square: $2^{63} \approx$ 1,000,000,000,000,000,000,000,000,000,000

Two endings Peasant became king

Peasant beheaded

Moral Be peasant or be King: beware of exponentials!

Jeopardy

Counting questions → Answer

\# {

n-bit sequences

Subsets of {1,….., n}

Functions: {1,….., n} to {0,1}

} = 2^n

? ← 2^{2^n}

Find a natural counting question whose answer is a double exponential

Find a natural counting question whose answer is a double exponential

2 solutions

Subsets

Functions

∃ more

Power set of S - set of subsets of S

$\mathbb{P}(S)$

$\mathbb{P}(\{a, b\}) = \{ \{\}, \{a\}, \{b\}, \{a,b\} \}$

$| \mathbb{P}(S) | = 2^{|S|}$

$| \mathbb{P}(\{a,b\}) | = 4 = 2^2 = 2^{|\{a,b\}|}$ ✅

$\mathbb{P}(S)$ is a set

What about power set of $\mathbb{P}(S)$?

Find a natural counting question whose answer is a double exponential

$\mathbb{P}(\ \mathbb{P}(S)\ )$ - set of subsets of $\mathbb{P}(S)$

$|\ \mathbb{P}(S)\ | = 2^{|S|}$

$|\ \mathbb{P}(\ \mathbb{P}(S)\ )\ | = 2^{|\mathbb{P}(S)|} = 2^{2^{|S|}}$

$\mathbb{P}(\{a, b\}) = \{\ \{\}, \ \{a\}, \{b\}, \{a,b\}\ \}$

$\mathbb{P}(\ \mathbb{P}(\{a, b\})\ ) = \mathbb{P}(\{\ \{\}, \ \{a\}, \{b\}, \{a,b\}\ \})$

$= \{\ \{\}, \{\ \{\}\ \}, \{\ \{a\}\ \}, \ldots, \{\ \{\}, \{a\}\ \}, \ldots, \{\{\},\{a\},\{b\},\{a,b\}\}\ \}$

$|\ \mathbb{P}(\ \mathbb{P}(\{a,b\})\ )\ | = 2^{|\mathbb{P}(\{a,b\})|} = 2^{2^{|\{a,b\}|}}$

$|\ \mathbb{P}(\ \mathbb{P}([n])\ )| = 2^{2^{n}}$

Double exponential

Solution 2: Boolean Functions

Functions from A to B

B^A

$\# = |B|^{|A|}$

x	f(x)
000	0
001	0
⋮	
111	1

2^n

2

2

⋮

2

n

2^{2^n}

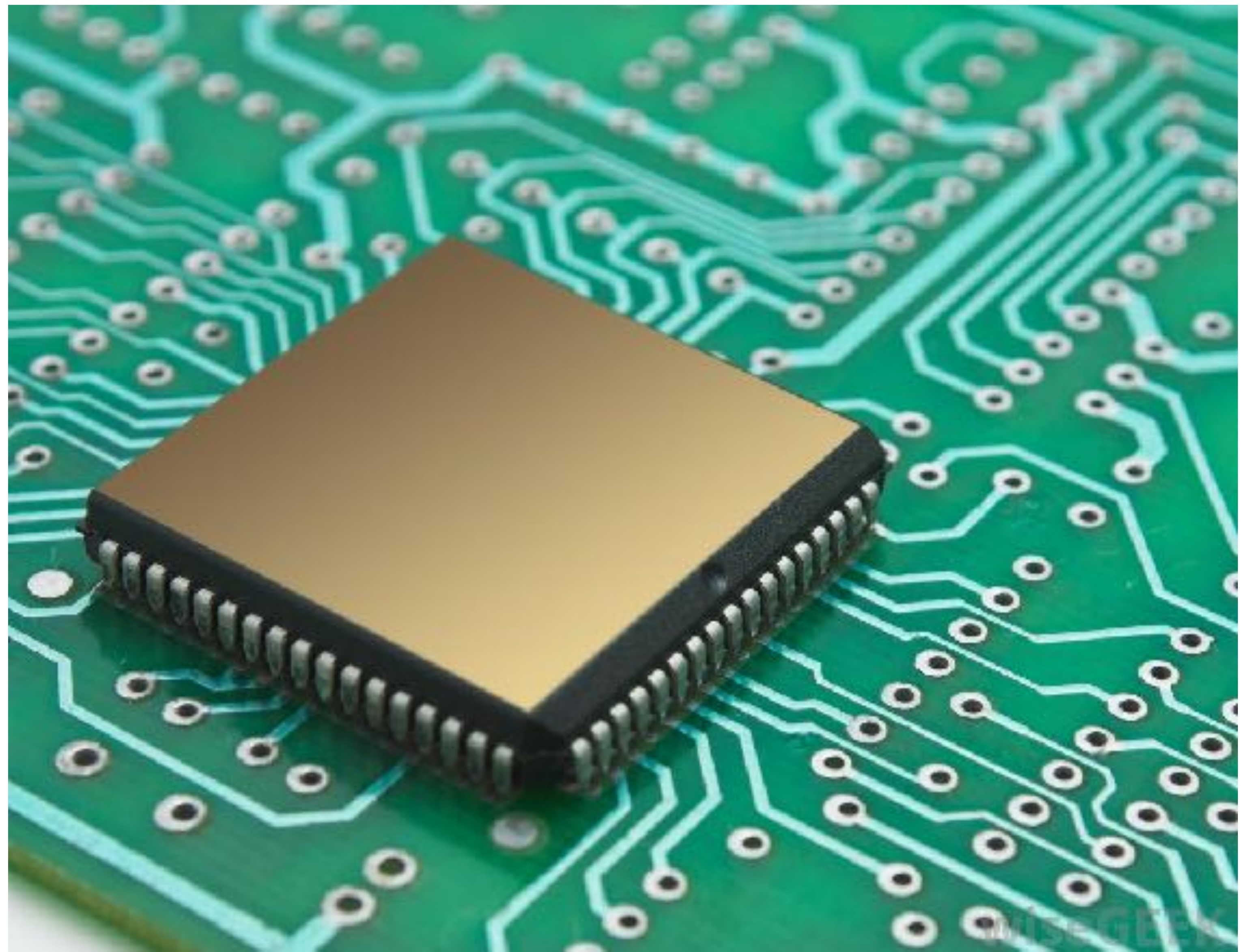
 # Boolean functions of n boolean (binary) variables

Functions from $\{0,1\}^n$ to $\{0,1\}$

$\{0,1\}^{\{0,1\}^n}$

$\# = |\{0,1\}|^{|\{0,1\}^n|} = 2^{2^n}$

Double exponential

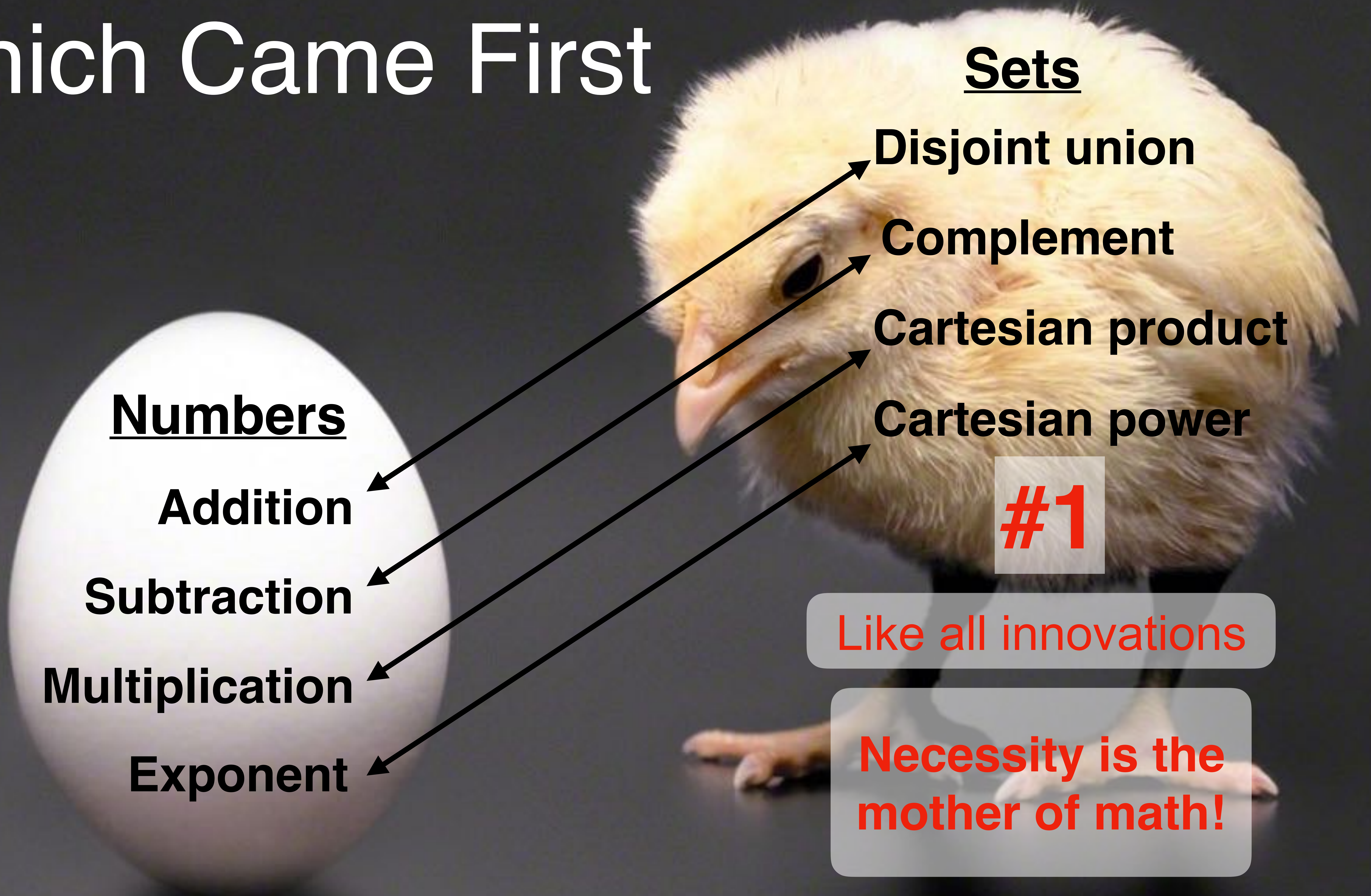


Circuit with n binary inputs, one binary output

Can implement 2^{2^n} functions

$2^{63} = ½ \cdot 2^{2^6}$

Which Came First



Numbers	**Sets**
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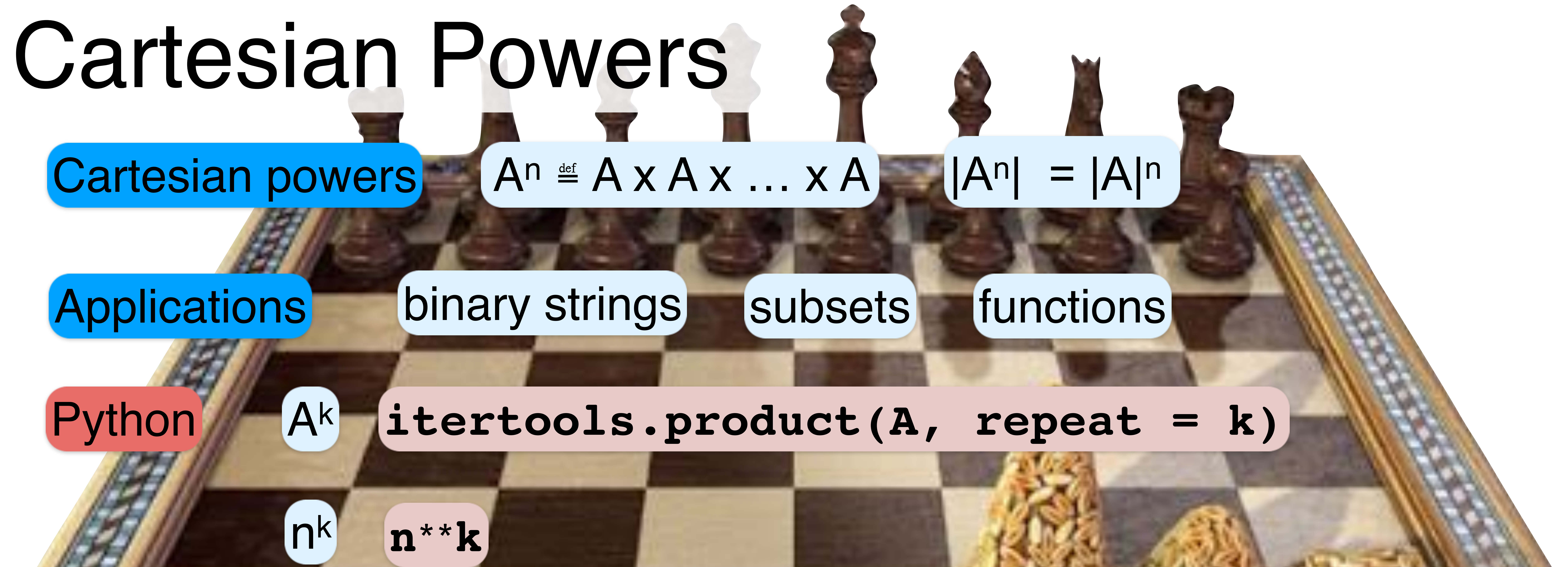
Exponent	Cartesian power

#1

Like all innovations

Necessity is the mother of math!

Cartesian Powers



Cartesian powers $A^n \overset{\text{def}}{=} A \times A \times \ldots \times A$ $|A^n| = |A|^n$

Applications binary strings subsets functions

Python A^k `itertools.product(A, repeat = k)`

n^k `n**k`

Variations