8
Continuous Time Model

Case 8

A company manufactures goods in the UK for sale in the USA. The investment to start production is 5 million pounds. Additional funds can be raised by borrowing British pounds at 16% to finance a hedging strategy. The rate of return demanded by investors, bearing in mind the risk involved, is 25%. The sales are predicted to generate 8 million dollars at the end of the year. The costs are 3 million pounds per year. The interest rates are 8% for dollars and 11% for pounds (here we assume continuous compounding). The current rate of exchange is 1.6 dollars to a pound. The volatility of the logarithmic return on the rate of exchange is estimated at $\sigma = 15\%$. The company pays 20% tax on earnings.

8.1 Shortcomings of Discrete Models

Discrete models have considerable advantages, not least the simplicity of the mathematics involved. Very little beyond elementary probability and linear algebra is needed to express important ideas. However, discrete models also have some serious disadvantages as they restrict the range of asset price movements and the set of time instants at which these movements may occur.

As a possible improvement, we shall consider a model obtained as a con-

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tinuous time limit from a sequence of suitably chosen binomial models. This development parallels the introduction of derivatives and differential equations at some point in the history of mathematics. Such tools give tremendous power in addition to more accurate modelling and elegance.

A mathematically precise introduction to continuous models is beyond the level of this book. While the presentation will be lacking full rigour, the facts are solid, and we shall be walking largely along the lines of complete proofs.

8.2 Continuous Time Limit

Our aim will be to reach the celebrated continuous Black–Scholes market model as a limit of a sequence of binomial models.

By $T > 0$ we denote the time horizon (measured in years). In the corresponding binomial model with N steps the duration of each time step will be $h = \frac{T}{N}$. Time instants t between 0 and T will be represented as multiples of h, so that $t = nh$, where $n = 0, \ldots, N$. For such t's the stock and risk-free asset prices in the N-step binomial model will be denoted by $S_N(t)$ and $A_N(t)$, respectively.

8.2.1 Choice of N-Step Binomial Model

In Chapter 6 we expressed the changes of the risk-free asset and stock prices in terms of single-step returns. Clearly, if the number of steps changes, so does the length of a single step, and so do the returns, which should be reflected in the notation. Using the new notational conventions, we shall write

$$\begin{aligned} A_N(t+h) &= (1+R_N)A_N(t), \\ S_N(t+h) &= (1+K_N(t))S_N(t), \end{aligned}$$

for $t = nh$, $n = 0, 1, 2, \ldots$, where the returns $K_N(t)$ are independent identically distributed random variables (see Appendix 10.3) such that

$$K_N(t) = \begin{cases} U_N & \text{if stock goes up in step } n, \\ D_N & \text{if stock goes down in step } n, \end{cases}$$

with $D_N < R_N < U_N$. In addition, it will be assumed that the probability of up/down stock price movements is $\frac{1}{2}$ for each step, so that

$$P(K_N(t) = U_N) = P(K_N(t) = D_N) = \frac{1}{2}.$$

Because option prices in the binomial model do not depend on the real-world probability, this simplifying assumption can be adopted without loss of generality.

Remark 8.1

The probability space on which the random variables are defined is often ignored by practically oriented minds. What really matters in applications is the probability distribution. Nevertheless, it is sometimes convenient to work with a concrete probability space. This can be realised in many different ways. Here is one possibility, which has the advantage of being simple but sufficiently rich that it can conveniently accommodate infinite sequences of independent random variables. Take $\Omega = [0,1]$ and equip it with probability P such that $P([a,b]) = b - a$ for any $0 \leq a \leq b \leq 1$ (the so-called uniform probability). Then $K_N(t) : \Omega \to \{U_N, D_N\}$ ($t = nh$, $h = \frac{T}{N}$) are defined for $\omega \in \Omega = [0,1]$ by

$$K_N^{\omega}(t) = \begin{cases} U_N \text{ if } \omega \in [\frac{k}{2^n}, \frac{k+1}{2^n}) \text{ for even } k, \\ D_N \text{ otherwise.} \end{cases} \tag{8.1}$$

Some simple (though tedious) work suffices to show that these random variables are independent and $P(K_N(t) = U_N) = P(K_N(t) = D_N) = \frac{1}{2}$.

When passing to the continuous time limit, it will become convenient to express the risk-free asset in terms of the continuously compounded interest rate r, so that

$$A(t) = e^{rt}A(0)$$

for any $t \geq 0$, where $e^{rh} = 1 + R_N$. We shall take $A(0) = 1$ for simplicity.

We define the single-step *logarithmic returns* by

$$k_N(t) = \ln(1 + K_N(t)) = \ln \frac{S_N(t+h)}{S_N(t)}$$

for $t = nh$, $n = 0, 2, \ldots, N-1$. These are independent identically distributed random variables such that

$$k_N(t) = \begin{cases} \ln(1+U_N) & \text{if stock goes up in step } n, \\ \ln(1+D_N) & \text{if stock goes down in step } n. \end{cases}$$

In general, the *logarithmic return* on stock between time instants $t < u$ is defined by

$$k_N(t,u) = \ln \frac{S_N(u)}{S_N(t)}.$$

We assume that the expectation and variance of the random variable $k_N(0,t)$ are of a special form:

$$\mathbb{E}(k_N(0,t)) = \mu t,$$
$$\operatorname{Var}(k_N(0,t)) = \sigma^2 t$$

for some $\mu \in \mathbb{R}$, $\sigma > 0$ independent of N.

By the properties of the logarithm, for each $t = nh$

$$k_N(0,t) = k_N(h) + k_N(2h) + \cdots + k_N(nh),$$

and since the terms in this sum are independent identically distributed random variables,

$$\mathbb{E}(k_N(0,t)) = n\mathbb{E}(k_N(h)),$$
$$\operatorname{Var}(k_N(0,t)) = n\operatorname{Var}(k_N(h)),$$

so

$$\mu = \frac{1}{h}\mathbb{E}(k_N(h)),$$

is the expected logarithmic return per unit time and

$$\sigma^2 = \frac{1}{h}\operatorname{Var}(k_N(h)),$$

is the variance of the logarithmic return per unit time. We call σ the ***volatility*** of the stock.

We can now express the single step returns by means of the parameters μ and σ:

$$1 + U_N = e^{\mu h + \sigma\sqrt{h}}, \quad 1 + D_N = e^{\mu h - \sigma\sqrt{h}}. \tag{8.2}$$

Exercise 8.1

Derive equalities (8.2).

Going back to stock prices, from (8.2) we get

$$S_N(h) = S(0)e^{\mu h + Y_1 \sigma\sqrt{h}},$$

where

$$Y_1 = \begin{cases} +1 & \text{with probability } \frac{1}{2}, \\ -1 & \text{with probability } \frac{1}{2}. \end{cases}$$

Extending this step by step, we arrive at

$$S_N(t) = S(0)e^{\mu t + \sigma W_N(t)} \tag{8.3}$$

for each $t = nh$, $n = 0, 1, \ldots, N$, where $h = \frac{T}{N}$ and

$$W_N(t) = \sqrt{h}(Y_1 + \cdots + Y_n), \tag{8.4}$$

for an infinite sequence of independent identically distributed random variables $Y_1, Y_2, \ldots$ such that

$$Y_n = \begin{cases} +1 & \text{with probability } \frac{1}{2}, \\ -1 & \text{with probability } \frac{1}{2}. \end{cases}$$

In particular, $W_N(0) = 0$. We call $W_N(t)$ a ***scaled random walk*** (with time step h and jumps $\pm\sqrt{h}$).

Remark 8.2

The scaled random walk W_N can be constructed for each N using the same sequence of random variables $Y_1, Y_2, \ldots$ defined on the same probability space Ω. For example, this can be the probability space in Remark 8.1. To this end we put

$$Y_n = \frac{\ln(1 + K_N(nh - h)) - \mu h}{\sigma\sqrt{h}}$$

for $n = 1, 2, \ldots$ with $K_N(nh - h)$ defined by (8.1).

8.2.2 Wiener Process

We would like to consider the limit as $N \to 0$ (that is, as $h = \frac{T}{N} \to 0$) of the scaled random walk $W_N(t)$. From (8.4) we know that $W_N(T)$ can be written as

$$W_N(T) = \sqrt{h}\,(Y_1 + \cdots + Y_N).$$

The expectation of each of the independent identically distributed random variables $Y_1, Y_2, \ldots$ is 0 and the variance is 1. The expectation of $Y_1 + \cdots + Y_N$ is therefore 0 and the variance is N. According to the Central Limit Theorem (see Appendix 10.3), the distribution of

$$\frac{W_N(T)}{\sqrt{T}} = \frac{Y_1 + \cdots + Y_N}{\sqrt{N}}$$

tends as $N \to \infty$ to the ***normal distribution*** with mean 0 and variance 1, in the sense that for all $a \leq b$

$$P\left(a \leq \frac{W_N(T)}{\sqrt{T}} \leq b\right) \to \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \quad \text{as } N \to \infty.$$

As a consequence, the limit distribution of $W_N(T)$ is normal with mean 0 and variance T.

A similar argument shows that the limit distribution of

$$W_N(t) - W_N(s) = \sqrt{h} \sum_{s \le nh < t} Y_n$$

is normal with mean 0 and variance $t - s$.

This motivates the next definition, where we introduce a process with such properties that it can be regarded as a limit (in a suitable sense) of a scaled random walk.

Definition 8.3

Wiener process (also known as *Brownian motion*) is a family of random variables $W(t)$ defined for $t \in [0, \infty)$ such that $W(0) = 0$, the increments $W(t) - W(s)$ have normal distribution with mean zero and variance $t - s$ for all $0 \le s < t$, and $W(t_n) - W(t_{n-1}), \ldots, W(t_2) - W(t_1)$ are independent for all sequences $0 \le t_1 < t_2 < \cdots < t_n$.

The existence of Wiener process, including the choice of a suitable probability space, is a matter of advanced considerations, beyond the scope of this text.

8.2.3 Black–Scholes Model

We have seen that in the N-step binomial model the stock price can be expressed as

$$S_N(t) = S(0)e^{\mu t + \sigma W_N(t)}, \tag{8.5}$$

where $W_N(t)$ is the scaled random walk. Replacing $W_N(t)$ by the Wiener process $W(t)$ (which corresponds to the limit as $N \to \infty$), we obtain a new model of stock prices, denoted by $S(t)$, defined for all real $t \ge 0$

$$S(t) = S(0)e^{\mu t + \sigma W(t)}. \tag{8.6}$$

Expression (8.6) is the essence of the famous *Black–Scholes model* of stock prices. It follows that

$$\ln S(t) = \ln S(0) + \mu t + \sigma W(t).$$

Since $W(t)$ has normal distribution $N(0, t)$, we can see that $\ln S(t)$ has normal distribution $N(\ln S(0) + \mu t, \sigma^2 t)$. For this reason the stock price $S(t)$ in the Black–Scholes model is said to have the *log normal distribution*.

The density of the distribution of $S(t)$ is shown in Figure 8.1 for $t = 10$, $S(0) = 1$, $\mu = 0$ and $\sigma = 0.1$. This can be compared with the discrete distribution in Figure 8.2.

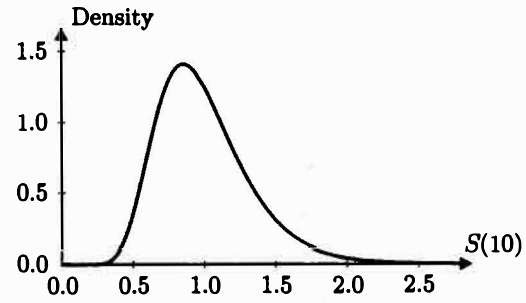

Figure 8.1 Density of the distribution of stock price $S(t)$ in the Black–Scholes model

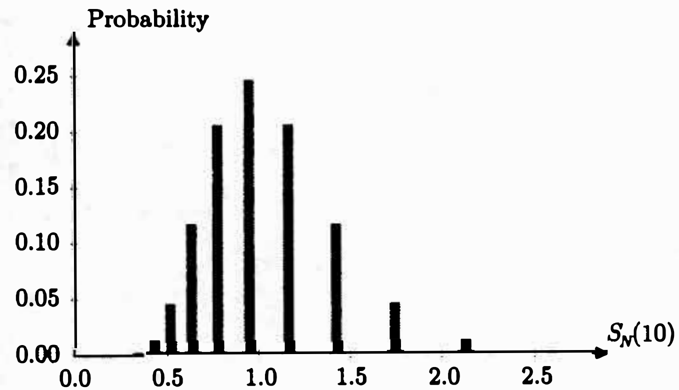

Figure 8.2 Distribution of stock price $S_N(t)$ in the binomial model with $N = 10$

Exercise 8.2

Show that

$$\mathbb{E}(S(t)) = S(0)e^{(\mu + \frac{1}{2}\sigma^2)t}.$$

Remark 8.4

According to Exercise 8.2,

$$\nu = \mu + \frac{1}{2}\sigma^2 \tag{8.7}$$

is the growth rate of the expected stock price. It is often used as a parameter instead of the expected logarithmic return μ when developing the Black–Scholes model, in which case (8.6) takes the form

$$S(t) = S(0)e^{(\nu - \frac{1}{2}\sigma^2)t + \sigma W(t)}.$$

8.3 Overview of Stochastic Calculus

This section provides an introduction to some of the mathematical tools used in continuous-time finance. We aim for an intuitive understanding based on properties established in the discrete setting and an informal treatment of the limit as the number of time steps goes to infinity, rather than precise constructions and proofs.

8.3.1 Itô Formula

Formulae (8.5) for the binomial approximation and (8.6) for the Black-Scholes model can be written as

$$S_N(t) = f(t, W_N(t)), \tag{8.8}$$

$$S(t) = f(t, W(t)), \tag{8.9}$$

where

$$f(t,x) = S(0)e^{\mu t + \sigma x}.$$

For any sufficiently regular function f, using the Taylor formula, we can write

$$\begin{aligned}\Delta f(t, W_N(t)) &= f_t(t, W_N(t))h + f_x(t, W_N(t))\Delta W_N(t) \\ &\quad + \frac{1}{2} f_{xx}(t, W_N(t))h + O(h^{3/2})\end{aligned} \tag{8.10}$$

for $t = nh$, $n = 0, 1, \ldots, N-1$, where the notation

$$f_t = \frac{\partial f}{\partial t}, \quad f_x = \frac{\partial f}{\partial x}, \quad f_{xx} = \frac{\partial^2 f}{\partial x^2}$$

is used for partial derivatives, and

$$\Delta f(t, W_N(t)) = f(t+h, W_N(t+h)) - f(t, W_N(t)),$$

$$\Delta W_N(t) = W_N(t+h) - W_N(t),$$

for increments, and where $O(h^{3/2})$ stands for terms of order $h^{3/2}$ or higher. We have used the fact that $\Delta W_N(t)$ has values $\pm\sqrt{h}$, so that $(\Delta W_N(t))^2 = h$ and $(\Delta W_N(t))^k = O(h^{3/2})$ for each $k \geq 3$.

Applying summation over time steps n in (8.10) such that $0 \leq nh < T$, we obtain

$$\begin{aligned} f(T, W_N(T)) &= f(0, W_N(0)) + \sum_{0 \leq nh < T} f_t(nh, W_N(nh))h \\ &\quad + \sum_{0 \leq nh < T} f_x(nh, W_N(nh))\Delta W_N(nh) \\ &\quad + \frac{1}{2} \sum_{0 \leq nh < T} f_{xx}(nh, W_N(nh))h + O(h^{1/2}). \end{aligned} \tag{8.11}$$

Observe that the sum of $N = \frac{T}{h}$ terms of order $O(h^{3/2})$ gives $O(h^{1/2})$.

The formula

$$\begin{aligned} f(T, W(T)) &= f(0, W(0)) + \int_0^T f_t(t, W(t))\mathrm{d}t \\ &\quad + \int_0^T f_x(t, W(t))\mathrm{d}W(t) + \frac{1}{2}\int_0^T f_{xx}(t, W(t))\mathrm{d}t \end{aligned} \tag{8.12}$$

corresponds to the limit as $N \to \infty$ of the expressions on both sides of the equality (8.11). The first and third integrals are the familiar Riemann integrals, while the second one is a special kind of integral called *stochastic* or *Itô integral*. Equality (8.12), called the *Itô formula*, is the cornerstone of *stochastic calculus*. For a precise definition of the stochastic integral and a proof of the Itô formula please refer to more advanced texts.

Remark 8.5

The term $\frac{1}{2}\int_0^T f_{xx}(t, W(t))\mathrm{d}t$, called the *Itô correction*, is specific to stochastic calculus. Suppose that instead of Wiener process $W(t)$ we use a differentiable function $z(t)$. Then

$$f(T, z(T)) = f(0, z(0)) + \int_0^T f_t(t, z(t))\mathrm{d}t + \int_0^T f_x(t, z(t))z'(t)\mathrm{d}t.$$

No term corresponding to the Itô correction appears in this case, It is convenient to write the Itô formula (8.12) in shorthand notation as

$$df(t, W(t)) = f_t(t, W(t))dt + f_x(t, W(t))dW(t) + \frac{1}{2}f_{xx}(t, W(t))dt, \tag{8.13}$$

omitting the integrals for brevity. In particular, the Itô formula for the composition of functions $h(t, W(t)) = g(t, f(t, W(t)))$ can be written as

$$\begin{aligned} dh(t, W(t)) &= g_t(t, W(t))dt + g_x(t, W(t))df(t, W(t)) \\ &\quad + \frac{1}{2}g_{xx}(t, W(t))[f_x(t, W(t))]^2 dt. \end{aligned} \tag{8.14}$$

The last formula will be used repeatedly in the following sections.

Going back to (8.8) with $f(t, x) = S(0)e^{\mu t + \sigma x}$, we can write (8.10) as

$$\Delta S_N(t) = \left(\mu + \frac{1}{2}\sigma^2\right) S_N(t)h + \sigma S_N(t)\Delta W_N(t) + O(h^{3/2}). \tag{8.15}$$

This can be viewed as a difference equation for the stock price $S_N(t)$ in the binomial model. Moreover, applying the Itô formula (8.13) to (8.9) with $f(t, x) = S(0)e^{\mu t + \sigma x}$, we obtain

$$dS(t) = (\mu + \frac{1}{2}\sigma^2)S(t)dt + \sigma S(t)dW(t). \tag{8.16}$$

This is a ***stochastic differential equation*** satisfied by the stock price $S(t)$ in the Black-Scholes model.

Remark 8.6

The stochastic differential equation (8.16) with the parameter $\nu = \mu + \frac{1}{2}\sigma^2$ from (8.7) becomes

$$dS(t) = \nu S(t)dt + \sigma S(t)dW(t).$$

Written in this elegant form, the equation is often the starting point when constructing the Black-Scholes model.

8.3.2 Expected Value of Itô Integral

The Itô integral $\int_0^t f(u, W(u))dW(u)$ corresponds to the limit of

$$\sum_{0 \le nh < t} f(nh, W_N(nh))\Delta W_N(nh)$$

when $N \to \infty$. The precise meaning of such a limit of a sequence of random variables requires careful treatment, not covered here.

Because of the independence of increments of the random walk, it follows that $W_N(nh)$ and $\Delta W_N(nh)$ are independent, and so are $f(nh, W_N(nh))$ and $\Delta W_N(nh)$ for each n. As a result,

$$\begin{aligned} &\mathbb{E}\left[\sum_{0 \le nh < t} f(nh, W_N(nh))\Delta W_N(nh)\right] \\ &\qquad = \sum_{0 \le nh < t} \mathbb{E}\left[f(nh, W_N(nh))\right] \mathbb{E}\left[\Delta W_N(nh)\right] = 0. \end{aligned}$$

In the limit as $N \to \infty$ this corresponds to

$$\mathbb{E}\int_0^t f(u, W(u))dW(u) = 0. \tag{8.17}$$

This is an informal argument, but the conclusion can be proved rigorously under suitable assumptions on f.

8.3.3 Girsanov Theorem

Our goal in this section will be to construct a probability in the Black-Scholes model that will play an analogous role to the martingale (risk-neutral) probability in the discrete case.

In the binomial model the discounted stock price is given by

$$\tilde{S}_N(t) = \frac{S_N(t)}{A_N(t)} = S(0)e^{(\mu - r)t + \sigma W_N(t)}$$

for $t = nh$, $n = 0, 1, \ldots, N$. The discounted stock price $\tilde{S}_N(t)$ is a martingale under the martingale probability on the binomial tree given by

$$p_* = \frac{R - D}{U - D} = \frac{e^{rh} - e^{\mu h - \sigma\sqrt{h}}}{e^{\mu h + \sigma\sqrt{h}} - e^{\mu h - \sigma\sqrt{h}}}.$$

A somewhat tedious computation using the Taylor formula, best performed with the aid of a symbolic algebra computer system, gives the following expansions in the powers of $\sqrt{h}$:

$$\begin{aligned} p_* &= \frac{e^{rh} - e^{\mu h - \sigma\sqrt{h}}}{e^{\mu h + \sigma\sqrt{h}} - e^{\mu h - \sigma\sqrt{h}}} \\ &= \frac{1}{2} - \frac{1}{2}b\sqrt{h} + O(h^{3/2}) = \frac{1}{2}e^{-b\sqrt{h} - \frac{1}{2}b^2 h + O(h^{3/2})}, \end{aligned} \tag{8.18}$$

$$\begin{aligned} 1 - p_* &= \frac{e^{\mu h + \sigma\sqrt{h}} - e^{rh}}{e^{\mu h + \sigma\sqrt{h}} - e^{\mu h - \sigma\sqrt{h}}} \\ &= \frac{1}{2} + \frac{1}{2}b\sqrt{h} + O(h^{3/2}) = \frac{1}{2}e^{+b\sqrt{h} - \frac{1}{2}b^2 h + O(h^{3/2})}, \end{aligned} \tag{8.19}$$

where

$$b = \frac{\mu - r + \frac{1}{2}\sigma^2}{\sigma} \tag{8.20}$$

is substituted for brevity.

Denote by P_{W_N} and $P^*_{W_N}$ the true probability and, respectively, the martingale probability that the random walk follows a particular path of W_N. Then P_{W_N} is the product of factors p and $1-p$, whereas $P^*_{W_N}$ is the product of factors p_* and $1-p_*$ corresponding to the 'up' and 'down' movements depending on whether $\Delta W_N(t) = +\sqrt{h}$ or $-\sqrt{h}$ for $t = nh$, $n = 0, 1, \ldots, N-1$. Because $p = 1 - p = \frac{1}{2}$ and using (8.18), (8.19) for p_* and $1 - p_*$, we get $P_{W_N} = 2^{-N}$ and

$$P^*_{W_N} = 2^{-N} e^{-bW_N(T) - \frac{1}{2}b^2T + O(h^{1/2})}.$$

An informal argument that adding $N = \frac{T}{h}$ terms of order $O(h^{3/2})$ gives a term or order $O(h^{1/2})$ applies here just like in the derivation of the Itô formula. As a result,

$$P^*_{W_N} = P_{W_N} e^{-bW_N(T) - \frac{1}{2}b^2T + O(h^{1/2})}. \tag{8.21}$$

From here, in the limit as $N \to \infty$ so that $h = \frac{T}{N} \to 0$, we can obtain a new probability P^* such that for any event A

$$P_*(A) = \mathbb{E}\left(e^{-bW(T) - \frac{1}{2}b^2T}\mathbf{I}_A\right), \tag{8.22}$$

where $\mathbf{I}_A$ is the indicator function of A, equal to 1 on A and 0 otherwise, and where $\mathbb{E}$ is the expectation under probability P. Formula (8.22), even though reached informally, can serve as a precise definition of probability P_*.

Applying the Itô formula (8.13) with $f(t, x) = S(0)e^{(\mu - r)t + \sigma x}$, we can verify that the discounted stock price process

$$\widetilde{S}(t) = \frac{S(t)}{A(t)} = S(0)e^{(\mu - r)t + \sigma W(t)}$$

in the Black–Scholes model satisfies

$$d\widetilde{S}(t) = \left(\mu - r + \frac{1}{2}\sigma^2\right)\widetilde{S}(t)dt + \sigma\widetilde{S}(t)dW(t) = \sigma\widetilde{S}(t)dW_*(t),$$

where

$$W_*(t) = bt + W(t) \tag{8.23}$$

with b given by (8.20). We know that $W(t)$ is a Wiener process under P. The following important and interesting proposition establishes a similar property of $W_*(t)$ under the new probability P_*.

Proposition 8.7 (special case of Girsanov theorem)

$W_*(t)$ defined by (8.23) is a Wiener process under the probability P_* defined by (8.22).

Remark 8.8

Proposition 8.7 is a special case of an important result known as the *Girsanov theorem*, which applies in a general setting when b is a stochastic process and not just a constant number. We shall not need this level of generality for our purposes.

Proof

Observe that if X is a random variable with normal distribution $N(0, t)$, then

$$\begin{aligned}
\mathbb{E}\left(e^{-bX - \frac{1}{2}b^2t}\mathbf{I}_{\{X + bt \le a\}}\right) &= \int_{\{x + bt \le a\}} e^{-bx - \frac{1}{2}b^2t}\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}dx \\
&= \int_{\{x + bt \le a\}} \frac{1}{\sqrt{2\pi t}}e^{-\frac{(x+bt)^2}{2t}}dx \\
&= \int_{\{y \le a\}} \frac{1}{\sqrt{2\pi t}}e^{-\frac{y^2}{2t}}dy = P(X \le a).
\end{aligned} \tag{8.24}$$

Let $0 = t_0 < t_1 < \cdots < t_n = T$ and $a_1, \ldots, a_n \in \mathbb{R}$. Since the increments $W(t_i) - W(t_{i-1})$ for $i = 1, \ldots, n$ have normal distribution $N(0, t_i - t_{i-1})$ and are independent under P,

$$\begin{aligned}
&P_*\left(\bigcap_{i=1}^{n}\{W_*(t_i) - W_*(t_{i-1}) \le a_i\}\right) \\
&= \mathbb{E}\left(e^{-bW(T) - \frac{1}{2}b^2T}\mathbf{I}_{\bigcap_{i=1}^{n}\{W_*(t_i) - W_*(t_{i-1}) \le a_i\}}\right) \\
&= \mathbb{E}\left(\prod_{i=1}^{n} e^{-b(W(t_i) - W(t_{i-1})) - \frac{1}{2}b^2(t_i - t_{i-1})}\mathbf{I}_{\{W(t_i) - W(t_{i-1}) + b(t_i - t_{i-1}) \le a_i\}}\right) \\
&= \prod_{i=1}^{n}\mathbb{E}\left(e^{-b(W(t_i) - W(t_{i-1})) - \frac{1}{2}b^2(t_i - t_{i-1})}\mathbf{I}_{\{W(t_i) - W(t_{i-1}) + b(t_i - t_{i-1}) \le a_i\}}\right) \\
&= \prod_{i=1}^{n} P(W(t_i) - W(t_{i-1}) \le a_i),
\end{aligned}$$

where the last equality holds by (8.24). It follows, in particular, that for each i

$$P_*\left(W_*(t_i) - W_*(t_{i-1}) \le a_i\right) = P(W(t_i) - W(t_{i-1}) \le a_i),$$

so the increments $W_*(t_i) - W_*(t_{i-1})$ have normal distribution $N(0, t_i - t_{i-1})$. Moreover, it follows that

$$P_*\left(\bigcap_{i=1}^{n}\{W_*(t_i) - W_*(t_{i-1}) \leq a_i\}\right) = \prod_{i=1}^{n} P(W(t_i) - W(t_{i-1}) \leq a_i)$$
$$= \prod_{i=1}^{n} P_*(W_*(t_i) - W_*(t_{i-1}) \leq a_i),$$

which means that the increments $W_*(t_i) - W_*(t_{i-1})$ for $i = 1, \ldots, n$ are independent under P_*. By Definition 8.3, $W_*(t)$ is therefore a Wiener process under P_*. □

Exercise 8.3

Show that the Black–Scholes stock price (8.6) can be written as

$$S(t) = S(0)e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_*(t)}. \tag{8.25}$$

Exercise 8.4

Show that

$$dS(t) = rS(t)dt + \sigma S(t)dW_*(t).$$

8.4 Options in Black–Scholes Model

We consider European options written on stock in the Black Scholes model. A connection will be made between the pricing and replication of such options, and the solution of a final value problem for a partial differential equation, the celebrated Black–Scholes equation. The option price will be expressed as the expectation under P_* of the discounted payoff. Explicit pricing formulae will be derived for European calls and puts in the Black–Scholes model.

8.4.1 Replicating Strategy

In an N-step binomial model consider a European option expiring at time T with payoff $g(S_N(T))$. A replicating strategy is a pair of processes $x_N(t), y_N(t)$, which represents the positions in stock and the risk-free asset, with the value of the strategy at time $t = nh$, $n = 0, \ldots, N$ given by

$$V_N(t) = x_N(t)S_N(t) + y_N(t)A_N(t),$$

such that:

- the strategy satisfies the self-financing condition

$$x_N(t)S_N(t) + y_N(t)A_N(t) = x_N(t-h)S_N(t) + y_N(t-h)A_N(t)$$

  for $t = nh$, $n = 1, \ldots, N$, or equivalently

$$\Delta V_N(t) = x_N(t)\Delta S_N(t) + y_N(t)\Delta A_N(t)$$

  for $t = nh$, $n = 0, \ldots, N-1$;
- the value of the strategy at time T matches the option payoff,

$$V_N(T) = g(S_N(T)).$$

Such replicating strategies in the binomial model are discussed in Chapter 6.

In the Black–Scholes model we consider a European option with expiry time T and payoff $g(S(T))$. By analogy with the binomial model, a *replicating strategy* will be a pair of processes $x(t), y(t)$ representing the positions in stock and the risk-free asset, with value at time $t \in [0, T]$ given by

$$V(t) = x(t)S(t) + y(t)A(t),$$

such that:

- the strategy satisfies the self-financing condition

$$dV(t) = x(t)dS(t) + y(t)dA(t) \tag{8.26}$$

  for $t \in [0, T]$;
- the value of the strategy at time T matches the option payoff,

$$V(T) = g(S(T)). \tag{8.27}$$

8.4.2 Black–Scholes Equation

The discounted stock price in the Black–Scholes model is denoted by $\widetilde{S}(t) = e^{-rt}S(t)$. If there is a replicating strategy $x(t), y(t)$ for an option with expiry time T and payoff $g(S(T))$, the option is said to be *attainable*. The discounted value of the replicating strategy will be denoted by $\widetilde{V}(t) = e^{-rt}V(t)$.

If $x(t)$ and $y(t)$ can be expressed as sufficiently regular functions of t and $W(t)$, then so can $V(t)$ and $\widetilde{V}(t)$, and the Itô formula can be applied for any of these functions. Thus,

$$\begin{aligned}
d\widetilde{V}(t) &= -re^{-rt}V(t)dt + e^{-rt}dV(t) \\
&= -re^{-rt}\left[x(t)S(t) + y(t)A(t)\right]dt + e^{-rt}\left[x(t)dS(t) + e^{-rt}y(t)dA(t)\right] \\
&= x(t)d\widetilde{S}(t) \\
&= \sigma x(t)\widetilde{S}(t)dW_*(t).
\end{aligned}$$

We know that $W_*(t)$ is a Wiener process under P_*. Thus, using (8.17), we obtain

$$\begin{aligned} V(0) = \widetilde{V}(0) &= \mathbb{E}_*\left[\widetilde{V}(0) + \int_0^T \sigma x(t)\widetilde{S}(t)\mathrm{d}W_*(t)\right] \\ &= \mathbb{E}_*\widetilde{V}(T) = \mathrm{e}^{-rT}\mathbb{E}_*V(T) = \mathrm{e}^{-rT}\mathbb{E}_*g(S(T)), \end{aligned}$$

where $\mathbb{E}_*$ is the expectation under P_*. A standard no-arbitrage argument shows that the price of the option must be equal to $V(0)$ and therefore to $\mathrm{e}^{-rT}\mathbb{E}_*g(S(T))$.

For this expression to be useful it is necessary to ensure that the option is attainable and that the positions $x(t), y(t)$ in the replicating strategy can be expressed as sufficiently regular functions of t and $W(t)$. This will be done for an arbitrary payoff $g(S(T))$ by representing the replicating strategy in terms of a solution to a final value problem for a partial differential equation, the Black–Scholes equation.

Let us take a sufficiently regular function $u(t,x)$ and put

$$f(t,x) = \mathrm{e}^{-rt}u(t, S(0)\mathrm{e}^{(r-\frac{1}{2}\sigma^2)t+\sigma x}),$$

so that, according to (8.25),

$$f(t, W_*(t)) = \mathrm{e}^{-rt}u(t, S(t)).$$

The Itô formula (8.13) applied to $f(t, W_*(t))$ gives

$$\mathrm{d}f(t, W_*(t)) = \left(f_t(t, W_*(t)) + \frac{1}{2}f_{xx}(t, W_*(t))\right)\mathrm{d}t + f_x(t, W_*(t))\mathrm{d}W_*(t).$$

If f satisfies the partial differential equation

$$f_t + \frac{1}{2}f_{xx} = 0, \tag{8.28}$$

then $\mathrm{d}f(t, W_*(t)) = f_x(t, W_*(t))\mathrm{d}W_*(t)$. Because $W_*(t)$ is a Wiener process under P_*, it follows by (8.17) that

$$\begin{aligned} f(0, W_*(0)) &= \mathbb{E}_*\left[f(0, W_*(0)) + \int_0^T f_x(t, W_*(t))\mathrm{d}W_*(t)\right] \\ &= \mathbb{E}_*f(T, W_*(T)). \end{aligned}$$

Written in terms of $u(t,x)$, the partial differential equation (8.28) becomes

$$u_t + rxu_x + \frac{1}{2}\sigma^2x^2u_{xx} = ru. \tag{8.29}$$

This is the famous *Black–Scholes equation*. We can conclude that for any solution $u(t,x)$ to this equation

$$u(0, S(0)) = f(0, W_*(0)) = \mathbb{E}_*f(T, W_*(T)) = \mathrm{e}^{-rT}\mathbb{E}_*u(T, S(T)).$$

In particular, the solution $u(t,x)$ to the Black–Scholes equation (8.29) that satisfies the final condition

$$u(T,x) = g(x) \tag{8.30}$$

plays an important role in pricing and replicating the option with payoff $g(S(T))$. The replicating strategy $x(t), y(t)$ can be expressed in terms of this solution. To this end, we apply the Itô formula (8.14) to $u(t, S(t))$, bearing in mind that $u(t,x)$ satisfies the Black–Scholes equation (8.29), $\mathrm{d}S(t) = rS(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W_*(t)$ and $\mathrm{d}A(t) = rA(t)\mathrm{d}t$:

$$\begin{aligned} \mathrm{d}u(t, S(t)) &= \left[u_t(t, S(t)) + \frac{1}{2}\sigma^2S(t)^2u_{xx}(t, S(t))\right]\mathrm{d}t + u_x(t, S(t))\mathrm{d}S(t) \\ &= r\left[u(t, S(t)) - S(t)u_x(t, S(t))\right]\mathrm{d}t + u_x(t, S(t))\mathrm{d}S(t) \\ &= \frac{u(t, S(t)) - S(t)u_x(t, S(t))}{A(t)}\mathrm{d}A(t) + u_x(t, S(t))\mathrm{d}S(t). \end{aligned}$$

It follows that the strategy

$$x(t) = u_x(t, S(t)), \quad y(t) = \frac{u(t, S(t)) - S(t)u_x(t, S(t))}{A(t)}$$

with value

$$V(t) = x(t)S(t) + y(t)A(t) = u(t, S(t))$$

for each $t \in [0, T]$ satisfies the self-financing condition (8.26) and the replication condition (8.27). This is a replicating strategy for the European option with expiry time T and payoff $g(S(T))$.

To complete the argument that the option is attainable, we need to know that there is a solution $u(t,x)$ to the Black–Scholes equation (8.29) with final condition (8.30). This can be obtained from general existence results for parabolic partial differential equations. Alternatively, it can be verified by direct computation that

$$u(t,x) = \mathrm{e}^{r(T-t)}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi(T-t)}}\mathrm{e}^{-\frac{y^2}{2(T-t)}}g\left(x\mathrm{e}^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma y}\right)\mathrm{d}y$$

is such a solution.

These conclusions can be summarised in the following important theorem.

Theorem 8.9 (Black–Scholes Equation)

Let $g(x)$ be a sufficiently regular function, and let $u(t,x)$ be a solution to the final value problem for the Black–Scholes equation

$$u_t(t,x) + rxu_x(t,x) + \frac{1}{2}\sigma^2 x^2 u_{xx}(t,x) = ru(t,x),$$
$$u(T,x) = g(x).$$

Then the European option with exercise time T and payoff $g(S(T))$ is attainable with replicating strategy

$$x(t) = u_x(t,S(t)), \quad y(t) = \frac{u(t,S(t)) - S(t)u_x(t,S(t))}{A(t)},$$

and the no arbitrage price of the option, which is equal to the value of the replicating strategy, is

$$V(0) = u(0,S(0)) = e^{-rT}\mathbb{E}_* g(S(T)),$$

where $\mathbb{E}_*$ is the expectation under P_*.

Remark 8.10

The expected logarithmic return μ does not appear in the Black–Scholes equation (8.29). This means that the option price does not depend on μ, a seemingly paradoxical result analogous to the fact that the true probability does not affect the option price in the binomial model. It is an important result, because in practice it is difficult to get a reliable estimate of μ. On the other hand, the interest rate r is known precisely, and good estimates for the volatility σ can be obtained.

8.4.3 Black–Scholes Formula

It follows from Theorem 8.9 that the time 0 price of a European call option with strike price X in the Black–Scholes model is

$$C_{\mathrm{E}}(0) = e^{-rT}\mathbb{E}_* \left(\max\{S(T) - X, 0\}\right).$$

Let us compute this expectation. From Exercise 8.3 we know that the Black–Scholes stock price can be written as

$$S(T) = S(0)e^{(r-\frac{1}{2}\sigma^2)T + \sigma W_*(T)}.$$

Since, by the Girsanov theorem, $W_*(T)$ has normal distribution under P_* with mean 0 and variance T,

$$\begin{aligned}
C_{\mathrm{E}}(0) &= \mathbb{E}_* \left(e^{-rT}\max\{S(T) - X, 0\}\right) \\
&= \mathbb{E}_* \left(e^{-rT}\max\{S(0)e^{(r-\frac{1}{2}\sigma^2)T + \sigma W_*(T)} - X, 0\}\right) \\
&= \int_{-d_-}^{\infty} \left(S(0)e^{\sigma x - \frac{1}{2}\sigma^2 T} - Xe^{-rT}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx \\
&= S(0)\int_{-d_+}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx - Xe^{-rT}\int_{-d_-}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx \\
&= S(0)N(d_+) - Xe^{-rT}N(d_-),
\end{aligned}$$

where

$$d_\pm = \frac{\ln\frac{S(0)}{X} + \left(r \pm \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \tag{8.31}$$

and where

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\,dy = \int_{-x}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\,dy \tag{8.32}$$

is the normal cumulative distribution function.

Exercise 8.5

Show that

$$N(d_-) = P_*(S(T) \geq X).$$

The choice of time 0 to compute the price of the option is arbitrary. In general, the option price can be computed at any time $t < T$, in which case the time remaining before the option is exercised will be $T - t$. Substituting t for 0 and $T - t$ for T in the above formulae, we obtain the following result.

Theorem 8.11 (Black–Scholes Formula)

The time t price of a European call with strike price X and exercise time T, where $t < T$, is given by

$$C_{\mathrm{E}}(t) = S(t)N(d_+(t)) - Xe^{-r(T-t)}N(d_-(t)) \tag{8.33}$$

with

$$d_\pm(t) = \frac{\ln\frac{S(t)}{X} + \left(r \pm \frac{1}{2}\sigma^2\right)(T - t)}{\sigma\sqrt{T - t}}. \tag{8.34}$$

Exercise 8.6

Derive the Black–Scholes formula

$$P_E(t) = Xe^{-r(T-t)}N(-d_-(t)) - S(t)N(-d_+(t)),$$

with $d_+(t)$ and $d_-(t)$ given by (8.34), for the price of a European put with strike price X and exercise time T.

Remark 8.12

Observe that the Black–Scholes formula does not contain μ. It is a property analogous to that in Remarks 6.17 and 8.10, and of similar practical significance: there is no need to know μ to work out the price of a European call or put option in the Black–Scholes model.

It is interesting to compare numerically the Black–Scholes formula for the price of a European call with the Cox–Ross–Rubinstein formula. Figure 8.3 shows the price of a European call with strike $X = 100$ on a stock with $S(0) = 100$, $\sigma = 0.3$ and $\mu = 20\%$. (Though μ is irrelevant for the Black–Scholes formula, it still features in the discrete time approximation (8.3).) The interest rate under continuous compounding is taken to be $r = 8\%$. The option price is computed in two ways as a function of the time T remaining before the option is exercised:

a) (*solid line*) from the Black–Scholes formula for T between 0 and 1;

b) (*dots*) using the Cox–Ross–Rubinstein formula with T increasing from 0 to 1 over $N = 10$ steps of duration $h = 0.1$ each; the discrete growth rates for each step are computed using formulae (8.2).

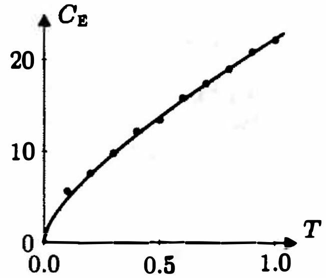

Figure 8.3 Call price in continuous and discrete time models as a function of time T remaining before the option is exercised

Even with as few as 10 steps there is remarkably good agreement between the discrete and continuous time formulae.

Exercise 8.7

Show that the replicating strategy for a European call is

$$x(t) = N(d_+(t)), \quad y(t) = -Xe^{-r(T-t)}N(d_-(t))$$

with $d_+(t)$ and $d_-(t)$ given by (8.34).

8.5 Risk Management

The writer of an option is exposed to risk. The premium due to the writer at time 0 is the expectation of discounted future payoffs under the risk-neutral probability. In the case of some unfavourable scenarios, the amounts due to the holder of the option can be substantial and indeed unlimited, for example, in the case of a call option. This risk to the writer needs to be addressed in a satisfactory manner.

8.5.1 Delta Hedging

The value of a European call or put option as given by the Black–Scholes formula clearly depends on the price of the underlying asset. This can be seen in a slightly broader context.

Consider a portfolio whose value depends on the current stock price $S = S(0)$ and is hence denoted by $V(S)$. Its dependence on S can be measured by the derivative $\frac{d}{dS}V(S)$, called the *delta* of the portfolio. For small price variations from S to $S + \Delta S$ the value of the portfolio will change by

$$\Delta V(S) \cong \frac{d}{dS}V(S) \times \Delta S.$$

The principle of *delta hedging* is based on embedding derivative securities in a portfolio, so that the extended portfolio value does not alter much when S varies. This can be achieved by ensuring that the delta of the portfolio is equal to zero. Such a portfolio is called *delta neutral*.

We take a portfolio composed of stock, money market account, and the hedged derivative security, its value given by

$$V(S) = xS + y + zH(S),$$

where the derivative security price is denoted by $H(S)$ and a bond with current value 1 is used in the risk-free position. Specifically, suppose that a single derivative security has been written, that is, $z = -1$. Then

$$\frac{d}{dS}V(S) = x - \frac{d}{dS}H(S).$$

The last term $\frac{d}{dS}H(S)$, which is the *delta* of the derivative security, can readily be computed in the Black-Scholes model of stock prices, so that an explicit formula for $H(S)$ is available.

Proposition 8.13

Denote the European call option price in the Black-Scholes model by $C_E(S)$. The delta of the option is given by

$$\frac{d}{dS}C_E(S) = N(d_+),$$

where $N(x)$ is the standard normal distribution function and d_+ is defined by (8.31).

Proof

The price $S = S(0)$ appears in three places in the Black-Scholes formula, see Theorem 8.11, so the differentiation requires a bit of work, with plenty of nice cancellations in due course, and is left to the reader, see Exercise 8.7. Bear in mind that the derivative $\frac{d}{dS}C_E(S)$ is computed at time $t = 0$. □

Exercise 8.8

Find a similar expression for the delta $\frac{d}{dS}P_E(S)$ of a European put option in the Black-Scholes model.

For the remainder of this section we shall consider a European call option within the Black-Scholes model. By Proposition 8.13 the portfolio $(x, y, z) = (N(d_+), y, -1)$, where the position in stock $N(d_+)$ is computed for the initial stock price $S = S(0)$, has delta equal to zero for any y. Consequently, its value

$$V(S) = N(d_+)S + y - C_E(S)$$

does not vary much under small changes of the stock price about the initial value. It is convenient to choose y so that the initial value of the portfolio is equal to zero. By the Black-Scholes formula for $C_E(S)$ this gives

$$y = -Xe^{-Tr}N(d_-),$$

with d_- given by (8.31).

Let us analyse the following example, which will subsequently be expanded and modified. Suppose that the risk-free rate is 8% and consider a 90-day call option with strike price $X = 60$ dollars written on a stock with current price $S = 60$ dollars. Assume that the stock volatility is $\sigma = 30\%$. The Black-Scholes formula gives the option price $C_E = 4.14452$ dollars, the delta of the option being equal to 0.581957.

Suppose that we write and sell 1,000 call options, cashing the premium of \$4,144.52. To construct the hedge we buy 581.96 shares for \$34,917.39, borrowing \$30,772.88. Our portfolio will be (x, y, z) with $x = 581.96$, $y = -30,772.88$, $z = -1,000$ and with total value zero. (While it might be more natural mathematically to consider a single option with $z = -1$, in practice options are traded in batches.)

We shall analyse the value of the portfolio after one day by considering some possible scenarios. The time to expiry will then be 89 days. Suppose that the stock volatility and the risk-free rate do not vary, and consider the following three scenarios of stock price movements:

1. The stock price remains unaltered, $S(\frac{1}{365}) = 60$ dollars. A single option is now worth \$4.11833, so our liability due to the short position in options is reduced. Our debt on the money market is increased by the interest due. The position in stock is worth the same as initially. The balance on day one is

stock	34,917.39
money	−30,779.62
options	−4,118.33
TOTAL	**19.45**

   Without hedging ($x = 0$, $y = 4,118.33$, $z = -1,000$) our wealth would have been \$27.10, that is, we would have benefited from the reduced value of the option and the interest due on the premium invested without risk.

2. The stock price goes up to $S(\frac{1}{365}) = 61$ dollars. A single option is now worth \$4.72150, which is more than initially. The unhedged (naked) position would have suffered a loss of \$576.07. On the other hand, for a holder of a delta neutral portfolio the loss on the options is almost completely balanced out

by the increase in stock value:

stock	35,499.35
money	−30,779.62
options	−4,721.50
TOTAL	**−1.77**

3. The stock price goes down to $S(\frac{1}{365}) = 59$ dollars. The value of the written options decreases, a single option now being worth \$3.55908. The value of the stock held decreases too. The portfolio brings a small loss:

stock	34,335.44
money	−30,779.62
options	−3,559.08
TOTAL	**−3.26**

In this scenario it would have been much better not to have hedged at all, since then we would have gained \$586.35.

It may come as a surprise that the hedging portfolio brings a profit when the stock price remains unchanged. As we shall see later in Exercise 8.12, a general rule is at work here.

Exercise 8.9

Find the stock price on day one for which the hedging portfolio attains its maximum value.

Exercise 8.10

Suppose that 50,000 puts with exercise date in 90 days and strike price $X = 1.80$ dollars are written on a stock with current price $S(0) = 1.82$ dollars and volatility $\sigma = 14\%$. The risk-free rate is $r = 5\%$. Construct a delta neutral portfolio and compute its value after one day if the stock price drops to $S(\frac{1}{365}) = 1.81$ dollars.

Going back to our example, let us collect the values V of the delta neutral portfolio for various stock prices after one day as compared to the values U of the unhedged position:

S	V	U
58.00	−71.35	1,100.22
58.50	−31.56	849.03
59.00	−3.26	586.35
59.50	13.69	312.32
60.00	19.45	27.10
60.50	14.22	−269.11
61.00	−1.77	−576.07
61.50	−28.24	−893.53
62.00	−64.93	−1,221.19

Now, let us see what happens if the stock price changes are considerable:

S	V	U
50	−2,233.19	3,594.03
55	−554.65	2,362.79
60	19.45	27.10
65	−481.60	−3,383.73
70	−1,765.15	−7,577.06

If we fear that such large changes might happen, the above hedge is not a satisfactory solution. If we do not hedge, at least we have a gamble with a positive outcome whenever the stock price goes down. Meanwhile, no matter whether the stock price goes up or down, the delta neutral portfolio may bring losses, though considerably smaller than the naked position.

Let us see what can happen if some other variables, in addition to the stock price, change after one day:

1. Suppose that the interest rate increases to 9% with volatility as before. Some loss will result from an increase in the option value. The interest on the cash loan due on day one is not affected because the new rate will only have an effect on the interest payable on the second day or later. The values of the hedging portfolio are given in the second column in the table below.
2. Now suppose that σ grows to 32%, with the interest rate staying at the original level of 8%. The option price will increase considerably, which is not compensated by the stock position even if the stock price goes up. The

results are given in the third column in the following table:

S	V	
	$r = 9\%, \sigma = 30\%$	$r = 8\%, \sigma = 32\%$
58.00	−133.72	−299.83
58.50	−97.22	−261.87
59.00	−72.19	−234.69
59.50	−58.50	−218.14
60.00	−55.96	−212.08
60.50	−64.38	−216.33
61.00	−83.51	−230.68
61.50	−113.07	−254.90
62.00	−152.78	−288.74

As we can see, in some circumstances delta hedging may be far from satisfactory. We need to improve the stability of hedging when the underlying asset price changes considerably and/or some other variables change simultaneously. In what follows, after introducing some theoretical tools, we shall return again to the current example.

Exercise 8.11

Find the value of the delta neutral portfolio in Exercise 8.10 if the risk-free rate of interest decreases to 3% on day one.

8.5.2 Greek Parameters

We shall define so-called *Greek parameters* describing the sensitivity of a portfolio with respect to the various variables determining the option price. The strike price X and expiry date T are fixed once the option is written, so we have to analyse the four remaining variables S, t, r, σ.

Let us write the value of a general portfolio containing stock and some contingent claims based on this stock as a function $V(S,t,\sigma,r)$ of these variables

and denote

$$
\begin{aligned}
\text{delta}_V &= \frac{\partial V}{\partial S},\\
\text{gamma}_V &= \frac{\partial^2 V}{\partial S^2},\\
\text{theta}_V &= \frac{\partial V}{\partial t},\\
\text{vega}_V &= \frac{\partial V}{\partial \sigma},\\
\text{rho}_V &= \frac{\partial V}{\partial r}.
\end{aligned}
$$

For small changes ΔS, Δt, $\Delta\sigma$, Δr of the variables we have the following approximate equality (by the Taylor formula):

$$
\begin{aligned}
\Delta V \cong{}& \text{delta}_V \times \Delta S + \text{theta}_V \times \Delta t + \text{vega}_V \times \Delta\sigma + \text{rho}_V \times \Delta r\\
&+ \frac{1}{2}\text{gamma}_V \times (\Delta S)^2.
\end{aligned}
$$

Hence, a way to immunise a portfolio against small changes of a particular variable is to ensure that the corresponding Greek parameter is equal to zero. For instance, to hedge against volatility movements we should construct a *vega neutral* portfolio, with vega equal to zero. To retain the benefits of delta hedging, we should design a portfolio with both delta and vega equal to zero (*delta-vega neutral*). A *delta-gamma neutral* portfolio will be immune against larger changes of the stock price. Examples of such hedging portfolios will be examined below.

The Black–Scholes formula allows us to compute the derivatives explicitly for a single option. For a European call we have

$$
\begin{aligned}
\text{delta}_{C_E} &= N(d_+),\\
\text{gamma}_{C_E} &= \frac{1}{S\sigma\sqrt{2\pi T}}e^{-\frac{d_+^2}{2}},\\
\text{theta}_{C_E} &= -\frac{S\sigma}{2\sqrt{2\pi T}}e^{-\frac{d_+^2}{2}} - rXe^{-rT}N(d_-),\\
\text{vega}_{C_E} &= \frac{S\sqrt{T}}{\sqrt{2\pi}}e^{-\frac{d_+^2}{2}},\\
\text{rho}_{C_E} &= TXe^{-rT}N(d_-).
\end{aligned}
$$

(The Greek parameters are computed at time $t = 0$.)

Remark 8.14

It is easy to see from the above that

$$\text{theta}_{C_E} + rS\,\text{delta}_{C_E} + \frac{1}{2}\sigma^2 S^2 \text{gamma}_{C_E} = rC_E.$$

This implies that the function u expressing the dependence of call price on t and S, that is, $C_E = u(t,S)$ satisfies the Black–Scholes equation:

$$\frac{\partial u}{\partial t} + rS\frac{\partial u}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} = ru.$$

Exercise 8.12

Show that a delta neutral portfolio with initial value zero hedging a single call option will gain in value with time if the stock price, volatility and risk-free rate remain unchanged.

Exercise 8.13

Derive formulae for the Greek parameters of a put option.

To show some possibilities offered by Greek parameters we consider hedging the position of a writer of European call options.

Delta-Gamma Hedging. The construction is based on making both delta and gamma zero. A portfolio of the form (x, y, z) is insufficient for this. Given the position in options, say $z = -1{,}000$, there remains only one parameter that can be adjusted, namely the position x in the underlying. This allows us to make the delta of the portfolio zero. To make the gamma also equal to zero an additional degree of freedom is needed. To this end we consider another option on the same underlying stock, for example, a call expiring after 60 days, $\widehat{T} = 60/365$, with strike price $\widehat{X} = 65$, and construct a portfolio $(x, y, z, \widehat{z})$, where $\widehat{z}$ is a position in the additional option. The other variables are as in the previous examples: $r = 8\%$, $\sigma = 30\%$, $S(0) = 60$.

Let us sum up all the information about the prices and selected Greek parameters (we also include vega, which will be used later):

option	time to expiry	strike price	option price	delta	gamma	vega
original	90/365	60	4.14452	0.581957	0.043688	11.634305
additional	60/365	65	1.37826	0.312373	0.048502	8.610681

We choose x and $\widehat{z}$ so that the delta and gamma of the portfolio are zero,

$$\text{delta}_V = x - 1{,}000\,\text{delta}_{C_E} + \widehat{z}\,\text{delta}_{\widehat{C}_E} = 0,$$
$$\text{gamma}_V = -1{,}000\,\text{gamma}_{C_E} + \widehat{z}\,\text{gamma}_{\widehat{C}_E} = 0,$$

and the money position y so that the value of the portfolio is zero,

$$V(S) = xS + y - 1{,}000C_E(S) + \widehat{z}\widehat{C}_E(S) = 0.$$

This gives the following system of equations:

$$x - 581.957 + 0.312373\,\widehat{z} = 0,$$
$$-43.688 + 0.048502\,\widehat{z} = 0,$$

with solution $x \cong 300.58$, $\widehat{z} \cong 900.76$. It follows that $y \cong -15{,}131.77$. That is, we take long positions in stock and the additional option, and a short cash position. (We already have a short position $z = -1{,}000$ in the original option.)

After one day, if stock goes up, the original option will become more expensive, increasing our liability, which will be set off by increases in the value of stock and the additional options held. The reverse happens if the stock price declines. Our money debt increases in either case by the interest due after one day. The values of the portfolio are given below (for comparison we also recall the values of the delta neutral portfolio):

$S(\frac{1}{365})$	delta-gamma	delta
58.00	−2.04	−71.35
58.50	0.30	−31.56
59.00	1.07	−3.26
59.50	0.81	13.69
60.00	0.02	19.45
60.50	−0.79	14.22
61.00	−1.11	−1.77
61.50	−0.49	−28.24
62.00	1.52	−64.93

We can see that we are practically safe within the given range of stock prices. For larger changes we are also in a better position as compared with delta hedging:

$S(\frac{1}{365})$	delta-gamma	delta
50	−614.08	−2,233.19
55	−78.22	−554.65
60	0.02	19.45
65	63.13	−481.60
70	440.81	−1,765.15

As predicted, a delta-gamma neutral portfolio offers better protection against stock price changes than a delta neutral one.

Delta-Vega Hedging. Next we shall hedge against an increase in volatility, while retaining cover against small changes in the stock price. This will be achieved by constructing a delta-vega neutral portfolio containing, as before, an additional option. The conditions imposed are

$$\text{delta}_V = x - 1,000\,\text{delta}_{C_E} + \hat{z}\,\text{delta}_{\hat{C}_E} = 0,$$
$$\text{vega}_V = -1,000\,\text{vega}_{C_E} + \hat{z}\,\text{vega}_{\hat{C}_E} = 0.$$

They lead to the system of equations

$$x - 581.957 + 0.312373\,\hat{z} = 0,$$
$$-1,1634.305 + 8.610681\,\hat{z} = 0,$$

with an approximate solution $x \cong 159.89$, $\hat{z} \cong 1,351.15$. The corresponding money position is $y \cong -7,311.12$.

Suppose that the volatility increases to $\sigma = 32\%$ on day one. Let us compare the results for delta-vega and delta hedging:

$S(1/365)$	delta-vega	delta
58.00	−5.90	−299.83
58.50	−12.81	−261.87
59.00	−16.05	−234.69
59.50	−14.99	−218.14
60.00	−9.06	−212.08
60.50	2.27	−216.33
61.00	19.52	−230.68
61.50	43.17	−254.90
62.00	73.62	−288.74

Exercise 8.14

Using the data in our ongoing example (stock price \$60, volatility 30%, interest rate 8%), construct a delta-rho neutral portfolio to hedge a short position of 1,000 call options expiring after 90 days with strike price \$60, taking as an additional component a call option expiring after 120 days with strike price \$65. Analyse the sensitivity of the portfolio value to stock price variations if the interest rate goes up to 9% after one day, comparing with the previous results. What will happen if the interest rate jumps to 15%?

The examples above illustrate the variety of possible hedging strategies. The choice between them depends on individual aims and preferences. We have not touched upon questions related to transaction costs or long term hedging. Nor have we discussed the optimality of the choice of an additional derivative instrument. Portfolios based on three Greek parameters would require yet another derivative security as a component. They could provide comprehensive cover, though their performance might deteriorate if the variables remain unchanged. In addition, they might prove expensive if transaction costs were included.

8.5.3 Value-at-Risk

For the purpose of measuring the risk involved in running a business it is common to adopt the intuitive understanding of risk as the size and likelihood of a possible loss.

Let us present the basic idea using a simple example. We buy a share of stock for $S(0) = 100$ dollars to sell it after one year. The selling price $S(1)$ is random. We shall suffer a loss if $S(1) < 100e^r$, where r is the risk-free rate under continuous compounding. (The purchase can either be financed by a loan, or, if the initial sum is already at our disposal, we take into account the foregone opportunity of a risk-free investment.) What is the probability of the loss being less than a given amount, for example,

$$P(100e^r - S(1) < 20) = ?$$

Let us reverse the question and fix the probability, 95% say. Now we seek an amount such that the probability of a loss not exceeding this amount is 95%. This is referred to as *Value-at-Risk* at 95% confidence level and denoted by VaR. (Other confidence levels can also be used.) So, VaR is an amount such that

$$P(100e^r - S(1) < \text{VaR}) = 95\%.$$

It should be noted that the majority of textbooks neglect the time value of money in this context, stating the definition of VaR only for $r = 0$.

Example 8.15

Suppose that the distribution of the stock price is log normal, the logarithmic return $k = \ln(S(1)/S(0))$ having normal distribution with mean $\mu = 12\%$ and standard deviation $\sigma = 30\%$. (Note that we use the actual mean return, not the risk-free one since we are working under the true probability rather than the martingale one.) With probability 95% the return will satisfy $k > \mu + x\sigma \cong -37.50\%$, where $N(x) \cong 5\%$, so $x \cong -1.645$. (Here $N(x)$ is the normal distribution function with mean 0 and variance 1.) Hence, with probability 95% the future price $S(1)$ will satisfy

$$S(1) > S(0)e^{\mu+x\sigma} \cong 68.83 \text{ dollars},$$

and so, given that $r = 8\%$,

$$\text{VaR} = S(0)e^{r} - S(0)e^{\mu+x\sigma} \cong 39.50 \text{ dollars}.$$

Exercise 8.15

Evaluate VaR at 95% confidence level for a one-year investment of \$1,000 into euros if the interest rate for risk-free investments in euros is $r_{\text{EUR}} = 4\%$ and the exchange rate from euros into US dollars follows the log normal distribution with $\mu = 1\%$ and $\sigma = 15\%$. Take into account the foregone opportunity of investing dollars without risk, given that the risk-free interest rate for dollars is $r_{\text{USD}} = 5\%$.

Exercise 8.16

Suppose that \$1,000 is invested in European call options on a stock with current price $S(0) = 60$ dollars. The options expire after 6 months with strike price $X = 40$ dollars. Assume that $\sigma = 30\%$, $r = 8\%$, and the expected logarithmic return on stock is 12%. Compute VaR after 6 months at 95% confidence level. Find the final wealth if the stock price grows at the expected rate. Find the stock price level that will be exceeded with 5% probability and compute the corresponding final payoff.

Case 8: Discussion

We discuss the problem from the point of view of controlling Value-at-Risk. This approach is often applied by banks to assess companies when they apply for loans.

First note that to satisfy the expectations of their investors the company should be able to achieve a profit of 1.25 million pounds a year to pay the dividend. A lower profit would mean a dividend lower than required, with any shortfall regarded as a loss from the point of view of the investors. The profit depends on the rate of exchange d at the end of the year, hence some risk emerges. (We assume that the other values will be as predicted.)

To begin with, suppose that no action is taken to manage the risk.

1. **Unhedged Position.** If the exchange rate d turns out to be 1.6 dollars to a pound at the end of the year, then the net earnings will be 1.6 million pounds, as shown in the following statement (all amounts in pounds):

sales	5,000,000
costs	−3,000,000
earnings before tax	2,000,000
tax	−400,000
earnings after tax	1,600,000
dividend	−1,250,000
result	350,000

The surplus income will be 0.35 million pounds.

However, if the exchange rate d becomes 2 dollars to a pound, the dividend will in fact have to be reduced and the investors will end up with a loss of 0.45 million pounds (taking the full dividend as a hypothetical cost):

sales	4,000,000
costs	−3,000,000
earnings before tax	1,000,000
tax	−200,000
earnings after tax	800,000
dividend	−1,250,000
result	−450,000

Let us compute VaR. We assume that the rate of exchange has log normal distribution of the form

$$d(t) = d(0)\exp\left((r_{\text{USD}} - r_{\text{GBP}} - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}N\right),$$

where N has the standard normal distribution.[1] With probability 95%, N will not exceed 1.6449, which corresponds to an exchange rate $d = d(1) \cong 1.9650$ dollars to a pound. The statement for this borderline rate of exchange will be as follows (all amounts rounded to the nearest pound):

sales	4,071,290
costs	−3,000,000
earnings before tax	1,071,290
tax	−214,258
earnings after tax	857,032
dividend	−1,250,000
result	−392,968

[1] See Etheridge (2002), Section 5.3.

As a result, VaR $\cong$ 392, 968 dollars. The final result as a function of the exchange rate d is

$$b(d) = 80\% \times \left(\frac{8,000,000}{d} - 3,000,000\right) - 1,250,000$$
$$= \frac{6,400,000}{d} - 3,650,000.$$

The break even exchange rate, which solves $b(d) = 0$, is approximately equal to 1.7534 dollars to a pound. In an optimistic scenario in which the pound weakens, for example, down to 1.5 dollars, the final balance will be about £616, 667.

The question is how to manage this risk exposure.

2. **Forward Contract.** The easiest solution would be to fix the exchange rate in advance by entering into a long forward contract. The forward rate is $d(0)e^{r_{USD}-r_{GBP}} = 1.6 \times e^{-3\%} \cong 1.5527$ dollars to a pound. As a result, the company can obtain the following statement with guaranteed surplus, but no possibility of further gains should the exchange rate become more favourable:

sales	5,152,273
costs	−3,000,000
earnings before tax	2,152,273
tax	−430,455
net income	1,721,818
dividend	−1,250,000
result	471,818

3. **Full Hedge with Options.** Options can be used to ensure that the rate of exchange is capped at a certain level, whilst the benefits associated with favourable exchange rate movements are retained. However, this may be costly because of the premium paid for options.

The company can buy call options on the pound. A European call to buy one pound with strike price 1.6 dollars to a pound will cost \$0.0669.[2] Suppose that the company buys 5 million of such options, paying £209, 069, an amount they have to borrow at 16%. The interest is tax deductible,

[2] For options on currencies the Black–Scholes formula applies with $r = r_{USD}$, $t = 1$, $S(0) = d(0)\exp(-r_{GBP})$.

making the loan less costly. Nevertheless, the final result is disappointing:

sales	5,000,000
costs	−3,000,000
earnings before interest and tax	2,000,000
interest	−33,451
earnings before tax	1,966,549
tax	−393,310
net income	1,573,239
loan repaid	−209,069
dividend	−1,250,000
result	114,170

If the exchange rate drops to 1.5 dollars to a pound, the options will not be exercised and the sum obtained from sales will reach £5, 333, 333, with a positive final result of £380, 837. This strategy leads to a better result than the hedge involving a forward contract only if the rate of exchange drops below 1.4687 dollars to a pound.

4. **Partial Hedge with Options.** To reduce the cost of options the company can hedge partially by buying call options to cover only a fraction of the dollar amount from sales. Suppose that the company buys 2, 500, 000 units of the same call option as above, paying half of the previous premium. Half of the revenue is then exposed to risk. To find VaR at 95% confidence level we assume that this sum is exchanged at 1.9650 dollars to a pound, as in the case of an unhedged position, the other half being exchanged at the exercise price:

sales	4,535,645
costs	−3,000,000
earnings before interest and tax	1,535,645
interest	−16,726
earnings before tax	1,518,919
tax	−303,784
net income	1,215,136
loan repaid	−104,534
dividend	−1,250,000
result	−139,399

If the exchange rate drops to 1.5 dollars to a pound, the investors will have a surplus of £498, 752.

5. **Combination of Options and Forward Contracts.** Finally, let us investigate what happens if the company hedges with both kinds of derivatives. Half of their position will be hedged with options. In the worst case scenario they will buy pounds for half of their dollar revenue at the rate of 1.6 dollars to a pound, the remaining half being exchanged at the forward rate of 1.5527 dollars to a pound. The outcome is shown below, where we summarise the resulting VaR for all strategies considered (the result below is equal to minus VaR):

strategy	1	2	3	4	5
result	$-392,968$	$471,818$	$114,170$	$-139,399$	$292,994$

These values are computed at 95% confidence level, corresponding to the exchange rate of 1.9887 dollars to a pound.

Clearly, VaR provides only partial information about possible outcomes of various strategies. Figure 8.4 shows the graphs of the final result as a function of the exchange rate d for each of the above strategies. The graphs are labelled by the strategy number as above. The strategy using a forward contract (strat-

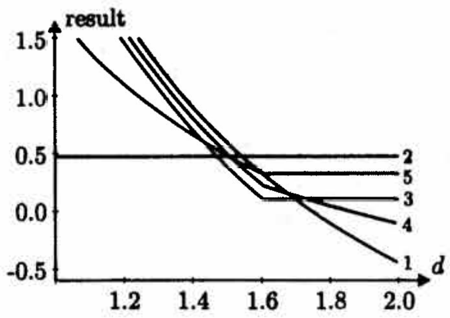

Figure 8.4 Comparison of various strategies

egy 2) appears to be the safest one. An adventurous investor who strongly believes that the pound will weaken considerably may prefer to remain uncovered (strategy 1). A variety of middle-of-the-road strategies are also available. The probability distribution of the exchange rate d should also be taken into account when examining the graphs.