Mathematical Finance

Ch01-1: A Simple Market Model

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Outline

Basic Notions and Assumptions

- Suppose that two assets are traded:
  1. one **risk-free** asset (e.g. a bank deposit or a bond issued by a government, a financial institution, or a company) and
  2. one **risky** security (e.g. typically, some stock; or a foreign currency, gold, a commodity or virtually any asset whose future price is unknown today).

- Throughout the Ch 1, we restrict the time scale, t, to two instants only:
  - today, $t = 0$, and
  - some future time, $t = T$ (say one year from now, then $T = 1$).
- [More refined and realistic situations will be studied in later chapters.]

- The position in risky securities can be specified as the number of shares of stock held by an investor.
- Denote: the price of one share at time t as $S(t)$ (note: $t = 0, T$).
  - The current stock price $S(0)$ is known to all investors, but
  - the future price $S(T)$ remains uncertain: it may go up as well as down.
- The difference $S(T) - S(0)$ as a fraction of the initial value represents the so-called *rate of return*, or briefly *return*:

$$K_S = \frac{S(T) - S(0)}{S(0)},$$

which is also uncertain.

- The risk-free position can be described as the amount held in a bank account, or, as an alternative, investors may choose to invest in bonds.
- Denote: the price of one bond at time t as $A(t)$.
    - The current bond price $A(0)$ is known to all investors, just like the current stock price.
    - However, in contrast to stock, the price $A(T)$ the bond will fetch at time T is also known with certainty.
        - e.g., $A(T)$ may be a payment guaranteed by the institution issuing bonds, in which case the bond is said to mature at time T with face value $A(T)$.
- The return on bonds is defined in a similar way as that on stock,

$$K_A = \frac{A(T) - A(0)}{A(0)}.$$

- [Chapters 2 and 9 give a detailed exposition of risk-free assets.]

- To build a mathematical model of a market of financial securities, the first stage is to specify a number of assumptions.
- The purpose here is to find a compromise between
  - the complexity of the real world and
  - the limitations and simplifications of a mathematical model, imposed in order to make it tractable.
- The assumptions reflect our current position on this compromise and will be modified in the future.

Assumption 1.1 (Randomness)

The future stock price $S(T)$ is *unknown* at time 0, thus a *random variable,* with at least two different values.

The future price $A(T)$ of the risk-free security is a *known* number, or say, is *deterministic*.

Assumption 1.2 (Positivity of Prices)

All stock and bond prices are strictly positive,

$$A(t) > 0 \quad \text{and} \quad S(t) > 0 \quad \text{for} \quad t = 0, T.$$

- The total wealth of an investor holding x stock shares and y bonds at a time instant $t = 0, T$ is

$$V(t) = xS(t) + yA(t).$$

- (1.1)

- The pair (x, y) is called a *portfolio*,
$V(t)$ being the *value* of this portfolio or, in other words, the *wealth* of the investor at time t.

- The jumps of asset prices between times 0 and T give rise to a change of the portfolio value:

$$V(T) - V(0) = x(S(T) - S(0)) + y(A(T) - A(0)).$$

- This difference (which may be positive, zero, or negative) as a fraction of the initial value represents the return on the portfolio,

$$K_V = \frac{V(T) - V(0)}{V(0)}.$$

- The returns on bonds or stock are particular cases of the return on a portfolio (with $x = 0$ or $y = 0$, respectively).
- *Note:* because $S(T)$ is a random variable, so is $V(T)$ as well as the corresponding returns K_S and K_V.
- The return K_A on a risk-free investment is deterministic.

Example 1.1

- Let $A(0) = 100$ and $A(T) = 110$ dollars.
- Then the return on an investment in bonds will be

$$K_A = 0.10 = 10\%.$$

- Also, let $S(0) = 50$ dollars and suppose that the random variable $S(T)$ can take two values,

$$S(T) = \begin{cases} 52 & \text{with probability } p, \\ 48 & \text{with probability } 1-p, \end{cases}$$

  for a certain $0 < p < 1$.
- The return on stock will then be

$$K_S = \begin{cases} 0.04 = 4\% & \text{if stock goes up,} \\ -0.04 = -4\% & \text{if stock goes down.} \end{cases}$$

Example 1.2

- Given the bond and stock prices in Example 1.1, the value at time 0 of a portfolio with $x = 20$ stock shares and $y = 10$ bonds is
$$V(0) = 2,000$$
dollars.
- The time T value of this portfolio will be
$$V(T) = \begin{cases} 2,140 & \text{if stock goes up,} \\ 2,060 & \text{if stock goes down,} \end{cases}$$
so the return on the portfolio will be
$$K_V = \begin{cases} 0.07 = 7\% & \text{if stock goes up,} \\ 0.03 = 3\% & \text{if stock goes down.} \end{cases}$$

Exercise

Exercise 1.1

Exercise 1.2

- To represent the risky and risk-free positions x and y in a portfolio,
  it is mathematically convenient and not too far from reality to allow arbitrary real numbers, including negative ones and fractions.
- This is reflected in the following assumption, which imposes no restrictions on the trading positions.

Assumption 1.3 (Divisibility, Liquidity and Short Selling)

An investor may hold any number x and y of stock shares and bonds, whether integer or fractional, negative, positive or zero. In general,

$$x, y \in \mathbb{R}$$

Divisibility

- The fact that one can hold a fraction of a share or bond is referred to as *divisibility*.
- Almost perfect divisibility is achieved in real world dealings whenever the volume of transactions is large as compared to the unit prices.

Liquidity

- The fact that no bounds are imposed on x or y is related to another market attribute, i.e., *liquidity*.
- It means that any asset can be bought or sold on demand at the market price in arbitrary quantities.

This is clearly a mathematical idealisation.

- In practice, however, there exist restrictions on the volume of trading, or large transactions may affect the prices.

[Modelling such effects requires sophisticated mathematics beyond the scope of this course.]

Long and short positions

- If the number of securities of a particular kind held in a portfolio is positive, we say that the investor has a *long position*.
- Otherwise, we say that a *short position* is taken or that the asset is *shorted*.

Short position in risk-free securities:

- A short position in risk-free securities may involve issuing and selling bonds,
  but in practice the same financial effect is more easily achieved by borrowing cash, the interest rate being determined by the bond prices.
- Repaying the loan with interest is referred to as *closing* the short position.

Short position in stock:

- A short position in stock can be realised by *short selling*.
- i.e., the investor borrows the stock, sells it, and uses the proceeds to make some other investment.
- The owner of the stock keeps all the rights to it.
  - In particular, she is entitled to receive any dividends due and may wish to sell the stock at any time.
- Because of this, the investor must always have sufficient resources to fulfil the resulting obligations and, in particular, to *close* the short position in risky assets, that is, to repurchase the stock and return it to the owner.
  - Similarly, the investor must always be able to close a short position in risk-free securities, by repaying the cash loan with interest.
- In view of this, we impose the following restriction.

Assumption 1.4 (Solvency)

The wealth of an investor must be non-negative at all times,

$$V(t) \geq 0 \quad \text{for} \quad t = 0, T.$$

- A portfolio satisfying this condition is called *admissible*.

- In the real world, the number of possible different prices is finite
    1. because they are quoted to within a specified number of decimal places and
    2. because there is only a certain final amount of money in the whole world, supplying an upper bound for all prices.

Assumption 1.5 (Discrete Unit Prices)

The future price $S(T)$ of a share of stock is a random variable taking only finitely many values.

- However, this will not be the case in Ch 8 (Continuous Time Model).

No-Arbitrage Principle

- In this section we are going to state the most fundamental assumption about the market.
- In brief, we shall assume that the market does not allow for risk-free profits with no initial investment. [**No free lunch!**]
  - A possibility of risk-free profits with no initial investment can emerge when market participants make a mistake. [**Arbitrage opportunities**.]

Example 1.3

- Suppose that
  - dealer A in New York offers to buy British pounds at a rate $d_A = 1.62$ dollars to a pound, while
  - dealer B in London sells them at a rate $d_B = 1.60$ dollars to a pound.
- If this were the case, the dealers would, in effect, be handing out free money. [But, how?]
- An investor with no initial capital could realise a profit of $d_A - d_B = 0.02$ dollars per each pound traded by taking simultaneously a short position with dealer A and a long position with dealer B.
- The demand for their generous services would quickly compel the dealers to adjust the exchange rates so that this profitable opportunity would disappear.

Exercise

Exercise 1.3

- The next example illustrates a situation when a risk-free profit could be realised without initial investment in our simplified framework of a single time step.

Example 1.4

- Suppose that
  - dealer A in New York offers to buy British pounds a year from now at a rate $d_A = 1.58$ dollars to a pound, while
  - dealer B in London would sell British pounds immediately at a rate $d_B = 1.60$ dollars to a pound.
- Suppose further that
  - dollars can be borrowed at an annual rate of 4%, and
  - British pounds can be invested in a bank account at 6%.
- This would also create an opportunity for a risk-free profit without initial investment, though perhaps not as obvious as before.

Think: how to employ this opportunity here?

To take advantage of it:

- For instance, an investor could borrow 10,000 dollars and convert them into 6,250 pounds, which could then be deposited in a bank account.
- After one year interest of 375 pounds would be added to the deposit, and the whole amount could be converted back into 10,467.50 dollars.
  (A suitable agreement would have to be signed with dealer A at the beginning of the year.)
- After paying back the dollar loan with interest of 400 dollars, the investor would be left with a profit of 67.50 dollars.

Comment:

- Apparently, one or both dealers have made a mistake in quoting their exchange rates, which can be exploited by investors.
- Once again, increased demand for their services will prompt the dealers to adjust the rates, reducing d_A and/or increasing d_B to a point when the profit opportunity disappears.

[End of Ex. 1.4]

- Thus, we shall make an assumption forbidding situations similar to the above example.

Assumption 1.6 (No-Arbitrage Principle)

There is no admissible portfolio (x, y) with initial value $V(0) = 0$ such that $V(T) \geq 0$ with probability 1 and $V(T) > 0$ with non-zero probability, where $V(0)$, $V(T)$ are given by (1.1).

- In other words, if the initial value of an admissible portfolio is zero, $V(0) = 0$ (no initial investment), and $V(T) \geq 0$ (no risk of a loss), then $V(T) = 0$ (no profit) with probability 1.
- This means that no investor can lock in a profit without risk and with no initial endowment.
- If a portfolio violating this principle did exist, we would say that an *arbitrage* opportunity was available.

Comments:

- Arbitrage opportunities rarely exist in practice.
- If and when they do, the gains are typically extremely small as compared to the volume of transactions, making them beyond the reach of small investors.
- In addition, they can be more subtle than the examples above.
- Situations when the No-Arbitrage Principle is violated are typically short-lived and difficult to spot.
- The activities of investors (called *arbitrageurs*) pursuing arbitrage profits effectively make the market free of arbitrage opportunities.

- The exclusion of arbitrage in the mathematical model is close enough to reality and turns out to be the most important and fruitful assumption.
- Arguments based on the No-arbitrage Principle are the main tools of financial mathematics.

One-Step Binomial Model

- In this section we restrict ourselves to a very simple example, in which the stock price $S(T)$ takes only two values.

Example 1.6

- Suppose that $S(0) = 100$ dollars and $S(T)$ can take two values,
$$S(T) = \begin{cases} 125 & \text{with probability } p, \\ 105 & \text{with probability } 1-p, \end{cases}$$
where $0 < p < 1$, while the bond prices are $A(0) = 100$ and $A(T) = 110$ dollars.
- Thus, the return K_S on stock will be 25% if stock goes up, or 5% if stock goes down.
- The risk-free return will be $K_A = 10\%$.
- The stock prices are represented as a tree in Figure 1.1.

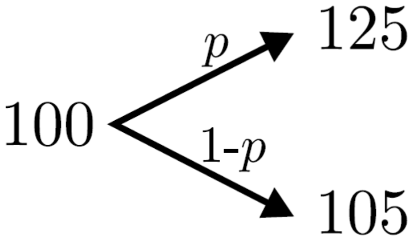

Figure: 1.1 One-step binomial tree of stock prices

- Note: both stock prices at time T happen to be higher than that at time 0.
  - Going 'up' or 'down' is relative to the other price at time T rather than to the price at time 0.
- In fact we could just as well allow the value in the 'up' scenario to be lower than the 'down' value.
  - e.g., if the asset is a currency (say ¥), some investors may regard a reduction in the exchange rate (say ¥/\$) as favourable.
    [An increase in an exchange rate means a decrease in the converse rate.]
- It is best to treat the references to 'up' and 'down' price movements as an abstract indication of two possible different outcomes.

In sum, 'up' and 'down' represent 'good' and 'bad' scenarios, respectively.

In general, the choice of stock and bond prices in a binomial model is constrained by the No-Arbitrage Principle.

- Suppose that the possible up and down stock prices at time T are

$$S(T) = \begin{cases} S^u(T) & \text{with probability } p \\ S^d(T) & \text{with probability } 1-p \end{cases}$$

where $S^d(T) < S^u(T)$ and $0 < p < 1$.

Proposition 1.7

Proposition 1.7

The restriction

$$\frac{S^d(T)}{S(0)} < \frac{A(T)}{A(0)} < \frac{S^u(T)}{S(0)}$$

has to be imposed on the model or else an arbitrage opportunity would arise.

Proof:

1)

- Suppose that $\frac{A(T)}{A(0)} \leq \frac{S^d(T)}{S(0)}$.
- In this case, at time 0:
  - borrow the amount $S(0)$ risk-free;
  - buy one share of stock for $S(0)$.
- This way, you will be holding a portfolio (x, y) with $x = 1$ shares of stock and $y = -\frac{S(0)}{A(0)}$ bonds.
- The time 0 value of this portfolio is

$$V(0) = 0.$$

- At time T the value will become

$$V(T) = \begin{cases} S^u(T) - \frac{S(0)}{A(0)}A(T) & \text{if stock goes up,} \\ S^d(T) - \frac{S(0)}{A(0)}A(T) & \text{if stock goes down.} \end{cases}$$

- The first of these two possible values is strictly positive, while the other one is non-negative, that is, $V(T)$ is a non-negative random variable such that $V(T) > 0$ with probability $p > 0$.
- The portfolio provides an arbitrage opportunity, violating the No-Arbitrage Principle (Assumption 1.6 (*or numbered as 1.5 in text.*)).

2)

- Now suppose that $\frac{A(T)}{A(0)} \geq \frac{S^u(T)}{S(0)}$, then at time 0:
  - sell short one share for $S(0)$.
  - invest the amount $S(0)$ risk-free.
- As a result, you will be holding a portfolio (x, y) with $x = -1$ and $y = \frac{S(0)}{A(0)}$, again of zero initial value,

$$V(0) = 0.$$

- The final value of this portfolio will be

$$V(T) = \begin{cases} -S^u(T) + \frac{S(0)}{A(0)}A(T) & \text{if stock goes up,} \\ -S^d(T) + \frac{S(0)}{A(0)}A(T) & \text{if stock goes down.} \end{cases}$$

which is non-negative, with the second value being strictly positive.

- Thus, $V(T)$ is a non-negative random variable such that $V(T) > 0$ with probability $1 - p > 0$.
- Once again, this indicates an arbitrage opportunity, contrary to the No-Arbitrage Principle.

[End of Proof]

- The common sense reasoning behind the above argument is straightforward:
- Buy cheap assets and sell (or sell short) expensive ones, pocketing the difference.

Risk and Return

- Let $A(0) = 100$ and $A(T) = 110$ dollars, as before, but $S(0) = 80$ dollars and

$$S(T) = \begin{cases} 100 & \text{with probability } 0.8, \\ 60 & \text{with probability } 0.2. \end{cases}$$

- Suppose that you have \$10,000 to invest in a portfolio.
- You decide to buy $x = 50$ shares, which fixes the risk-free investment at $y = 60$.
- Then

$$V(T) = \begin{cases} 11,600 & \text{if stock goes up,} \\ 9,600 & \text{if stock goes down,} \end{cases}$$

$$K_V = \begin{cases} 0.16 = 16\% & \text{if stock goes up,} \\ -0.04 = -4\% & \text{if stock goes down.} \end{cases}$$

- The *expected return*, that is, the mathematical expectation of the return on the portfolio is

$$E(K_V) = 16\% \times 0.8 - 4\% \times 0.2 = 12\%,$$

that is, 12%.

- The *risk* of this investment is defined to be the standard deviation of the random variable K_V:

$$\sigma_V = \sqrt{(16\% - 12\%)^2 \times 0.8 + (-4\% - 12\%)^2 \times 0.2} = 8\%,$$

that is 8%.

Compare this with investments in just one type of security.

1)

- If $x = 0$, then $y = 100$, that is, the whole amount is invested risk-free.
- In this case the return is known with certainty to be $K_A = 10\%$ and the risk as measured by the standard deviation is zero, $\sigma_A = 0$.

2)

- If $x = 125$ and $y = 0$, the entire amount being invested in stock, then

$$V(T) = \begin{cases} 12,500 & \text{if stock goes up,} \\ 7,500 & \text{if stock goes down,} \end{cases}$$

and $E(K_S) = 15\%$ with $\sigma_S = 20\%$.

- Given the choice between two portfolios with the same expected return, any investor would obviously prefer that involving lower risk.
- Similarly, if the risk levels were the same, any investor would opt for higher return.

However, in the case in hand higher return is associated with higher risk.

- In such circumstances the choice depends on individual preferences,
    - to be discussed in Chapter 3, where we shall also consider portfolios consisting of several risky securities, and
    - see the power of portfolio selection and portfolio diversification as tools for reducing risk while maintaining the expected return.

Exercise

Exercise 1.4