1
A Simple Market Model

Case 1

A UK company is preparing to purchase a piece of equipment in the US for \$100,000 in a year's time, the price guaranteed by the producer to remain unchanged. Considering the current exchange rate of 1.62 dollars to a pound, the manager of the company has reserved £64,000 in the budget to become available at the time of purchase. Analyse this decision.

1.1 Basic Notions and Assumptions

Suppose that two assets are traded: one risk-free and one risky security. The former can be thought of as a bank deposit or a bond issued by a government, a financial institution, or a company. The risky security will typically be some stock. It may also be a foreign currency, gold, a commodity or virtually any asset whose future price is unknown today.

Throughout this chapter we restrict the time scale to two instants only: today, $t = 0$, and some future time, $t = T$, say one year from now. More refined and realistic situations will be studied in later chapters.

The position in risky securities can be specified as the number of shares of stock held by an investor. The price of one share at time $t = 0, T$ will be denoted by $S(t)$. The current stock price $S(0)$ is known to all investors, but

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the future price $S(T)$ remains uncertain: it may go up as well as down. The difference $S(T) - S(0)$ as a fraction of the initial value represents the *return* on the stock,

$$K_S = \frac{S(T) - S(0)}{S(0)},$$

which is also uncertain.

The dynamics of stock prices will be discussed in Chapters 6, 7 and 8.

The risk-free position can be described as the amount held in a bank account. As an alternative to keeping money in a bank, investors may choose to invest in bonds. The price of one bond at time $t = 0, T$ will be denoted by $A(t)$. The current bond price $A(0)$ is known to all investors, just like the current stock price. However, in contrast to stock, the price $A(T)$ the bond will fetch at time T is also known with certainty, being guaranteed by the institution issuing bonds. The bond is said to mature at time T with face value $A(T)$. The return on bonds is defined in a similar way as that on stock,

$$K_A = \frac{A(T) - A(0)}{A(0)},$$

and it is called the risk-free return.

Chapters 2 and 9 give a detailed exposition of risk-free assets.

Our task is to build a mathematical model of a market of financial securities. A crucial first stage is concerned with the properties of the mathematical objects involved. This is done below by specifying a number of assumptions, the purpose of which is to find a compromise between the complexity of the real world and the limitations and simplifications of a mathematical model, imposed in order to make it tractable.

We have already observed that the future price $A(T)$ of the risk-free security is deterministic, that is, it is a number known in advance. Meanwhile, the future stock price $S(T)$, unknown at time 0, is a random variable with at least two different values. In reality the number of possible different values of $S(T)$ is finite because they are quoted to within a specified number of decimal places and because there is only a certain final amount of money in the whole world, supplying an upper bound for all prices. Therefore it is natural to require the future price $S(T)$ of a share of stock to be a random variable taking only finitely many values. However, in Chapter 8 this will not be the case.

All stock and bond prices must be strictly positive,

$$A(t) > 0 \quad \text{and} \quad S(t) > 0 \quad \text{for } t = 0, T.$$

The total wealth of an investor holding x stock shares and y bonds at a time instant $t = 0, T$ is

$$V(t) = xS(t) + yA(t). \tag{1.1}$$

The pair (x, y) is called a *portfolio*, $V(t)$ being the *value* of this portfolio or, in other words, the *wealth* of the investor at time t.

The jumps of asset prices between times 0 and T give rise to a change in the portfolio value

$$V(T) - V(0) = x(S(T) - S(0)) + y(A(T) - A(0)).$$

This difference as a fraction of the initial value represents the **return** on the portfolio,

$$K_V = \frac{V(T) - V(0)}{V(0)}.$$

The returns on bonds or stock are particular cases of the return on a portfolio (with $x = 0$ or $y = 0$, respectively). Note that because $S(T)$ is a random variable, so is $V(T)$ as well as the corresponding returns K_S and K_V. The return K_A on a risk-free investment is deterministic.

Example 1.1

Let $A(0) = 100$ and $A(T) = 110$ dollars. Then the return on an investment in bonds will be

$$K_A = 0.10 = 10\%.$$

Also, let $S(0) = 50$ dollars and suppose that the random variable $S(T)$ can take two values,

$$S(T) = \begin{cases} 52 & \text{with probability } p, \\ 48 & \text{with probability } 1 - p, \end{cases}$$

for some $0 < p < 1$. The return on stock will then be

$$K_S = \begin{cases} 4\% & \text{if stock goes up,} \\ -4\% & \text{if stock goes down.} \end{cases}$$

Example 1.2

Given the bond and stock prices in Example 1.1, the value at time 0 of a portfolio with $x = 20$ stock shares and $y = 10$ bonds is

$$V(0) = 2,000$$

dollars. The time T value of this portfolio will be

$$V(T) = \begin{cases} 2,140 & \text{if stock goes up,} \\ 2,060 & \text{if stock goes down,} \end{cases}$$

so the return on the portfolio will be

$$K_V = \begin{cases} 7\% & \text{if stock goes up,} \\ 3\% & \text{if stock goes down.} \end{cases}$$

Exercise 1.1

Let $A(0) = 90$, $A(T) = 100$, $S(0) = 25$ dollars and let

$$S(T) = \begin{cases} 30 & \text{with probability } p, \\ 20 & \text{with probability } 1-p, \end{cases}$$

where $0 < p < 1$. For a portfolio with $x = 10$ shares and $y = 15$ bonds calculate $V(0)$, $V(T)$ and K_V.

Exercise 1.2

Given the same bond and stock prices as in Exercise 1.1, find a portfolio whose value at time T is

$$V(T) = \begin{cases} 1,160 & \text{if stock goes up,} \\ 1,040 & \text{if stock goes down.} \end{cases}$$

What is the value of this portfolio at time 0?

It is mathematically convenient and not too far from reality to allow arbitrary real numbers, including negative ones and fractions, to represent the risky and risk-free positions x and y in a portfolio, so in general no restrictions are imposed on the positions

$$x, y \in \mathbb{R}.$$

The fact that one can hold a fraction of a share or bond is referred to as ***divisibility***. Almost perfect divisibility is achieved in real world dealings whenever the volume of transactions is large as compared to the unit prices.

The fact that no bounds are imposed on x or y is related to another market attribute known as ***liquidity***. It means that any asset can be bought or sold on demand at the market price in arbitrary quantities. This is clearly a mathematical idealisation because in practice there may exist restrictions on the volume of trading, or large transactions may affect the prices. Modelling such effects requires sophisticated mathematics beyond the scope of this book.

If the number of securities of a particular kind held in a portfolio is positive, we say that the investor has a ***long position***. Otherwise, we say that a ***short position*** is taken or that the asset is ***shorted***. A short position in risk-free securities may involve issuing and selling bonds, but in practice the same financial effect is more easily achieved by borrowing cash. Repaying the loan with interest is referred to as ***closing*** the short position.

A short position in stock can be realised by ***short selling***. This means that the investor borrows the stock, sells it, and uses the proceeds to make some other investment. The owner of the stock keeps all the rights to it. In particular, she is entitled to receive any dividends due and may wish to sell the stock at any time. Because of this, the investor must always have sufficient resources to fulfil the resulting obligations and, in particular, to ***close*** the short position in risky assets, that is, to repurchase the stock and return it to the owner. Similarly, the investor must always be able to close a short position in risk-free securities, by repaying the cash loan with interest. To reflect these practical restrictions sometimes it is assumed that the wealth of an investor must be non-negative at all times, $V(t) \geq 0$.

1.2 No-Arbitrage Principle

In this section we are going to formulate the most fundamental assumption about the market. In brief, we shall assume that the market does not allow risk-free profits with no initial investment. Profits of this kind may happen when some market participants make a mistake.

Example 1.3

Suppose that dealer A in New York offers to buy British pounds at a rate $d_A = 1.62$ dollars to a pound, while dealer B in London sells them at a rate $d_B = 1.60$ dollars to a pound. If this were the case, the dealers would, in effect, be handing out free money. An investor with no initial capital could realise a profit of $d_A - d_B = 0.02$ dollars per each pound traded by taking simultaneously a short position with dealer B and a long position with dealer A. The demand for their generous services would quickly compel the dealers to adjust the exchange rates so that this profitable opportunity would disappear.

Exercise 1.3

On 19 July 2002 dealer A in New York and dealer B in London used the following rates to change currency, namely euros (€), British pounds (£)

and US dollars ($):

dealer A	buy	sell
€1.0000	$1.0202	$1.0284
£1.0000	$1.5718	$1.5844

dealer B	buy	sell
€1.0000	£0.6324	£0.6401
$1.0000	£0.6299	£0.6375

Spot a chance of a risk-free profit without initial investment.

The next example illustrates a situation when a risk-free profit could be realised without initial investment in our simplified framework of a single time step.

Example 1.4

Suppose that dealer A in New York offers to buy British pounds a year from now at a rate $d_A = 1.58$ dollars to a pound, while dealer B in London would sell British pounds immediately at a rate $d_B = 1.60$ dollars to a pound. Suppose further that dollars can be borrowed at an annual rate of 4%, and British pounds can be invested in a bank account at 6%. This would also create an opportunity for a risk-free profit without initial investment, though perhaps not as obvious as before.

For instance, an investor could borrow 10,000 dollars and convert them into 6,250 pounds, which could then be deposited in a bank account. After one year interest of 375 pounds would be added to the deposit, and the whole amount could be converted back into 10,467.50 dollars. (A suitable agreement would have to be signed with dealer A at the beginning of the year.) After paying back the dollar loan with interest of 400 dollars, the investor would be left with a profit of 67.50 dollars.

Apparently, one or both dealers have made a mistake in quoting their exchange rates, which can be exploited by investors. Once again, increased demand for their services will prompt the dealers to adjust the rates, reducing d_A and/or increasing d_B to a point when the profit opportunity disappears.

In the previous examples the profit was certain even though no initial investment was required. One should also consider the slightly more subtle possibility of making a profit that is no longer certain, but carries no risk of loss and requires no initial investment. Suppose a lottery has a hundred million tickets

with just one award, a car. One would not expect these tickets to be offered for free even though in practice the chance of a profit is very slim indeed. A similar situation arises when a seller of some goods offers an additional free bonus of this kind (a scratch card where no proof of purchase is necessary). In reality, such a free bonus is never free, the premium being included in the price of the goods.

We shall make an assumption excluding situations similar to those mentioned above.

Assumption 1.5 (No-Arbitrage Principle)

There is no portfolio (x,y) with initial value $V(0) = 0$ such that $V(T) \geq 0$ with probability 1 and $V(T) > 0$ with non-zero probability, where $V(0), V(T)$ are given by (1.1).

In other words, if the initial value of a portfolio is zero, $V(0) = 0$ (no initial investment), and $V(T) \geq 0$ (no risk of a loss), then $V(T) = 0$ (no profit) with probability 1. This means that no investor can lock in a profit without risk and with no initial endowment. If a portfolio violating this principle did exist, we would say that an ***arbitrage*** opportunity were available.

Arbitrage opportunities rarely exist in practice. If and when they do, the gains are typically small as compared to the volume of transactions, making them beyond the reach of small investors. In addition, they can be more subtle than the examples above. Situations when the No-Arbitrage Principle is violated are typically short-lived and difficult to spot. The activities of investors (called arbitrageurs) pursuing arbitrage profits effectively make the market free of arbitrage opportunities.

The exclusion of arbitrage in the mathematical model is close enough to reality and turns out to be a critically important and fruitful assumption. Arguments based on the No-Arbitrage Principle are the main tools of financial mathematics.

1.3 One-Step Binomial Model

In this section we restrict ourselves to a very simple example, in which the stock price $S(T)$ takes only two different values. Despite its simplicity, this situation is sufficiently interesting to convey the flavour of the theory to be developed later on.

Example 1.6

Suppose that $S(0) = 100$ dollars and $S(T)$ can take two values,

$$S(T) = \begin{cases} 125 & \text{with probability } p, \\ 105 & \text{with probability } 1-p, \end{cases}$$

where $0 < p < 1$, while the bond prices are $A(0) = 100$ and $A(T) = 110$ dollars. Thus, the return K_S on stock will be 25% if stock goes up, or 5% if stock goes down. The risk-free return will be $K_A = 10\%$. The stock prices are represented as a tree in Figure 1.1.

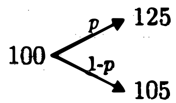

Figure 1.1 One-step binomial tree of stock prices

Observe that both stock prices at time T happen to be higher than that at time 0. Going 'up' or 'down' is relative to the other price at time T rather than to the price at time 0. In fact we could just as well allow the value in the 'up' scenario to be lower than the 'down' value. For instance, if the asset is a currency, some investors may regard a reduction in the exchange rate as favourable. In fact an increase in an exchange rate means a decrease in the converse rate. It is best to treat the references to 'up' and 'down' price movements as an abstract indication of two possible different outcomes.

In general, the choice of stock and bond prices in a binomial model is constrained by the No-Arbitrage Principle. Suppose that the possible up and down stock prices at time T are

$$S(T) = \begin{cases} S^u(T) & \text{with probability } p, \\ S^d(T) & \text{with probability } 1-p, \end{cases}$$

where $S^d(T) < S^u(T)$ and $0 < p < 1$.

Proposition 1.7

The restriction

$$\frac{S^d(T)}{S(0)} < \frac{A(T)}{A(0)} < \frac{S^u(T)}{S(0)},$$

has to be imposed on the model or else an arbitrage opportunity would arise.

Proof

Suppose that $\frac{A(T)}{A(0)} \le \frac{S^d(T)}{S(0)}$. In this case, at time 0:

- borrow the amount $S(0)$ risk free;
- buy one share of stock for $S(0)$.

This way, you will be holding a portfolio (x, y) with $x = 1$ shares of stock and $y = -\frac{S(0)}{A(0)}$ bonds. The time 0 value of this portfolio is

$$V(0) = 0.$$

At time T the value will become

$$V(T) = \begin{cases} S^u(T) - \frac{S(0)}{A(0)}A(T) & \text{if stock goes up,} \\ S^d(T) - \frac{S(0)}{A(0)}A(T) & \text{if stock goes down.} \end{cases}$$

The first of these two possible values is strictly positive, while the other one is non-negative, that is, $V(T)$ is a non-negative random variable such that $V(T) > 0$ with probability $p > 0$. The portfolio provides an arbitrage opportunity, violating the No-Arbitrage Principle (Assumption 1.5).

Now suppose that $\frac{A(T)}{A(0)} \ge \frac{S^u(T)}{S(0)}$. If this is the case, then at time 0:

- sell short one share for $S(0)$;
- invest the amount $S(0)$ risk free.

As a result, you will be holding a portfolio (x, y) with $x = -1$ and $y = \frac{S(0)}{A(0)}$, again of zero initial value,

$$V(0) = 0.$$

The final value of this portfolio will be

$$V(T) = \begin{cases} -S^u(T) + \frac{S(0)}{A(0)}A(T) & \text{if stock goes up,} \\ -S^d(T) + \frac{S(0)}{A(0)}A(T) & \text{if stock goes down,} \end{cases}$$

which is non-negative, with the second value being strictly positive. Thus, $V(T)$ is a non-negative random variable such that $V(T) > 0$ with probability $1 - p > 0$. Once again, this indicates an arbitrage opportunity, contrary to the No-Arbitrage Principle (Assumption 1.5). □

The common sense reasoning behind the above argument is straightforward: buy cheap assets and sell (or sell short) expensive ones, pocketing the difference.

1.4 Risk and Return

Let $A(0) = 100$ and $A(T) = 110$ dollars, as before, but $S(0) = 80$ dollars and

$$S(T) = \begin{cases} 100 & \text{with probability } 0.8, \\ 60 & \text{with probability } 0.2. \end{cases}$$

Suppose that you have \$10,000 to invest in a portfolio. You decide to buy $x = 50$ shares, which fixes the risk-free investment at $y = 60$. Then

$$V(T) = \begin{cases} 11,600 & \text{if stock goes up,} \\ 9,600 & \text{if stock goes down,} \end{cases}$$

$$K_V = \begin{cases} 16\% & \text{if stock goes up,} \\ -4\% & \text{if stock goes down.} \end{cases}$$

The *expected return*, that is, the mathematical expectation of the return on the portfolio is

$$E(K_V) = 16\% \times 0.8 - 4\% \times 0.2 = 12\%.$$

The *risk* of this investment is defined to be the standard deviation of the random variable K_V:

$$\sigma_V = \sqrt{(16\% - 12\%)^2 \times 0.8 + (-4\% - 12\%)^2 \times 0.2} = 8\%.$$

Let us compare this with investments in just one type of security. If $x = 0$, then $y = 100$, that is, the whole amount is invested risk free, then the return is known with certainty to be $K_A = 10\%$ and the risk as measured by the standard deviation is zero, $\sigma_A = 0$.

On the other hand, if $x = 125$ and $y = 0$, the entire amount being invested in stock, then

$$V(T) = \begin{cases} 12,500 & \text{if stock goes up,} \\ 7,500 & \text{if stock goes down,} \end{cases}$$

and $E(K_S) = 15\%$ with $\sigma_S = 20\%$.

Given the choice between two portfolios with the same expected return, any investor would obviously prefer that involving lower risk. Similarly, if the risk levels were the same, any investor would opt for higher expected return. However, in the case in hand, higher expected return is associated with higher risk. In such circumstances the choice depends on individual preferences. These issues will be discussed in Chapter 3, where we shall also consider portfolios consisting of several risky securities. The emerging picture will show the power of portfolio selection and portfolio diversification as tools for managing risk versus expected return.

Exercise 1.4

For the above stock and bond prices, design a portfolio with initial wealth of \$10,000 split fifty-fifty between stock and bonds. Compute the expected return and risk as measured by standard deviation.

1.5 Forward Contracts

A *forward contract* is an agreement to buy or sell a risky asset at a specified future time, known as the *delivery date*, for a price F fixed at the present moment, called the *forward price*. An investor who agrees to buy the asset is said to *enter into a long forward contract* or to *take a long forward position*. If an investor agrees to sell the asset, we speak of a *short forward contract* or a *short forward position*. No money is paid at the time when a forward contract is exchanged.

Example 1.8

Suppose that the forward price is \$80. If the market price of the asset (the *spot* price) turns out to be \$84 on the delivery date, then the long forward contract holder will gain \$4 by buying the asset for \$80, while the holder of a short forward contract will suffer a loss of \$4. However, if the spot price turns out to be \$75, then the holder of a long forward contract will have to pay \$80 for the asset, suffering a loss of \$5, whereas the short contract holder will gain \$5. In either case the loss of one party will be the gain of the other.

In general, the party holding a long forward contract with delivery date T will benefit if the future asset price $S(T)$ rises above the forward price F. If the asset price $S(T)$ falls below the forward price F, then the holder of a long forward contract will suffer a loss. The payoff for a long forward position is $S(T) - F$ (which can be positive, negative or zero). For a short forward position the payoff is $F - S(T)$.

Apart from stock and bonds, a portfolio held by an investor may contain forward contracts, in which case it will be described by a triple (x, y, z). Here x is the number of shares of stock and y the number of bonds, as before, and z is the number of forward contracts (positive for a long forward position and negative for a short position). Because no payment is due when a forward contract is exchanged, the initial value of such a portfolio is simply

$$V(0) = xS(0) + yA(0). \tag{1.2}$$

On the delivery date the value of the portfolio will become

$$V(T) = xS(T) + yA(T) + z(S(T) - F). \tag{1.3}$$

Having included forward contracts in the portfolio, we need to extend the No-Arbitrage Principle accordingly.

Assumption 1.9 (No-Arbitrage Principle)

There is no portfolio (x, y, z) that includes a position z in forward contracts and has initial value $V(0) = 0$ such that $V(T) \geq 0$ with probability 1 and $V(T) > 0$ with non-zero probability, where $V(0), V(T)$ are given by (1.2) and (1.3).

The No-Arbitrage Principle determines the forward price F. For simplicity we shall consider the simplest case when the risky security involves no cost of carry. A typical example is a stock paying no dividends. By contrast, a commodity will usually involve storage costs, whereas a foreign currency will earn interest, which can be regarded as negative cost of carry.

A long forward position guarantees that the asset can be bought for the forward price F at delivery. Alternatively, the asset can be bought now and held until delivery. However, if the initial cash outlay is to be zero, the purchase must be financed by a loan. The loan with interest, which will need to be repaid at the delivery date, is a candidate for the forward price. The following proposition shows that this is indeed the case.

Proposition 1.10

If the risky security involves no cost of carry, then the forward price must be

$$F = S(0)\frac{A(T)}{A(0)} = S(0)(1 + K_A) \tag{1.4}$$

or an arbitrage opportunity would exist otherwise.

Proof

Suppose that $F > S(0)(1 + K_A)$. Then, at time 0:

- borrow the amount $S(0)$;
- buy the asset for $S(0)$;
- enter into a short forward contract with forward price F and delivery date T.

The resulting portfolio $(1, -\frac{S(0)}{A(0)}, -1)$ consisting of stock, a risk-free position, and a short forward contract has initial value $V(0) = 0$. Then, at time T:

- close the short forward position by selling the asset for F;
- close the risk-free position by paying $\frac{S(0)}{A(0)}A(T) = S(0)(1 + K_A)$.

The final value of the portfolio,

$$V(T) = F - S(0)(1 + K_A) > 0$$

will be your profit, violating the No-Arbitrage Principle (Assumption 1.9).

On the other hand, if $F < S(0)(1 + K_A)$, then at time 0:

- sell short the asset for $S(0)$;
- invest this amount risk free;
- take a long forward position in stock with forward price F and delivery date T.

The initial value of this portfolio $(-1, \frac{S(0)}{A(0)}, 1)$ is $V(0) = 0$. Subsequently, at time T:

- receive the amount $\frac{S(0)}{A(0)}A(T) = S(0)(1+K_A)$ from the risk-free investment;
- buy the asset for F, closing the long forward position, and return the asset to the owner.

Your arbitrage profit will be

$$V(T) = S(0)(1 + K_A) - F > 0,$$

which once again violates the No-Arbitrage Principle (Assumption 1.9). It follows that the forward price must be $F = S(0)(1 + K_A)$. □

Exercise 1.5

Let $A(0) = 100$, $A(1) = 112$ and $S(0) = 34$ dollars. Is it possible to find an arbitrage opportunity if the forward price of stock is $F = 38.60$ dollars with delivery date 1?

Remark 1.11

Forward contracts make it possible to place Example 1.3 on a rigorous footing within the No-Arbitrage Principle framework. The price offered by either of the dealers A, B is a promise to exchange at rate d_A or d_B, respectively, these quotations being forward rates for instantaneous delivery. An arbitrage portfolio would be of the form $(x, y, z_1, z_2) = (0, 0, -1, 1)$ with the last two entries representing the positions z_1 taken with dealer A and z_2 with dealer B.

1.6 Call and Put Options

Let $A(0) = 100$, $A(T) = 110$, $S(0) = 100$ dollars and

$$S(T) = \begin{cases} 120 & \text{with probability } p, \\ 80 & \text{with probability } 1-p, \end{cases}$$

where $0 < p < 1$.

A *call option* with *strike price* \$100 and *exercise time* T is a contract giving the holder the right (but no obligation) to purchase a share of stock for \$100 at time T.

If the stock price falls below the strike price, the option will be worthless. There would be little point in buying a share for \$100 if its market price is \$80, and no one would want to exercise the right. Otherwise, if the share price rises to \$120, which is above the strike price, the option will bring a profit of \$20 to the holder, who is entitled to buy a share for \$100 at time T and may sell it immediately at the market price of \$120. This is known as *exercising* the option. The option may just as well be exercised simply by collecting the difference of \$20 between the market price of stock (the *spot* price) and the strike price. In practice, the latter is often the preferred method because no stock needs to change hands.

As a result, the payoff of the call option, that is, its value at time T is a random variable

$$C(T) = \begin{cases} 20 & \text{if stock goes up,} \\ 0 & \text{if stock goes down.} \end{cases}$$

In general

$$C(T) = \max(S(T) - X, 0),$$

where X is the strike price. Meanwhile, $C(0)$ will denote the value of the option at time 0, that is, the price for which the option can be bought or sold today.

Remark 1.12

At first sight a call option might resemble a long forward position. Both involve buying an asset at a future date for a price fixed in advance. An essential difference is that the holder of a long forward contract is committed to buying the asset for the fixed price, whereas the owner of a call option has the right but no obligation to do so. Another difference is that an investor will need to pay a premium to purchase a call option, whereas no payment is due when exchanging a forward contract.

In a market in which options are available, it is possible to invest in a portfolio (x, y, z) consisting of x shares of stock, y bonds and z options. The time 0 value of such a portfolio is

$$V(0) = xS(0) + yA(0) + zC(0). \tag{1.5}$$

At time T it will be worth

$$V(T) = xS(T) + yA(T) + zC(T). \tag{1.6}$$

The No-Arbitrage Principle needs, once again, to be extended to cover portfolios of this kind.

Assumption 1.13 (No-Arbitrage Principle)

There is no portfolio (x, y, z) that includes a position z in call options and has initial value $V(0) = 0$ such that $V(T) \geq 0$ with probability 1 and $V(T) > 0$ with non-zero probability, where $V(0), V(T)$ are given by (1.5) and (1.6).

We need to understand how to compute the price $C(0)$ of the call option at time 0 consistent with the absence of arbitrage opportunities. Because the holder of a call option has a certain right, but never an obligation, it is reasonable to expect that $C(0)$ will be positive: one needs to pay a premium to acquire this right. We shall see that the option price $C(0)$ can be found in two steps:

Step 1
Construct an investment in x stocks and y bonds such that the value of the investment at time T is the same as that of the option,

$$xS(T) + yA(T) = C(T),$$

no matter whether the stock price $S(T)$ goes up to \$120 or down to \$80. This is known as *replicating* the option.

Step 2
Compute the time 0 value of the investment in stock and bonds. It will be shown in Proposition 1.14 that it must be equal to the option price,

$$xS(0) + yA(0) = C(0),$$

because an arbitrage opportunity would exist otherwise. This step will be referred to as *pricing* or *valuing* the option.

Step 1 (Replicating the Option)
The time T value of the investment in stock and bonds will be

$$xS(T)+yA(T)=\begin{cases} x120+y110 & \text{if stock goes up,}\\ x80+y110 & \text{if stock goes down.}\end{cases}$$

The equality $xS(T)+yA(T)=C(T)$ between the two random variables can be written as

$$\begin{cases} x120+y110=20,\\ x80+y110=\ \ 0.\end{cases}$$

The first of these equations covers the case when the stock price goes up to \$120. The other one applies when stock drops down to \$80. Because we want the value of the investment in stock and bonds at time T to match exactly that of the option *no matter whether the stock price goes up or down*, these two equations are to be satisfied simultaneously. Solving for x and y, we find that

$$x=\frac{1}{2},\quad y=-\frac{4}{11}.$$

To replicate the option we need to buy $\frac{1}{2}$ a share of stock and take a short position of $-\frac{4}{11}$ in bonds (or borrow $\frac{4}{11}\times 100=\frac{400}{11}$ dollars in cash).

Step 2 (Pricing the Option)
We can compute the value of the investment in stock and bonds at time 0:

$$xS(0)+yA(0)=\frac{1}{2}\times 100-\frac{4}{11}\times 100\cong 13.6364$$

dollars. The following proposition shows that this must be equal to the price of the option.

Proposition 1.14

If the option can be replicated by investing in a portfolio (x,y) of stock and bonds, then $C(0)=xS(0)+yA(0)$, or else an arbitrage opportunity would exist.

Proof

Suppose that $C(0)>xS(0)+yA(0)$. If this is the case, then at time 0:

- issue and sell one option for $C(0)$ dollars;
- take a long position in the portfolio (x,y), which costs $xS(0)+yA(0)$ (for a call option this involves buying shares and borrowing cash).

The balance of these transactions is positive, $C(0)-xS(0)-yA(0)>0$. Invest this amount risk free. The resulting portfolio has initial value $V(0)=0$. Subsequently, at time T:

- if stock goes up, then settle the option by paying the difference of $S^u(T)-X$ between the spot price and the strike price; you will pay nothing if stock goes down; the cost to you will be $C(T)$, which covers both possibilities;
- close the position in stock and bonds, receiving the amount $xS(T)+yA(T)$.

The cash balance of these transactions will be zero, $-C(T)+xS(T)+yA(T)=0$, regardless of whether stock goes up or down (the portfolio replicates the option). But you will be left with the initial risk-free investment of $C(0)-xS(0)-yA(0)$ plus interest, thus realising an arbitrage opportunity, contrary to the No-Arbitrage Principle (Assumption 1.13).

On the other hand, if $C(0)<xS(0)+yA(0)$, then, at time 0:

- buy one option for $C(0)$ dollars;
- take a short position in the portfolio (x,y) (which involves buying bonds and short selling stock).

The cash balance of these transactions is positive, $-C(0)+xS(0)+yA(0)>0$, and can be invested risk free. In this way you will have constructed a portfolio with initial value $V(0)=0$. Subsequently, at time T:

- if stock goes up, then exercise the option, receiving the difference of $S^u(T)-X$ between the spot price and the strike price; you will receive nothing if stock goes down; your income will be $C(T)$, which covers both possibilities;
- close the position in stock and bonds, paying $xS(T)+yA(T)$.

The cash balance of these transactions will be zero, $C(T)-xS(T)-yA(T)=0$, regardless of whether stock goes up or down. But you will be left with an arbitrage profit resulting from the risk-free investment of $-C(0)+xS(0)+yA(0)$ plus interest, contradicting the No-Arbitrage Principle (Assumption 1.13). □

Here we can see once again that the arbitrage strategy follows a common-sense pattern: sell (or sell short if necessary) expensive securities and buy inexpensive ones, as long as all your financial obligations arising in the process can be discharged regardless of what happens in the future.

Proposition 1.14 implies that today's price of the option must be

$$C(0)=\frac{1}{2}S(0)-\frac{4}{11}A(0)\cong 13.6364$$

dollars. Anyone who would offer to sell the option for less or to buy it for more than this price would be creating an arbitrage opportunity, which amounts to handing out free money.

Remark 1.15

Note that the probabilities p and $1-p$ of stock going up or down are irrelevant in pricing or replicating the option. This is a remarkable feature, and by no means a coincidence.

Remark 1.16

Options may appear to be superfluous in a market in which they can be replicated by stock and bonds. In the simplified one-step model this is indeed a valid objection. However, in a situation involving multiple time steps (or continuous time) replication becomes a much more onerous task. It requires rebalancing the positions in stock and bonds at every time instant at which there is a change in their prices, resulting in considerable management and transaction costs. In some cases it may not even be possible to replicate an option precisely. This is why the majority of investors prefer to buy or sell options, replication being normally undertaken by specialised dealers and institutions.

Exercise 1.6

Let the bond and stock prices $A(0)$, $A(T)$, $S(0)$, $S(T)$ be as above. Compute the price $C(0)$ of a call option with exercise time T and a) strike price \$90, b) strike price \$110.

Exercise 1.7

Let the prices $A(0)$, $S(0)$, $S(T)$ be as above. Compute the price $C(0)$ of a call option with strike price \$100 and exercise time T if a) $A(T) = 105$ dollars, b) $A(T) = 115$ dollars.

A ***put*** option with strike price \$100 and exercise time T gives the right (but no obligation) to ***sell*** one share of stock for \$100 at time T. This kind of option is worthless if the stock goes up, but it brings a profit otherwise, the payoff being

$$P(T) = \begin{cases} 0 & \text{if stock goes up,} \\ 20 & \text{if stock goes down,} \end{cases}$$

given that the possible values of $S(T)$ are the same as above. In general,

$$P(T) = \max(X - S(T), 0),$$

where X is the strike price. The notion of a portfolio can be extended to allow positions in put options, denoted by z, as before.

The replicating and pricing procedure for puts follows the same pattern as for call options. In particular, the price $P(0)$ of the put option is equal to the time 0 value of a replicating investment in stock and bonds.

Remark 1.17

There is some similarity between a put option and a short forward position: both involve selling an asset for a fixed price at a certain time in the future. However, an essential difference is that the holder of a short forward contract is committed to selling the asset for the fixed price, whereas the owner of a put option has the right but no obligation to sell. Moreover, an investor who wants to buy a put option will have to pay for it, whereas no payment is involved when a forward contract is exchanged.

Exercise 1.8

Let the bond and stock prices $A(0)$, $A(T)$, $S(0)$, $S(T)$ be as above. Formulate a version of the No-Arbitrage Principle for portfolios containing puts and compute the price $P(0)$ of a put option with strike price \$100.

The general properties of options and forward contracts will be discussed in Chapters 4 and 5. In Chapters 6 and 7 the pricing and replicating schemes will be extended to more complicated though still discrete market models, as well as to other financial instruments.

1.7 Foreign Exchange

Foreign currency is an important example of a risky security. The basic novelty as compared to stock is that each unit of foreign currency held in a portfolio generates additional risk-free income. Here we assume that the currency is kept in a bank account or invested in risk-free bonds. The ability to generate additional income, also present in the stock market if dividends are paid to the shareholders, needs to be accounted for when considering derivative securities written on such assets.

First we begin with an adjustment to formula (1.4) for forward prices. Note that to have one unit of foreign currency at time T it is sufficient to buy a fraction of the unit, namely $\frac{A_f(0)}{A_f(T)} = \frac{1}{1+K_f}$, where the subscript f designates foreign currency bond prices and the return on them. Therefore, a candidate for the forward price (more precisely, *forward rate of exchange*) is

$$F = S(0)\frac{1+K_{\mathrm{h}}}{1+K_{\mathrm{f}}}, \tag{1.7}$$

where $1+K_{\mathrm{h}} = \frac{A_{\mathrm{h}}(T)}{A_{\mathrm{h}}(0)}$, the subscript h indicating the home currency.

Exercise 1.9

Give an arbitrage proof of formula (1.7).

Exercise 1.10

Suppose that the return on dollar bonds is $K_{\$} = 5\%$, the present exchange rate is $S(0) = 1.6$ dollars to a pound, and the forward rate for delivery date $T = 1$ is $F = 1.50$ dollars to a pound. How much should a sterling bond cost today if it promises to pay £100 at time 1?

Our next task is to find prices of options written on foreign currency. This is best illustrated by an example.

Example 1.18

Consider pound sterling as a security on the US dollar market. Suppose that the return on US dollar bonds is $K_{\$} = 5\%$ and on British pound bonds it is $K_{£} = 3\%$. Consider a call option on pound sterling with strike price $X = 1.64$ dollars to a pound within a binomial model with $S(0) = 1.62$, $S^{\mathrm{u}}(T) = 1.84$ and $S^{\mathrm{d}}(T) = 1.46$ dollars to a pound. The replication step requires finding x, y such that

$$\begin{aligned} x(1+K_{£})S^{\mathrm{u}}(T) + y(1+K_{\$}) &= S^{\mathrm{u}}(T) - X, \\ x(1+K_{£})S^{\mathrm{d}}(T) + y(1+K_{\$}) &= 0. \end{aligned}$$

Note the difference as compared to the replication of stock options, where the factor $1+K_{£}$ representing the income generated by the underlying security (here pound sterling) was absent. The solution of this system of equations is $x = 0.5110$ pounds and $y = -0.7318$ dollars, and consequently

$$C(0) = xS(0) + y = 0.0960$$

dollars.

Exercise 1.11

Find the price of a put written on pound sterling with strike $X = 1.71$ dollars to a pound using the data of Example 1.18.

1.8 Managing Risk with Options

The availability of options and other derivative securities extends the possible investment scenarios. Suppose that your initial wealth is \$1,000 and compare the following two investments in the setup of Section 1.6:

- buy 10 shares; at time T they will be worth

$$10 \times S(T) = \begin{cases} 1,200 & \text{if stock goes up,} \\ 800 & \text{if stock goes down;} \end{cases}$$

or

- buy $1,000/13.6364 \cong 73.3333$ call options; in this case your final wealth will be

$$73.3333 \times C(T) \cong \begin{cases} 1,466.67 & \text{if stock goes up,} \\ 0.00 & \text{if stock goes down.} \end{cases}$$

If stock goes up, the investment in options will produce a much higher return than shares, namely about 46.67%. However, it will be disastrous otherwise: you will lose all your money. Meanwhile, when investing in shares, you would just gain or lose 20%. Without specifying the probabilities we cannot compute the expected returns or standard deviations. Nevertheless, one would readily agree that investing in options is more risky than in stock. This can be exploited by adventurous investors.

Exercise 1.12

In the above setting, find the final wealth of an investor whose initial capital of \$1,000 is split fifty-fifty between stock and options.

Options can also be employed to reduce risk. Consider an investor planning to purchase stock in the future. The share price today is $S(0) = 100$ dollars, but the investor will only have funds available at a future time T, when the share price will become

$$S(T) = \begin{cases} 160 & \text{with probability } p, \\ 40 & \text{with probability } 1-p, \end{cases}$$

for some $0 < p < 1$. Assume, as before, that $A(0) = 100$ and $A(T) = 110$ dollars, and compare the following two strategies:

- wait until time T, when the funds become available, and purchase the stock for $S(T)$;

or

- at time 0 borrow money to buy a call option with strike price \$100; then, at time T repay the loan with interest and purchase the stock, exercising the option if the stock price goes up.

The investor will be open to considerable risk if she chooses to follow the first strategy. On the other hand, following the second strategy, she will need to borrow $C(0) \cong 31.8182$ dollars to pay for the option. At time T she will have to repay \$35 to clear the loan and may use the option to purchase the stock, hence the cost of purchasing one share will be

$$S(T) - C(T) + 35 = \begin{cases} 135 & \text{if stock goes up,} \\ 75 & \text{if stock goes down.} \end{cases}$$

Clearly, the risk is reduced, the spread between these two figures being narrower than before.

Exercise 1.13

Compute the risk as measured by the standard deviation of the cost of buying one share with and without the option if a) $p = 0.25$, b) $p = 0.5$, c) $p = 0.75$.

Exercise 1.14

Show that the risk (as measured by the standard deviation of the cost of buying one share) of the above strategy involving an option is half of that when no option is purchased, no matter what the probability $0 < p < 1$ is.

If two options are bought, then the risk will be reduced to nil:

$$S(T) - 2 \times C(T) + 70 = 110 \text{ with probability 1.}$$

This strategy turns out to be equivalent to a long forward contract, since the forward price of the stock is exactly \$110 (see Section 1.5). It is also equivalent to borrowing money to purchase a share for \$100 today and repaying \$110 to clear the loan at time T.

Case 1: Discussion

The first task is to recognise the nature of the problem. Since the UK company will have £64,000 at their disposal in a year's time to purchase the piece of equipment for \$100,000, the dollar price being guaranteed by the producer, the source of concern is the unknown exchange rate at the time of purchase. The amount budgeted for may turn out insufficient to purchase the equipment if the exchange rate falls below 1.5625 dollars to a pound.

To ensure that the project is successful a financial package needs to be designed to hedge the interest rate risk. Having followed this chapter, we are familiar with some tools to achieve this goal. It will be necessary to collect some additional market data to apply these tools.

The first approach is to employ forward contracts to convert pounds into dollars at the end of the year. The forward price F of pound sterling can be found using the methods discussed in Section 1.7. In order to do so, we need to look up the pound sterling and US dollar bond prices, or, equivalently, the corresponding risk-free returns. Suppose that these turn out to be $K_\$ = 5\%$ and $K_£ = 3\%$. This gives $F = 1.6515$ dollars to a pound. Converting £64,000 at this rate will give \$105,693.20. The company will be able to purchase the piece of equipment, leaving it with a surplus of \$5,693.20, irrespective of what happens to the exchange rate.

As an alternative to forward contracts, since the company will need to convert their currency into dollars, in effect selling pound sterling, we could employ put options on pound sterling.

This requires building a model of future random exchange rates. At this point all we can do is to use the single step binomial model. We know the spot rate $S(0) = 1.62$ dollars to a pound and the risk-free returns $K_\$ = 5\%$ and $K_£ = 3\%$. The two future rates $S^{\mathrm{u}}(1), S^{\mathrm{d}}(1)$ need to be calibrated to match market data. To this end we can use the prices of some traded options on pound sterling. Suppose that calls with strikes 1.6 and 1.7 dollars to a pound, selling at 1.1178 and 0.0624 dollars to a pound, are the most liquid (and therefore considered the most representative) options traded. Since these call prices can be expressed in terms of $S^{\mathrm{u}}(1), S^{\mathrm{d}}(1)$ just like in Example 1.18, we obtain a system of two equations, which can be solved to get $S^{\mathrm{u}}(1) \cong 1.8126$ and $S^{\mathrm{d}}(1) \cong 1.4273$ dollars to a pound.

Having calibrated the model, we need to decide the strike rate X for the put option to be used to hedge the interest rate risk. The strike X affects the option price P, which we shall denote by $P(X)$ for clarity. We need to purchase 64,000 puts to convert the available sum of £64,000 into dollars. This will cost $P(X) \times 64,000$ dollars, which we need to borrow. Suppose that the company's bank quotes a 10% interest rate for a US dollar loan. At the end of the year we shall need to repay $P(X) \times 64,000 \times 1.1$ dollars in addition to the price of the piece of equipment. This gives an equation for the strike price:

$$100,000 + P(X) \times 64,000 \times 1.1 = X \times 64,000.$$

The solution is $X \cong 1.6680$ pounds to a dollar. The price of a put is $P(X) \cong 0.0959$ dollars. We need to borrow $P(X) \times 64,000 \cong 6,135.32$ dollars to purchase the options, and will have to repay \$6,748.85 including interest at the end of the year. If the exchange rate goes up, we shall sell pound sterling at the market price with an additional profit of $64,000 \times 1.8126 - 106,748.85 \cong 9,259.81$ dollars. If the rate goes down, we shall exercise the options and break even.

The strategy involving options does not look as attractive as forward contracts. Hedging with puts will also enable the company to purchase the piece of equipment irrespective of any changes in the exchange rate, but the end financial result is unpredictable, depending on the fluctuations of the rate. A practical problem could be that put options with the ideal strike rate $X \cong 1.6680$ pounds to a dollar may not be available via exchange trading, necessitating an over-the-counter (OTC) transaction. Another obvious weakness of this approach to hedging with options is connected with the simplicity of the model of exchange rates. The case is certainly worth revisiting when we acquire better models. On the other hand, when using forwards, we did not need to adopt any particular model of interest rates, so the result is much more robust.