6
Binomial Model

Case 6

We are tempted to choose a pension scheme involving risky assets in addition to the risk-free investment. One of the possibilities available within the framework of Chapter 3 would be a portfolio with average annual return of 9% and standard deviation 8.5%. We should be concerned about the accuracy of these parameters. Even if these are correct, there is some danger related to the performance of our investment. Particular scenarios far from expected values may occur. We would like to analyse the behaviour of our portfolio with a time grid more refined than annual. As a first step we could construct a model of quarterly movements; then an extension to monthly steps should be straightforward. Having built such a model, we would like to consider methods of taming the risk.

6.1 Definition of the Model

In the previous chapters we introduced random variables $S(t)$, $0 < t \leq T$ representing future stock prices, with $S(0) > 0$ given. Here we restrict ourselves to evenly spaced discrete time instants $t = 0, h, 2h, \ldots, nh, \ldots$ with a horizon $T = Nh$, where $h > 0$ is the length of a single step. With a slight abuse of notation it is convenient to write $S(n)$ instead of $S(nh)$.

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Our first goal is to build a specific concrete model of random variables $S(n)$. This model is very simple on the one hand, but on the other hand it is quite flexible and very suitable for practical applications.

6.1.1 Single Step

We give a complete exposition of the model already discussed in Chapter 1. To define $S(1)$ we need a probability space. Since we allow $S(1)$ to take only two values, it is sufficient to consider $\Omega = \{\mathrm{u}, \mathrm{d}\}$ equipped with a field $\mathcal{F}$ of all subsets of Ω and a probability P determined by a single number p such that

$$P(\mathrm{u}) = p, \quad P(\mathrm{d}) = 1 - p,$$

(see Appendix 10.3). Of course, other choices of Ω are possible.

Now we let

$$S(1) : \Omega \to (0, +\infty)$$

be given by

$$S(1) = S(0)(1 + K),$$

where the rate of return is random and has the form

$$K^\omega = \begin{cases} U & \text{if } \omega = \mathrm{u}, \\ D & \text{if } \omega = \mathrm{d}, \end{cases}$$

where $-1 < D < U$. Then

$$S^\omega(1) = \begin{cases} S(0)(1 + U) & \text{if } \omega = \mathrm{u}, \\ S(0)(1 + D) & \text{if } \omega = \mathrm{d}. \end{cases}$$

6.1.2 Two Steps

We assume that the second step has the same form as the first one, the return in the second step being independent of the first return. This requires extending the probability space. To make room for two independent random variables, each with two values U, D, it is convenient to take $\Omega_1 = \Omega_2 = \{\mathrm{u}, \mathrm{d}\}$ and define $\Omega = \Omega_1 \times \Omega_2 = \{\mathrm{uu}, \mathrm{ud}, \mathrm{du}, \mathrm{ud}\}$ (where for better clarity we employ simplified notation writing uu rather than (u, u), for example) with the field $\mathcal{F}$ of all subsets of Ω and probability determined by

$$P(\mathrm{uu}) = p^2, \quad P(\mathrm{ud}) = P(\mathrm{du}) = p(1 - p), \quad P(\mathrm{dd}) = (1 - p)^2.$$

Then

$$K^\omega(1) = \begin{cases} U & \text{if } \omega = \mathrm{ud} \text{ or } \omega = \mathrm{uu}, \\ D & \text{if } \omega = \mathrm{du} \text{ or } \omega = \mathrm{dd}, \end{cases}$$

$$K^\omega(2) = \begin{cases} U & \text{if } \omega = \mathrm{uu} \text{ or } \omega = \mathrm{du}, \\ D & \text{if } \omega = \mathrm{ud} \text{ or } \omega = \mathrm{dd}, \end{cases}$$

are independent. We define

$$S(1) = S(0)(1 + K(1)),$$
$$S(2) = S(1)(1 + K(2)) = S(0)(1 + K(1))(1 + K(2)).$$

At time 2 we thus have three possible stock prices

$$S^\omega(2) = \begin{cases} S(0)(1 + U)^2 & \text{if } \omega = \mathrm{uu}, \\ S(0)(1 + U)(1 + D) & \text{if } \omega = \mathrm{ud} \text{ or } \omega = \mathrm{du}, \\ S(0)(1 + D)^2 & \text{if } \omega = \mathrm{dd}. \end{cases}$$

The ud and du scenarios give the same result.

6.1.3 General Case

For N steps we take $\Omega = \{\mathrm{u}, \mathrm{d}\}^N$ with the field $\mathcal{F}$ of all subsets of Ω and probability

$$P(\omega) = p^k (1 - p)^{N-k}$$

whenever there are k symbols u (and then $N - k$ symbols d) in the sequence $\omega = \omega_1 \cdots \omega_N$ with $\omega_1, \ldots, \omega_N \in \{\mathrm{u}, \mathrm{d}\}$. The return $K(n)$ is defined by means of the n-th coordinate of the sequence ω:

$$K^\omega(n) = \begin{cases} U & \text{if } \omega_n = \mathrm{u}, \\ D & \text{if } \omega_n = \mathrm{d}. \end{cases}$$

Then, inductively, extending the two-step scheme, we write

$$S(n + 1) = S(n)(1 + K(n + 1)).$$

For example,

$$S^\omega(3) = \begin{cases} S(0)(1 + U)^3 & \text{if } \omega = \mathrm{uuu}, \\ S(0)(1 + U)^2(1 + D) & \text{if } \omega = \mathrm{duu} \text{ or } \omega = \mathrm{udu} \text{ or } \omega = \mathrm{uud}, \\ S(0)(1 + U)(1 + D)^2 & \text{if } \omega = \mathrm{udd} \text{ or } \omega = \mathrm{dud} \text{ or } \omega = \mathrm{ddu}, \\ S(0)(1 + D)^3 & \text{if } \omega = \mathrm{ddd}. \end{cases}$$

There are fewer stock prices at time n than scenarios. The emerging tree representing the stock prices is called recombining and has the form (here we take $N = 3$)

$$
\begin{array}{ccccccc}
S(0) & & S(1) & & S(2) & & S(3) \\
\hline
 & & & & & & S(0)(1+U)^3 \\
 & & & & & \overset{p}{\nearrow} & \\
 & & & & S(0)(1+U)^2 & & \\
 & & & \overset{p}{\nearrow} & & \searrow_{1-p} & \\
 & & S(0)(1+U) & & & & S(0)(1+U)^2(1+D) \\
 & \overset{p}{\nearrow} & & \searrow_{1-p} & & \overset{p}{\nearrow} & \\
S(0) & & & & S(0)(1+U)(1+D) & & \\
 & \searrow_{1-p} & & \overset{p}{\nearrow} & & \searrow_{1-p} & \\
 & & S(0)(1+D) & & & & S(0)(1+U)(1+D)^2 \\
 & & & \searrow_{1-p} & & \overset{p}{\nearrow} & \\
 & & & & S(0)(1+D)^2 & & \\
 & & & & & \searrow_{1-p} & \\
 & & & & & & S(0)(1+D)^3
\end{array}
$$

6.1.4 Flow of Information

As time goes by the price movements gradually reveal information about the scenario ω realised in the financial world. We restrict our attention to the case $N = 3$, which is sufficient to explain the ideas. At the beginning each element of

$$\Omega = \{\text{uuu}, \text{uud}, \text{udu}, \text{udd}, \text{duu}, \text{dud}, \text{ddu}, \text{ddd}\},$$

can be realised as a future scenario. After one step we shall have more information. If the return on the stock turns out to be U, then the scope is restricted to

$$B_{\text{u}} = \{\text{uuu}, \text{uud}, \text{udu}, \text{udd}\}.$$

Otherwise, if the return is D, then the scenario must be among the elements of

$$B_{\text{d}} = \{\text{duu}, \text{dud}, \text{ddu}, \text{ddd}\}.$$

The knowledge of the first two returns allows us to tell which of the following four sets the scenario belongs to:

$$B_{\text{uu}} = \{\text{uuu}, \text{uud}\}, \quad B_{\text{ud}} = \{\text{udu}, \text{udd}\},$$
$$B_{\text{du}} = \{\text{duu}, \text{dud}\}, \quad B_{\text{dd}} = \{\text{ddu}, \text{ddd}\}.$$

After each piece of information is revealed, our uncertainty is reduced.

Mathematically, we have so-called partitions of Ω, meaning that $\Omega = B_{\text{u}} \cup B_{\text{d}}$, or $\Omega = B_{\text{uu}} \cup B_{\text{ud}} \cup B_{\text{du}} \cup B_{\text{dd}}$, where the components are pairwise disjoint. This motivates the following general definition.

Definition 6.1

A family $\mathcal{P} = \{B_i\}$ of events is a *partition* of Ω if $B_i \neq \emptyset$, $B_i \cap B_j = \emptyset$ for $i \neq j$ and $\Omega = \bigcup_i B_i$.

Notice that $B_{\text{u}} = B_{\text{uu}} \cup B_{\text{ud}}$, $B_{\text{d}} = B_{\text{du}} \cup B_{\text{dd}}$. We say that $\mathcal{P}_2 = \{B_{\text{uu}}, B_{\text{ud}}, B_{\text{du}}, B_{\text{dd}}\}$ is a finer partition than $\mathcal{P}_1 = \{B_{\text{u}}, B_{\text{d}}\}$.

Definition 6.2

Given two partitions $\mathcal{P}$, $\mathcal{P}'$, we say that $\mathcal{P}'$ is *finer* than $\mathcal{P}$ if each element of $\mathcal{P}$ can be represented as a union of some elements of $\mathcal{P}'$.

Note that the random variables $S(1)$, $S(2)$ are constant on the components of the corresponding partition $\mathcal{P}_1$, $\mathcal{P}_2$, respectively. For example, $S(1)$ is constant and equal to $S(0)(1+U)$ on B_{u}. To complete the picture, the same can be said about $S(0)$ being constant on the whole Ω, with $\mathcal{P}_0 = \{\Omega\}$, and $S(3)$ being constant on each of the (single-element) components of $\mathcal{P}_3 = \{B_{\text{uuu}}, \ldots, B_{\text{ddd}}\}$, where

$$B_{\text{uuu}} = \{\text{uuu}\}, \quad B_{\text{uud}} = \{\text{uud}\}, \quad B_{\text{udu}} = \{\text{udu}\}, \quad B_{\text{udd}} = \{\text{udd}\},$$
$$B_{\text{duu}} = \{\text{duu}\}, \quad B_{\text{dud}} = \{\text{dud}\}, \quad B_{\text{ddu}} = \{\text{ddu}\}, \quad B_{\text{ddd}} = \{\text{ddd}\}.$$

Definition 6.3

In general, for any integer $n \leq N$ and any $v_1, \ldots, v_n \in \{\text{u}, \text{d}\}$ we define $B_{v_1 \ldots v_n}$ to be the set consisting of all $\omega \in \Omega$ such that the first n elements in the sequence $\omega = \omega_1 \cdots \omega_N$ are $\omega_1 = v_1, \ldots, \omega_n = v_n$. The partition $\mathcal{P}_n$ is defined as the family of all such subsets $B_{v_1 \ldots v_n}$ of Ω.

6.1.5 Filtration

A common framework for analysing the flow of information is based on using fields of sets instead of partitions. Given a finite partition $\mathcal{P}$, we consider the family of sets $\mathcal{F}$ consisting of the empty set $\emptyset$ and all possible unions of components in $\mathcal{P}$. The resulting family $\mathcal{F}$ of sets is a field of subsets of Ω (see Appendix 10.3), referred to as the *field extending partition* $\mathcal{P}$.

Remark 6.4

If Ω is infinite, a further extension of the notion of a field is necessary (allowing countable unions), but here we have no need for this.

Definition 6.5

We denote by $\mathcal{F}_n$ the field extending the partition $\mathcal{P}_n$.

For example, for $N = 3$

$$\begin{aligned}
\mathcal{F}_0 &= \{\emptyset, \Omega\},\\
\mathcal{F}_1 &= \{\emptyset, \Omega, B_{\mathrm{u}}, B_{\mathrm{d}}\},\\
\mathcal{F}_2 &= \{\emptyset, \Omega, B_{\mathrm{uu}}, B_{\mathrm{ud}}, B_{\mathrm{du}}, B_{\mathrm{dd}}, B_{\mathrm{uu}} \cup B_{\mathrm{ud}}, B_{\mathrm{uu}} \cup B_{\mathrm{du}},\\
&\qquad B_{\mathrm{uu}} \cup B_{\mathrm{dd}}, B_{\mathrm{ud}} \cup B_{\mathrm{du}}, B_{\mathrm{ud}} \cup B_{\mathrm{dd}}, B_{\mathrm{du}} \cup B_{\mathrm{dd}},\\
&\qquad \Omega \setminus B_{\mathrm{uu}}, \Omega \setminus B_{\mathrm{ud}}, \Omega \setminus B_{\mathrm{du}}, \Omega \setminus B_{\mathrm{dd}}\} \quad \text{(16 subsets altogether)},\\
\mathcal{F}_3 &= 2^{\Omega} \quad \text{(all subsets of } \Omega).
\end{aligned}$$

We can see that

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3.$$

This is an example of a *filtration*, which in general is a sequence of fields such that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$.

Remark 6.6

The family $\mathcal{F}_n$ consists of all sets A such that for every scenario $\omega \in \Omega$ it is possible to tell whether or not $\omega \in A$ by knowing the stock prices up to and including time n only (that is, knowing only the first n elements in the sequence $\omega = \omega_1 \cdots \omega_N$).

6.2 Option Pricing

6.2.1 Investment Strategies

We assume that, in addition to trading in stock, which follows the binomial model, we can buy or sell units of so-called *money market account*, a risk-free security whose prices are determined by the interest rate R in the following (deterministic) way:

$$A(n) = A(0)(1+R)^n.$$

Suppose that $U = 10\%$, $D = -10\%$ and $R = 5\%$ with $S(0) = A(0) = 100$ dollars.

Suppose further that we have at our disposal a certain amount of money, say $V(0) = 300$ dollars, which will be the initial value of our investment. We can use this amount to prepare a portfolio consisting of a position $x(1)$ in stock and $y(1)$ in the money market account. We may decide to buy $x(1) = 2$ shares, and will be left with $V(0) - x(1)S(0)$ to invest risk free, so $y(1) = \frac{V(0)-x(1)S(0)}{A(0)} = 1$. The equality

$$V(0) = x(1)S(0) + y(1)A(0)$$

must hold. The value of the portfolio at time 1 will be

$$\begin{aligned}
V^{\mathrm{u}}(1) &= x(1)S^{\mathrm{u}}(1) + y(1)A(1) = 325 \quad \text{on } B_{\mathrm{u}},\\
V^{\mathrm{d}}(1) &= x(1)S^{\mathrm{d}}(1) + y(1)A(1) = 285 \quad \text{on } B_{\mathrm{d}}.
\end{aligned}$$

This is the amount at our disposal for the next step. We assume that there is no withdrawal or injection of external funds, and call such a strategy *self-financing*.

In state u at time 1 we may decide to reduce the position in stock to $x^{\mathrm{u}}(2) = 1$, hence

$$y^{\mathrm{u}}(2) = \frac{V^{\mathrm{u}}(1) - x^{\mathrm{u}}(2) \times S^{\mathrm{u}}(1)}{A(1)} = 2.0476.$$

At time 2 this portfolio will take one of two possible values

$$\begin{aligned}
V^{\mathrm{uu}}(2) &= x^{\mathrm{u}}(2)S^{\mathrm{uu}}(2) + y^{\mathrm{u}}(2)A(2) = 346.75 \quad \text{on } B_{\mathrm{uu}},\\
V^{\mathrm{ud}}(2) &= x^{\mathrm{u}}(2)S^{\mathrm{ud}}(2) + y^{\mathrm{u}}(2)A(2) = 324.75 \quad \text{on } B_{\mathrm{ud}}.
\end{aligned}$$

If we find ourselves in state d at time 1, we may choose $x^{\mathrm{d}}(2) = 3$, so

$$y^{\mathrm{d}}(2) = \frac{V^{\mathrm{d}}(1) - x^{\mathrm{d}}(2) \times S^{\mathrm{d}}(1)}{A(1)} = 0.1429.$$

At time 2 we shall have

$$\begin{aligned}
V^{\mathrm{du}}(2) &= x^{\mathrm{d}}(2)S^{\mathrm{du}}(2) + y^{\mathrm{d}}(2)A(2) = 312.75 \quad \text{on } B_{\mathrm{du}},\\
V^{\mathrm{dd}}(2) &= x^{\mathrm{d}}(2)S^{\mathrm{dd}}(2) + y^{\mathrm{d}}(2)A(2) = 258.75 \quad \text{on } B_{\mathrm{dd}}.
\end{aligned}$$

We can proceed further in this fashion.

In general, we assume that we build a *strategy*, that is, a sequence of portfolios $(x(n), y(n))$ according to the following principles:

- The strategy is *self-financing*, that is

$$x(n)S(n) + y(n)A(n) = x(n+1)S(n) + y(n+1)A(n),$$

which means that rebalancing the portfolio at time n concerns the choice of $x(n+1)$, and then the value of $y(n+1)$ follows.
- The decision about the position $x(n+1), y(n+1)$ to be held during step $n+1$ is taken on the basis of the information available at time n. Mathematically, this means that $x(n+1)$ and $y(n+1)$ are constant on each of the elements of the partition $\mathcal{P}_n$. We say that such sequences $x(n), y(n)$ are *predictable*.

The value of a strategy is given by

$$V(0) = x(1)S(0) + y(1)A(0),$$
$$V(n) = x(n)S(n) + y(n)A(n) \quad \text{for } n > 0.$$

6.2.2 Single Step

We begin with the case $N = 1$. We shall extend the procedure presented for calls and puts in Section 1.6 to European contingent claim with payoff of the general form $f(S(1))$ for some function f. The value of such an option at time $t = 0, 1$ will be denoted by $H(t)$. At the exercise time

$$H(1) = f(S(1)).$$

As in Section 1.6, the option can be priced by replicating the two possible values $H^{\mathrm{u}}(1), H^{\mathrm{d}}(1)$ of $H(1)$ and computing the time 0 value of the replicating portfolio. This requires solving the system of equations

$$\begin{cases} x(1)S^{\mathrm{u}}(1) + y(1)A(1) = H^{\mathrm{u}}(1), \\ x(1)S^{\mathrm{d}}(1) + y(1)A(1) = H^{\mathrm{d}}(1), \end{cases}$$

so

$$x(1) = \frac{H^{\mathrm{u}}(1) - H^{\mathrm{d}}(1)}{S^{\mathrm{u}}(1) - S^{\mathrm{d}}(1)} = \frac{H^{\mathrm{u}}(1) - H^{\mathrm{d}}(1)}{S(0)(U-D)}$$

which is the replicating position in stock, called the *delta* of the option, and

$$\begin{aligned} y(1) &= \frac{1}{A(1)}\left[H^{\mathrm{u}}(1) - x(1)S^{\mathrm{u}}(1)\right] \\ &= \frac{1}{A(1)}\frac{H^{\mathrm{d}}(1)S^{\mathrm{u}}(1) - H^{\mathrm{u}}(1)S^{\mathrm{d}}(1)}{S^{\mathrm{u}}(1) - S^{\mathrm{d}}(1)} \\ &= \frac{1}{A(0)(1+R)}\frac{H^{\mathrm{d}}(1)(1+U) - H^{\mathrm{u}}(1)(1+D)}{U-D}. \end{aligned}$$

As a consequence of the No-Arbitrage Principle, the price of the claim must be

$$H(0) = x(1)S(0) + y(1)A(0).$$

After substituting the above expressions for $x(1), y(1)$ and rearranging, we get

$$H(0) = \frac{1}{1+R}\left(H^{\mathrm{u}}(1)\frac{R-D}{U-D} + H^{\mathrm{d}}(1)\frac{U-R}{U-D}\right). \tag{6.1}$$

The coefficients $\frac{R-D}{U-D}$ and $\frac{U-R}{U-D}$ add up to one. Moreover, they are both greater than 0 and less than 1 if $D < R < U$, a condition for the lack of arbitrage in the binomial model; see Proposition 1.7. These coefficients can be regarded as probabilities and it is customary to use the notation

$$p_* = \frac{R-D}{U-D}.$$

Then

$$\begin{aligned} H(0) &= \frac{1}{1+R}(H^{\mathrm{u}}(1)p_* + H^{\mathrm{d}}(1)(1-p_*)) \\ &= \frac{1}{1+R}\mathbb{E}_*\left[H(1)\right] = \frac{1}{1+R}\mathbb{E}_*\left[f(S(1))\right], \end{aligned}$$

where the asterisk indicates that the probabilities $p_*, 1-p_* \in (0,1)$ are used to compute the expectation.

Proposition 6.7

$$\mathbb{E}_*\left[K(1)\right] = R.$$

Proof

$$\mathbb{E}_*\left[K(1)\right] = Up_* + D(1-p_*) = U\frac{R-D}{U-D} + D\frac{U-R}{U-D} = R.$$

□

Definition 6.8

A probability $(p_*, 1-p_*)$ such that $p_*, 1-p_* \in (0,1)$ under which the expected return $\mathbb{E}_*[K(1)]$ is equal to the risk-free rate R is called *risk-neutral*.

It is convenient to introduce the notation

$$\widetilde{S}(1) = \frac{S(1)}{1+R}, \quad \widetilde{V}(1) = \frac{V(1)}{1+R}, \quad \widetilde{H}(1) = \frac{H(1)}{1+R},$$

calling $\widetilde{S}(1), \widetilde{V}(1), \widetilde{H}(1)$ the *discounted values* of $S(1), V(1), H(1)$.

Proposition 6.9

Under the risk-neutral probability

$$\mathbb{E}_*[\widetilde{S}(1)] = S(0), \quad \mathbb{E}_*[\widetilde{V}(1)] = V(0), \quad \mathbb{E}_*[\widetilde{H}(1)] = H(0).$$

Proof

From Proposition 6.7 we have $\mathbb{E}_*[K(1)] = R$. It follows that

$$\begin{aligned}\mathbb{E}_*[\widetilde{S}(1)] &= \frac{1}{1+R}\mathbb{E}_*[S(1)] = \frac{S(0)}{1+R}\mathbb{E}_*[1+K(1)]\\ &= \frac{S(0)}{1+R}(1+\mathbb{E}_*[K(1)]) = S(0).\end{aligned}$$

Next, since $x(1)$ and $y(1)$ are deterministic (their values are decided at time 0),

$$\begin{aligned}\mathbb{E}_*[\widetilde{V}(1)] &= \mathbb{E}_*[x(1)\widetilde{S}(1) + y(1)A(0)]\\ &= x(1)\mathbb{E}_*[\widetilde{S}(1)] + y(1)A(0)\\ &= x(1)S(0) + y(1)A(0) = V(0).\end{aligned}$$

The prices of the claim H are just the values of the replicating portfolio so the result follows. □

Exercise 6.1

Show that the price of a call option grows with U, the other variables being kept constant. Analyse the impact of a change of D on the option price.

Exercise 6.2

Find a formula for the price $C_{\mathrm{E}}(0)$ of a European call option if $R = 0$ and $S(0) = X = 1$. Compute the price for $U = 0.05$ and $D = -0.05$, and also for $U = 0.01$ and $D = -0.19$. Draw a conclusion about the relationship between the variance of the return on stock and that on the option.

Exercise 6.3

Find the initial value of the portfolio replicating a call option if proportional transaction costs are incurred whenever the underlying stock is sold. (No transaction costs apply when the stock is bought.) Compare this value with the case free of such costs. Assume that $S(0) = X = 100$, $U = 0.1$, $D = -0.1$ and $R = 0.05$, admitting transaction costs at $c = 2\%$ (the seller receiving 98% of the stock value).

Exercise 6.4

Let $S(0) = 75$ and let $U = 0.2$ and $D = -0.1$. Suppose that you can borrow money at 12%, but the rate for deposits is lower at 8%. Find the values of the replicating portfolios for a put and a call. Is the answer consistent with the put and call prices following from Proposition 6.9?

6.2.3 General Setting

Example 6.10 (Exotic Option)

Consider an interesting example of an option with exercise time $N = 2$ and payoff

$$C(2) = \max\left(0, \frac{S(1)+S(2)}{2} - X\right),$$

a European call option with the average stock price as the underlying security. It is an example of a *path-dependent* or *exotic* option. For $U = 10\%$, $D = -10\%$, $S(0) = 100$, $X = 90$,

$$C(2) = \begin{cases} C^{\mathrm{uu}}(2) = 25.5,\\ C^{\mathrm{ud}}(2) = 14.5,\\ C^{\mathrm{du}}(2) = \ \ 4.5,\\ C^{\mathrm{dd}}(2) = \ \ 0.0.\end{cases}$$

With $R = 5\%$ we find $p_* = 0.75$ and then

$$\begin{aligned}C^{\mathrm{u}}(1) &= \frac{1}{1+R}(p_*C^{\mathrm{uu}}(2) + (1-p_*)C^{\mathrm{ud}}(2)) = 21.67,\\ C^{\mathrm{d}}(1) &= \frac{1}{1+R}(p_*C^{\mathrm{du}}(2) + (1-p_*)C^{\mathrm{dd}}(2)) = 3.21.\end{aligned}$$

This gives

$$C(0) = \frac{1}{1+R}(p_*C^{\mathrm{u}}(1) + (1-p_*)C^{\mathrm{d}}(1)) = 6.24.$$

The single-step method is applied three times here, starting from the terminal values and going backwards in time.

Inspired by the example, we give a general definition.

Definition 6.11

A *path-dependent* European option with exercise time N and underlying asset S is a contingent claim with payoff of the form $H(N) = f(S(1), \ldots, S(N))$ available to the holder only at time N, where f is a function of N variables. A *path-independent* European contingent claim has payoff of the form $H(N) = f(S(N))$ with function f of a single variable.

Particular cases are plain vanilla call and put options with exercise price X, given by $H(N) = f(S(N))$ where $f(x) = (x - X)^+$ for call and $f(x) = (X - x)^+$ for put.

European options can be priced in the binomial model by consecutive backward replication, or equivalently by computing the discounted expected payoff in each single-step subtree. The value obtained in each single-step phase must be the current option price, or otherwise we would see arbitrage opportunities. To formalise this statement we first discuss the No-Arbitrage Principle in the multi-step setting.

We extend the notion of a strategy $x(n), y(n)$ by allowing a third component $z(n)$, as indicated in Section 1.6, representing the position in the contingent claim we are going to price. The value of such a strategy is given by

$$\begin{aligned} V(0) &= x(1)S(0) + y(1)A(0) + z(1)H(0), \\ V(n) &= x(n)S(n) + y(n)A(n) + z(n)H(n) \quad \text{for } n > 0. \end{aligned}$$

The strategy is called *admissible* if $V(n) \geq 0$ for all n.

Definition 6.12

A strategy is an *arbitrage opportunity* if it is admissible, self-financing, predictable, $V(0) = 0$, and such that $V^\omega(n) > 0$ for some n for at least one scenario $\omega \in \Omega$.

We repeat the fundamental assumption:

Assumption 6.13 (No-Arbitrage Principle)

There are no arbitrage opportunities.

Theorem 6.14

The initial value $V(0)$ of an (x, y)-strategy replicating a contingent claim $H(N)$ must be equal to the market price $H(0)$ of that claim.

Proof

Suppose $V(0) > H(0)$. Then we build an (x, y, z)-strategy $(-x(n), -y(n) + \frac{V(0)-H(0)}{A(0)}, 1)$, which has zero initial value, and is worth $A(N)\frac{V(0)-H(0)}{A(0)} > 0$ at time N since $x(N)S(N) + y(N)A(N) = H(N)$. This is a contradiction with the No-Arbitrage Principle. The case $V(0) < H(0)$ can be dealt with similarly. □

6.2.4 Two Steps

We restrict our attention to European claims with path-independent payoff. For $N = 2$ we begin with

$$H(2) = f(S(2)),$$

which has three possible values. For each of the three subtrees in Figure 6.1 we use the one-step replication procedure as described above.

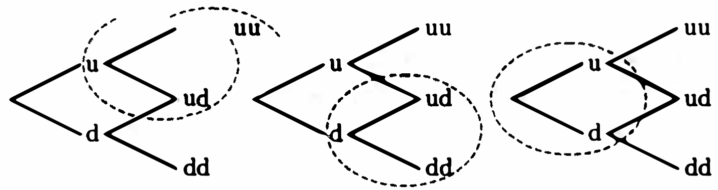

Figure 6.1 Subtrees in the two-step binomial tree

The claim price $H(1)$ has two values

$$H(1) = \begin{cases} \frac{1}{1+R}\left[p_* f(S^{uu}(2)) + (1-p_*) f(S^{ud}(2))\right] & \text{on } B_u, \\ \frac{1}{1+R}\left[p_* f(S^{du}(2)) + (1-p_*) f(S^{dd}(2))\right] & \text{on } B_d, \end{cases}$$

found by the one-step procedure applied to the two subtrees at nodes u and d. This gives

$$\begin{aligned} H(1) &= \frac{1}{1+R}\left[p_* f(S(1)(1+U)) + (1-p_*) f(S(1)(1+D))\right] \\ &= g(S(1)), \end{aligned}$$

where

$$g(x) = \frac{1}{1+R}\left[p_* f(x(1+U)) + (1-p_*) f(x(1+D))\right].$$

As a result, $H(1)$ can be regarded as a derivative security expiring at time 1 with payoff defined by the function g. (Though it cannot be exercised at time 1, the derivative security can be sold for $H(1) = g(S(1))$.) This means that the one-step procedure can be applied once again to the single subtree at the root of the tree. We have, therefore,

$$H(0) = \frac{1}{1+R}\left[p_* g(S(0)(1+U)) + (1-p_*)\, g(S(0)(1+D))\right].$$

It follows that

$$\begin{aligned} H(0) &= \frac{1}{1+R}\left[p_* g(S^{\mathrm{u}}(1)) + (1-p_*)\, g(S^{\mathrm{d}}(1))\right] \\ &= \frac{1}{(1+R)^2}\left[p_*^2 f(S^{\mathrm{uu}}(2)) + 2p_*(1-p_*)\, f(S^{\mathrm{ud}}(2)) + (1-p_*)^2 f(S^{\mathrm{dd}}(2))\right]. \end{aligned}$$

The last expression in square brackets is the expectation of $f(S(2))$ under the risk-neutral probability P_* defined as

$$P_*(\mathrm{uu}) = p_*^2, \quad P_*(\mathrm{ud}) = P_*(\mathrm{du}) = p_*(1-p_*), \quad P_*(\mathrm{dd}) = (1-p_*)^2.$$

This shows the validity of the following result.

Theorem 6.15

The expectation of the discounted payoff computed with respect to the risk-neutral probability is equal to the present value of the European claim

$$H(0) = \mathbb{E}\left((1+R)^{-2} f(S(2))\right).$$

Exercise 6.5

Let $S(0) = 120$ dollars and let $U = 0.2, D = -0.1, R = 0.1$. Consider a call option with strike price $X = 120$ dollars and exercise time $T = 2$. Find the option price and the replicating strategy.

Exercise 6.6

Using the data in the previous exercise, find the price of a call and the replicating strategy if a 15 dollar dividend is paid at time 1.

6.2.5 Several Steps

The extension of the results above to a multi-step model is straightforward. Beginning with the payoff at the final step, we proceed backwards, solving the one-step problem repeatedly. Here is the procedure for the three-step model:

$$\begin{aligned} H(3) &= f(S(3)), \\ H(2) &= \frac{1}{1+R}\left[p_* f(S(2)(1+U)) + (1-p_*) f(S(2)(1+D))\right] \\ &= g(S(2)), \\ H(1) &= \frac{1}{1+R}\left[p_* g(S(1)(1+U)) + (1-p_*) g(S(1)(1+D))\right] \\ &= h(S(1)), \\ H(0) &= \frac{1}{1+R}\left[p_* h(S(0)(1+U)) + (1-p_*) h(S(0)(1+D))\right], \end{aligned}$$

where

$$\begin{aligned} g(x) &= \frac{1}{1+R}\left[p_* f(x(1+U)) + (1-p_*) f(x(1+D))\right], \\ h(x) &= \frac{1}{1+R}\left[p_* g(x(1+U)) + (1-p_*) g(x(1+D))\right]. \end{aligned}$$

It follows that

$$\begin{aligned} H(0) &= \frac{1}{1+R}\left[p_* h(S^{\mathrm{u}}(1)) + (1-p_*) h(S^{\mathrm{d}}(1))\right] \\ &= \frac{1}{(1+R)^2}\left[p_*^2 g(S^{\mathrm{uu}}(2)) + 2p_*(1-p_*) g(S^{\mathrm{ud}}(2)) + (1-p_*)^2 g(S^{\mathrm{dd}}(2))\right] \\ &= \frac{1}{(1+R)^3}\left[p_*^3 f(S^{\mathrm{uuu}}(3)) + 3p_*^2(1-p_*) f(S^{\mathrm{uud}}(3))\right. \\ &\qquad \left. +3p_*(1-p_*)^2 f(S^{\mathrm{udd}}(3)) + (1-p_*)^3 f(S^{\mathrm{ddd}}(3))\right]. \end{aligned}$$

The emerging pattern is this: each term in the square bracket is characterised by the number k of upward stock price movements. This number determines the power of p_* and the choice of the payoff value. The power of $1-p_*$ is the number of downward price movements, equal to $3-k$ in the last expression, and $N-k$ in general, where N is the number of steps. The coefficients in front of each term give the number of scenarios (paths through the tree) that lead to the corresponding payoff, equal to $\binom{N}{k} = \frac{N!}{k!(N-k)!}$, the number of k-element combinations out of N elements. For example, there are three paths through the 3-step tree leading to the price $S^{\mathrm{udd}}(3)$, namely udd, dud, ddu.

As a result, in the N-step model

$$H(0) = \frac{1}{(1+R)^N}\sum_{k=0}^{N}\binom{N}{k} p_*^k (1-p_*)^{N-k} f\left(S(0)(1+U)^k(1+D)^{N-k}\right). \quad (6.2)$$

The expectation of $f(S(N))$ under the risk-neutral probability

$$P_*(\omega) = p_*^k(1-p_*)^{N-k},$$

where k is the number of upward stock price movements in the scenario $\omega \in \Omega$, can readily be recognised in this formula. The result can be summarised as follows.

Theorem 6.16

The value of a European path-independent contingent claim with payoff $f(S(N))$ in the N-step binomial model is the expectation of the discounted payoff under the risk-neutral probability:

$$H(0) = \mathbb{E}_*\Big((1+R)^{-N} f(S(N))\Big).$$

Remark 6.17

There is no need to know the actual probability p to compute $H(0)$. This remarkable property of the option price is important in practice, as the value of p may be difficult to estimate from market data. Instead, the formula for $H(0)$ features p_*, the risk-neutral probability, which may have nothing in common with p, but is easy to compute from the formula

$$p_* = \frac{R-D}{U-D}.$$

6.2.6 Cox–Ross–Rubinstein Formula

The payoff for a call option with strike price X satisfies $f(x) = 0$ for $x \leq X$, which reduces the number of terms in (6.2). The summation starts with the least m such that

$$S(0)(1+U)^m(1+D)^{N-m} > X.$$

Hence

$$\begin{aligned} C_{\mathrm{E}}(0) &= (1+R)^{-N}\sum_{k=m}^{N}\binom{N}{k}p_*^k(1-p_*)^{N-k}\Big(S(0)(1+U)^k(1+D)^{N-k} - X\Big)q^k \\ &= S(0)\sum_{k=m}^{N}\binom{N}{k}q^k(1-q)^{N-k} - (1+R)^{-N}X\sum_{k=m}^{N}\binom{N}{k}p_*^k(1-p_*)^{N-k}, \end{aligned}$$

where

$$q = p_*\frac{1+U}{1+R}, \quad 1-q = (1-p_*)\frac{1+D}{1+R}.$$

A similar formula can be derived for put options, either directly or using put-call parity.

These important results are summarised in the following theorem, in which $\Phi(m,N,p)$ denotes the cumulative binomial distribution with N trials and probability p of success in each trial,

$$\Phi(m,N,p) = \sum_{k=0}^{m}\binom{N}{k}p^k(1-p)^{N-k}.$$

Theorem 6.18 (Cox–Ross–Rubinstein Formula)

In the binomial model, the price of a European call and put option with strike price X to be exercised after N time steps is given by

$$\begin{aligned} C_{\mathrm{E}}(0) &= S(0)\left[1-\Phi(m-1,N,q)\right] - (1+R)^{-N}X\left[1-\Phi(m-1,N,p_*)\right], \\ P_{\mathrm{E}}(0) &= -S(0)\Phi(m-1,N,q) + (1+R)^{-N}X\Phi(m-1,N,p_*). \end{aligned}$$

Exercise 6.7

Let $S(0) = 50$, $R = 5\%$, $U = 30\%$ and $D = -10\%$. Find the price of a European call and put with strike price $X = 60$ to be exercised after $N = 3$ time steps.

Exercise 6.8

Let $S(0) = 50$, $R = 0.5\%$, $U = 1\%$ and $D = -1\%$. Find m and the price $C_{\mathrm{E}}(0)$ of a European call option with strike $X = 60$ to be exercised after $N = 50$ time steps.

6.3 American Claims

An American option can be exercised at any time n such that $0 \leq n \leq N$, with payoff $f(S(n))$. The function f is the same for each n. Of course, it can be exercised only once. The price of an American claim at time n will be denoted by $H_{\mathrm{A}}(n)$.

To begin with, we shall analyse an American claim expiring after 2 time steps. Unless it has already been exercised, at expiry it will be worth

$$H_{\mathrm{A}}(2) = f(S(2)),$$

where we have three possible values depending on the values of $S(2)$. At time 1 the holder will have the choice to exercise immediately, with payoff $f(S(1))$, or to wait until time 2, when the value of the claim will become $f(S(2))$. The value of an option which is not exercised at time 1 can be computed by treating $f(S(2))$ as a one-step European contingent claim to be priced at time 1, which gives

$$\frac{1}{1+R}\left[p_* f(S(1)(1+U)) + (1-p_*) f(S(1)(1+D))\right].$$

In effect, the holder has the choice between this or the immediate payoff $f(S(1))$. The value of the American claim at time 1 will therefore be the higher of the two,

$$\begin{aligned} H_A(1) &= \max\left\{ f(S(1)), \frac{1}{1+R}\left[p_* f(S(1)(1+U)) + (1-p_*) f(S(1)(1+D))\right]\right\} \\ &= f_1(S(1)) \end{aligned}$$

(a random variable with two values), where

$$f_1(x) = \max\left\{ f(x), \frac{1}{1+R}\left[p_* f(x(1+U)) + (1-p_*) f(x(1+D))\right]\right\}.$$

A similar argument gives the value at time 0,

$$H_A(0) = \max\left\{ f(S(0)), \frac{1}{1+R}\left[p_* f_1(S(0)(1+U)) + (1-p_*) f_1(S(0)(1+D))\right]\right\}.$$

Example 6.19

To illustrate the above procedure we consider an American put option with strike price $X = 80$ expiring at time 2 on a stock with initial price $S(0) = 80$ in a binomial model with $U = 0.1$, $D = -0.05$ and $R = 0.05$. (We consider a put, as we know that there is no difference between American and European call options, see Theorem 5.6.) The stock values are

n	0	1	2
			96.80
		88.00 <	
$S(n)$	80.00 <		83.60
		76.00 <	
			72.20

The price of the American put will be denoted by $P_A(n)$ for $n = 0, 1, 2$. At expiry the payoff will be positive only in the scenario with two downward stock

price movements,

n	0	1	2
			0.00
		? <	
$P_A(n)$	? <		0.00
		? <	
			7.80

At time 1 the option writer can choose between exercising the option immediately or waiting until time 2. In the up state at time 1 both the immediate payoff and the value of the option if not exercised are zero. In the down state the immediate payoff is 4, while the value of the option if not exercised is $1.05^{-1} \times \frac{1}{3} \times 7.8 \cong 2.48$. The option holder will choose the higher value (exercising the option in the down state at time 1). This gives the time 1 values of the American put,

n	0	1	2
			0.00
		0.00 <	
$P_A(n)$	? <		0.00
		4.00 <	
			7.80

At time 0 the choice is, once again, between the payoff, which is zero, or the value of the option if not exercised at that time, which is $1.05^{-1} \times \frac{1}{3} \times 4 \cong 1.27$ dollars. Taking the higher of the two completes the tree of option prices,

n	0	1	2
			0.00
		0.00 <	
$P_A(n)$	1.27 <		0.00
		4.00 <	
			7.80

For comparison, the price of a European put is $P_E(0) = 1.05^{-1} \times \frac{1}{3} \times 2.48 \cong 0.79$, clearly less than the American put price $P_A(0) \cong 1.27$.

This can be generalised, leading to the following definition.

Definition 6.20

An *American derivative security* or *contingent claim* with payoff function f expiring at time N is a sequence of random variables defined by backward induction:

$$
\begin{aligned}
H_A(N) &= f(S(N)),\\
H_A(N-1) &= \max\left\{f(S(N-1)), \frac{1}{1+R}[p_* f(S(N-1)(1+U))\right.\\
&\qquad \left. + (1-p_*)f(S(N-1)(1+D))]\right\} =: f_{N-1}(S(N-1)),\\
H_A(N-2) &= \max\left\{f(S(N-2)), \frac{1}{1+R}[p_* f_{N-1}(S(N-2)(1+U))\right.\\
&\qquad \left. + (1-p_*)f_{N-1}(S(N-2)(1+D))]\right\} =: f_{N-2}(S(N-2)),\\
&\vdots\\
H_A(1) &= \max\left\{f(S(2)), \frac{1}{1+R}[p_* f_2(S(1)(1+U))\right.\\
&\qquad \left. + (1-p_*)f_2(S(1)(1+D))]\right\} =: f_1(S(1)),\\
H_A(0) &= \max\left\{f(S(0)), \frac{1}{1+R}[p_* f_1(S(0)(1+U))\right.\\
&\qquad \left. + (1-p_*)f_1(S(0)(1+D))]\right\}.
\end{aligned}
$$

Exercise 6.9

Compute the value of an American put expiring at time 3 with strike price $X = 62$ on a stock with initial price $S(0) = 60$ in a binomial model with $U = 0.1$, $D = -0.05$ and $R = 0.03$.

Exercise 6.10

Compare the prices of an American call and a European call with strike price $X = 120$ expiring at time 2 on a stock with initial price $S(0) = 120$ in a binomial model with $U = 0.2$, $D = -0.1$ and $R = 0.1$.

Example 6.21

The last exercise can be modified to show that the equality of European and American call prices may not hold if a dividend is paid. Suppose that a dividend of 14 is paid at time 2. Otherwise, we shall use the same data as in Exercise 6.10.

The ex-dividend stock prices are

n	0		1		2
					158.80
			144.00	<	
$S(n)$ ex-div	120.00	<			115.60
			108.00	<	
					83.20

The corresponding European and American call values will be

n	0		1		2
					38.80 38.80
			23.52 **24.00**	<	
$C_E(n)$ $C_A(n)$	14.25 14.55	<			0.00 0.00
			0.00 0.00	<	
					0.00 0.00

The American call should be exercised early in the up state at time 1 with payoff 24 dollars (bold figures), which is more than the value of holding the option to expiry. As a result, the price of the American call is higher than that of the European call.

Exercise 6.11

Compute the prices of European and American puts with strike price $X = 14$ expiring at time 2 on a stock with $S(0) = 12$ in a binomial model with $U = 0.1$, $D = -0.05$ and $R = 0.02$, assuming that a dividend of 2 is paid at time 1.

6.4 Martingale Property

Looking into the future, we can see plenty of uncertainty and would like to have some guidance for our investment decisions. One possibility is to estimate the expected future price of the risky security. To this end we need a model: here we have at our disposal the binomial one, so we continue with the assumption that the prices follow this model.

Example 6.22

Our views will evolve with time as we gather some information and our perspective changes. To focus our attention consider a binomial model with $S(0) = 100$, $U = 10\%$, $D = -10\%$, $R = 5\%$ and $p = 0.8$.

Time step 0. Let us look just one step ahead into the future and compute the expected stock price at time 1, discounted back to time 0:

$$\mathbb{E}(\tilde{S}(1)) = \frac{pS^{\mathrm{u}}(1) + (1-p)S^{\mathrm{d}}(1)}{1+R} = 100.95.$$

We can see that the average growth in stock is above the risk-free rate.

Time step 1. We keep our time 0 perspective, working with discounted values and thinking in terms of 'what if' questions, covering all possible future developments. Two cases need to be analysed:

- The stock goes up to $S^{\mathrm{u}}(1) = 110$, so the scenarios still remaining are in B_{u}. Working within this set, we evaluate the time 2 expected discounted price

$$\frac{pS^{\mathrm{uu}}(2) + (1-p)S^{\mathrm{ud}}(2)}{(1+R)^2} = 105.76.$$

- The stock goes down to $S^{\mathrm{d}}(1) = 90$, the future lies in the set B_{d}, and the expected discounted price is

$$\frac{pS^{\mathrm{du}}(2) + (1-p)S^{\mathrm{dd}}(2)}{(1+R)^2} = 86.53.$$

We have obtained a random variable on Ω, constant on each element of the partition $\mathcal{P}_1$, and denoted by

$$\mathbb{E}(\tilde{S}(2)|\mathcal{P}_1) = \begin{cases} 105.75 & \text{on } B_{\mathrm{u}}, \\ 85.53 & \text{on } B_{\mathrm{d}}. \end{cases}$$

We call it the *conditional expectation* of $\tilde{S}(2)$ given $\mathcal{P}_1$.

Time step 2. Moving forward one more step in time, we encounter four possibilities:

$$\begin{aligned}
\frac{pS^{\mathrm{uuu}}(3) + (1-p)S^{\mathrm{uud}}(3)}{(1+R)^3} &= 110.80 \quad \text{on } B_{\mathrm{uu}}, \\
\frac{pS^{\mathrm{udu}}(3) + (1-p)S^{\mathrm{udd}}(3)}{(1+R)^3} &= 90.65 \quad \text{on } B_{\mathrm{ud}}, \\
\frac{pS^{\mathrm{duu}}(3) + (1-p)S^{\mathrm{dud}}(3)}{(1+R)^3} &= 90.65 \quad \text{on } B_{\mathrm{du}}, \\
\frac{pS^{\mathrm{ddu}}(3) + (1-p)S^{\mathrm{ddd}}(3)}{(1+R)^3} &= 74.17 \quad \text{on } B_{\mathrm{dd}}.
\end{aligned}$$

In fact there are just three distinct values to consider given that $S^{\mathrm{udu}}(3) = S^{\mathrm{duu}}(3)$ and $S^{\mathrm{udd}}(3) = S^{\mathrm{dud}}(3)$, so that the expected values on B_{ud} and B_{du} coincide. We have a random variable which is constant on each element of the partition $\mathcal{P}_2$, called the conditional expectation of $\tilde{S}(3)$ given $\mathcal{P}_2$:

$$\mathbb{E}(\tilde{S}(3)|\mathcal{P}_2) = \begin{cases} 110.80 & \text{on } B_{\mathrm{uu}}, \\ 90.65 & \text{on } B_{\mathrm{ud}} \cup B_{\mathrm{du}}, \\ 74.17 & \text{on } B_{\mathrm{dd}}. \end{cases}$$

Definition 6.23

If X is a random variable on Ω and $\mathcal{P}$ is a partition, then the *conditional expectation* $\mathbb{E}(X|\mathcal{P})$ is defined to be a random variable that is constant on each set $A \in \mathcal{P}$ with value $\mathbb{E}(X|A)$, the conditional expectation of X given event A (see Appendix 10.3).

Remark 6.24

Conditional expectation is often expressed in terms of a field rather than a partition. Namely, if $\mathcal{F}$ is a field extending a partition, then the conditional expectation of X given $\mathcal{F}$ is defined by

$$\mathbb{E}(X|\mathcal{F}) = \mathbb{E}(X|\mathcal{P}).$$

Exercise 6.12

Show that $\mathbb{E}_*(\tilde{S}(2)|\mathcal{P}_1) = \tilde{S}(1)$ and $\mathbb{E}_*(\tilde{S}(3)|\mathcal{P}_2) = \tilde{S}(2)$, where the expectation is computed under the risk-neutral probability P_*.

This is not a coincidence as we shall demonstrate next.

Theorem 6.25

For each $n = 0, \ldots, N-1$

$$\mathbb{E}_*(\tilde{S}(n+1)|\mathcal{P}_n) = \tilde{S}(n).$$

Proof

Since $S(n)$ is constant on any $A \in \mathcal{P}_n$ and $K(n+1)$ is independent of A,

$$\begin{aligned}\mathbb{E}_*(\tilde{S}(n+1)|A) &= \frac{1}{(1+R)^{n+1}}\mathbb{E}_*(S(n+1)|A) \\ &= \frac{1}{(1+R)^{n+1}}S(n)\mathbb{E}_*(1+K(n+1)) \\ &= \frac{1}{(1+R)^{n+1}}S(n)\,(1+p_*U+(1-p_*)D) \\ &= \frac{1}{(1+R)^{n+1}}S(n)(1+R) \\ &= \tilde{S}(n).\end{aligned}$$

It follows that

$$\mathbb{E}_*(\tilde{S}(n+1)|\mathcal{P}_n) = \tilde{S}(n).$$

□

Remark 6.26

A process $X(n)$ such that $\mathbb{E}(X(n+1)|\mathcal{P}_n) = X(n)$ for each n is called a *martingale.* According to Theorem 6.25 the discounted price process $\tilde{S}(n)$ is a martingale under the probability P_*. Because of this, P_* is often referred to as *martingale probability.*

Theorem 6.27

The discounted value $\tilde{V}(n)$ of a self-financing strategy $x(n), y(n)$ is a martingale under P_*.

Proof

The random variables $x(n+1), y(n+1)$ are constant on any $B \in \mathcal{P}_n$, so

$$\begin{aligned}\mathbb{E}_*(\tilde{V}(n+1)|B) &= \frac{1}{(1+R)^{n+1}}\mathbb{E}_*(x(n+1)S(n+1)+y(n+1)A(n+1)|B) \\ &= x(n+1)\mathbb{E}_*(\tilde{S}(n+1)|B)+y(n+1) \\ &= x(n+1)\tilde{S}(n)+y(n+1) \\ &= \frac{1}{(1+R)^{n}}(x(n+1)S(n)+y(n+1)A(n)) \\ &= \tilde{V}(n),\end{aligned}$$

implying that

$$\mathbb{E}_*(\tilde{V}(n+1)|\mathcal{P}_n) = \tilde{V}(n),$$

as required. □

Corollary 6.28

Since the value process $H(n)$ of a European contingent claim is equal to the value process $V(n)$ of the replicating strategy, it follows that discounted claim prices $\tilde{H}(n)$ also form a martingale under P_*.

Remark 6.29

Applying the general property

$$\mathbb{E}(\mathbb{E}(Y|\mathcal{P})) = \mathbb{E}(Y),$$

valid for any random variable Y and partition $\mathcal{P}$, we can derive the following formula

$$H(0) = \mathbb{E}_*((1+R)^{-N}H(N)),$$

so we have an alternative proof of Theorem 6.16.

6.5 Hedging

6.5.1 European Options

Let us put ourselves in the position of an option writer who has sold a European call with strike price $X = 105$ and exercise time $N = 3$ for the no-arbitrage price resulting from the binomial model with $S(0) = 100$ dollars and $U = 20\%$, $D = -10\%, R = 10\%$. The option price can be computed using the CRR formula, which gives $C_{\mathrm{E}}(0) = 23.3075$ dollars.

The writer of the option is exposed to risk. If the stock keeps going up during each of the three steps, reaching 172.80 dollars at time 3, it will be necessary to pay 67.80 dollars to the option holder. Of course, if the stock goes down, the writer may be able to keep the premium charged for the option. In order to make their position safe the writer can hedge the option by means of a replicating strategy.

It is possible, but not necessary, to construct the whole tree of replicating portfolios by working backwards from the payoff. It would not be hard to do it in the case in hand, but for large trees, with many time instants, it would be wasteful of computing resources. Instead, we can work a step at a time starting at time 0 and following the actual stock price changes without the need to analyse all scenarios.

The first portfolio to be constructed at time 0 requires finding the call prices at time 1, for which we again use the CRR formula,

$$C_{\mathrm{E}}(1) = \begin{cases} 33.9394 & \text{on } B_{\mathrm{u}}, \\ 9.0358 & \text{on } B_{\mathrm{d}}, \end{cases}$$

and then computing the stock position

$$x(1) = \frac{C_{\mathrm{E}}^{\mathrm{u}}(1) - C_{\mathrm{E}}^{\mathrm{d}}(1)}{S^{\mathrm{u}}(1) - S^{\mathrm{d}}(1)} = 0.8301$$

and the money market position

$$y(1) = C_{\mathrm{E}}(0) - x(1)S(0) = -59.7045.$$

In practice, the writer would charge a bit more for the option than $C_{\mathrm{E}}(0) = 23.31$ dollars to be compensated for the service.

Next, we wait till time 1 and check the stock price. Suppose that it has gone up to $S^{\mathrm{u}}(1)$. The stock position will need to be rebalanced to become

$$x^{\mathrm{u}}(2) = \frac{C_{\mathrm{E}}^{\mathrm{uu}}(2) - C_{\mathrm{E}}^{\mathrm{ud}}(2)}{S^{\mathrm{uu}}(2) - S^{\mathrm{ud}}(2)} = 0.9343,$$

so we have to purchase more stock. The money market position will become

$$y^{\mathrm{u}}(2) = y(1) + \frac{1}{1+R}(x(1) - x^{\mathrm{u}}(2))S^{\mathrm{u}}(1) = -71.0744.$$

We then wait till time 2 and check the stock price again. Suppose it has gone down to $S^{\mathrm{ud}}(2)$, so we rebalance the portfolio to

$$x^{\mathrm{ud}}(3) = \frac{C_{\mathrm{E}}^{\mathrm{udu}}(3) - C_{\mathrm{E}}^{\mathrm{udd}}(3)}{S^{\mathrm{udu}}(3) - S^{\mathrm{udd}}(3)} = 0.7593,$$

$$y^{\mathrm{ud}}(3) = y^{\mathrm{u}}(2) + \frac{1}{(1+R)^2}(x^{\mathrm{u}}(2) - x^{\mathrm{ud}}(3))S^{\mathrm{ud}}(2) = -55.4470.$$

At the next time instant the option will be exercised. Suppose the stock has gone up again and reached $S^{\mathrm{udu}}(3) = 129.60$. We are ready to close all positions:

- pay the option holder $-(S^{\mathrm{udu}}(3) - X) = 24.60$ dollars;
- pay $-y^{\mathrm{ud}}(3)(1+R)^3 = 73.80$ dollars to close the money market position (with interest);
- sell $y^{\mathrm{ud}}(3) = 0.7593$ of a share worth $S^{\mathrm{udu}}(3) = 129.60$ dollars each, and receive $0.7593 \times 129.60 = 98.40$ dollars.

The balance of these three transactions is zero.

We have analysed the udu scenario in some detail, but it is not hard to convince oneself that a similar pattern can be followed no matter which scenario unfolds. As a writer of the option, we have been able to hedge all risk.

Exercise 6.13

Compute the hedging strategy along scenario duu for the same European call as above.

6.5.2 American Options

We are using the same data as in Section 6.5.1 and consider an American put with strike price $X = 100$ and expiry time $N = 2$. Building hedging positions is more difficult than in the European case since we cannot use the CRR formula and we have to compute the whole tree of option prices (on the left we give the stock prices so that we can see the cases where exercising the option prevails):

stock				option			
$S(0)$	$S(1)$	$S(2)$	$S(3)$	$P_{\mathrm{A}}(0)$	$P_{\mathrm{A}}(1)$	$P_{\mathrm{A}}(2)$	$P_{\mathrm{A}}(3)$
			172.8				0.0000
		144				0.0000	
	120		129.6		0.2571		0.0000
100		108		3.1861		0.8485	
	90		97.2		10.0000		2.8000
		81				19.0000	
			72.9				27.1000

Having written and sold one option for 3.1861 dollars, we compute the stock position

$$x(1) = \frac{P_{\mathrm{A}}^{\mathrm{u}}(1) - P_{\mathrm{A}}^{\mathrm{d}}(1)}{S^{\mathrm{u}}(1) - S^{\mathrm{d}}(1)} = -0.3248$$

in the hedging portfolio, which indicates the number of shares to be sold short. The money market position includes the premium received for the option and the short-selling proceeds,

$$y(1) = P_{\mathrm{A}}(0) - x(1)S(0) = 35.6624.$$

We wait till time 1 holding this portfolio. Suppose that the stock has gone down to $S^{\mathrm{d}}(1)$. The holder of the option has the choice to exercise it or not. If the option is exercised, the writer must pay $X - S^{\mathrm{d}}(1) = 10$ dollars. Closing the positions in the hedging portfolio will provide $x(1)S^{\mathrm{d}}(1) + y(1)(1+R) = 10$

dollars, so the writer will break even. If the holder decides not to exercise at this point, then the writer will need to rebalance the portfolio to

$$x^{d}(2) = \frac{P_A^{du}(2) - P_A^{dd}(2)}{S^{du}(2) - S^{dd}(2)} = -0.6723,$$
$$y^{d}(2) = y(1) + \frac{1}{1+R}(x(1) - x^{d}(2))S^{d}(1) = 64.0955,$$

which will be held till time 2. Suppose that the stock has gone down again and become $S^{dd}(2)$ at time 2. If the holder exercises the option, the writer can liquidate the portfolio, receiving $x^{d}(2)S^{dd}(2) + y^{d}(2)(1+R)^2 = 23.1010$ dollars, more than enough to cover the payoff of 19 dollars. If the option is not exercised, then the writer can rebalance the portfolio again as follows:

$$x^{dd}(3) = \frac{P_A^{ddu}(3) - P_A^{ddd}(3)}{S^{ddu}(3) - S^{ddd}(3)} = -1.0000,$$
$$y^{dd}(3) = y^{d}(2) + \frac{1}{(1+R)^2}\left(x^{d}(2) - x^{dd}(3)\right)S^{dd}(2) = 86.0339.$$

When time 3 is reached, suppose that the stock price turns out to be $S^{ddd}(3)$. At this time the holder will exercise the option and the writer will have $x^{dd}(3)S^{ddd}(3) + y^{dd}(3)(1+R)^3 = 41.6111$ dollars from liquidating the hedging portfolio, once again more than enough to cover the payoff of 27.10 dollars.

Observe that if the option holder does not exercise the option when the stock price turns out to be $S^{d}(1)$ at time 1, then this hedging strategy provides more cash than necessary for the writer to cover the payoff when the option is exercised at a later time. The holder would do much better exercising when the stock is $S^{d}(1)$ at time 1, receiving the payoff of 10 dollars. If the holder then still wants to have a put option that can be exercised at time 2 or 3, it would be possible to purchase one for just

$$\frac{1}{1+R}\left(p_* P_A^{du}(2) + (1-p_*)P_A^{dd}(2)\right) = 6.2718,$$

keeping the difference of 3.7282 dollars.

Though we have analysed only the ddd scenario, a similar pattern can be followed to construct a hedging portfolio along any other scenario.

Exercise 6.14

Compute the hedging strategy along scenario udd for the above American put. Should the holder exercise the option early along this scenario or keep it till expiry?

Case 6: Discussion

First we build a binomial tree of quarterly portfolio movements consistent with the expected annual return of 9% and standard deviation 8.5%. Taking three months (quarter of a year) as the time step, we need to find U, D such that $S(4) = S(0)(1+K(1))(1+K(2))(1+K(3))(1+K(4))$, where the $K(n)$ are independent $\{U, D\}$-valued random variables. To this end we formulate the conditions

$$\mathbb{E}(S(4)) = S(0)(1+9\%),$$
$$\mathrm{Var}\left(\frac{S(4)-S(0)}{S(0)}\right) = (8.5\%)^2,$$

which represent a system of equations in three variables p, U, D. There is too much flexibility, and the simplest solution is to postulate $p = 0.5$. Then we get $U = 6.16\%$, $D = -1.80\%$ (all values here and in what follows are approximate). The quarterly risk-free return $R = 1.23\%$ can be computed from the annual interest rate of 5%. This gives the risk-neutral probability $p_* = 0.76$. We feed the binomial tree with $S(0) = 100$ dollars and have

time	0	1	2	3	4
unit price					\$127.00
				\$119.63	
			\$112.69		\$117.48
		\$106.16		\$110.66	
	\$100		\$104.24		\$108.67
		\$98.20		\$102.37	
			\$96.43		\$100.52
				\$94.69	
					\$92.99

Consider the first phase of building up our portfolio (the first 40 years): Suppose that at the end of the first quarter we will be investing 1,000 dollars (this sum of money will vary and we just consider an exemplary amount). We are going to analyse the various possibilities from the perspective of the beginning of the quarter, comparing the number of units we would purchase at the spot price, the forward price, and the prices resulting from using call options.

The forward price is $F(0,1) = 101.20$ dollars, so we can take the long forward position for $1,000/101.2 = 9.88$ units. The number of units we shall own at the end of the quarter is known in this case, their value being random, which will have an impact on the subsequent steps.

If we consider a call with strike price equal to the average price after the first quarter, equal to 102.18 dollars, the cost of buying a single option will be $C = 2.99$ dollars at the beginning of the period and 3.03 dollars at the end. We have to buy the option at the beginning of the quarter, but the investment will be performed at the end, so we have to borrow the money. We assume that we can borrow at the risk-free rate, an acceptable assumption since we are investing some of our money risk free anyway and would be 'borrowing' from that part of the fund.

We need to decide how many calls to buy (additionally, one could investigate other strike prices). Consider, for example, 5 options. Then in the 'up' case we would buy 5 units at the strike price and spend the remainder of the 1,000 dollars to buy risky asset units at the spot price. In the 'down' case all units will be purchased at the spot price. For instance, in the 'up' state our 1000 dollars will be converted into the following number of units:

$$5 + (1,000 - 15.14 - 5 \times 102.18)/106.2 = 9.46.$$

A summary of the results is given below:

	unhedged		forward		with 5 calls	
	units	worth	units	worth	units	worth
up	9.42	\$1,000	9.88	\$1,048.70	9.46	\$1,004.75
down	10.18	\$1,000	9.88	\$970.08	10.03	\$984.86

The strategy involving calls does not appear attractive.

Next, we discuss the use of derivatives to reduce the risk involved in the second phase of our plan, the final 20 years, when we will be gradually liquidating our position. We begin the second phase with a capital of 1,182,000 dollars accumulated over 40 years (see Case 2). Suppose we pay ourselves equal amounts at the end of each quarter (for simplicity, the growth assumption will be applied on an annual basis), so during the first year these payments are at 27,500 dollars per quarter, which gives an annual amount of 110,000 dollars (approximately 50% of the annual salary at the end of year 40).

We restrict out attention to a single year, so we consider a 4-step binomial tree. Suppose the initial unit price is again 100 dollars and the tree parameters are as above. Hypothetically, assume that all the money is invested in risky assets, so we have 11,820 units.

Consider a strategy where at each node of the tree we buy a certain number of puts with exercise times at the end of the subsequent quarter, with strike prices equal to the expected unit price at the end of the quarter, that is, 102.18, 104.40, 106.68, 109.00 dollars, respectively. The payoffs of these single-step puts are

time	1	2	3	4
put payoff				\$0
			\$0	
		\$0		\$0
	\$0		\$0	
		\$0.1584		\$0.3304
	\$3.9795		\$4.3102	
		\$7.9739		\$8.4777
			\$11.9849	
				\$16.0142

The corresponding put prices are

time	0	1	2	3
put price				\$0
			\$0	
		\$0.0374		\$0.0780
	\$0.9391		\$1.0171	
		\$2.0010		\$2.2490
			\$6.0690	
				\$10.1534

Along each scenario we compute the number of units left after some are sold to make the pension payments and put purchases, and some are bought for the payoffs of the options exercised. For instance, at node d at the end of the first quarter the number of units will be

$$11,820 - \frac{27,500}{98.20} - \frac{5,000 \times 0.9391}{100} + \frac{5,000 \times 3.9795}{98.20} = 11,696.$$

We continue in this fashion. The setting is path dependent: there will be 16 values after 4 steps. Multiplying the number of units by the final asset price, we find the terminal wealth. It turns out optimal to use 8,855 put options if the worst scenario value is to be maximised.

no of puts	0	8,855
worst value	\$992,029.50	\$1,158,097.77
best value	\$1,380,541.07	\$1,369,584.85
average	\$1,174,733.86	\$1,212,799.96

An alternative would be to buy an American option expiring at the end of the fourth quarter. Here we should discuss various possibilities concerned with the strike price and the number of options. If the strike price is greater than 100.56 dollars, it is optimal to exercise the option at time 0, so this case is not interesting. If the strike is 100 dollars, the option will cost 0.42515 dollars, and will be exercised in the d scenario at time 1. The optimal number of American puts that maximises the worst case result turns out to be 51.253, in which case this result is 1,059,206.41 dollars. It is better to buy puts with a lower strike price. For example, if the strike is 95 dollars, with 41,016 puts the corresponding result becomes 1,074,404.40 dollars (here the options are exercised at the end of step 4 in the dddd scenario), which is approximately the best we can get.