Lecture 1: Financial Time Series and Their Characteristics

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Outline

1. Introduction
2. Asset returns
3. Behavior of financial return data
4. Distributional properties of returns

Financial time series analysis

★Data ! UNCERTAINTY

- Financial time series (FTS) analysis is concerned with theory and practice of the valuation of financial assets over time, such as stocks, bonds, or bank deposits, etc.
- Different from other T.S. analysis?
    - Not exactly, but with an added uncertainty. For example, FTS must deal with the changing business and economic environment and the fact that volatility is not directly observed.
    - As a result of the added uncertainty, statistical theory and methods play an important role in financial time series analysis.

Objective of the course

1. Theory   2. Software

- to provide some basic knowledge of financial time series data such as skewness, heavy tails, and measure of dependence between asset returns;
- to introduce some statistical tools useful for analyzing these series;
- to gain experience in financial applications of various econometric methods.

Outline

Returns

Most financial studies involve return, instead of prices, of assets:

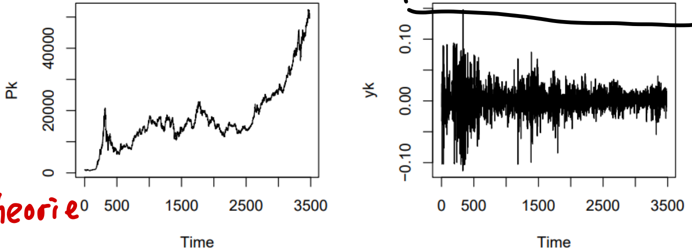
- Return of an asset is a complete and scale-free summary of the investment opportunity;
  - \$ 1 change in a \$ 10 stock is more significant than \$ 1 change in a \$ 100 stock.
- Asset returns exhibit more attractive statistical properties than asset prices themselves.

Undesired statistical properties of Prices

- Daily closing price at time index k (left, P_k) of the WIG index (main summary index of Warsaw Stock Exchange) 1991-2007
- Log returns (right, y_k) of the WIG index, where $y_k = \log(P_k/P_{k-1})$

Undesired statistical properties of Prices

Return is stationary



Usual Theorie

Exponential Equation

- The price increases exponentially with time, but mathematical tools (e.g. correlation, regression) work most naturally with linear functions.
- The price displays unit-root behavior and thus cannot be modelled as stationary series.

Undesired statistical properties of Prices

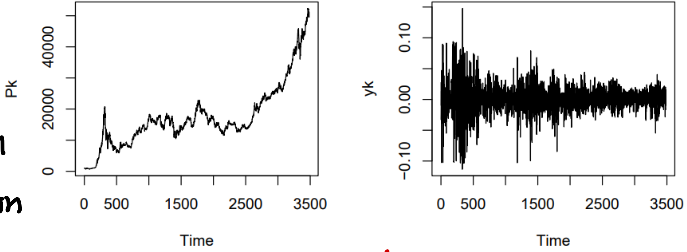
Random Variable

↓

矩估计

Id: Mean

Expectation



Step1: Price Data → Return rate

[Indirect] [Direct]

- The mean value of an exponentially-increasing time series has no obvious meaning.
- The derivative of an exponential function is exponential, so day-to-day changes in price have the same unfortunate properties.

One-period simple return

How to compute the return

Let P_t denote the price of an asset at time t.

Holding an asset from time $t-1$ to t, the value of the asset changes from P_{t-1} to P_t. Assuming that no dividends are paid over the period for now.

- One-period simple net return or simple return

Simple Return Rate

$$\boxed{R_t = \frac{P_t - P_{t-1}}{P_{t-1}} = \frac{P_t}{P_{t-1}} - 1,}$$

One-period simple return

- One-period simple net return

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}},$$

Often we write $R_t = 100R_t\%$, as $100R_t$ is the percentage of the gain with respect to the initial capital P_{t-1}.
The returns for less risky assets such as bonds can be even smaller in a short period and are often quoted in basis points, which is $10,000R_t$.

One-period simple return

- One-period simple gross return

$$R_t + 1 = \frac{P_t}{P_{t-1}},$$

It is the ratio of the new market value at the end of the holding period over the initial market value.

Multiple-period simple return

The holding period for an investment may be more than one time unit. For $k \geq 1$,

- *k*-period simple net return:

$$R_t(k) = \frac{P_t - P_{t-k}}{P_{t-k}}$$

- *k*-period simple gross return:

$$\frac{P_t}{P_{t-k}} = R_t(k) + 1$$

Multiple-period simple return

- The k-period simple gross return may be expressed in terms of one-period simple gross return,

$$\frac{P_t}{P_{t-k}} = \frac{P_t}{P_{t-1}} \times \frac{P_{t-1}}{P_{t-2}} \times \cdots \times \frac{P_{t-k+1}}{P_{t-k}}$$

- The k-period simple gross return is just the product of the k one-period simple gross returns,

$$\begin{aligned} R_t(k) + 1 &= \frac{P_t}{P_{t-k}} \\ &= (R_t + 1)(R_{t-1} + 1)...(R_{t-k+1} + 1) \end{aligned}$$

Multiple-period returns

If all one-period returns $R_t, ..., R_{t-k+1}$ are small,

$$R_t(k) \approx R_t + R_{t-1} + ... + R_{t-k+1}.$$

This is a useful approximation when the time unit is small (such as a day, an hour or a minute).

Example

Suppose the daily closing prices of a stock are

Day	1	2	3	4	5
Price	37.84	38.49	37.12	37.60	36.30

- What is the simple net return from day 1 to day 2?
- What is the simple net return from day 1 to day 5?

Example

- What is the simple net return from day 1 to day 2?

$$R_2 = \frac{38.49 - 37.84}{37.84} = 0.017.$$

- What is the simple net return from day 1 to day 5?

$$R_5(4) = \frac{36.30 - 37.84}{37.84} = -0.041.$$

- Verify that

$$1 + R_5(4) = (1 + R_2)(1 + R_3)(1 + R_4)(1 + R_5)$$

Annualized (average) return

- When the investment horizon is longer than 1 year ($k > 1$) it is customary to report the returns as annualized (average) returns.
- The annualized return is computed by its geometric mean of the k one-period simple gross returns involved.

Geometric vs. Arithmetic Mean

- arithmetic mean

$$\frac{1}{k}\sum_{i=1}^{k} x_i.$$

- geometric mean:

$$(\prod_{i=1}^{k} x_i)^{1/k} = \sqrt[k]{x_1 x_2 \cdots x_k}.$$

It is easier to compute arithmetic average than geometric mean.

Annualized (average) return

The annualized return is computed by its geometric mean of the k one-period simple gross returns involved:

$$Annualized[R_t(k)] = \left[\prod_{j=0}^{k-1}(1 + R_{t-j})\right]^{1/k} - 1,$$

which can be approximated by,

$$Annualized[R_t(k)] \approx \frac{1}{k}\sum_{j=0}^{k-1} R_{t-j}. \tag{1}$$

Derivation of (1)

It is noted that,

$$\left[\prod_{j=0}^{k-1}(1+R_{t-j})\right]^{1/k} - 1 = \exp\left[\frac{1}{k}\sum_{j=0}^{k-1}\ln(1+R_{t-j})\right] - 1$$

Since $\frac{1}{k}\sum_{j=0}^{k-1}\ln(1+R_{t-j})$ is always (very) close to 0, we can use a first-order Taylor series expansion to approximate $\exp(x)$ around 0 by $x+1$ (why?).

Derivation of (1)

Therefore,

$$\exp\left[\frac{1}{k}\sum_{j=0}^{k-1}\ln(1+R_{t-j})\right]-1\approx\frac{1}{k}\sum_{j=0}^{k-1}\ln(1+R_{t-j})$$

Further by Taylor expansion around 0,
$\ln(x+1)\approx\ln(1)+x+\cdots$. Therefore,

$$\frac{1}{k}\sum_{j=0}^{k-1}\ln(1+R_{t-j})\approx\frac{1}{k}\sum_{j=0}^{k-1}R_{t-j}$$

Arithmetic vs. Geometric mean

When we calculate the average return, why should we use the geometric mean, instead of the arithmetic mean?

Example 1:

Period	Return
Period 1	100%
Period 2	-50%

If we calculate the arithmetic mean, it is $(100\% - 50\%)/2 = 25\%$.

Is the average return 25%?

Arithmetic vs. Geometric mean

Example 1:

Period	Return	
Period 1	100%	\$1 investment $\Rightarrow$ \$2
Period 2	-50%	\$2 investment $\Rightarrow$ \$1

It seems that the average return should be 0%.

Arithmetic vs. Geometric mean

Example 1:

Period	Return
Period 1	100%
Period 2	-50%

If we calculate the geometric mean, it is
$\{(1 + 100\%) \times (1 - 50\%)\}^{1/2} - 1 = 0\%$

Arithmetic vs. Geometric mean

Example 2:

Period	Return
Period 1	100%
Period 2	-50%

If we calculate the geometric mean, it is
$\{(1 + 100\%) \times (1 - 50\%)\}^{1/2} - 1 = 0\%$

Arithmetic vs. Geometric mean

Example 2:

Period	Portfolio A	Portfolio B
Period 1	12%	50%
Period 2	-3%	-40%
Period 3	8%	30%
Period 4	15%	70%
Period 5	0%	10%
Period 6	4%	-50%

Returns of portfolio A tend to be lower but more stable;
Returns of portfolio B tend to be more volatile;

Arithmetic Mean

Portfolio A:

$$(12\% - 3\% + 8\% + 15\% + 0\% + 4\%)/6$$
$$= 36\%/6 = 6\%$$

Portfolio B:

$$(50\% - 40\% + 30\% + 70\% + 10\% - 50\%)/6$$
$$= 70\%/6 \approx 11.67\%$$

It appears that portfolio B offers a substantially higher returns than portfolio A.

Arithmetic mean can be misleading!

Geometric mean

Portfolio A:

$$[1.12 \times 0.97 \times 1.08 \times 1.15 \times 1.00 \times 1.04]^{(1/6)} - 1$$
$$= 1.05809 - 1 = 5.81\%$$

Portfolio B:

$$[1.5 \times 0.6 \times 1.3 \times 1.7 \times 1.1 \times 0.5]^{(1/6)} - 1$$
$$= 1.01508 - 1 = 1.51\%$$

Portfolio A provides a higher return.

Arithmetic and Geometric mean

- Geometric mean is more reliable, which is the compounded returns over the entire investment horizon.
- It's harder to recover from the negative returns. If you lose 50%, you can't gain that back with 50% return. Instead, you need to earn 100% to just get back to even.
- Arithmetic mean: $0.5 - 0.5 = 0$; Geometric mean: $((1 - 0.5) \times (1 + 1))^{1/2} - 1 = 0$.
- Arithmetic mean throws out these compound effects that geometric mean captures correctly.

Geometric mean

- Arithmetic average tends to overstate the true return;
- The more volatile the return stream, the more important it is to use geometric averages.

货币基金

- 货币基金是聚集社会闲散资金，由基金管理人运作，基金托管人保管资金的一种开放式基金，专门投向无风险的货币市场工具，区别于其他类型的开放式基金，具有高安全性、高流动性、稳定收益性，具有“准储蓄”的特征。
- 货币基金资产主要投资于短期货币工具（一般期限在一年以内，平均期限120天），如国债、央行票据、商业票据、银行定期存单、政府短期债券、企业债券（信用等级较高）、同业存款等短期有价证券。

七日年化

- 7日年化收益率是货币基金近7天的平均收益率水平
- 2021年9月2日，兴业银行理财产品添利1号七日年化收益率为2.882%，是指8月27日-9月2日这七天的平均年化收益率
- 7日年化收益率=7日总收益率（%)/7$\times$ 365
- 因为基金的收益每天都在变化，7日年化收益率也是每日更新。

1年期人民币存款利率

- 1年期人民币存款利率是本金存银行1年的收益率
- 目前1年期人民币存款利率是1.75%。即如果将1万元存入银行，那么一年后可以拿到175元利息。
- 如果人民币存款利率没有调整，1年期人民币存款利率是固定不变的。

万份收益

- 每万份基金单位收益，即投资1万元当日获利的金额
- 2021年9月2日，兴业银行理财产品添利1号的万份收益为0.7943，是指投资1万元当日获利的金额为0.7943元。

Effects of compounding

- For a bank deposit account, the quoted interest rate often refers to as 'simple interest'.
- For example, an interest rate of 5% payable every six months will be quoted as a **simple interest** of 10% per annum in the market.

Effects of compounding

- Assume that the initial deposit is \$1.00 and the quoted simple interest rate per annum is 10%.
- If the bank pays interest 2 times in a year, then the interest rate for each payment is 5%. The **compounded return** is,

$$\$1 \times \left(1 + \frac{0.1}{2}\right)^2 = 10.25\%$$

- The compound return 10.25% is greater than the quoted annual rate of 10%. This is due to the earning from 'interest-on-interest' in the second six-month period.

Effects of compounding

- More generally, if the bank pays interest m times in a year. For example, the account holder is paid every quarter when m = 4, every month when m = 12, and every day when m = 365. annual rate
- The interest rate for each payment is $\boxed{10\%/m.}$ The gross return at the end of one year (net value of the deposit) becomes,

$$\$1 \times \left(1 + \frac{0.1}{m}\right)^m$$

Table 1: Illustration of Effects of Compounding: Time Interval is 1 Year and Interest Rate Is 10% per annum.

Type	m(payment)	Int.	Net
Annual	1	0.1	\$1.10000
Semi-Annual	2	0.05	\$1.10250
Quarterly	4	0.025	\$1.10381
Monthly	12	0.0083	\$1.10471
Weekly	52	0.1/52	\$1.10506
Daily	365	0.1/365	\$1.10516
Continuously	∞		\$1.10517

The last number is obtained by exp(0.1).

Effects of compounding

- Suppose m continues to increase, and the earnings are paid continuously eventually. Then the gross return at the end of one year is

$$\lim_{m\to\infty}(1+\frac{r}{m})^m = \exp(r),$$

where $e = 2.71828183...$ as $m \to \infty$.

Effects of compounding

More generally, the net asset value A of continuous compounding is

$$A = \lim_{m\to\infty} P\left(1 + \frac{r}{m}\right)^{mn} \approx P\exp(r \times n),$$

where r is the interest rate per annum, P is the initial capital, and n is the number of years.

理论价值

$$\boxed{P = A\exp(-r \times n),}$$

where P is referred to as the present value of an asset that is worth A dollars n years from now.

Continuously compounded return

Simple Return ⇒ Compound Return

- The continuously compounded return or log return is the natural logarithm of the simple gross return of an asset.

$t-(t-1)$

- One period continuously compounded return or log return

$\prod r_{k-1}$

simple

$P_t = P_{t-1} \exp(r \cdot 1)$

$r_t = \ln(\frac{P_t}{P_{t-1}})$

$$r_t = \ln(1 + R_t) = \ln \frac{P_t}{P_{t-1}} = p_t - p_{t-1},$$

where $p_t = \ln P_t$ is called the log price and $1 + R_t$ is the one-period gross return.

k period log return

- Consider k period log return

r_t : arithmetic sum of log return

$$
\begin{aligned}
& r_t(k) \\
= \; & \ln[1 + R_t(k)] \\
= \; & \ln[(1 + R_t)(1 + R_{t-1}) \cdots (1 + R_{t-k+1})] \\
= \; & \ln(1 + R_t) + \ln(1 + R_{t-1}) + \cdots + \ln(1 + R_{t-k+1}) \\
= \; & \boxed{r_t + r_{t-1} + \cdots + r_{t-k+1}}
\end{aligned}
$$

- The continuously compounded multiperiod return is simply the sum of continuously compounded one-period returns involved.

Log returns

- When the values are small, the log returns and the simple returns are approximately the same,

$$\boxed{r_t = \ln(1 + R_t) \approx R_t.}$$

  However, $r_t < R_t$.

- The log return r_t is also called continuously compounded return due to its close link with the concept of compound rates or interest rates.

Example 2

Use the previous daily prices.

Day	1	2	3	4	5
Price	37.84	38.49	37.12	37.60	36.30

- What is the log return from day 1 to day 2?
  $r_2 = \log(38.49) - \log(37.84) = 0.017.$
- What is the log return from day 1 to day 5?
  $r_5(4) = \log(36.3) - \log(37.84) = -0.042.$
- It is easy to verify $r_5(4) = r_2 + \cdots + r_5$.

Portfolio Return

The simple net return of a portfolio consisting of n assets is a weighted average of the simple net returns of the assets involved, where the weight on each asset is the percentage of the portfolio's value invested in that asset.

Let p be a portfolio that places weight w_i on asset i, then the simple return of p at time t is

$$R_{p,t} = \sum_{i=1}^{n} w_i R_{i,t}, \qquad 1 + R_{p,t} = \sum_{i=1}^{n} w_i (1 + R_{i,t}),$$

where $R_{i,t}$ is the simple return of asset i.

Example 3

An investor holds stocks of IBM, Microsoft and Citi-Group. Assume that her capital allocation is 30%, 30% and 40%. The monthly simple returns of these three stocks are 1.42%, 3.37% and 2.20%, respectively. What is the mean simple return of her stock portfolio in percentage?

Answer:

$$E(R_t) = 0.3 \times 1.42 + 0.3 \times 3.37 + 0.4 \times 2.20 = 2.32.$$

Portfolio Return

The continuously compounded returns of a portfolio do not have the above convenient property. If the simple returns R_{it} are all small in magnitude, then we have

$$r_{p,t} \approx \sum_{i=1}^{n} w_i r_{i,t},$$

where $r_{p,t}$ is the continuously compounded return of the portfolio at time t.

Adjusting for dividends (Total Returns)

If an asset pays a dividend, D_t, sometime between months $t-1$ and t, the returns are now defined as follows,

$$R_t = \frac{P_t + D_t}{P_{t-1}} - 1 = \frac{P_t - P_{t-1}}{P_{t-1}} + \frac{D_t}{P_{t-1}},$$

where

- $\frac{P_t - P_{t-1}}{P_{t-1}}$ is referred as the capital gain, and
- $\frac{D_t}{P_{t-1}}$ is referred to as the dividend yield.

Also we have $r_t = \log(P_t + D_t) - \log(P_{t-1})$.

Adjusting for dividends (Total Returns)

If an asset pays a dividend, D_t, sometime between months $t-1$ and t, the returns are now defined as follows,

$$R_t(k) = \frac{P_t + D_t + \ldots + D_{t-k+1}}{P_{t-k}} - 1,$$

$$r_t(k) = r_t + \ldots + r_{t-k+1} = \sum_{j=0}^{k-1} \log\left(\frac{P_{t-j} + D_{t-j}}{P_{t-j-1}}\right),$$

Adjusting for inflation (Real Returns)

- Let CPI_t be the consumer price index at time period t and π_t be the CPI inflation, $\pi_t = (CPI_t - CPI_{t-1})/CPI_{t-1}$.
- If we consider price inflation, the real returns can be computed by

$$1 + R_t^{Real} = \frac{P_t/CPI_t}{P_{t-1}/CPI_{t-1}} = \frac{P_t/P_{t-1}}{CPI_t/CPI_{t-1}} = \frac{1+R_t}{1+\pi_t},$$

- This identity also show the following important approximate relationship

$$1 + R_t = (1 + R_t^{Real})(1 + \pi_t) \Rightarrow R_t \approx R_t^{Real} + \pi_t$$

Adjusting for inflation (Real Returns)

Note that

$$r_t^{Real} = \log(1 + R_t^{Real}) = \log\left(\frac{P_t}{P_{t-1}}\frac{CPI_{t-1}}{CPI_t}\right),$$
$$= r_t - \pi_t^c,$$

where $\pi_t^c = \log(1 + \pi_t)$.

Therefore,

$$r_t = r_t^{Real} + \pi_t^c.$$

Excess Returns

- Excess Returns: the difference between the asset's return and the return on some reference asset.
- $Z_t = R_t - R_t^*$, where R_t^* is the simple return of the reference asset;
- $z_t = r_t - r_t^*$, where r_t^* is the log return of the reference asset;

Excess Returns

The commonly used reference rates are,

- LIBOR rates (London Interbank Offered Rate: the average interest rate that leading banks in London charge when lending to other banks);
- log returns of a riskless asset (e.g., yields of short-term government bonds such as the 3-month US treasury bills);
- log returns of market portfolio (e.g. the S&P 500 index or CRSP value-weighted index;

Simple return and log return

Let R_t be the simple return and r_t be the log return.

$$r_t = \log(1 + R_t),$$
$$R_t = e^{r_t} - 1.$$

If the returns are in percentage, then

$$r_t = 100 \times \log\left(1 + \frac{R_t}{100}\right),$$
$$R_t = [\exp(r_t/100) - 1] \times 100.$$

Simple return and log return

Temporal aggregation of the returns produces

$$1 + R_t(k) = (1 + R_t)(1 + R_{t-1}) \cdots (1 + R_{t-k+1}),$$

$$r_t(k) = r_t + r_{t-1} + \cdots + r_{t-k+1}.$$

These two relations are important in practice, e.g. obtain annual returns from monthly returns.

Example 4

- If the monthly log returns of an asset are 4.46%, -7.34% and 10.77%, then what is the corresponding quarterly log return?
  Answer: (4.46 - 7.34 + 10.77)% = 7.89%.
- If the monthly simple net returns of an asset are 4.46%, -7.34% and 10.77%, then what is the corresponding quarterly simple net return?
  Answer:
  $R = (1 + 0.0446)(1 - 0.0734)(1 + 0.1077) - 1 = 1.0721 - 1 = 0.0721 = 7.21\%.$

Zero-coupon bond

- Bonds are quoted in annualized yields.
- A so-called zero-coupon bond is a bond bought at a price lower than its face value (also called par value or principal), with the face value repaid at the time of maturity.
- It does not make periodic interest payments (i.e. coupons), hence the term ‘zero-coupon’.

Bond yield

- Bond yield is the return an investor will receive by holding a bond to maturity.
- The common types of bond yield are the current yield and yield to maturity (YTM).

Current yield

- The current yield for the coupon bonds:

$$\text{Current yield} = \frac{\text{Annual interest payments}}{\text{Market price of the bond}} \times 100\%$$

  Example: If an investor paid \$90 for a bond with face value of \$100, also known as par value, and the bond paid a coupon rate of 5% per annum, then the current yield of the bond is,

$$(0.05 \times 100)/90 \times 100\% = 5.56\%$$

- Current yield does not include any capital gains or losses of the investment.

Current yield

- The current yield for the zero-coupon bonds?

$$\text{Current yield} = \left(\frac{\text{Face value}}{\text{Market price of the bond}} \right)^{1/k} - 1,$$

where k denotes time to maturity in years.
Example: If an investor purchased a zero-coupon bond with face value \$100 for \$90 and the bond will mature in 2 years, then the yield is,

$$(100/90)^{1/2} - 1 = 5.41\%$$

The yield to maturity (YTM)

The yield to maturity (YTM) is defined as the constant interest rate (discount rate) that makes the present value of a bond's cash flows equal to its price. YTM is sometimes referred to as the Internal Rate of Return (IRR).

The yield to maturity (YTM)

Suppose that the bond holder will receive k payments between purchase and maturity. The yield to maturity y is calculated by,

$$P = \frac{C_1}{1+y} + \frac{C_2}{(1+y)^2} + \cdots + \frac{C_k + F}{(1+y)^k},$$

where P is the price of the bond, F is the face value and C_i is the ith cash flow of coupon payment.

$y = \text{Annual rate} \times \frac{\text{pay pr.}}{1 \text{ yr}}$

Example

Consider an 8% coupon, 30-year maturity bond with par value of $1,000 paying 60 semiannual coupon payments of $40 each. The coupon bond is currently selling at $1,276.76. What is the yield to maturity?

$$\$1,276.76 = \sum_{t=1}^{60} \frac{40}{(1+r)^t} + \frac{1000}{(1+r)^{60}}$$

We have $\boxed{r = 3\%}$ per half year.

Example 3

Consider an 8% coupon, 30-year maturity bond with par value of $\$1,000$ paying 60 semiannual coupon payments of $40 each. The coupon bond is currently selling at $\$1,276.76$. What is the current yield?

It would be $80/1,276.76 = .0627$, or 6.27% per year.

Bond yields and prices

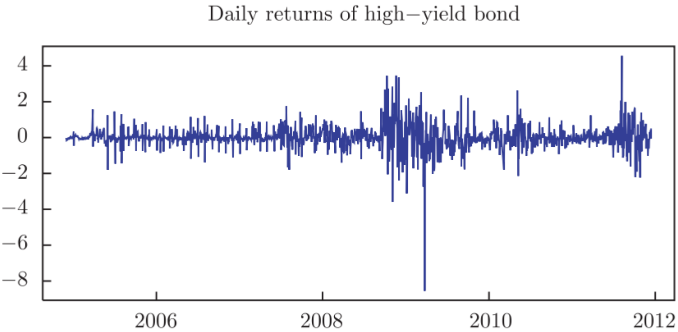


Figure 1: Time series of the daily returns of high-yield bonds in November 29, 2004 - December 10, 2014.

Bond yields and prices

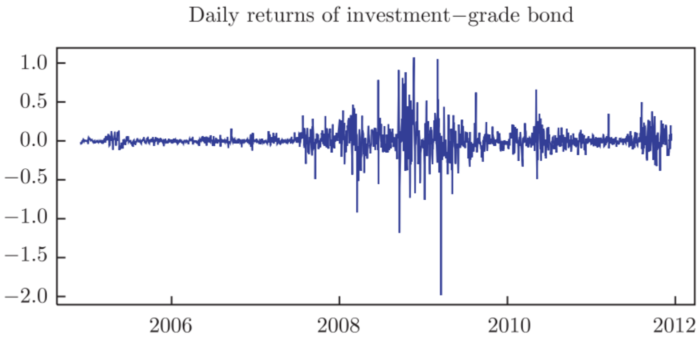


Figure 2: Time series of the daily returns of investment-grade bonds in November 29, 2004 - December 10, 2014.

Yield spread

The Yield spread is an excess yield defined as the difference between the yield of a bond and the yield of a reference bond such as a US treasury bill with a similar maturity.

Bond yields and prices

Two baskets of high-yield bonds and investment-grade bonds (i.e. the bonds with relatively low risk of default) with an average duration of 4.4 years each.

value ↑
⇓
risk ↑

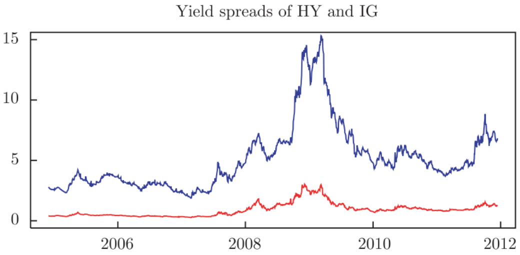


Figure 3: Time series of the yield spreads (the difference) of high-yield bonds(blue curve) and investment-grade bonds(red curve) over the Treasury bond in November 29, 2004 - December 10, 2014.

Bond yields and prices

- The high-yield bonds have higher yields than the investment grade bonds, but have higher volatility too (about 3 times).
- The yield spreads widened significantly in a period after the financial crisis following Lehman Brothers filing bankrupt protection on September 15, 2008, reflecting higher default risks in corporate bonds.

Outline

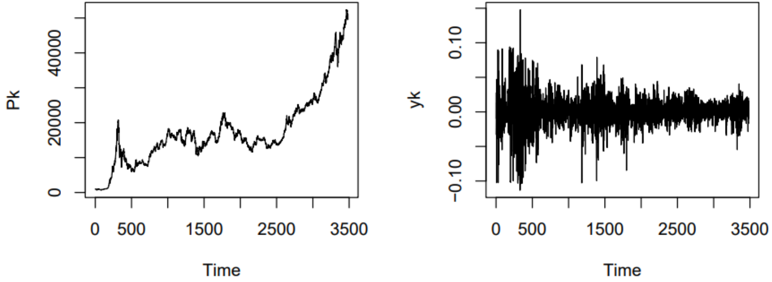
Stylized facts in financial log-return series

- Stationarity: mean-reverting behavior
- Volatility clustering
- Asymmetry
- Leverage effect
- Fat tail

Stylized facts in financial log-return series

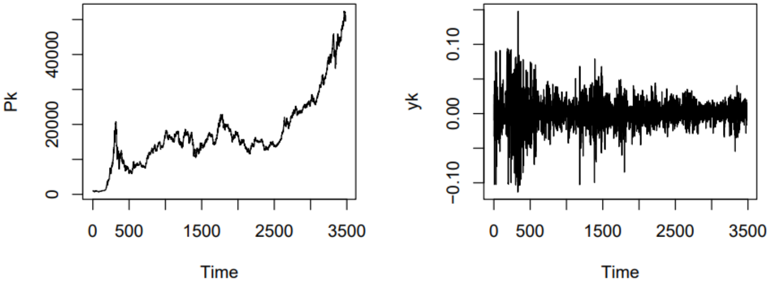
Example 1. Daily closing values (left, P_k) and log returns (right, $y_k = \log(P_k/P_{k-1})$) of the WIG index (main summary index of Warsaw Stock Exchange) 1991-2007.

Stationarity



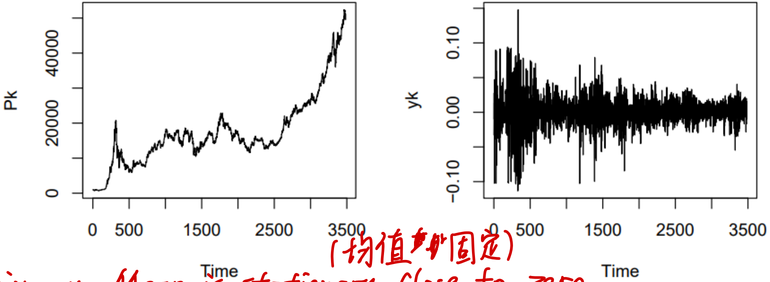
The log returns show clear mean-reverting behavior: the returns fluctuate around a constant level, which seems to be very close to zero;

Volatility clustering



The log returns show some large spikes (jumps) that represent unusually large (in absolute value) daily movements (e.g. 15%);

Volatility clustering



(均值都固定)

Stationary: Mean is stationary, close to zero

Volatility Clustering: high → high, low → low (大起大落)

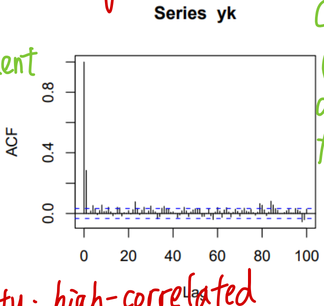
The log returns show volatility clustering: periods of high (low) volatility are followed by periods of high (low) volatility;

Volatility clustering

ACF: Corr(X,y)

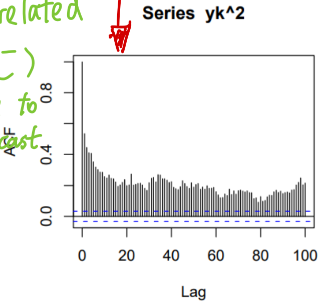
$Corr(r_t, r_{t-1})$

Independent



Correlated

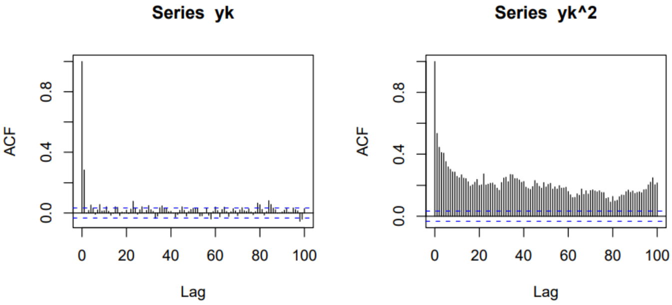
able to forecast



Volatility: high-correlated

Left plot: the series y_k is uncorrelated, here with the exception of lag 1 (typically, log-return series are uncorrelated with the exception of the first few lags);

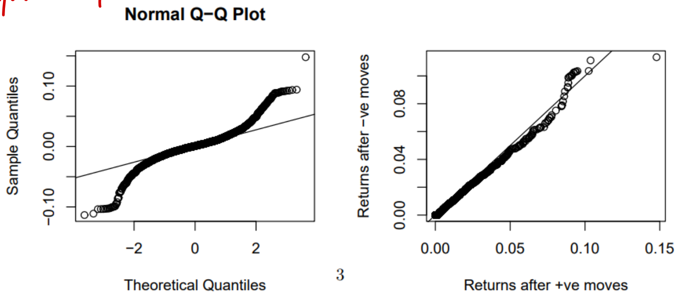
Volatility clustering



Right plot: the squared series y_k^2 is strongly auto-correlated even for very large lags. In this example it is not obvious that the auto-correlation of y_k^2 decays to zero at all.

Asymmetry

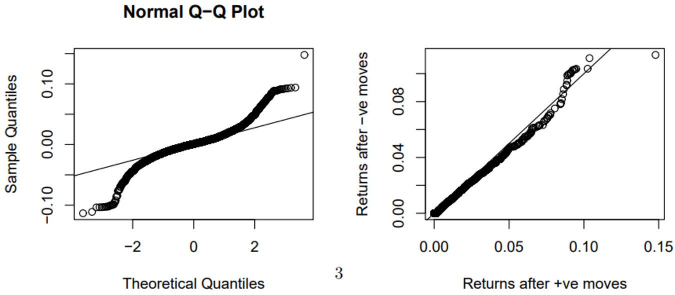
quantile - quantile point



3

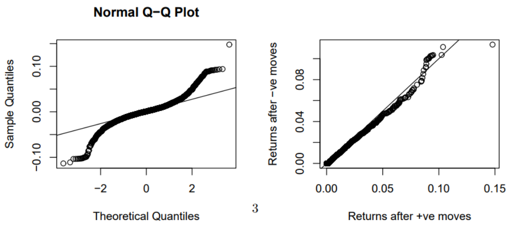
A Q-Q plot is a scatterplot created by plotting two sets of quantiles against one another. If both sets of quantiles came from the same distribution, we should see the points forming a line that's roughly straight.

Asymmetry



For more details of Q-Q plot, refer to `http://data.library.virginia.edu/understanding-q-q-plots` .

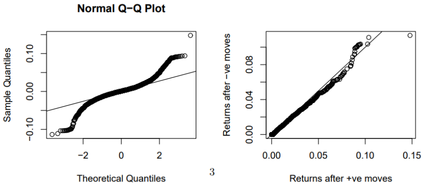
Asymmetry



Left plot: Q-Q plot of the marginal distribution of y_k against the standard normal;

- The distribution of return is often negatively skewed, reflecting the fact that the downturns of financial markets are often much steeper than the recoveries.

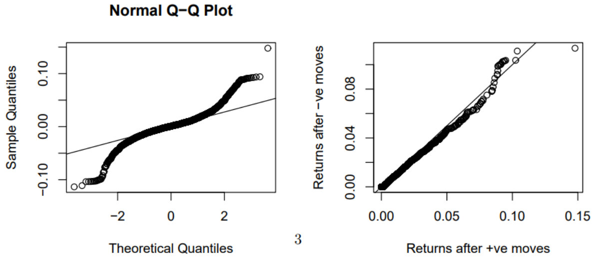
Leverage effect



Right plot: sample quantiles of the two conditional distributions plotted against each other;

- The series y_k responds differently to its own positive and negative movements, or in other words the conditional distribution of $|y_k|\big|\{y_{k-1} > 0\}$ is different from that of $|y_k|\big|\{y_{k-1} < 0\}$;

Leverage effect



Right plot: sample quantiles of the two conditional distributions plotted against each other;

- "leverage effect ": market responds differently to "good " and "bad " news;

Fat tails

Definition 1

A random variable is said to have fat tails if it exhibits more extreme outcomes than a normally distributed random variable with the same mean and variance.

The mean-variance model assumes normality.

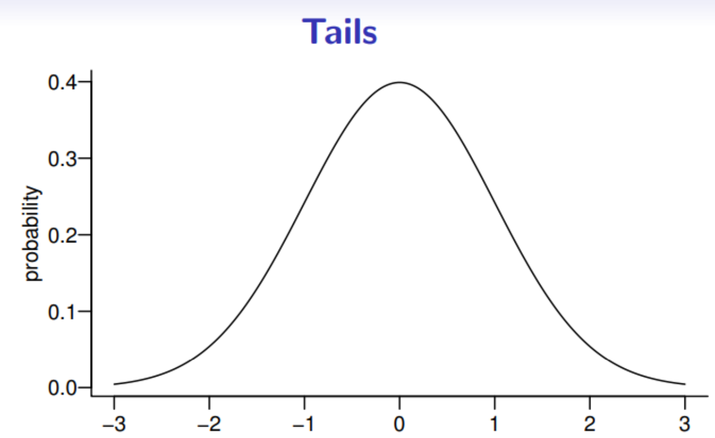
Fat tails

- The tails are the extreme left and right parts of a distribution;
- If the tails are fat, there is a higher probability of extreme outcomes than one would get from the normal distribution with the same mean and variance;
- Also implies that there is a lower probability of non-extreme outcomes;
- Probabilities are between zero and one so the area under the distribution is one;

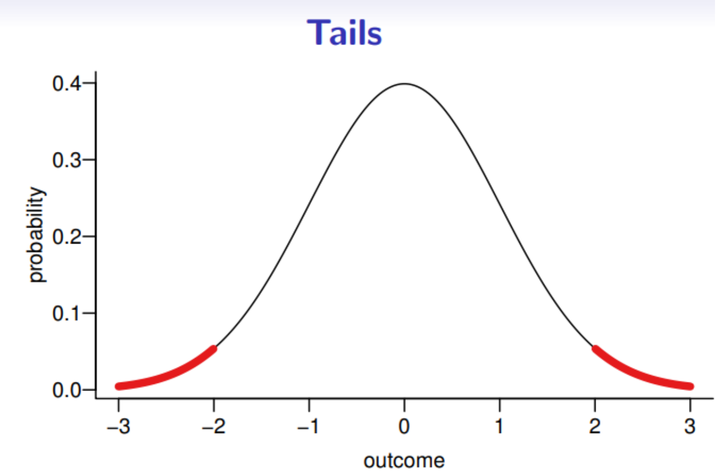
The Student-t distribution

- The degrees of freedom, (ν), of the Student-t; distribution indicate how fat the tails are;
- $\nu = \infty$ implies the normal;
- $\nu < 2$ implies superfat tails;
- For a typical stock, $3 < \nu < 5$;
- The Student-t is convenient when we need a fat tailed distribution;

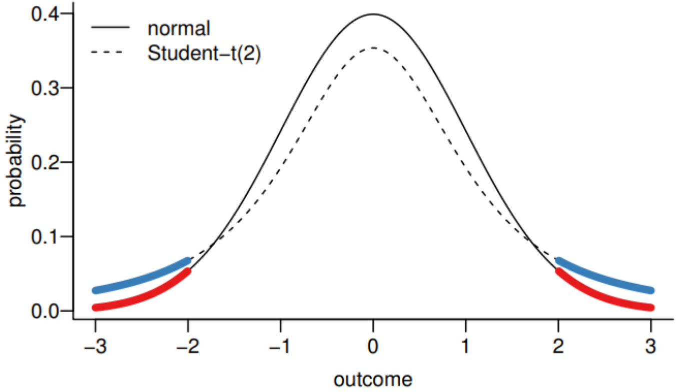
Normal distribution



Normal distribution



Tails



normal
Student–t(2)
probability
0.0
0.1
0.2
0.3
0.4
–3
–2
–1
0
1
2
3
outcome

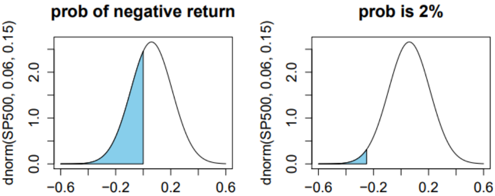
Example 1:

- Assume the annual returns on the S&P500 are normally distributed with mean 6% and standard deviation 15%. $SP500 \sim N(0.06, 0.15^2)$.

What is the chance of losing money on a given year?

$$\Pr(SP500 < 0) = 0.34$$

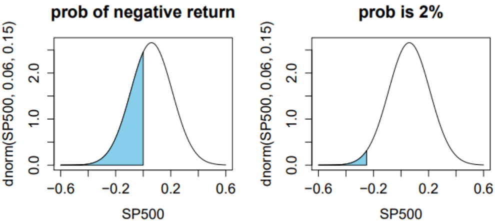
In R: pnorm(0,0.06,0.15)



- Assume the annual returns on the S&P500 are normally distributed with mean 6% and standard deviation 15%. $SP500 \sim N(0.06, 0.15^2)$. What is the value that there's only a 2% chance of losing that or more?

$$\Pr(SP500 < -0.25) = 0.02$$

In R: qnorm(0.02,0.06,0.15)



Probability of extreme outcomes

暴跌

- On October 19, 1987, a date that subsequently became known as “Black Monday”, the Dow Jones Industrial Average plummeted 508 points, losing 22.6% of its total value. The S&P 500 dropped 20.4%, falling from 282.7 to 225.06. This was the greatest loss Wall Street had ever suffered on a single day.

Example 2:

- Prior to the 1987 crash, monthly S&P500 returns (r_t) followed (approximately) a normal with mean μ and standard deviation equal to σ. How extreme was the crash of x under the normal assumption?

$$r_t \sim N(\mu, \sigma^2)$$

In R: $pnorm(x, \mu, \sigma)$.

Probability of extreme outcomes

- Some return outcomes and probabilities of daily S&P 500 returns assuming normality, 1929-2009 (*Financial Risk Forecasting*, by Jon Danielsson):

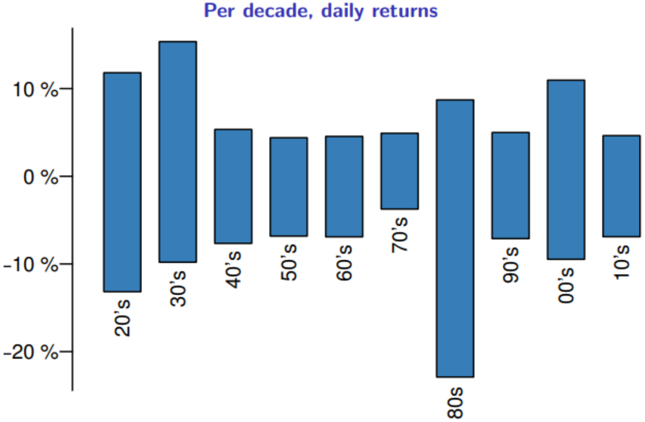
Returns above or below	Probability
1%	0.865
2%	0.035
3%	0.00393
5%	2.74×10^{-6}
15%	2.7×10^{-43}
23%	2.23×10^{-97}

Probability of extreme outcomes

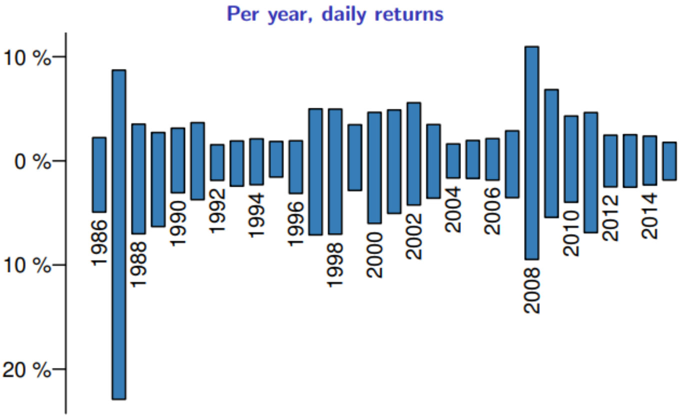
Fat Tail: Underestimate risks

- If S&P 500 returns were normally distributed, the probability of a one-day crash of 23% would be 2.23×10^{-97}! In other words, the crash is supposed to happen once every 10^{95} years (accounting for weekends and holidays.)
- Scientists generally assume that the earth is about 10^{7} years old and the universe 10^{13} years old.

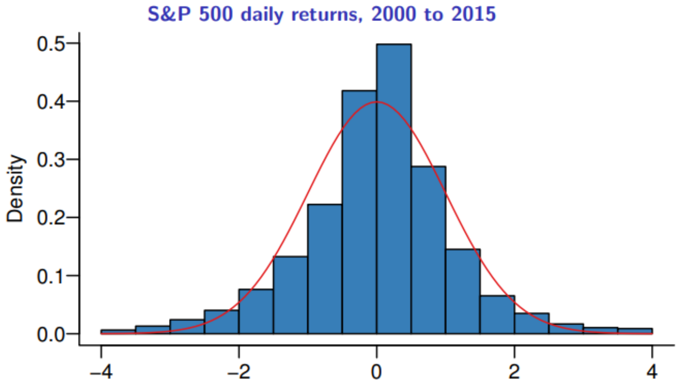
Max and min of S&P 500 returns



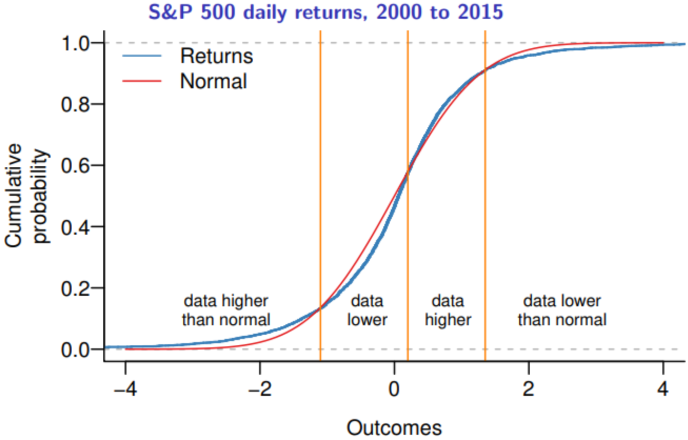
Max and min of S&P 500 returns



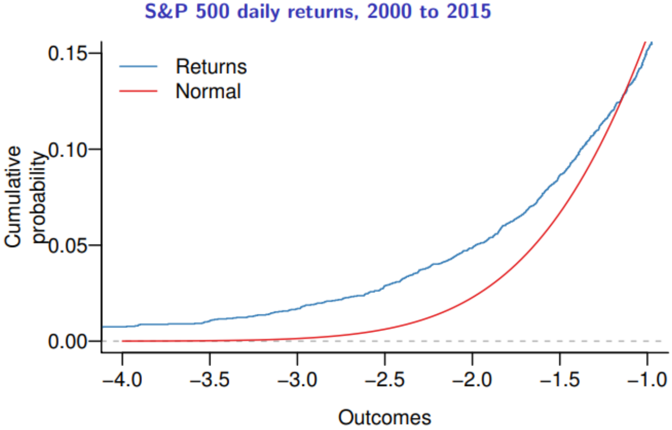
Empirical density vs. normal



Empirical density vs. normal



Empirical density vs. normal

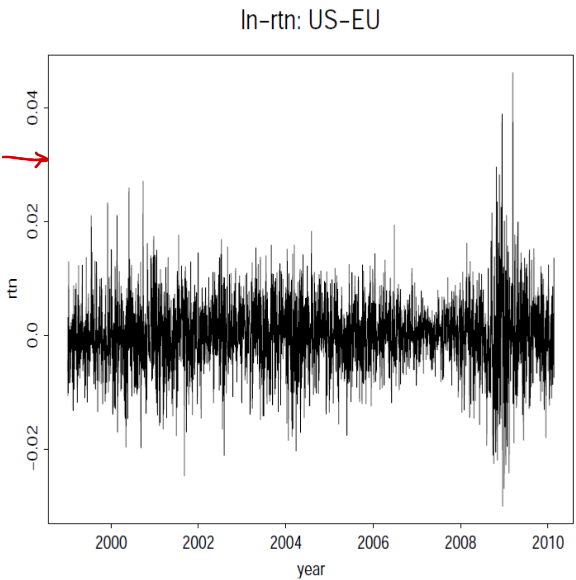


Daily log returns of FX (Dollar vs Euro)

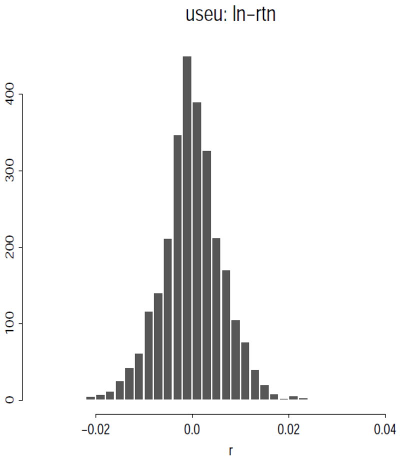
Bond

Stock

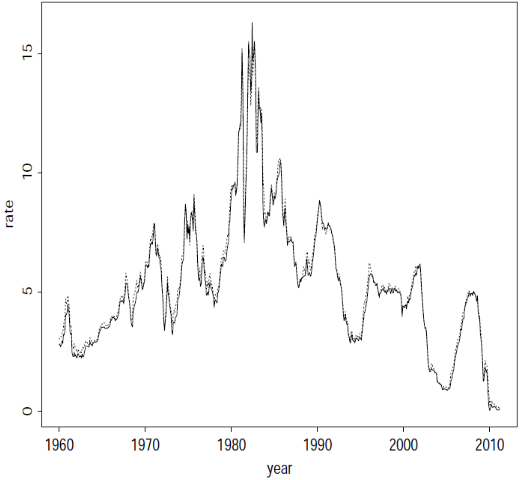
Exchange



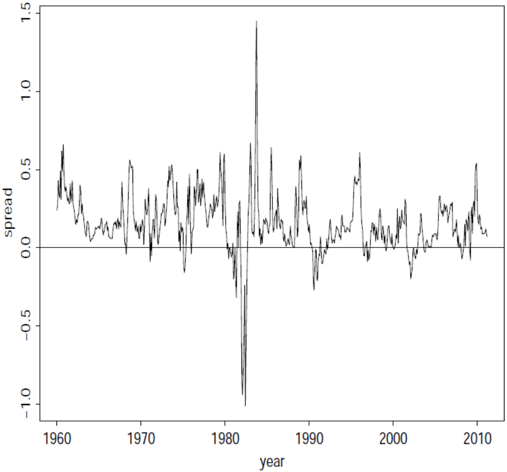
Histogram of daily log returns of FX (Dollar vs Euro)



Monthly US interest rates: 3m & 6m TB



Spread of monthly US interest rates: 3m & 6m TB



Outline

Moments of Distributions

Key: What is the distribution of

$$(r_{it}; i = 1, \cdots, N;\ t = 1, \cdots, T)?$$

Some theoretical properties:

Moments of a random variable X with density $f(x)$:
$l-$th moment

Moment information

lth moment EX $m'_l = E(X^l) = \int_{-\infty}^{\infty} x^l f(x) dx.$

lth central moment = E(X-μx)

The first moment is the mean or expectation of X, μ_x.

Moments of Distributions

2th central moment = $E(X-\mu X)^2$ 方差

3th central moment = skewness

- $l-$th central moment

4th moment

$$m_l = E(X-\mu_x)^l = \int_{-\infty}^{\infty} (x-\mu_x)^l f(x) dx,$$

- The second central moment is the variance $\sigma_x^2 = E(X-\mu_x)^2$, where σ_x is the standard deviation.
- The variance measures how much the random variable jumps around from the mean.

Skewness

- The third central moment is the skewness of the random variable, a measure of the extent of symmetry.

$$S(x) = E\left[\frac{(X-\mu_x)^3}{\sigma_x^3}\right]$$

- Skewness measures the degree of asymmetry of a distribution around its mean.
    - Positive skewness indicates a distribution with an asymmetric tail extending toward more positive values.
    - Negative skewness indicates a distribution with an asymmetric tail extending toward more negative values.

Higher moments

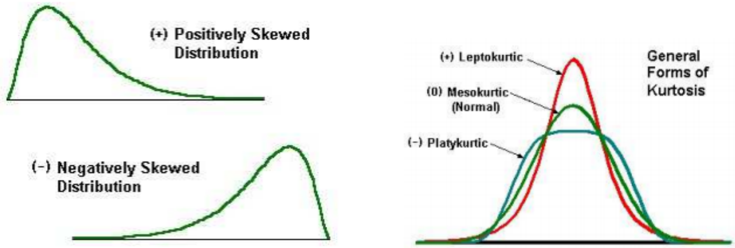


Figure 4: Skewness and kurtosis of the distribution

不对称性

峭度, 陡度

Kurtosis

- The fourth central moment is the kurtosis, a measure of how much mass in the tails of the distribution.

$$K(x) = E\left[\frac{(X-\mu_x)^4}{\sigma_x^4}\right].$$

- The quantity $K(x) - 3$ is called the excess kurtosis because $K(x) = 3$ for a normal distribution. Thus, the excess kurtosis of a normal random variable is zero.

leptokurtic

- A distribution with positive excess kurtosis is called leptokurtic.
- A leptokurtic distribution tends to have a distinct peak near the mean, declines rather rapidly, and has fatter tails, implying that the distribution puts more mass on the tails of its support than a normal distribution does.
- In practice, this means that a random sample from such a distribution tends to contain more extreme values. For example, the Student's t-distribution.

platykurtic

- A distribution with negative excess kurtosis is called platykurtic.
- A platykurtic distribution tends to have a flat top near the mean rather than a sharp peak and has thinner tails, for example, the continuous or discrete uniform distributions.

Higher moments

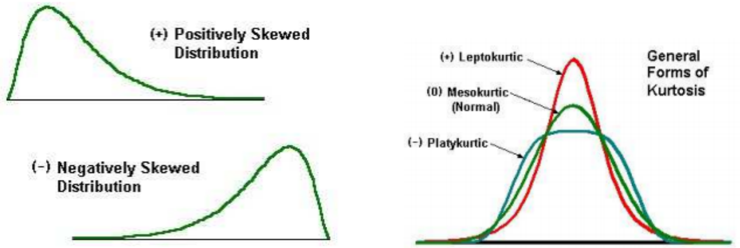


Figure 5: Skewness and kurtosis of the distribution

Estimation of Mean and Variance

Data: $\{x_1, \cdots, x_T\}$.

- sample mean:

$$\hat{\mu}_x = \frac{1}{T}\sum_{t=1}^{T} x_t,$$

- sample variance:

$$\hat{\sigma}_x^2 = \frac{1}{T-1}\sum_{t=1}^{T}(x_t - \hat{\mu}_x)^2,$$

Estimation of skewness and kurtosis

- sample skewness:

$$\hat{S}(x) = \frac{1}{(T-1)\hat{\sigma}_x^3} \sum_{t=1}^{T} (x_t - \hat{\mu}_x)^3,$$

- sample kurtosis:

$$\hat{K}(x) = \frac{1}{(T-1)\hat{\sigma}_x^4} \sum_{t=1}^{T} (x_t - \hat{\mu}_x)^4.$$

- Under normality assumption,

一般采用95%置信区间

$$\hat{S}(x) \sim N(0, \frac{6}{T}), \quad \hat{K}(x) - 3 \sim N(0, \frac{24}{T}).$$

Hypothesis test

①控制置信区间(size)

②检验

- Type I error: reject a true null hypothesis
  - The size of a test is the probability of falsely rejecting the null hypothesis. That is, it is the probability of making a Type I error.
- Type II error: fail to reject a false null hypothesis
  - The power of a test $=$ Pr(reject $H_0|H_1$ is true). As the power increases, there is a decreasing probability of a type II error.

Significance Tests for Unknown Mean

显著性检验：z检验(σ√, μX)

1. Given an asset return series $\{r_1, ..., r_T\}$, to test the population mean μ_r is equal to a specified value μ_0.

If the population standard deviation σ_r is known, the test statistic is defined as

$$z = \frac{\hat{\mu}_r - \mu_0}{\sigma_r/\sqrt{T}} \sim N(0, 1),$$

where $\hat{\mu}_r$ is the sample mean and T is the sample size.

Decision rule: Reject H_0 of a symmetric distribution if $|z| > Z_{1-\alpha/2}$ or p-value is less than α, where $Z_{1-\alpha/2}$ is the $(1-\alpha/2)$th quantile of the standard normal distribution.

One-sample t-test for Unknown Mean

2. Given an asset return series $\{r_1, ..., r_T\}$, to test the population mean μ_r is equal to a specified value μ_0.

If the population standard deviation σ_r is unknown, the test statistic is given by,

$$t = \frac{\hat{\mu}_r - \mu_0}{\hat{\sigma}_r / \sqrt{T}} \sim t(T-1)$$

where $\hat{\sigma}_r$ is the sample standard deviation.

Hypothesis tests

3. Given an asset return series $\{r_1, ..., r_T\}$, to test the skewness of the returns, we consider the null hypothesis

$$H_0 : S(r) = 0 \text{ versus } H_a : S(r) \neq 0.$$

The t-ratio statistic of the sample skewness is,

$$t = \frac{\hat{S}(r)}{\sqrt{6/T}} \sim N(0, 1)$$

if normality holds.

Decision rule: Reject H_0 of normal tails if $|t| > Z_{1-\alpha/2}$ or p-value is less than α

Hypothesis tests

3. To test the excess kurtosis of the return series, we consider the hypotheses

$$H_0 : K(r) - 3 = 0 \text{ versus } H_a : K(r) - 3 \neq 0.$$

The test statistic is given by,

$$t = \frac{\hat{K}(r) - 3}{\sqrt{24/T}} \sim N(0, 1)$$

if normality holds.

Decision rule: Reject H_0 of normal tails if $|t| > Z_{1-\alpha/2}$ or p-value is less than α

Hypothesis tests

4. A joint test (Jarque-Bera test):

$$JB = \frac{\hat{S}^2(r)}{6/T} + \frac{[\hat{K}(r) - 3]^2}{24/T} \sim \chi^2_2,$$

if normality holds, where χ^2_2 denotes a chi-squared distribution with 2 degrees of freedom.

Decision rule: Reject H_0 of normality if $JB > \chi^2_2(\alpha)$ or p-value is less than α.

Stock Returns (simple returns) are not Normal

Stock returns (R_t) are not completely modeled by normal distributions because

- a normally distributed random variable can take any value between $-\infty$ and ∞, the model implies the possibility of unlimited losses, but liability is usually limited; $R_t \geq -1$ since you can lose no more than your investment;

Stock Returns (simple returns) are not Normal

Stock returns (R_t) are not completely modeled by normal distributions because

- Multi-period returns are not normal because $1 + R_t(k) = (1 + R_t)(1 + R_{t-1}) \cdots (1 + R_{t-k+1})$ is not normal — sums of normals are normal but not so with products.
- Empirical data suggests that returns show leptokurtosis, fatter tails than expected with a normal distribution.

Normal and lognormal distributions

- A better assumption is that the log returns r_t are normally distributed with mean μ and variance σ^2.
- Recall that the log return is $r_t = \log(1 + R_t)$. Thus, we assume that $\log(1 + R_t)$ is $N(\mu, \sigma^2)$ so that $1 + R_t$ is an exponential and therefore positive and thus $R_t \geq -1$. This solves the first problem.
- Recall that $\log(1 + R_t(k)) = r_t + \cdots + r_{t-k+1}$. Since sum of a finite number of independent normal random variables is normal, normality of single-period log returns implies normality of multiple-period log returns.

Normal and lognormal distributions

Definition 2

Y is lognormal if $X = \log(Y)$ is normal.

If the log returns r_t of an asset are $i.i.d.$ as normal with mean μ and variance σ^2, the simple return R_t are then $i.i.d.$ lognormal random variables with mean and variance given by,

$$\mathrm{E}(R_t) = \exp(\mu + \frac{\sigma^2}{2}) - 1,$$
$$\mathrm{Var}(R_t) = \exp(2\mu + \sigma^2)[\exp(\sigma^2) - 1].$$

Normal and lognormal distributions

Proposition 1

If $X \sim N(\mu, \sigma^2)$*, then* $Y = \exp(X)$ *is lognormal with mean and variance*

$$E(Y) = \exp(\mu + \frac{\sigma^2}{2}),$$

$$Var(Y) = \exp(2\mu + \sigma^2)[\exp(\sigma^2) - 1].$$

Proof of Proposition 1

Let $Z \sim N(0,1)$. $Y = \exp(\sigma Z + \mu)$ is lognormal(μ, σ^2).

$$
\begin{aligned}
F_Y(y) &= P(Y \leq y) = P(\log Y \leq \log y) \\
&= P\left(Z \leq \frac{\log y - \mu}{\sigma}\right) = \Phi\left(\frac{\log y - \mu}{\sigma}\right), y > 0
\end{aligned}
$$

$$
f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{\phi\left(\frac{\log y - \mu}{\sigma}\right)}{\sigma y}, y > 0
$$

Proof of Proposition 1

These permit us to work out a formula for the moments of Y. First of all, for any positive integer k,

$$E(Y^k) = \int_0^\infty y^k f_Y(y)dy = \int_0^\infty \frac{y^k \phi\left(\frac{\log y - \mu}{\sigma}\right)}{\sigma y} dy$$

Proof of Proposition 1

hence after making the substitution $y = \exp(\sigma z + \mu)$, so that $dy = \sigma \exp(\sigma z + \mu) dz$, we find

$$
\begin{aligned}
\underline{\mathrm{E}(Y^k)} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{z^2}{2} + k\sigma z + k\mu\right) dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(z - k\sigma)^2}{2} + \frac{k^2\sigma^2}{2} + k\mu\right) dz \\
&\underline{= e^{\frac{k^2\sigma^2}{2} + k\mu},}
\end{aligned}
$$

where we use the fact that $\int_{-\infty}^{\infty} \phi(z - a) dz = 1$.

Proof of Proposition 1

Since

$$E(Y^k) = \exp(\frac{k^2\sigma^2}{2} + k\mu),$$

In particular, we have

$$E(Y) = \exp(\frac{\sigma^2}{2} + \mu)$$
$$E(Y^2) = \exp(2\sigma^2 + 2\mu)$$
$$\mathrm{Var}(Y) = E(Y^2) - (E(Y))^2 = \exp(\sigma^2 + 2\mu)(\exp(\sigma^2) - 1)$$

Normal and lognormal distributions

If the simple return R_t of an asset is lognormally distributed with the mean μ_R and variance σ_R^2, the mean and variance of the corresponding log return r_t are

$$E(r_t) = \log \frac{\mu_R + 1}{\sqrt{1 + \frac{\sigma_R^2}{(1+\mu_R)^2}}}, \quad \mathrm{Var}(r_t) = \log \left[1 + \frac{\sigma_R^2}{(1+\mu_R)^2} \right].$$

这是非线性关系
不能用线性关系衡量

Z曲线理论：
一般：货币贬值有助于出口.
实际：货币贬值影响抵消或超过出口量↑影响

Normal and lognormal distributions

Proposition 2

If Y is lognormal with mean μ_y and variance σ_y^2, then $X = \log(Y)$ is normal with mean and variance

$$E(X) = \log \frac{\mu_y}{\sqrt{1 + \frac{\sigma_y^2}{\mu_y^2}}}, \quad Var(X) = \log \left[1 + \frac{\sigma_y^2}{\mu_y^2}\right].$$

Pearson correlation coefficient

Consider two variables X and Y.

- Correlation coefficient:

$$\rho = \frac{Cov(X, Y)}{\sigma_X \sigma_Y},$$

  where σ_x is the standard deviation of X.
- It measures the strength of linear dependence between X and Y, and lies between -1 and 1.
- If $Y = a + bX$ then $\rho = \pm 1$.
- If X and Y are independent, then $\rho = 0$.

Pearson correlation coefficient

Consider two variables X and Y.

- Correlation coefficient:

$$\rho = \frac{Cov(X,Y)}{\sigma_X \sigma_Y},$$

  where σ_x is the standard deviation of X.
- Correlation does not imply causation.Some pairs of variables are destined to have high correlation by chance.

Correlation does not imply causation

Pirates cause global warming?

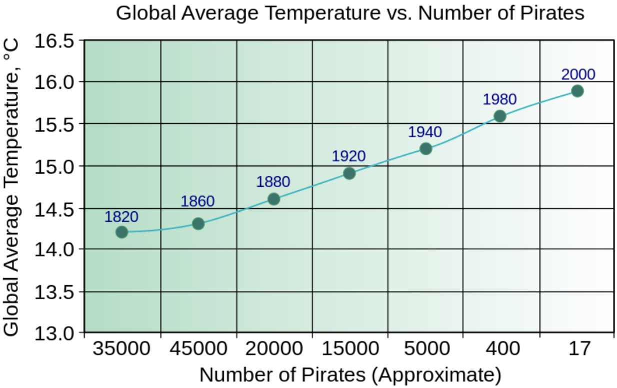


Figure 6: pirates cause global warming?

Pearson correlation coefficient

Consider two variables X and Y.

- Correlation coefficient:

$$\rho = \frac{Cov(X, Y)}{\sigma_X \sigma_Y},$$

  where σ_x is the standard deviation of X.
- Linear correlation is the appropriate measure of dependence if asset returns follow a multivariate normal (or elliptical) distribution.

Drawbacks of Pearson correlation coefficient

- ρ requires both $Var(X)$ and $Var(Y)$ exist.
- $\rho = 0$ does not imply independence. Only if X and Y are bivariate normal does $\rho = 0$ imply independence.
- $\rho = 0$ is not invariant under nonlinear strictly increasing transformation.
- ρ focuses on linear dependence and is not robust to outliers.
- The actual range of ρ can be much smaller than $[-1, 1]$.

Concordance measure

- Concordance measures have the useful property of being invariant to increasing transformations of X and Y.
- Since the linear correlation ρ is not invariant to increasing transformations of X and Y, it does not measure concordance.
- Two common measures of concordance are Kendall's tau statistic and Spearman's rho statistic.

Concordance

Let (X_i, Y_i) and (X_j, Y_j) denote two observations from a vector (X, Y) of continuous random variables.

- Loosely, two random variables are concordant if large values of one random variable are associated with large values of the other random variable.
- More formally, (X_i, Y_i) and (X_j, Y_j) are concordant if $(X_i - X_j)(Y_i - Y_j) > 0$.

Correlation ≠ causations

↳ For large value, X↑ Y↑

Disconcordance

Let (X_i, Y_i) and (X_j, Y_j) denote two observations from a vector (X, Y) of continuous random variables.

- Two random variables are disconcordant if large values of one random variable are associated with small values of the other random variable.
- More formally, (X_i, Y_i) and (X_j, Y_j) are discordant if $(X_i - X_j)(Y_i - Y_j) < 0$.

Kendall's tau

Kendall's tau: Let $\{(X_1, Y_1), \dots, (X_n, Y_n)\}$ denote a random sample of n observations from a vector (X, Y) of continuous random variables. There are $\binom{n}{2}$ distinct pairs (X_i, Y_i) and (X_j, Y_j) of observations in the sample, and each pair is either concordant or discordant.

Let c denote the number of concordant pairs and d the number of discordant pairs. Then Kendall's tau for the sample is defined as

$$\boxed{\rho_\tau} = \frac{c-d}{c+d} = \boxed{\frac{c-d}{\binom{n}{2}}}$$

Spearman's rho

- Let $F_X(x)$ and $F_Y(y)$ be the cumulative distribution function of X and Y .

$$\rho_s = \rho(F_X(X), F_Y(Y)).$$

That is, the correlation coefficient of probability-transformed variables. It is just the correlation coefficient of the ranks of the data.