Financial economics

Lecture 08. Portfolio Choice

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Schedule

- The process of personal portfolio selection
- The trade-off between expected return and risk
  - One riskless asset + one risky asset
  - One riskless asset + two risky assets
- Efficient diversification with many risky assets

Introduction

- How should you invest your wealth optimally?
  - Portfolio selection
- Your wealth portfolio contains
  - Stock, bonds, shares of unincorporated businesses, houses, pension benefits, insurance policies, and all liabilities

Portfolio selection

- There are general principles to guide you, but the implementation will depend such factors as your
  - Age, existing wealth, existing and target level of education, health, future earnings potential, consumption preferences, risk preferences, life goals, your children's educational needs, obligations to older family members, and a host of other factors

Portfolio selection

- The study of how people should invest their wealth
- Process of trading off risk & expected return to find the best portfolio of assets & liabilities
  - Narrower: consider only securities
    - Which industry? Which firm? How much?
  - Wider: house purchase, insurance, debt
  - Broad: human capital, education
- Target: to manage your wealth portfolio efficiently
- Although there are some general rules for portfolio selection that apply to virtually everyone, there is no single portfolio or portfolio strategy that is best for everyone.
- The best strategy depends on an individual's personal circumstances (age, family status, occupation, income, wealth, etc)

The life cycle

- The risk exposure you should accept depends upon your age
- Case 1: The real asset
  - A young couple starting a family should buy a house and take out a mortgage loan
  - An older couple about to retire may sell their house and invest the proceeds in some asset that will provide a steady stream of income for as long as they live
- Case 2: Optimal insurance policy

Same in all other respects	**Life insurance**	**Life annuity**
Parent with dependent children	Need when children young	Depends
Single without dependents	No need	Need

Time horizons

- Planning horizon
  - The total length of time for which one plans
  - For retirement and living: A 25-year-old who expects to live to age 85, the planning horizon would be 60 years
  - For the education of a three-year-old child: college at age 18, the planning horizon is 15 years
- Decision horizon (under the control of the individual)
  - The length of time between decisions to revise a portfolio/How often one reviews one's portfolio
  - Some people at regular intervals, e.g., once a month (when they pay their bills) or once a year (when they file income tax forms)
  - A sudden event may trigger one's reviewing their portfolio
- Trading horizon (not under the control of the individual)
  - The shortest possible time interval over which investors revise their portfolios
  - Determined by the structure of the markets in the economy (when the exchanges are open or organized off-exchange markets exist)

Risk tolerance

- Your tolerance for bearing risk is a major determinant of portfolio choices
  - Influenced by one's capacity to bear risk---maintain their standard of living in the face of adverse movements in the market value of their investment portfolio
    - one's age, family status, job status, wealth, and other attributes that affect
  - Influenced by one's attitude toward risk (risk averse vs. risk loving)
- In the analysis of portfolio selection, do not distinguish one's capacity to bear risk or one's attitude toward risk

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Trade-off between expected return and risk

- The target is to find the portfolio that offers the investors
  - The highest expected rate of return for any degree of risk they are willing to tolerate.
- Two steps for portfolio choice
  1. Find the optimal combination of risky assets
  2. Mix this optimal risky-asset portfolio with the riskless asset
- We start with the second step

Trade-off between expected return and risk

- Assume a world with a single risky asset and a single riskless asset
  - The risky asset is, in the real world, a portfolio of risky assets
  - The risk-free asset is a default-free bond
    - with the same maturity as the investor's decision horizon (or trading horizon if no specific investor)
    - Offers a perfectly predictable rate of return in terms of the unit of account (e.g., currency) selected for the analysis

Combining the riskless asset and a single risky asset

- Suppose you have $100,000 to invest and choose between
  - Riskless asset: the interest rate of 0.06 per year
  - Risky asset: expected rate of return of 0.14 per year and a standard deviation of 0.2.
- How much of your $100,000 should you invest in the risky asset?

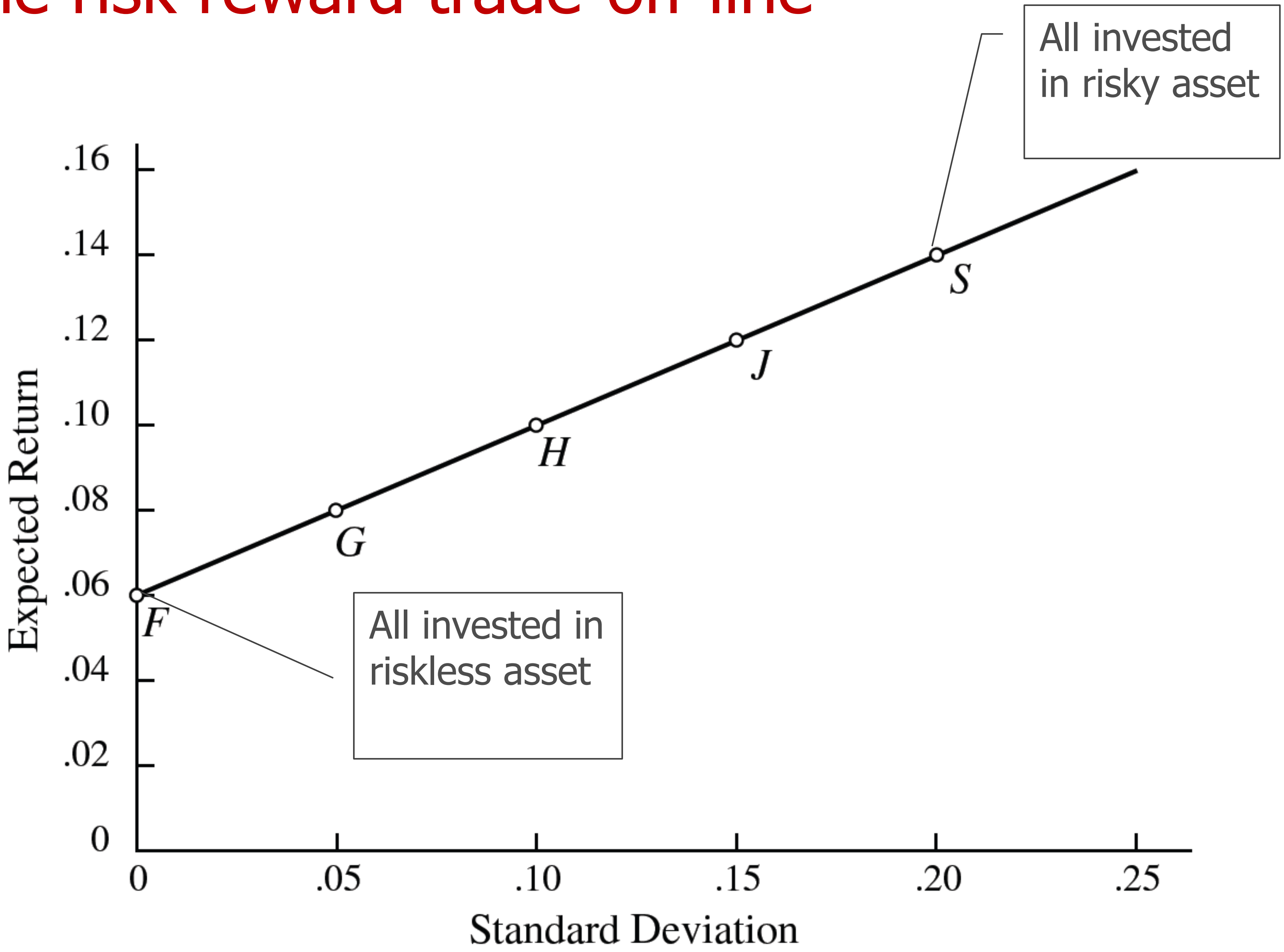
Combining the riskless asset and a single risky asset

- We examine all possible risk-return combinations

Portfolio	% invested in risky asset	% invested in riskless asset	Expected rate of return $E(r)$	Standard deviation σ
F	0	100	0.06	0.00
G	0.25	0.75	0.08	0.05
H	0.50	0.50	0.10	0.1
J	0.75	0.25	0.12	0.15
S	1	0	0.14	0.20

- $E(\alpha r_1 + \beta r_2) = \alpha E(r_1) + \beta E(r_2)$
- $Var(\alpha r_1 + \beta r_2) = \alpha^2 Var(r_1) + \beta^2 Var(r_2) + 2\alpha\beta cov(r_1, r_2)$, $cov(r_1, r_2) = 0$ if r_1 and r_2 are independent
  - Riskless return and risky return are independent

The risk-reward trade-off line



Identify composition for any point

- Suppose we want to identify the portfolio that has an expected rate of return of 0.09
- From the trade-off line, it is somewhere between G and H.
- But what is the proportion invested in risky assets and what is the standard deviation of the portfolio?
- We need a formula (Return -> portfolio composition)

Step 1: Relate portfolio's expected return to the proportion invested in the risky asset

- Let ω denote the proportion of the $100,000 investment to be allocated to the risky asset.
- The remaining proportion, $1-\omega$, is invested in riskless asset
- The expected rate of return on any portfolio is given by

$$E(r) = \omega E(r_s) + (1-\omega) r_f$$

$$= r_f + \omega\left[E(r_s) - r_f\right]$$

$$= 0.06 + \omega(0.14 - 0.06)$$

$$= 0.06 + 0.08\omega$$

  - $E(r_s)$ is the expected rate of return on the risky asset
  - r_f is the riskless rate.

$$E(r) = r_f + \omega[E(r_s) - r_f]$$

- The portfolio is expected to earn a riskless rate (r_f) plus a risk premium component ($\omega[E(r_s) - r_f]$).
- a risk premium component depends on the risk premium on the risky asset ($E(r_s) - r_f$) and the proportion invested in the risky asset (ω)

- To find a portfolio composition whose expected rate of return is 0.09,

$$E(r) = r_f + \omega\left[E(r_s) - r_f\right]$$

$$0.09 = 0.06 + 0.08\omega$$

$$\omega = 0.375$$

- The portfolio is a mix of 37.5% risky asset and 62.5% riskless asset.

Step 2: Relate the portfolio standard deviation to the proportion invested in risky asset

- Denoting the standard deviation of the risky asset σ_s, the portfolio's standard deviation is

$$\sigma = \sigma_s \omega = 0.2\omega$$

- The standard deviation corresponds to an expected rate of return of 0.09

$$\sigma = 0.2 \times 0.375 = 0.075$$

Step 3: Relate the portfolio expected rate of return to its standard deviation

$$\sigma = \sigma_s \omega \;\rightarrow \omega = \frac{\sigma}{\sigma_s}$$

$$E(r) = r_f + \omega\left[E(r_s) - r_f\right] = r_f + \frac{\left[E(r_s) - r_f\right]}{\sigma_s}\sigma$$

The portfolio's rate of return expressed as a function of its standard deviation is a straight line:

$$E(r) = 0.06 + \frac{[0.14 - 0.06]}{0.2}\sigma = 0.6 + 0.4\sigma$$

- Intercept: $r_f = 0.06$
- Slope: $\frac{\left[E(r_s) - r_f\right]}{\sigma_s} = 0.4$
  - Measures the extra expected rate the market offers for each unit of extra risk an investor is willing to bear.

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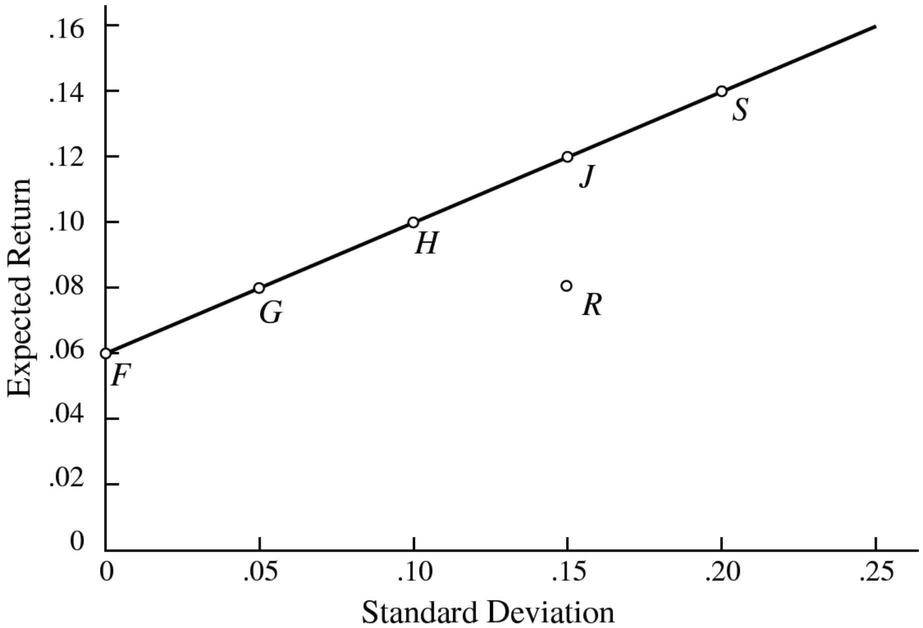
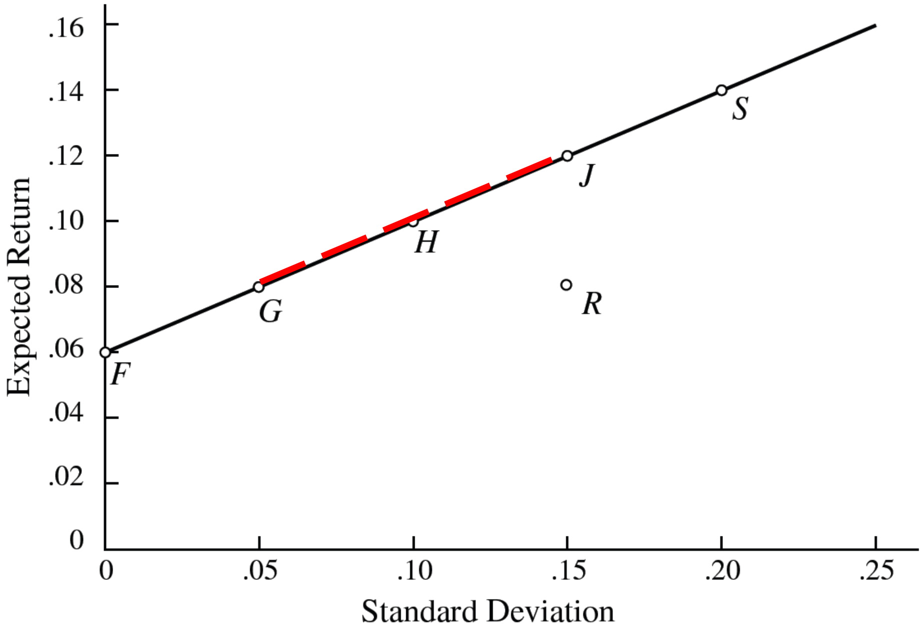
Portfolio efficiency

- Risky asset 2 (R)
  - Expected rate of return 0.08
  - Standard deviation 0.15
- For an investor whose target return is 0.08, is investing all money in R a wise choice?
  - By investing all your money in R, you bear all risk in R (0.15)



Portfolio efficiency

- R is inefficient because one could achieve the same expected rate of return but lower risk (G).
- Or one could achieve higher expected rate of return while bearing the same level of risk (J) or lower level of risk (points between G and J
- What about holding a mix of two risky assets and the riskless asset?



Portfolio efficiency

- An efficient portfolio is defined as the portfolio that offers the investor the highest expected rate of return at a specific risk
  - Or the lowest risk at a specific expected rate of return
- We now investigate more than one risky asset in a portfolio
  - First, we consider the risk and return combinations attainable by mixing only risky assets 1 and 2
  - Second, we add riskless asset.

Portfolio of two risky assets

- A proportion ω in risky asset 1 and $1-\omega$ in risky asset 2
- Expected rate of return

$$E(r) = \omega E(r_1) + (1-\omega)E(r_2)$$

  - Weighted average of expected returns of assets
  - Weights are the proportions invested in the assets
- Variance

$$\sigma^2 = \omega^2\sigma_1^2 + (1-\omega)^2\sigma_2^2 + 2\omega(1-\omega)\rho_{1,2}\sigma_1\sigma_2$$

  - $\rho_{1,2}$ is the correlation coefficient between r_1 and r_2.
- Compared to combining a risky asset with a riskless asset

$$E(r) = \omega E(r_s) + (1-\omega)r_f$$

$$\sigma = \sigma_s\omega$$

The benefit of diversification

$$\sigma^2 = \omega^2\sigma_1^2 + (1-\omega)^2\sigma_2^2 + 2\omega(1-\omega)\rho_{1,2}\sigma_1\sigma_2$$

Because $\rho_{1,2} \leq 1$,

$$\sigma^2 \leq \omega^2\sigma_1^2 + (1-\omega)^2\sigma_2^2 + 2\omega(1-\omega)\sigma_1\sigma_2$$

$$\sigma^2 \leq [\omega\sigma_1 + (1-\omega)\sigma_2]^2$$

$$\sigma \leq \omega\sigma_1 + (1-\omega)\sigma_2$$

As long as $\rho_{1,2} \neq 1$ or $\rho_{1,2} < 1$,

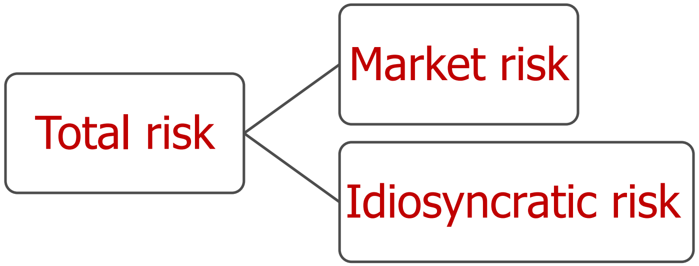
$$\sigma < \omega\sigma_1 + (1-\omega)\sigma_2$$

By diversifying between asset 1 and asset 2,

- Obtain weighted average of expected returns: $E(r) = \omega E(r_1) + (1-\omega)E(r_2)$
- Bear risk lower than weighted average of the risks

How to understand $\rho_{1,2}$?

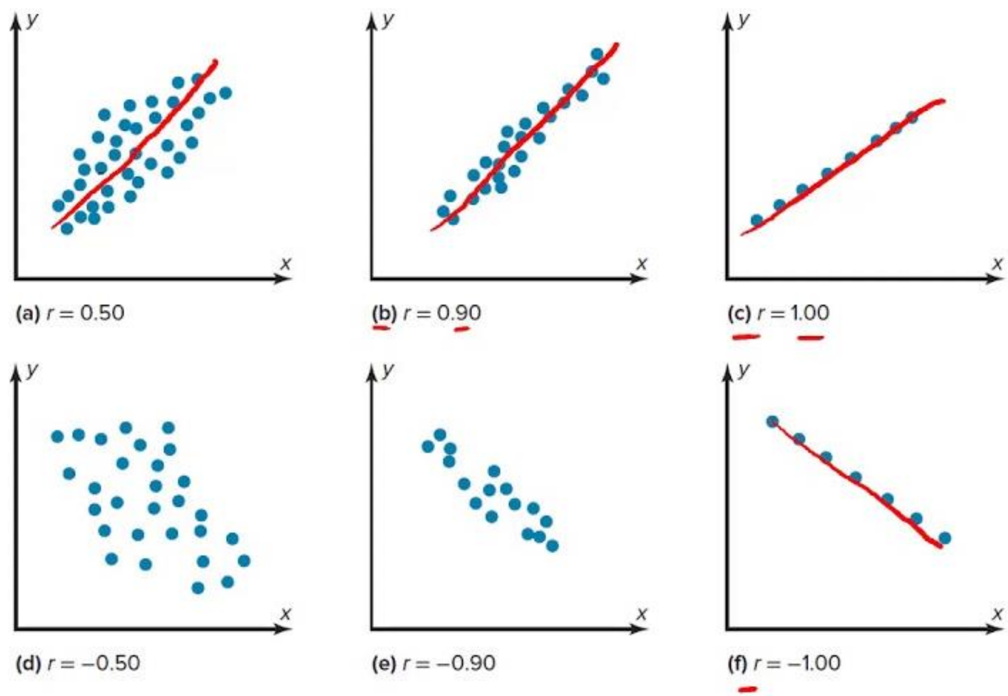
- Mathematically, $\rho_{1,2}$ is the correlation coefficient between r_1 and r_2
- Measures how returns of two risky asset correlate
- Buy why would they correlate? --- Market risk/Systematic risk
  - Due to factors common to both assets: macro economy, government regulation, etc
- The part of risk that does not correlate --- Idiosyncratic risk/Non-systematic risk
  - Due to factors unique to individual asset: firm management, etc

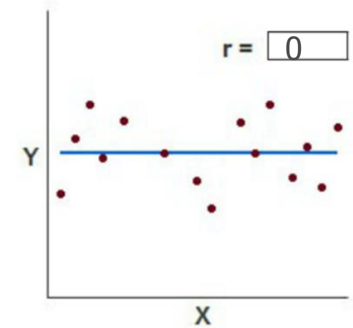


  - $\rho_{1,2} = 1$
    - Main risk comes from the market risk

How to understand $\rho_{1,2}$?

- Scatter plot r_1 on x-axis and r_1 on y-axis (r indicates $\rho_{1,2}$ in the plots)





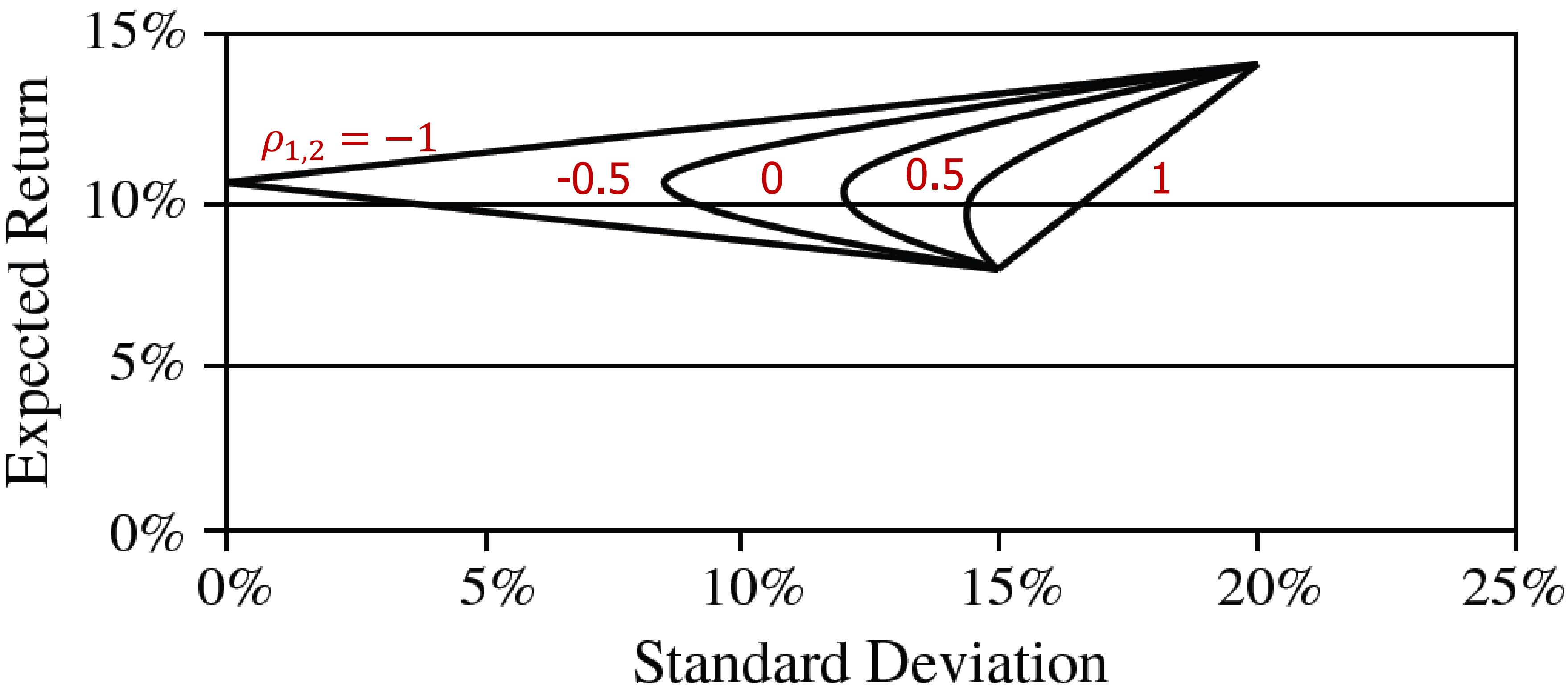
Portfolio of two risky assets

	Risky Asset 1	Risky asset 2
$E(r)$	0.14	0.08
σ	0.2	0.15

- Conditional on alternative assumptions about the correlation coefficient

Correlation ($\rho_{1,2}$)	prop invested in risky asset 1 (ω)	prop invested in risky asset 1 ($1-\omega$)	Expected return $E(r)$	Standard deviation σ
1	0	1	0.08	0.15 (min σ)
	0.25	0.75	0.95	0.1625
	0.50	0.50	0.11	0.175
	0.75	0.25	0.125	0.1875
0.5	23.08	76.92	0.0938	0.1441
	0.25	0.75	0.095	0.1442
	0.50	0.50	0.11	0.1521
	0.75	0.25	0.125	0.1718
0	0.25	0.75	0.095	0.1231
	36	64	0.1016	0.12
	0.50	0.50	0.11	0.125
	0.75	0.25	0.125	0.1546
-0.5	0.25	0.75	0.095	0.0976
	40.54	59.46	0.1043	0.0854
	0.50	0.50	0.11	0.0901
	0.75	0.25	0.125	0.1352
-1	0.25	0.75	0.95	0.0625
	42.86	57.14	0.1057	0
	0.50	0.50	0.11	0.025
	0.75	0.25	0.125	0.1125
	1	0	0.14	0.2

Risk-reward trade-off curve: risky assets only



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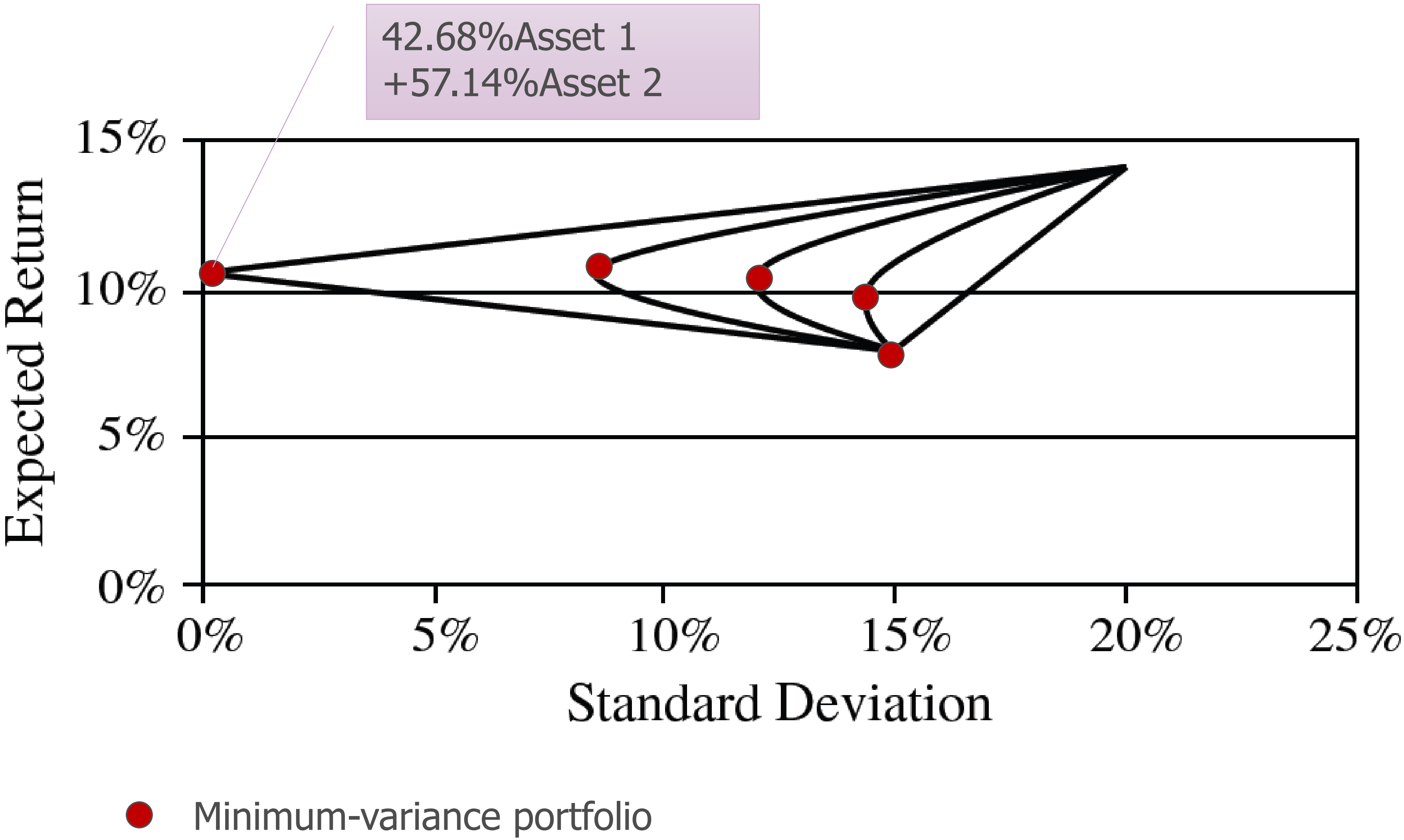
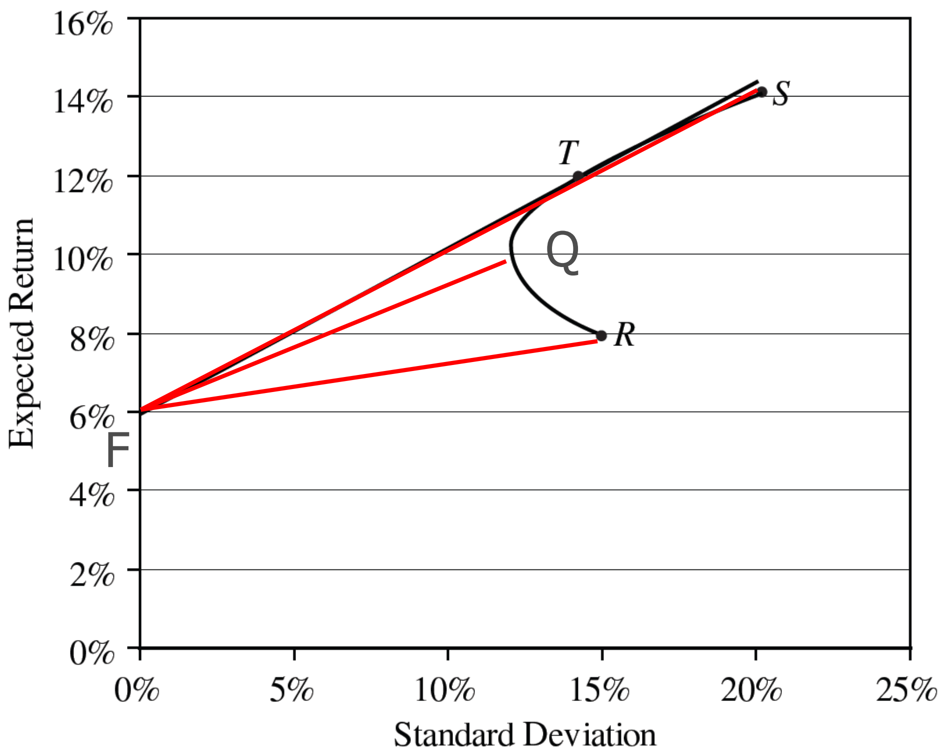


Minimum-variance portfolio

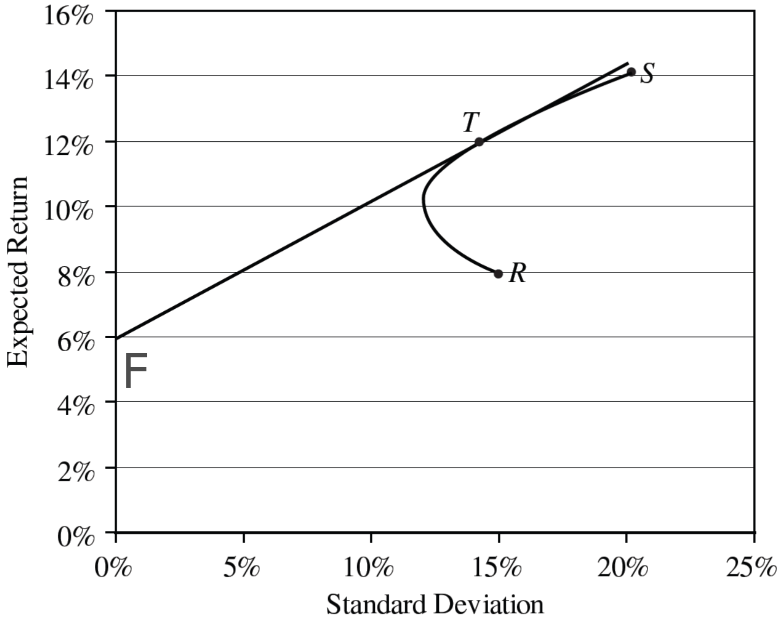
- $\min_{\omega} \omega^2\sigma_1^2 + (1-\omega)^2\sigma_2^2 + 2\omega(1-\omega)\rho_{1,2}\sigma_1\sigma_2$
- FOC. $2\omega\sigma_1^2 - 2(1-\omega)\sigma_2^2 + (2-4\omega)\rho_{1,2}\sigma_1\sigma_2 = 0$
- Solve $\omega^* = \frac{\sigma_2^2-\rho_{1,2}\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2-2\rho_{1,2}\sigma_1\sigma_2}$

Combining riskless asset with risky assets

- Assume $\rho_{1,2} = 0$
- Line FS: risk-return trade-off line of portfolio combining riskless asset and risky asset 1
- Line FR: risk-return trade-off line of portfolio combining riskless asset and risky asset 2
- Line FQ: risk-return trade-off line of portfolio combining riskless asset and a portfolio Q consisting of risky asset 1 and 2



The optimal combination of risky assets



- Tangency point T
  - **Tangency portfolio**
- Compared to other trade-off lines (e.g., FS, FR, and FQ), FT provides higher expected rate of return for any level of risk one is willing to tolerate.

The optimal combination of risky assets

- How to solve for portfolio T?
  - Maximize slope of the line connecting F and points on the curve SR.

$$\max_{\omega} \frac{E(r_T) - r_f}{\sigma_T}$$

$$\text{s.t.}, \mathrm{E}(r_T) = \omega E(r_1) + (1-\omega) E(r_2)$$

$$\sigma_T^2 = \omega^2 \sigma_1^2 + (1-\omega)^2 \sigma_2^2 + 2\omega(1-\omega)\rho_{1,2}\sigma_1\sigma_2$$

- This is a straightforward, albeit very tedious, calculus problem and the solution can be shown to be

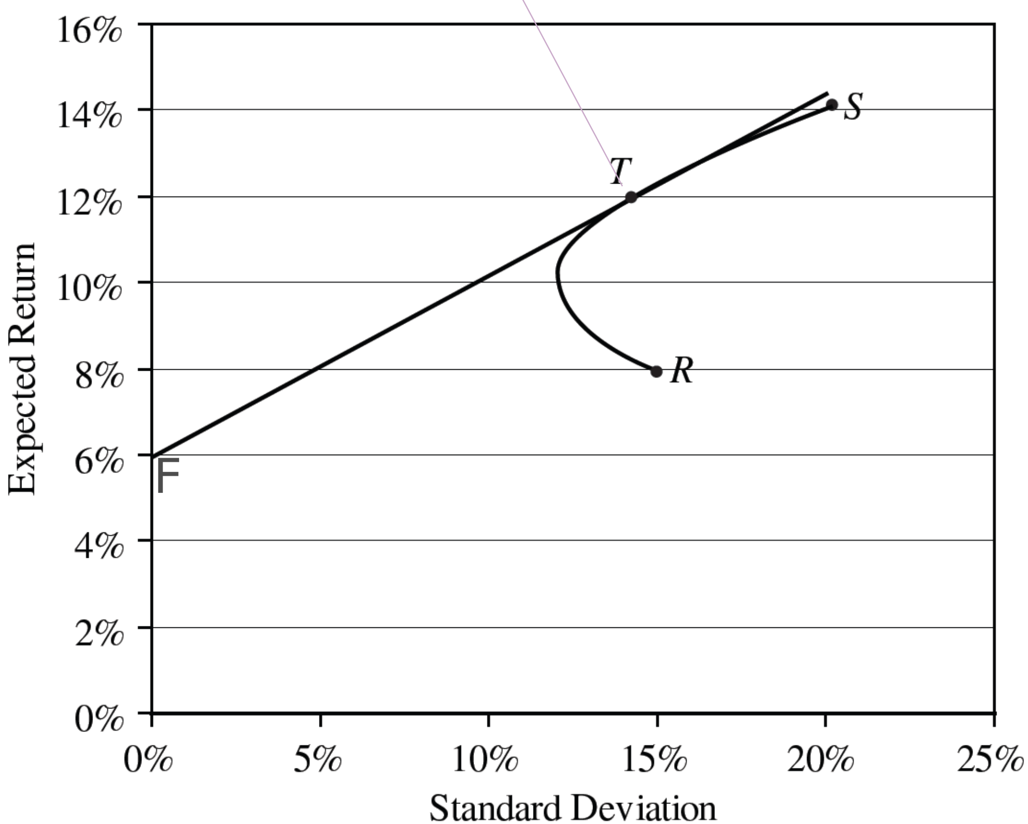
$$\omega = \frac{[E(r_1) - r_f]\sigma_2^2 - [E(r_2) - r_f]\rho_{1,2}\sigma_1\sigma_2}{[E(r_1) - r_f]\sigma_2^2 + [E(r_2) - r_f]\sigma_1^2 - [E(r_1) - r_f + E(r_2) - r_f]\rho_{1,2}\sigma_1\sigma_2}$$

The optimal combination of risky assets

- Therefore, we can solve for the optimal combination of risky assets (the tangency portfolio)

69.23% Asset 1
+30.77% Asset 2

$E(r_T) = 0.122$
$\sigma_T = 0.146$



- New trade-off line FT

$$E(r) = r_f + \omega\left[E(r_T) - r_f\right]$$

$$= r_f + \frac{\left[E(r_T) - r_f\right]}{\sigma_T}\sigma$$

$$= 0.06 + \frac{[0.122 - 0.06]}{0.146}\sigma$$

$$= 0.06 + 0.42\sigma$$

- Compare to the old trade-off line

$$E(r) = 0.6 + 0.4\sigma$$

- Given the same risk level (σ), the new line provides higher return

Selecting the preferred portfolio

- For an investor whose target return is 0.08, what is the optimal portfolio?

$$E(r) = \omega E(r_T) + (1 - \omega)r_f$$

$$0.08 = 0.122\omega + 0.06(1 - \omega)$$

$$\omega = 0.32$$

- The standard deviation of this portfolio is

$$\sigma = \omega\sigma_T = 0.32 \times 0.146 = 0.047$$

Asset	**Weight**
Riskless asset	0.68
Risky asset	0.32
Risky asset 1	0.22 (=0.32*69.23%)
Risky asset 2	0.10 (=0.32*30.77%)

*The numbers are rounding.

Selecting the preferred portfolio

- What about in the old trade-off line?

$$E(r) = \omega E(r_1) + (1 - \omega) r_f$$

$$0.08 = 0.14\omega + 0.06(1 - \omega)$$

$$\omega = 0.25$$

- The standard deviation of this portfolio is

$$\sigma = \omega \sigma_T = 0.25 \times 0.2 = 0.05 > 0.047$$

- Given target expected rate of return, combining tangency portfolio with riskless asset brings lower risk

Selecting the preferred portfolio

- For an investor whose highest risk tolerable is 0.08, what is the optimal portfolio?

$$\sigma = \omega\sigma_T = 0.146\omega = 0.08$$

$$\omega = 0.55$$

- The expected rate of return of this portfolio is

$$E(r) = \omega E(r_T) + (1 - \omega)r_f$$

$$= 0.55 \times 0.122 + 0.06 \times (1 - 0.55) = 0.094$$

Asset	**Weight**
Riskless asset	0.45
Risky asset	0.55
Risky asset 1	0.38 (=0.55*69.23%)
Risky asset 2	0.17 (=0.55*30.77%)

*The numbers are rounding.

Selecting the preferred portfolio

- What about in the old trade-off line?

$$\sigma = \omega\sigma_T = 0.2\omega = 0.08$$

$$\omega = 0.4$$

- The expected rate of return of this portfolio is

$$E(r) = \omega E(r_T) + (1-\omega)r_f$$

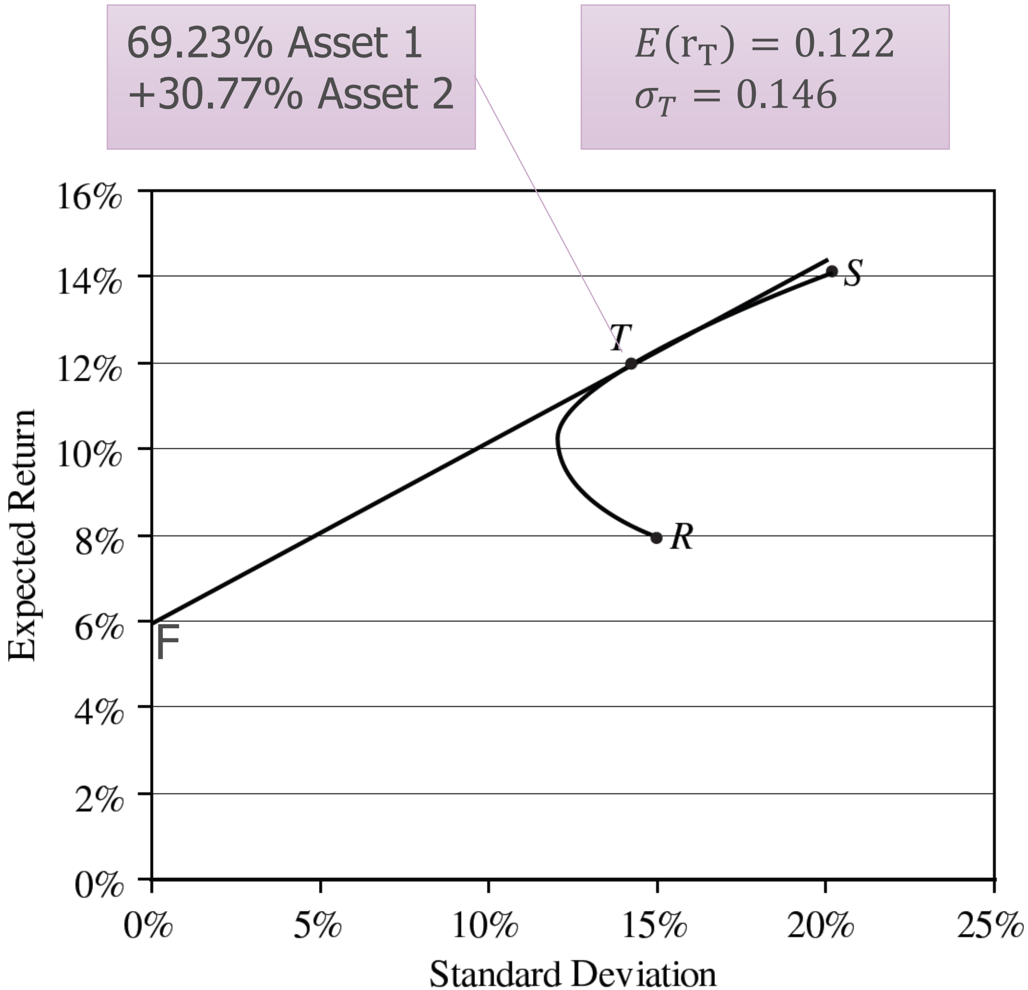
$$= 0.4 \times 0.14 + 0.06 \times (1-0.4) = 0.092 < 0.094$$

- Given tolerable risk, combining tangency portfolio with riskless asset brings higher expected rate of return

The optimal combination of risky assets

- To find the optimal combination of risky assets,
  - Step 1: consider portfolio constructed from the risky assets only
  - Step 2: find the tangency portfolio of risky assets to combine with the risky assets
- We do not need to know
  - Anything about investors' preference
- We need to know the
  - Return distributions of the two risky assets
    - Expected rate of return
    - Standard deviations of the rate of return
    - The correlation between the two rate of return

Selecting the preferred portfolio



- All investors (with the same beliefs for the return distributions) choose preferred portfolio along the same trade-off line FT, i.e., the combination of riskless asset and the tangency portfolio.
- Whether one prefer portfolios near F (low risk) or T (high risk) depends on one's preference, including risk tolerance.

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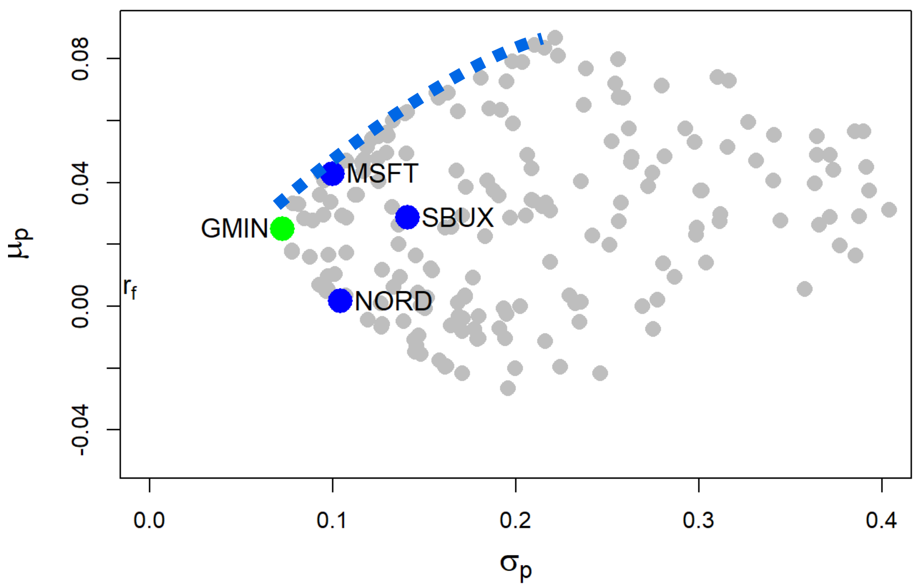
Portfolio of many (n) risky assets

- The idea is roughly the same to find the optimal portfolio
  - Step 1: consider portfolio constructed from the n risky assets only
    - What factors are important?
      - Expected rate of return for every risky asset: $r_1, r_2, \dots, r_n$
      - Risk of every risky asset: $\sigma_1, \sigma_2, \dots, \sigma_n$
      - Correlation between every two risky assets: $\rho_{1,2}, \rho_{1,3}, \dots, \rho_{n,n-1}$ (a total of $\frac{n(n-1)}{2}$ $\rho' s$)

Portfolio of many risky assets

- The idea is roughly the same to find the optimal portfolio
  - Step 1: consider portfolio constructed from the risky assets only
    - **Efficient portfolio frontier**: the set of portfolios of risky assets offering the highest possible expected rate of return for any given standard deviation



Portfolio of many risky assets

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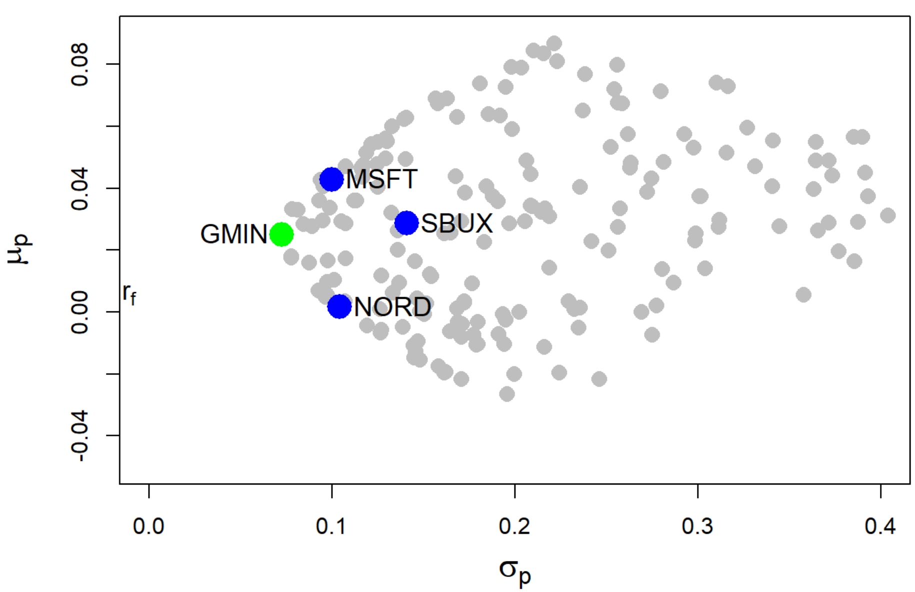

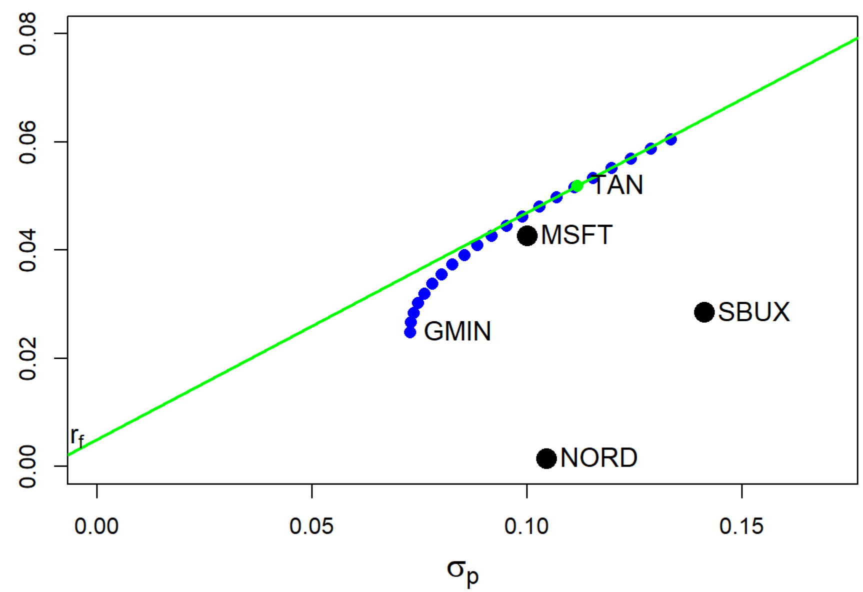

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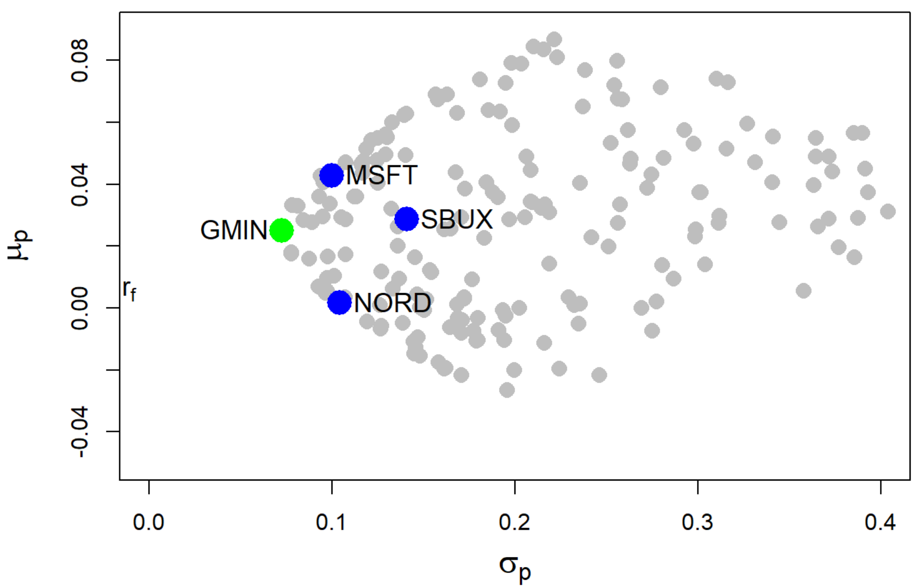
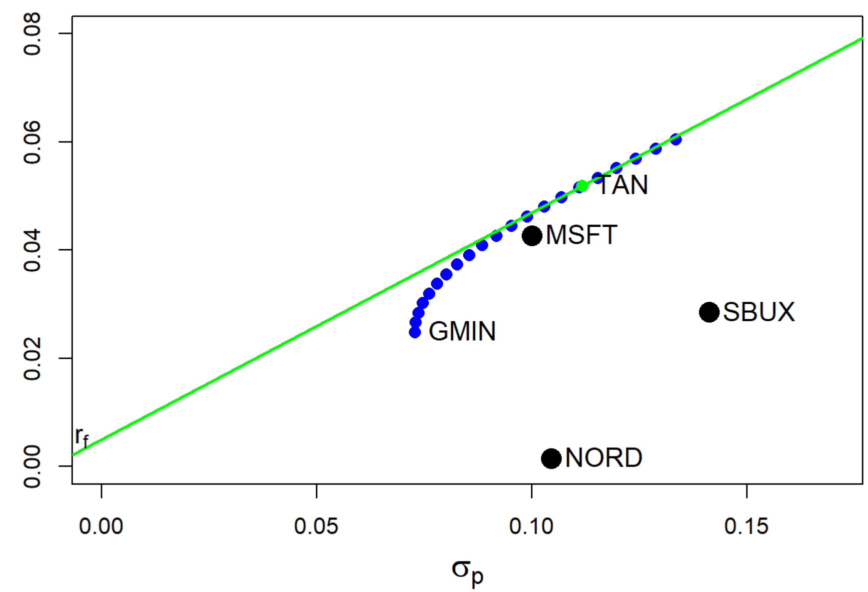




Diversification with many risky assets

- Consider n uncorrelated assets with identical expected rate of return ($\mathrm{E}(r)$) and standard deviation (σ)
- If we construct an equally weighted portfolio of these assets with $\omega_i = \frac{1}{n}$, what are the mean and the variance of this portfolio?

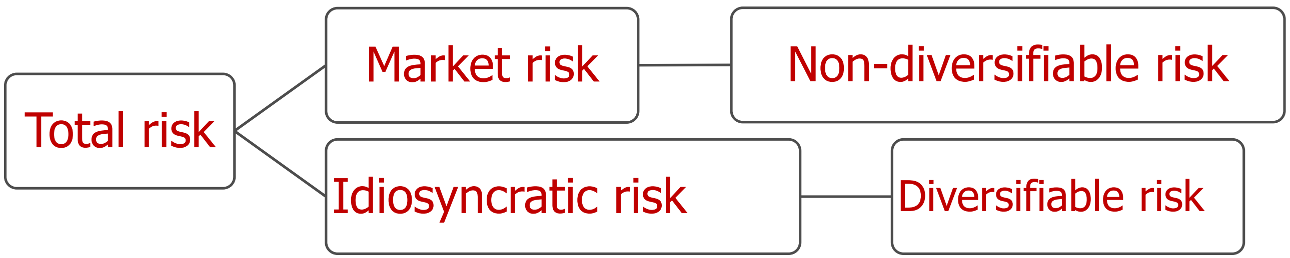
$$E(r_p) = \sum_{i=1}^{n} \omega_i E(r) = E(r)$$

$$\sigma_p^2 = \sum_{i=1}^{n} \omega_i^2 \sigma^2 = \frac{1}{n}\sigma^2 < \sigma^2$$

- Investors are better off by diversifying their portfolio among risky assets: same expected rate of return, lower risk.
  - Don’t put all your eggs in one basket
- As $n \to \infty$, $\sigma_p^2 \to 0$. No risk at all!

Diversification

- Portfolio of *less than (positively) perfectly correlated* assets always offer at least as good risk-return opportunities than the individual component assets on their own.



- Idiosyncratic risks are diversified (eliminated through diversification).
- Market risk is not diversifiable.
- An asset's contribution to a portfolio's risk is determined by its relations with all other assets in the portfolio
- All relations are due to market risk.