$|\mathbb{Q}| = |\mathbb{N}|$

Proof.
First prove that $|\{q : q \in \mathbb{Q}, q > 0\}| = |\mathbb{N}|$
1) $q \in \mathbb{N}$ *and* $q > 1$
There exists a series of unique prime numbers $p_1 ... p_n$ and exponents $r_1 ... r_n$ such that

$$q = p_1^{r_1} ... p_n^{r_n}$$

2) $q \notin \mathbb{N}$ *and* $q > 1$
There exists a series of unique prime numbers $p_1 ... p_n, q_1 ... q_m$ and exponents $r_1 ... r_n, s_1 ... s_m$ such that

$$q = \frac{p_1^{r_1} ... p_n^{r_n}}{q_1^{s_1} ... q_m^{s_m}}$$

Define $f : \{q \in \mathbb{Q}, q > 0\} \to \mathbb{N}$
$f(1) = 1$, $f(q) = p_1^{2r_1} ... p_n^{2r_n}$ if $q \in \mathbb{N} \backslash \{1\}$ and
$f(q) = p_1^{2r_1} ... p_n^{2r_n} q_1^{2s_1 - 1} ... q_m^{2s_m - 1}$ if $q \in \mathbb{Q} \backslash \mathbb{N}$.
For injection, $f(q_1) = f(q_2) \Rightarrow q_1 = q_2$ since f is unique.
For surjection, $\forall m = p_1^{r_1} ... p_n^{r_n} q_1^{s^1} ... q_m^{s_m} \in \mathbb{N}$
Without loss of generality, suppose that $r_1, ..., r_n$ are positive even numbers, while $s_1, ..., s_n$ are positive odd numbers.
Thus, $\exists q \in \mathbb{Q}$ such that $f(q = \frac{p_1^{r_1/2} ... p_n^{r_n/2}}{q_1^{(s^1+1)/2} ... q_m^{(s_m+1)/2}}) = m$
Thus, f is a bijection.
According to symmetry, $g : \{q \in \mathbb{Q}, q < 0\} \to \mathbb{N}$ is also a bijection.
Define $h : \mathbb{Q} \to \mathbb{Z}$ by $h(x) = \begin{cases} 0, \ if \ x = 0 \\ f(x), \ if \ x > 0 \\ g(x), \ if \ x < 0 \end{cases}$
Thus h is a bijection.
Thus, $|\mathbb{Q}| = |\mathbb{Z}| \Rightarrow |\mathbb{Q}| = |\mathbb{N}|$

$\square$