Linear Programming II

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Outline

- Another linear programming example – l1 regression
- Seidel's 2-dimensional linear programming algorithm
- Ellipsoid algorithm, and continued discussion of simplex algorithm

L1 Regression

- Input: n x d matrix A with n larger than d, and n x 1 vector b
- Find x with Ax = b
- Unlikely an x exists, so instead compute $\min_{x} \sum_{i=1,\ldots,n} | A_i \cdot x - b_i |$
- Solve with linear programming? How to handle the absolute values?
- Create variables s_i for i = 1, …, n with $s_i \geq 0$
  - Also have variables $x_1, \ldots, x_d$
- Add constraints $A_i \cdot x - b_i \leq s_i$ and $-(A_i \cdot x - b_i) \leq s_i$ for i = 1, …, n
- What should the objective function be?
- min $\sum_{i=1,\ldots,n} s_i$

Simple Linear Regression

$y = f(x) + \varepsilon$ ε epsilon error term

- Goal: given $(y_i, x_i),\ i = 1, \ldots, n$, estimate β_0, β_1
- ε_i is the error term; can always assume $E(\varepsilon) = 0$.
- Minimize errors - how do we define that?
- One criterion is least squares:

if $E(\varepsilon) \neq 0$
absorb this by $f(x)$

$$\min_{\beta_0, \beta_1} \sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2$$

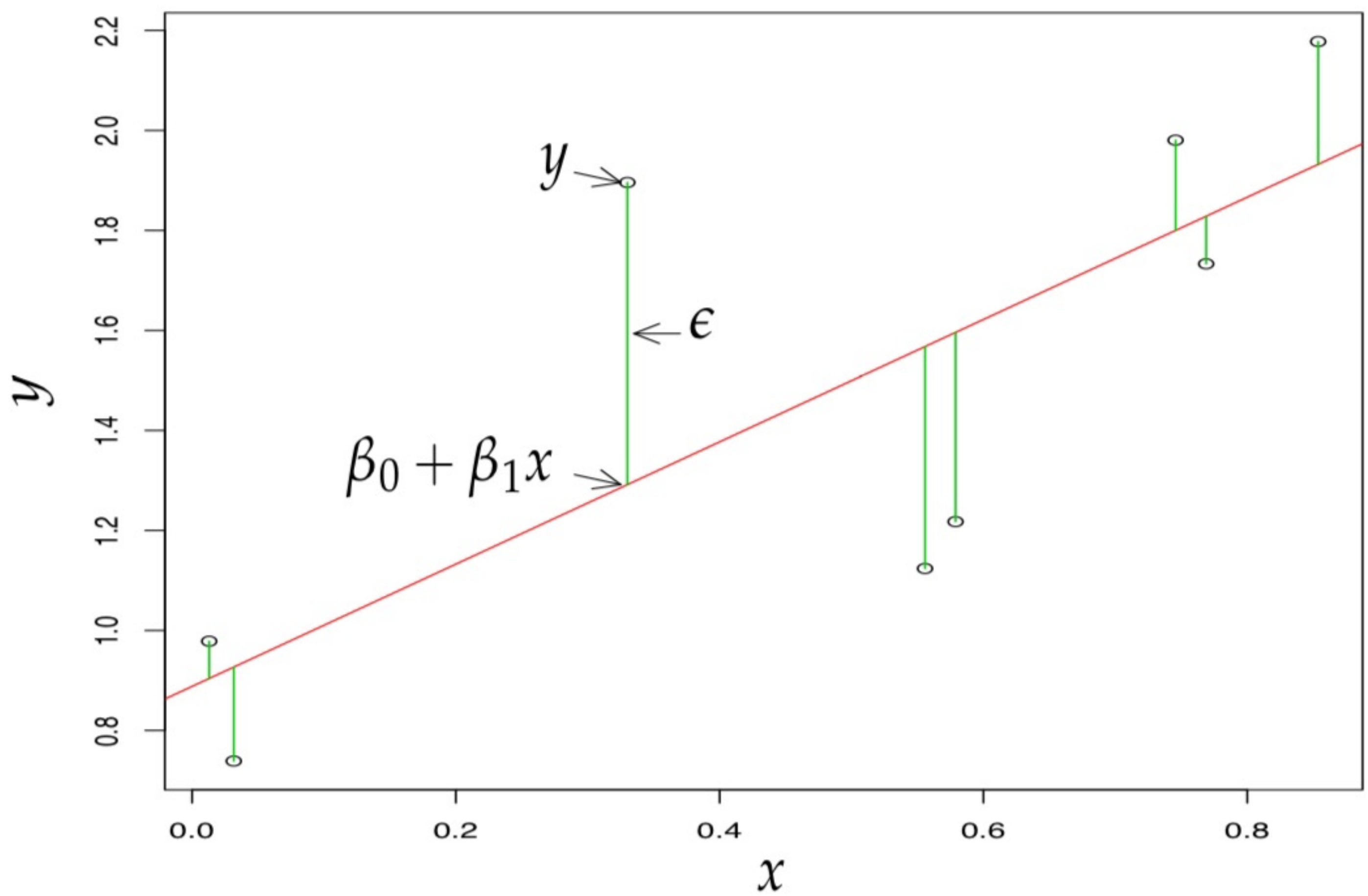
minimize error of least squares.
square ε 保证 always positive
n data point $\varepsilon_i \to \varepsilon_n$

$\because y = \beta_0 + \beta_1 x + \varepsilon$

$\therefore$

find β_0 β_1 $\to$ model ε 最小

Least Squares Estimate



Matrix Notation

Let

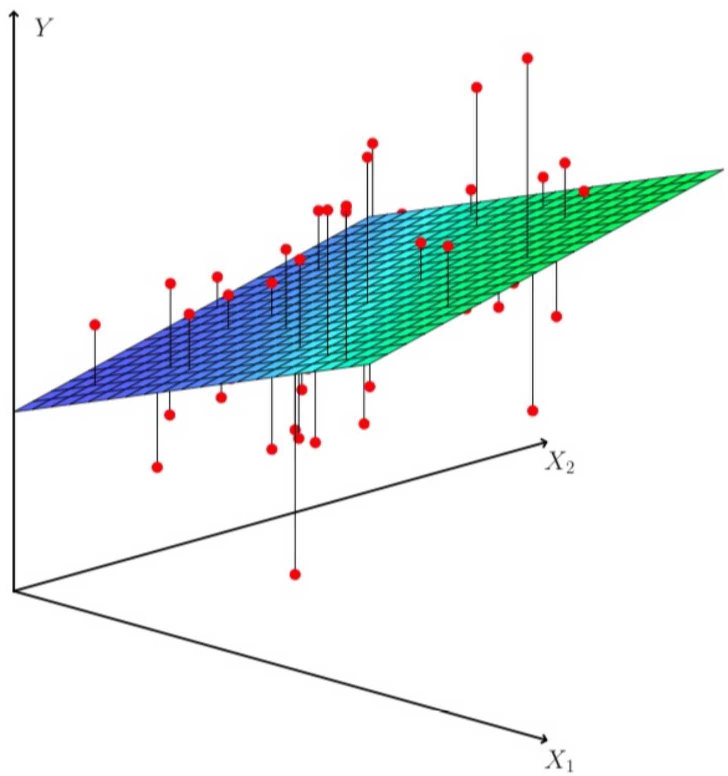
$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad X = \begin{pmatrix} 1 & x_{11} & \cdots & x_{1p} \\ \vdots & \vdots & x_{ij} & \vdots \\ 1 & x_{n1} & \cdots & x_{np} \end{pmatrix}$$

$n \times 1$ $\qquad$ $n \times (p+1)$

Least Squares Estimate

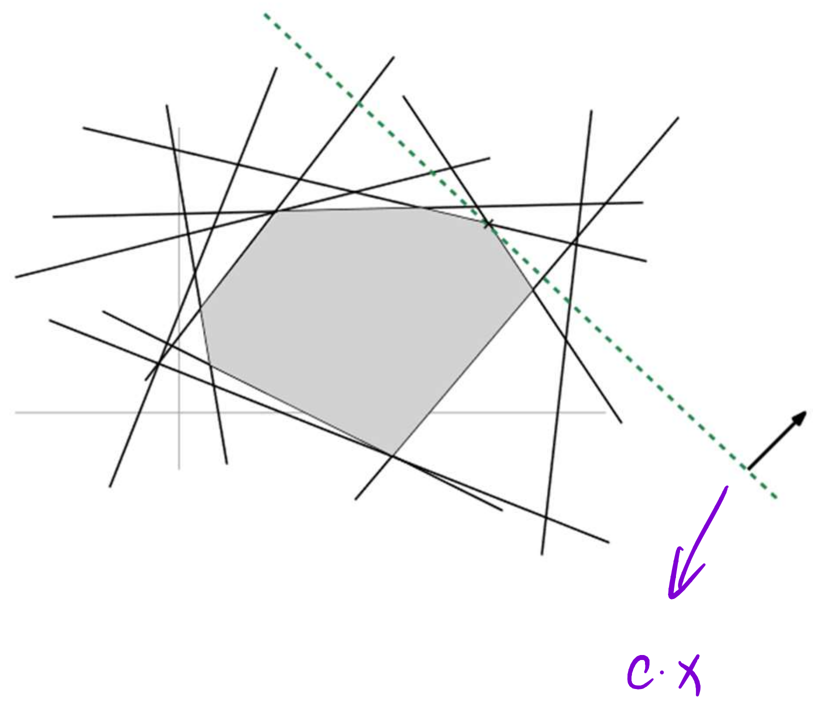
We estimate the parameters using least squares, i.e.

$$\text{arg min}_{\beta_0,\beta_1,\dots,\beta_p} \frac{1}{n}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_{i1} - \cdots - \beta_p x_{ip})^2$$



Seidel's 2-Dimensional Algorithm

- Variables x_1, x_2
- Constraints $a_1 \cdot x \leq b_1, \ldots, a_m \cdot x \leq b_m$
- Maximize $c \cdot x$
- Start by making sure the program has bounded objective function value



What if the LP is unbounded?

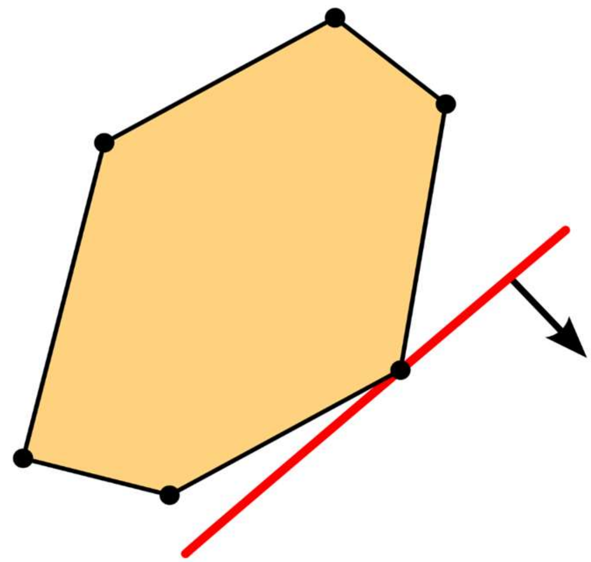
- Add constraints $-\mathrm{M} \leq \mathrm{x}_1 \leq \mathrm{M}$ and $-\mathrm{M} \leq \mathrm{x}_2 \leq \mathrm{M}$ for a large value M

- How large should M be?

- Maximum, if it were bounded, occurs at the intersection of two constraints $\mathrm{ax}_1 + \mathrm{bx}_2 = \mathrm{c}$ and $\mathrm{ex}_1 + \mathrm{fx}_2 = \mathrm{d}$
  $\begin{bmatrix} \text{a b} \\ \text{e f} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \text{c} \\ \text{d} \end{bmatrix}$

- If a, b, e, f, c, d are specified with L bits, can show $|\mathrm{x}_1|, |\mathrm{x}_2|$ specified with O(L) bits
- Can evaluate the objective function on each of the 4 corners of the box to find two constraints $\mathrm{c}_1, \mathrm{c}_2$ which give the maximum

What Convexity Tells Us



- Maximizing a linear function over the feasible region finds a tangent point

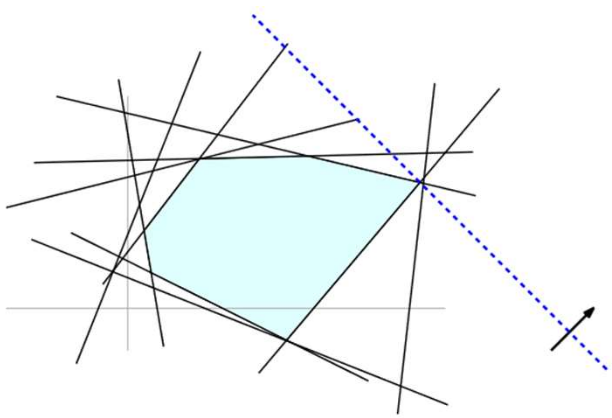
- What's a super naïve $O(m^3)$ time algorithm?

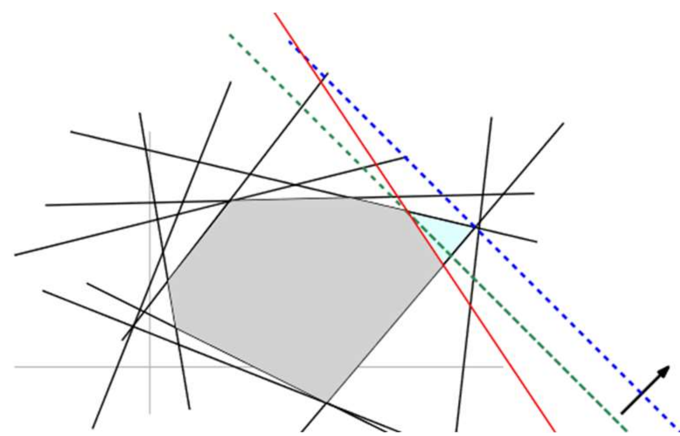
- Find the intersection of each pair of constraints, compute its objective function value, and make sure this point is feasible for all constraints

- What's a less naïve $O(m^2)$ time algorithm?

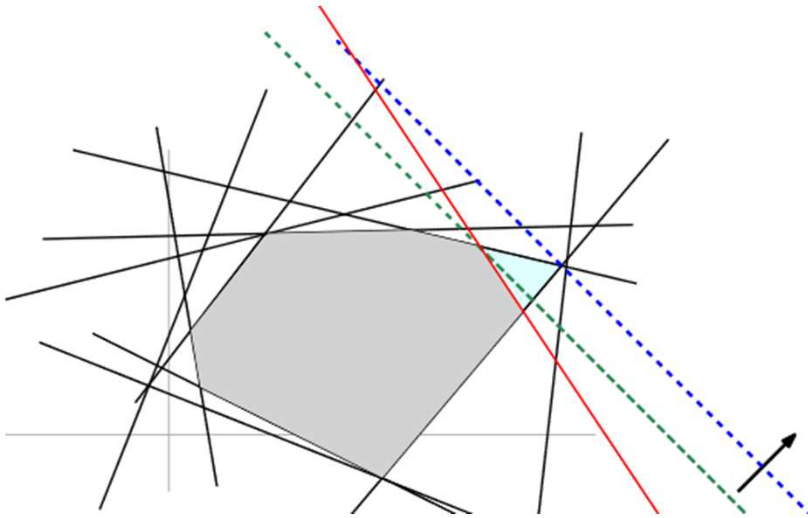
An $O(m^2)$ Time Algorithm

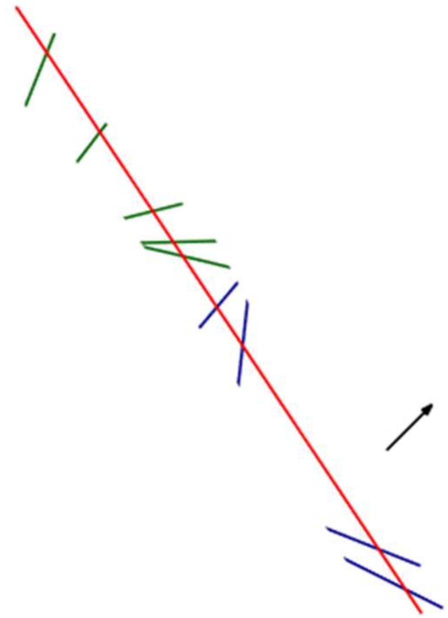
- Order the constraints $a_1 \cdot x \leq b_1, \ldots, a_m \cdot x \leq b_m, c_1, c_2$
- Recursively find optimum point x^* of $a_2 \cdot x \leq b_2, \ldots, a_m \cdot x \leq b_m, c_1, c_2$
- If $a_1 x^* \leq b_1$, then x^* is overall optimum
- Otherwise, new optimum intersects the line $a_1 x^* = b_1$
- Need to solve a 1-dimensional problem





1-Dimensional Problem





- Takes O(m) time to solve
- Note: new optimum might not be determined by one of the two constraints determining the old optimum

An $O(m^2)$ Time Algorithm

- Recursively find optimum point x^* of $a_2 \cdot x \leq b_2, \ldots, a_m \cdot x \leq b_m, c_1, c_2$
- If $a_1 x^* \leq b_1$, then x^* is still optimal
- Otherwise, new optimum intersects the line $a_1 \cdot x = b_1$
- Solve a 1-dimensional problem in O(m) time
- T(m) = T(m-1) + O(m) = $O(m^2)$ time
- Can we get O(m) time?

1. sort $O(m \log m)$

2. recursive start from $a_2 x \le b_2$ $T(m-1)$

3. 检查 ① $a_1 \cdot x^* \le b_1$ x^* optimal $O(1)$

② $a_1 \cdot x = b_1 \rightarrow$ line 上面 find optimal $O(m)$

4. recursive. $T(m) = T(m-1) + O(m)$

$$T(m) = T(m-1) + O(m)$$

$$= [T(m-2) + O(m-1)] + O(m)$$

$$= T(m-2) + O(m-1) + O(m)$$

$$= T(2) + O(3) + O(4) + O(m-1) + O(m)$$

$= O(m) \cdot m$ m 个 $O(m)$ 的和.

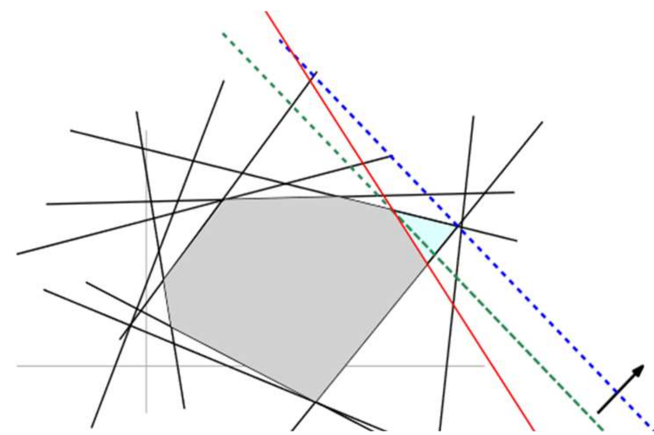
$$m \cdot O(m) = O(m^2)$$

Seidel's O(m) Time Algorithm

- Order constraints randomly: $a_{i_1} \cdot x \leq b_{i_1}, \ldots, a_{i_m} \cdot x \leq b_{i_m}, c_1, c_2$
  - Leave c_1, c_2 at the end
- Recursively find the optimum x^* of $a_{i_2} \cdot x \leq b_{i_2}, \ldots, a_{i_m} \cdot x \leq b_{i_m}, c_1, c_2$
- Case 1: If $a_{i_1} \cdot x^* \leq b_{i_1}$, then x^* is overall optimum
  - O(1) time
- Case 2: If $a_{i_1} \cdot x^* > b_{i_1}$, then we need to intersect the line $a_{i_1} \cdot x = b_{i_1}$ with each other line $a_{i_j} \cdot x = b_{i_j}$ and solve a 1-dimensional problem in O(m) time

Backwards Analysis

- Let x^* be the optimum point of $a_{i_2} \cdot x \leq b_{i_2}, \ldots, a_{i_m} \cdot x \leq b_{i_m}, c_1, c_2$
- What is the chance that $a_{i_1} \cdot x^* > b_{i_1}$?
- Suppose the optimum x' of $a_{i_1} \cdot x \leq b_{i_1}, \ldots, a_{i_m} \cdot x \leq b_{i_m}, c_1, c_2$ is the intersection of two constraints $a_{i_j} \cdot x = b_{i_j}$ and $a_{i_{j'}} \cdot x = b_{i_{j'}}$
- If we've seen these two constraints, then the new constraint $a_{i_1} \cdot x \leq b_{i_1}$ can't change the optimum. Otherwise, optimum would change
- Expected time for processing the last constraint is at most (1-2/m) · O(1) + (2/m) · O(m) = O(1)



Backwards Analysis

- We process the randomly ordered constraints in reverse order:

$$a_{i_1} \cdot x \le b_{i_1}, \ldots, a_{i_m} \cdot x \ \le b_{i_m}, c_1, c_2$$

- When processing the last constraint of:

$$a_{i_j} \cdot x \le b_{i_j}, \ldots, a_{i_m} \cdot x \ \le b_{i_m}, c_1, c_2$$

  the expected amount of time is

  (1-2/(m-j+1)) · O(1) + (2/(m-j+1)) · O(m-j+1) = O(1)

- The expected total time to process m constraints is $\sum_j O(1) = O(m)$, as desired!
- Formally, let T(m) be the expected time to process all m constraints

  T(m) ≤ (1-2/m) O(1) + (2/m) · $O(m)$ + T(m-1)

  = O(1) + T(m-1)

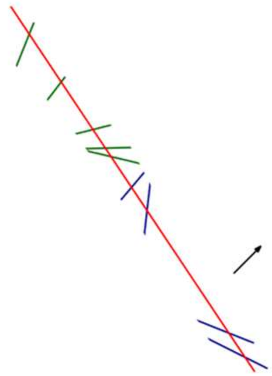
  = O(m). Also add initial constant time for finding c_1, c_2

What if the LP is Infeasible?

- Let j be the largest index for which $a_{i_j} \cdot x \le b_{i_j}, \dots, a_{i_m} \cdot x \le b_{i_m}, c_1, c_2$ is infeasible. That is, $a_{i_{j+1}} \cdot x \le b_{i_{j+1}}, \dots, a_{i_m} \cdot x \le b_{i_m}, c_1, c_2$ is feasible

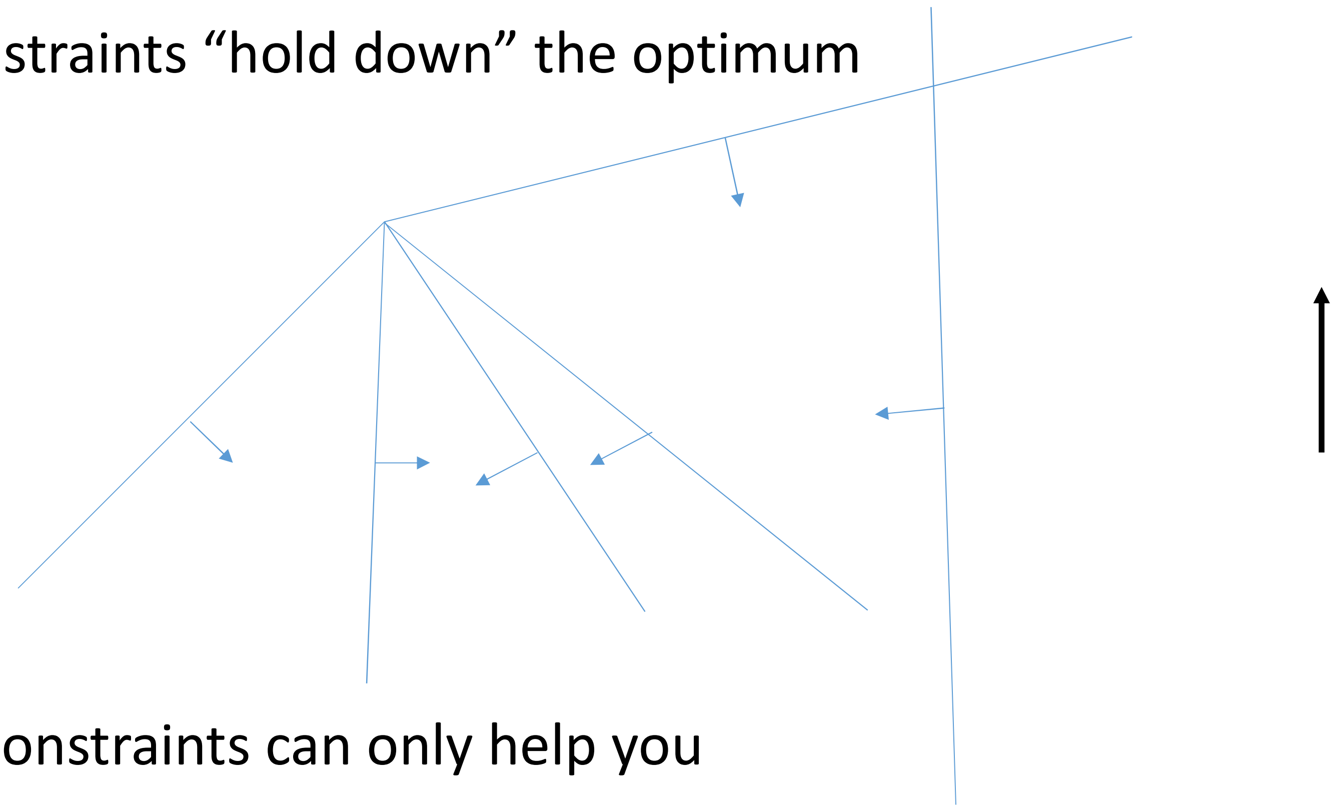
- Since $a_{i_{j+1}} \cdot x \le b_{i_{j+1}}, \dots, a_{i_m} \cdot x \le b_{i_m}, c_1, c_2$ is randomly ordered, we spend an expected O(m) time to process such constraints

- When processing $a_{i_j} \cdot x \le b_{i_j}$ we will find the constraints are infeasible in O(m) time when solving the 1-dimensional problem



What If More than 2 lines Intersect at a Point?

- 2 of the constraints "hold down" the optimum



- Additional constraints can only help you

Higher Dimensions?

- The probability that our optimum changes is now at most d/m instead of 2/m

- When we find a violated constraint, we need to find a new optimum

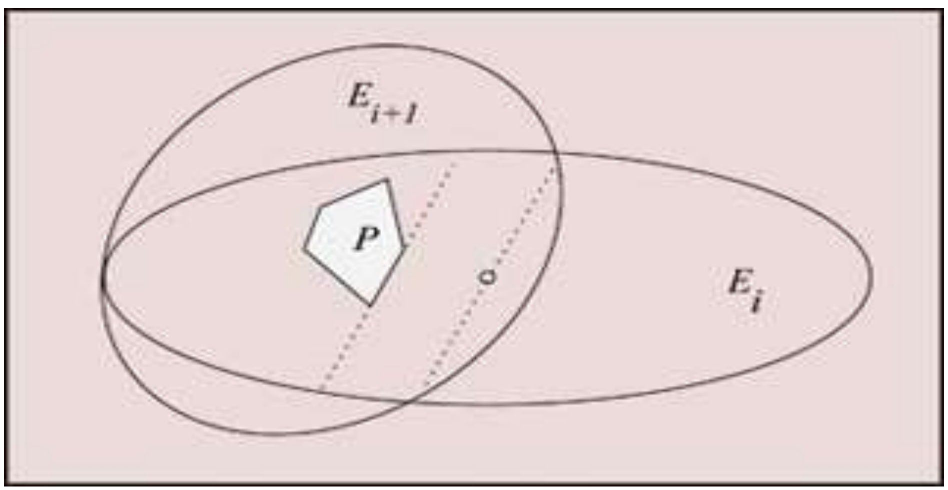
- New optimum inside this hyperplane
    - Project each constraint into this hyperplane
    - Solve a (d-1)-dimensional linear program on m-1 constraints to find optimum
    - Time is $d^{O(d)}m$

Ellipsoid Algorithm

Solves feasibility problem

Replace objective function with constraint, do binary search

Replace “minimize $x_1 + x_2$" with $x_1 + x_2 \leq \lambda$



Can handle exponential number of constraints if there’s a separation oracle

Karmarkar’s Algorithm

- Works with feasible points but doesn’t go corner to corner
- Moves in interior of the feasible region – “interior point method”