Financial Economics

Lecture 02. Time Value of Money

(supplement: annuities)

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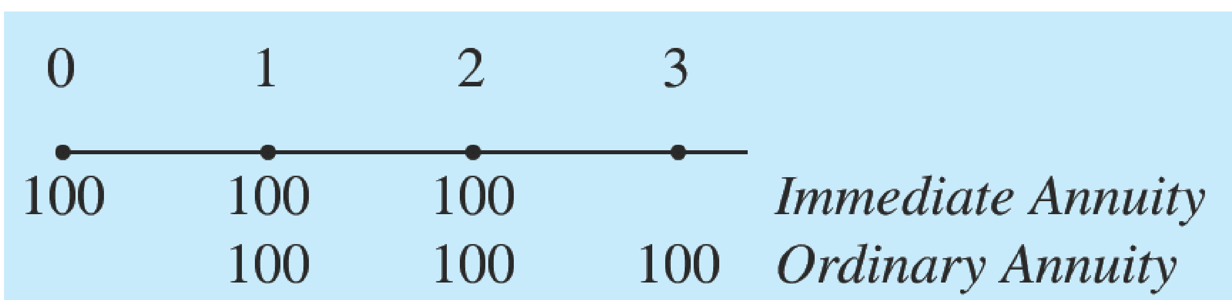
Outline

- Future Value and Compounding
- Present Value and Discounting
- More about PV and FV
- Annuities
  - Present Value and Future Value of annuities
  - Perpetual Annuities and Growing annuities
- Capital Budgeting Decision Rule

Annuities

- A level stream of cash flows or payments
  - all subsequent cash flows are separated by exactly one period
  - all periods are of equal length
  - the term structure of interest is flat
  - all cash flows have the same (nominal) value
- Financial analysts use several annuities with differing assumptions about the first payment. We will examine just two:
  - *Immediate Annuity:* Cash flows start immediately
  - *Ordinary annuity:* Cash flows start at the end of the current period



Annuity Formula Notation

- PV = the present value of the annuity
- i = interest rate to be earned over the life of the annuity
- n = the number of payments
- pmt = the periodic payment

Ordinary Annuity: Present value

$$PV = \frac{pmt}{(1+i)^1} + \frac{pmt}{(1+i)^2} + \frac{pmt}{(1+i)^3} + \cdots + \frac{pmt}{(1+i)^{n-1}} + \frac{pmt}{(1+i)^n}$$

$$PV = pmt * \{\frac{1}{(1+i)^1} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \cdots + \frac{1}{(1+i)^{n-1}} + \frac{1}{(1+i)^n}\}$$

$$PV * (1+i) = pmt * (1+i) * \{\frac{1}{(1+i)^1} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \cdots + \frac{1}{(1+i)^{n-1}} + \frac{1}{(1+i)^n}\}$$

$$PV * (1+i) = pmt * \{\frac{1}{(1+i)^0} + \frac{1}{(1+i)^1} + \frac{1}{(1+i)^2} + \cdots + \frac{1}{(1+i)^{n-2}} + \frac{1}{(1+i)^{n-1}} + [\frac{1}{(1+i)^n} - \frac{1}{(1+i)^n}]\}$$

$$= pmt * \frac{1}{(1+i)^0} + pmt * \{\frac{1}{(1+i)^1} + \frac{1}{(1+i)^2} + \cdots + \frac{1}{(1+i)^{n-2}} + \frac{1}{(1+i)^{n-1}} + \frac{1}{(1+i)^n}\} - pmt \frac{1}{(1+i)^n}$$

$$\Rightarrow PV * (1+i) = pmt * \frac{1}{(1+i)^0} + PV - pmt \frac{1}{(1+i)^n}$$

$$\Rightarrow PV * (1+i) + PV = pmt - pmt \frac{1}{(1+i)^n} \Rightarrow PV = \frac{pmt * \{1 - \frac{1}{(1+i)^n}\}}{i}$$

$$\boxed{\Rightarrow PV = \frac{pmt}{i} * \left(1 - \frac{1}{(1+i)^n}\right)}$$

Ordinary Annuity: Present value

- Present value factor of an ordinary annuity of $\$1$ per period for n periods at an interest rate of i

$$PV = \frac{1-(1+i)^{-n}}{i}$$

- Present value of an ordinary annuity of $\$pmt$ per period for n periods at an interest rate of i

$$PV = pmt \times \frac{1-(1+i)^{-n}}{i}$$

- Example: how much would you have to put into a fund earning an interest rate of 10% per year to be able to take out $100 per year for the next three years?
  - The present value factor of an ordinary annuity of $\$1$ per period for 3 periods at an interest rate of 10% is $\frac{1-(1+0.1)^{-3}}{0.1} = 2.4869$
  - The present value is $\$100 \times 2.4869 = \$248.69$

Ordinary Annuity: Future value

$$PV = pmt \times \frac{1-(1+i)^{-n}}{i}$$

$$FV = PV \times (1+i)^n = pmt \times \frac{1-(1+i)^{-n}}{i} \times (1+i)^n = pmt \times \frac{(1+i)^n - 1}{i}$$

Ordinary Annuity: Future value

- Future value factor of an ordinary annuity of $\$1$ per period for n periods at an interest rate of i

$$FV = \frac{(1+i)^n - 1}{i}$$

- Future value of an ordinary annuity of $\$pmt$ per period for n periods at an interest rate of i

$$FV = pmt \times \frac{(1+i)^n - 1}{i}$$

- Example: Suppose you intend to save $100 each year for the next three years. How much will you have accumulated at the end of that time if the interest rate is 10% per year?
  - The future value factor of an ordinary annuity of $\$1$ per period for 3 periods at an interest rate of 10% is $\frac{(1+0.1)^3-1}{0.1} = 3.6410$
  - The future value is $\$100 \times 3.6410 = \$364.10$

Ordinary Annuity Formula

- Given any three out of PV, i, n, and pmt, can compute the last
  - Present value: $PV = pmt \times \frac{1-(1+i)^{-n}}{i}$
  - Payment: $pmt = \frac{PV \times i}{1-(1+i)^{-n}}$
  - Number of periods: $n = -\frac{\ln(1-PV \times i/pmt)}{\ln(1+i)}$
  - Interest rate: no close-form solution, use numerical method
- Similar formulas go with the future value

Present Value Annuity Tables

Formula: $PV = [1 - 1 / (1 + i)^n] / i$

n / i	1%	2%	3%	4%	5%	6%	7%	8%	9%	10%	11%	12%	13%	14%	15%
1	0.9901	0.9804	0.9709	0.9615	0.9524	0.9434	0.9346	0.9259	0.9174	0.9091	0.9009	0.8929	0.8850	0.8772	0.8696
2	1.9704	1.9416	1.9135	1.8861	1.8594	1.8334	1.8080	1.7833	1.7591	1.7355	1.7125	1.6901	1.6681	1.6467	1.6257
3	2.9410	2.8839	2.8286	2.7751	2.7232	2.6730	2.6243	2.5771	2.5313	2.4869	2.4437	2.4018	2.3612	2.3216	2.2832
4	3.9020	3.8077	3.7171	3.6299	3.5460	3.4651	3.3872	3.3121	3.2397	3.1699	3.1024	3.0373	2.9745	2.9137	2.8550
5	4.8534	4.7135	4.5797	4.4518	4.3295	4.2124	4.1002	3.9927	3.8897	3.7908	3.6959	3.6048	3.5172	3.4331	3.3522
6	5.7955	5.6014	5.4172	5.2421	5.0757	4.9173	4.7665	4.6229	4.4859	4.3553	4.2305	4.1114	3.9975	3.8887	3.7845
7	6.7282	6.4720	6.2303	6.0021	5.7864	5.5824	5.3893	5.2064	5.0330	4.8684	4.7122	4.5638	4.4226	4.2883	4.1604
8	7.6517	7.3255	7.0197	6.7327	6.4632	6.2098	5.9713	5.7466	5.5348	5.3349	5.1461	4.9676	4.7988	4.6389	4.4873
9	8.5660	8.1622	7.7861	7.4353	7.1078	6.8017	6.5152	6.2469	5.9952	5.7590	5.5370	5.3282	5.1317	4.9464	4.7716
10	9.4713	8.9826	8.5302	8.1109	7.7217	7.3601	7.0236	6.7101	6.4177	6.1446	5.8892	5.6502	5.4262	5.2161	5.0188
11	10.3676	9.7868	9.2526	8.7605	8.3064	7.8869	7.4987	7.1390	6.8052	6.4951	6.2065	5.9377	5.6869	5.4527	5.2337
12	11.2551	10.5753	9.9540	9.3851	8.8633	8.3838	7.9427	7.5361	7.1607	6.8137	6.4924	6.1944	5.9176	5.6603	5.4206
13	12.1337	11.3484	10.6350	9.9856	9.3936	8.8527	8.3577	7.9038	7.4869	7.1034	6.7499	6.4235	6.1218	5.8424	5.5831
14	13.0037	12.1062	11.2961	10.5631	9.8986	9.2950	8.7455	8.2442	7.7862	7.3667	6.9819	6.6282	6.3025	6.0021	5.7245
15	13.8651	12.8493	11.9379	11.1184	10.3797	9.7122	9.1079	8.5595	8.0607	7.6061	7.1909	6.8109	6.4624	6.1422	5.8474
16	14.7179	13.5777	12.5611	11.6523	10.8378	10.1059	9.4466	8.8514	8.3126	7.8237	7.3792	6.9740	6.6039	6.2651	5.9542
17	15.5623	14.2919	13.1661	12.1657	11.2741	10.4773	9.7632	9.1216	8.5436	8.0216	7.5488	7.1196	6.7291	6.3729	6.0472
18	16.3983	14.9920	13.7535	12.6593	11.6896	10.8276	10.0591	9.3719	8.7556	8.2014	7.7016	7.2497	6.8399	6.4674	6.1280
19	17.2260	15.6785	14.3238	13.1339	12.0853	11.1581	10.3356	9.6036	8.9501	8.3649	7.8393	7.3658	6.9380	6.5504	6.1982
20	18.0456	16.3514	14.8775	13.5903	12.4622	11.4699	10.5940	9.8181	9.1285	8.5136	7.9633	7.4694	7.0248	6.6231	6.2593
21	18.8570	17.0112	15.4150	14.0292	12.8212	11.7641	10.8355	10.0168	9.2922	8.6487	8.0751	7.5620	7.1016	6.6870	6.3125
22	19.6604	17.6580	15.9369	14.4511	13.1630	12.0416	11.0612	10.2007	9.4424	8.7715	8.1757	7.6446	7.1695	6.7429	6.3587
23	20.4558	18.2922	16.4436	14.8568	13.4886	12.3034	11.2722	10.3711	9.5802	8.8832	8.2664	7.7184	7.2297	6.7921	6.3988
24	21.2434	18.9139	16.9355	15.2470	13.7986	12.5504	11.4693	10.5288	9.7066	8.9847	8.3481	7.7843	7.2829	6.8351	6.4338
25	22.0232	19.5235	17.4131	15.6221	14.0939	12.7834	11.6536	10.6748	9.8226	9.0770	8.4217	7.8431	7.3300	6.8729	6.4641
26	22.7952	20.1210	17.8768	15.9828	14.3752	13.0032	11.8258	10.8100	9.9290	9.1609	8.4881	7.8957	7.3717	6.9061	6.4906
27	23.5596	20.7069	18.3270	16.3296	14.6430	13.2105	11.9867	10.9352	10.0266	9.2372	8.5478	7.9426	7.4086	6.9352	6.5135
28	24.3164	21.2813	18.7641	16.6631	14.8981	13.4062	12.1371	11.0511	10.1161	9.3066	8.6016	7.9844	7.4412	6.9607	6.5335
29	25.0658	21.8444	19.1885	16.9837	15.1411	13.5907	12.2777	11.1584	10.1983	9.3696	8.6501	8.0218	7.4701	6.9830	6.5509
30	25.8077	22.3965	19.6004	17.2920	15.3725	13.7648	12.4090	11.2578	10.2737	9.4269	8.6938	8.0552	7.4957	7.0027	6.5660

Perpetual Annuities

- Recall the annuity formula:

$$PV = pmt \times \frac{1-(1+i)^{-n}}{i}$$

- Let $n \rightarrow infinity$ with $i > 0$:

$$PV = \frac{pmt}{i}$$

- Example: If you want to set up a scholarship fund to award the well-performed student with $1,000 per year, forever, how much is the initial fund, given the interest rate of 10% per year?
  - The present value of a level perpetuity of $1,000 at an interest rate of 10% is $\frac{\$1{,}000}{0.1} = \$10{,}000$

Growing Annuities

- What if the cash flows from a perpetual annuity grow at a constant rate?
- The present value of a growth perpetuity

$$PV = \frac{C_1}{i - g}$$

  - C_1 is the first year's cash flow and g is the growth rate of annuity
- Example: if the scholarship increases by 1% per year?
  - The present value of a level perpetuity of $1,000 at an interest rate of 10% and a growth rate of 1% is $\frac{\$1{,}000}{0.1-0.01} = \$11{,}111$