Properties of Binomial Coefficients

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\binom{5}{3} = \frac{5!}{3!\,2!} = \frac{5!}{2!\,3!} = \binom{5}{2}$$

Two proofs

Algebraic

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n!}{(n-k)!\,k!} = \binom{n}{n-k}$$

$$\binom{n}{k} = \binom{n}{n-k}$$

Combinatorial

$$\binom{[n]}{k} \longleftrightarrow \binom{[n]}{n-k}$$

Complement bits to create 1-1 correspondence

$$\binom{[4]}{3} \longleftrightarrow \binom{[4]}{4-3}$$

$$\begin{Bmatrix} 1110 \\ 1101 \\ 1011 \\ 0111 \end{Bmatrix} \longleftrightarrow \begin{Bmatrix} 0001 \\ 0010 \\ 0100 \\ 1000 \end{Bmatrix}$$

$$\binom{n}{k} = \left| \binom{[n]}{k} \right| = \left| \binom{[n]}{n-k} \right| = \binom{n}{n-k}$$

$$\binom{n}{k} = \frac{n}{k} \cdot \binom{n-1}{k-1}$$

$$\binom{5}{3} = \frac{5!}{3!\,2!} = 10 = \frac{5}{3} \cdot 6 = \frac{5}{3}\binom{4}{2}$$

Recursive definition

choose one of these locations to be 2

choose location of 2

choose k locations of k non-zeros (1's & 2)

$$\binom{n}{k} \cdot k = n \cdot \binom{n-1}{k-1}$$

from remaining n-1 locations choose locations of the k-1 1's

Number of length-n ternary strings with k-1 1's and one 2

n=4 k=3 { 0112, 1012, 2110,… }

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

$$\tbinom{3}{0} + \tbinom{3}{1} + \tbinom{3}{2} + \tbinom{3}{3} = 1 + 3 + 3 + 1 = 8 = 2^3$$

Combinatorial proof

$$2^{[n]} = \biguplus_{i=0}^{n} \binom{[n]}{i}$$

$$2^n = |2^{[n]}| = \sum_{i=0}^{n} \left| \binom{[n]}{i} \right| = \sum_{i=0}^{n} \binom{n}{i}$$

Algebraic proof: next video

Outside the circle

\# subsets of [n] of size ≤ n-1

\# n-bit sequences with ≤ n-1 1's

n = 3

$$\underbrace{000}_{1},\ \underbrace{001,\ 010,\ 100}_{3},\ \underbrace{011,\ 101,\ 110}_{3}$$

1 + 3 + 3 = 7

Two ways

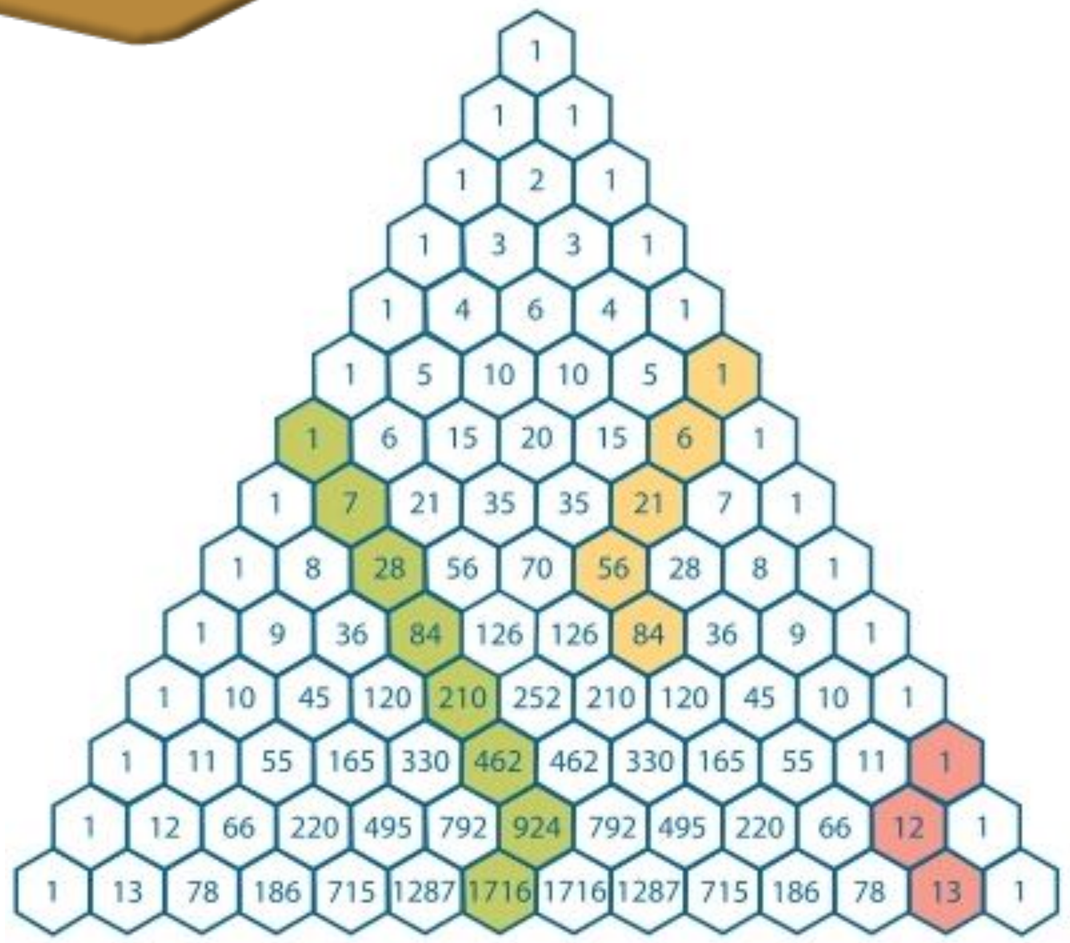
$$\sum_{i=0}^{n-1} \binom{n}{i}$$

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n \qquad \sum_{i=0}^{n-1} \binom{n}{i} = 2^n - \binom{n}{n} = 2^n - 1$$

$2^3 - 1 = 7$

Hockey Stick Identity

$$\sum_{i=0}^{n} \binom{i+k-1}{k-1} = \binom{n+k}{k}$$



$$\binom{k-1}{k-1} + \binom{k}{k-1} + \ldots + \binom{n+k-1}{k-1} = \binom{n+k}{k}$$

k=3, n=2

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} = 1 + 3 + 6 = 10 = \binom{5}{3}$$

Combinatorial Proof

$$\binom{k-1}{k-1} + \binom{k}{k-1} + \ldots + \binom{n+k-1}{k-1} = \binom{n+k}{k}$$

A = { binary sequences of length n+k with k 1's }

$$|A| = \binom{n+k}{k}$$

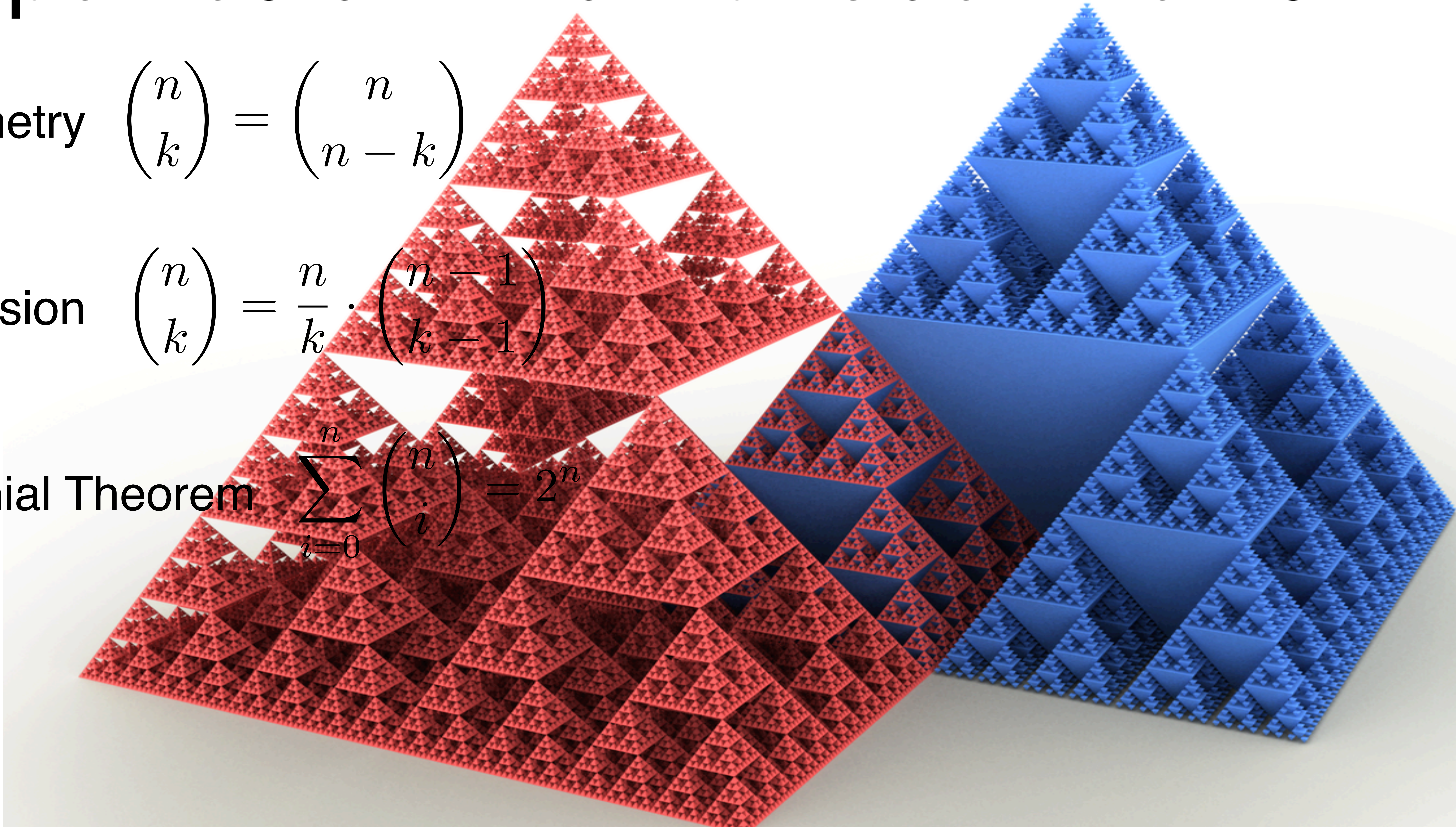
A_i = { (n+k)-bit sequences with k 1's where the last 1 is at location i}

$$|A_i| = \binom{i-1}{k-1}$$

Example

$$A = A_k \uplus A_{k+1} \uplus \ldots \uplus A_{n+k}$$

Properties of Binomial Coefficients



Symmetry $$\binom{n}{k} = \binom{n}{n-k}$$

Recursion $$\binom{n}{k} = \frac{n}{k} \cdot \binom{n-1}{k-1}$$

Binomial Theorem $$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

Next: Pascal Triangle and Binomial Theorem