

Haitham Abdel Razaq Moh'd Almatani 407920
 Themistoklis Dimaridis 355835
 Kirill Beskorovainyi 451420

Aufgabe 1:

a) i)

$$z^3 = -8i$$

Wir bringen erstmal die Zahl $-8i = 0 + -8i$ in Eulerdarstellung, der Form $z = re^{i\phi}$.

- $r = \sqrt{x^2 + y^2} = \sqrt{0^2 + (-8)^2} = \sqrt{8^2} = 8$

- $\phi = \arg(z) = \arctan\left(\frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}\right)$

In unserem Fall der Winkel $\phi = -\frac{\pi}{2}$, da $x = 0$ und $y < 0$.

Also gilt:

$$-8i = re^{i\phi}$$

$$-8i = 8e^{-\frac{\pi}{2}i}$$

Das heißt also:

$$z^3 = -8i$$

$$\Leftrightarrow z^3 = 8e^{-\frac{\pi}{2}i}$$

$$\Rightarrow r^3 = 8 \quad 3\phi = -\frac{\pi}{2} + 2k\pi, k \in \mathbb{Z}$$

$$r = \sqrt[3]{8} = 2 \quad \phi = -\frac{\pi}{6} + \frac{2k\pi}{3}, k \in \mathbb{Z}$$

Für $k = 0, 1, 2$ bekommen wir unsere 3 verschiedenen Lösungen wie folgendes:

$$k = 0 \quad : \quad y_0 = -\frac{\pi}{6} \text{ und somit } z_0 = 2e^{-\frac{\pi}{6}i}$$

$$k = 1 \quad : \quad y_1 = -\frac{\pi}{6} + \frac{2\pi}{3} = \frac{3\pi}{6} = \frac{\pi}{2} \text{ und somit } z_1 = 2e^{\frac{\pi}{2}i}$$

$$k = 2 \quad : \quad y_2 = -\frac{\pi}{6} + \frac{2 \times 2\pi}{3} = -\frac{\pi}{6} + \frac{4\pi}{3} = \frac{7\pi}{6} \text{ und somit } z_2 = 2e^{\frac{7\pi}{6}i}$$

Allgemeine Polardarstellung:

$$z = r(\cos(\phi) + i\sin(\phi))$$

Somit sehen unsere Lösungen wie folgt aus:

$$\begin{aligned} z_0 &= 2(\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6})) \\ z_1 &= 2(\cos(\frac{\pi}{6}) + i\sin(\frac{\pi}{6})) \\ z_2 &= 2(\cos(\frac{7\pi}{6}) + i\sin(\frac{7\pi}{6})) \\ \text{a) } & \quad \text{ii) } \end{aligned}$$

$$\begin{aligned} z^4 + 2(\sqrt{12} - 2i)z^2 + 8 - 4\sqrt{12}i &= 0 \\ z^4 + 2 \times z^2(\sqrt{12} - 2i) + (\sqrt{12} - 2i)^2 &= 0 \end{aligned}$$

Laut der binomischen Formel: $(a + b)^2 = a^2 + 2ab + b^2$ können wir die Gleichung umformen:

$$\begin{aligned} (z^2 + \sqrt{12} - 2i)^2 &= 0 \\ \Leftrightarrow z^2 + \sqrt{12} - 2i &= 0 \\ \Leftrightarrow z^2 &= -\sqrt{12} + 2i \end{aligned}$$

$-\sqrt{12} + 2i$ bringen wir zunächst in Eulerdarstellung.

$$r = \sqrt{x^2 + y^2} = \sqrt{(-12)^2 + 2^2} = \sqrt{12 + 4} = \sqrt{16} = 4$$

$$\phi = \arg(z) = \arctan\left(\frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}\right) + \pi, \text{ da } x < 0.$$

$$\begin{aligned} \text{Also } \arctan\left(\frac{2}{-\sqrt{12}}\right) + \pi &= \arctan\left(\frac{2}{-2\sqrt{3}}\right) + \pi = \arctan\left(-\frac{\sqrt{3}}{3}\right) + \pi = \\ -\frac{\pi}{6} + \pi &= \frac{5\pi}{6} \end{aligned}$$

Also gilt:

$$-\sqrt{12} + 2i = 4e^{i\frac{5\pi}{6}}$$

$$z^2 \text{ in Eulerdarstellung: } z^2 = (re^{i\phi})^2 = r^2e^{i2\phi}$$

Also gilt:

$$\begin{aligned} z^2 &= -\sqrt{12} + 2i \\ \Leftrightarrow r^2e^{i2\phi} &= 4e^{i\frac{5\pi}{6}} \\ \Rightarrow r^2 = 4 \quad \text{und} \quad 2\phi &= \frac{5\pi}{6} + 2k\pi, \quad k \in \mathbb{Z} \\ r &= 2 \quad \phi = \frac{5\pi}{12} + k\pi, \quad k \in \mathbb{Z} \end{aligned}$$

Also für $k = 0, 1$ bekommen wir unsere 2 verschiedenen Lösungen wie folgendes:

$$\begin{aligned} k = 0: \quad \phi_0 &= \frac{5\pi}{12} \text{ und somit } z_0 = 2e^{\frac{5\pi}{12}i} \\ k = 1: \quad \phi_1 &= \frac{5\pi}{12} + \pi = \frac{17\pi}{12} \text{ und somit } z_1 = 2e^{\frac{17\pi}{12}i} \end{aligned}$$

Allgemeine Polardarstellung:

$$z = r(\cos(\phi) + i\sin(\phi))$$

Somit sehen unsere Lösungen wie folgt aus:

$$\begin{aligned} z_0 &= 2(\cos(\frac{5\pi}{12}) + i\sin(\frac{5\pi}{12})) \\ z_1 &= 2(\cos(\frac{17\pi}{12}) + i\sin(\frac{17\pi}{12})) \\ \text{b)} \end{aligned}$$

$$z_1 = \sqrt{2}e^{-\frac{\pi}{4}i}$$

z_1 in allgemeine Polardarstellung:

$$z_1 = re^{i\phi}$$

Also gilt: $re^{i\phi} = \sqrt{2}e^{-\frac{\pi}{4}i}$

$$\Rightarrow r = \sqrt{2} \quad \phi = -\frac{\pi}{4}$$

Also ist z_1 in Polardarstellung:

$$\begin{aligned} z_1 &= r(\cos(\phi) + i\sin(\phi)) \\ z_1 &= \sqrt{2}(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})) \\ z_1 &= \sqrt{2}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) \\ z_1 &= 1 - i \end{aligned}$$

Also $z_1 = 1 - i$ in kartesische Darstellung.

c)

Sei $z = e^{\frac{5\pi}{12}i}$ in Eulerdarstellung.

Allgemein gilt: $z = re^{i\phi}$

Also ist $re^{i\phi} = e^{\frac{5\pi}{12}i}$

$\Rightarrow r = 1$ und $\phi = \frac{5\pi}{12} = \frac{\pi}{4} + \frac{\pi}{6}$

Für die Polardarstellung gilt:

$$\begin{aligned} z &= r(\cos(\phi) + i\sin(\phi)) \\ &= 1 \times (\cos(\frac{5\pi}{12}) + i\sin(\frac{5\pi}{12})) \\ z &= (\cos(\frac{\pi}{4} + \frac{\pi}{6}) + i\sin(\frac{\pi}{4} + \frac{\pi}{6})) \end{aligned}$$

Allgemein gilt für die Additionstheoreme:

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

Mithilfe also der Additionstheorie haben wir:

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, \sqrt{3}\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \Rightarrow \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$z = \cos\left(\frac{\pi}{4}\right) \times \cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{4}\right) \times \sin\left(\frac{\pi}{6}\right) + i \left(\sin\left(\frac{\pi}{4}\right) \times \cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{4}\right) \times \sin\left(\frac{\pi}{6}\right) \right)$$

$$z = \left(\frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} \right) - \left(\frac{\sqrt{2}}{2} \times \frac{1}{2} \right) + i \left(\frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2} \right)$$

$$z = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} + i \left(\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} \right)$$

$$z = \frac{\sqrt{6} - \sqrt{2}}{4} + i \left(\frac{\sqrt{6} + \sqrt{2}}{4} \right) \text{ in kartesische Darstellung}$$

Aufgabe 2:

$$p(z) = z^4 + z^3 + 3z^2 + 4z - 4$$

a)

$$\begin{aligned} p(2i) &= (2i)^4 + (2i)^3 + 3(2i)^2 + 4(2i) - 4 \\ &= 2^4 \cdot i^4 + 2^3 \cdot i^3 + 3 \cdot 4i^2 + 8i - 4 \\ &= 16(1) + 8(-1) + 12(-1) + 8i - 4 \\ &= 16 - 12 - 4 + 8i - 8i \\ &= 0 \end{aligned}$$

b)

$$q(z) = z^2 + 4$$

$$\begin{array}{r}
z^4 + z^3 + 3z^2 + 4z - 4 = (z^2 + 4)(z^2 + z - 1) \\
\underline{-z^4 \qquad -4z^2} \\
z^3 - z^2 + 4z \\
\underline{-z^3 \qquad -4z} \\
-z^2 - 4 \\
z^2 + 4 \\
\underline{ } \\
0
\end{array}$$

c)

Bestimmung der Nullstellen von $z^2 + z - 1$ mit Anwendung der $p - q$ Formel:

$$\begin{array}{ll}
z^2 + 4 = 0 & z^2 + z - 1 = 0 \\
z^2 = -4 & z_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q} \\
z = \pm\sqrt{4} & = -\frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + 1} \\
z = \pm\sqrt{4i^2} & = -\frac{1}{2} \pm \sqrt{\frac{5}{4}} \\
z = \pm 2i & -\frac{1}{2} \pm \frac{\sqrt{5}}{2}
\end{array}$$

Also die komplexe Linearfaktorzerlegung von $p(z)$:

$$p(z) = z^4 + z^3 + 3z^2 + 4z - 4 = (z - 2i)(z + 2i)\left(z - \left(-\frac{1}{2} + \frac{\sqrt{5}}{2}\right)\right)\left(z - \left(-\frac{1}{2} - \frac{\sqrt{5}}{2}\right)\right)$$

d)

Die reelle Zerlegung von $p(z)$:

$$p(z) = z^4 + z^3 + 3z^2 + 4z - 4 = (z^2 + 4)\left(z - \left(-\frac{1}{2} + \frac{\sqrt{5}}{2}\right)\right)\left(z - \left(-\frac{1}{2} - \frac{\sqrt{5}}{2}\right)\right)$$

Aufgabe 3:

a)

$$f(x) = \frac{3x - 1}{x^2 - 4x + 13}$$

Wir müssen erstmal die Nullstellen des Nenners, $x^2 - 4x + 13$, bestimmen.

$$x^2 - 4x + 13 = 0$$

$$\begin{aligned}
\text{p-q Formel: } x_{1,2} &= -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q} \\
&= 2 \pm \sqrt{2^2 - 13} \\
&= 2 \pm \sqrt{-9} \\
&= 2 \pm \sqrt{9i^2} \\
&= 2 \pm 3i
\end{aligned}$$

Also $f(x) = \frac{3x-1}{x^2-4x+13} = \frac{3x-1}{(x-(2+3i))(x-(2-3i))}$ Also der Ansatz für komplexe Partialbruchzerlegung von $f(x)$ lautet:

$$\frac{3x-1}{x^2-4x+13} = \frac{A}{x-(2+3i)} + \frac{B}{x-(2-3i)}$$

b)

$$\begin{aligned}
f(x) &= \frac{2+i}{x-(3+2i)} + \frac{2-i}{x-(3-2i)} \\
&= \frac{(2+i)(x-(3-2i)) + (2-i)(x-(3+2i))}{(x-(3+2i))(x-(3-2i))} \\
&= \frac{(2+i)(x-3+2i) + (2-i)(x-3-2i)}{(x-3-2i)(x-3+2i)} \\
&= \frac{2x-6+4i+xi-3i-2+2x-6-4i-xi+3i-2}{x^2-3x+2xi-3x+9-6i-2xi+6i+4}
\end{aligned}$$