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Aufgabe 1:

a) i)

$$z^3 = -8i$$

Wir bringen erstmal die Zahl $-8i = 0 + -8i$ in Eulerdarstellung, der Form $z = re^{i\phi}$.

- $r = \sqrt{x^2 + y^2} = \sqrt{0^2 + (-8)^2} = \sqrt{8^2} = 8$
- $\phi = arg(z) = arctan\left(\frac{Im(z)}{Re(z)}\right)$

In unserem Fall der Winkel $\phi = -\frac{\pi}{2}$, da $x = 0$ und $y < 0$.

Also gilt:

$$-8i = re^{i\phi}$$

$$-8i = 8e^{-\frac{\pi}{2}i}$$

Das heißt also:

$$z^3 = -8i$$

$$<=> z^3 = 8e^{-\frac{\pi}{2}i}$$

$$=> r^3 = 8 \qquad 3\phi = -\frac{\pi}{2} + 2k\pi,\ k \in \mathbb{Z}$$

$$r = \sqrt[3]{8} = 2 \qquad \phi = -\frac{\pi}{6} + \frac{2k\pi}{3},\ k \in \mathbb{Z}$$

Für $k = 0, 1, 2$ bekommen wir unsere 3 verschiedenen Lösungen wie folgendes:

$k = 0$: $y_0 = -\frac{\pi}{6}$ und somit $z_0 = 2e^{-\frac{\pi}{6}i}$

$k = 1$: $y_1 = -\frac{\pi}{6} + \frac{2\pi}{3} = \frac{3\pi}{6} = \frac{\pi}{2}$ und somit $z_1 = 2e^{\frac{\pi}{2}i}$

$k = 2$: $y_2 = -\frac{\pi}{6} + \frac{2\times 2\pi}{3} = -\frac{\pi}{6} + \frac{4\pi}{3} = \frac{7\pi}{6}$ und somit $z_2 = 2e^{\frac{7\pi}{6}}i$

Allgemeine Polardarstellung:

$$z = r(cos(\phi) + isin(\phi))$$

Somit sehen unsere Lösungen wie folgt aus:

$z_0 = 2(cos\left(-\frac{\pi}{6}\right) + isin\left(-\frac{\pi}{6}\right))$

$z_1 = 2(cos\left(\frac{\pi}{6}\right) + isin\left(\frac{\pi}{6}\right))$

$z_2 = 2(cos\left(\frac{7\pi}{6}\right) + isin\left(\frac{7\pi}{6}\right))$

a) ii)

$$z^4 + 2(\sqrt{12} - 2i)z^2 + 8 - 4\sqrt{12}i = 0$$
$$z^4 + 2 \times z^2(\sqrt{12} - 2i) + (\sqrt{12} - 2i)^2 = 0$$

Laut der binomischen Formel: $(a+b)^2 = a^2 + 2ab + b^2$ können wir die Gleichung umformen:

$$(z^2 + \sqrt{12} - 2i)^2 = 0$$
$$<=> z^2 + \sqrt{12} - 2i = 0$$
$$<=> z^2 = -\sqrt{12} + 2i$$

$-\sqrt{12} + 2i$ bringen wir zunächst in Eulerdarstellung.
$r = \sqrt{x^2 + y^2} = \sqrt{(-12)^2 + 2^2} = \sqrt{12 + 4} = \sqrt{16} = 4$
$\phi = arg(z) = arctan\left(\frac{Im(z)}{Re(z)}\right) + \pi$, da $x < 0$.

Also $arctan\left(\frac{2}{-\sqrt{12}}\right) + \pi = arctan\left(\frac{2}{-2\sqrt{3}}\right) + \pi = arctan\left(-\frac{\sqrt{3}}{3}\right) + \pi = -\frac{\pi}{6} + \pi = \frac{5\pi}{6}$

Also gilt:

$$-\sqrt{12} + 2i = 4e^{i\frac{5\pi}{6}}$$

z^2 in Eulerdarstellung: $z^2 = (re^{i\phi})^2 = r^2 e^{i2\phi}$
Also gilt:

$$z^2 = -\sqrt{12} + 2i$$
$$<=> r^2 e^{i2\phi} = 4e^{i\frac{5\pi}{6}}$$
$$=> r^2 = 4 \qquad \text{und} \qquad 2\phi = \frac{5\pi}{6} + 2k\pi,\ k \in \mathbb{Z}$$
$$r = 2 \qquad \phi = \frac{5\pi}{12} + k\pi,\ k \in \mathbb{Z}$$

Also für $k = 0, 1$ bekommen wir unsere 2 verschiedene Lösungen wie folgendes:

$k = 0$: $\phi_0 = \frac{5\pi}{12}$ und somit $z_0 = 2e^{\frac{5\pi}{12}i}$

$k = 1$: $\phi_1 = \frac{5\pi}{12} + \pi = \frac{17\pi}{12}$ und somit $z_1 = 2e^{\frac{17\pi}{12}i}$

Allgemeine Polardarstellung:

$$z = r(cos(\phi) + isin(\phi))$$

Somit sehen unsere Lösungen wie folgt aus:

$z_0 = 2(cos(\frac{5\pi}{12} + isin(\frac{5\pi}{12})$
$z_1 = 2(cos(\frac{17\pi}{12} + isin(\frac{17\pi}{12})$
b)

$$z_1 = \sqrt{2}e^{-\frac{\pi}{4}i}$$

z_1 in allgemeine Polardarstellung:

$$z_1 = re^{i\phi}$$

Also gilt: $re^{i\phi} = \sqrt{2}e^{-\frac{\pi}{4}i}$

$$=> r = \sqrt{2} \qquad \phi = -\frac{\pi}{4}$$

Also ist z_1 in Polardarstellung:

$$z_1 = r(cos(\phi) + isin(\phi))$$

$$z_1 = \sqrt{2}(cos\left(-\frac{\pi}{4}\right) + isin\left(-\frac{\pi}{4}\right))$$

$$z_1 = \sqrt{2}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$

$$z_1 = 1 - i$$

Also $z_1 = 1 - i$ in kartesische Darstellung.
c)
Sei $z = e^{\frac{5\pi}{12}i}$ in Eulerdarstellung.
Allgemein gilt: $z = re^{i\phi}$
Also ist $re^{i\phi} = e^{\frac{5\pi}{12}i}$
$=> r = 1$ und $\phi = \frac{5\pi}{12} = \frac{\pi}{4} + \frac{\pi}{6}$
Für die Polardarstellung gilt:

$$z = r(cos(\phi) + isin(\phi))$$

$$1 \times (cos\left(\frac{5\pi}{12}\right) + isin\left(\frac{5\pi}{12}\right))$$

$$z = \left(cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) + isin\left(\frac{\pi}{4} + \frac{\pi}{6}\right)\right)$$

Allgemein gilt für die Additionstheoreme:

$cos(x + y) = cos(x)cos(y) - sin(x)sin(y)$

$sin(x + y) = sin(x)cos(y) + cos(x)sin(y)$

Mithilfe also der Additionstheorie haben wir:

$$cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2},\ \sqrt{3}sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} => sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$z = cos\left(\frac{\pi}{4}\right) \times cos\left(\frac{\pi}{6}\right) - sin\left(\frac{\pi}{4}\right) \times sin\left(\frac{\pi}{6}\right) + i\left(sin\left(\frac{\pi}{4}\right) \times cos\left(\frac{\pi}{6}\right) + cos\left(\frac{\pi}{4}\right) \times sin\left(\frac{\pi}{6}\right)\right)$$

$$z = \left(\frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2} \times \frac{1}{2}\right) + i\left(\frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2}\right)$$

$$z = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} + i\left(\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}\right)$$

$$z = \frac{\sqrt{6} - \sqrt{2}}{4} + i\left(\frac{\sqrt{6} + \sqrt{2}}{4}\right) \text{ in kartesische Darstellung}$$

Aufgabe 2:

$$p(z) = z^4 + z^3 + 3z^2 + 4z - 4$$

a)

$$\begin{aligned} p(2i) &= (2i)^4 + (2i)^3 + 3(2i)^2 + 4(2i) - 4 \\ &= 2^4 \cdot i^4 + 2^3 \cdot i^3 + 3 \cdot 4i^2 + 8i - 4 \\ &= 16(1) + 8(-1) + 12(-1) + 8i - 4 \\ &= 16 - 12 - 4 + 8i - 8i \\ &= 0 \end{aligned}$$

b)

$$q(z) = z^2 + 4$$

$$
\begin{array}{l}
z^4 + z^3 + 3z^2 + 4z - 4 = \left(z^2+4\right)\left(z^2+z-1\right) \\
\underline{-z^4 \qquad -4z^2} \\
\qquad z^3 - z^2 + 4z \\
\qquad \underline{-z^3 \qquad -4z} \\
\qquad\qquad -z^2 \qquad -4 \\
\qquad\qquad \underline{z^2 \qquad +4} \\
\qquad\qquad\qquad 0
\end{array}
$$

c)

Bestimmung der Nullstellen von z^2+z-1 mit Anwendung der $p-q$ Formel:

$$
\begin{aligned}
z^2+4&=0 & z^2+z-1&=0\\
z^2&=-4 & z_{1,2}&=-\frac{p}{2}\pm\sqrt{\left(\frac{p}{2}\right)^2-q}\\
z&=\pm\sqrt{4} & &=-\frac{1}{2}\pm\sqrt{\left(\frac{1}{2}\right)^2+1}\\
z&=\pm\sqrt{4i^2} & &=-\frac{1}{2}\pm\sqrt{\frac{5}{4}}\\
z&=\pm 2i & &-\frac{1}{2}\pm\frac{\sqrt{5}}{2}
\end{aligned}
$$

Also die komplexe Linearfaktorzerlegung von $p(z)$:

$$p(z)=z^4+z^3+3z^2+4z-4=(z-2i)(z+2i)(z-\left(-\frac{1}{2}+\frac{\sqrt{5}}{2}\right))(z-\left(-\frac{1}{2}-\frac{\sqrt{5}}{2}\right))$$

d)

Die reelle Zerlegung von $p(z)$:

$$p(z)=z^4+z^3+3z^2+4z-4=(z^2+4)(z-\left(-\frac{1}{2}+\frac{\sqrt{5}}{2}\right))(z-\left(-\frac{1}{2}-\frac{\sqrt{5}}{2}\right))$$