Multinomials

Multinomial Coefficients

Beyond binary - ternary alphabets

$$k_1 + k_2 + k_3 = n$$

$$\# \{1,\ 2,\ 3\} \text{ sequences with } \begin{cases} k_1 \ \text{1's} \\ k_2 \ \text{2's} \\ k_3 \ \text{3's} \end{cases}$$

$$\underbrace{\binom{n}{k,n-k}}_{\text{sum to n}} = \binom{n}{k}$$

$$\binom{n}{k_1}\binom{n-k_1}{k_2} = \frac{n!}{k_1!\cdot\cancel{(n-k_1)!}} \cdot \frac{\cancel{(n-k_1)!}}{k_2!\cdot\underbrace{(n-k_1-k_2)!}_{k_3}} = \frac{n!}{k_1!\cdot k_2!\cdot k_3!} \triangleq \binom{n}{k_1,k_2,k_3}$$

k_1 location of 1's

k_2 location of 2's out of n-k_1 locations left

(location of k_3 3's is determined)

symmetric in k_1, k_2, k_3

Simple Example

sequences over {1,2,3,4}

digit	1	2	3	4
# times	1	4	4	2

length 11

31222334243

$$\binom{11}{1,4,4,2} = \frac{11!}{1!\cdot 4!\cdot 4!\cdot 2!}$$

$$= \frac{11\cdot 10\cdot 9\cdot \cancel{8}\cdot 7\cdot \cancel{6}\cdot 5\cdot \cancel{4!}}{\cancel{4!}(\cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1)(\cancel{2}\cdot 1)}$$

$$= 11\cdot 10\cdot 9\cdot 7\cdot 5 = 34,650$$

MISSISSIPPI

anagrams = ?

sequences over {M,I,S,P}

letter	M	I	S	P
# times	1	4	4	2

$\underbrace{\hphantom{1 \quad 4 \quad 4 \quad 2}}_{\text{length 11}}$ (SISSISIPPIM)

Same as sequences over {1,2,3,4} in previous slide

$$\binom{11}{1,4,4,2} = 34,650$$

$\uparrow \uparrow \uparrow \uparrow$

M I S P

Students in Class

10 students

3 classes: morning, afternoon, evening

Any number of students in each class 3^{10}

6 morning

3 afternoon

1 evening

$$\underbrace{\binom{10}{6,3,1}}_{\text{morning students, afternoon students, evening students}} = \frac{10!}{6!\cdot 3!\cdot 1!} = \frac{10\cdot \overset{3}{\cancel{9}}\cdot \overset{4}{\cancel{8}}\cdot 7}{\cancel{3}\cdot \cancel{2}} = 840$$

Multinomial Theorem

$$(a_1 + a_2 + ... + a_m)^n = \sum_{\substack{k_1 + k_2 + ... + k_m = n \\ k_1, k_2, ..., k_m \geq 0}} \binom{n}{k_1, k_2, ..., k_m} \prod_{t=1}^{m} a_t^{k_t}$$

$$(a+b+c)^2$$

$$(a+b+c)^2 = \sum_{\substack{i+j+k=2 \\ i,j,k \geq 0}} \binom{2}{i,j,k} a^i b^j c^k$$

$$= \tbinom{2}{2,0,0} a^2 + \tbinom{2}{0,2,0} b^2 + \tbinom{2}{0,0,2} c^2 + \tbinom{2}{1,1,0} ab + \tbinom{2}{1,0,1} ac + \tbinom{2}{0,1,1} bc$$

$$= a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$$

Sum of Multinomials

Recall binomial identity $$2^n = \sum_{i=0}^{n} \binom{n}{i}$$

Similarly for multinomials

$$m^n = (1+1+...+1)^n = \sum_{\substack{k_1+...+k_m=n \\ k_1,...,k_m \geq 0}} \binom{n}{k_1, k_2, ..., k_m}$$

$$3^2 = 9 = \underbrace{\binom{2}{2,0,0}}_{1} + \underbrace{\binom{2}{0,2,0}}_{1} + \underbrace{\binom{2}{0,0,2}}_{1} + \underbrace{\binom{2}{1,1,0}}_{2} + \underbrace{\binom{2}{1,0,1}}_{2} + \underbrace{\binom{2}{0,1,1}}_{2}$$

Students in Class

2 students

3 classes: morning, afternoon, evening $\qquad 3^2 = 9$

Broken by class

$$\underbrace{\binom{2}{2,0,0}}_{1} + \underbrace{\binom{2}{0,2,0}}_{1} + \underbrace{\binom{2}{0,0,2}}_{1} + \underbrace{\binom{2}{1,1,0}}_{2} + \underbrace{\binom{2}{1,0,1}}_{2} + \underbrace{\binom{2}{0,1,1}}_{2} = 9$$

same as last slide

Next: Application