Generalized Gauss-Markov Theorem.

Assumption.
Y=$X\beta$+e
where $E[e|X]$=0 $var[e|X]$=Ω $E[Y^2|X] <\infty$ $E[X^2]<\infty$
X^TX is invertible

Under the given assumption and Ω>0, let $\tilde{\beta}$ be a linear unbiased estimator of β then we have

$$var[\tilde{\beta} \mid X] \geq (X^T\Omega^{-1}X)^{-1}$$

Proof.

let $\tilde{\beta}$=CY where C is a function of X.

E[$\tilde{\beta}$|X]=E[CY|X]

=E[C(Xβ+e)|X]

=E[CXβ+Ce|X]

=CXβ

=β

Thus CX=I

var[$\tilde{\beta}$|X]=var[(Ce)|X]=Cvar[e|X]C^T=$C\Omega C^T$

Without loss of generality, let C=$(X^T\Omega^{-1}X)^{-1}X^T\Omega^{-1} + D$

CX=I $\Rightarrow$ DX=0

Note that Ω is positive semi-definite and symmetry,

Thus,

var[$\tilde{\beta}$|X]=$C\Omega C^T$

=$((X^T\Omega^{-1}X)^{-1}X^T\Omega^{-1} + D)\Omega(D^T + (\Omega^{-1})^TX(X^T\Omega^{-1}X)^{-1})$

=$(X^T\Omega^{-1}X)^{-1}X^T\Omega^{-1}X(X^T\Omega^{-1}X)^{-1} + D\Omega D^T$

=$(X^T\Omega^{-1}X)^{-1} + D\Omega D^T$ $\geq$$(X^T\Omega^{-1}X)^{-1}$ $\square$