$\forall x,\ y \in \mathbb{R},\ x < y,\ \exists i \in \mathbb{R}\backslash\mathbb{Q}$ such that $x < i < y$

Proof. ‘
Case 1: $x, y \in \mathbb{Q}$, choose $i = x + \frac{\sqrt{2}}{2}(y - x)\ \in \mathbb{R}\backslash\mathbb{Q}$,
Case 2: $x,\ y\ \notin \{x,\ y \in \mathbb{Q}\}$, Obviously $\exists m,\ n \in \mathbb{R},\ x < m < n < y$
If $m,\ n \in \mathbb{Q}$, then by Case 1, $\exists i \in \mathbb{R}\backslash\mathbb{Q}$ such that $x < m < i < n < y\ \Rightarrow x < i < y$
If $m,\ n\ \notin \{m,\ n \in \mathbb{Q}\}$ Then choose $i = m$ or $i = n$ such that $i \in \mathbb{R}\backslash\mathbb{Q}$ $\square$