# 5 Options: General Properties

## Case 5

European call and put options with strike price 98 euros on a German company G with current stock price 99 euros cost 4.145 and 2.738 euros, respectively. The stock of a British company B costs 66 pounds, and European call and put options on B with strike 65 pounds and exercise time in one month cost 2.55 and 1.34 pounds, respectively. A European call option on the pound with exercise price equal to the current rate, 1.5 euros to a pound, costs 14.40 euros and a put costs 13.16 euros for 1000 units. All these options have exercise time in one month. The world is ideal here, no transaction costs, long and short positions can be taken at the same prices since we assume that the reader is employed by a large financial institution for the purpose of spotting any instances of mispricing.

## 5.1 Definitions

In Chapter 1 we have seen simple examples of call and put options in a one-step discrete time setting. Here we give the definitions and then establish some fundamental properties of options. These properties are universal in the sense that they are valid in any market model, and the arguments are based on the No-Arbitrage Principle alone.

M. Capiński, T. Zastawniak, *Mathematics for Finance*,
Springer Undergraduate Mathematics Series,

A *European call option* is a contract giving the holder the right to buy an asset, called the *underlying*, for a price $X$ fixed in advance, known as the *exercise price* or *strike price*, at a specified future time $T$, called the *exercise* or *expiry time*. A *European put option* gives the right to sell the underlying asset for the strike price $X$ at the exercise time $T$.

An *American call* or *put option* gives the right to buy or, respectively, to sell the underlying asset for the strike price $X$ at any time between now and a specified future time $T$, called the *expiry time*. In other words, an American option can be exercised at any time up to and including expiry.

The term 'underlying asset' has quite general scope. Apart from typical assets such as stocks, commodities or foreign currency, there are options on stock indices, interest rates, or even on the snow level at a ski resort. Some underlying assets may be impossible to buy or sell. The option is then cleared in cash in a fashion which resembles settling a bet. For example, the holder of a European call option on the Standard and Poor Index (see page 107) with strike price 800 will gain if the index turns out to be 815 on the exercise date. The writer of the option will have to pay the holder an amount equal to the difference $815 - 800 = 15$ multiplied by a fixed sum of money, say by \$100. No payment will be due if the index turns out to be lower than 800 on the exercise date.

An option is determined by its payoff, which for a European call is

$$\begin{cases} S(T) - X & \text{if } S(T) > X, \\ 0 & \text{otherwise.} \end{cases}$$

This payoff is a random variable, contingent on the price $S(T)$ of the underlying on the exercise date $T$. (This explains why options are often referred to as *contingent claims*.) It is convenient to use the notation

$$x^+ = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

for the *positive part* of a real number $x$. Then the payoff of a European call option can be written as $(S(T) - X)^+$. For a put option the payoff is $(X - S(T))^+$.

Since the payoffs are always non-negative and with positive probability they are strictly positive, a premium must be paid to buy an option. If no premium had to be paid, an investor purchasing an option could under no circumstances lose money and would in fact make a profit whenever the payoff turned out to be positive. This would be contrary to common sense and in particular to the No-Arbitrage Principle. The premium is the market price of the option.

Establishing bounds and some general properties for option prices is the primary goal of the present chapter. The prices of calls and puts will be denoted by $C_E, P_E$ for European options and $C_A, P_A$ for American options, respectively. The same constant interest rate $r$ will apply for lending and borrowing money without risk, and continuous compounding will be used.



## Example 5.1

On 20 October 2007 European calls on Cadbury Schweppes PLC stock with strike price 640 pence to be exercised on 21 December 2007 traded at 22.5 pence at the Euronext London International Financial Futures Exchange (LIFFE). Suppose that the purchase of such an option was financed by a loan at 5.23% compounded continuously, so that $22.5e^{0.0523 \times \frac{2}{12}} \cong 22.7$ pence would have to be paid back on the exercise date. The investment would bring a profit if the stock price turned out to be higher than $640 + 22.7 = 662.7$ pence on the exercise date.

### *Exercise 5.1*

Find the stock price on the exercise date for a European put option with strike price \$36 and exercise date in three months to produce a profit of \$3 if the option is bought for \$4.50, financed by a loan at 12% compounded continuously.

The gain of an option holder (writer) is the payoff modified by the premium $C_E$ or $P_E$ paid (received) for the option. At time $T$ the gain of the holder of a European call is $(S(T) - X)^+ - C_E e^{rT}$, where the time value of the premium is taken into account. In other words, we include the opportunity cost of an alternative risk-free investment. For the holder of a European put the gain is $(X - S(T))^+ - P_E e^{rT}$. These gains are illustrated in Figure 5.1. For the writer of an option the gains are $C_E e^{rT} - (S(T) - X)^+$ for a call and $P_E e^{rT} - (X - S(T))^+$ for a put option. Note that the potential loss for the holder of a call or put is always limited to the premium paid. For the writer the loss can be much higher, even unbounded in the case of a call option.

### *Exercise 5.2*

Find the expected gain (or loss) for a holder of a European call option with strike price \$90 to be exercised in 6 months if the stock price on the exercise date may turn out to be \$87, \$92 or \$97 with probability $\frac{1}{3}$ each, given that the option is bought for \$8, financed by a loan at 9% compounded continuously.

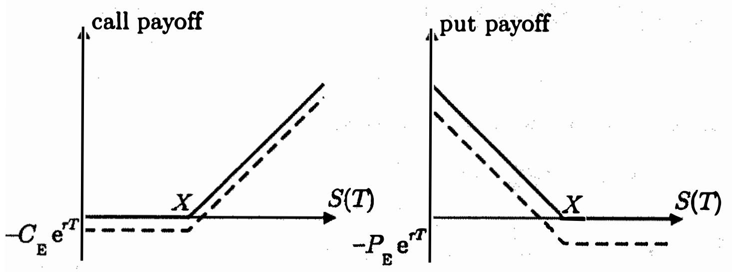


**Figure 5.1** Payoffs (solid lines) and gains (broken lines) for the holder of a European call or put option

## 5.2 Put-Call Parity

In this section we shall make an important link between the prices of European call and put options.

Consider a portfolio constructed by writing and selling one put and buying one call option, each with the same strike price $X$ and exercise date $T$. Adding the payoffs of the long position in the call and the short position in the put, we obtain the payoff of a long forward contract with forward price $X$ and delivery time $T$. Indeed, if $S(T) \geq X$, then the call will pay $S(T) - X$ and the put will be worthless. If $S(T) < X$, then the call will be worth nothing and the writer of the put will need to pay $X - S(T)$. In either case, the value of the portfolio will be $S(T) - X$ at expiry, the same as for the long forward position, see Figure 5.2. As a result, the current value of such a portfolio of options should be that of the forward contract, which is $S(0) - Xe^{-rT}$ if the interest rate $r$ is constant, see Remark 4.7.

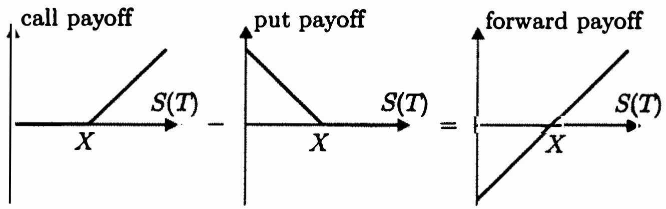


**Figure 5.2** Long forward payoff constructed from a call and a put

This motivates the theorem below. Even though the theorem follows from the above intuitive argument, we shall give another proof with a view to possible generalisations.



### Theorem 5.2 (Put-Call Parity)

For a stock that pays no dividends the following relation holds between the prices of European call and put options, each with exercise price $X$ and exercise time $T$:

$$C_E - P_E = S(0) - Xe^{-rT}. \tag{5.1}$$

### Proof

Suppose that

$$C_E - P_E > S(0) - Xe^{-rT}. \tag{5.2}$$

In this case an arbitrage strategy can be constructed as follows. At time 0:

- buy one share for $S(0)$;
- buy one put option for $P_E$;
- write and sell one call option for $C_E$;
- invest the sum $C_E - P_E - S(0)$ (borrow, if negative) in the money market at the interest rate $r$.

The balance of these transactions is 0. Then, at time $T$:

- close out the money market position, collecting (paying, if negative) the sum $(C_E - P_E - S(0))e^{rT}$;
- sell the share for $X$ either by exercising the put if $S(T) \leq X$ or settling the short position in calls if $S(T) > X$.

The balance will be $(C_E - P_E - S(0))e^{rT} + X$, which is positive by (5.2), contradicting the No-Arbitrage Principle.

Now suppose that

$$C_E - P_E < S(0) - Xe^{-rT}. \tag{5.3}$$

Then the following reverse strategy will result in arbitrage. At time 0:

- sell short one share for $S(0)$;
- write and sell a put option for $P_E$;
- buy one call option for $C_E$;
- invest the sum $S(0) - C_E + P_E$ (borrow, if negative) in the money market at the interest rate $r$.

The balance of these transactions is 0. At time $T$:

- close out the money market position, collecting (paying, if negative) the sum $(S(0) - C_E + P_E)e^{rT}$;
- buy one share for $X$ either by exercising the call if $S(T) > X$ or settling the short position in puts if $S(T) \leq X$, and close the short position in stock.

The balance will be $(S(0) - C_E + P_E)e^{rT} - X$, positive by (5.3), once again contradicting the No-Arbitrage Principle. □

### Exercise 5.3

Suppose that a stock paying no dividends is trading at \$15.60 a share. European calls on the stock with strike price \$15 and exercise date in three months are trading at \$2.83. The interest rate is $r = 6.72\%$, compounded continuously. What is the price of a European put with the same strike price and exercise date?

### Exercise 5.4

European call and put options with strike price \$24 and exercise date in six months are trading at \$5.09 and \$7.78. The price of the underlying stock is \$20.37 and the interest rate is 7.48%. Find an arbitrage opportunity.

## Remark 5.3

We can make a simple but powerful observation based on Theorem 5.2: the difference between European calls and puts prices does not depend on any variables that are absent in the put-call parity relation (5.1). As an example, consider the expected return on stock. If the price of a call were to increase with the expected return, which on first sight seems consistent with intuition because higher stock prices mean higher payoffs on calls, then the price of a put should also increase. The latter, however, contradicts common sense because higher stock prices mean lower payoffs on puts. Because of this, one could conjecture that put and call prices should be independent of the expected return on stock. We shall see that this is indeed the case once pricing formulae are developed for calls and puts.

Following the argument presented at the beginning of this section, we can reformulate put-call parity as follows:

$$C_E - P_E = V_X(0), \tag{5.4}$$

where $V_X(0)$ is the value of a long forward contract, see (4.8). Note that if $X$ is equal to the theoretical forward price $S(0)e^{rT}$ of the asset, then the value of the forward contract is zero, $V_X(0) = 0$, and so $C_E = P_E$. Formula (5.4) allows us to generalise put-call parity by drawing on the relationships established in Remark 4.7. Namely, if the stock pays a dividend between times 0 and $T$, then



$V_X(0) = S(0) - \mathrm{div}_0 - Xe^{-rT}$, where $\mathrm{div}_0$ is the present value of the dividend. It follows that

$$C_E - P_E = S(0) - \mathrm{div}_0 - Xe^{-rT}. \tag{5.5}$$

If dividends are paid continuously at a rate $r_{\mathrm{div}}$, then

$$V_X(0) = S(0)e^{-r_{\mathrm{div}}T} - Xe^{-rT},$$

so

$$C_E - P_E = S(0)e^{-r_{\mathrm{div}}T} - Xe^{-rT}. \tag{5.6}$$

### Exercise 5.5

Outline an arbitrage proof of (5.5).

### Exercise 5.6

Outline an arbitrage proof of (5.6).

### Exercise 5.7

For the data in Exercise 4.5, find the strike price for European calls and puts to be exercised in six months such that $C_E = P_E$.

For American options put-call parity gives only some estimates, rather than a strict equality, involving put and call prices.

## Theorem 5.4 (American Put-Call Parity Estimates)

The prices of American put and call options with the same strike price $X$ and expiry time $T$ on a stock that pays no dividends satisfy

$$S(0) - Xe^{-rT} \geq C_A - P_A \geq S(0) - X.$$

## Proof

Suppose that the first inequality fails to hold, that is,

$$C_A - P_A - S(0) + Xe^{-rT} > 0.$$

Then we can write and sell a call, and buy a put and a share, financing the transactions in the money market. If the holder of the American call chooses to exercise it at time $t \le T$, then we shall receive $X$ for the share and settle the money market position, ending up with the put and a positive amount

$$X + (C_A - P_A - S(0))e^{rt} = (Xe^{-rt} + C_A - P_A - S(0))e^{rt} \ge (Xe^{-rT} + C_A - P_A - S(0))e^{rt} > 0.$$

If the call option is not exercised at all, we can sell the share for $X$ by exercising the put at time $T$ and close the money market position, also ending up with a positive amount

$$X + (C_A - P_A - S(0))e^{rT} > 0.$$

Now suppose that

$$C_A - P_A - S(0) + X < 0.$$

In this case we can write and sell a put, buy a call and sell short one share, investing the balance in the money market. If the American put is exercised at time $t \le T$, then we can withdraw $X$ from the money market to buy a share and close the short sale. We shall be left with the call option and a positive amount

$$(-C_A + P_A + S(0))e^{rt} - X > Xe^{rt} - X \ge 0.$$

If the put is not exercised at all, then we can buy a share for $X$ by exercising the call at time $T$ and close the short position in stock. On closing the money market position, we shall also end up with a positive amount

$$(-C_A + P_A + S(0))e^{rT} - X > Xe^{rT} - X > 0.$$

The theorem, therefore, holds by the No-Arbitrage Principle. □

*Exercise 5.8*

Modify the proof of Theorem 5.4 to show that

$$S(0) - Xe^{-rT} \ge C_A - P_A \ge S(0) - \mathrm{div}_0 - X$$

for a stock paying a dividend between time 0 and the expiry time $T$, where $\mathrm{div}_0$ is the value of the dividend discounted to time 0.

*Exercise 5.9*

Modify the proof of Theorem 5.4 to show that

$$S(0) - Xe^{-rT} \ge C_A - P_A \ge S(0)e^{-r_{\mathrm{div}}T} - X$$

for a stock paying dividends continuously at a rate $r_{\mathrm{div}}$.



## 5.3 Bounds on Option Prices

First of all, we note the obvious inequalities

$$C_E \le C_A, \quad P_E \le P_A, \tag{5.7}$$

for European and American options with the same strike price $X$ and expiry time $T$. They hold because an American option gives at least the same rights as the corresponding European option.

Figure 5.3 shows a scenario of stock prices in which the payoff of a European call is zero at the exercise time $T$, whereas that of an American call will be positive if the option is exercised at an earlier time $t < T$ when the stock price $S(t)$ is higher than $X$. Nevertheless, it does not necessarily follow that the inequalities in (5.7) can be replaced by strict ones; see Section 5.3.2, where it is shown that $C_E = C_A$ for call options on an asset that pays no dividends.

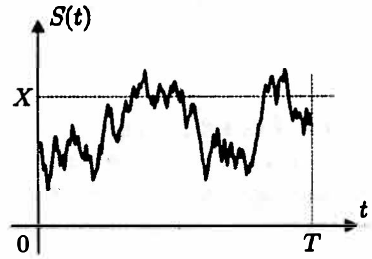


**Figure 5.3** Scenario in which an American call can bring a positive payoff, but a European call cannot

*Exercise 5.10*

Prove (5.7) by an arbitrage argument.

It is also obvious that the price of a call or put option has to be non-negative because an option of this kind offers the possibility of a future gain with no liability. Therefore,

$$C_E \ge 0, \quad P_E \ge 0.$$

Similar inequalities are of course valid for the more valuable American options. In what follows we are going to discuss some further simple bounds for the prices of European and American options.

### 5.3.1 European Options

We shall establish some upper and lower bounds on the prices of European call and put options.

On the one hand, observe that

$$C_E < S(0).$$

If the reverse inequality were satisfied, that is, if $C_E \geq S(0)$, then we could write and sell the option and buy the stock, investing the balance on the money market. On the exercise date $T$ we could then sell the stock for $\min(S(T), X)$, settling the call option. Our arbitrage profit would be $(C_E - S(0))e^{rT} + \min(S(T), X) > 0$. This proves that $C_E < S(0)$. On the other hand, we have the lower bound

$$S(0) - Xe^{-rT} \leq C_E,$$

which follows immediately by put-call parity, since $P_E \geq 0$. Moreover, put-call parity implies that

$$P_E < Xe^{-rT},$$

since $C_E < S(0)$, and

$$-S(0) + Xe^{-rT} \leq P_E,$$

since $C_E \geq 0$.

These results are summarised in the following proposition and illustrated in Figure 5.4. The shaded areas correspond to option prices that satisfy the bounds.

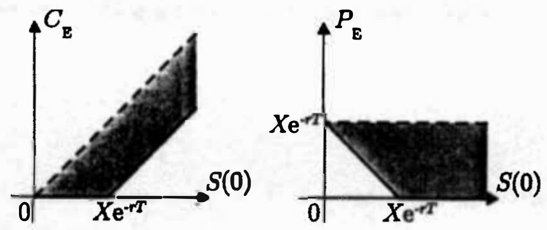


**Figure 5.4** Bounds on European call and put prices

#### Proposition 5.5

The prices of European call and put options on a stock paying no dividends satisfy the inequalities

$$\max\{0, S(0) - Xe^{-rT}\} \leq C_E < S(0),$$
$$\max\{0, -S(0) + Xe^{-rT}\} \leq P_E < Xe^{-rT}.$$



For dividend-paying stock the bounds are

$$\max\{0, S(0) - \mathrm{div}_0 - Xe^{-rT}\} \leq C_E < S(0) - \mathrm{div}_0,$$
$$\max\{0, -S(0) + \mathrm{div}_0 + Xe^{-rT}\} \leq P_E < Xe^{-rT}.$$

#### *Exercise 5.11*

Prove the above bounds on option prices for dividend-paying stock.

#### *Exercise 5.12*

For dividend-paying stock, sketch the regions of call and put prices determined by the bounds.

### 5.3.2 Calls on Non-Dividend-Paying Stock

Consider European and American call options with the same strike price $X$ and expiry time $T$. We know that $C_A \geq C_E$, since the American option gives more rights than its European counterpart. If the underlying stock pays no dividends, then $C_E \geq S(0) - Xe^{-rT}$ by Proposition 5.5. It follows that $C_A > S(0) - X$ if $r > 0$. Because the price of the American option is greater than the payoff, the option will sooner be sold than exercised at time 0.

The choice of 0 as the starting time is of course arbitrary. Replacing 0 by any given $t < T$, we can show by the same argument that the American option will not be exercised at time $t$. This means that the American option will in fact never be exercised prior to expiry. This being so, it should be equivalent to the European option. In particular, their prices should be equal, leading to the following theorem.

#### Theorem 5.6

The prices of European and American call options on a stock that pays no dividends are equal, $C_A = C_E$, whenever the strike price $X$ and expiry time $T$ are the same for both options.

#### Proof

We already know that $C_A \geq C_E$, see (5.7) and Exercise 5.10. If $C_A > C_E$, then write and sell an American call and buy a European call, investing the balance $C_A - C_E$ at the interest rate $r$. If the American call is exercised at time $t \leq T$, then borrow a share and sell it for $X$ to settle your obligation as writer of the call option, investing $X$ at rate $r$. Then, at time $T$ you can use the European call to buy a share for $X$ and close your short position in stock. Your arbitrage profit will be $(C_A - C_E)e^{rT} + Xe^{r(T-t)} - X > 0$. If the American option is not exercised at all, you will end up with the European option and an arbitrage profit of $(C_A - C_E)e^{rT} > 0$. This proves that $C_A = C_E$. □

Theorem 5.6 may seem counter-intuitive at first sight. While it is possible to gain $S(t) - X$ by exercising an American call option if $S(t) > X$ at time $t < T$, this is not so with a European option, which cannot be exercised at time $t < T$. It might, therefore, appear that the American call option should be more valuable than the European one. Nevertheless, there is no contradiction. Even though a European call option cannot be exercised at time $t < T$, it can be sold for at least $S(t) - X$.

The situation is different for dividend-paying stock. Example 6.21 in the next chapter shows a case in which an American call option is worth more than its European counterpart and should be exercised prior to expiry, at least in some scenarios.

On the other hand, it often happens that an American put should be exercised prematurely even if the underlying stock pays no dividends, as in the following example.

### Example 5.7

Suppose that the stock price is \$10, the strike price of an American put expiring in one year is \$80, and the interest rate is 16%. Exercising the option now, we can gain \$70, which can be invested at 16% to become \$81.20 after one year. As will be shown in (5.8), the value of a put option cannot possibly exceed the strike price, so we are definitely better off by exercising the option early.

## 5.3.3 American Options

First we consider options on non-dividend-paying stock. In this case, the price of an American call is equal to that of a European call, $C_A = C_E$, see Theorem 5.6, so it must satisfy the same bounds as in Proposition 5.5. For an American put we have

$$-S(0) + X \le P_A$$

because $P_A$ cannot be less than the payoff of the option at time 0. This gives a sharper lower bound than that for a European put. However, the upper bound



has to be relaxed as compared to a European put. Namely,

$$P_A < X. \tag{5.8}$$

Indeed, if $P_A \ge X$, then the following arbitrage strategy could be constructed: Write and sell an American put for $P_A$ and invest this amount at the interest rate $r$. If the put is exercised at time $t \le T$, then a share of the underlying stock will have to be bought for $X$ and can then be sold for $S(t)$. The final cash balance will be positive, $P_Ae^{rt} - X + S(t) > 0$. If the option is not exercised at all, the final balance will also be positive, $P_Ae^{rT} > 0$, at expiry. These results can be summarised as follows.

### Proposition 5.8

The prices of American call and put options on a stock paying no dividends satisfy the inequalities

$$\max\{0, S(0) - Xe^{-rT}\} \le C_A < S(0),$$
$$\max\{0, -S(0) + X\} \le P_A < X.$$

Next we consider options on dividend-paying stock. The lower bounds for European options imply

$$S(0) - \text{div}_0 - Xe^{-rT} \le C_E \le C_A,$$

$$-S(0) + \text{div}_0 + Xe^{-rT} \le P_E \le P_A.$$

But because the price of an American option cannot be less than its payoff at any time, we also have $S(0) - X \le C_A$ and $X - S(0) \le P_A$. Moreover, the upper bounds $C_A < S(0)$ and $P_A < X$ can be established in a similar manner as for non-dividend-paying stock. All these inequalities can be summarised as follows: for dividend-paying stock

$$\max\{0, S(0) - \text{div}_0 - Xe^{-rT}, S(0) - X\} \le C_A < S(0),$$
$$\max\{0, -S(0) + \text{div}_0 + Xe^{-rT}, -S(0) + X\} \le P_A < X.$$

#### *Exercise 5.13*

Prove by an arbitrage argument that $C_A < S(0)$ for an American call on dividend-paying stock.

## 5.4 Variables Determining Option Prices

The option price depends on a number of variables. These can be variables characterising the option, such as the strike price $X$ or expiry time $T$, variables describing the underlying asset, for example, the current price $S(0)$ or dividend rate $r_{\text{div}}$, variables connected with the market as a whole such as the risk-free rate $r$, and of course the running time $t$.

We shall analyse option prices as functions of one of the variables, keeping the remaining variables constant. This is a significant simplification because usually a change in one variable is accompanied by changes in some or all of the other variables. Nevertheless, even the simplified situation will provide interesting insights.

### 5.4.1 European Options

**Dependence on the Strike Price.** We shall consider options on the same underlying asset and with the same exercise time $T$, but with different values of the strike price $X$. The call and put prices will be denoted by $C_{\mathrm{E}}(X)$ and, respectively, $P_{\mathrm{E}}(X)$ to emphasise their dependence on $X$. All remaining variables such as the exercise time $T$, running time $t$ and the underlying asset price $S(0)$ will be kept fixed for the time being.

#### Proposition 5.9

If $X' < X''$, then

$$C_{\mathrm{E}}(X') \geq C_{\mathrm{E}}(X''),$$
$$P_{\mathrm{E}}(X') \leq P_{\mathrm{E}}(X'').$$

This means that $C_{\mathrm{E}}(X)$ is a decreasing and $P_{\mathrm{E}}(X)$ an increasing function of $X$.

These inequalities are obvious. The right to buy at a lower price is more valuable than the right to buy at a higher price. Similarly, it is better to sell an asset at a higher price than at a lower one. However, at the exercise time the prices are equal to the payoffs and can be zero for various exercise prices. We shall see that the same may hold prior to exercise in some models.

*Exercise 5.14*

Give a rigorous arbitrage argument to prove the inequalities in Proposition 5.9.



#### Proposition 5.10

If $X' < X''$, then

$$C_{\mathrm{E}}(X') - C_{\mathrm{E}}(X'') \leq e^{-rT}(X'' - X'),$$
$$P_{\mathrm{E}}(X'') - P_{\mathrm{E}}(X') \leq e^{-rT}(X'' - X').$$

#### Proof

By put-call parity (5.1)

$$C_{\mathrm{E}}(X') - P_{\mathrm{E}}(X') = S(0) - X'e^{-rT},$$
$$C_{\mathrm{E}}(X'') - P_{\mathrm{E}}(X'') = S(0) - X''e^{-rT}.$$

Subtracting, we get

$$(C_{\mathrm{E}}(X') - C_{\mathrm{E}}(X'')) + (P_{\mathrm{E}}(X'') - P_{\mathrm{E}}(X')) = (X'' - X')e^{-rT}.$$

Since, by Proposition 5.9, both terms on the left-hand side are non-negative, neither of them can exceed the right-hand side. □

#### Remark 5.11

The inequalities in Proposition 5.10 mean that the call and put prices as functions of the strike price satisfy the Lipschitz condition with constant $e^{-rT} < 1$,

$$|C_{\mathrm{E}}(X'') - C_{\mathrm{E}}(X')| \leq e^{-rT}|X'' - X'|,$$
$$|P_{\mathrm{E}}(X'') - P_{\mathrm{E}}(X')| \leq e^{-rT}|X'' - X'|.$$

In particular, the slope of the graph of the option price as a function of the strike price is less than 45°. This is illustrated in Figure 5.5 for a call option.

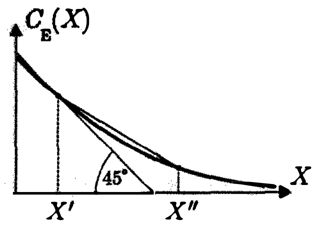


**Figure 5.5** Lipschitz property of call prices $C_{\mathrm{E}}(X)$

The reader may suspect that in Proposition 5.10 we should have strict inequalities. However, at exercise the option price is the same as the payoff, and if $S(T) \geq X''$, then we will have equality (with $T = 0$). Later we shall see that the equality may also hold for small $T$ in some models.

## Proposition 5.12

For any $\alpha \in [0,1]$ and any $X', X''$

$$C_E(\alpha X' + (1-\alpha)X'') \le \alpha C_E(X') + (1-\alpha)C_E(X''),$$
$$P_E(\alpha X' + (1-\alpha)X'') \le \alpha P_E(X') + (1-\alpha)P_E(X'').$$

In other words, $C_E(X)$ and $P_E(X)$ are convex functions of $X$.

## Proof

For brevity, we put $X = \alpha X' + (1-\alpha)X''$. Suppose that

$$C_E(X) > \alpha C_E(X') + (1-\alpha)C_E(X'').$$

We can write and sell an option with strike price $X$, and purchase $\alpha$ options with strike price $X'$ and $1-\alpha$ options with strike price $X''$, investing the balance $C_E(X) - (\alpha C_E(X') + (1-\alpha)C_E(X'')) > 0$ on the money market. If the option with strike price $X$ is exercised at expiry, then we shall have to pay $(S(T)-X)^+$. We can raise the amount $\alpha\,(S(T)-X')^+ + (1-\alpha)\,(S(T)-X'')^+$ by exercising $\alpha$ calls with strike $X'$ and $1-\alpha$ calls with strike $X''$. In this way we will realise an arbitrage profit because of the following inequality, which is easy to verify (the details are left to the reader, see Exercise 5.15):

$$(S(T)-X)^+ \le \alpha\,(S(T)-X')^+ + (1-\alpha)\,(S(T)-X'')^+ . \qquad (5.9)$$

Convexity for put options follows from that for calls by put-call parity (5.1). Alternatively, an arbitrage argument can be given along similar lines as for call options. □

### *Exercise 5.15*

Verify inequality (5.9).

## Remark 5.13

According to Proposition 5.12, $C_E(X)$ and $P_E(X)$ are convex functions of $X$. Geometrically, this means that if two points on the graph of the function are joined with a straight line, then the graph of the function between the two points will lie below the line. This is illustrated in Figure 5.6 for call prices.

**Dependence on the Underlying Asset Price.** The current price $S(0)$ of the underlying asset is given by the market and cannot be altered. However, we can consider an option on a portfolio consisting of $x$ shares, worth $S = xS(0)$.



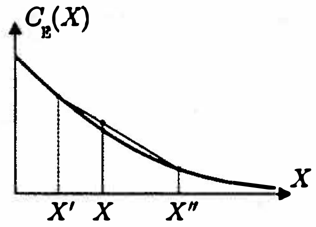


**Figure 5.6** Convexity of call prices $C_E(X)$

(Typically, options are traded with packages of shares as the underlying assets.) The payoff of a European call with strike price $X$ on such a portfolio to be exercised at time $T$ is $(xS(T)-X)^+$ and for a put it is $(X - xS(T))^+$. We shall study the dependence of option prices on $S$. Assuming that all remaining variables are fixed, we shall denote the call and put prices by $C_E(S)$ and $P_E(S)$.

## Remark 5.14

The functions $C_E(S)$ and $P_E(S)$ are important because they should reflect the dependence of option prices on very sudden changes of the price of the underlying such that the remaining variables remain almost unaltered. The claim of the next proposition is obvious from the economic point of view. For instance, a call becomes more valuable if the stock goes up because a positive final payoff becomes more likely.

## Proposition 5.15

If $S' < S''$, then

$$C_E(S') \le C_E(S''),$$
$$P_E(S') \ge P_E(S''),$$

that is, $C_E(S)$ is a non-decreasing function and $P_E(S)$ a non-increasing function of $S$.

## Proof

Suppose that $C_E(S') > C_E(S'')$ for some $S' < S''$, where $S' = x'S(0)$ and $S'' = x''S(0)$. We can write and sell a call on a portfolio with $x'$ shares and buy a call on a portfolio with $x''$ shares, the two options sharing the same strike price $X$ and exercise time $T$, and we can invest the balance $C_E(S') - C_E(S'') > 0$ on the money market. Since $x' < x''$, the payoffs satisfy $(x'S(T)-X)^+ \le$

$(x''S(T) - X)^+$. The payoff of the option bought is sufficient to cover that of the option sold. We shall be left with no less than the balance $C_E(S') - C_E(S'') > 0$ plus interest as our arbitrage profit.

The inequality for puts follows by a similar arbitrage argument. □

### Exercise 5.16

Prove the inequality in Proposition 5.15 for put options.

### Proposition 5.16

Suppose that $S' < S''$. Then

$$C_E(S'') - C_E(S') \leq S'' - S',$$
$$P_E(S') - P_E(S'') \leq S'' - S'.$$

### Proof

We employ put-call parity (5.1):

$$C_E(S'') - P_E(S'') = S'' - Xe^{-rT},$$
$$C_E(S') - P_E(S') = S' - Xe^{-rT}.$$

Subtracting, we get

$$(C_E(S'') - C_E(S')) + (P_E(S') - P_E(S'')) = S'' - S'.$$

Both terms on the left-hand side are non-negative by Proposition 5.15, so each is less than or equal to the right-hand side. □

### Remark 5.17

A consequence of Proposition 5.16 is that the slope of the straight line joining two points on the graph of the call or put price as a function of $S$ is no greater than 45°. This is illustrated in Figure 5.7 for call options. In other words, the call and put prices $C_E(S)$ and $P_E(S)$ satisfy the Lipschitz condition with constant 1,

$$|C_E(S'') - C_E(S')| \leq |S'' - S'|,$$
$$|P_E(S'') - P_E(S')| \leq |S'' - S'|.$$



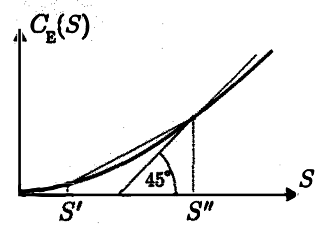


**Figure 5.7** Lipschitz property of call prices $C_E(S)$

### Proposition 5.18

Let $S' < S''$ and let $\alpha \in [0, 1]$. Then

$$C_E(\alpha S' + (1-\alpha)S'') \leq \alpha C_E(S') + (1-\alpha)C_E(S''),$$
$$P_E(\alpha S' + (1-\alpha)S'') \leq \alpha P_E(S') + (1-\alpha)P_E(S'').$$

This means that the call and put prices are convex functions of $S$.

### Proof

We put $S = \alpha S' + (1-\alpha)S''$ for brevity. Let $S' = x'S(0)$, $S'' = x''S(0)$ and $S = xS(0)$, so $x = \alpha x' + (1-\alpha)x''$. Suppose that

$$C_E(S) > \alpha C_E(S') + (1-\alpha)C_E(S'').$$

We write and sell a call on a portfolio with $x$ shares, and purchase $\alpha$ calls on a portfolio with $x'$ shares and $1-\alpha$ calls on a portfolio with $x''$ shares, investing the balance $C_E(S) - \alpha C_E(S') - (1-\alpha)C_E(S'')$ on the money market. If the option sold is exercised at time $T$, then we shall have to pay $(xS(T) - X)^+$. We can cover this liability by the options held since

$$(xS(T) - X)^+ \leq \alpha (x'S(T) - X)^+ + (1-\alpha)(x''S(T) - X)^+,$$

and so this is an arbitrage strategy.

The inequality for put options can be proved by a similar arbitrage argument or using put-call parity. □

## 5.4.2 American Options

In general, American options have similar properties to their European counterparts. One difficulty is the absence of put-call parity; we only have the weaker estimates in Theorem 5.4. In addition, we have to take into account the possibility of early exercise.

**Dependence on the Strike Price.** We shall denote the call and put prices by $C_A(X)$ and $P_A(X)$ to emphasise the dependence on $X$, keeping any other variables fixed.

The following proposition is obvious for the same reasons as for European options: higher strike price makes the right to buy less valuable and the right to sell more valuable.

## Proposition 5.19

If $X' < X''$, then

$$C_A(X') \geq C_A(X''),$$
$$P_A(X') \leq P_A(X'').$$

### *Exercise 5.17*

Give a rigorous arbitrage proof of Proposition 5.19.

## Proposition 5.20

Suppose that $X' < X''$. Then

$$C_A(X') - C_A(X'') \leq e^{-rT}(X'' - X'),$$
$$P_A(X'') - P_A(X') \leq X'' - X'.$$

## Proof

The inequality for calls follows from Theorem 5.6 and Proposition 5.10.

Suppose that $X' < X''$, but $P_A(X'') - P_A(X') > X'' - X'$. Let us write and sell a put with strike $X''$, buy a put with strike $X'$ and invest the balance $P_A(X'') - P_A(X')$ in the money market. If the written option is exercised at time $t \leq T$, we shall have to pay $X'' - S(t)$. To do so, we can exercise the option with strike $X'$, receiving the payoff $X' - S(t)$. Added to the amount invested in the money market with interest accumulated up to time $t$, it gives

$$\begin{aligned} X' - S(t) + (P_A(X'') - P_A(X'))e^{rt} &> X' - S(t) + (X'' - X')e^{rt} \\ &\geq X'' - S(t). \end{aligned}$$

The positive surplus above $X'' - S(t)$ would be our arbitrage profit. □



## Proposition 5.21

For any $\alpha \in [0,1]$ and any $X', X''$

$$C_A(\alpha X' + (1-\alpha)X'') \leq \alpha C_A(X') + (1-\alpha)C_A(X''),$$
$$P_A(\alpha X' + (1-\alpha)X'') \leq \alpha P_A(X') + (1-\alpha)P_A(X'').$$

## Proof

The inequality for calls follows directly from Theorem 5.6 and Proposition 5.12.

For brevity, we put $X = \alpha X' + (1-\alpha)X''$. Suppose that

$$P_A(X) > \alpha P_A(X') + (1-\alpha)P_A(X'').$$

We write and sell a put with strike price $X$ and buy $\alpha$ puts with strike price $X'$ and $(1-\alpha)$ puts with strike price $X''$, investing the positive balance of these transactions in the money market. If the written option is exercised at time $t \leq T$, then the payoffs of the options held will bring enough money to cover the payment since

$$(X - S(t))^+ \leq \alpha(X' - S(t))^+ + (1-\alpha)(X'' - S(t))^+.$$

The investment in the money market will provide an arbitrage profit. □

**Dependence on the Underlying Asset Price.** Once again, we shall consider options on a portfolio of $x$ shares. The prices of American calls and puts on such a portfolio will be denoted by $C_A(S)$ and $P_A(S)$, where $S = xS(0)$ is the value of the portfolio, all remaining variables being fixed. The payoffs at time $t$ are $(xS(t) - X)^+$ for calls and $(X - xS(t))^+$ for puts.

## Proposition 5.22

If $S' < S''$, then

$$C_A(S') \leq C_A(S''),$$
$$P_A(S') \geq P_A(S'').$$

## Proof

For calls, this is an immediate consequence of Theorem 5.6 and Proposition 5.15.

Suppose that $P_A(S') < P_A(S'')$ for some $S' < S''$, where $S' = x'S(0)$ and $S'' = x''S(0)$. We can write and sell a put on a portfolio with $x''$ shares and buy a put on a portfolio with $x'$ shares, both options having the same strike price $X$ and expiry time $T$. The balance $P_A(S'') - P_A(S')$ of these transactions can be invested in the money market. If the written option is exercised at time $t \le T$, then we can meet the liability by exercising the other option immediately. Indeed, since $x' < x''$, the payoffs satisfy $(X - x'S(t))^+ \ge (X - x''S(t))^+$. The investment in the money market provides an arbitrage profit. □

### Proposition 5.23

Suppose that $S' < S''$. Then

$$C_A(S'') - C_A(S') \le S'' - S',$$
$$P_A(S') - P_A(S'') \le S'' - S'.$$

### Proof

The inequality for calls follows from Theorem 5.6 and Proposition 5.16.

For puts, suppose that $P_A(S') - P_A(S'') > S'' - S'$ for some $S' < S''$, where $S' = x'S(0)$ and $S'' = x''S(0)$. We buy $x'' - x' > 0$ shares, buy a put on a portfolio of $x''$ shares, and sell a put on a portfolio of $x'$ shares, both puts with the same strike price $X$ and expiry $T$. The cash balance of these transactions is $-(S'' - S') - P_A(S'') + P_A(S') > 0$. Should the holder of the put on $x'$ shares choose to exercise it at time $t \le T$, we shall need to pay $(X - x'S(t))^+$. By selling $x'' - x'$ shares and exercising the option on $x''$ shares at time $t$, we can raise enough cash to settle this liability:

$$(x'' - x')\, S(t) + (X - x''S(t))^+ \ge (X - x'S(t))^+,$$

since $x'' > x'$. If the put on $x'$ shares is not exercised at all, we need to take no action. In any case, the initial profit plus interest is ours to keep, resulting in an arbitrage opportunity. □

### Proposition 5.24

Let $S' < S''$ and let $\alpha \in [0, 1]$. Then

$$C_A(\alpha S' + (1-\alpha)S'') \le \alpha C_A(S') + (1-\alpha)C_A(S''),$$
$$P_A(\alpha S' + (1-\alpha)S'') \le \alpha P_A(S') + (1-\alpha)P_A(S'').$$

### Proof

The inequality for calls follows from Theorem 5.6 and Proposition 5.18.



Let $S = \alpha S' + (1-\alpha)S''$ and let $S' = x'S(0)$, $S'' = x''S(0)$ and $S = xS(0)$. Suppose that

$$P_A(S) > \alpha P_A(S') + (1-\alpha)P_A(S'').$$

We can write and sell a put on a portfolio with $x$ shares, and purchase $\alpha$ puts on a portfolio with $x'$ shares and $1-\alpha$ puts on a portfolio with $x''$ shares, all three options sharing the same strike price $X$ and expiry time $T$. The positive balance $P_A(S) - \alpha P_A(S') - (1-\alpha)P_A(S'')$ of these transactions can be invested in the money market. If the written option is exercised at time $t \le T$, then we shall have to pay $(X - xS(t))^+$, where $x = \alpha x' + (1-\alpha)x''$. This is an arbitrage strategy because the other two options cover the liability:

$$(X - xS(t))^+ \le \alpha (X - x'S(t))^+ + (1-\alpha)(X - x''S(t))^+.$$

□

**Dependence on the Expiry Time.** For American options we can also formulate a general result on the dependence of their prices on the expiry time $T$. To emphasise this dependence, we shall now write $C_A(T)$ and $P_A(T)$ for the prices of American calls and puts, assuming that all other variables are fixed.

### Proposition 5.25

If $T' < T''$, then

$$C_A(T') \le C_A(T''),$$
$$P_A(T') \le P_A(T'').$$

### Proof

Suppose that $C_A(T') > C_A(T'')$. We write and sell one option expiring at time $T'$ and buy one with the same strike price but expiring at time $T''$, investing the balance on the money market. If the written option is exercised at time $t \le T'$, we can exercise the other option immediately to cover our liability. The positive balance $C_A(T') - C_A(T'') > 0$ invested on the money market will be our arbitrage profit.

The argument is the same for puts. □

## 5.5 Time Value of Options

The following convenient terminology is often used. We say that at time $t$ a call option with strike price $X$ is

- *in the money* if $S(t) > X$,
- *at the money* if $S(t) = X$,
- *out of the money* if $S(t) < X$.

Similarly, for a put option we say that it is

- *in the money* if $S(t) < X$,
- *at the money* if $S(t) = X$,
- *out of the money* if $S(t) > X$.

Also convenient, though less precise, are the terms *deep in the money* and *deep out of the money*, which mean that the difference between the two sides in the respective inequalities is considerable.

An American option in the money will bring a positive payoff if exercised immediately, whereas an option out of the money will not. We use the same terms for European options, though their meaning is different: even if the option is currently in the money, it may no longer be so on the exercise date, when the payoff may well turn out to be zero. A European option in the money is no more than a promising asset.

## Definition 5.26

At time $t \le T$ the *intrinsic value* of a call option with strike price $X$ is equal to $(S(t) - X)^+$. The intrinsic value of a put option with the same strike price is $(X - S(t))^+$.

We can see that the intrinsic value is zero for options out of the money or at the money. Options in the money have positive intrinsic value. The price of an option at expiry $T$ coincides with the intrinsic value. The price of an American option prior to expiry may be greater than the intrinsic value because of the possibility of future gains. The price of a European option prior to the exercise time may be greater or smaller than the intrinsic value.

## Definition 5.27

The *time value* of an option is the difference between the price of the option and its intrinsic value, that is,

$$\begin{aligned} &C_E(t) - (S(t) - X)^+ && \text{for a European call,} \\ &P_E(t) - (X - S(t))^+ && \text{for a European put,} \\ &C_A(t) - (S(t) - X)^+ && \text{for an American call,} \\ &P_A(t) - (X - S(t))^+ && \text{for an American put.} \end{aligned}$$



## Example 5.28

Let us examine some typical data. Suppose that the current price of stock is \$125.23 per share. An American call option with strike price \$110 is in the money and has \$15.23 intrinsic value. The option price must be at least equal to the intrinsic value, since the option may be exercised immediately. Typically, the price will be higher than the intrinsic value because of the possibility of future gains. On the other hand, a put option with strike price \$110 will be out of the money and its intrinsic value will be zero. The positive price of the put is entirely due to the possibility of future gains. Similar relationships for other strike prices can be seen in the table.

| Strike Price | Intrinsic Value | | Option Price | | Time Value | |
|---|---|---|---|---|---|---|
| | Call | Put | Call | Put | Call | Put |
| 110 | 15.23 | 0.00 | 18.40 | 2.84 | 3.17 | 2.84 |
| 120 | 5.23 | 0.00 | 12.27 | 6.46 | 7.04 | 6.46 |
| 130 | 0.00 | 4.77 | 6.78 | 9.64 | 6.78 | 4.41 |

The time value of a European call as a function of $S$ is shown in Figure 5.8. It can never be negative, and for large values of $S$ it exceeds the difference $X - Xe^{-rT}$. This is because of the inequality $C_E(S) \ge S - Xe^{-rT}$, see Proposition 5.5.

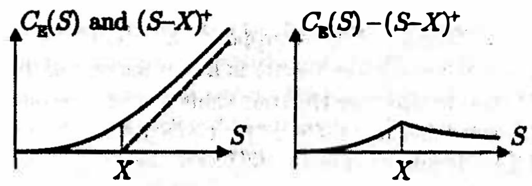


**Figure 5.8** Time value $C_E(S) - (S - X)^+$ of a European call option

The market value of a European put may be lower than its intrinsic value, that is, the time value may be negative, see Figure 5.9. This may be so only if the put option is in the money, $S < X$, and it should be deep in the money. For a European option we have to wait until the exercise time $T$ to realise the payoff. The risk that the stock price will rise above $X$ in the meantime may be considerable, which reduces the value of the option.

The time value of an American call option is the same as that of a European call (if there are no dividends) and Figure 5.8 applies. For an American put, a typical graph of the time value is shown in Figure 5.10.

Figures 5.8, 5.9 and 5.10 also illustrate the following assertion.

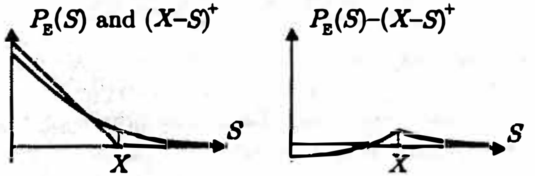


**Figure 5.9** Time value $P_E(S) - (X-S)^+$ of a European put option

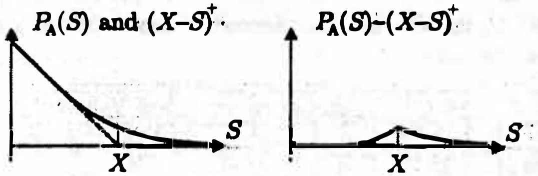


**Figure 5.10** Time value $P_A(S) - (X-S)^+$ of an American put option

### Proposition 5.29

For any European or American call or put option with strike price $X$, the time value attains its maximum at $S = X$.

### Proof

We shall present an argument for European calls. For $S \leq X$ the intrinsic value of the option is zero. Since $C_E(S)$ is a non-decreasing function of $S$ by Proposition 5.15, this means that the time value is a non-decreasing function of $S$ for $S \leq X$. On the other hand, $C_E(S'') - C_E(S') \leq S'' - S'$ for any $S' < S''$ by Proposition 5.16. It follows that $C_E(S'') - (S'' - X)^+ \leq C_E(S') - (S' - X)^+$ if $X \leq S' < S''$, which means that the time value is a non-increasing function of $S$ for $S \geq X$. As a result, the time value has a maximum at $S = X$.

The proof for other options is similar. □

*Exercise 5.18*

Prove Proposition 5.29 for put options.



## Case 5: Discussion

If we lock into the strike rate of 1.5 euros to a pound for the currency options available, take a long forward position in stock B with delivery price 65 pounds, and take a short forward position in stock G with delivery price 98 euros, assuming that these forward contracts are available, this will generate a profit at exercise time. We would then be able to sell a share of G for 98 euros, pay 97.50 euros to buy 65 pounds, and purchase B for 65 pounds. We would be left with a profit of 0.50 euros on each share.

However, going forward is free only at the forward prices. Here we have to analyse the costs of taking the above forward positions. These costs, as we know, are the differences between the corresponding option prices. The call price minus the put price is the value of the long forward position. So going long forward in stock B at delivery price 65 pounds will cost $(2.55 - 1.34) \times 1.5 = 1.82$ euros. A short forward contract in G generates $4.145 - 2.738$ euros. To complete the position we have to go long forward in the pound, which costs $65 \times (0.0144 - 0.01316) = 0.0805$ euros. The balance is the cost of 0.4885 euros. This is lower that the future profit, but this cost is paid today, so we face the question of finding the euro interest rate. In practice, this information is readily available, but here it can be extracted from put-call parity on stock G, namely $r = -\frac{1}{T}\ln(\frac{G(0)-C+P}{X}) = 4.994\%$. Borrowing 0.4884 euros at this rate (we live in an ideal world) gives the cost of 0.4905 after a month, so there is an arbitrage profit.

An acute reader may have started the analysis of the case with investigating put-call relationships for all three securities. The pound interest rate generated by stock B is 3.8832%, and then the forward position in the currency should cost 0.00138, as compared to 0.00124 implied by the currency option prices. So call-put parity is violated and arbitrage can be realised in a simpler way following the proof of Theorem 5.2.