7
General Discrete Time Models

Case 7

Turning back again to our personal pension fund, we know that investing in risky securities has an advantage of high return, even though facing the risk is not desired. Risk can be managed, at least partially, by means of derivative securities. We have employed options written on one stock. However, portfolio theory taught us about the advantages of investing in many risky assets. This reduces risk as a result of diversification, but we would have to cope with many underlying assets. Here, the first problem is concerned with building an appropriate model. As a first step in this direction, we could tackle just two stocks.

7.1 Trinomial Tree Model

A natural generalisation of the binomial tree model extends the range of possible values of the one-step returns K to three. The idea is to allow the price not only to move up or down, but also to take an intermediate value at any given step. A natural probability space would be $\Omega = \{\mathrm{u}, \mathrm{m}, \mathrm{d}\}$ with the probabilities p, q, $1-p-q$ of outcomes $\mathrm{u}, \mathrm{m}, \mathrm{d}$ determined by the choice of two numbers p, q such that $0 < p, q, p+q < 1$.

M. Capiński, T. Zastawniak, *Mathematics for Finance*,
Springer Undergraduate Mathematics Series,

Condition 7.1

The return K is a random variable of the form

$$K^{\omega} = \begin{cases} U & \text{if } \omega = \mathrm{u}, \\ M & \text{if } \omega = \mathrm{m}, \\ D & \text{if } \omega = \mathrm{d}, \end{cases}$$

where $D < M < U$.

This means that U and D are the returns corresponding to the upward and downward price movements, as before, whereas M is the return on the intermediate price movement. As in the case of the binomial tree model, if there is more than one risky asset, the condition $D < M < U$ may need to be relaxed.

Condition 7.2

The one-step return R on a risk-free investment is the same at each time step and

$$D < R < U.$$

Since $S(1)/S(0) = 1 + K$, Condition 7.1 implies that $S(1)$ takes three different values,

$$S^{\omega}(1) = \begin{cases} S^{\mathrm{u}}(1) = S(0)(1+U) & \text{if } \omega = \mathrm{u}, \\ S^{\mathrm{m}}(1) = S(0)(1+M) & \text{if } \omega = \mathrm{m}, \\ S^{\mathrm{d}}(1) = S(0)(1+D) & \text{if } \omega = \mathrm{d}. \end{cases}$$

Exercise 7.1

Prove that the condition $D < R < U$ is equivalent to the absence of arbitrage.

7.1.1 Pricing

In the binomial tree model we successfully used the risk-neutral probability for contingent claim pricing. In the trinomial model the condition $\mathbb{E}_*(K) = R$ (cf. Definition 6.8) for a risk-neutral probability $(p_*, q_*, 1 - p_* - q_*)$ becomes

$$p_*(U - R) + q_*(M - R) + (1 - p_* - q_*)(D - R) = 0. \tag{7.1}$$

This is one equation with two variables. In general, it can have none or infinitely many solutions such that $p_*, q_*, 1 - p_* - q_* \in (0, 1)$.

Exercise 7.2

Show that the condition $D < R < U$ guarantees that there are solutions to (7.1) such that all three probabilities $p_*, q_*, 1 - p_* - q_*$ belong to $(0, 1)$.

The triple $(p_*, q_*, 1 - p_* - q_*)$, regarded as a vector in $\mathbb{R}^3$, is orthogonal to the vector with coordinates $(U - R, M - R, D - R)$ representing the possible one-step gains (or losses) expressed by means of returns for an investor holding a single share of stock, the purchase of which was financed by a cash loan. This means that $(p_*, q_*, 1 - p_* - q_*)$ lies on the intersection of the triangle $\{(a, b, c) : a, b, c \geq 0, a + b + c = 1\}$ and the plane orthogonal to the gains vector $(U - R, M - R, D - R)$, as in Figure 7.1. Condition 7.2 guarantees that the intersection is non-empty, since the line containing the vector $(U - R, M - R, D - R)$ does not pass through the positive octant. In this case there are infinitely many risk-neutral probabilities, the intersection being a line segment.

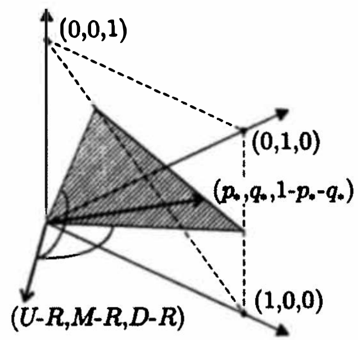

Figure 7.1 Geometric interpretation of risk-neutral probabilities p_*, q_*

Another interpretation of condition (7.1) for the risk-neutral probability is illustrated in Figure 7.2. If masses p_*, q_* and $1 - p_* - q_*$ are attached at the points with coordinates U, M and D on the real axis, then the centre of mass will be at R.

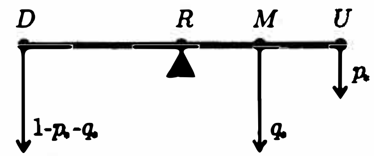

Figure 7.2 Barycentric interpretation of risk-neutral probabilities p_*, q_*

Exercise 7.3

Let $U = 0.2$, $M = 0$, $D = -0.1$ and $R = 0$. Find all risk-neutral probabilities.

In Chapter 6 we discussed the pricing of derivatives in the binomial model by two methods: replication or the discounted expected payoff under the risk-neutral probability. In the case of the trinomial model there are many different risk-neutral probabilities, which may produce different values for the corresponding discounted expected payoff. However, an option trading simultaneously at two different prices would create an instant arbitrage opportunity. The replication method, on the other hand, would also lead to difficulties. A contingent claim $H(1)$ on a probability space $\Omega = \{u, m, d\}$ takes three values $H^u(1)$, $H^m(1)$, $H^d(1)$. If we seek a portfolio (x, y) with value $V(1)$ equal to $H(1)$, this will yield three simultaneous equations

$$\begin{cases} xS^u(1) + yA(0)(1+R) = H^u(1), \\ xS^m(1) + yA(0)(1+R) = H^m(1), \\ xS^d(1) + yA(0)(1+R) = H^d(1), \end{cases}$$

in two variables x, y, which, in general, may have no solution. Pricing by replication would fail in such cases.

7.1.2 Two Assets

As we have seen in Chapter 6, the binomial model is well suited for modelling the price movements of a single asset. However, it is insufficient if an additional risky asset is involved. Working with a single time step, denote by $S_1(n)$, $S_2(n)$, $n = 0, 1$, the prices of two stocks. If we insist on describing the price movements by $S_j(1) = S_j(0)(1+K_j)$, $j = 1, 2$, with each K_j being non-constant and defined on the same probability space $\Omega = \{u, d\}$, then the random variables K_j must be perfectly correlated with correlation 1 or -1.

Exercise 7.4

Prove this fact.

Perfectly correlated returns are definitely not acceptable in a realistic model. Working in a trinomial model changes everything. Suppose that

$$\begin{cases} K_j^u = U_j, \\ K_j^m = M_j, \\ K_j^d = D_j. \end{cases}$$

Then, provided some technical conditions are satisfied, we can solve the pricing problem. Replication of a contingent claim $H(1)$ is now feasible. For a portfolio (x_1, x_2, y) consisting of three assets, with x_j expressing the position in stock j, we obtain the system of three equations

$$\begin{cases} x_1 S_1^u(1) + x_2 S_2^u(1) + yA(0)(1+R) = H^u(1), \\ x_1 S_1^m(1) + x_2 S_2^m(1) + yA(0)(1+R) = H^m(1), \\ x_1 S_1^d(1) + x_2 S_2^d(1) + yA(0)(1+R) = H^d(1), \end{cases} \tag{7.2}$$

in three unknowns x_1, x_2, y, so the system can have a single solution. The risk-neutral probability approach is also feasible. If we require that both returns should satisfy the condition $\mathbb{E}_*(K_j) = R$, we end up with a system of two equations for p^*, q^*:

$$\begin{cases} p_*(U_1 - R) + q_*(M_1 - R) + (1 - p_* - q_*)(D_1 - R) = 0, \\ p_*(U_2 - R) + q_*(M_2 - R) + (1 - p_* - q_*)(D_2 - R) = 0. \end{cases} \tag{7.3}$$

Proposition 7.3

There is a unique solution to the system of equations (7.3) determining the risk-neutral probability if and only if the replication system (7.2) has a unique solution for each claim.

Exercise 7.5

Give a proof of Proposition 7.3.

Example 7.4

Assume $S_1(0) = S_2(0) = 30$, and $U_1 = 20\%$, $M_1 = 10\%$, $D_1 = -10\%$, $U_2 = -5\%$, $M_2 = 15\%$, $D_2 = 10\%$. Assume that $R = 5\%$. Consider a call option on S_1 with strike price 32 so that $C^u(1) = 4$, $C^m(1) = 1$, $C^d(1) = 0$.

The replicating portfolio is $x_1 = 0.25926$, $x_2 = -0.37037$, $y = 4.97354$. The risk-neutral probabilities are $p_* = 0.16667$, $q_* = 0.38889$, $1 - p_* - q_* = 0.44444$.

We can compute the price of the option by taking the initial value of the replicating portfolio or the discounted expected payoff using the risk-neutral probabilities. Each method gives $C(0) = 1.6402$.

However, the existence of unique solutions to the above systems is insufficient as the following example shows.

Example 7.5

Take the same data as in the preceding example with different returns for stock S_2, namely let $U_2 = 10\%$, $M_2 = 5\%$, $D_2 = -20\%$. Both systems have unique solutions but the numbers $p_* = 2$, $q_* = -0.83333$, $1 - p_* - q_* = -0.166667$ do not represent probabilities. The replicating portfolio is now $x_1 = 1.55556$, $x_2 = -1.11111$, $y = -14.5032$ and the option price given by each method is the same but negative: $C(0) = -1.2698$. This of course violates the No-Arbitrage Principle.

Later in this chapter we prove two theorems that give a proper foundation for the pricing problem, consistent with the No-Arbitrage Principle.

Exercise 7.6

Suppose the returns are $U_1 = 15\%$, $M_1 = 5\%$, $D_1 = -10\%$, $U_2 = 20\%$, $M_2 = 12\%$, $D_2 = -20\%$, $R = 5\%$. Find the risk-neutral probabilities.

7.2 General Model

This section is devoted to describing the so-called primary market consisting of basic securities (also called underlying securities) such as stocks, in addition to the money market account. This will then be extended to include contingent claims (derivative securities). Stock prices are given exogenously and derivatives prices will be obtained as endogenous quantities within the model of underlying assets.

As in Chapter 6 we assume that time runs in steps of fixed length h. For various time-dependent quantities we shall simplify the notation by writing n in place of the time $t = nh$ of the nth step. For example, we shall write $A(n)$ in place of $A(nh)$.

Suppose that m risky assets are traded. These will be referred to as stocks. Their prices at time $n = 0, 1, 2, \ldots$ are denoted by $S_1(n), \ldots, S_m(n)$. In addition, investors have at their disposal a risk-free asset. Unless stated otherwise, we take the initial level of the risk-free investment to be one unit of the home currency, $A(0) = 1$. Because the money market account can be manufactured using bonds (see Chapter 2), we shall frequently refer to a risk-free investment as a position in bonds, finding it convenient to think of $A(n)$ as the bond price at time n.

The positions in risky assets number $1, \ldots, m$ will be denoted by $x_1, \ldots, x_m$, respectively, and the risk-free position by y. The wealth of an investor holding a portfolio of such positions at time n will be

$$V(n) = \sum_{j=1}^{m} x_j S_j(n) + yA(n).$$

We introduce some natural assumptions.

Assumption 7.6 (Randomness)

The future stock prices $S_1(n), \ldots, S_m(n)$ are random variables for any $n = 1, 2, \ldots$. The future bond prices $A(n)$ for any $n = 1, 2, \ldots$ are known numbers.

Assumption 7.7 (Positivity of Prices)

All stock and bond prices are strictly positive,

$$S(n) > 0 \quad \text{and} \quad A(n) > 0 \quad \text{for } n = 0, 1, 2, \ldots .$$

Assumption 7.8 (Divisibility, Liquidity and Short Selling)

An investor may buy, sell and hold any number x_k of stock shares of each kind $k = 1, \ldots, m$ and take any risk-free position y (whether integer or fractional, negative, positive or zero) meaning that

$$x_1, \ldots, x_m, y \in \mathbb{R}.$$

Assumption 7.9 (Discrete Unit Prices)

For each $n = 0, 1, 2, \ldots$ the share prices $S_1(n), \ldots, S_m(n)$ are random variables taking only finitely many values. Mathematically this can be obtained by requiring that the probability space Ω be finite.

7.2.1 Investment Strategies

The positions held by an investor in the risky and risk-free assets can be altered at any time step by selling some assets and investing the proceeds in other assets. In real life, cash can be taken out of the portfolio for consumption or injected from other sources. Nevertheless, we shall assume that no consumption or injection of funds takes place in our model to keep things as simple as possible.

Decisions made by an investor on when to rebalance the portfolio and how many assets to buy or sell are based on the information currently available. We are going to exclude the unlikely possibility that investors could foresee the future, as well as the somewhat more likely (but illegal) one that they will act on insider information. However, all the historical information about the market, up to and including the time instant when a particular trading decision is executed, will be freely available.

Example 7.10

Let $m = 2$ and suppose that

$$\begin{aligned} S_1(0) &= 60, & S_1(1) &= 65, & S_1(2) &= 75, \\ S_2(0) &= 20, & S_2(1) &= 15, & S_2(2) &= 25, \\ A(0) &= 100, & A(1) &= 110, & A(2) &= 121, \end{aligned}$$

in a certain market scenario. At time 0 an initial wealth of $V(0) = 3{,}000$ dollars is invested in a portfolio consisting of $x_1(1) = 20$ shares of stock number one, $x_2(1) = 65$ shares of stock number two, and $y(1) = 5$ bonds. Our notational convention will be to use 1 rather than 0 as the argument to denote the positions $x_1(1)$, $x_2(1)$ and $y(1)$ created at time 0 so as to reflect the fact that this portfolio will be held over the first time step. At time 1 this portfolio will be worth $V(1) = 20 \times 65 + 65 \times 15 + 5 \times 110 = 2{,}825$ dollars. At that time the number of assets can be altered by buying or selling some of them, as long as the total value remains \$2,825. For example, we could form a new portfolio consisting of $x_1(2) = 15$ shares of stock one, $x_2(2) = 94$ shares of stock two, and $y(2) = 4$ bonds, which will be held during the second time step. The value of this portfolio will be $V(2) = 15 \times 75 + 94 \times 25 + 4 \times 121 = 3{,}959$ dollars at time 2, when the positions in stock and bonds can be adjusted once again, as long as the total value remains \$3,959, and so on. However, if no adjustments are made to the original portfolio, then it will be worth \$2,825 at time 1 and \$3,730 at time 2.

Definition 7.11

A *portfolio* is a vector $(x_1(n), \ldots, x_m(n), y(n))$ indicating the number of shares and bonds held by an investor between times $n-1$ and n. A sequence of portfolios indexed by $n = 1, 2, \ldots$ is called an *investment strategy*. The *wealth* of an investor or the *value of the strategy* at time $n \geq 1$ is

$$V(n) = \sum_{j=1}^{m} x_j(n) S_j(n) + y(n) A(n).$$

At time $n = 0$ the *initial wealth* is given by

$$V(0) = \sum_{j=1}^{m} x_j(1) S_j(0) + y(1) A(0).$$

Strategies employing the underlying assets only will sometimes be referred to as (x, y)-strategies.

The contents of a portfolio can be adjusted by buying or selling some assets at any time step, as long as the current value of the portfolio remains unaltered.

Definition 7.12

An investment strategy is called *self-financing* if the portfolio constructed at time $n \geq 1$ to be held over the next time step $n+1$ is financed entirely by the current wealth $V(n)$, that is,

$$\sum_{j=1}^{m} x_j(n+1) S_j(n) + y(n+1) A(n) = V(n). \tag{7.4}$$

Example 7.13

Let the stock and bond prices be as in Example 7.10. Suppose that an initial wealth of $V(0) = 3{,}000$ dollars is invested by purchasing $x_1(1) = 18.22$ shares of the first stock, short selling $x_2(1) = -16.81$ shares of the second stock, and buying $y(1) = 22.43$ bonds. The time 1 value of this portfolio will be $V(1) = 18.22 \times 65 - 16.81 \times 15 + 22.43 \times 110 = 3{,}399.45$ dollars. The investor will benefit from the drop in the price of the shorted stock. This example illustrates the fact that portfolios containing fractional or negative numbers of assets are allowed.

As we have already assumed, we do not impose any restrictions on the numbers $x_1(n), \ldots, x_m(n), y(n)$. The fact that they can take non-integer values is referred to as *divisibility*. Negative $x_j(n)$ means that stock number j is *sold short* (in other words, a short position is taken in stock j), negative $y(n)$ corresponds to borrowing cash (taking a short position in the money market, for example, by issuing and selling a bond). The absence of any bounds on the size of these numbers means that the market is *liquid*, that is, any number of assets of each type can be purchased or sold at any time.

In practice some measures to control short selling may be implemented by stock exchanges. Typically, investors are required to deposit a certain percentage of the short sale as a collateral to cover possible losses. If their losses exceed the deposit, the position must be closed. The deposit creates a burden on the portfolio, particularly if it earns no interest for the investor. However, restrictions of this kind may not concern dealers who work for major financial institutions holding a large number of shares deposited by smaller investors. These shares may be borrowed internally in lieu of short selling.

Example 7.14

We continue assuming that stock prices follow the scenario in Example 7.10. Suppose that 20 shares of the first stock are sold short, $x_1(1) = -20$. The investor will receive $20 \times 60 = 1,200$ dollars in cash, but has to pay a security deposit of, say 50%, that is, \$600. One time step later she will suffer a loss of $20 \times 65 - 1,200 = 100$ dollars. This is subtracted from the deposit and the position can be closed by withdrawing the balance of $600 - 100 = 500$ dollars. On the other hand, if 60 shares of the second stock are shorted, that is, $x_2(1) = -60$, then the investor will make a profit of $1,200 - 60 \times 15 = 300$ dollars after one time step. The position can be closed with final wealth $600 + 300 = 900$ dollars. In both cases the final balance should be reduced by $600 \times 0.1 = 60$ dollars, the interest that would have been earned on the amount deposited, had it been invested in the money market.

An investor constructing a portfolio at time n has no knowledge of future stock prices. Investment decisions can be based on the information provided by the market to date and the resulting beliefs concerning the future. As in the case of the binomial model in Chapter 6, this can be formalised in terms of partitions. The Definition 6.1 of a partition extends, without any changes, to the case of an arbitrary finite probability space Ω.

The partition capturing the knowledge provided by the prices of several stocks up to and including time n can be constructed as follows. Take any $\omega_1 \in \Omega$ and consider the prices of all stocks at all times $0, 1, \ldots, n$ along scenario ω_1. Then consider the set B_1 of all scenarios $\omega \in \Omega$ such that the prices of all stocks at all times $0, 1, \ldots, n$ along scenario ω are the same as those along scenario ω_1. Next take any $\omega_2 \in \Omega \setminus B_1$ and repeat this procedure to obtain another subset B_2 of Ω. Repeat this again until all elements of Ω are exhausted. The resulting sets $B_1, B_2, \ldots$ form a partition of Ω, which will be denoted by $\mathcal{P}_n$.

We are now ready to give a formal definition of strategies that do not involve any knowledge of future stock prices.

Definition 7.15

An investment strategy is called *predictable* if for each $n = 0, 1, 2, \ldots$ the portfolio $(x_1(n+1), \ldots, x_m(n+1), y(n+1))$ constructed at time n is constant on each of the components of partition $\mathcal{P}_n$.

The next proposition shows that the position in the risk-free asset in a predictable self financing strategy is determined by the initial wealth and the positions in the risky assets.

Proposition 7.16

Given the initial wealth $V(0)$ and a predictable sequence $(x_1(n), \ldots, x_m(n))$, $n = 1, 2, \ldots$ of positions in risky assets, it is always possible to find a sequence $y(n)$ of risk-free positions such that $(x_1(n), \ldots, x_m(n), y(n))$ is a predictable self-financing investment strategy.

Proof

Put

$$y(1) = \frac{V(0) - x_1(1)S_1(0) - \cdots - x_m(1)S_m(0)}{A(0)}$$

and then compute

$$V(1) = x_1(1)S_1(1) + \cdots + x_m(1)S_m(1) + y(1)A(1).$$

Next, let

$$y(2) = \frac{V(1) - x_1(2)S_1(1) - \cdots - x_m(2)S_m(1)}{A(1)},$$

finding

$$V(2) = x_1(2)S_1(2) + \cdots + x_m(2)S_m(2) + y(2)A(2),$$

and so on. This clearly defines a self-financing strategy. The strategy is predictable because $y(n+1)$ can be expressed in terms of stock and bond prices up to time n:

$$y(n+1) = \frac{V(n) - x_1(n+1)S_1(n) - \cdots - x_m(n+1)S_m(n)}{A(n)},$$

which completes the proof. □

Exercise 7.7

Find the number of bonds $y(1)$ and $y(2)$ held by an investor during the first and second steps of a predictable self-financing investment strategy with initial value $V(0) = 200$ dollars and risky asset positions

$$x_1(1) = 35.24, \quad x_1(2) = -40.50,$$
$$x_2(1) = 24.18, \quad x_2(2) = \quad 10.13,$$

if the prices of assets follow the scenario in Example 7.10. Also find the time 1 value $V(1)$ and time 2 value $V(2)$ of this strategy.

Example 7.17

Once again, suppose that the stock and bond prices follow the scenario in Example 7.10. If an amount $V(0) = 100$ dollars were invested in a portfolio with $x_1(1) = -12$, $x_2(1) = 31$ and $y(1) = 2$, then it would lead to insolvency, since the time 1 value of this portfolio is negative, $V(1) = -12 \times 65 + 31 \times 15 + 2 \times 110 = -95$ dollars .

Such a portfolio might be impossible to construct in practice. No short position would be allowed unless it can be closed at any time and in any scenario (if necessary, by selling other assets in the portfolio to raise cash).

Definition 7.18

A strategy is called ***admissible*** if it is self-financing, predictable, and for each $n = 0, 1, 2, \ldots$

$$V(n) \geq 0$$

with probability 1.

Exercise 7.8

Consider a market consisting of one risk-free asset with $A(0) = 10$ and $A(1) = 11$ dollars, and one risky asset such that $S(0) = 10$ and $S(1) = 13$ or 9 dollars. On the x, y plane draw the set of all portfolios (x, y) such that the one-step strategy involving risky position x and risk-free position y is admissible.

7.2.2 The Principle of No Arbitrage

We are ready to formulate the fundamental principle underlying all mathematical models in finance. It generalises the simplified one-step version of the No-Arbitrage Principle in Chapter 1 to models with several time steps and several risky assets. Whereas the notion of a portfolio is sufficient to state the one-step version, in the general setting we need to use a sequence of portfolios forming an admissible investment strategy. This is because investors can adjust their positions at each time step.

Assumption 7.19 (No-Arbitrage Principle)

There is no admissible strategy such that $V(0) = 0$ and $V(n) > 0$ with positive probability for some $n = 1, 2, \ldots$.

Exercise 7.9

Show that the No-Arbitrage Principle would be violated if there was a self-financing predictable strategy with initial value $V(0) = 0$ and final value $V(2) \geq 0$, such that $V(1) < 0$ with positive probability.

The strategy in Exercise 7.9 is not admissible since $V(1)$ may be negative. In fact, this assumption is not essential for the formulation of the No-Arbitrage Principle. An admissible strategy realising an arbitrage opportunity can be found whenever there is a predictable self-financing strategy such that $V(0) = 0$ and $0 \neq V(n) \geq 0$ for some $n > 0$.

Exercise 7.10

Consider a market with one risk-free asset and one risky asset that follows the binomial tree model. Suppose that whenever stock goes up, you know that it will go down at the next step. Find a self-financing (but not necessarily predictable) strategy with $V(0) = 0$, $V(1) \geq 0$ and $0 \neq V(2) \geq 0$.

This exercise indicates that predictability is an essential assumption in the No-Arbitrage Principle. An investor who could foresee the future behaviour of stock prices (here, if stock goes down at one step, you can predict what it will do at the next step) would always be able to find a suitable investment strategy to ensure a risk-free profit.

Exercise 7.11

Consider a market with a risk-free asset such that $A(0) = 100$, $A(1) =$

110, $A(2) = 121$ dollars and a risky asset, the price of which can follow three possible scenarios,

Scenario	$S(0)$	$S(1)$	$S(2)$
ω_1	100	120	144
ω_2	100	120	96
ω_3	100	90	96

Is there an arbitrage opportunity if a) there are no restrictions on short selling, and b) no short selling of the risky asset is allowed?

Exercise 7.12

Given the bond and stock prices in Exercise 7.11, is there an arbitrage strategy if short selling of stock is allowed, but the number of units of each asset in a portfolio must be an integer?

Exercise 7.13

Given the bond and stock prices in Exercise 7.11, is there an arbitrage strategy if short selling of stock is allowed, but transaction costs of 5% of the transaction volume apply whenever stock is traded.

7.3 Fundamental Theorems of Asset Pricing

We restrict ourselves to a simple version for a single-step model. The essential ideas needed for the general case are covered by the proofs presented below. These proofs are elementary but not easy. To some extent they go beyond the average level of difficulty elsewhere in this book.

7.3.1 First Fundamental Theorem

The first fundamental theorem of mathematical finance, which we are now going to formulate and prove, provides an equivalent condition for the absence of arbitrage. This theorem generalises the results proved for the binomial model in Chapter 6, where the lack of arbitrage was shown to be equivalent to $D < R < U$, which in turn is equivalent to the risk-neutral probability $p_* = \frac{R-D}{U-D}$ satisfying $0 < p_* < 1$.

Here we assume that Ω has n elements $\omega_1, \ldots, \omega_n$ and a probability on Ω is described by means of a vector $(p_1, \ldots, p_n)$ of positive numbers with $\sum_{j=1}^n p_j = 1$ such that p_i is the probability of scenario ω_i for $i = 1, \ldots, n$.

Definition 7.20

We say that probability $(p_1^*, \ldots, p_n^*)$ is risk-neutral if $0 < p_i < 1$ for each $i = 1, \ldots, n$ and

$$\mathbb{E}_*(S_i(1)) = S_i(0)(1+R),$$

for all i, where $\mathbb{E}_*$ denotes expectation computed under probability $(p_1^*, \ldots, p_n^*)$.

Theorem 7.21

There is no arbitrage if and only if there exists a risk-neutral probability $(p_1^*, \ldots, p_n^*)$.

Proof

Assume first the existence of a risk-neutral probability $(p_1^*, \ldots, p_n^*)$ and consider any portfolio with $V(0) = 0$ and $V(1) \geq 0$. Compute

$$\begin{aligned}
\mathbb{E}_*(V(1)) &= \sum_{j=1}^n \left(\sum_{i=1}^m x_i S_i^{\omega_j}(1) \right) p_j^* = \sum_{i=1}^m x_i \left(\sum_{j=1}^n S_i^{\omega_j}(1) p_j^* \right) \\
&= \sum_{i=1}^m x_i \mathbb{E}_*(S_i(1)) = (1+R) \sum_{i=1}^m x_i S_i(0) \\
&= (1+R)V(0) = 0.
\end{aligned}$$

This implies that $V(1) = 0$ for all ω since for a non-negative random variable to have zero expectation the random variable must be identically zero (within a finite probability space context). This means that it would be impossible to find an arbitrage opportunity.

For the reverse implication, assume the absence of arbitrage. The set V consisting of all discounted portfolio gains

$$G = \sum_{i=1}^n x_i \left(\frac{S_i(1)}{1+R} - S_i(0) \right), \quad \text{where } x_1, \ldots, x_n \in \mathbb{R},$$

is a vector space, which can be identified with a subspace of $\mathbb{R}^n$ by identifying G with the vector $(G^{\omega_1}, \ldots, G^{\omega_n})$ in $\mathbb{R}^n$.

We claim that each of the elements G of V has the following property: if $G^\omega \geq 0$ for all $\omega \in \Omega$, then $G^\omega = 0$ for all $\omega \in \Omega$. Indeed, suppose that

$$G^\omega = \sum_{i=1}^n x_i \left(\frac{S_i^\omega(1)}{1+R} - S_i(0) \right) \geq 0$$

for all $\omega \in \Omega$. Assume zero initial wealth $V(0) = 0$ for a portfolio consisting of positions $x_1, \ldots, x_n$ in the risky assets and a suitable money market position y. The discounted final value of this portfolio will be

$$\begin{aligned}
\widetilde{V}(1) &= \frac{1}{1+R} \sum_{i=1}^n x_i S_i(1) + y \\
&= \frac{1}{1+R} \sum_{i=1}^n x_i S_i(1) + y - V(0) \\
&= \sum_{i=1}^n x_i \left(\frac{S_i(1)}{1+R} - S_i(0) \right) = G \geq 0.
\end{aligned}$$

Hence to avoid arbitrage, $V(1)$ must be identically equal to zero, and the same can be said about G, which proves the claim.

We consider the set

$$A = \{(q_1, \ldots, q_n) : \sum_{i=1}^n q_i = 1, q_i \geq 0\}$$

of all probabilities on Ω. We shall find a vector $(p_1^*, \ldots, p_n^*)$ in A with strictly positive coordinates, and orthogonal to V. The coordinates of such a vector will give the required probabilities p_i^*, since orthogonality means the inner product of $(p_1^*, \ldots, p_n^*)$ and $(G^{\omega_1}, \ldots, G^{\omega_n})$ will be zero, that is,

$$\mathbb{E}_*(G) = \sum_{i=1}^n p_i^* G^{\omega_i} = 0$$

for all G. In particular, if $x_i = 1$ with $x_k = 0$ for $k \neq i$, then $G = \frac{S_i(1)}{1+R} - S_i(0)$; so $\mathbb{E}_*(S_i(1)) = S(0)(1+R)$ for all $i = 1, \ldots, m$. The existence of such a vector $(p_1^*, \ldots, p_n^*)$ follows from Lemma 7.22 below. Indeed, A is a convex compact set in $\mathbb{R}^n$, $A \cap V = \emptyset$. According to the Lemma, there is a $z \in \mathbb{R}^n$ orthogonal to V and such that $\sum_{i=1}^n z_i a_i > 0$ for all $a \in A$. For any $j = 1, \ldots, n$ it follows that $z_j > 0$ by taking $a \in A$ such that $a_j = 1$ (and then $a_k = 0$ for all $k \neq j$). Putting

$$p_j^* = \frac{z_j}{\sum_{i=1}^n z_i}$$

completes the proof of the theorem. □

Lemma 7.22

If $A \subset \mathbb{R}^n$ is convex and compact and V is a vector subspace of $\mathbb{R}^n$ disjoint with A, then there exists $z \in \mathbb{R}^n$ such that $\langle z, a \rangle = \sum_{i=1}^n z_i a_i > 0$ for all $a \in A$, and $\langle z, v \rangle = \sum_{i=1}^n z_i v_i = 0$ for all $v \in V$. (Here $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^n$).

Proof

Consider the set

$$A - V = \{a - v : a \in A, v \in V\}.$$

Since A and V are disjoint, $0 \notin A - V$.

Next, we can see that $A - V$ is convex and closed since both A and V are. In more detail, if $x, y \in A - V$, $x = a - v$, $y = b - w$, then $\alpha x + (1-\alpha) y = \alpha a + (1-\alpha) b - [\alpha v + (1-\alpha) w] \in A - V$, which proves convexity. To show that $A - V$ is closed, take a sequence $x_n \in A - V$ convergent to some $x \in \mathbb{R}^n$. Then $x_n = a_n - v_n$, the sequence a_n has a convergent subsequence $a_{n_k} \to a$ since A is compact, so v_{n_k} also converges to some $v \in \mathbb{R}^n$. The limit of the sequence x_n must be the same as the limit of x_{n_k} so $x = a - v \in A - V$ as claimed.

Now take z to be the element of $A - V$ closest to 0. For any other element c of $A - V$ we have $|c| \geq |z| > 0$ (recall that $0 \notin A - V$). Existence of such an element z follows from the fact that the mapping $x \mapsto |x|$ is continuous, it is strictly positive on $A - V$, and since this set is closed, the minimum is attained and is different from 0. Consider the triangle with vertices $0, z, c$. We shall prove that $\langle z, c \rangle \geq \langle z, z \rangle = |z|^2 > 0$. The idea is best explained by Figure 7.3. We have

$$\langle z, c \rangle = |z|\,|c| \cos \alpha = |z|\,|c| \frac{|z|}{|y|} = |z|^2 \frac{|c|}{|y|},$$

where the triangle $0, z, y$ has a right angle at z, so it is sufficient to see that $|c| \geq |y|$. Suppose that $|c| < |y|$. The line segment joining c and z is contained in the convex set $A - V$. But the line intersects the interior of the ball with the centre at 0 and radius $|z|$. So this segment contains a point with distance to 0 strictly smaller than $|z|$, a contradiction with the definition of z.

This gives $\langle z, a - v \rangle = \langle z, a \rangle - \langle z, v \rangle \geq \langle z, z \rangle$ for each $v \in V$. However, V is a vector space, so $nv \in V$ for each n, which makes the inequality impossible unless $\langle z, v \rangle = 0$. This in turn yields $\langle z, a \rangle \geq \langle z, z \rangle > 0$, completing the proof. □

7.3.2 Second Fundamental Theorem

In Section 7.1 we have seen that in the single-step trinomial model with one stock it may be impossible to find a portfolio replicating a given contingent

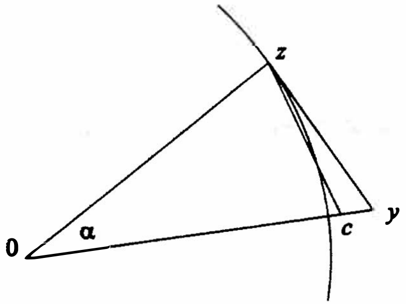

Figure 7.3 Segment z, c intersects the ball of radius $|z|$ and centre at 0

claim. The binomial model, on the other hand, allows for all claims to be replicated. The trinomial model for two stocks may also have this property.

Definition 7.23

A model of a market is called complete if for each European contingent claim there exists a replicating portfolio.

For pricing purposes, replication is a desired property. The pricing formulae derived in simple cases (of binomial trees) show that the price of a claim can be expressed by means of the expectation with respect to the risk-neutral probability. A unique risk-neutral probability gives a unique price, and it can be expected that this will be consistent with the possibility of replication. The following theorem closes this circle of ideas.

Theorem 7.24

A model is arbitrage-free and complete if and only if there exists a unique risk-neutral probability $(p_1^*, \ldots, p_n^*)$.

Proof

The absence of arbitrage implies the existence of a risk-neutral probability $(p_1^*, \ldots, p_n^*)$. We shall show that in a complete model such a vector is unique. We fix any $\omega \in \Omega$ and consider a claim with payoff $H(1) = \mathbb{I}_{\{\omega\}}$, the indicator function of the singleton $\{\omega\}$. By completeness, there exists a portfolio (x, y) such that $H(1) = xS(1) + y(1+R)$. Take the expectation under $(p_1^*, \ldots, p_n^*)$

on both sides of this equality to get

$$\begin{aligned} p_j^* &= \mathbb{E}_*(\mathbb{I}_{\{\omega\}}) = \mathbb{E}_*(xS(1) + y(1+R)) \\ &= x\mathbb{E}_*(S(1)) + y(1+R) = xS(0)(1+R) + y(1+R), \end{aligned} \tag{7.5}$$

since $\mathbb{E}_*(S(1)) = S(0)(1+R)$ for any risk-neutral probability. This shows that the risk-neutral probability is unique.

For the reverse implication, assume $(p_1^*, \ldots, p_n^*)$ is a unique risk-neutral probability and suppose there is a claim $H(1)$ that cannot be replicated. Our goal is to construct a risk-neutral probability $(q_1^*, \ldots, q_n^*)$ different from $(p_1^*, \ldots, p_n^*)$. Consider the set $V = \{V(1) : (x, y) \in R^{n+1}\}$ of values at time 1 of all portfolios, so that $H(1) \notin V$. These values form a vector subspace of $\mathbf{R}^n$ (with the random variable $V(1)$ identified with the vector $(V^{\omega_1}(1), \ldots, V^{\omega_n}(1))$). This subspace contains the vector $\mathbf{1} = (1, \ldots, 1)$ because if $x = 0$ and $y = \frac{1}{1+R}$, then $V^{\omega}(1) = 1$ for all scenarios $\omega \in \Omega$. We equip $\mathbf{R}^n$ with the inner product $\langle z, w \rangle = \sum_{i=1}^n z_j w_j p_j^* = \mathbb{E}_*(zw)$ and use Lemma 7.22 with $A = \{C\}$ to get a $z \in \mathbf{R}^n$ such that $\langle z, C \rangle > 0$ and $\langle z, v \rangle = 0$ for all $v \in V$. The latter, in particular, implies, that $\langle z, \mathbf{1} \rangle = \mathbb{E}_* z = 0$, while the former condition guarantees that $z \neq 0$. Now we can define a risk-neutral probability different from $(p_1^*, \ldots, p_n^*)$:

$$q_j^* = \left(1 + \frac{z_j}{2a}\right) p_j^*$$

where $a = \max_{\omega_j \in \Omega} |z_j|$. This is a probability since

$$\begin{aligned} \sum_{j=1}^n q_j^* &= \sum_{j=1}^n p_j^* + \sum_{j=1}^n \frac{z_j}{2a} p_j^* \\ &= 1 + \frac{1}{2a}\mathbb{E}_*(z) = 1. \end{aligned}$$

Clearly $-a \leq z_j$ so the number $1 + \frac{z_j}{2a}$ is always positive and for at least one value of j it is different from 1 since the vector z is non-zero, thus $q_j^* \neq p_j^*$ for at least one j and $0 < q_j^*$ for all j. Finally, we check the risk-neutral property of $(q_1^*, \ldots, q_n^*)$:

$$\begin{aligned} \sum_{j=1}^n S^{\omega_j}(1) q_j^* &= \mathbb{E}_*\left(\left(1 + \frac{z}{2a}\right) S(1)\right) \\ &= \mathbb{E}_*(S(1)) + \frac{1}{2a}\mathbb{E}_*(zS(1)) \\ &= S(0)(1+R) \end{aligned}$$

since $S(1)$ belongs to V (take $x = 1$, $y = 0$) so $\mathbb{E}_*(zS(1)) = \langle z, S(1) \rangle = 0$, completing the proof. □

7.4 Extended Models

The model of the primary market is based on trading m stocks and the money market account. Here we want to admit contingent claims among tradable assets, which requires a suitable extension of the model.

Suppose that, in addition to the primary assets, there are k contingent claims of European type with the same exercise date N. These are determined by payoffs $H_i(N)$ for $i = 1, \ldots, k$. The contingent claims can be traded, so we need to add them to our portfolios. The positions in contingent claims will be represented by $z_1, \ldots, z_k$. Other than that, we maintain a similar setup as for the primary market.

Assumption 7.25 (Divisibility, Liquidity and Short Selling)

An investor may buy, sell and hold any number of contingent claims (whether integer or fractional, negative, positive or zero). In general,

$$z_1, \ldots, z_k \in \mathbb{R}.$$

Definition 7.26

A *portfolio* is a vector

$$(x_1(n), \ldots, x_m(n), y(n), z_1(n), \ldots, z_k(n))$$

indicating the positions in the primary securities, the money market account and the derivative securities held by an investor between times $n-1$ and n. A sequence of portfolios indexed by $n = 1, 2, \ldots$ is called an *extended investment strategy*. The *wealth* of an investor or the *value of the extended strategy* at time $n \geq 1$ is

$$V(n) = \sum_{j=1}^{m} x_j(n)S_j(n) + y(n)A(n) + \sum_{i=1}^{k} z_i(n)H_i(n).$$

At time $n = 0$ the *initial wealth* is given by

$$V(0) = \sum_{j=1}^{m} x_j(1)S_j(0) + y(1)A(0) + \sum_{i=1}^{k} z_i(1)H_i(0).$$

Definition 7.27

An extended investment strategy is called *self-financing* if the portfolio constructed at time $n \geq 1$ to be held over the next time step $n+1$ is financed

entirely by the current wealth $V(n)$, that is,

$$\sum_{j=1}^{m} x_j(n+1)S_j(n) + y(n+1)A(n) + \sum_{i=1}^{k} z_i(n+1)H_i(n) = V(n).$$

Definition 7.28

An extended investment strategy is called *predictable* if for each $n = 0, 1, 2, \ldots$ the portfolio

$$(x_1(n+1), \ldots, x_m(n+1), y(n+1), z_1(n+1), \ldots, z_k(n+1))$$

constructed at time n is constant on each component of partition $\mathcal{P}_n$ (see Section 7.2.1 for the construction of $\mathcal{P}_n$).

Definition 7.29

An extended strategy is called *admissible* if it is self-financing, predictable, and for each $n = 0, 1, 2, \ldots$

$$V(n) \geq 0$$

with probability 1.

The No-Arbitrage Principle extends without any major modifications.

Assumption 7.30 (No-Arbitrage Principle)

There is no admissible extended strategy such that $V(0) = 0$ and $V(n) > 0$ with positive probability for some n.

What we mean by this assumption is this: the processes of contingent claim values $H_i(n)$ have to be such that the extended market model is free of arbitrage opportunities.

Example 7.31

Consider the single-step binomial model of Section 1.3 ($S(0) = 100$ dollars; $U = 0.2$, $D = -0.2$, $R = 0.1$), which, as we know, is free of arbitrage. Extend it by a call with strike price $X = 100$ dollars, so $k = 1$, $H_1(T) = (S(T) - 100)^+$. With $H_1(0) = \mathbb{E}_*(H_1(T)) \cong 13.6364$ dollars the extended model is also free of arbitrage. However, adding $H_2(T) = (S(T) - 90)^+$, $H_2(0) = 19.4545$ creates arbitrage opportunities since the replication price is in fact $\mathbb{E}_*(H_2(T)) \cong 20.4545$ dollars.

When the primary market is free of arbitrage and complete, we can use the theory developed above to obtain unique option prices. For arbitrary contingent claim payoffs $H_i(N)$ we can find primary replicating strategies, and the values of such strategies give the contingent claim values $H_i(n)$. It is obvious that the extended market will then satisfy the No-Arbitrage Principle. In particular, we have the following pricing formulae: $H_i(0) = \mathbb{E}_*(H_i(N)/A(N))$.

On the other hand, for an incomplete primary model, there may be many extensions consistent with the No-Arbitrage Principle. In this case the derivative prices are, in general, non-unique.

Case 7: Discussion

In view of the fundamental theorems, we shall build a trinomial model for two stocks with a unique risk-neutral probability. Suppose that the current stock prices are $S_1(0) = 30$ and $S_2(0) = 50$. The data within the setup of portfolio theory is concerned with expected returns, their standard deviations and the correlation coefficient. Suppose that the expected returns are $\mu_1 = 9\%$, $\mu_2 = 6\%$ with risk-free rate $R = 5\%$, the standard deviations are $\sigma_1 = 11\%$, $\sigma_2 = 13.5\%$ and the correlation coefficient is $\rho_{12} = -0.36$. The parameters needed for the model are 6 returns and 2 probabilities, which leaves some room since the data give 5 equations. For simplicity we assume that the real-life probabilities are $\frac{1}{3}$ in each scenario, and we still have some flexibility left, so the choice proposed below is non-unique: we assume that

$$\begin{aligned} U_1 &= 22\%, & M_1 &= 10\%, & D_1 &= -5\%, \\ U_2 &= -5\%, & M_2 &= 15\%, & D_1 &= 8\%, \end{aligned}$$

which gives the risk-neutral probabilities

$$p_* = 0.29948, \quad q_* = 0.12760, \quad 1 - p_* - q_* = 0.57292.$$

Suppose that portfolio theory suggests a simple choice of weights, namely: $w_1 = w_2 = 0.5$ (with obvious modifications of the subsequent analysis for any other values). These weights imply that for each share of stock S_2 we have to buy approximately 1.67 shares of stock S_1. The expected return on our portfolio is 7.5%. To insure its future values against unfavourable price movements we consider put options.

Step 1. First we analyse the idea of buying two put options, one for each stock. Suppose we have flexibility as far as the strike price is concerned, and let

$$X_1 = S_1(0)(1 + 7.5\%) = 32.25, \quad X_2 = S_2(0)(1 + 7.5\%) = 53.75$$

so as to guarantee the growth of each stock in line with the expected (required) overall return. It is easy to see that the put prices are

$$P_1 = 2.05, \quad P_2 = 1.78,$$

and the cost of hedging, bearing in mind the weights and the resulting need to cover 1.67 shares of stock S_1 for each share of S_2, gives the cost at $1.67 \times 2.05 + 1.78 = 5.19$. This has to be paid at the beginning of the period. Suppose we borrow the amount risk free (using for this the risk-free part of our investment fund), so that the cost at the end of the period will be 5% more, namely 5.45, approximately. Within the model, the payoff of the portfolio with the put options is

$$\begin{aligned} V_1^{\mathrm{u}} &= 1.67 \times 36.60 + 53.75 = 114.75, \\ V_1^{\mathrm{m}} &= 1.67 \times 33.00 + 57.50 = 112.50, \\ V_1^{\mathrm{d}} &= 1.67 \times 32.25 + 54.00 = 105.25, \end{aligned}$$

(the put on stock S_1 is exercised in the 'down' scenario, and the put on stock S_2 in the 'up' scenario). The average payoff (using the real-world probabilities of $\frac{1}{3}$) is 110.83, so subtracting the cost of options we get the expected return of $\mu_{V_1} = 3.51\%$. This does not look appealing as it is lower than the risk-free rate.

Step 2. Consider a so-called basket option, whose payoff is a function of both stock prices. Here we take a put on the portfolio with strike 107.5 (the initial value is $30 \times 1.67 + 50 = 100$, so the strike is chosen in the same spirit as before). So we have the payoff of the option in the form

$$\max\{107.5 - (1.67 \times S_1(T) + S_2(T)), 0\}$$

with values 0, 0, 6 at the three scenarios.

The price of the option is 3.27. Increased by 5%, this gives the cost of hedging at the end of the period equal to 3.44. The portfolio with an added basket put has payoffs

$$\begin{aligned} V_2^{\mathrm{u}} &= 1.67 \times 36.60 + 47.50 = 108.50, \\ V_2^{\mathrm{m}} &= 1.67 \times 33.00 + 57.50 = 112.50, \\ V_2^{\mathrm{d}} &= 1.67 \times 28.50 + 54.00 + 6 = 107.5, \end{aligned}$$

with expected value 109.5. After subtracting the cost of hedging, the expected return is 6.06%.

The latter is a better result than the previous one. This is a consequence of the low price of the basket option. (Note that in more sophisticated models, like the Black–Scholes model which we are about to present, the problem of pricing basket options is difficult with no closed-form solutions.)

The above result should be compared with the unhedged position, where the expected return of 7.5% is higher, but the possible final values of the portfolio are more risky, since in the worst case scenario d we would get only 101.5, with the same amounts in the other two. The choice between hedged and unhedged portfolios depends on the individual preferences of an investor.