A proof of Minkowski's inequality

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Suppose that $n \in \mathbb{N}$, and let $a_k, b_k \in \mathbb{R}, 1 \leq k \leq n$. If $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ then

$$\left[\sum_{k=1}^{n} |a_k + b_k|^p\right]^{\frac{1}{p}} \leq \left[\sum_{k=1}^{n} |a_k|^p\right]^{\frac{1}{p}} + \left[\sum_{k=1}^{n} |b_k|^p\right]^{\frac{1}{p}}$$

Proof.
We first show the $H\ddot{o}lder's\ inequality$ under assumption.

$$\sum_{k=1}^{n} |a_k b_k| \leq \left[\sum_{k=1}^{n} |a_k|^p\right]^{\frac{1}{p}} \left[\sum_{k=1}^{n} |b_k|^p\right]^{\frac{1}{p}}$$

Lemma 1. *If $A, B > 0$ and $t < 1$, then $A^t B^{1-t} \leq tA + (1-t)B$*

Proof.

Define $f(A) = tA + (1-t)B - A^t B^{1-t}$

let $\frac{\partial}{\partial A} f = t - tA^{t-1} B^{1-t} = 0$

Notice $\frac{\partial^2}{\partial A^2} f = -t(t-1)A^{t-2} B^{1-t} > 0$

Thus f has a minimum at $A = B$ $\square$

Thus

$$\begin{aligned}
\frac{\sum_{k=1}^{n} |a_k b_k|}{\left[\sum_{k=1}^{n} |a_k|^p\right]^{\frac{1}{p}} \left[\sum_{k=1}^{n} |b_k|^p\right]^{\frac{1}{p}}} &= \sum_{k=1}^{n} \frac{(|a_k|^p)^{\frac{1}{p}}}{\left[\sum_{k=1}^{n} |a_k|^p\right]^{\frac{1}{p}}} \frac{(|b_k|^q)^{\frac{1}{q}}}{\left[\sum_{k=1}^{n} |b_k|^q\right]^{\frac{1}{q}}} \\
&\leq \sum_{k=1}^{n} \frac{(|a_k|^p)}{p\left[\sum_{k=1}^{n} |a_k|^p\right]} + \frac{(|b_k|^q)}{q\left[\sum_{k=1}^{n} |b_k|^q\right]} \\
&= \frac{1}{p} + \frac{1}{q} = 1
\end{aligned}$$

We now prove Minkowski's inequality.
By triangle inequality,

$$\begin{aligned}
\sum_{k=1}^{n} |a_k + b_k|^p &\leq \sum_{k=1}^{n} |a| \, |a_k + b_k|^{p-1} + \sum_{k=1}^{n} |b| \, |a_k + b_k|^{p-1} \\
&\leq \left[\sum_{k=1}^{n} |a_k|^p\right]^{\frac{1}{p}} \left[\sum_{k=1}^{n} |a_k + b_k|^{q(p-1)}\right]^{\frac{1}{q}} + \left[\sum_{k=1}^{n} |b_k|^p\right]^{\frac{1}{p}} \left[\sum_{k=1}^{n} |a_k + b_k|^{q(p-1)}\right]^{\frac{1}{q}} \\
&= (\left[\sum_{k=1}^{n} |a_k|^p\right]^{\frac{1}{p}} + \left[\sum_{k=1}^{n} |b_k|^p\right]^{\frac{1}{p}}) \left[\sum_{k=1}^{n} |a_k + b_k|^{q(p-1)}\right]^{\frac{1}{q}} \quad (*)
\end{aligned}$$

Note that $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow pq = p + q$
Then

$$(*) = (\left[\sum_{k=1}^{n} |a_k|^p\right]^{\frac{1}{p}} + \left[\sum_{k=1}^{n} |b_k|^p\right]^{\frac{1}{p}}) \left[\sum_{k=1}^{n} |a_k + b_k|^p\right]^{\frac{1}{q}}$$

That's to say,

$$\sum_{k=1}^{n} |a_k + b_k|^p \leq (\left[\sum_{k=1}^{n} |a_k|^p\right]^{\frac{1}{p}} + \left[\sum_{k=1}^{n} |b_k|^p\right]^{\frac{1}{p}}) \left[\sum_{k=1}^{n} |a_k + b_k|^p\right]^{\frac{1}{q}}$$

Thus,

$$\left[\sum_{k=1}^{n} |a_k + b_k|^p\right]^{\frac{1}{p}} \leq \left[\sum_{k=1}^{n} |a_k|^p\right]^{\frac{1}{p}} + \left[\sum_{k=1}^{n} |b_k|^p\right]^{\frac{1}{p}}$$

$\square$