# Linear Programming I

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# Outline

- Definition of linear programming and examples
- A linear program to solve max flow and min-cost max flow
- A linear program to solve minimax-optimal strategies in games
- Algorithms for linear programming

# Example

- There are 168 hours in a week. Want to allocate our time between
  - studying (S)
  - going to parties (P)
  - everything else (E)
- To survive: $\mathrm{E} \geq 56$
- For sanity: $\mathrm{P} + \mathrm{E} \geq 70$
- To pass courses: $\mathrm{S} \geq 60$
- If party a lot, need to study or eat more: $2S + E - 3P \geq 150$
- Is there a *feasible* solution? Yes, S = 80, P = 20, E = 68
- Happiness is 2P + E. Find a feasible solution maximizing this *objective function*

# Linear Program

- This is called a *linear program (LP)*

- All constraints are linear in our variables

- Objective function is linear

- Don't allow $\mathrm{S} \cdot \mathrm{E} \geq 100$, that's a polynomial program. Much harder.

# Formal Definition

- Given:
  - n variables $x_1, \dots, x_n$
  - m linear inequalities in these variables
    - E.g., $3x_1 + 4x_2 \leq 6$, $0 \leq x_1$, $x_1 \leq 3$

- Goal:
  - Find values for the $x_i$'s that satisfy constraints and maximize objective
  - In the feasibility problem just satisfy the constraints

  - What would happen if we allowed strict inequalities $x_1 < 3$?
  - max $x_1$

# Time Allocation Problem

- Variables: S, P, E
- Objective: Maximize 2P + E subject to
- Constraints: S + P + E = 168

$$\mathrm{E} \geq 56$$

$$\mathrm{S} \geq 60$$

$$2S + E - 3P \geq 150$$

$$P + E \geq 70$$

$$\mathrm{P} \geq 0$$

# Operations Research Problem

| | labor | materials | pollution |
|---|---|---|---|
| plant 1 | 2 | 3 | 15 |
| plant 2 | 3 | 4 | 10 |
| plant 3 | 4 | 5 | 9 |
| plant 4 | 5 | 6 | 7 |

- Required to make at least 400 cars at plant 3
- Have 3300 hours of labor and 4000 units of material
- At most 12000 units of pollution
- Maximize number of cars made

What are the variables?

$x_1, x_2, x_3, x_4$ denote the number of cars at plant i

What's our objective?

maximize $x_1 + x_2 + x_3 + x_4$

| | labor | materials | pollution |
|---|---|---|---|
| plant 1 | 2 | 3 | 15 |
| plant 2 | 3 | 4 | 10 |
| plant 3 | 4 | 5 | 9 |
| plant 4 | 5 | 6 | 7 |

Make at least 400 cars at plant 3
3300 hours of labor and 4000 units of material
At most 12000 units of pollution
Maximize number of cars made

Note: linear programming does not give an integral solution (NP-hard)

What are the variables?

$x_1, x_2, x_3, x_4$ denote the number of cars at plant i

What's our objective?

maximize $x_1 + x_2 + x_3 + x_4$

Constraints: $x_i \geq 0$ for all i

$$x_3 \geq 400$$
$$2x_1 + 3x_2 + 4x_3 + 5x_4 \leq 3300$$
$$3x_1 + 4x_2 + 5x_3 + 6x_4 \leq 4000$$
$$15x_1 + 10x_2 + 9x_3 + 7x_4 \leq 12000$$

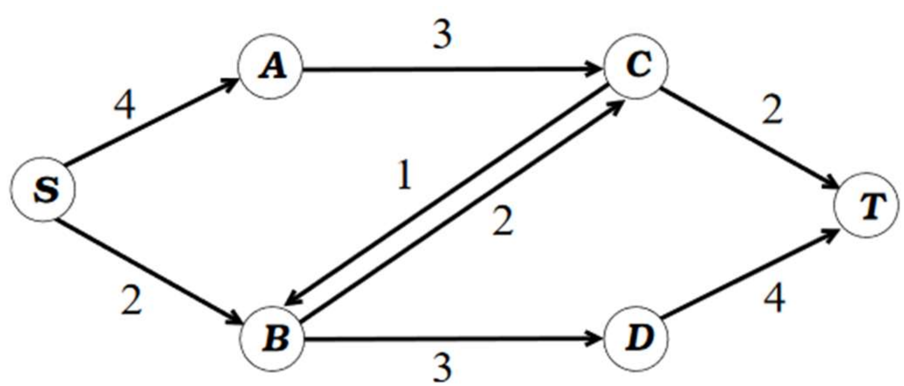
# Modeling Network Flow

Variables: $f_{uv}$ for each edge (u,v), representing positive flow

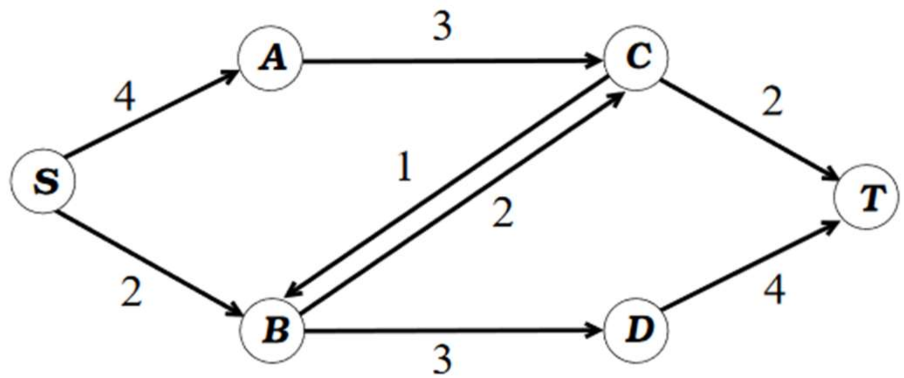
Objective: maximize $\sum_u f_{ut} - \sum_u f_{tu}$

Constraints: For all edges (u,v) $0 \leq f_{uv} \leq c(u,v)$ (capacity constraints)

For all $v \notin \{s,t\}, \sum_u f_{uv} = \sum_u f_{vu}$ (flow conservation)



# Modeling Network Flow




In this case, our LP is: maximize $f_{ct} + f_{dt}$ subject to the constraints:

$0 \leq f_{sa} \leq 4$, $0 \leq f_{ac} \leq 3$, etc.

$f_{sa} = f_{ac}$, $f_{sb} + f_{cb} = f_{bc} + f_{bd}$, $f_{ac} + f_{bc} = f_{cb} + f_{ct}$, $f_{bd} = f_{dt}$.

# Min Cost Max Flow

- Edge (u,v) has a capacity c(u,v) and a cost w(u,v)
- Find a max s-t flow of least total cost, where the cost of flow f is

$$\sum_{(u,v)\in E} w(u,v) f_{uv}$$

- *How to solve this?*
- Solution 1: Solve for a maximum flow f

  Add a constraint that flow must equal the flow of f

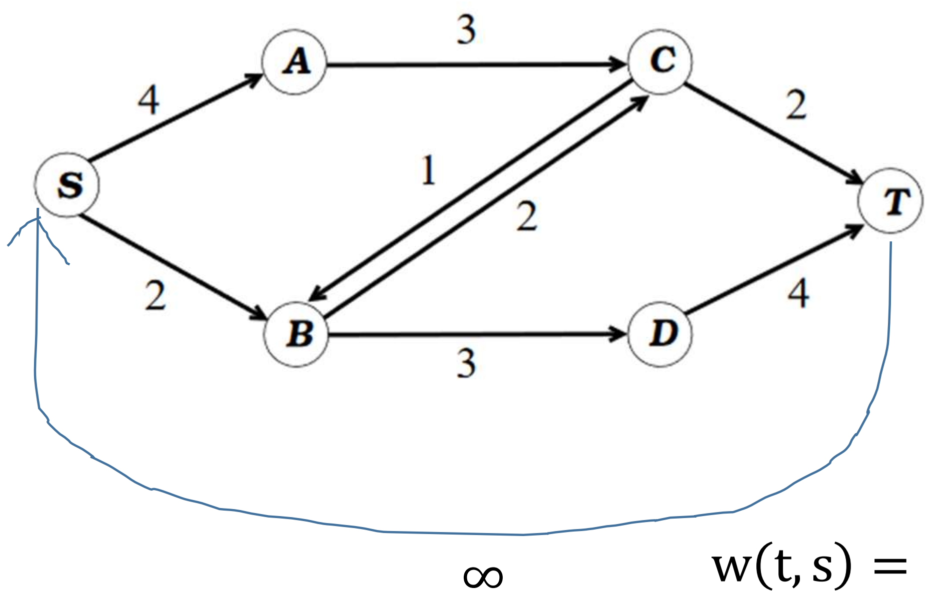
  Minimize $\sum_{(u,v)\in E} w(u,v) f_{uv}$ also subject to original constraints
- Solution 2: Add an edge (t,s) of infinite capacity and very negative cost

  Minimizing cost automatically maximizes flow

# Min Cost Max Flow

$$\min \sum_{(u,v)\in E} w(u,v) f_{uv}$$

$$f_{ts} = f_{sa} + f_{sb}$$




$$f_{ct} + f_{dt} = f_{ts}$$

$$w(t,s) = -\infty$$

# Zero Sum Games

Row payoffs:

$$\boxed{\begin{matrix} 20 & -10 & 5 \\ 5 & 10 & -10 \\ -5 & 0 & 10 \end{matrix}}$$

- Given a zero-sum game with n rows and n columns, compute a minimax optimal strategy for row player

- *What are the variables?*
  - Probabilities $p_1, \ldots, p_n$ on our actions
  - Linear constraints: $\sum_{i=1,\ldots,n} p_i = 1$ and $p_i \geq 0$ for all i
  - Maximize the minimum expected payoff, over all column pure strategies

- *How to maximize a minimum with a linear program?*
- Create new “dummy variable” v to represent minimum

# Zero Sum Games

- $R_{i,j}$ represents payoff to row player with row player action i and column player action j
- Variables: $p_1, \ldots, p_n$ and v
- Objective: maximize v
- Constraints:
  - $p_i \geq 0$ for all i, and $\sum_i p_i = 1$
  - For all columns j, $\sum_i p_i R_{ij} \geq v$

# Linear Programs in Standard Form?

- Many different ways to write the same LP
- Use vector notation, so $c^T x = \sum_{i=1,\ldots,d} c_i x_i$ if there are d variables
- Any LP can be written in the following form:
- Max $c^T x$

  Subject to $Ax \leq b$

  $x \geq 0$

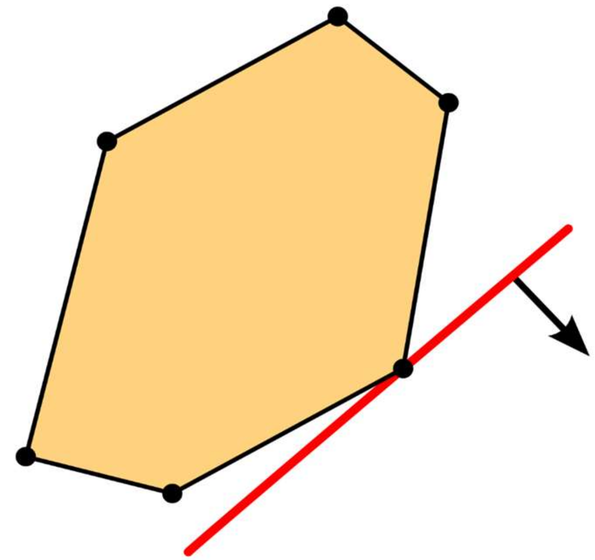
How to handle equality constraints $d^T x$ = e?

How to convert min $c^T x$ to a maximization?

How to handle an unconstrained variable $x_i$ which could be positive or negative?

Substitute $x_i = y_i - z_i$ , $y_i \geq 0, z_i \geq 0$, everywhere

# Facts about Linear Programs



- Consider the LP
- Max $c^T x$

  Subject to $Ax \leq b$

  $x \geq 0$
- Think of maximizing $c^T x$ over the set $Ax \leq b, x \geq 0$
- What does the set $Ax \leq b, x \geq 0$ look like?
  - Each row is a *halfspace,* cutting $R^d$ into two pieces by a hyperplane
  - The intersection of halfspaces could be empty
    - Then the LP is *infeasible*
  - Could be unbounded
  - Could be bounded and then we call it the *feasible region*
- Maximizing $c^T x$ moves the hyperplane with normal vector c until it is tangent to the feasible region

# Convexity Properties

- Feasible region $\mathrm{Ax} \leq \mathrm{b}$, $\mathrm{x} \geq 0$ is convex
  - If p and q are in the feasible region, then so is the line segment joining p and q. Why?

- Proof by pictures, e.g., convex polygon in two dimensions
- Formally, since $\mathrm{Ap} \leq \mathrm{b}$ and $\mathrm{Aq} \leq \mathrm{b}$, for any $\lambda \in [0,1]$,
  - $\lambda \mathrm{Ap} \leq \lambda \mathrm{b}$ and $(1-\lambda)\mathrm{Aq} \leq (1-\lambda)\mathrm{b}$
  - So $\mathrm{A}(\lambda \mathrm{p} + (1-\lambda)\mathrm{q}) \leq \mathrm{b}$
  - Also $\lambda \mathrm{p} \geq 0$ and $(1-\lambda)\mathrm{q} \geq 0$ since $\mathrm{p} \geq 0$ and $\mathrm{q} \geq 0$, so $\lambda \mathrm{p} + (1-\lambda)\mathrm{q} \geq 0$

- More generally, intersections of convex sets are convex
- Max $\mathrm{c}^{\mathrm{T}}\mathrm{x}$ occurs at a vertex. Can we just enumerate all vertices?

# Algorithms for Linear Programming

- Simplex Algorithm
  - Practical, but exponential time in the worst-case

- Ellipsoid Algorithm
  - First polynomial time algorithm, but slow in practice

- Karmarkar's Algorithm (interior point)
  - Polynomial time algorithm and competitive in practice

- Software: LINDO, CPLEX, Solver (in Excel)

# Time Allocation Problem

- Variables: S, P, E
- Objective: Maximize 2P + E subject to
- Constraints: S + P + E = 168

$$\mathrm{E} \geq 56$$
$$\mathrm{S} \geq 60$$
$$2S + E - 3P \geq 150$$
$$P + E \geq 70$$
$$\mathrm{P} \geq 0$$

Substitute S = 168-P-E, so two variables P and E, want to maximize 2P+E.

# Intuition for Linear Programming

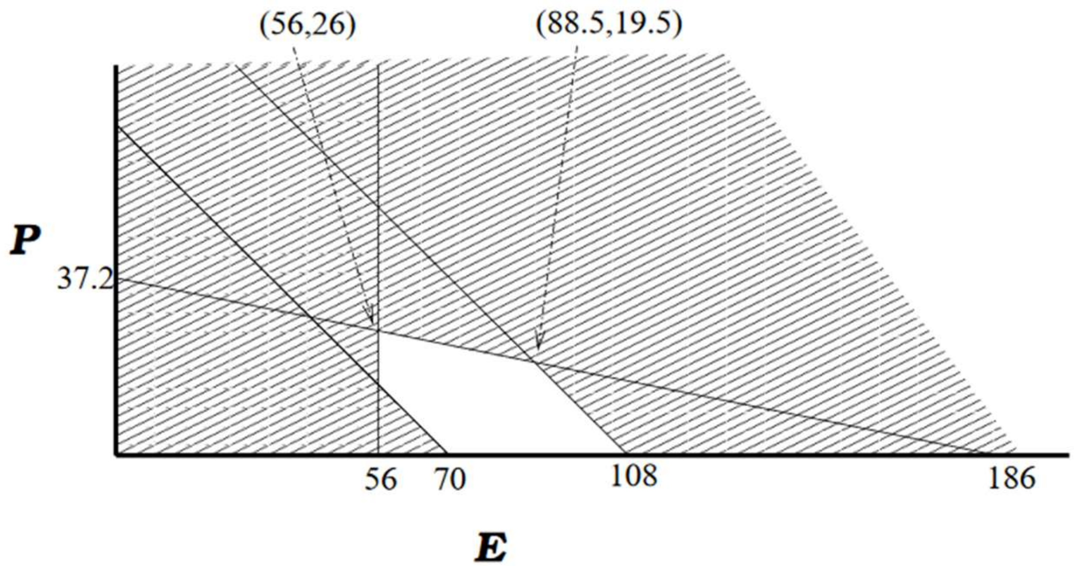
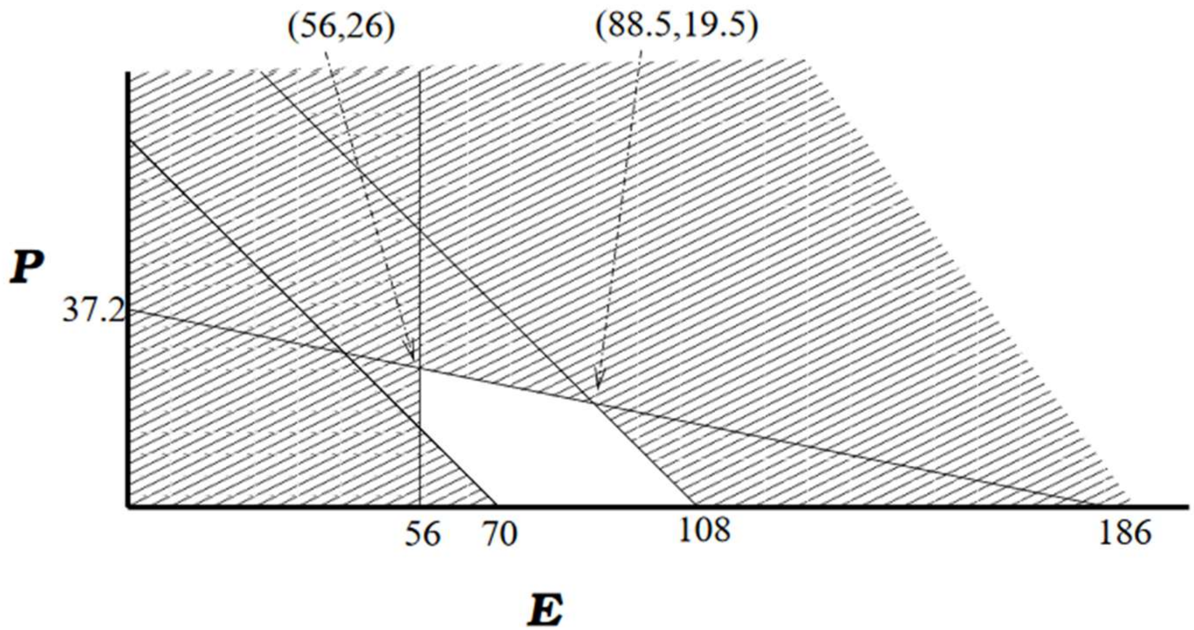



Figure 13.1: Feasible region for our time-planning problem. The constraints are: $E \geq 56$; $P + E \geq 70$; $P \geq 0$; $S \geq 60$ which means $168 - P - E \geq 60$ or $P + E \leq 108$; and finally $2S - 3P + E \geq 150$ which means $2(168 - P - E) - 3P + E \geq 150$ or $5P + E \leq 186$.

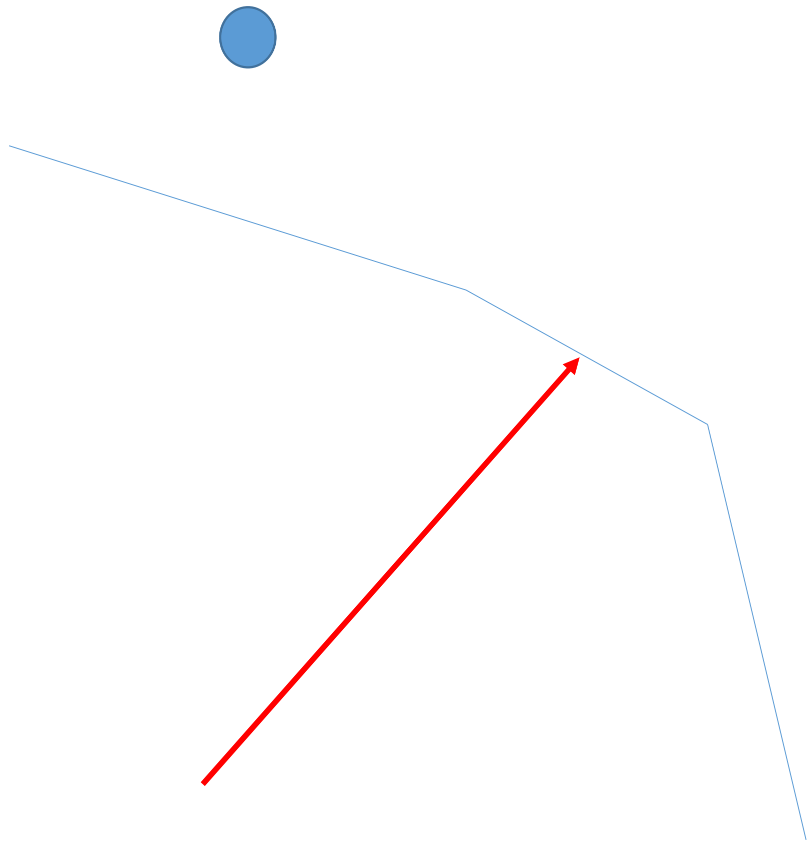
Maximizing P occurs at (56,26). Maximizing 2P+E occurs at (88.5, 19.5)

# Simplex Algorithm




Start at vertex of the feasible region (polyhedron in high dimensions)
Look at cost of objective function at each neighbor
Move to neighbor of maximum cost
Always make progress, but could take exponential time (in high dimensions)

# Simplex Algorithm



Get stuck in local maximum?

No, since feasible set is convex

# Other Annoyances I

- How to start at a vertex of the feasible region?
- Max $c^T x$
  Subject to $Ax \leq b$
  $x \geq 0$
- What if it's not even feasible?
- Introduce "slack" variable s. Consider:
- $\min s$
  subject to $Ax \leq b + s \cdot 1^m$
  $x \geq 0, s \geq 0, s \leq \max_i -b_i$
- Feasible. Can run simplex starting at x = $0^n$ and s = $\max_i -b_i$
- If original LP is feasible, minimum achieved when s = 0, and x that is output is a vertex in the feasible region of original LP

# Other Annoyances II

- What if the feasible region is unbounded?
  - Ok, as long as objective function is bounded

- What if objective function is unbounded?
  - Output $\infty$, how to detect this?

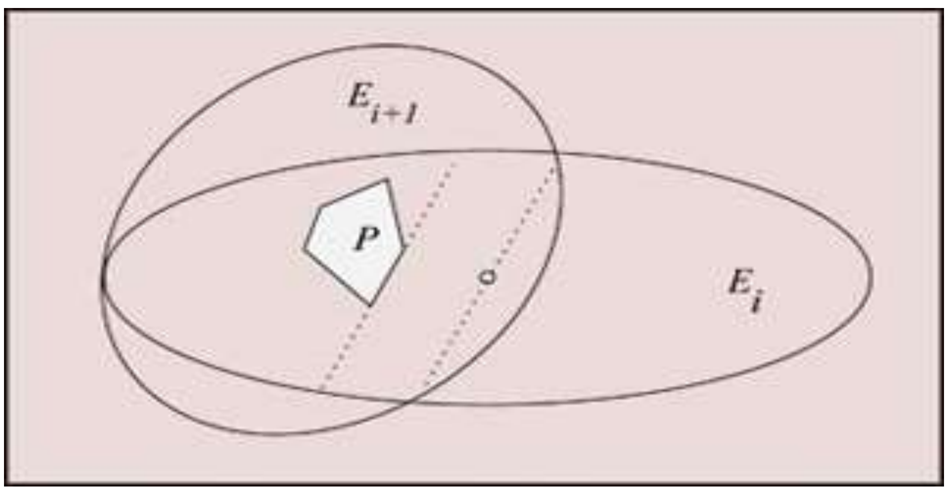
- Many ways
  - see one based on duality in a few lectures
  - include constraints $-M \leq x_i \leq M$ for all i, for a very large value M
  - can efficiently find M to ensure if solution is finite, still find the optimum

# Ellipsoid Algorithm

Solves feasibility problem

Replace objective function with constraint, do binary search

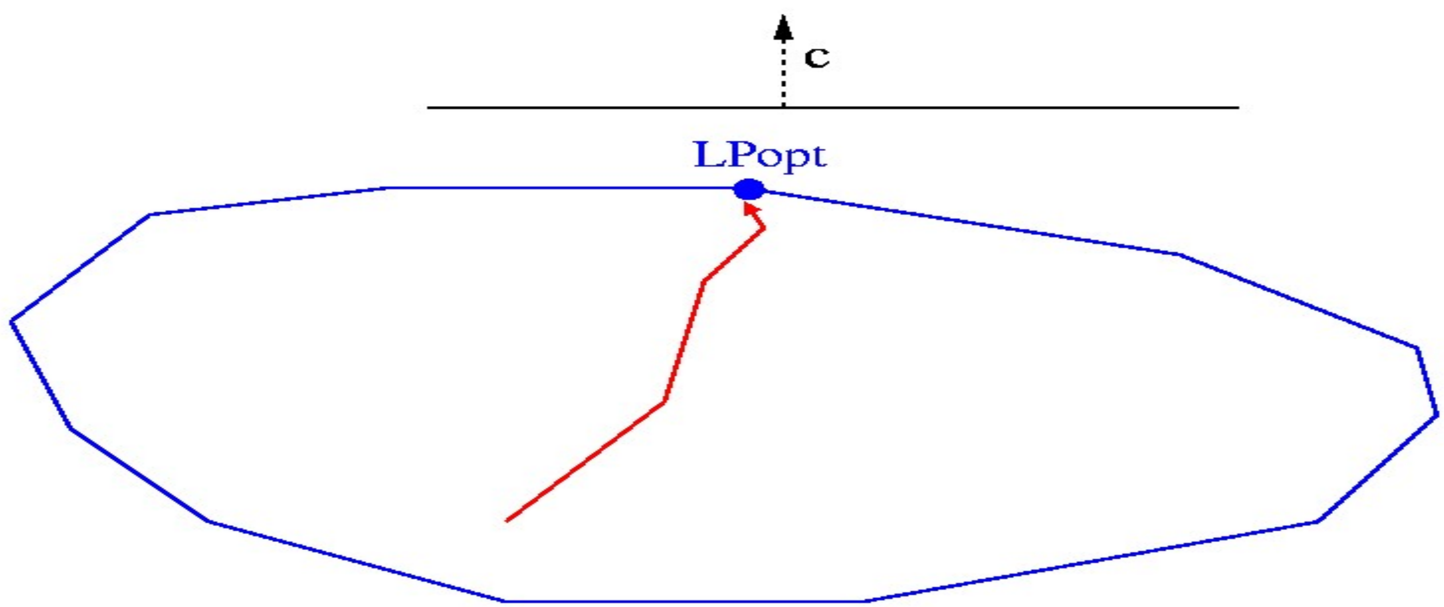
Replace “minimize $x_1 + x_2$" with $x_1 + x_2 \leq \lambda$




Can handle exponential number of constraints if there’s a separation oracle

# Karmarkar’s Algorithm

- Works with feasible points but doesn’t go corner to corner
- Moves in interior of the feasible region – “interior point method”



这个幻灯片中的极小极大最优策略是指在零和博弈中，列玩家（$q$）想要选择一个分布 $q^*$，最大化他在所有可能的行玩家策略 $p$ 下的预期收益。换句话说，列玩家的目标是：

$$\max_q \min_p V_C(p,q)$$

其中 $V_C(p,q)$ 是列玩家在策略 $p$ 和 $q$ 下的预期收益。

这个幻灯片提出了一个断言（Claim）：
$$\max_q \min_p V_C(p,q) = -\min_q \max_p V_R(p,q)$$

这里 $V_R(p,q)$ 是行玩家在策略 $p$ 和 $q$ 下的预期收益。

证明（Proof）分为以下几步：

1. 首先，根据零和博弈的定义，我们知道一个玩家的收益加上另一个玩家的收益等于零：
$$V_R(p,q) + V_C(p,q) = 0$$
因此：
$$V_C(p,q) = -V_R(p,q)$$
2. 接着，使用这个关系来重写列玩家的目标函数：
$$\max_q \min_p V_C(p,q) = \max_q \min_p -V_R(p,q)$$
3. 考虑到 $-V_R(p,q)$ 是 $V_R(p,q)$ 的相反数，我们可以将最小化 $-V_R(p,q)$ 转化为最大化 $V_R(p,q)$ 的相反数：
$$\max_q \min_p -V_R(p,q) = \max_q -\max_p V_R(p,q)$$
4. 然后，将最外面的最大化（对于 $q$）转变为最小化的相反数：
$$\max_q -\max_p V_R(p,q) = -\min_q \max_p V_R(p,q)$$

这就完成了证明。

这个证明告诉我们，在零和博弈中，列玩家（通常是最小化玩家）通过选择最小化行玩家最大收益的策略，实际上是在最大化自己的最小收益。这是因为行玩家的最大收益是列玩家的最大损失，而列玩家希望最小化这个损失。

在这个特定的幻灯片中，它还说明了如果列玩家采取了他的极小极大策略 $q^*$，则可以确保行玩家的收益不会超过一个特定的上界 $ub$，这是行玩家预期收益的上界。而行玩家的策略 $p^*$ 同样可以保证自己获得一个下界 $lb$ 的收益。在理想情况下，这两个值是相等的，这个共同值被称为博弈的价值。

证明：