Mathematical Finance

Ch2: Risk-Free Assets

Instructor: Yu Wang[1]

[1]WISE
Xiamen University

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Outline

1. Time Value of Money
   - Simple Interest
   - Periodic Compounding
   - Streams of Payments
   - Continuous Compounding
   - How to Compare Compounding Methods

2. Money Market
   - Zero-Coupon Bonds
   - Coupon Bonds
   - Money Market Account

Time Value of Money

Questions:

Whether \$100 to be received after one year is worth equivalently to the same amount today?

If the answer is "no", what are the possible reasons?

- It is a fact of life that \$100 to be received after one year is worth less than the same amount today.

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Reasons:

1. (main) Money due in the future or locked in a fixed-term account cannot be spent right away.
    - One would therefore expect to be compensated for postponed consumption.
2. Prices may rise in the meantime (e.g., inflation), and the amount will not have the same purchasing power as it would have at present.
3. There is always a risk, even if a negligible one, that the money will never be received. (called default or credit risk)
    - Whenever a future payment is uncertain due to the possibility of default, its value today will be reduced to compensate for the risk.

But, this chapter considers situations free from such (default or credit) risk.

- Recall: generic examples of risk-free assets are a bank deposit or a bond.

- The way in which money changes its value in time, often referred to as the ***time value of money***, is a complex but fundamental important issue in finance.
- Concerned mainly with two questions:
    - What is the future value of an amount invested or borrowed today?
    - What is the present value of an amount to be paid or received at a certain time in the future?
- The answers depend on various factors to be discussed in this chapter.

Outline

Simple Interest

Future value

- Suppose that an amount is paid into a bank account, where it is to earn *interest*.
- The *future value* of this investment consists of
  - the initial deposit, called the *principal* and denoted by P, plus
  - all the interest earned.

- To begin with, we shall consider the simple case when interest is attracted only by the principal, which remains unchanged during the period of investment.
  - e.g., the interest earned may be paid out in cash, credited to another account attracting no interest, or credited to the original account after some longer period.

- After one year ($t = 1$) the interest earned will be rP, where $r > 0$ is the *interest rate*.
    - The value of the investment will thus become $V(1) = P + rP = (1 + r)P$.
- After two years ($t = 2$) the investment will grow to $V(2) = (1 + 2r)P$.
- Consider a fraction of a year ($0 < t < 1$):
  e.g., Interest is typically calculated on a daily basis:
    - The interest earned in one day will be $\frac{1}{365}rP$.
    - After n days the interest will be $\frac{n}{365}rP$ and the total value of the investment will become $V(\frac{n}{365}) = (1 + \frac{n}{365}r)P$.

- This motivates the rule of *simple interest*:
  the value of the investment at time t, denoted by $V(t)$, is given by

$$V(t) = (1 + tr)P,$$

  - (2.1) where time t, expressed in years, can be an arbitrary non-negative real number; see Figure 2.1.
- Obviously, the equality: $V(0) = P$.
- The number $1 + rt$ is called the *growth factor*.
  - Here we assume that the interest rate r is constant.
- If the principal P is invested at time s, rather than at time 0, then the value at time $t \geq s$ will be

$$V(t) = (1 + (t - s)r)P.$$

  - (2.2)

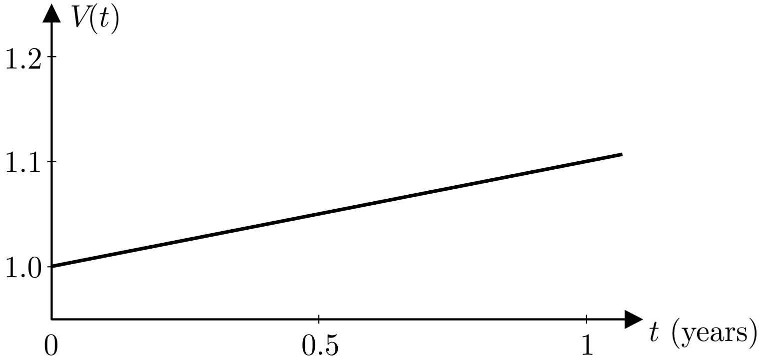

Figure: 2.1 Principal attracting simple interest at 10% ($r = 0.1$, $P = 1$)

Note:

- Throughout this course the unit of time will be one year.
- Thus, we shall transform any period expressed in other units (days, weeks, months) into a fraction of a year.

Example 2.1

- Consider a deposit of \$150 held for 20 days and attracting simple interest at a rate of 8%.
- This gives $t = \frac{20}{365}$ and $r = 0.08$.
- After 20 days the deposit will grow to $V(\frac{20}{365}) = (1 + \frac{20}{365} \times 0.08) \times 150 \cong 150.66$.

Remark 2.2

- An important example of employing simple interest is provided by the **LIBOR** rate (London Interbank Offered Rate).
  - *[In China, we have **SHIBOR**.]*
- This is the rate of interest valid for transactions between the largest London banks.
- LIBOR rates are quoted for various short periods of time up to one year, and have become reference values for a variety of transactions.
  - e.g., the rate for a typical commercial loan may be formulated as a particular LIBOR rate plus some additional margin.
  - *[The LPR (Loan Prime Rate) plays a similar role of reference/benchmark rate in China's (mortgage) loan market.]*

- The *return* on an investment commencing at time s and terminating at time t will be denoted by $K(s,t)$. Thus,

$$K(s,t) = \frac{V(t) - V(s)}{V(s)}.$$

  - (2.3)

- In the case of simple interest

$$K(s,t) = (t-s)r,$$

  which clearly follows from (2.2).

- In particular, the interest rate is equal to the return over one year,

$$K(0,1) = r.$$

Interest rate v.s. Return:

- As a general rule,
  - interest rates will always refer to a period of one year, facilitating the comparison between different investments, independently of their actual maturity time.

  By contrast,
  - the return reflects both the interest rate and the length of time the investment is held.

Exercise

Exercise 2.1

Exercise 2.2

Exercise 2.3

Exercise 2.4

Simple Interest

Present value

- The last exercise is concerned with an important general problem:
  Find the initial sum whose value at time t is given.
- In the case of simple interest the answer is easily found by solving (2.1) for the principal, obtaining

$$V(0) = V(t)(1+rt)^{-1}.$$

- (2.4)

- $V(0)$ here is called the *present* or *discounted value* of $V(t)$ and
- $(1+rt)^{-1}$ is the *discount factor*.

Example 2.2 (*Perpetuity*)

A *perpetuity* is a sequence of payments of a fixed amount to be made at equal time intervals and continuing indefinitely (or infinitely) into the future.

- Suppose that payments of an amount C are to be made once a year, the first payment due a year hence.
- This can be achieved by depositing

$$P = \frac{C}{r}$$

  in a bank account to earn simple interest at a constant rate r.
- Such a deposit will indeed produce a sequence of interest payments amounting to $C = rP$ payable every year.

Comments:

- In practice simple interest is used only for short-term investments and for certain types of loans and deposits.
- It is not a realistic description of the value of money in the longer term.
- In the majority of cases the interest already earned can be reinvested to attract even more interest, producing a higher return than that implied by (2.1).
- This will be analysed in detail in what follows.

Outline

Periodic Compounding

- Again, suppose that an amount P is deposited in a bank account, attracting interest at a constant rate $r > 0$.
- However, we assume that the interest earned will now be added to the principal periodically (e.g., annually, semi-annually, quarterly, monthly, or perhaps even on a daily basis).
- Subsequently, interest will be attracted not just by the original deposit, but also by all the interest earned so far.
- In these circumstances we shall talk of *discrete* or *periodic compounding*.

Example 2.3

In the case of monthly compounding

- the first interest payment of $\frac{r}{12}P$ will be due after one month, increasing the principal to $(1+\frac{r}{12})P$, all of which will attract interest in the future.
- The next interest payment, due after two months, will thus be $\frac{r}{12}(1+\frac{r}{12})P$, and the capital will become $(1+\frac{r}{12})^2P$.
  $\vdots$
- After one year it will become $(1+\frac{r}{12})^{12}P$,
- after n months it will be $(1+\frac{r}{12})^nP$, and
- after t years $(1+\frac{r}{12})^{12t}P$.

The last formula admits t equal to a whole number of months, that is, a multiple of $\frac{1}{12}$.

Periodic Compounding

Future value

- In general, if m interest payments are made per annum, the time between two consecutive payments measured in years will be $\frac{1}{m}$, the first interest payment being due at time $\frac{1}{m}$.
- Each interest payment will increase the principal by a factor of $1 + \frac{r}{m}$.
- Given the interest rate r unchanged, after t years the *future value* of an initial principal P will become

$$V(t) = \left(1 + \frac{r}{m}\right)^{tm} P,$$

  - (2.5) because of tm interest payments during this period.
- In this formula t must be a whole multiple of the period $\frac{1}{m}$.
- The number $\left(1 + \frac{r}{m}\right)^{tm}$ is the *growth factor*.

- Note that in each period we apply simple interest with the starting amount changing from period to period.
- Formula (2.5) can be equivalently written in a recursive way:

$$V(t + \frac{1}{m}) = V(t)(1 + \frac{r}{m})$$

with $V(0) = P$.

The exact value of the investment may sometimes need to be known at time instants between interest payments.

- e.g., what is the value after 10 days of a deposit of \$100 subject to monthly compounding at 12%?
  - One possible answer is \$100, since the first interest payment would be due only after one whole month.
  - This suggests that (2.5) should be extended to arbitrary values of t by means of a step function with steps of duration $\frac{1}{m}$, as shown in Figure 2.2.

Later on, Remark 2.21 shows that the extension consistent with the No-Arbitrage Principle should use the right-hand side of (2.5) for all $t \geq 0$.

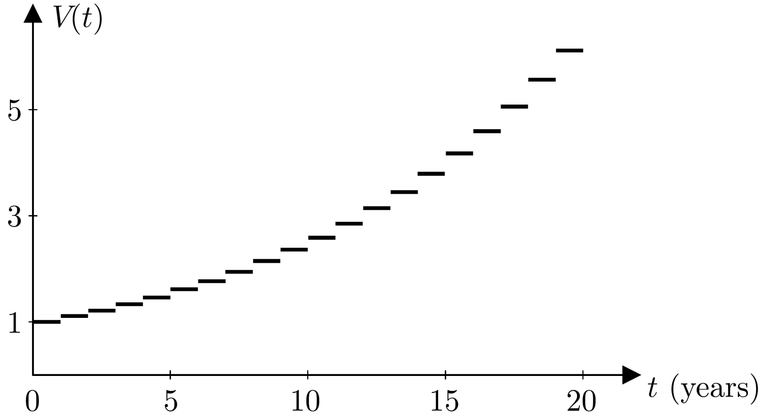

Figure: 2.2 Annual compounding at 10% ($m = 1,\ r = 0.1,\ P = 1$)

Exercise

Exercise 2.5

Exercise 2.6

Exercise 2.7

Proposition 2.5

- The future value $V(t)$ increases if any one of the parameters m, t, r or P increases, the others remaining unchanged.

Proof

Recall: (2.5)

$$V(t) = \left(1 + \frac{r}{m}\right)^{tm} P$$

1) It is immediately obvious from (2.5) that $V(t)$ increases if t, r or P increases.

2) To show that $V(t)$ increases as the compounding frequency m increases, we need to verify that if $m < k$, then

$$\left(1 + \frac{r}{m}\right)^{tm} < \left(1 + \frac{r}{k}\right)^{tk}.$$

- This clearly reduces to

$$\left(1+\frac{r}{m}\right)^m < \left(1+\frac{r}{k}\right)^k,$$

which can be verified directly (using the binomial formula):

$$\begin{aligned}\left(1+\frac{r}{m}\right)^m &= 1+r+\frac{1-\frac{1}{m}}{2!}r^2+\cdots+\frac{\left(1-\frac{1}{m}\right)\times\cdots\times\left(1-\frac{m-1}{m}\right)}{m!}r^m\\ &\le 1+r+\frac{1-\frac{1}{k}}{2!}r^2+\cdots+\frac{\left(1-\frac{1}{k}\right)\times\cdots\times\left(1-\frac{m-1}{k}\right)}{m!}r^m\\ &< 1+r+\frac{1-\frac{1}{k}}{2!}r^2+\cdots+\frac{\left(1-\frac{1}{k}\right)\times\text{ÂůÂůÂů}\times\left(1-\frac{k-1}{k}\right)}{k!}r^k\\ &= \left(1+\frac{r}{k}\right)^k.\end{aligned}$$

- The first inequality holds because each term of the sum on the left-hand side is no greater than the corresponding term on the right-hand side.
- The second inequality is true because the sum on the right-hand side contains $k - m$ additional positive terms as compared to the sum on the left-hand side.

[End of Proof]

Note: In both equalities we use the binomial formula

$$(a+b)^m = \sum_{i=0}^{m} \frac{m!}{i!(m-i)!} a^i b^{m-i}.$$

Exercise

Exercise 2.8

Exercise 2.9

Periodic Compounding

Present value

- The last exercise touches upon the problem of finding the present value of an amount payable at some future time instant in the case when periodic compounding applies.
- Here the formula for the *present* or *discounted value* of $V(t)$ is

$$V(0) = V(t)(1+\frac{r}{m})^{-tm},$$

the number $(1+\frac{r}{m})^{-tm}$ being the *discount factor*.

Remark 2.6

- Fix the terminal value $V(t)$ of an investment.
- It is an immediate consequence of Proposition 2.5 that the present value increases if any one of the factors r, t, m decreases, the other ones remaining unchanged.

Exercise

Exercise 2.10

- One often requires the value $V(t)$ of an investment at an intermediate time $0 < t < T$, given the value $V(T)$ at some fixed future time T.
- This can be achieved by computing the present value of $V(T)$, taking it as the principal, and running the investment forward up to time t.
- Under periodic compounding with frequency m and interest rate r, this obviously gives

$$V(t) = (1 + \frac{r}{m})^{-(T-t)m} V(T).$$

- (2.6)

Periodic Compounding

Return

- To find the return on a deposit attracting interest compounded periodically we use the general formula (2.3) and readily arrive at

$$K(s,t) = \frac{V(t) - V(s)}{V(s)} = (1 + \frac{r}{m})^{(t-s)m} - 1.$$

- In particular, $K(0, \frac{1}{m}) = \frac{r}{m}$, which provides a simple way of computing the interest rate given the return.

Exercise

Exercise 2.11

Exercise 2.12

Remark 2.7

- The return on a deposit subject to periodic compounding is *not* additive.
- Take, for simplicity, $m = 1$.
- Then

$$K(0,1) = K(1,2) = r,$$
$$K(0,2) = (1+r)^2 - 1 = 2r + r^2,$$

and clearly $K(0,1) + K(1,2) \neq K(0,2)$.

Outline

Streams of Payments

An *annuity* is a sequence of finitely many payments of a fixed amount due at equal time intervals.

- Suppose that payments of an amount C are to be made once a year for n years, the first one due a year hence.
- Assuming that annual compounding applies, we shall find the present value of such a stream of payments.
- We compute the present values of all payments and add them up to get

$$\frac{C}{1+r}+\frac{C}{(1+r)^2}+\frac{C}{(1+r)^3}+\cdots+\frac{C}{(1+r)^n}.$$

- It is sometimes convenient to introduce the following piece of notation:

$$PA(r, n) = \frac{1}{1+r} + \frac{1}{(1+r)^2} + \text{ÂůÂůÂů} + \frac{1}{(1+r)^n},$$

  \- (2.7) called the *present value factor for an annuity*.

- It allows us to express the present value of an annuity in a concise form:

$$PA(r, n) \times C.$$

- The expression for $PA(r, n)$ can be simplified by using the formula (i.e., the summation for geometric sequence)

$$a + qa + q^2 a + \cdots + q^{n-1} a = a\frac{1-q^n}{1-q}.$$

- (2.8)

- In our case $a = \frac{1}{1+r}$ and $q = \frac{1}{1+r}$, hence

$$PA(r, n) = \frac{1-(1+r)^{-n}}{r}.$$

- (2.9)

Remark 2.8 (Replicate an annuity)

Note that an initial bank deposit of

$$P = PA(r,n) \times C = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \text{ÂůÂůÂů} + \frac{C}{(1+r)^n}$$

attracting interest at a rate r compounded annually would produce a stream of n annual payments of C each.

- A deposit of $C(1+r)^{-1}$ would grow to C after one year, which is just what is needed to cover the first annuity payment.
- A deposit of $C(1+r)^{-2}$ would become C after two years to cover the second payment, and so on.
- Finally, a deposit of $C(1+r)^{-n}$ would deliver the last payment of C due after n years.

Example 2.9 (An application of Remark 2.8)

- Consider a loan of \$1,000 to be paid back in 5 equal instalments due at yearly intervals.
- The instalments include both the interest payable each year calculated at 15% of the current outstanding balance and the repayment of a fraction of the loan.
- A loan of this type is called an *amortised loan*.
- The amount of each instalment can be computed as

$$\frac{1,000}{PA(15\%, 5)} \cong 298.32.$$

- This is because the loan is equivalent to an annuity from the point of view of the lender.

Exercise

Exercise 2.13

Exercise 2.14

Exercise 2.15

Exercise 2.16

Revisit the present value of a perpetuity

- Recall that a *perpetuity* is an infinite sequence of payments of a fixed amount C occurring at the end of each year.
- The formula for the present value of a perpetuity can be obtained from (2.7) in the limit as $n \to \infty$:

$$\lim_{n\to\infty} PA(r,n) \times C = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \text{ÂůÂůÂů} = \frac{C}{r}.$$

- (2.10)

- The limit amounts to taking the sum of a geometric series.

Remark 2.10

- The present value of a perpetuity is given by the same formula as in Example 2.2, even though periodic compounding has been used in place of simple interest.
- In both cases the annual payment C is exactly equal to the interest earned throughout the year, and the amount remaining to earn interest in the following year is always $\frac{C}{r}$.
- Nevertheless, periodic compounding allows us to view the same sequence of payments in a different way:
  - The present value $\frac{C}{r}$ of the perpetuity is decomposed into infinitely many parts, as in (2.10), each responsible for producing one future payment of C.

Remark 2.11

- Formula (2.9) for the annuity factor is easier to memorise in the following way, using the formula for a perpetuity:
  - The sequence of n payments of $C = 1$ can be represented as the difference between two perpetuities, one starting now and the other after n years.
  - (Cutting off the tail of a perpetuity, we obtain an annuity.)
- In doing so we need to compute the present value of the latter perpetuity.
- This can be achieved by means of the discount factor $(1+r)^{-n}$.
- Hence,

$$PA(r,n) = \frac{1}{r} - \frac{1}{r} \times \frac{1}{(1+r)^n} = \frac{1-(1+r)^{-n}}{r}.$$

Exercise

Exercise 2.17

Outline

Continuous Compounding

Recall: Formula (2.5) for the future value at time t of a principal P attracting interest at a rate $r > 0$ compounded m times a year:

$$V(t) = \left(1 + \frac{r}{m}\right)^{tm} P,$$

which can be rewritten as

$$V(t) = \left[\left(1 + \frac{r}{m}\right)^{\frac{m}{r}}\right]^{tr} P.$$

- In the limit as $m \to \infty$, we obtain

$$V(t) = e^{tr} P,$$

- (2.11) where e is the base of natural logarithms:

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$$

- This is known as *continuous compounding*.
- The corresponding *growth factor* is e^{tr}.
- A typical graph of $V(t)$ is shown in Figure 2.3.

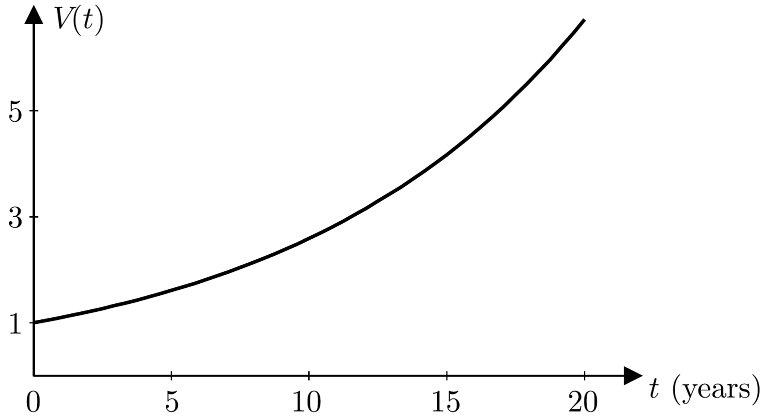

Figure: 2.3 Continuous compounding at 10% ($r = 0.1$, $P = 1$)

- The derivative of $V(t) = e^{tr}P$ is

$$V'(t) = re^{tr}P = rV(t).$$

- In the case of continuous compounding the rate of the growth is proportional to the current wealth.

- Formula (2.11) is a good approximation of the case of periodic compounding when the frequency m is large.
- It is simpler and lends itself more readily to transformations than the formula for periodic compounding.

Exercise

Exercise 2.18

Exercise 2.19

Proposition 2.12

Proposition 2.12

Continuous compounding produces higher future value than periodic compounding with any frequency m, given the same initial principal P and interest rate r.

Proof

- It suffices to verify that

$$e^{tr} > \left[(1+\frac{r}{m})^{\frac{m}{r}}\right]^{rt} = (1+\frac{r}{m})^{tm}.$$

- The inequality holds because the sequence $(1+\frac{r}{m})^{\frac{m}{r}}$ is increasing and converges to e as $m \nearrow \infty$.

[End of Proof]

Exercise

Exercise 2.20

- The present value under continuous compounding is obviously given by

$$V(0) = V(t)e^{-tr}.$$

- In this case the *discount factor* is e^{-tr}.
- Given the terminal value $V(T)$, we clearly have

$$V(t) = e^{-r(T-t)}V(T).$$

- (2.12)

Exercise

Exercise 2.21

Exercise 2.22

- The return $K(s,t)$ defined by (2.3) on an investment subject to continuous compounding fails to be additive, just like in the case of periodic compounding.
- It proves convenient to introduce the *logarithmic return*

$$k(s,t) = ln\frac{V(t)}{V(s)}.$$

- (2.13)

Proposition 2.13

Proposition 2.13

The logarithmic return is additive,

$$k(s,t) + k(t,u) = k(s,u).$$

Proof

- This is an easy consequence of (2.13):

$$
\begin{aligned}
k(s,t)+k(t,u) &= ln\frac{V(t)}{V(s)} + ln\frac{V(u)}{V(t)} \\
&= ln\frac{V(t)}{V(s)}\frac{V(u)}{V(t)} \\
&= ln\frac{V(u)}{V(s)} \\
&= k(s,u).
\end{aligned}
$$

[End of Proof]

- If $V(t)$ is given by (2.11), then $k(s,t) = r(t-s)$, which enables us to recover the interest rate

$$r = \frac{k(s,t)}{t-s}.$$

Exercise

Exercise 2.23

Outline

Compare Compounding Methods

- As we have already noticed, frequent compounding will produce a higher future value than less frequent compounding if the interest rates and the initial principal are the same.
  ($m \nearrow \Longrightarrow V(t) \nearrow$, if r, P are the same.)
- We shall consider the general circumstances in which one compounding method will produce either the same or higher future value than another method, given the same initial principal.
  (Note: here m and r are allowed to be different for different compounding methods.)

Example 2.14

- Suppose that certificates promising to pay \$120 after one year can be purchased or sold now, or at any time during this year, for \$100.
- This is consistent with a constant interest rate of 20% under annual compounding.
- If an investor decided to sell such a certificate half a year after the purchase, what price would it fetch?
- Suppose it is \$110, a frequent first guess based on halving the annual profit of \$20.

- However, this turns out to be too high a price, leading to the following arbitrage strategy:
  1. Borrow \$1,000 to buy 10 certificates for \$100 each.
  2. After six months sell the 10 certificates for \$110 each and buy 11 new certificates for \$100 each.
     (The balance of these transactions is nil.)
  3. After another six months sell the 11 certificates for \$110 each, cashing \$1,210 in total, and pay \$1,200 to clear the loan with interest.
     The balance of \$10 would be the arbitrage profit.

- A similar argument shows that the certificate price after six months cannot be too low, say, \$109.

- The price of a certificate after six months is related to the interest rate under semi-annual compounding:
  - If this rate is r, then the price is $100(1+\frac{r}{2})$ dollars and vice versa.
- Arbitrage will disappear if the corresponding growth factor $(1+\frac{r}{2})^2$ over one year is equal to the growth factor 1.2 under annual compounding,

$$(1+\frac{r}{2})^2 = 1.2,$$

  which gives $r \cong 0.1909$, or 19.09%.
- If so, then the certificate price after six months should be $100(1+\frac{0.1909}{2}) \cong 109.54$ dollars.

- Thus, the growth factors over a fixed period, typically one year, can be used to compare any two compounding methods.

- Method of Comparison (1)

Definition 2.15

Definition 2.15

We say that two compounding methods are *equivalent* if the corresponding growth factors over a period of one year are the same.
If one of the growth factors exceeds the other, then the corresponding compounding method is said to be *preferable*.

Example 2.16

- Semi-annual compounding at 10% is equivalent to annual compounding at 10.25%.
- Indeed, in the former case the growth factor over a period of one year is
$$\left(1+\frac{0.1}{2}\right)^2 = 1.1025,$$
which is the same as the growth factor in the latter case.
- Both are preferable to monthly compounding at 9%, for which the growth factor over one year is only
$$\left(1+\frac{0.09}{12}\right)^{12} \cong 1.0938.$$

- We can freely switch from one compounding method to another equivalent method by recalculating the interest rate.
- In the chapters to follow we shall normally use either annual or continuous compounding.

Exercise

Exercise 2.24

Exercise 2.25

- Instead of comparing the growth factors, it is often convenient to compare the so-called effective rates as defined below.

- Method of Comparison (2)

Definition 2.17

Definition 2.17

For a given compounding method with interest rate r the *effective rate* r_e is one that gives the same growth factor over a one year period under annual compounding.

In particular,

1. in the case of periodic compounding with frequency m and rate r the effective rate r_e satisfies

$$\left(1+\frac{r}{m}\right)^m = 1 + r_e.$$

2. In the case of continuous compounding with rate r

$$e^r = 1 + r_e.$$

Example 2.18

- In the case of semi-annual compounding at 10% the effective rate is 10.25%, see Example 2.16.

Proposition 2.19

Proposition 2.19

Two compounding methods are equivalent if and only if the corresponding effective rates r_e and r'_e are equal, $r_e = r'_e$.

The compounding method with effective rate r_e is preferable to the other method if and only if $r_e > r'_e$.

Proof

- This is because the growth factors over one year are $1 + r_e$ and $1 + r_e'$, respectively.

[End of Proof]

Example 2.20

- In Exercise 2.8 we have seen that daily compounding at 15% is preferable to semi-annual compounding at 15.5%.
- The corresponding effective rates r_e and r_e' can be found from

$$1 + r_e = \left(1 + \frac{0.15}{365}\right)^{365} \cong 1.1618,$$

$$1 + r_e' = \left(1 + \frac{0.155}{2}\right)^{2} \cong 1.1610.$$

- This means that r_e is about 16.18% and r_e' about 16.10%.

Remark 2.21

- Recall: formula (2.5) for periodic compounding, i.e.,

$$V(t) = \left(1 + \frac{r}{m}\right)^{tm} P,$$

  admits only time instants t being whole multiples of the compounding period $\frac{1}{m}$.
- An argument similar to that in Example 2.14 shows that the appropriate no-arbitrage value of an initial sum P at any time $t \geq 0$ should be $\left(1 + \frac{r}{m}\right)^{tm} P$.
- A reasonable extension of (2.5) is therefore to use the right-hand side for all $t \geq 0$ rather than just for whole multiples of $\frac{1}{m}$.

From now on we shall always use this extension.

- In terms of the effective rate r_e the future value can be written as
$$V(t) = (1 + r_e)^t P,$$
for all $t \geq 0$. [Note: $1 + r_e = \left(1 + \frac{r}{m}\right)^m$]
- This applies both to continuous compounding and to periodic compounding extended to arbitrary times as in Remark 2.21.
- Proposition 2.19 implies that, given the same initial principal,
    - equivalent compounding methods will produce the same future value for all times $t \geq 0$.
    - a compounding method preferable to another one will produce a higher future value for all $t > 0$.

Remark 2.22

- Simple interest does not fit into the scheme for comparing compounding methods.
    - In this case of simple interest the future value $V(t)$ is a linear function of time t,
      whereas
    - it is an exponential function if either continuous or periodic compounding applies.

Exercise

Exercise 2.26

Exercise 2.27

Money Market

- The money market consists of risk-free (more precisely, default-free) securities.
- An example is a *bond*, which is a financial security promising the holder a sequence of guaranteed future payments.
- Risk-free means here that these payments will be delivered with certainty.
  - (Nevertheless, even in this case risk cannot be completely avoided, since the market prices of such securities may fluctuate unpredictably; see Chapter 9.)
- There are many kinds of bonds like treasury bills and notes, treasury, mortgage and debenture bonds, commercial papers, and others with various particular arrangements concerning the issuing institution, maturity, number of payments, embedded rights and guarantees.

Outline

Zero-Coupon Bonds

- The simplest case of a bond is a *zero-coupon bond*, which involves just a single payment.
- The issuing institution (e.g., a government, a bank or a company) promises to exchange the bond for a certain amount of money F (i.e., the *face value*) on a given day T (i.e., the *maturity date*).
- Typically, the life span of a zero-coupon bond is up to one year, the face value being some round figure, for example 100.
- In effect, the person or institution who buys the bond is lending money to the bond writer.

Given the interest rate, the present value of such a bond can easily be computed.

- Suppose that a bond with face value $F = 100$ dollars is maturing in one year, and the annual compounding rate r is 12%.
- Then the present value of the bond should be

$$V(0) = F(1 + r)^{-1} \cong 89.29$$

  dollars.

- In reality, the opposite happens:
  - Bonds are freely traded and their prices are determined by market forces, whereas the interest rate is implied by the bond prices,

$$r = \frac{F}{V(0)} - 1.$$

- (2.14)

- This formula gives the implied annual compounding rate.
  - e.g., if a one-year bond with face value \$100 is being traded at \$91, then the implied rate is 9.89%.

- For simplicity, we shall consider *unit bonds* with face value equal to one unit of the home currency, $F = 1$.

- Typically, a bond can be sold at any time prior to maturity at the market price.
- This price at time t is denoted $B(t, T)$. In particular,
  - $B(0, T)$ is the current (time 0) price of the bond, and
  - $B(T, T) = 1$ is equal to the face value.

- Again, these prices determine the interest rates by applying formulas (2.6) and (2.12) with $V(t) = B(t, T)$, $V(T) = 1$.
  - e.g., the implied annual compounding rate satisfies the equation
    $$B(t, T) = (1 + r)^{-(T-t)}.$$
    Note:
    - (2.6) is $V(t) = (1 + \frac{r}{m})^{-(T-t)m} V(T)$, and
    - (2.12) is $V(t) = e^{-r(T-t)} V(T)$.

- The last formula has to be suitably modified if a different compounding method is used.
  - Using periodic compounding with frequency m, we need to solve the equation

$$B(t,T) = \left(1 + \frac{r}{m}\right)^{-m(T-t)}.$$

  - In the case of continuous compounding the equation for the implied rate satisfies

$$B(t,T) = e^{-r(T-t)}.$$

- Of course all these different implied rates are equivalent to one another, since the bond price does not depend on the compounding method used.

Remark 2.23

- In general, the implied interest rate may depend on the trading time t as well as on the maturity time T.
- This is an important issue to be discussed in Chapter 9.
- For the time being, we adopt the simplifying assumption that the interest rate remains constant throughout the period up to maturity.

Exercise

Exercise 2.28

Exercise 2.29

Note:

- $B(0, T)$ is the discount factor and
- $B(0, T)^{-1}$ is the growth factor for each compounding method.

These universal factors are all needed to compute the time value of money, without resorting to the corresponding interest rates.

However, interest rates are useful because they are more intuitive.

- For an average bank customer the information that a one-year \$100 bond is selling for \$92.59 may not be as clear as the equivalent statement that a deposit will earn 8% interest if kept for one year.

Outline

Coupon Bonds

Bonds promising a sequence of payments are called *coupon bonds*.

- These payments consist of
  - the face value due at maturity, and
  - *coupons* paid regularly (typically annually, semi-annually, or quarterly), with the last coupon due at maturity.
- The assumption of constant interest rates allows us to compute the price of a coupon bond by discounting all the future payments.

Example 2.24

- Consider a bond with face value $F = 100$ dollars maturing in five years, $T = 5$, with coupons of $C = 10$ dollars paid annually, the last one at maturity.
- This means a stream of payments of 10, 10, 10, 10, 110 dollars at the end of each consecutive year.
- Given the continuous compounding rate r, say 12%, we can find the price of the bond:

$$V(0) = 10e^{-r} + 10e^{-2r} + 10e^{-3r} + 10e^{-4r} + 110e^{-5r} \cong 90.27$$

dollars.

Exercise

Exercise 2.30

Exercise 2.31

Example 2.25

- We continue Example 2.24.
- After one year, once the first coupon is cashed, the bond becomes a four-year bond worth

$$V(1) = 10e^{-r} + 10e^{-2r} + 10e^{-3r} + 110e^{-4r} \cong 91.78$$

  dollars.
- Observe that the total wealth at time 1 is

$$V(1) + C = V(0)e^{r}.$$

- Six months later the bond will be worth

$$V(1.5) = 10e^{-0.5r} + 10e^{-1.5r} + 10e^{-2.5r} + 110e^{-3.5r} \cong 97.45$$

  dollars.

- After four years the bond will become a zero-coupon bond with face value \$110 and price

$$V(4) = 110e^{-r} \cong 97.56$$

dollars.

[End of Example]

- An investor may choose to sell the bond at any time prior to maturity.
- The price at that time can once again be found by discounting all the payments due at later times.

Exercise

Exercise 2.32

Exercise 2.33

Remark 2.26

The practice with respect to quoting bond prices between coupon payments is somewhat complicated.

- The present value of future payments, called the *dirty price*, is the price the bond would fetch when sold between coupon payments.
- *Accrued interest* accumulated since the last coupon payment is evaluated by applying the simple interest rule.
- The *clean price* is then quoted by subtracting accrued interest from the dirty price.

- The coupon can be expressed as a fraction of the face value.
- Assuming that coupons are paid annually, we shall write $C = iF$, where i is called the *coupon rate*.

Proposition 2.27

Proposition 2.27

Whenever coupons are paid annually, the coupon rate is equal to the interest rate for annual compounding if and only if the price of the bond is equal to its face value.
In this case we say that the bond is trading *at par*.

Proof

- To avoid cumbersome notation we restrict ourselves to an example.
- Suppose that annual compounding with $r = i$ applies, and consider a bond with face value $F = 100$ maturing in three years, $T = 3$.
- Then the price of the bond is

$$
\begin{aligned}
V(0) &= \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{F+C}{(1+r)^3} \\
&= \frac{rF}{1+r} + \frac{rF}{(1+r)^2} + \frac{F(1+r)}{(1+r)^3} \\
&= \frac{rF}{1+r} + \frac{rF}{(1+r)^2} + \frac{F}{(1+r)^2} \\
&= \frac{rF}{1+r} + \frac{F(1+r)}{(1+r)^2} = F.
\end{aligned}
$$

- Conversely, note that

$$\frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{F+C}{(1+r)^3}$$

is one-to-one as a function of r (in fact, a strictly decreasing function), so it assumes the value F exactly once, and we know this happens for $r = i$.

[End of Proof]

Remark 2.28

- If a bond sells below the face value (i.e., traded at discount or below par), it means that the implied interest rate is higher than the coupon rate
  - (since the price of a bond decreases when the interest rate goes up).
- If the bond price is higher than the face value (i.e., traded at premium or above par), it means that the interest rate is lower than the coupon rate.
- This is important information in real circumstances, where the bond price is determined by the market and gives an indication of the level of interest rates.

Exercise

Exercise 2.34

Outline

Money Market Account

- An investment in the money market can be realised by means of a financial intermediary (typically an investment bank), who buys and sells bonds on behalf of its customers (thus reducing transaction costs).
- The risk-free position of an investor is given by the level of his or her account with the bank.
- It is convenient to think of this account as a tradable asset, (which is indeed the case,) since the bonds themselves are tradable.
    - A long position in the money market involves buying the asset, that is, investing money.
    - A short position amounts to borrowing money.

1) First, consider an investment in a zero-coupon bond closed prior to maturity.

- An initial amount $A(0)$ invested in the money market makes it possible to purchase $A(0)/B(0,T)$ bonds.
- The value of each bond will fetch

$$B(t,T) = e^{-(T-t)r} = e^{rt}e^{-rT} = e^{rt}B(0,T)$$

at time t.

- As a result, the investment will reach

$$A(t) = \frac{A(0)}{B(0,T)}B(t,T) = A(0)e^{rt}$$

at time $t \leq T$.

Exercise

Exercise 2.35

Exercise 2.36

Exercise 2.37

- The investment in a bond has a finite time horizon.
- It will be terminated with $A(T) = A(0)e^{rT}$ at the maturity time T of the bond.
- To extend the position in the money market beyond T one can reinvest the amount $A(T)$ into a bond newly issued at time T, maturing at $T' > T$.
- Taking $A(T)$ as the initial investment with T playing the role of the starting time, we have

$$A(t') = A(T)e^{r(t'-T)} = A(0)e^{rt'}$$

for $T \le t' \le T'$.

- By repeating this argument, we readily arrive at the conclusion that an investment in the money market can be prolonged for as long as required, the formula

$$A(t) = A(0)e^{rt}$$

- (2.15) being valid for all $t \geq 0$.

Exercise

Exercise 2.38

2) We can also consider coupon bonds as a tool to manufacture an investment in the money market.

- Suppose that the first coupon C is due at time t.
- At time 0 we buy $A(0)/V(0)$ coupon bonds.
- At time t we cash the coupon and sell the bond for $V(t)$, receiving the total sum $C + V(t) = V(0)e^{rt}$ (see Example 2.25).
- Because the interest rate is constant, this sum of money is certain.
- In this way we have effectively created a zero-coupon bond with face value $V(0)e^{rt}$ maturing at time t.
- It means that the scheme worked out above for zero-coupon bonds applies to coupon bonds as well, resulting in the same formula (2.15) for $A(t)$.

Exercise

Exercise 2.39

- As we have seen, under the assumption that the interest rate is constant, the function $A(t)$ does not depend on the way the money market account is constructed, that is, it neither depends on the types of bonds selected for investment nor on the method of extending the investment beyond the maturity of the bonds.

- Throughout most of this course we shall assume $A(t)$ to be deterministic and known.
- Indeed, we assume that $A(t) = e^{rt}$ (implying that $A(0) = 1$), where r is a constant interest rate.
- Variable interest rates and a random money market account will be studied in Chapter 9.

Case and Discussion

Case 2

- Consider a do-it-yourself pension fund based on regular savings invested in a bank account attracting interest at 5% per annum. When you retire after 40 years, you want to receive a pension equal to 50% of your final salary and payable for 20 years. Your earnings are assumed to grow at 2% annually, and you want the pension payments to grow at the same rate.

Case and Discussion

Discussion

It is clear that we have to save some money on a regular basis.The simplest method would be to put away the same amount each year. However, since some growth is assumed, this fixed amount might be relatively large as compared to the salary income in the early years. So we formulate the following question: what fixed percentage of your salary should you be paying into this pension fund?

For simplicity we assume that by salary we mean the annual salary. (The reader is encouraged to analyse the version with monthly payments.) We also make the bold assumption that the interest rate will remain constant. It will be variable, certainly, but during such a long term the fluctuations will have an averaging effect, lending some credibility to the result of our calculations (which, nevertheless, have to be treated as crude estimates).

In the solution to Exercise 2.17, formula (2.9) for the present value of an annuity is extended to the case of payments growing at a constant rate. For simplicity, suppose that all payments are made annually at the end of each year. Let S be the initial salary and let x denote the percentage of the salary to be invested in the pension fund. Then the present value of the savings will be

$$V(0) = \sum_{n=1}^{40} \frac{xS(1+g)^n}{(1+r)^n},$$

where $g = 2\%$ is the growth rate and $r = 5\%$ is the interest rate.

Employ (2.8) with $q = \frac{1+g}{1+r}$ to get

$$V(0) = xS(0)GAF(r, g, N)$$

with the *growing annuity factor* given by

$$GAF(r, g, N) = \frac{1+g}{r-g}\left(1 - \frac{(1+g)^N}{(1+r)^N}\right),$$

where N is the number of years. In the case in hand

$$GAF(5\%, 2\%, 40) = 23.34.$$

After 40 years you will accumulate the amount $V(40) = V(0)(1+r)^{40}$, which then becomes the starting capital for the retirement period.

The final salary will be $S(1+g)^{40}$ after 40 years, and we want the initial pension to be 50% of that, growing at rate g in the following years. The pension will also be a growing annuity with present value (the âĂŸpresentâĂŹ here being the end of year 40) $\frac{1}{2}S(1+g)^{40}GAF(r,g,20)$, which must be equal to the accumulated capital $V(40)$. This gives an equation for x,

$$xS(1+r)^{40}GAF(r,g,40) = \frac{1}{2}S(1+g)^{40}GAF(r,g,20),$$

with solution $x = 10.05\%$. It would be sufficient to pay just over 10% of the salary into this pension scheme to achieve its objectives.

However, in the course of 60 years the rates r and g will certainly fluctuate. It is therefore interesting to discuss the sensitivity of the solution to changes in those rates. The value of x for a range of values of r and g are shown in Table 2.1.

	$r = 4\%$	$r = 5\%$	$r = 6\%$
$g = 1\%$	9.69%	7.24%	5.24%
$g = 2\%$	13.70%	10.05%	7.34%
$g = 3\%$	18.62%	13.78%	10.14%

Table: Table 2.1 Percentage x of salary paid into the pension scheme

Observe that x depends quite strongly on the premium rate of interest $r - g$ above the growth rate g, but much less so on g (or r) itself when the value of $r - g$ is fixed. (For example, examine the values of x on the diagonal, corresponding to $r - g = 3\%$.) While r and g will tend to increase or decrease together with the rate of inflation, the difference $r - g$ can be expected to remain relatively stable over the years, lending some justification to the numerical results obtained under the manifestly false assumption of constant rates.