Financial Econometrics - Part V

Factor Models in Finance

Li Chen

lichen812@xmu.edu.cn

WISE, Xiamen University

October 29, 2022

Factor Model

Suppose that there are k assets and T time periods. Let r_{it} be the return of asset i in the time period t. A general form for the factor model is

$$r_{it} = \alpha_i + \beta_{i1} f_{1t} + ... + \beta_{im} f_{mt} + \epsilon_{it} = \alpha_i + b_i' f_t + \epsilon_{it}, \qquad (1)$$

for $t = 1, 2, ..., T$, $i = 1, 2, ..., k$, where α_i is the intercept term, $f_t = (f_{1t}, ..., f_{mt})'$ are m common factors, $b_i = (\beta_{i1}, ..., \beta_{im})'$ is called the vector of factor loadings for asset i, ϵ_{it} is the specific factor of asset i.

Factor Model

The factor vector $f_t = (f_{1t}, ..., f_{mt})'$ is assumed to be an m-dimensional stationary process such that

$$E(f_t) = \mu_f, \quad (2)$$

$$Cov(f_t) = \Sigma_f. \quad (3)$$

Factor Model

At time t, we can write the model as

$$r_t = \alpha + Bf_t + \epsilon_t, \tag{4}$$

where $r_t = (r_{1t}, ..., r_{kt})'$ is a $k \times 1$ vector, $\alpha = (\alpha_1, ..., \alpha_k)'$, B is a $k \times m$ factor loading matrix, and $\epsilon_t = (\epsilon_{1t}, ..., \epsilon_{kt})'$ for which $Cov(\epsilon_t) = D = diag(\sigma_1^2, ..., \sigma_k^2)$. Then, we have the covariance matrix

$$Cov(r_t) = B\Sigma_f B' + D. \tag{5}$$

Factor Model

For fixed i, we can write the model as

$$R_i = \alpha_i 1_T + F b_i' + E_i, \tag{6}$$

where $R_t = (r_{11}, ..., r_{1T})'$ is a $T \times 1$ vector, $1_T = (1, ..., 1)'$ is a T-dimensional vector of ones, F is a $T \times m$ matrix whose tth row is f_t', and $E_i = (\epsilon_{i1}, ..., \epsilon_{iT})'$ for which $Cov(E_i) = \sigma^2 I_T$.

Capital Asset Pricing Model

Capital Asset Pricing Model (CAPM)

$$r_{it} = \alpha_i + \beta_i r_{mt} + \epsilon_{it}, \tag{7}$$

for $i = 1, 2, ..., m$, $t = 1, 2, ..., T$. In the model, r_{it} is the excess return of asset i, r_{mt} is the excess return of the market, and β_i is the well-known β (Beta) for the asset.

Question: How to interpret α_i and β_i in the CAPM?

Capital Asset Pricing Model

- $\beta > 1$: Aggressive stock: Exhibits greater volatility than the market portfolio.
- $\beta = 1$: Tracking stock: Tracks the market portfolio exactly.
- $0 < \beta < 1$: Conservative stock: Exhibits less volatility than the market portfolio.
- $\beta = 0$: Independence: Independent of the market.
- $-1 < \beta < 0$: Imperfect hedge: Moves in the opposite direction to the market portfolio.
- $\beta = -1$: Perfect hedge: Moves in the exact opposite direction to the market portfolio.

Fama-French three Factor Model

In the Fama-French (1992, 1993) method, combinations of portfolios are constructed to take account of the observed fact that

- Small stocks have higher average returns than large stocks;
- Value stocks have higher average returns than growth stocks;

Therefore, they established the Fama-French three Factor Model (FF3FM) as

$$r_{it} = \alpha_i + \beta_i r_{mt} + \gamma_i SMB_t + \theta_i HML_t + \epsilon_{it}, \tag{8}$$

for $i = 1, 2, ..., m,\ t = 1, 2, ..., T$.

Fama-French three Factor Model

In the model,

- r_{it} is the excess return of asset i;
- r_{mt} is the excess return of the market;
- SMB_t is the performance of small stocks relative to large stocks
  - Small firms are more susceptible to changes in economic conditions;
  - SMB is the return on a portfolio long in small stocks and short in large stocks.
- HML_t is the performance of value stocks to growth stocks
  - High book value relative to market value increases the likelihood of financial distress;
  - HML is the return on a portfolio long in value stocks and short in growth stocks.

Question: How to interpret γ_i and θ_i in the FF3FM?

Principal Component Analysis

Suppose that $r = (r_1, ..., r_k)'$ is a k-dimensional vector of random variables with covariance matrix Σ_r. A principal component analysis (PCA) is concerned with using a few linear combinations of r_i to explain the structure of Σ_r.
e.g. if r denotes the monthly return of k assects, then PCA can be used to study the main source of variations of these k asset returns. PCA is one of the methods of dimension-reduction.

Principal Component Analysis

PCA steps:

1. Find the variance-covariance matrix of r_t, denoted as Σ_r;
2. Compute the eigenvalues and eigenvectors of Σ_r;
3. Find the first L largest eigenvalues $\lambda_1, ..., \lambda_L$ and their corresponding eigenvectors $w = (w_1, ..., w_L)$;
4. Finally, $f_t = w' r_t$.