# Mathematical Finance

## Ch01-2: A Simple Market Model

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# Outline

# Forward Contracts

A *forward contract* is an agreement to buy or sell a risky asset at a specified future time, known as the *delivery date*, for a price $F$ fixed at the present moment, called the *forward price*.

- An investor who agrees to buy the asset is said to *enter into a long forward contract* or to *take a long forward position*.
- If an investor agrees to sell the asset, we speak of a *short forward contract* or a *short forward position*.
- No money is paid at the time when a forward contract is exchanged.

# Example 1.8

- Suppose that the forward price is \$80.
- If the market price of the asset turns out to be \$84 on the delivery date, then
    - the holder of a long forward contract will buy the asset for \$80 and can sell it immediately for \$84, cashing the difference of \$4.
    - the party holding a short forward position will have to sell the asset for \$80, suffering a loss of \$4.
- However, if the market price of the asset turns out to be \$75 on the delivery date, then
    - the party holding a long forward position will have to buy the asset for \$80, suffering a loss of \$5.
    - Meanwhile, the party holding a short position will gain \$5 by selling the asset above its market price.
- In either case the loss of one party is the gain of the other. [**Zero-sum Game!!**]

In general,

- the party holding a long forward contract with delivery date $T$ will benefit if the future asset price $S(T)$ rises above the forward price $F$.
- If the asset price $S(T)$ falls below the forward price $F$, then the holder of a long forward contract will suffer a loss.

The payoff (which can be positive, negative or zero):

- for a long forward position: $S(T) - F$;
- for a short forward position: $F - S(T)$.

- Apart from stock and bonds, a portfolio held by an investor may contain forward contracts, in which case it will be described by a triple $(x, y, z)$. Here, as before,
  - $x$: the number of shares of stock,
  - $y$: the numbers of bonds, while
  - $z$: the number of forward contracts (positive for a long forward position and negative for a short position).
- Because no payment is due when a forward contract is exchanged, the initial value of such a portfolio is simply

$$V(0) = xS(0) + yA(0).$$

- (1.2)

- On the delivery date, the value of the portfolio will become

$$V(T) = xS(T) + yA(T) + z(S(T) - F).$$

- (1.3)

- Having included forward contracts in the portfolio, we need to extend the No-Arbitrage Principle accordingly.

### Assumption 1.9 (No-Arbitrage Principle)

There is no portfolio $(x, y, z)$ that includes a position $z$ in forward contracts and has initial value $V(0) = 0$ such that $V(T) \geq 0$ with probability 1 and $V(T) > 0$ with non-zero probability, where $V(0)$, $V(T)$ are given by (1.2) and (1.3).

- The forward price $F$ is determined by the No-Arbitrage Principle.
- For simplicity, we shall first consider the simplest case for the risky security with no carrying costs.
  - A typical example of such an asset is a stock paying no dividend.
  - (By contrast, a commodity will usually involve storage costs, while a foreign currency will earn interest, which can be regarded as a negative carrying cost.)

- A long forward position guarantees that the asset will be bought for the forward price $F$ at delivery.
- Alternatively, the asset can be bought now and held until delivery.
  - However, if the initial cash outlay is to be zero, the purchase must be financed by a loan.
  - The loan with interest, which will need to be repaid at the delivery date, is a candidate for the forward price.

The following proposition shows that this is indeed the case.

# Proposition 1.10

## Proposition 1.10

If the risky security involves no cost of carry, then the forward price must be

$$F = S(0)\frac{A(T)}{A(0)} = S(0)(1 + K_A)$$

- (1.4)

or an arbitrage opportunity would exist otherwise.

Proof

1)

- Suppose that $F > S(0)(1 + K_A)$.
- Then, at time 0:
  - borrow the amount $S(0)$;
  - buy the asset for $S(0)$ dollars;
  - enter into a short forward contract with forward price $F$ and delivery date $T$.
- The resulting portfolio $(1, -\frac{S(0)}{A(0)}, -1)$ consisting of stock, a risk-free position, and a short forward contract has initial value $V(0) = 0$.

- Then, at time $T$:
  - close the short forward position by selling the asset for $F$;
  - close the risk-free position by paying $\frac{S(0)}{A(0)}A(T) = S(0)(1 + K_A)$ dollars.
- The final value of the portfolio,

$$V(T) = F - S(0)(1 + K_A) > 0,$$

will be your arbitrage profit, violating the No-Arbitrage Principle (Assumption 1.9).

2)

- On the other hand, if $F < S(0)(1+K_A)$, then at time 0:
  - sell short the asset for $S(0)$;
  - invest this amount risk-free;
  - take a long forward position in stock with forward price $F$ dollars and delivery date $T$.
- The initial value of this portfolio $(-1, \frac{S(0)}{A(0)}, 1)$ is also $V(0) = 0$.
- Subsequently, at time $T$:
  - receive the amount $\frac{S(0)}{A(0)}A(T) = S(0)(1+K_A)$ from the risk-free investment;
  - buy the asset for $F$, closing the long forward position, and return the asset to the owner.

- Your arbitrage profit will be

$$V(T) = S(0)(1 + K_A) - F > 0,$$

  which once again violates the No-Arbitrage Principle (Assumption 1.9).
- It follows that the forward price must be $F = S(0)(1 + K_A)$.

*[End of Proof]*

# Exercise

Exercise 1.5

# Remark 1.11

- Forward contracts make it possible to place Example 1.3 on a rigorous footing within the No-Arbitrage Principle framework.
- The price offered by either of the dealers $A, B$ is a promise to exchange at rate $d_A$ or $d_B$, respectively, these quotations being forward rates for instantaneous delivery.
- An arbitrage portfolio would be of the form $(x, y, z_1, z_2) = (0, 0, -1, 1)$ with the last two entries representing the positions $z_1$ taken with dealer $A$ and $z_2$ with dealer $B$.

# Outline

# Call Option

- Let $A(0) = 100$, $A(T) = 110$, $S(0) = 100$ dollars and

$$S(T) = \begin{cases} 120 & \text{with probability } p \\ 80 & \text{with probability } 1 - p \end{cases}$$

where $0 < p < 1$.

**Definition: Call option/Call**

A *call option* with *strike price* or *exercise price* $X$ (say \$100) and *exercise time* $T$ is a contract giving the holder the right (but no obligation) to purchase a share of stock for \$100 at time $T$.

- If the stock price falls below the strike price, i.e., $S(T) < X$, the option will be worthless.
  - There would be little point in buying a share for \$100 if its market price is \$80, and no-one would want to exercise the right.
- Otherwise, if the share price rises to \$120, which is above the strike price (i.e., $S(T) > X$), the option will bring a profit of \$20 to the holder,
  - who is entitled to buy a share for \$100 at time $T$ and may sell it immediately at the market price of \$120.
  - known as *exercising* the option.
- Alternatively, the option may also be exercised simply by collecting the difference of \$20 between the market price of stock and the strike price.
  - In practice, the latter is often the preferred method because no stock needs to change hands.

- As a result, the payoff of the call option, that is, its value at time $T$ is a random variable

$$C(T) = \begin{cases} 20 & \text{if stock goes up,} \\ 0 & \text{if stock goes down,} \end{cases}$$

- In general

$$C(T) = \max(S(T) - X, 0),$$

 where $X$ is the strike price.

- Meanwhile, $C(0)$ will denote the value of the option at time 0, that is, the price for which the option can be bought or sold today.

# Remark 1.12

- At first sight, a call option may resemble a long forward position.
    - Both involve buying an asset at a future date for a price fixed in advance.
- An essential difference is that
    - the holder of a long forward contract is committed to buying the asset for the fixed price, whereas
    - the owner of a call option has the right but no obligation to do so.
- Another difference is that
    - an investor will need to pay a premium to purchase a call option, whereas
    - no payment is due when exchanging a forward contract.

- In a market in which options are available, it is possible to invest in a portfolio $(x, y, z)$ consisting of $x$ shares of stock, $y$ bonds and $z$ options.
- The time 0 value of such a portfolio is

$$V(0) = xS(0) + yA(0) + zC(0).$$

  - (1.5)
- At time $T$ it will be worth

$$V(T) = xS(T) + yA(T) + zC(T).$$

  - (1.6)

Just like in the case of portfolios containing forward contracts, the No-Arbitrage Principle needs to be extended to cover portfolios consisting of stock, bonds and options.

**Assumption 1.13 (No-Arbitrage Principle)**

There is no portfolio $(x, y, z)$ that includes a position $z$ in call options and has initial value $V(0) = 0$ such that $V(T) \geq 0$ with probability 1 and $V(T) > 0$ with non-zero probability, where $V(0)$, $V(T)$ are given by (1.5) and (1.6).

- Our task will be to find the time 0 price $C(0)$ of the call option consistent with the assumptions about the market and, in particular, with the absence of arbitrage opportunities.
- Because the holder of a call option has a certain right, but never an obligation, it is reasonable to expect that $C(0)$ will be positive:
  - one needs to pay a premium to acquire this right.

The option price $C(0)$ can be found in two steps:

**Step 1**

- Construct an investment in $x$ stocks and $y$ bonds such that the value of the investment at time $T$ is the same as that of the option,

$$xS(T) + yA(T) = C(T),$$

  no matter whether the stock price $S(T)$ goes up to \$120 or down to \$80.
- known as *replicating* the option.

## Step 2

- Compute the time 0 value of the investment in stock and bonds.
- It will be shown (in Proposition 1.14) that it must be equal to the option price,

$$xS(0) + yA(0) = C(0),$$

  because an arbitrage opportunity would exist otherwise.
- referred to as *pricing* or *valuing* the option.

# Example

## Step 1 (Replicating the Option)

- The time $T$ value of the investment in stock and bonds will be

$$xS(T) + yA(T) = \begin{cases} x120 + y110 & \text{if stock goes up,} \\ x80 + y110 & \text{if stock goes down,} \end{cases}$$

- Thus, the equality $xS(T) + yA(T) = C(T)$ between two random variables can be written as

$$\begin{cases} x120 + y110 = 20, \\ x80 + y110 = 0. \end{cases}$$

  - The first of these equations covers the case when the stock price goes up to \$120, whereas
  - the second equation corresponds to the case when it drops to \$80.

- Because we want the value of the investment in stock and bonds at time $T$ to match exactly that of the option *no matter whether the stock price goes up or down*, these two equations are to be satisfied simultaneously.
- Solving for $x$ and $y$, we find that

$$x = \frac{1}{2}, \quad y = -\frac{4}{11}.$$

- To replicate the option we need to buy $\frac{1}{2}$ a share of stock and take a short position of $-\frac{4}{11}$ in bonds (or borrow $\frac{4}{11} \times 100 = \frac{400}{11}$ dollars in cash).

**Step 2 (Pricing the Option)**

- We can compute the value of the investment in stock and bonds at time 0:

$$xS(0) + yA(0) = \frac{1}{2} \times 100 - \frac{4}{11} \times 100 \cong 13.6364$$

  dollars.

- The following proposition shows that this must be equal to the price of the option $C(0)$.

# Proposition 1.14

## Proposition 1.14

If the option can be replicated by investing in the a portfolio $(x, y)$ of stock and bonds, then $C(0) = xS(0) + yA(0)$, or else an arbitrage opportunity would exist.

Proof

1)

- Suppose that $C(0) > xS(0) + yA(0)$.
- If this is the case, then at time 0:
  - issue and sell one option for $C(0)$ dollars;
  - take a long position in the portfolio $(x, y)$, which costs $xS(0) + yA(0)$ (Note: for a call option this involves buying shares and borrowing cash).
    [For the above example, borrow $\frac{4}{11} \times 100 = \frac{400}{11}$ dollars in cash (or take a short position $y = -\frac{4}{11}$ in bonds by selling them); and purchase $x = \frac{1}{2}$ shares of stock for $xS(0) = \frac{1}{2} \times 100 = 50$ dollars.]

- The cash balance of these transactions is positive, $C(0) - xS(0) - yA(0) > 0$.
- Invest this amount risk-free.
- The resulting portfolio consisting of shares, risk-free investments and a call option has initial value $V(0) = 0$.

- Subsequently, at time $T$:
  - if stock goes up, then settle the option by paying the difference of $S^u(T) - X$ between the market price of one share and the strike price (that is, \$20 in the above example);
    you will pay nothing if stock goes down;
    the cost to you will be $C(T)$, which covers both possibilities;
  - close the position in stock and bonds, receiving the amount $xS(T) + yA(T)$.
    [For the above example, repay the loan with interest (or close your short position $y = -\frac{4}{11}$ in bonds); this will cost you $\frac{4}{11} \times 110 = 40$ dollars. And sell the stock for $\frac{1}{2}S(T)$ obtaining either $\frac{1}{2} \times 120 = 60$ dollars if the price goes up, or $\frac{1}{2} \times 80 = 40$ dollars if it goes down.]

- The cash balance of these transactions will be zero, $-C(T) + xS(T) + yA(T) = 0$, regardless of whether stock goes up or down.
- But you will be left with the initial risk-free investment of $C(0) - xS(0) - yA(0)$ plus interest, thus realising an arbitrage opportunity, contrary to the No-Arbitrage Principle (Assumption 1.13).

2)

- On the other hand, if $C(0) < xS(0) + yA(0)$, then, at time 0:
  - Buy one option for $C(0)$ dollars;
  - take a short position in the portfolio $(x, y)$ (which involves buying bonds and short selling stock).
    [For the above example, buy $\frac{4}{11}$ bonds for $\frac{4}{11} \times 100 = \frac{400}{11}$ dollars, and sell short $x = \frac{1}{2}$ shares of stock for $\frac{1}{2} \times 100 = 50$ dollars.]
- The cash balance of these transactions is positive, $-C(0) + xS(0) + yA(0) > 0$, and can be invested risk-free.
- In this way you will have constructed a portfolio with initial value $V(0) = 0$.

- Subsequently, at time $T$:
  - if stock goes up, then exercise the option, receiving the difference of $S^u(T) - X$ between the market price of one share and the strike price (that is, \$20 in the above example);
    you will receive nothing if stock goes down;
    your income will be $C(T)$, which covers both possibilities;
  - close the position in stock and bonds, paying $xS(T) + yA(T)$.
    [For the above example, sell the bonds for $\frac{4}{11}A(T) = \frac{4}{11} \times 110 = 40$ dollars; and close the short position in stock, paying $\frac{1}{2}S(T)$, that is, $\frac{1}{2} \times 120 = 60$ dollars if the price goes up, or $\frac{1}{2} \times 80 = 40$ dollars if it goes down.]

- The cash balance of these transactions will be zero, $C(T) - xS(T) - yA(T) = 0$, regardless of whether stock goes up or down.
- But you will be left with an arbitrage profit resulting from the risk-free investment of $-C(0) + xS(0) + yA(0)$ plus interest, again a contradiction with the No-Arbitrage Principle (Assumption 1.13).

*[End of Proof]*

- Here we see once more that the arbitrage strategy follows a common sense pattern:
- Sell (or sell short if necessary) expensive securities and buy cheap ones, as long as all your financial obligations arising in the process can be discharged, regardless of what happens in the future.

- Proposition 1.14 implies that today’ s price of the option must be
$$C(0) = \frac{1}{2}S(0) - \frac{4}{11}A(0) \cong 13.6364$$
dollars.
- Anyone who would offer to sell the option for less or to buy it for more than this price would be creating an arbitrage opportunity, which amounts to handing out free money.

# Remark 1.15

- Note that the probabilities $p$ and $1 - p$ of stock going up or down are irrelevant in pricing and replicating the option.
- This is a remarkable feature of the theory, and by no means a coincidence.

# Remark 1.16 (Read it yourself)

- Options may appear to be superfluous in a market in which they can be replicated by stock and bonds.
  - In the simplified one-step model this is indeed a valid objection.
- However, in a situation involving multiple time steps (or continuous time) replication becomes a much more onerous task.
  - It requires rebalancing the positions in stock and bonds at every time instant at which there is a change in their prices, resulting in considerable management and transaction costs.
- In some cases it may not even be possible to replicate an option precisely.
- This is why the majority of investors prefer to buy or sell options, replication being normally undertaken only by specialised dealers and institutions.

# Exercise

Exercise 1.6

Exercise 1.7

# Outline

# Put Option

**Definition: Put option/Put**

A *put* option with strike price $X$ (say \$100) and exercise time $T$ gives the right to sell one share of stock for \$100 at time $T$.

- Put option is worthless if the stock goes up, but it brings a profit otherwise, the payoff being

$$P(T) = \begin{cases} 0 & \text{if stock goes up,} \\ 20 & \text{if stock goes down,} \end{cases}$$

given that the possible values of $S(T)$ are the same as above.

- In general,

$$P(T) = \max(X - S(T), 0)$$

  where $X$ is the strike price.
- The notion of a portfolio may be extended to allow positions in put options, denoted by $z$, as before.

- The replicating and pricing procedure for puts follows the same pattern as for call options.
- In particular, the price $P(0)$ of the put option is equal to the time 0 value of a replicating investment in stock and bonds.

# Remark 1.17

- There is some similarity between a put option and a short forward position:
  both involve selling an asset for a fixed price at a certain time in the future.
- However, an essential difference is that
  - the holder of a short forward contract is committed to selling the asset for the fixed price, whereas
  - the owner of a put option has the right but no obligation to sell.
- Moreover,
  - an investor who wants to buy a put option will have to pay for it, whereas
  - no payment is involved when a forward contract is exchanged.

# Exercise

Exercise 1.8

- The general properties of options and forward contracts will be discussed in Chapters 4 and 5.
- In Chapters 6 and 7 the pricing and replicating schemes will be extended to more complicated though still discrete market models, as well as to other financial instruments.

Note:

- A common feature of these new securities is that their payoffs depend on the stock prices.
- Because of this they are called *derivative securities* or *contingent claims*.

# Foreign Exchange

- Foreign currency is an important example of a risky security.
- The basic novelty as compared to stock is that each unit of foreign currency held in a portfolio generates additional risk-free income.
  - Here we assume that the currency is kept in a bank account or invested in risk-free bonds.
- The ability to generate additional income, also present in the stock market if dividends are paid to the shareholders, needs to be accounted for when considering derivative securities written on such assets.

1) Forward price:

- First we begin with adjustment to formula (1.4) (Recall: $F = S(0)\frac{A(T)}{A(0)} = S(0)(1+K_A)$) for forward prices.
- Note that to have one unit of foreign currency at time $T$ it is sufficient to buy a fraction of the unit at time 0, namely $\frac{A_f(0)}{A_f(T)} = \frac{1}{1+K_f}$, where the subscript $f$ designates foreign currency bond prices and the return on them.
- Therefore, a candidate for the forward price (more precisely, *forward rate of exchange*) is

$$F = S(0)\frac{1+K_h}{1+K_f},$$

- (1.7) where $1+K_h = \frac{A_h(T)}{A_h(0)}$, the subscript $h$ indicating the home currency.

- Note: Proof of formula (1.7) is left in Exercise 1.9.

# Exercise

Exercise 1.9

Exercise 1.10

2) Prices of options:

- Our next task is to find prices of options written on foreign currency.
- This is best illustrated by an example.

# Example 1.18

- Consider pound sterling as a security on the US dollar market.
- Suppose that the return on US dollar bonds is $K_{\$} = 5\%$ and on British pound bonds it is $K_{¿} = 3\%$.
- Consider a call option on pound sterling with strike price $X = 1.64$ dollars to a pound within a binomial model with $S(0) = 1.62$, $S^u(T) = 1.84$ and $S^d(T) = 1.46$ dollars to a pound.
- The replication step requires finding $x, y$ such that

$$
\begin{aligned}
x(1 + K_{£})S^u(T) + y(1 + K_{\$}) &= S^u(T) - X, \\
x(1 + K_{£})S^d(T) + y(1 + K_{\$}) &= 0.
\end{aligned}
$$

- Note the difference as compared to the replication of stock options, where the factor $1 + K_{£}$ representing the income generated by the underlying security (here pound sterling) was absent.
- The solution of this system of equations is $x = 0.5110$ pounds and $y = -0.7318$ dollars, and consequently

$$C(0) = xS(0) + y = 0.0960$$

  dollars.

Finding the price of a put is left in Exercise 1.11 below.

# Exercise

Exercise 1.11

# Managing Risk with Options

**1)** The availability of options and other derivative securities extends the possible investment scenarios.

- Recall and use the setup of Section 1.6:
  Let $A(0) = 100$, $A(T) = 110$, $S(0) = 100$ dollars and

$$S(T) = \begin{cases} 120 & \text{with probability } p \\ 80 & \text{with probability } 1 - p \end{cases}$$

where $0 < p < 1$; $C(0) \simeq 13.6364$.

- Suppose that your initial wealth is \$1,000 and compare the following two investments:
    - buy 10 shares; at time $T$ they will be worth

$$10 \times S(T) = \begin{cases} 1,200 & \text{if stock goes up,} \\ 800 & \text{if stock goes down,} \end{cases}$$

    or

    - buy $1,000/13.6364 \cong 73.3333$ options; in this case your final wealth will be

$$73.3333 \times C(T) \cong \begin{cases} 1,466.67 & \text{if stock goes up,} \\ 0.00 & \text{if stock goes down.} \end{cases}$$

- If stock goes up, the investment in options will produce a much higher return than shares, namely about 46.67%.
- However, it will be disastrous otherwise: you will lose all your money.
- Meanwhile, when investing in shares, you would gain just 20% or lose 20%.
- Without specifying the probabilities we cannot compute the expected returns or standard deviations.
- Nevertheless, one would readily agree that investing in options is more risky than in stock.
- This can be exploited by adventurous investors.

# Exercise

Exercise 1.12

**2)** Options can also be employed to reduce risk.

- Consider an investor planning to purchase stock in the future.
- The share price today is $S(0) = 100$ dollars, but the investor will only have funds available at a future time $T$, when the share price will become

$$S(T) = \begin{cases} 160 & \text{with probability } p, \\ 40 & \text{with probability } 1 - p, \end{cases}$$

for some $0 < p < 1$.

- Assume, as before, that $A(0) = 100$ and $A(T) = 110$ dollars, and compare the following two strategies:
  1. wait until time $T$, when the funds become available, and purchase the stock for $S(T)$;
     or
  2. at time 0 borrow money to buy a call option with strike price \$100;
     then, at time $T$ repay the loan with interest and purchase the stock, exercising the option if the stock price goes up.

- The investor will be open to considerable risk if she chooses to follow the first strategy.
- On the other hand, following the second strategy, she will need to borrow $C(0) \cong 31.8182$ dollars to pay for the option.
- At time $T$ she will have to repay \$35 to clear the loan and may use the option to purchase the stock, hence the cost of purchasing one share will be

$$S(T) - C(T) + 35 = \begin{cases} 135 & \text{if stock goes up,} \\ 75 & \text{if stock goes down,} \end{cases}$$

- Clearly, the risk is reduced, the spread between these two figures being narrower than before.

# Exercise

Exercise 1.13

Exercise 1.14

- If two options are bought, then the risk will be reduced to nil:

$$S(T) - 2 \times C(T) + 70 = 110 \text{ with probability } 1.$$

- This strategy turns out to be equivalent to a long forward contract, since the forward price of the stock is exactly \$110 (see Section 1.5, Proposition 1.10).
- It is also equivalent to borrowing money to purchase a share for \$100 today and repaying \$110 to clear the loan at time $T$.

# Case and Discussion

## Case 1

- A UK company is preparing to purchase a piece of equipment in the US for \$100,000 in a year's time, the price guaranteed by the producer to remain unchanged.
- Considering the current exchange rate of 1.62 dollars to a pound, the manager of the company has reserved £64,000 in the budget to become available at the time of purchase.
- Analyse this decision.

# Case and Discussion

## Discussion

The first task is to recognise the nature of the problem. Since the UK company will have £64,000 at their disposal in a year's time to purchase the piece of equipment for \$100,000, the dollar price being guaranteed by the producer, the source of concern is the unknown exchange rate at the time of purchase. The amount budgeted for may turn out insufficient to purchase the equipment if the exchange rate falls below 1.5625 dollars to a pound.

To ensure that the project is successful a financial package needs to be designed to hedge the interest rate risk. Having followed this chapter, we are familiar with some tools to achieve this goal. It will be necessary to collect some additional market data to apply these tools.

The first approach is to employ forward contracts to convert pounds into dollars at the end of the year. The forward price $F$ of pound sterling can be found using the methods discussed in Section 1.7. In order to do so, we need to look up the pound sterling and US dollar bond prices, or, equivalently, the corresponding risk-free returns. Suppose that these turn out to be $K_{\$} = 5\%$ and $K_{¿} = 3\%$. This gives $F = 1.6515$ dollars to a pound. Converting £64,000 at this rate will give \$105,693.20. The company will be able to purchase the piece of equipment, leaving it with a surplus of \$5,693.20, irrespective of what happens to the exchange rate.

As an alternative to forward contracts, since the company will need to convert their currency into dollars, in effect selling pound sterling, we could employ put options on pound sterling.

This requires building a model of future random exchange rates. At this point all we can do is to use the single step binomial model. We know the spot rate $S(0) = 1.62$ dollars to a pound and the risk-free returns $K_{\$} = 5\%$ and $K_{¿} = 3\%$. The two future rates $S^u(1)$, $S^d(1)$ need to be calibrated to match market data. To this end we can use the prices of some traded options on pound sterling. Suppose that calls with strikes 1.6 and 1.7 dollars to a pound, selling at 1.1178 and 0.0624 dollars to a pound, are the most liquid (and therefore considered the most representative) options traded. Since these call prices can be expressed in terms of $S^u(1)$, $S^d(1)$ just like in Example 1.18, we obtain a system of two equations, which can be solved to get $S^u(1) \cong 1.8126$ and $S^d(1) \cong 1.4273$ dollars to a pound.

Having calibrated the model, we need to decide the strike rate $X$ for the put option to be used to hedge the interest rate risk. The strike $X$ affects the option price $P$, which we shall denote by $P(X)$ for clarity. We need to purchase 64,000 puts to convert the available sum of £64,000 into dollars. This will cost $P(X) \times 64,000$ dollars, which we need to borrow. Suppose that the company's bank quotes a 10% interest rate for a US dollar loan. At the end of the year we shall need to repay $P(X) \times 64,000 \times 1.1$ dollars in addition to the price of the piece of equipment. This gives an equation for the strike price:

$$100,000 + P(X) \times 64,000 \times 1.1 = X \times 64,000.$$

The solution is $X \cong 1.6680$ pounds to a dollar. The price of a put is $P(X) \cong 0.0959$ dollars. We need to borrow $P(X) \times 64,000 \cong 6,135.32$ dollars to purchase the options, and will have to repay \$6,748.85 including interest at the end of the year. If the exchange rate goes up, we shall sell pound sterling at the market price with an additional profit of $64,000 \times 1.8126 - 106,748.85 \cong 9,259.81$ dollars. If the rate goes down, we shall exercise the options and break even.

The strategy involving options does not look as attractive as forward contracts. Hedging with puts will also enable the company to purchase the piece of equipment irrespective of any changes in the exchange rate, but the end financial result is unpredictable, depending on the fluctuations of the rate. A practical problem could be that put options with the ideal strike rate $X \cong 1.6680$ pounds to a dollar may not be available via exchange trading, necessitating an over-the-counter (OTC) transition. Another obvious weakness of this approach to hedging with options is connected with the simplicity of the model of exchange rates. The case is certainly worth revisiting when we acquire better models. On the other hand, when using forwards, we did not need to adopt any particular model of interest rates, so the result is much more robust.