4
Forward and Futures Contracts

Case 4

In Cases 2 and 3 we considered a personal pension scheme. Including risky assets in the portfolio improves the expected return in the long term, helping to reduce pension contributions. An inevitable drawback is the risk involved. You would like to explore some instruments available in financial markets to reduce the risk while sustaining the advantages of your pension scheme.

4.1 Forward Contracts

In Section 1.5 we introduced the notion of a forward contract. Let us just recall that this is an agreement to sell or buy the underlying asset at a specified time for a price determined at the initiation of the contract.

4.1.1 Underlying Asset

The market price of the underlying asset at time $t \in [0, T]$ will be denoted by $S(t)$. It is assumed to be strictly positive for all t. We take $t = 0$ to be the present time, $S(0)$ being the spot price of the underlying, known to all investors. The future prices $S(t)$ for $t > 0$ remain unknown, in general.

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Mathematically, $S(t)$ can be represented as a positive random variable on a probability space Ω, that is,

$$S(t) : \Omega \to (0, \infty).$$

The probability space Ω consists of all feasible price movement 'scenarios' $\omega \in \Omega$. We shall write $S^{\omega}(t)$ to denote the price at time t if the market follows scenario $\omega \in \Omega$. The spot price $S(0)$ is simply a positive number, but it can be thought of as a constant random variable. The unknown future prices $S(t)$ for $t > 0$ are non-constant random variables. This means that for each $t > 0$ there are at least two scenarios $\omega, \omega' \in \Omega$ such that $S^{\omega}(t) \neq S^{\omega'}(t)$.

The probability space and the probability distributions of all $S(t)$ are the main ingredients of a market model. In this chapter the discussion does not require specifying any particular model, being robust enough to be valid in any model. The results concerning forward and futures contracts are *model-independent*, which is an important feature.

Holding the underlying asset may involve some intermediate cash flow. For example, if $S(t)$ represents a stock, a dividend may be paid to the shareholders. This is also the case with currencies, which generate interest if deposited at the corresponding rate. On the other hand, in the case of commodities, storage costs may apply. In general, we apply the term *cost of carry* for the cash flow involved in maintaining a long or short position in the underlying asset.

Consider a forward contract exchanged at time 0 with delivery time T and forward price $F(0,T)$. No payment is made by either party at time 0. At delivery, the party with a long forward position will receive (or pay if negative) the amount of $S(T) - F(0,T)$ by buying the underlying asset for $F(0,T)$ and selling it for the market price $S(T)$. The payoff for the short forward position will be $F(0,T) - S(T)$, see Figure 4.1.

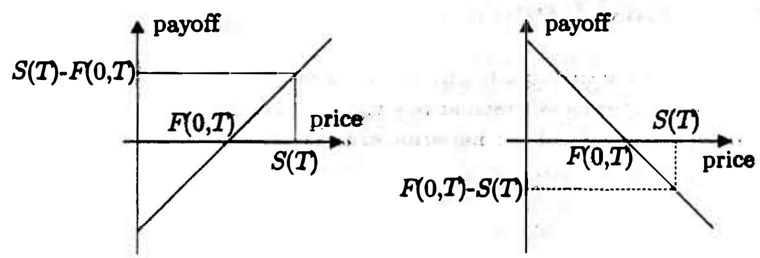

Figure 4.1 Payoff for long and short forward positions at delivery

If the contract is initiated at time $t < T$ rather than 0, then we shall write $F(t,T)$ for the forward price, the payoff at the delivery being $S(T) - F(t,T)$ for a long forward position and $F(t,T) - S(T)$ for a short position.

4.1.2 Forward Price

The No-Arbitrage Principle (Assumption 1.9) makes it possible to obtain formulae for the forward prices of assets of various kinds. Recall Proposition 1.10, where we established the formula

$$F(0,T) = S(0)\frac{A(T)}{A(0)}$$

under the assumption of zero cost of carry.

We also assumed that the future value of the money market account is known at time 0. This amounts to assuming that the interest rate is constant during the whole time period. For example, using continuous compounding we can write $A(t) = A(0)e^{rt}$, so the above formula takes the intuitively clear form

$$F(0,T) = S(0)e^{rT}.$$

Note that $e^{rT} = \frac{1}{B(0,T)}$, where $B(0,T)$ denotes the price of a unit zero-coupon bond maturing at time T. This motivates the following version of Proposition 1.10, where is no longer necessary to assume a constant rate of interest. In addition, we take an arbitrary time t prior to delivery time T as the time when the forward contract is initiated, with the corresponding forward price $F(t,T)$.

Theorem 4.1

For an underlying asset with no cost of carry, if a forward contract is initiated at time $t \leq T$, then the forward price must be

$$F(t,T) = \frac{S(t)}{B(t,T)}. \tag{4.1}$$

Proof

Suppose that $F(t,T) > S(t)/B(t,T)$. In this case, at time t:

- sell $S(t)/B(t,T)$ zero-coupon bonds, receiving the amount $S(t)$;
- buy one share for $S(t)$;
- take a short forward position, that is, agree to sell one share for $F(t,T)$ at time T.

Then, at time T:

- sell the stock for $F(t,T)$ using the forward contract;
- pay the face value $1 for each unit bond sold.

This will bring a risk-free profit of

$$F(t,T) - \frac{S(t)}{B(t,T)} > 0,$$

contrary to the No-Arbitrage Principle.

Next, suppose that $F(t,T) < S(t)/B(t,T)$. In this case construct the opposite strategy to the one above. Namely, at time t:

- sell short one share for $S(t)$;
- buy $S(t)/B(t,T)$ zero-coupon bonds;
- enter into a long forward contract with forward price $F(t,T)$.

Then, at time T:

- receive the face value \$1 for each unit bond held, collecting $S(t)/B(t,T)$ dollars;
- buy the stock for $F(t,T)$ using the forward contract;
- close out the short position in stock by returning it to the owner.

You will end up with a positive amount

$$\frac{S(t)}{B(t,T)} - F(t,T) > 0,$$

again a contradiction with the No-Arbitrage Principle. □

Remark 4.2

Consider an S-maturity bond $B(t,S)$ as an underlying asset. The forward price of the bond is $F(t,T) = B(t,S)/B(t,T)$ for delivery at time T, where $t \leq T \leq S$. In particular, $F(0,S) = 1$, which agrees with the fact that the bond will have value 1 with certainty at time S.

In a market with restrictions on short sales the inequality $F(t,T) < S(t)/B(t,T)$ does not necessarily lead to arbitrage opportunities.

Exercise 4.1

Suppose that $S(0) = 17$ dollars, $F(0,1) = 18$ dollars, $r = 8\%$, and short selling requires a 30% security deposit attracting interest at $d = 4\%$. Is there an arbitrage opportunity? Find the highest rate d for which there is no arbitrage opportunity.

Exercise 4.2

Suppose that the price of stock on 1 April 2000 turns out to be 10% lower than it was on 1 January 2000. Assuming that the risk-free rate is constant at $r = 6\%$, what is the percentage drop of the forward price on 1 April 2000 as compared to that on 1 January 2000 for a forward contract with delivery on 1 October 2000?

Remark 4.3

In the case considered here we always have $F(t,T) = S(t)/B(t,T) > S(t)$ since $B(t,T) < 1$. The difference $F(t,T) - S(t)$, which is called the *basis*, converges to 0 as $t \to T$ since $B(t,T) \to B(T,T) = 1$.

Including Dividends. We shall generalise the formula for the forward price to cover assets that generate income during the lifetime of the forward contract. The income may be in the form of dividends or interest. We shall also cover the case when the asset involves some costs (called the cost of carry), such as storage or insurance, by treating the costs as negative income.

Suppose that the stock will pay a dividend div at an intermediate time t between initiating the forward contract and delivery. At time t the stock price will drop by the amount of the dividend paid. The formula for the forward price, which involves the present stock price, can be modified by subtracting the present value of the dividend.

Theorem 4.4

The forward price of a stock paying dividend div at time t, where $0 < t < T$, is

$$F(0,T) = [S(0) - B(0,t)\mathrm{div}]\frac{1}{B(0,T)}. \tag{4.2}$$

Proof

Suppose that

$$F(0,T) > [S(0) - B(0,t)\mathrm{div}]\frac{1}{B(0,T)}.$$

We shall construct an arbitrage strategy. At time 0:

- enter into a short forward contract with forward price $F(0,T)$ and delivery time T;
- borrow $S(0)$ dollars by issuing $S(0)/B(0,T)$ zero-coupon bonds;
- buy one share;
- enter into div $\frac{B(0,t)}{B(0,T)}$ long forward contracts on a bond maturing at time T with delivery time t and forward price $\frac{B(0,T)}{B(0,t)}$.

At time t:

- cash the dividend div and invest it in bonds $B(t,T)$, buying div $\frac{B(0,t)}{B(0,T)}$ bonds at the forward price $\frac{B(0,T)}{B(0,t)}$.

At time T:

- sell the share for $F(0,T)$;
- pay $S(0)/B(0,T)$ to the bondholders;
- receive the amount div $\frac{B(0,t)}{B(0,T)}$ from the bonds purchased at time t.

The final balance will be positive:

$$F(0,T) - \frac{S(0)}{B(0,T)} + \text{div}\frac{B(0,t)}{B(0,T)} > 0,$$

a contradiction with the No-Arbitrage Principle. The case when $F(0,T) < [S(0) - B(0,t)\text{div}]\frac{1}{B(0,T)}$ can be dealt with similarly by inverting all positions. □

The formula can easily be generalised to the case when dividends are paid more than once:

$$F(0,T) = [S(0) - \text{div}_0]B(0,T), \tag{4.3}$$

where div_0 is the present value of all dividends due during the lifetime of the forward contract.

Exercise 4.3

Consider a stock whose price on 1 January is \$120 and which will pay a dividend of \$1 on 1 July 2000 and \$2 on 1 October 2000. The interest rate is 12%. Is there an arbitrage opportunity if on 1 January 2000 the forward price for delivery of the stock on 1 November 2000 is \$131? If so, compute the arbitrage profit.

Exercise 4.4

Suppose that the risk-free rate is 8%. However, as a small investor, you can invest money at 7% only and borrow at 10%. Does either of the strategies in the proof of Theorem 4.4 give an arbitrage profit if $F(0,1) = 89$ and $S(0) = 83$ dollars, and a \$2 dividend is paid in the middle of the year, that is, at time 1/2?

Dividend Yield. Dividends are often assumed to be paid continuously at a specified rate, rather than at discrete time instants. For example, in a case of a highly diversified portfolio of stocks it is better to avoid analysing frequent payments scattered throughout the year. Another example is foreign currency, attracting interest at the corresponding rate.

Recall that in Section 1.7 we gave a formula for the forward price of foreign currency:

$$F(0,T) = S(0)\frac{1+K_h}{1+K_f} = S(0)\frac{A_h(T)/A_h(0)}{A_f(T)/A_f(0)},$$

where $A_h(t)$, $A_f(t)$ denote the home and foreign currency money market accounts and where K_h, K_f are the corresponding returns. Using the above approach and employing bonds denominated in the home and foreign currencies, we can write this formula as

$$F(0,T) = S(0)\frac{B_f(0,T)}{B_h(0,T)}.$$

Next, suppose that a stock pays dividends continuously at a rate $r_{div} > 0$, called the (continuous) ***dividend yield***. If the dividends are reinvested in the stock, then an investment in one share held at time 0 will increase to become $e^{r_{div}T}$ shares at time T. (The situation is similar to continuous compounding.) Consequently, in order to have one share at time T we should begin with $e^{-r_{div}T}$ shares at time 0. This observation is used in the arbitrage proof below.

Theorem 4.5

The forward price for stock paying dividends continuously at a rate r_{div} is

$$F(0,T) = S(0)e^{-r_{div}T}\frac{1}{B(0,T)}. \tag{4.4}$$

Proof

Suppose that

$$F(0,T) > S(0)e^{-r_{div}T}\frac{1}{B(0,T)}.$$

In this case, at time 0:

- enter into a short forward contract;
- borrow the amount $S(0)e^{-r_{div}T}$ to buy $e^{-r_{div}T}$ shares.

Between time 0 and T collect the dividends paid continuously, reinvesting them in the stock. At time T you will have 1 share. At that time:

- sell the share for $F(0,T)$, closing out the short forward position;
- pay $S(0)e^{-r_{div}T}/B(0,T)$ to clear the loan with interest.

The final balance $F(0,T) - S(0)e^{-r_{div}T}/B(0,T) > 0$ will be your arbitrage profit. The argument in the case when

$$F(0,T) < S(0)e^{-r_{div}T}\frac{1}{B(0,T)}$$

is similar and involves the opposite positions. □

In general, if the contract is initiated at time $t < T$, then

$$F(t,T) = S(t)e^{-r_{div}(T-t)}\frac{1}{B(t,T)}. \tag{4.5}$$

Exercise 4.5

A US importer of German cars wants to arrange a forward contract to buy euros in half a year. The interest rates for investments in US dollars and euros are $r_\$ = 4\%$ and $r_€ = 3\%$, respectively, the current exchange rate being 0.9834 euros to a dollar. What is the forward price of euros expressed in dollars (that is, the forward exchange rate)?

4.1.3 Value of Forward Contract

Every forward contract has value zero when initiated. As time goes by, the price of the underlying asset may change. Along with it, the value of the forward contract will vary and will no longer be zero, in general. In particular, the value of a long forward contract will be $S(T) - F(0,T)$ at delivery, which may turn out to be positive, zero or negative. We shall derive formulae to capture the changes in the value of a forward contract.

Suppose that the forward price $F(t,T)$ for a forward contract initiated at time t, where $0 < t < T$, is higher than $F(0,T)$. This is good news for an investor with a long forward position initiated at time 0. At time T such an investor will gain $F(t,T) - F(0,T)$ as compared to an investor entering into a new long forward contract at time t with the same delivery date T. To find the value of the original forward position at time t all we have to do is to discount this gain back to time t. This discounted amount would be received (or paid, if negative) by the investor with a long position should the forward contract initiated at time 0 be closed out at time t, that is, prior to delivery T. This intuitive argument needs to be supported by a rigorous arbitrage proof.

Theorem 4.6

For any t such that $0 \le t \le T$ the time t value of a long forward contract with forward price $F(0,T)$ is given by

$$V(t) = [F(t,T) - F(0,T)]B(t,T). \tag{4.6}$$

Proof

At time t the holder of the long forward position may simultaneously enter the short position at no cost, which is done with the forward price prevailing at that time, namely $F(t,T)$. The final payoff of the combined position is

$$S(T) - F(0,T) - [S(T) - F(t,T)] = F(t,T) - F(0,T)$$

which is non-random, that is, risk free. Therefore its time t value is the discounted T value, with $B(t,T)$ as the discount factor. □

For a stock paying no dividends formula (4.6) gives

$$V(t) = \left[\frac{S(t)}{B(t,T)} - \frac{S(0)}{B(0,T)}\right]B(t,T) = S(t) - S(0)\frac{B(t,T)}{B(0,T)}. \tag{4.7}$$

The message is this: if the stock price grows at the same rate as a risk-free investment, then the value of the forward contract will be zero. The growth above the risk-free rate results in a gain for the holder of a long forward position.

Remark 4.7

Consider a contract with delivery price X rather than $F(0,T)$. The value of this contract at time t will be given by (4.6) with $F(0,T)$ replaced by X,

$$V_X(t) = [F(t,T) - X]B(t,T).$$

Such a contract may have non-zero value initially. In the case of a stock paying no dividends

$$V_X(0) = [F(0,T) - X]B(0,T) = S(0) - XB(0,T). \tag{4.8}$$

Exercise 4.6

Suppose that the price of a stock is \$45 at the beginning of the year, the risk-free rate is 6%, assumed constant, and a \$2 dividend is to be paid after half a year. For a long forward position with delivery in one year, find its value after 9 months if at that time the stock price turns out to be a) \$49, b) \$51.

4.2 Futures

One of the two parties to a forward contract will be losing money. There is always some risk of default by the party suffering a loss. Futures contracts are designed to eliminate such risk.

Just like a forward contract, a *futures contract* involves an underlying asset, let it be a stock with price $S(t)$, and a specified time of delivery T. In addition to the usual stock prices, the market dictates the so-called *futures prices* $f(t,T)$ for each $t \leq T$. These prices are unknown at time 0, except for $f(0,T)$, and we shall treat them as random variables.

As in the case of a forward contract, it costs nothing to initiate a futures position. The difference lies in the cash flow during the lifetime of the contract. A long forward contract involves just a single payment $S(T) - F(0,T)$ at delivery. A futures contract involves a random cash flow, known as *marking to market*. Namely, for prescribed time instants $t_1 < t_2 < \cdots < t_N = T$, typically these will be consecutive trading days, the holder of a long futures position will receive at time t_n, $n = 1, \ldots, N$ the amount

$$f(t_n,T) - f(t_{n-1},T)$$

if positive, or will have to pay it if negative. The opposite payments apply for a short futures position. The following two conditions are imposed:

1) the futures price at delivery is $f(T,T) = S(T)$;
2) for each time $t_n \leq T$, $n = 1, \ldots, N$ the futures price $f(t_n,T)$ is such that it costs nothing to enter a futures contract at that time.

4.2.1 Pricing

We shall show that in some circumstances the forward and the futures prices are the same. Let r be the risk-free rate under continuous compounding.

Theorem 4.8

If the interest rate is constant, then $f(0,T) = F(0,T)$.

Proof

Suppose for simplicity that marking to market is performed at just one intermediate time instant $0 < t < T$. The argument below can easily be extended to cover more frequent marking to market.

Suppose $f(0,T) > F(0,T)$. If so, then at time 0:

- take a long forward position (at no cost);
- open a fraction $e^{-r(T-t)}$ of a short futures position (at no cost).

At time t:

- pay the amount $e^{-r(T-t)}[f(t,T) - f(0,T)]$ as a result of marking to market;
- borrow $e^{-r(T-t)}[f(t,T) - f(0,T)]$ risk free;
- increase the short futures position to 1 contract (at no cost).

Note that the difference $f(t,T) - f(0,T)$ can be negative in which case 'pay' means 'receive' and 'borrow' means 'invest'. Then at time T:

- close the risk-free investment paying $f(t,T) - f(0,T)$;
- close the short futures position paying $S(T) - f(t,T)$ (since $S(T) = f(T,T)$);
- close the long forward position receiving $S(T) - F(0,T)$.

The final balance will be $f(0,T) - F(0,T) > 0$, which means that the strategy is an arbitrage opportunity, a contradiction.

The case when $f(0,T) < F(0,T)$ can be dealt with similarly by taking the opposite positions. □

The above argument applies to any time t so that $f(t,T) = F(t,T)$. The construction cannot be performed if the interest rate changes unpredictably. However if interest rate movements are known in advance, then the argument can be suitably modified and the equality between the futures and forward prices remains valid.

In an economy with constant interest rates r we obtain a simple structure of futures prices,

$$f(t,T) = S(t)e^{r(T-t)} \tag{4.9}$$

if the stock pays no dividends. The futures prices are random, but this is caused entirely by the randomness of the underlying asset price. If the futures prices depart from the values given by (4.9), this reflects the market's view of future interest rate changes.

Example 4.9

Consider three scenarios with the same stock price at maturity of the contract. Let $T = 2/365$ be two days from now with marking-to-market days $t_1 = 1/365$ and $t_2 = T$.

1. The stock price goes up at the rate r. Then $S(t) = S(0)e^{rt}$,

$$f(t,T) = S(0)e^{rt}e^{r(T-t)} = S(0)e^{rT}$$

for all t, and marking-to-market payments are zero.

2. The price goes up more than in scenario 1, so on day one $S(t_1) > S(0)e^{rt_1}$. Then
$$f(t_1, T) = S(t_1)e^{r(T-t_1)} > f(0, T) = S(0)e^{rT},$$
with positive cash flow to the holder of a long futures position. Next, $S(T) = S(0)e^{rT} = f(T, T)$, which means that a negative cash flow on day two
$$f(T, T) - f(t_1, T) = S(0)e^{rT} - S(t_1)e^{r(T-t_1)} = -(f(t_1, T) - f(0, 0)),$$
exactly offsetting the payment received on day one.
3. The price on day one is below $S(0)e^{rt_1}$ so first we have a negative payment followed by a positive payment of the same size given by the above formulae.

The difference in the timing of the balancing payments is crucial. Due to the time value of money the long position in scenario 2 is better than in scenario 3.

This example also shows an important benchmark for the profitability of a futures position. An investor who wants to take advantage of a predicted increase in the price of stock above the risk-free rate should enter into a long futures position. A short futures position will bring a profit should the stock price go down or increase below the risk-free rate.

4.2.2 Margins

To ensure that the obligations involved in a futures position are fulfilled, certain practical regulations are enforced. Each investor entering into a futures contract has to pay a deposit, called the *initial margin*, which is kept by the clearing house as collateral. In the case of a long futures position the amount $f(t_n, T) - f(t_{n-1}, T)$ is added to the deposit if positive or subtracted if negative at each time step n, typically once a day. (The opposite amount is added or subtracted for a short futures position.) Any excess that builds up above the initial margin can be withdrawn by the investor. On the other hand, if the deposit drops below a certain level, called the *maintenance margin*, the clearing house will issue a *margin call*, requesting the investor to make a payment and restore the deposit to the level of the initial margin. A futures position can be closed at any time, in which case the deposit will be returned to the investor. In particular, the futures position will be closed immediately by the clearing house if the investor fails to respond to a margin call. As a result, the risk of default is eliminated.

Example 4.10

Suppose that the initial margin is set at 10% and the maintenance margin at 5% of the futures price. The table below shows a scenario with futures prices

$f(t, T)$. The columns labelled 'margin 1' and 'margin 2' show the deposit at the beginning and at the end of each day, respectively. The 'payment' column contains the amounts paid to top up the deposit (negative numbers) or withdrawn (positive numbers).

t-days	$f(t, T)$	cash flow	margin 1	payment	margin 2
0	140	opening:	0	−14	14
1	138	− 2	12	0	12
2	130	− 8	4	− 9	13
3	140	+10	23	+ 9	14
4	150	+10	24	+ 9	15
		closing:	15	+15	0
			total:	10	

On day 0 a futures position is opened and a 10% deposit paid. On day 1 the futures price drops by \$2, which is subtracted from the deposit. On day 2 the futures price drops further by \$8, triggering a margin call because the deposit falls below 5%. The investor has to pay \$9 to restore the deposit to the 10% level. On day 3 the forward price increases and \$9 is withdrawn, leaving a 10% margin. On day 4 the forward price goes up again, allowing the investor to withdraw another \$9. At the end of the day the investor decides to close the position, collecting the balance of the deposit. The total of all payments is \$10, the increase in the futures price between day 0 and 4.

Remark 4.11

An important feature of the futures market is liquidity. This is possible due to standardisation and the presence of a clearing house. Only futures contracts with particular delivery dates are traded. Moreover, futures contracts on commodities such as gold or timber specify standardised delivery arrangements as well as standardised physical properties of the assets. The clearing house acts as an intermediary, matching the total of a large number of short and long futures positions of various sizes. The clearing house also maintains the margin deposit for each investor to eliminate the risk of default. This is in contrast to forward contracts negotiated directly between two parties.

An investor believing that the price of certain stock will go up can invest in the stock directly but can also take a long futures position. The investment required for the latter is the initial margin which is just a percentage of the stock price. If the prediction is correct, since the money engaged is smaller, the result is more impressive. This is best illustrated by an example.

Example 4.12

Let $S(0) = 100$ dollars, $r = 6\%$, $T = 0.5$ and let marking to market be restricted to just one instant $t_1 = 0.25$, equal to the investment horizon. If the initial deposit is 20% of the price, the investor can open 5 futures positions. Suppose $S(t_1) = 108$ dollars and then $f(t_1, T) \cong 106.39$ dollars, which together with $f(0,T) \cong 103.05$ dollars gives a marking-to-market gain of about \$16.73. This is more than double the gain of \$8 from a direct investment in stock.

4.2.3 Hedging with Futures

One relatively simple way to reduce an exposure to stock price variations is to enter a forward contract. However, a contract of this kind may not be readily available, not to mention the risk of default. Another possibility is to use the futures market, which is well-developed, liquid and protected from the risk of default.

Example 4.13

Let $S(0) = 100$ dollars and let the risk-free rate be constant at $r = 8\%$. Assume that marking to market takes place once a month. Suppose that we wish to sell the stock after 3 months. To hedge the exposure to stock price variations we enter into one short futures contract on the stock with the same maturity. The payments resulting from marking to market are invested (or borrowed), attracting interest at the risk-free rate. The results for two different stock price scenarios are displayed below. The column labelled 'm2m' represents the payments due to marking to market and the last column shows the interest accrued up to the delivery date.

Scenario 1

t-months	$S(t)$	$f(t,3/12)$	m2m	interest
0	100	102.02		
1	102	103.37	−1.35	−0.02
2	101	101.67	+1.69	+0.01
3	105	105.00	−3.32	0.00
		total:	−2.98	−0.01

In this scenario we sell the stock for \$105.00, but marking to market brings losses, reducing the sum to $105.00 - 2.98 - 0.01 = 102.01$ dollars. Note that if the marking-to-market payments were not invested at the risk-free rate, then the realised sum would be $105.00 - 2.98 = 102.02$ dollars, that is, exactly equal to the futures price $f(0, 3/12)$.

Scenario 2

t-months	$S(t)$	$f(t,3/12)$	m2m	interest
0	100	102.02		
1	98	99.31	+2.70	+0.04
2	97	97.65	+1.67	+0.01
3	92	92.00	+5.65	0.00
		total:	+10.02	+0.05

In this case we sell the stock for \$92.00 and benefit from marking to market along with the interest earned, bringing the final sum to $92.00 + 10.02 + 0.05 = 102.07$ dollars. Without the interest the final sum would be $92.00 + 10.02 = 102.02$ dollars, once again exactly the futures price $f(0, 3/12)$.

In reality the calculations in Example 4.13 are slightly more complicated because of the presence of the initial margin, which we have neglected for simplicity. Some limitations come from the standardisation of futures contracts. As a result, a difficulty may arise in matching the terms of the contract to our needs. The exercise dates for futures are typically certain fixed days in four specified months in a year, for example, the third Friday in March, June, September and December. If we want to close out our investment at the end of April, we will need to hedge with futures contracts with delivery date beyond the end of April.

Example 4.14

Suppose we wish to sell stock after 2 months and we hedge using futures with delivery in 3 months (we work in the same scenarios as in Example 4.13).

Scenario 1

t-months	$S(t)$	$f(t,3/12)$	m2m	interest
0	100	102.02		
1	102	103.37	−1.35	−0.01
2	101	101.67	+1.69	0.00
		total:	+0.34	−0.01

We sell the stock for \$101.00, which together with marking to market and interest will give \$101.33.

Scenario 2

t-months	$S(t)$	$f(t,3/12)$	m2m	interest
0	100	102.02		
1	98	99.31	+2.70	+0.02
2	97	97.65	+1.67	0.00
		total:	+4.37	+0.02

In this case we sell the stock for \$97.00, and together with marking to market and interest obtain \$101.39.

We almost hit the target, which is the futures price $f(0,2) \cong 101.34$ dollars, that is, the value of \$100 compounded at the risk-free rate.

Remark 4.15

The difference between the spot and futures prices is called the *basis* (as for forward contracts):

$$b(t,T) = S(t) - f(t,T).$$

(Sometimes the basis is defined as $f(t,T) - S(t)$.) The basis converges to zero as $t \to T$, since $f(T,T) = S(T)$. In a market with constant interest rates it is given explicitly by

$$b(t,T) = S(t)(1 - e^{r(T-t)}),$$

being negative for $t < T$. If the asset pays dividends at a rate $r_{\text{div}} > r$, then:

$$b(t,T) = S(t)(1 - e^{(r-r_{\text{div}})(T-t)}).$$

Going back to the problem of designing a hedge, suppose that we wish to sell an asset at time $t < T$. To protect ourselves against a decrease in the asset price, at time 0 we can short a futures contract with futures price $f(0,T)$. At time t we shall receive $S(t)$ from selling the asset plus the cash flow $f(0,T) - f(t,T)$ due to marking to market (for simplicity, we neglect any intermediate cash flow, assuming that t is the first instance when marking to market takes place), that is, we obtain

$$f(0,T) + S(t) - f(t,T) = f(0,T) + b(t,T).$$

The price $f(0,T)$ is known at time 0, so the risk involved in the hedging position will be related to the unknown level of the basis. This uncertainty is mainly concerned with unknown future interest rates.

If the goal of a hedger is to minimise risk, it may be best to use a certain optimal hedge ratio, that is, to enter into N futures contracts, with N not necessarily equal to the number of shares of the underlying asset held. To see

this compute the risk as measured by the variance of the basis $b_N(t,T) = S(t) - Nf(t,T)$:

$$\text{Var}(b_N(t,T)) = \sigma^2_{S(t)} + N^2\sigma^2_{f(t,T)} - 2N\sigma_{S(t)}\sigma_{f(t,T)}\rho_{S(t)f(t,T)},$$

where $\rho_{S(t)f(t,T)}$ is the correlation coefficient between the spot and futures prices, and $\sigma_{S(t)}, \sigma_{f(t,T)}$ are the standard deviations. The variance is a quadratic function in N and has a minimum at

$$N = \rho_{S(t)f(t,T)}\frac{\sigma_{S(t)}}{\sigma_{f(t,T)}},$$

which is the optimal hedge ratio.

Exercise 4.7

Find the optimal hedge ratio if the interest rates are constant.

4.2.4 Index Futures

A stock exchange index is a weighted average of a selection of stock prices with weights proportional to the market capitalisation of stocks. An index of this kind will be approximately proportional to the value of the market portfolio (see Chapter 3) if the chosen set of stocks is large enough. For example, the Standard and Poor Index S&P500 is computed using 500 stocks, representing about 80% of trade at the New York Stock Exchange. For the purposes of futures markets the index can be treated as a security. This is because the index can be identified with a portfolio, even though in practice transaction costs would impede trading in this portfolio. The futures prices $f(t,T)$, expressed in index points, are assumed to satisfy the same conditions as before. Marking to market is given by the difference $f(t_n,T) - f(t_{n-1},T)$ multiplied by a fixed amount (\$500 for futures on S&P500). If the number of stocks included in the index is large, it is possible and convenient to assume that the index is an asset with dividends paid continuously.

Exercise 4.8

Suppose that the value of a stock exchange index is 13,500, the futures price for delivery in 9 months is 14,100 index points, and the interest rate is 8%. Find the dividend yield.

Our goal in this section is to study applications of index futures for hedging based on the Capital Asset Pricing Model introduced in Chapter 3. As we know, see (3.25), the expected return on a portfolio over a time step of length t is given by

$$\mu_V = R + \beta_V(\mu_M - R),$$

where β_V is the beta factor of the portfolio, μ_M is the expected return on the market portfolio and R is the risk-free return over a single time step τ. By $V(0)$ we denote the initial value of the portfolio. We assume for simplicity that the index is equal to the value of the market portfolio, so that the futures prices are given by

$$f(t,T) = M(t)e^{r(T-t)},$$

$M(t)$ being the value of the market portfolio and r the continuously compounded rate such that $e^{rt} = 1 + R$.

We can form a new portfolio with value $\widetilde{V}(0)$ by supplementing the original portfolio with N short futures contracts on the index with delivery time T. The initial value of the new portfolio is the same as the value $V(0)$ of the original portfolio, since it costs nothing to initiate a futures contract. At the first marking-to-market instant t the value of the new portfolio will be

$$\widetilde{V}(t) = V(t) - N(f(t,T) - f(0,T)).$$

The return on the new portfolio over the period $[0,t]$ will be

$$K_{\widetilde{V}} = \frac{\widetilde{V}(t) - \widetilde{V}(0)}{\widetilde{V}(0)} = \frac{V(t) - N(f(t,T) - f(0,T)) - V(0)}{V(0)}.$$

We shall show that the beta factor $\beta_{\widetilde{V}}$ of the new portfolio can be modified arbitrarily by a suitable choice of the futures position N.

Proposition 4.16

For any given number a, if

$$N = (\beta_V - a)\frac{V(0)e^{tr}}{f(0,T)},$$

then $\beta_{\widetilde{V}} = a$.

Proof

We shall compute the beta factor from the definition:

$$\begin{aligned}\beta_{\widetilde{V}} &= \mathrm{Cov}(K_{\widetilde{V}}, K_M)/\sigma_M^2 \\ &= \mathrm{Cov}(K_V, K_M)/\sigma_M^2 - \frac{1}{V(0)}\mathrm{Cov}(N(f(t,T) - f(0,T)), K_M)/\sigma_M^2,\end{aligned}$$

where K_M is the return on the market portfolio and K_V the return on the portfolio without futures. Since $\mathrm{Cov}(f(0,T), K_M) = 0$ and covariance is linear with respect to each argument,

$$\mathrm{Cov}(N(f(t,T) - f(0,T)), K_M) = N\mathrm{Cov}(f(t,T), K_M).$$

Inserting the futures price $f(t,T) = M(t)e^{r(T-t)}$, we have

$$\mathrm{Cov}(f(t,T), K_M) = e^{r(T-t)}\mathrm{Cov}(M(t), K_M).$$

Again by the linearity of covariance in each argument

$$\mathrm{Cov}(M(t), K_M) = M(0)\mathrm{Cov}\left(\frac{M(t) - M(0)}{M(0)}, K_M\right) = M(0)\sigma_M^2.$$

Subsequent substitutions give

$$\beta_{\widetilde{V}} = \beta_V - \frac{NM(0)e^{r(T-t)}}{V(0)} = \beta_V - N\frac{f(0,T)}{V(0)e^{rt}},$$

which implies the asserted property. □

Corollary 4.17

If $a = 0$, then $\mu_{\widetilde{V}} = R$.

Example 4.18

Suppose that the index drops from $M(0) = 890$ down to $M(t) = 850$, that is, by 4.49% within one time step. Suppose further that the risk-free rate for the period of length t is $r = 1\%$, that is, $1 + R = e^{tr} = 1.1$. This means that the futures prices on the index (with delivery after 3 steps of length t) are

$$\begin{aligned}f(0,3t) &= M(0)(1+R)^3 = 890 \times 1.01^3 \cong 916.97, \\ f(t,3t) &= M(t)(1+R)^2 = 850 \times 1.01^2 \cong 867.09.\end{aligned}$$

Consider a portfolio with $\beta_V = 1.5$ and initial value $V(0) = 100$ dollars. This portfolio will have negative expected return

$$\begin{aligned}\mu_V &= R + (\mu_M - R)\beta_V \\ &\cong 1\% + (-4.49\% - 1\%)1.5 \cong -7.24\%.\end{aligned}$$

To construct a new portfolio with $\beta_{\widetilde{V}} = 0$ we can supplement the original portfolio with

$$N = \beta_V\frac{(1+R)V(0)}{f(0,3t)} \cong 1.5 \times \frac{1.01 \times 100}{916.97} \cong 0.1652$$

short forward contracts on the index with delivery after 3 steps.

Suppose that the actual return on the original portfolio during the first time step happens to be equal to the expected return. This gives $V(t) \cong 92.76$ dollars. Marking to market gives a payment of

$$-N(f(t,3t) - f(0,3t)) \cong -0.1652 \times (867.09 - 916.97) \cong 8.24$$

dollars due to the holder of $N \cong 0.1652$ short forward contracts. This makes the new portfolio worth

$$\tilde{V}(t) = V(t) - N(f(t,3t) - f(0,3t)) \cong 92.76 + 8.24 = 101.00$$

dollars at time t, matching the risk-free growth exactly.

Exercise 4.9

Perform the same calculations in the case when the index increases from 890 to 920.

Remark 4.19

The ability to adjust the beta factor of a portfolio is valuable to investors who may wish either to reduce or to magnify the systematic risk. For example, suppose that an investor is able to design a portfolio with superior average performance to that of the market. By entering into a futures position such that the beta factor of the resulting portfolio is zero, the investor will be hedged against adverse movements of the market. This is crucial in the event of recession, so that the superior performance of the portfolio as compared to the market can be turned into a profit despite a decline in the market. On the other hand, should the market show some growth, the expected return on the hedged portfolio will be reduced by comparison because the futures position will result in a loss.

It needs to be emphasised that this type of hedging with futures works only on average. In particular, setting the beta factor to zero will not make the investment risk free.

Let us conclude this chapter with a surprising application of index futures.

Example 4.20

In emerging markets short sales are rarely available. This was the case in Poland in the late 1990's. However, index futures were traded. Due to the fact that one of the indices (WIG20) was composed of 20 stocks only, it was possible to

manufacture a short sale of any stock among those 20 by entering into a short futures position on the index, combined with purchasing a suitable portfolio of the remaining 19 stocks. With a larger number of stocks comprising the index the transaction costs would have been too high to make such a construction practical.

Case 4: Discussion

First of all, we should discuss the choice between forward and futures contracts. Futures are more likely to be available, but margin maintenance may be causing difficulties. In addition to the initial deposit, we might face short term liquidity problems, so we consider forward contracts.

In the first phase we are building up our portfolio by purchasing assets. Hence, we should be worried about price increases. To hedge this risk we should take long positions. For instance, consider some stock, which we wish to include in our portfolio, with current price $S(0) = 100$ dollars. Taking a long forward position with maturity T corresponding to the nearest time when we will be buying more shares, we commit ourselves to buying the stock for $S(0)e^{rT}$. Let $T = 0.5$, $r = 5\%$, so the forward price is 102.53 dollars. This is less than the price related to our average growth assumptions, which for $\mu = 10\%$ (with continuous compounding) is $S(0)e^{\mu T} = 105.13$, approximately. Thus, from the funds devoted to the stock investment we would be able to buy more units at the forward price or save some money. Namely, if we plan to invest 1000 dollars in stock, using the prices based on the expected growth we would buy 9.51 shares, but these at the forward price would cost just 975.31 dollars.

Unfortunately, if the price $S(T)$ is lower than $S(0)e^{rT}$, which may happen even if the long term expected growth is at the assumed level, we would not benefit from this, since the forward contract is binding. To have more flexibility, we have to learn about more sophisticated instruments.

In the second phase, when we will be gradually liquidating our portfolio by paying ourselves a pension, a short position in forwards or futures would preserve our return at the level of the risk-free rate. Given a share worth $S(0)$ (we moved our clock, and 0 is now some time in the future), we can sell it for the forward price $S(0)e^{rT}$. This is favourable should the market price drop, unfavourable otherwise, but on the whole makes little sense. If we are to earn the risk-free rate, we can invest risk free.