# Financial Econometrics - Part IV
# Nonlinear Autoregressive Models

Li Chen

lichen812@xmu.edu.cn

WISE, Xiamen University

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# Threshold Autoregressive (TAR) Model

To capture asymmetric features in financial data, we need to establish Threshold Autoregressive (TAR) models. For example, a TAR(1) model is constructed as

$$X_t = \begin{cases} \alpha_1 X_{t-1} + u_t, & X_{t-1} < c \\ \alpha_2 X_{t-1} + u_t, & X_{t-1} \geq c, \end{cases} \tag{1}$$

for some threshold value $c$. In a compact form, we write the model as

$$X_t = \alpha_1 X_{t-1} 1(X_{t-1} < c) + \alpha_2 X_{t-1} 1(X_{t-1} \geq c) + u_t. \tag{2}$$

Question: How to estimate $\alpha_1, \alpha_2$ and $c$ ?

# Threshold Autoregressive (TAR) Model

Why TAR model is able to capture asymmetric features in financial data? We consider the following example.

$$X_t = \begin{cases} -1.5X_{t-1} + u_t, & X_{t-1} < 0 \\ 0.5X_{t-1} + u_t, & X_{t-1} \geq 0. \end{cases} \tag{3}$$

If $X_{t-1} < 0$, then there is a very strong tendency for $X_t$ to jump to a positive value.

If $X_{t-1} > 0$, then it will tend to take multiple time periods for $X_t$ to reduce to a negative value.

Therefore, there are more positive observations than negative observations.

# Threshold Autoregressive (TAR) Model

The Momentum Threshold Autoregressive (MTAR) Model

$$X_t = \begin{cases} \alpha_1 X_{t-1} + u_t, & \Delta X_{t-1} < 0 \\ \alpha_2 X_{t-1} + u_t, & \Delta X_{t-1} \geq 0. \end{cases} \tag{4}$$

# Self-Exciting Threshold Autoregressive (SETAR) Model

We consider a more general case as

$$X_t = \alpha_0^{(s)} + \sum_{j=1}^{p} \alpha_j^{(s)} X_{t-j} + h^{(s)} u_t \text{ if } X_{t-d} \in (r_{s-1}, r_s], \qquad (5)$$

for $s = 1, 2, ..., M$, where $\{s\}_{s=1}^{M}$ are different states.

Since the threshold or regime change is generated by $\{X_t\}$ itself rather than by other variables, we call it a Self-Exciting Threshold Autoregressive model.

# Self-Exciting Threshold Autoregressive (SETAR) Model

Example: Managed Foreign Exchange Rates with Floors and Ceilings

$$X_t = \begin{cases} a + u_t^{(1)} & \text{if } X_{t-1} \leq a, \\ X_{t-1} + u_t^{(2)} & \text{if } a < X_{t-1} < b, \\ b + u_t^{(3)} & \text{if } X_{t-1} \geq b. \end{cases} \tag{6}$$

# Self-Exciting Threshold Autoregressive (SETAR) Model

Example: U.S. quarterly real GNP from 1947:2 to 1991:1.

$$X_t = \begin{cases} -0.015 - 1.076X_{t-1} + \epsilon_{1t}, & X_{t-1} \leq X_{t-2} \leq 0, \\ 0.630X_{t-1} - 0.756X_{t-2} + \epsilon_{2t}, & X_{t-1} > X_{t-2}, X_{t-2} \leq 0, \\ 0.006 + 0.438X_{t-1} + \epsilon_{3t}, & X_{t-1} \leq X_{t-2}, X_{t-2} > 0, \\ 0.433X_{t-1} + \epsilon_{4t}, & X_{t-1} > X_{t-2} > 0. \end{cases} \tag{7}$$

Remarks:

(i) Regime 1: a recession period in which the economy changed from a contraction to an even worse one. The negative explosive nature of the regression function in this regime indicates the economy usually recovers quickly from the recession.

# Self-Exciting Threshold Autoregressive (SETAR) Model

Example: U.S. quarterly real GNP from 1947:2 to 1991:1.

$$X_t = \begin{cases} -0.015 - 1.076X_{t-1} + \epsilon_{1t}, & X_{t-1} \leq X_{t-2} \leq 0, \\ 0.630X_{t-1} - 0.756X_{t-2} + \epsilon_{2t}, & X_{t-1} > X_{t-2}, X_{t-2} \leq 0, \\ 0.006 + 0.438X_{t-1} + \epsilon_{3t}, & X_{t-1} \leq X_{t-2}, X_{t-2} > 0, \\ 0.433X_{t-1} + \epsilon_{4t}, & X_{t-1} > X_{t-2} > 0. \end{cases} \tag{8}$$

Remarks:

(ii) Regime 2: a period in which the economy was in contraction but improving. Here, the regression function tends to be positive, suggesting that the economy is more likely to grow continuously out of recession once a recovery has started.

# Self-Exciting Threshold Autoregressive (SETAR) Model

Example: U.S. quarterly real GNP from 1947:2 to 1991:1.

$$X_t = \begin{cases} -0.015 - 1.076X_{t-1} + \epsilon_{1t}, & X_{t-1} \le X_{t-2} \le 0, \\ 0.630X_{t-1} - 0.756X_{t-2} + \epsilon_{2t}, & X_{t-1} > X_{t-2}, X_{t-2} \le 0, \\ 0.006 + 0.438X_{t-1} + \epsilon_{3t}, & X_{t-1} \le X_{t-2}, X_{t-2} > 0, \\ 0.433X_{t-1} + \epsilon_{4t}, & X_{t-1} > X_{t-2} > 0. \end{cases} \tag{9}$$

Remarks:

(iii) Regime 3: a period in which the economy was reasonable but the growth declined.

(iv) Regime 4: an expansion period in which the economy became stronger.

# Smooth Transition AR (STAR) Model

Teräsvirta (1994, JASA)

$$X_t = \left(\alpha_{00} + \sum_{j=1}^{p} \alpha_{0j} X_{t-j}\right) + F\left(\frac{X_{t-d} - c}{\gamma}\right)\left(\alpha_{10} + \sum_{j=1}^{p} \alpha_{1j} X_{t-j}\right) + u_t, \tag{10}$$

where $d$ is the delay parameter, $c$ and $\gamma$ are parameters representing the location and scale of regime transition, and $F(.)$ is a smooth transition function, which is usually a continuous cumulative distribution function.

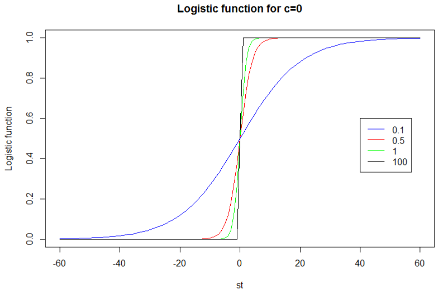
Examples of $F(.)$ include the N(0,1) CDF, the logistic function, and the exponential CDF.

# Smooth Transition AR (STAR) Model

Logistic function

$$F(z) = \frac{1}{1 + e^{-\gamma(z-c)}}, \quad -\infty < z < \infty. \tag{11}$$

where $c$ determines the location of regime transition, while $\gamma$ determines the smoothness of the regime shift.

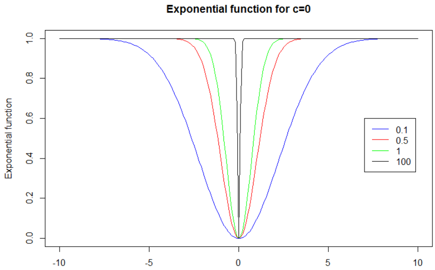


# Smooth Transition AR (STAR) Model

Exponential function

$$F(z) = 1 - \frac{1}{e^{-\gamma(z-c)^2}}, \quad -\infty < z < \infty. \tag{12}$$

where $c$ determines the location of regime transition, while $\gamma$ determines the smoothness of the regime shift.



# Markov-Chain Regime Switching Model

Suppose the time series $\{S_t\}$ follows a Markov chain on a finite number of states, say, $\{1, 2, ..., M\}$ with transition probability

$$Pr\,(\ S_t = j\ |S_{t-1} = i\ ) = p_{ij}, \quad i, j = 1, ..., M; \tag{13}$$

such that $\sum_{j=1}^{M} p_{ij} = 1$ for all $i = 1, 2, ..., M$.

Remarks:

(i) The state variable $S_t$ is not observable.

(ii) Here, the transition probabilities $p_{ij}$ are assumed to be constant over time.

# Markov-Chain Regime Switching Model

Example: A Two-Stage Regime Switching Model

$$X_t = \begin{cases} a_0 + \sum_{j=1}^{p} a_j X_{t-j} + u_{1t} & \text{if } S_t = 1, \\ b_0 + \sum_{j=1}^{p} b_j X_{t-j} + u_{2t} & \text{if } S_t = 2, \end{cases} \tag{14}$$

where the state variable $S_t$ assumes values $\{1, 2\}$ and follows afirst order Markov chain with transition probabilities

$$Pr\,(\ S_t = 2\ |S_{t-1} = 1\ ) = p_{12}, Pr\,(\ S_t = 1\ |S_{t-1} = 2\ ) = p_{21}, \tag{15}$$

The innovations $\{u_{1t}\}$ and $\{u_{2t}\}$ are i.i.d. with mean zero and variances $\sigma_1^2$ and $\sigma_2^2$ respectively.

# Functional Coefficient Autoregressive Model

Priestley (1988), and Chen and Tsay (1993a)

$$X_t = a_0(X_{t-d}) + \sum_{j=1}^{p} a_j(X_{t-d})X_{t-j} + u_t. \tag{16}$$

Remarks:

(i) This is a semiparametric model because the functional forms for the $a_j(.)$ are not parametrized. Obviously, the FCAR model is more flexible than the SETAR model. Most nonlinear time series models discussed earlier are special cases of the FCAR model.

(ii) When $a_j(.)$ is a smooth function for all j = 0,1,...,p, this model can be considered as a special case of threshold autoregressive model with an infinite number of threshold states which vary continuously.

# Functional Coefficient Autoregressive Model

Priestley (1988), and Chen and Tsay (1993a)

$$X_t = a_0(X_{t-d}) + \sum_{j=1}^{p} a_j(X_{t-d})X_{t-j} + u_t. \tag{17}$$

Remarks:

(iii) It can be used to model the processes where mean-reverting speeds are different.

(iv) It can be used to test whether the factor in the CAMP model is time-varying.

# Time-varying Volatility Model

Suppose that

$$r_t = c + \sigma(t/T)\epsilon_t, \tag{18}$$

for $t = 1, 2, ..., T$, where $\sigma(t/T)$ is a function defined on [0,1] and $E[\epsilon_t] = 0$, $E[\epsilon_t^2] = 1$. To estimate the nonparametric function, we have

$$(r_t - c)^2 = \sigma^2(t/T) + \sigma^2(t/T)(\epsilon_t^2 - 1) = \sigma(t/T)^2 + \eta_t, \tag{19}$$

where $\eta_t = \epsilon_t^2 - 1$.