Properties of measurable sets

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Define: If E is any subset of $\mathbb{R}^d$, the outer measure of E is $m_*(E) = inf\{\sum_{j=1}^{\infty} |Q_j| : E \subset \bigcup_{j=1}^{\infty} Q_j, Q_j \in Q\}$

Define: A subset E of $\mathbb{R}^d$ is Lebesgue measurable if for any $\varepsilon > 0$, there exists an open set O with $E \subset O$ and $m_*(O \backslash E) < \varepsilon$

If E is measurable, we define its measure $m(E)$ by $m(E) = m_*(E)$

Property 1: Every open set in $\mathbb{R}^d$ is measurable.

Proof.

let $E \subset E$ □

Property 2: If $m_*(E) = 0$, then E is measurable.

Proof.

$m_*(E) = 0 \Rightarrow inf\{m_*(O) : E \subset O\} = 0 \Rightarrow \forall \varepsilon > 0,\ \exists O\ such\ that\ E \subset O,\ m_*(O) < \varepsilon$

Then $m_*(O \backslash E) \leq m_*(O) < \varepsilon$ □

Property 3: Let $\{E_j\}_{j=1}^{\infty}$ be a family of measurable sets, then $\bigcup_{j=1}^{\infty} E_j$ is measurable.

Proof.

$\forall j, \exists O_j, m_*(O_j \backslash E_j) < \frac{\varepsilon}{2^{-j}}$

$\bigcup_{j=1}^{\infty} E_j \subset \bigcup_{j=1}^{\infty} O_j$

We now prove $\bigcup_{j=1}^{\infty} O_j \backslash \bigcup_{j=1}^{\infty} E_j \subset \bigcup_{j=1}^{\infty}(O_j \backslash E_j)$

Suppose $\forall x \in \bigcup_{j=1}^{\infty} O_j \backslash \bigcup_{j=1}^{\infty} E_j$

That's to say, $x \in \bigcup_{j=1}^{\infty} O_j$ and $x \in \bigcap_{j=1}^{\infty} E_j^c$

Thus, $x_0 \in O_{j_0}$ and $\forall j, x_0 \in E_j^c$

Thus, $x_0 \in O_{j_0} \bigcap E_{j_0}^c \Rightarrow x \in \bigcup_{j=1}^{\infty}(O_j \backslash E_j)$

$m_*(\bigcup_{j=1}^{\infty} O_j \backslash \bigcup_{j=1}^{\infty} E_j) \leq m_*(\bigcup_{j=1}^{\infty}(O_j \backslash E_j)) \leq \sum_{j=1}^{\infty} m_*(O_j \backslash E_j) < \varepsilon$ $\square$

Property 4: Closed sets are measurable.

Proof.

Lemma 1. *If F is closed, K is compact, and $K \bigcap F = \varnothing$, then $d(K, F) > 0$*

Proof. Suppose $\exists x \in K, d(x, F) = 0$

Then $inf_{y \in F}\{|x - y|\} = 0$

That's to say, $\exists y_n \in F,\ s.t.\ y_n \to x$

Thus, $x \in F$ $\square$

(1) F is compact:

Choose O, $F \subset O$. Let $m_*(O) \leq m_*(F) + \varepsilon/2$

Then $O \backslash F = O \bigcap F^c$, which is open

Thus $O \bigcap F^c = \bigcup_{j=1}^{\infty} Q_j$

For $N \in \mathbb{N}^+, K = \bigcup_{j=1}^{N} Q_j$ is compact and $d(K, F) > 0$

$K \bigcup F \subset O \Rightarrow m_*(O) \geq m_*(K) + m_*(F) = m_*(F) + \sum_{j=1}^{N} m_*(Q_j)$

$\sum_{j=1}^{N} m_*(Q_j) \leq m_*(O) - m_*(F) \leq \varepsilon/2$

let $N \to \infty$, then $m_*(O \backslash F) = m_*(\bigcup_{j=1}^{\infty} Q_j) \leq \sum_{j=1}^{\infty} m_*(Q_j) \leq \varepsilon/2 < \varepsilon$

(2) F is closed:

then for every closed ball $B(0, N) : \{x \in \mathbb{R}^d : |x| \leq N\}$

$\bigcup_{N=1}^{\infty} B(0, N) = \mathbb{R}^d$

$F \bigcap \mathbb{R}^d = F \bigcap \bigcup_{N=1}^{\infty} B(0, N) = \bigcup_{N=1}^{\infty}(F \bigcap B(0, N))$

Note that $F \bigcap B(0, N)$ is compact, and use Property 3, then F is measurable.

$\square$

Property 5: Let E be measurable, then E^c is measurable.

Proof.

$\forall n \in \mathbb{N}$, choose O_n such that $E \subset O_n$ and $m_*(O_n \backslash E) \leq \frac{1}{n}$

Note that O_n^c is closed, let $S = \bigcup_{n=1}^{\infty} O_n^c$

$E^c \backslash S \subset O \backslash E \Rightarrow m_*(E^c \backslash S) \leq m_*(O \backslash E) \leq \frac{1}{n},\ \forall n$

Note that $E^c = (E^c \backslash S) \bigcup S$, where $E^c \backslash S$ and S are both measurable.

Thus E^c is measurable. $\square$

Property 6: Let $\{E_j\}_{j=1}^{\infty}$ be a family of measurable sets, then $\bigcap_{j=1}^{\infty} E_j$ is measurable.

Proof.

Note that $\bigcap_{j=1}^{\infty} E_j = (\bigcup_{j=1}^{\infty} E_j^c)^c$ $\square$