# Financial Econometrics - Part III
# GARCH Type Models

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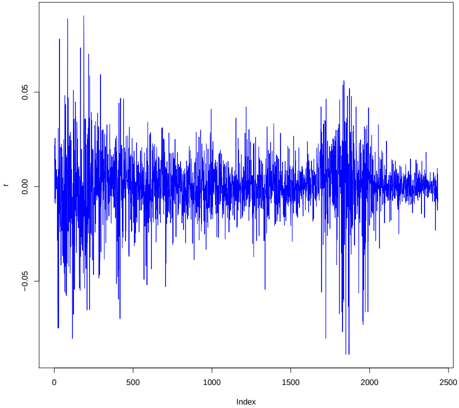
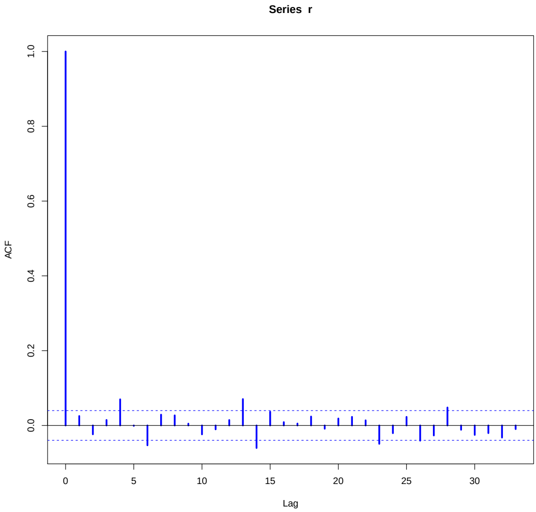
# ARCH effect - the phenomenon of volatility clustering

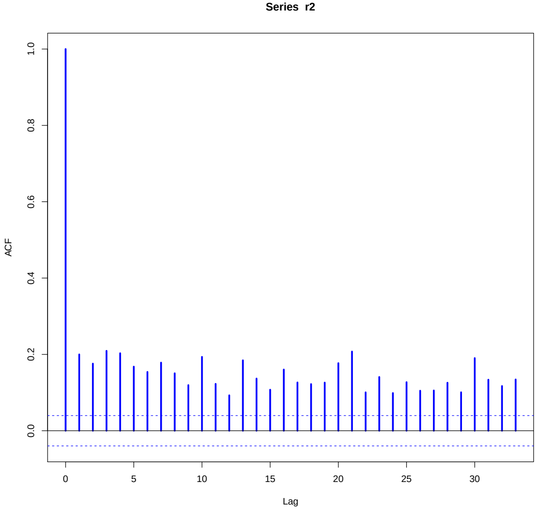
ARCH: autoregressive conditional heteroskedasticity.



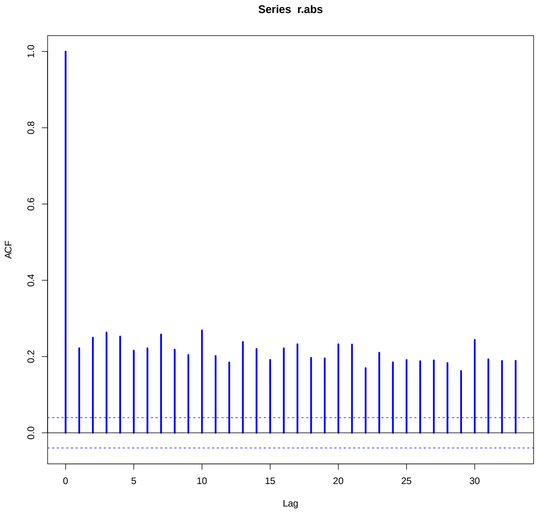
# ARCH effect - ACFs of $r$



# ARCH effect - ACFs of $r^2$



# ARCH effect - ACFs of $|r|$



# ARCH models

Let $P_t$ denote the price and $r_t = \log(P_t/P_{t-1})$ be the log-return at time $t$. Then

$$r_t = \mu_t + e_t, \tag{1}$$

where $\mu_t = E\left[r_t | F_{t-1}\right]$ denotes the conditional mean of the return, $e_t$ is a diffusion term which may be modeled as

$$e_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim i.i.d.(0, 1), \tag{2}$$

where $\sigma_t = Var\left[r_t | F_{t-1}\right] > 0$ is determined by the information available before time $t$, and $\epsilon_t$ is assumed to be independent of $\sigma_t$.

# ARCH models

A simple specification for the volatility function $\sigma_t$ is the autoregressive conditional heteroskedastic (ARCH) model

$$\sigma_t^2 = a_0 + a_1 e_{t-1}^2 + a_2 e_{t-2}^2 + ... + a_p e_{t-p}^2, \tag{3}$$

where $a_0 > 0, a_j \geq 0 (1 \leq j \leq p)$ are constants, and $p$ is a positive integer.

This model allows for capturing the effect of volatility clustering.

# ARCH(1) model

We consider the ARCH(1) model as a special example.

$$e_t = \sigma_t \epsilon_t, \quad (4)$$

$$\sigma_t^2 = a_0 + a_1 e_{t-1}^2, \quad (5)$$

where $a_0 > 0$, $a_1 \geq 0$. In particular, by the law of iterated expectation (LIE), we have

$$E\left[e_t\right] = 0, \quad (6)$$

$$Var\left[e_t\right] = \frac{a_0}{1 - a_1}, \quad (7)$$

where $0 \leq a_1 < 1$.

# ARCH(1) model

Suppose that $\epsilon_t \sim i.i.d.N(0,1)$, then

$$E(e_t^4|F_{t-1}) = 3[E(e_t^2|F_{t-1})]^2 = 3(a_0 + a_1 e_{t-1}^2)^2. \tag{8}$$

Therefore,

$$E(e_t^4) = E[E(e_t^4|F_{t-1})] = 3E(a_0+a_1 e_{t-1}^2)^2 = 3E(a_0^2+2a_0 a_1 e_{t-1}^2+a_1^2 e_{t-1}^4) \tag{9}$$

Assume that $m_4 = E(e_t^4)$ is stationary, then

$$m_4 = \frac{3a_0^2(1+a_1)}{(1-a_1)(1-3a_1^2)}. \tag{10}$$

# ARCH(1) model

To ensure that $m_4 > 0$, we have $0 \leq a_1^2 < \frac{1}{3}$. Then, the Kurtosis of $e_t$ is

$$\frac{E(e_t^4)}{[Var(e_t)^2]^2} = 3\frac{a_0^2(1+a_1)}{(1-a_1)(1-3a_1^2)} \times \frac{(1-a_1)^2}{a_0^2} = 3\frac{1-a_1^2}{1-3a_1^2} > 3. \tag{11}$$

This result implies that $e_t$ has positive excess kurtosis, and it is more likely to see outliers for $e_t$.

# ARCH(1) model

How to forecast volatility based on an estimated ARCH(1) model? Suppose that we had estimated an ARCH(1) model

$$e_t = \sigma_t \epsilon_t, \tag{12}$$

$$\sigma_t^2 = a_0 + a_1 e_{t-1}^2, \tag{13}$$

using $F_T$, what are the one-step ahead forecast $E[\sigma_{T+1}^2 | F_T]$ and two-step ahead forecast $E[\sigma_{T+2}^2 | F_T]$ given the information set $F_T$?

# GARCH models

Bollerslev (1986) developed the GARCH(p,q) model as

$$r_t = \mu + e_t, \tag{14}$$

$$e_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim i.i.d.(0, 1) \tag{15}$$

$$\sigma_t^2 = a_0 + \underbrace{a_1 e_{t-1}^2 + ... + a_p e_{t-p}^2}_{\text{ARCH components}} + \underbrace{b_1 \sigma_{t-1}^2 + ... + b_q \sigma_{t-q}^2}_{\text{GARCH components}}, \tag{16}$$

where $a_0 > 0, a_i \geq 0, b_j \geq 0$, for $1 \leq i \leq p, 1 \leq j \leq q$, and

$$\sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j < 1. \tag{17}$$

# GARCH models

It is easy to show that

$$E[e_t] = 0, \tag{18}$$

$$Var[e_t] = \frac{a_0}{1 - a_1 - ... - a_p - b_1 - ... - b_q} \tag{19}$$

$$= \frac{a_0}{1 - \sum_{i=1}^{p} a_i - \sum_{j=1}^{q} b_j}, \tag{20}$$

which denote the long-run variance.

# Residual diagnostics

How to verify that a particular GARCH(p,q) model has captured all the ARCH effects?

We define $\widehat{\epsilon}_t = \frac{\widehat{e}_t}{\widehat{\sigma}_t}$ as the standardized residuals. If there is no ARCH effects in the standardized residuals, then GARCH(p,q) has captured all the ARCH effects. Otherwise, we still have ARCH effects remained in $\widehat{\epsilon}_t$ and we need to improve our current model.

# GARCH-in-Mean

To account for risk premium, we propose a GARCH-in-Mean model

$$r_t = \mu + g(\sigma_t) + e_t, \tag{21}$$

$$e_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim i.i.d.(0,1) \tag{22}$$

$$\sigma_t^2 = a_0 + \sum_{i=1}^{p} a_i e_{t-i}^2 + \sum_{j=1}^{q} b_j \sigma_{t-j}^2, \tag{23}$$

where $g(x)$ represents the form of compensational return for risks. In particular, $g(x) = \theta_0 + \theta_1 x$ (see Engle, Lilien, and Robins 1987) or $g(x) = \alpha_0 + \alpha_1 \sqrt{x}$ .

# TGARCH model

Leverage effect: negative shocks are more likely to cause higher volatility than positive shocks.
To account for the leverage effect, we propose a Threshold GARCH (TGARCH) model

$$r_t = \mu + e_t, \tag{24}$$

$$e_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim i.i.d.(0,1) \tag{25}$$

$$\sigma_t^2 = a_0 + \sum_{i=1}^{p} a_i e_{t-i}^2 + \sum_{i=1}^{p} \gamma_i e_{t-i}^2 1(e_{t-i} < 0) + \sum_{j=1}^{q} b_j \sigma_{t-j}^2, \tag{26}$$

where 1(.) is an indicator function, which can also be replaced by a dummy variable.

# Dummy variable in Financial Econometrics

Suppose we want to see whether there exists Monday effect in the stock market. We define a dummy variable

$$D_t = \begin{cases} 1 & \text{Mondays;} \\ 0 & \text{Other days.} \end{cases} \tag{27}$$

Then we run the regression

$$r_t = \mu + \theta D_t + e_t, \tag{28}$$

and test

$$\mathbb{H}_0 : \theta = 0 \quad \text{vs} \quad \mathbb{H}_1 : \theta \neq 0. \tag{29}$$

# IGARCH model

An Integrated GARCH model (IGARCH(p,q)) can be written as

$$r_t = \mu + e_t, \tag{30}$$

$$e_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim i.i.d.(0,1) \tag{31}$$

$$\sigma_t^2 = a_0 + \sum_{i=1}^{p} a_i e_{t-i}^2 + \sum_{j=1}^{q} b_j \sigma_{t-j}^2, \tag{32}$$

where $\sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j = 1$.

# Value at Risk (VaR)

Value at risk (VaR) is a measure of the risk of loss for investments. It estimates how much a set of investments might lose (with a given probability), given normal market conditions, in a set time period such as a day.

$$VaR = \mu - \sigma_t \Phi^{-1}(\alpha). \tag{33}$$

For example, we invest \$10000 in an asset with monthly return $\mu = 0.01$ and $\sigma_t = 0.05$. What is the 5% VaR for this investment?

# Application - Shanghai Stock Exchange Composite Index

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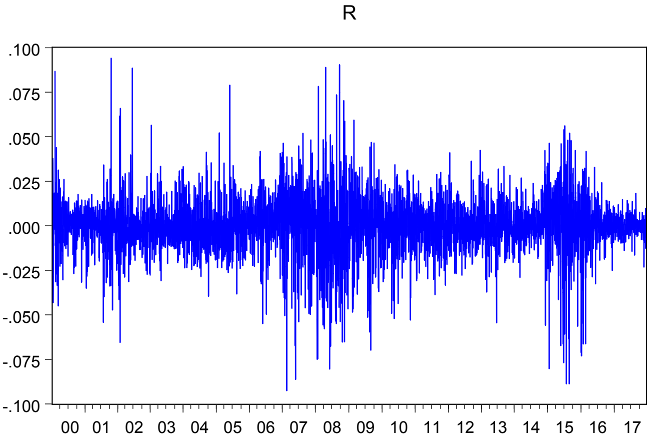


# Application - Shanghai Stock Exchange Composite Index

ACFs and PACFs of $r_t$.

**Correlogram of R**

Date: 03/26/18 Time: 17:01
Sample: 1/04/2000 12/15/2017
Included observations: 4348

| Autocorrelation | Partial Correlation | | AC | PAC | Q-Stat | Prob |
|---|---|---|---|---|---|---|
| | | 1 | 0.023 | 0.023 | 2.3915 | 0.122 |
| | | 2 | -0.023 | -0.024 | 4.7777 | 0.092 |
| | | 3 | 0.022 | 0.023 | 6.9181 | 0.075 |
| | | 4 | 0.058 | 0.056 | 21.496 | 0.000 |
| | | 5 | -0.003 | -0.004 | 21.525 | 0.001 |
| | | 6 | -0.049 | -0.047 | 31.905 | 0.000 |
| | | 7 | 0.028 | 0.028 | 35.292 | 0.000 |
| | | 8 | 0.011 | 0.005 | 35.856 | 0.000 |
| | | 9 | 0.002 | 0.006 | 35.879 | 0.000 |
| | | 10 | -0.003 | 0.001 | 35.928 | 0.000 |
| | | 11 | 0.016 | 0.012 | 36.992 | 0.000 |
| | | 12 | 0.022 | 0.018 | 39.109 | 0.000 |
| | | 13 | 0.052 | 0.055 | 51.071 | 0.000 |
| | | 14 | -0.020 | -0.023 | 52.894 | 0.000 |
| | | 15 | 0.043 | 0.044 | 60.936 | 0.000 |
| | | 16 | 0.010 | 0.002 | 61.392 | 0.000 |
| | | 17 | 0.006 | 0.004 | 61.535 | 0.000 |
| | | 18 | 0.026 | 0.029 | 64.583 | 0.000 |
| | | 19 | -0.025 | -0.028 | 67.329 | 0.000 |
| | | 20 | 0.020 | 0.017 | 69.135 | 0.000 |

# Application - Shanghai Stock Exchange Composite Index

Fitted model:

$$r_t = 0.000172 + 0.0594 r_{t-4} - 0.0485 r_{t-6} + 0.0536 r_{t-13} + \widehat{e}_t.$$

Dependent Variable: R
Method: Least Squares
Date: 03/26/18 Time: 17:01
Sample (adjusted): 1/24/2000 12/15/2017
Included observations: 4335 after adjustments

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| C | 0.000172 | 0.000241 | 0.713331 | 0.4757 |
| R(-4) | 0.059365 | 0.015131 | 3.923277 | 0.0001 |
| R(-6) | -0.048464 | 0.015135 | -3.202070 | 0.0014 |
| R(-13) | 0.053601 | 0.015092 | 3.551530 | 0.0004 |

| | | | |
|---|---|---|---|
| R-squared | 0.008764 | Mean dependent var | 0.000185 |
| Adjusted R-squared | 0.008078 | S.D. dependent var | 0.015941 |
| S.E. of regression | 0.015877 | Akaike info criterion | -5.446996 |
| Sum squared resid | 1.091724 | Schwarz criterion | -5.441114 |
| Log likelihood | 11810.36 | Hannan-Quinn criter. | -5.444920 |
| F-statistic | 12.76440 | Durbin-Watson stat | 1.957144 |
| Prob(F-statistic) | 0.000000 | | |

# Application - Shanghai Stock Exchange Composite Index

ACFs and PACFs of $\widehat{e}_t$: the above model fully describes the serial correlation in $r_t$.

Date: 03/26/18 Time: 17:07
Sample: 1/24/2000 12/15/2017
Included observations: 4335

| Autocorrelation | Partial Correlation | | AC | PAC | Q-Stat | Prob |
|---|---|---|---|---|---|---|
| | | 1 | 0.021 | 0.021 | 1.9794 | 0.159 |
| | | 2 | -0.022 | -0.022 | 4.0616 | 0.131 |
| | | 3 | 0.023 | 0.024 | 6.2942 | 0.098 |
| | | 4 | -0.000 | -0.002 | 6.2945 | 0.178 |
| | | 5 | -0.004 | -0.003 | 6.3638 | 0.272 |
| | | 6 | 0.002 | 0.002 | 6.3805 | 0.382 |
| | | 7 | 0.027 | 0.027 | 9.6245 | 0.211 |
| | | 8 | 0.004 | 0.003 | 9.6971 | 0.287 |
| | | 9 | 0.001 | 0.002 | 9.6989 | 0.375 |
| | | 10 | 0.003 | 0.002 | 9.7321 | 0.464 |
| | | 11 | 0.014 | 0.014 | 10.564 | 0.480 |
| | | 12 | 0.020 | 0.020 | 12.385 | 0.415 |
| | | 13 | 0.004 | 0.003 | 12.444 | 0.492 |
| | | 14 | -0.019 | -0.020 | 13.989 | 0.451 |
| | | 15 | 0.040 | 0.040 | 21.086 | 0.134 |
| | | 16 | 0.003 | 0.001 | 21.137 | 0.173 |
| | | 17 | -0.004 | -0.002 | 21.213 | 0.217 |
| | | 18 | 0.028 | 0.026 | 24.697 | 0.133 |
| | | 19 | -0.019 | -0.022 | 26.274 | 0.123 |
| | | 20 | 0.015 | 0.018 | 27.300 | 0.127 |

# Application - Shanghai Stock Exchange Composite Index

ARCH effect test for $\widehat{e}_t$:

Heteroskedasticity Test: ARCH

| | | | |
|---|---|---|---|
| F-statistic | 67.53716 | Prob. F(5,4324) | 0.0000 |
| Obs*R-squared | 313.6590 | Prob. Chi-Square(5) | 0.0000 |

Test Equation:
Dependent Variable: RESID^2
Method: Least Squares
Date: 03/26/18 Time: 17:09
Sample (adjusted): 2/14/2000 12/15/2017
Included observations: 4330 after adjustments

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| C | 0.000135 | 1.16E-05 | 11.62568 | 0.0000 |
| RESID^2(-1) | 0.106296 | 0.015176 | 7.004173 | 0.0000 |
| RESID^2(-2) | 0.076921 | 0.015206 | 5.058687 | 0.0000 |
| RESID^2(-3) | 0.131147 | 0.015120 | 8.673930 | 0.0000 |
| RESID^2(-4) | 0.086107 | 0.015206 | 5.662844 | 0.0000 |
| RESID^2(-5) | 0.064002 | 0.015176 | 4.217325 | 0.0000 |

| | | | |
|---|---|---|---|
| R-squared | 0.072439 | Mean dependent var | 0.000252 |
| Adjusted R-squared | 0.071366 | S.D. dependent var | 0.000658 |
| S.E. of regression | 0.000634 | Akaike info criterion | -11.88830 |
| Sum squared resid | 0.001737 | Schwarz criterion | -11.87946 |
| Log likelihood | 25744.16 | Hannan-Quinn criter. | -11.88518 |
| F-statistic | 67.53716 | Durbin-Watson stat | 1.976638 |
| Prob(F-statistic) | 0.000000 | | |

# Application - Shanghai Stock Exchange Composite Index

ACFs and PACFs of $\widehat{e}_t^2$.

**Correlogram of E2**

Date: 03/26/18 Time: 17:11
Sample: 1/04/2000 12/15/2017
Included observations: 4335

| Autocorrelation | Partial Correlation | | AC | PAC | Q-Stat | Prob |
|---|---|---|---|---|---|---|
| | | 1 | 0.162 | 0.162 | 113.93 | 0.000 |
| | | 2 | 0.139 | 0.116 | 197.66 | 0.000 |
| | | 3 | 0.181 | 0.148 | 340.09 | 0.000 |
| | | 4 | 0.148 | 0.093 | 434.70 | 0.000 |
| | | 5 | 0.126 | 0.064 | 503.39 | 0.000 |
| | | 6 | 0.122 | 0.055 | 568.20 | 0.000 |
| | | 7 | 0.135 | 0.069 | 647.60 | 0.000 |
| | | 8 | 0.121 | 0.051 | 711.65 | 0.000 |
| | | 9 | 0.087 | 0.013 | 744.78 | 0.000 |
| | | 10 | 0.146 | 0.080 | 838.06 | 0.000 |
| | | 11 | 0.106 | 0.027 | 886.82 | 0.000 |
| | | 12 | 0.088 | 0.014 | 920.21 | 0.000 |
| | | 13 | 0.134 | 0.063 | 998.65 | 0.000 |
| | | 14 | 0.116 | 0.038 | 1057.4 | 0.000 |
| | | 15 | 0.084 | 0.006 | 1088.4 | 0.000 |
| | | 16 | 0.111 | 0.037 | 1142.5 | 0.000 |
| | | 17 | 0.094 | 0.013 | 1181.0 | 0.000 |
| | | 18 | 0.103 | 0.029 | 1226.8 | 0.000 |
| | | 19 | 0.088 | 0.014 | 1260.6 | 0.000 |
| | | 20 | 0.128 | 0.056 | 1332.4 | 0.000 |

# Application - Shanghai Stock Exchange Composite Index

An estimated AR(13)-ARCH(1) model.

Dependent Variable: R
Method: ML - ARCH (Marquardt) - Normal distribution
Date: 03/26/18 Time: 17:20
Sample (adjusted): 1/24/2000 12/15/2017
Included observations: 4335 after adjustments
Convergence achieved after 9 iterations
Presample variance: backcast (parameter = 0.7)
GARCH = C(5) + C(6)*RESID(-1)^2

| Variable | Coefficient | Std. Error | z-Statistic | Prob. |
|---|---|---|---|---|
| C | -5.72E-06 | 0.000214 | -0.026773 | 0.9786 |
| R(-4) | 0.043504 | 0.009935 | 4.378838 | 0.0000 |
| R(-6) | -0.040643 | 0.009946 | -4.086304 | 0.0000 |
| R(-13) | 0.065531 | 0.009860 | 6.646338 | 0.0000 |
| Variance Equation | | | | |
| C | 0.000192 | 3.22E-06 | 59.73762 | 0.0000 |
| RESID(-1)^2 | 0.259946 | 0.016300 | 15.94767 | 0.0000 |

| | | | |
|---|---|---|---|
| R-squared | 0.008175 | Mean dependent var | 0.000185 |
| Adjusted R-squared | 0.007488 | S.D. dependent var | 0.015941 |
| S.E. of regression | 0.015881 | Akaike info criterion | -5.506681 |
| Sum squared resid | 1.092373 | Schwarz criterion | -5.497858 |
| Log likelihood | 11941.73 | Hannan-Quinn criter. | -5.503566 |
| Durbin-Watson stat | 1.956590 | | |

# Application - Shanghai Stock Exchange Composite Index

Fitted AR(13)-ARCH(1) model:

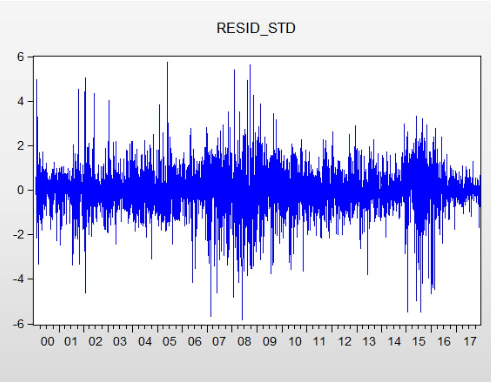
$$r_t = 0.00000572 + 0.0435 r_{t-4} - 0.0406 r_{t-6} + 0.0655 r_{t-13} + \widehat{e}_t \tag{34}$$

$$\widehat{e}_t = \widehat{\sigma}_t \widehat{z}_t \tag{35}$$

$$\widehat{\sigma}_t^2 = 0.000192 + 0.25 \widehat{e}_{t-1}^2. \tag{36}$$

# Application - Shanghai Stock Exchange Composite Index

Time series plot of the standardized residuals $\widehat{z}_t = \frac{\widehat{e}_t}{\widehat{\sigma}_t}$. Are there any ARCH effects left in $z_t$?



# Application - Shanghai Stock Exchange Composite Index

ACFs and PACFs of $\hat{z}_t^2$.

Date: 03/26/18 Time: 17:37
Sample: 1/04/2000 12/15/2017
Included observations: 4335

| Autocorrelation | Partial Correlation | | AC | PAC | Q-Stat | Prob |
|---|---|---|---|---|---|---|
| | | 1 | -0.024 | -0.024 | 2.5599 | 0.110 |
| | | 2 | 0.111 | 0.110 | 55.962 | 0.000 |
| | | 3 | 0.137 | 0.144 | 137.85 | 0.000 |
| | | 4 | 0.083 | 0.082 | 167.88 | 0.000 |
| | | 5 | 0.097 | 0.076 | 208.98 | 0.000 |
| | | 6 | 0.105 | 0.079 | 256.80 | 0.000 |
| | | 7 | 0.090 | 0.063 | 291.94 | 0.000 |
| | | 8 | 0.097 | 0.062 | 332.91 | 0.000 |
| | | 9 | 0.040 | -0.001 | 339.86 | 0.000 |
| | | 10 | 0.122 | 0.075 | 404.47 | 0.000 |
| | | 11 | 0.079 | 0.047 | 431.71 | 0.000 |
| | | 12 | 0.063 | 0.022 | 448.99 | 0.000 |
| | | 13 | 0.117 | 0.067 | 508.20 | 0.000 |
| | | 14 | 0.098 | 0.061 | 549.73 | 0.000 |
| | | 15 | 0.063 | 0.018 | 566.74 | 0.000 |
| | | 16 | 0.087 | 0.028 | 599.73 | 0.000 |
| | | 17 | 0.085 | 0.034 | 631.40 | 0.000 |
| | | 18 | 0.074 | 0.022 | 655.40 | 0.000 |
| | | 19 | 0.062 | 0.008 | 672.32 | 0.000 |
| | | 20 | 0.082 | 0.024 | 701.83 | 0.000 |

# Application - Shanghai Stock Exchange Composite Index

LM test for ARCH effects in $\widehat{z}_t$.

Heteroskedasticity Test: ARCH

| | | | |
|---|---|---|---|
| F-statistic | 41.10445 | Prob. F(5,4324) | 0.0000 |
| Obs*R-squared | 196.4692 | Prob. Chi-Square(5) | 0.0000 |

Test Equation:
Dependent Variable: WGT_RESID^2
Method: Least Squares
Date: 03/26/18 Time: 17:22
Sample (adjusted): 2/14/2000 12/15/2017
Included observations: 4330 after adjustments

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| C | 0.661510 | 0.047550 | 13.91186 | 0.0000 |
| WGT_RESID^2(-1) | -0.055653 | 0.015163 | -3.670228 | 0.0002 |
| WGT_RESID^2(-2) | 0.093102 | 0.015131 | 6.153016 | 0.0000 |
| WGT_RESID^2(-3) | 0.139554 | 0.015048 | 9.273716 | 0.0000 |
| WGT_RESID^2(-4) | 0.085782 | 0.015131 | 5.669350 | 0.0000 |
| WGT_RESID^2(-5) | 0.075759 | 0.015163 | 4.996241 | 0.0000 |

| | | | |
|---|---|---|---|
| R-squared | 0.045374 | Mean dependent var | 1.000373 |
| Adjusted R-squared | 0.044270 | S.D. dependent var | 2.486314 |
| S.E. of regression | 2.430656 | Akaike info criterion | 4.615584 |
| Sum squared resid | 25546.57 | Schwarz criterion | 4.624416 |
| Log likelihood | -9986.740 | Hannan-Quinn criter. | 4.618702 |
| F-statistic | 41.10445 | Durbin-Watson stat | 1.991375 |
| Prob(F-statistic) | 0.000000 | | |

# Application - Shanghai Stock Exchange Composite Index

Improve the ARCH(1) model to GARCH(1,1).

Dependent Variable: R
Method: ML - ARCH (Marquardt) - Normal distribution
Date: 03/26/18 Time: 17:38
Sample (adjusted): 1/24/2000 12/15/2017
Included observations: 4335 after adjustments
Convergence achieved after 14 iterations
Presample variance: backcast (parameter = 0.7)
GARCH = C(5) + C(6)*RESID(-1)^2 + C(7)*GARCH(-1)

| Variable | Coefficient | Std. Error | z-Statistic | Prob. |
|---|---|---|---|---|
| C | 0.000248 | 0.000161 | 1.547554 | 0.1217 |
| R(-4) | 0.023306 | 0.014816 | 1.573036 | 0.1157 |
| R(-6) | -0.048096 | 0.015718 | -3.059952 | 0.0022 |
| R(-13) | 0.022440 | 0.015302 | 1.466447 | 0.1425 |
| Variance Equation | | | | |
| C | 1.30E-06 | 2.11E-07 | 6.163245 | 0.0000 |
| RESID(-1)^2 | 0.071911 | 0.004205 | 17.10038 | 0.0000 |
| GARCH(-1) | 0.925702 | 0.003904 | 237.1380 | 0.0000 |

| | | | |
|---|---|---|---|
| R-squared | 0.006466 | Mean dependent var | 0.000185 |
| Adjusted R-squared | 0.005778 | S.D. dependent var | 0.015941 |
| S.E. of regression | 0.015895 | Akaike info criterion | -5.751623 |
| Sum squared resid | 1.094255 | Schwarz criterion | -5.741329 |
| Log likelihood | 12473.64 | Hannan-Quinn criter. | -5.747989 |
| Durbin-Watson stat | 1.955749 | | |

# Application - Shanghai Stock Exchange Composite Index

The fitted GARCH(1,1) model is

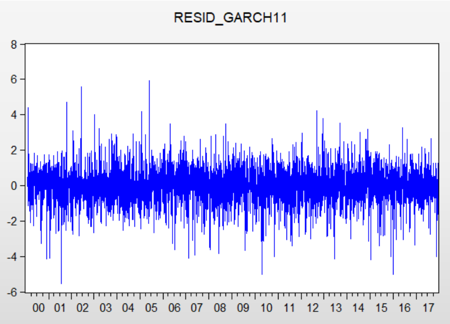
$$r_t = r_t = 0.000248 + 0.0233 r_{t-4} - 0.0481 r_{t-6} + 0.0224 r_{t-13} + \widehat{e}_t, \tag{37}$$

$$\widehat{e}_t = \widehat{\sigma}_t \widehat{z}_t, \tag{38}$$

$$\widehat{\sigma}_t^2 = 0.000000130 + 0.0719 \widehat{e}_{t-1}^2 + 0.9257 \widehat{\sigma}_{t-1}^2. \tag{39}$$

# Application - Shanghai Stock Exchange Composite Index

Standardized residuals of $\widehat{z}_t = \frac{\widehat{e}_t}{\widehat{\sigma}_t}$ for the GARCH(1,1) model.



# Application - Shanghai Stock Exchange Composite Index

ACFs and PACFs for $\widehat{z}_t^2$.

Date: 03/26/18 Time: 17:43
Sample: 1/04/2000 12/15/2017
Included observations: 4335

| Autocorrelation | Partial Correlation | | AC | PAC | Q-Stat | Prob |
|---|---|---|---|---|---|---|
| | | 1 | -0.012 | -0.012 | 0.6533 | 0.419 |
| | | 2 | 0.010 | 0.010 | 1.0822 | 0.582 |
| | | 3 | 0.007 | 0.008 | 1.3147 | 0.726 |
| | | 4 | 0.007 | 0.007 | 1.5408 | 0.819 |
| | | 5 | 0.001 | 0.001 | 1.5432 | 0.908 |
| | | 6 | -0.011 | -0.011 | 2.0671 | 0.913 |
| | | 7 | 0.004 | 0.004 | 2.1518 | 0.951 |
| | | 8 | 0.001 | 0.001 | 2.1582 | 0.976 |
| | | 9 | -0.007 | -0.007 | 2.3623 | 0.984 |
| | | 10 | 0.011 | 0.011 | 2.8543 | 0.985 |
| | | 11 | -0.013 | -0.012 | 3.5631 | 0.981 |
| | | 12 | 0.014 | 0.014 | 4.4662 | 0.973 |
| | | 13 | 0.003 | 0.003 | 4.4994 | 0.985 |
| | | 14 | -0.017 | -0.017 | 5.6943 | 0.974 |
| | | 15 | -0.028 | -0.029 | 9.1562 | 0.869 |
| | | 16 | -0.017 | -0.017 | 10.348 | 0.848 |
| | | 17 | 0.004 | 0.004 | 10.403 | 0.886 |
| | | 18 | -0.017 | -0.015 | 11.601 | 0.867 |
| | | 19 | -0.009 | -0.008 | 11.933 | 0.888 |
| | | 20 | 0.010 | 0.010 | 12.413 | 0.901 |

# Application - Shanghai Stock Exchange Composite Index

LM test for ARCH effects in $\widehat{z}_t$.

Heteroskedasticity Test: ARCH

| | | | |
|---|---|---|---|
| F-statistic | 0.310678 | Prob. F(5,4324) | 0.9068 |
| Obs*R-squared | 1.554987 | Prob. Chi-Square(5) | 0.9066 |

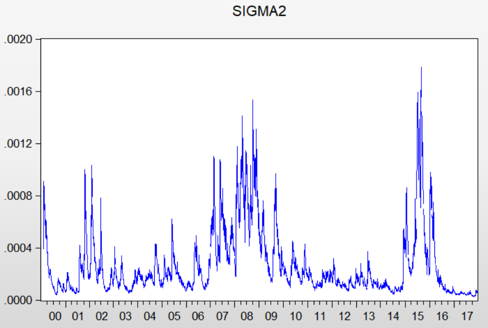
Test Equation:
Dependent Variable: WGT_RESID^2
Method: Least Squares
Date: 03/26/18 Time: 17:47
Sample (adjusted): 2/14/2000 12/15/2017
Included observations: 4330 after adjustments

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| C | 0.989804 | 0.046574 | 21.25219 | 0.0000 |
| WGT_RESID^2(-1) | -0.012402 | 0.015207 | -0.815502 | 0.4148 |
| WGT_RESID^2(-2) | 0.009746 | 0.015209 | 0.640814 | 0.5217 |
| WGT_RESID^2(-3) | 0.007632 | 0.015208 | 0.501857 | 0.6158 |
| WGT_RESID^2(-4) | 0.007316 | 0.015209 | 0.481065 | 0.6305 |
| WGT_RESID^2(-5) | 0.000775 | 0.015208 | 0.050974 | 0.9593 |

| | | | |
|---|---|---|---|
| R-squared | 0.000359 | Mean dependent var | 1.002905 |
| Adjusted R-squared | -0.000797 | S.D. dependent var | 2.088657 |
| S.E. of regression | 2.089489 | Akaike info criterion | 4.313101 |
| Sum squared resid | 18878.43 | Schwarz criterion | 4.321933 |
| Log likelihood | -9331.864 | Hannan-Quinn criter. | 4.316219 |
| F-statistic | 0.310678 | Durbin-Watson stat | 1.982333 |
| Prob(F-statistic) | 0.906797 | | |

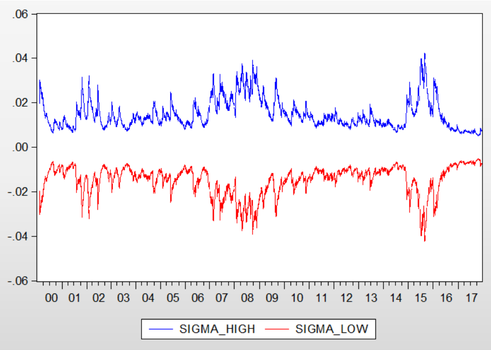
# Application - Shanghai Stock Exchange Composite Index

Estimated sequence of $\widehat{\sigma}_t^2$.



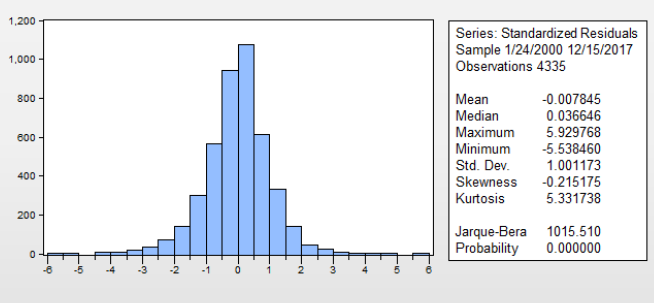
# Application - Shanghai Stock Exchange Composite Index

Upper and lower bounds of $[-\widehat{\sigma}_t, \widehat{\sigma}_t]$.



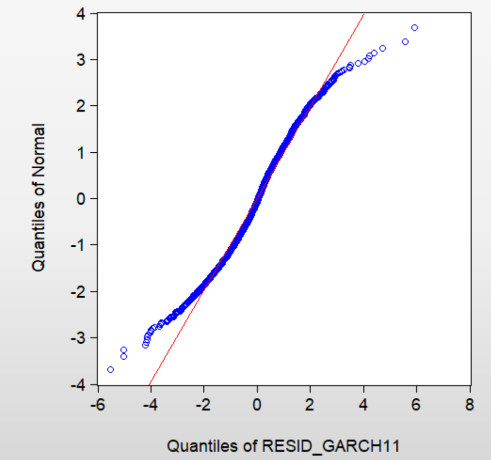
# Application - Shanghai Stock Exchange Composite Index

Descriptive statistics of $\widehat{z}_t$. Does it follow a normal distribution?



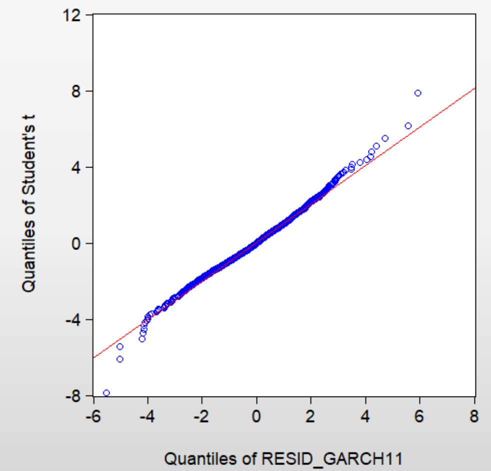
# Application - Shanghai Stock Exchange Composite Index

Quantile-Quantile (QQ) plot of $\widehat{z}_t$ versus normal distribution.



# Application - Shanghai Stock Exchange Composite Index

Quantile-Quantile (QQ) plot of $\widehat{z}_t$ versus t-distribution.



# Application - Shanghai Stock Exchange Composite Index

An estimated Threshold GARCH model to reveal the leverage effects.

Dependent Variable: R
Method: ML - ARCH (Marquardt) - Normal distribution
Date: 03/26/18 Time: 18:05
Sample (adjusted): 1/24/2000 12/15/2017
Included observations: 4335 after adjustments
Convergence achieved after 15 iterations
Presample variance: backcast (parameter = 0.7)
GARCH = C(5) + C(6)*RESID(-1)^2 + C(7)*RESID(-1)^2*(RESID(-1)<0) + C(8)*GARCH(-1)

| Variable | Coefficient | Std. Error | z-Statistic | Prob. |
|---|---|---|---|---|
| C | 0.000160 | 0.000171 | 0.931907 | 0.3514 |
| R(-4) | 0.025209 | 0.014896 | 1.692313 | 0.0906 |
| R(-6) | -0.047172 | 0.015791 | -2.987199 | 0.0028 |
| R(-13) | 0.024038 | 0.015287 | 1.572456 | 0.1158 |
| Variance Equation | | | | |
| C | 1.37E-06 | 2.17E-07 | 6.348414 | 0.0000 |
| RESID(-1)^2 | 0.059776 | 0.004970 | 12.02695 | 0.0000 |
| RESID(-1)^2*(RESID(-1)<0) | 0.023809 | 0.006027 | 3.950193 | 0.0001 |
| GARCH(-1) | 0.925154 | 0.003875 | 238.7735 | 0.0000 |

| | | | |
|---|---|---|---|
| R-squared | 0.006710 | Mean dependent var | 0.000185 |
| Adjusted R-squared | 0.006022 | S.D. dependent var | 0.015941 |
| S.E. of regression | 0.015893 | Akaike info criterion | -5.753061 |
| Sum squared resid | 1.093987 | Schwarz criterion | -5.741297 |
| Log likelihood | 12477.76 | Hannan-Quinn criter. | -5.748908 |
| Durbin-Watson stat | 1.955901 | | |

# Application - Shanghai Stock Exchange Composite Index

The fitted TGARCH model is

$$r_t = 0.00000160 + 0.0252 r_{t-4} - 0.0472 r_{t-6} + 0.0240 r_{t-13} + \widehat{e}_t, \tag{40}$$

$$\widehat{e}_t = \widehat{\sigma}_t \widehat{z}_t, \tag{41}$$

$$\widehat{\sigma}_t^2 = 0.000000137 + 0.0578 \widehat{e}_{t-1}^2 + 0.0238 \widehat{e}_{t-1}^2 1(\widehat{e}_{t-1} < 0) + 0.925 \widehat{\sigma}_{t-1}^2. \tag{42}$$

# Application - Shanghai Stock Exchange Composite Index

An estimated GARCH-in-Mean model to detect risk premium.

Dependent Variable: R
Method: ML - ARCH (Marquardt) - Normal distribution
Date: 03/26/18 Time: 18:07
Sample (adjusted): 1/24/2000 12/15/2017
Included observations: 4335 after adjustments
Convergence achieved after 19 iterations
Presample variance: backcast (parameter = 0.7)
GARCH = C(6) + C(7)*RESID(-1)^2 + C(8)*GARCH(-1)

| Variable | Coefficient | Std. Error | z-Statistic | Prob. |
|---|---|---|---|---|
| GARCH | 1.111361 | 1.272002 | 0.873710 | 0.3823 |
| C | 9.34E-05 | 0.000242 | 0.385995 | 0.6995 |
| R(-4) | 0.023693 | 0.014878 | 1.592454 | 0.1113 |
| R(-6) | -0.047769 | 0.015721 | -3.038506 | 0.0024 |
| R(-13) | 0.022747 | 0.015315 | 1.485288 | 0.1375 |
| Variance Equation | | | | |
| C | 1.30E-06 | 2.17E-07 | 5.999042 | 0.0000 |
| RESID(-1)^2 | 0.072061 | 0.004215 | 17.09705 | 0.0000 |
| GARCH(-1) | 0.925579 | 0.003924 | 235.8831 | 0.0000 |

| | | | |
|---|---|---|---|
| R-squared | 0.005848 | Mean dependent var | 0.000185 |
| Adjusted R-squared | 0.004929 | S.D. dependent var | 0.015941 |
| S.E. of regression | 0.015902 | Akaike info criterion | -5.751340 |
| Sum squared resid | 1.094937 | Schwarz criterion | -5.739576 |
| Log likelihood | 12474.03 | Hannan-Quinn criter. | -5.747187 |
| Durbin-Watson stat | 1.953118 | | |

# Application - Shanghai Stock Exchange Composite Index

The fitted GARCH-M model is

$$r_t = 0.00000934 + 0.0237 r_{t-4} - 0.0478 r_{t-6} + 0.0227 r_{t-13} + 1.11 \widehat{\sigma}_t^2 + \widehat{e}_t \tag{43}$$

$$\widehat{e}_t = \widehat{\sigma}_t \widehat{z}_t, \tag{44}$$

$$\widehat{\sigma}_t^2 = 0.000000130 + 0.0721 \widehat{e}_{t-1}^2 + 0.9258 \widehat{\sigma}_{t-1}^2. \tag{45}$$