3
Portfolio Management

Case 3

The case in Chapter 2 involved a pension fund invested in a risk-free bank account. However, you may be willing to accept some reasonable risk in the hope that this could lead to a more profitable pension scheme. Consider investing in some risky securities in addition to the bank account.

3.1 Risk and Return

An investment in a risky security always carries the burden of possible losses or poor performance. In this chapter we analyse the advantages of spreading the investment among several securities to keep the inevitable risk under control.

In Chapter 1 we considered one risky security in a single time step. Here, we shall assume a similar simple single time step setup but consider many risky securities. The length of the time step can be arbitrary, and our typical example will be one year. We shall keep the same notation as in Chapter 1, namely 0 for the beginning and T for the end of the time period.

M. Capiński, T. Zastawniak, *Mathematics for Finance*,
Springer Undergraduate Mathematics Series,

3.1.1 Expected Return

Suppose we make a single-period investment in some stock with known current price $S(0)$. The future price $S(T)$ is unknown, hence assumed to be a random variable

$$S(T) : \Omega \to [0, +\infty)$$

on a probability space Ω. The return

$$K = \frac{S(T) - S(0)}{S(0)}$$

is a random variable with expected value $\mathbb{E}(K) = \mu_K = \mu$. We introduce the convention of using the Greek letter μ for expectations of various random returns, with subscripts indicating the context, if necessary. By the linearity of mathematical expectation

$$\mu = \frac{\mathbb{E}(S(T)) - S(0)}{S(0)}.$$

The relationships between the prices and returns can be written as

$$S(T) = S(0)(1 + K),$$
$$\mathbb{E}(S(T)) = S(0)(1 + \mu),$$

which indicates the possibility of reversing the approach: given the returns we can find the prices.

Example 3.1

Suppose that the probability space Ω is finite. We can consider the elements of Ω to be the possible economic scenarios, writing $\Omega = \{\omega_1, \ldots, \omega_N\}$. The possible values of $S(T)$ form a vector $(S^{\omega_1}(T), \ldots, S^{\omega_N}(T))$ in $\mathbb{R}^N$. It is natural to equip Ω with the σ-field $\mathcal{F} = 2^\Omega$ of all subsets. To define a probability measure $P : \mathcal{F} \to [0, 1]$ it is sufficient to prescribe its values on single element sets, $P(\{\omega_i\}) = p_i$, by choosing $p_i \in [0, 1]$ such that $\sum_{i=1}^N p_i = 1$. Denoting the vector of probabilities by $p = (p_1, \ldots, p_N)$, we can write the expected price at the end of the period as

$$\mathbb{E}(S(T)) = \sum_{i=1}^N S^{\omega_i}(T) p_i = \langle S(T), p \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of vectors in $\mathbb{R}^N$. The return

$$K^{\omega_i} = \frac{S^{\omega_i}(T) - S(0)}{S(0)}, \quad K = (K^{\omega_1}, \ldots, K^{\omega_N})$$

has expected value

$$\mathbb{E}(K) = \langle K, p \rangle = \sum_{i=1}^N K^{\omega_i} p_i.$$

Example 3.2

Now we consider some special kinds of investment, which can be cast into the general framework of portfolios. The following considerations are motivated by the practical regulations encountered in some financial markets:

1. Margin purchases (leveraged purchases). Suppose an investment in stock is partially financed by a loan obtained at an interest rate $r_l \geq r$, higher that the rate r for risk-free investments. That is, we buy one share for $S(0)$, investing $wS(0)$ of our own money and borrowing $(1 - w)S(0)$, $w \in [0, 1]$.
2. Short selling. We borrow one share, paying some percentage of its value as collateral, say $cS(0)$, where $c \in [0, 1]$. This collateral attracts interest at a lower rate r_c than that available for risk-free investments, $0 \leq r_c \leq r$. We sell the stock and can invest the proceeds at rate r. At the end of the period we repurchase and return the stock, and recover the collateral. (For mathematical convenience we often take $c = 0$, a valid assumption for some large investors.)

Exercise 3.1

Write down the formulae for the return and expected return on a) a leveraged purchase, and b) a short position in stock.

3.1.2 Standard Deviation as Risk Measure

First of all, we need to identify a suitable quantity to measure risk. An investment in zero-coupon bonds held to maturity gives a return free of risk, 6% say, in which case the measure of risk should be equal to zero. If the return on an investment is, say 11% or 13%, depending on the market scenario, then the risk is clearly lower as compared with an investment returning 2% or 22%, respectively.

However, the spread of return values can hardly be used to measure risk because it ignores the probabilities. If the return rate is 22% with probability 0.99 and 2% with probability 0.01, the risk can be considered quite small, whereas the same rates of return occurring with probability 0.5 each would indicate a rather more risky investment.

The uncertainty is understood as the scatter of returns around some reference point. A natural candidate for the reference value is the expected return. The extent of scatter can be conveniently measured by standard deviation. This notion takes care of two aspects of risk:

1) the distances between possible values and the reference point;

2) the probabilities of attaining these values.

Definition 3.3

By (the measure of) ***risk*** we mean the standard deviation

$$\sigma_K = \sqrt{\mathrm{Var}(K)}$$

of the return K,

$$\mathrm{Var}(K) = \mathbb{E}(K - \mu)^2 = \mathbb{E}(K^2) - \mu^2$$

being the variance of the return.

Exercise 3.2

Compute the risk as measured by the standard deviations $\sigma_{K_1}, \sigma_{K_2}, \sigma_{K_3}$ for each of the following three investment projects, where the returns K_1, K_2, K_3 depend on the market scenario:

Scenario	Probability	Return K_1	Return K_2	Return K_3
ω_1	0.25	12%	11%	2%
ω_2	0.75	12%	13%	22%

Which of these is the most risky and the least risky project?

Exercise 3.3

Consider two scenarios, ω_1 with probability $\frac{1}{4}$ and ω_2 with probability $\frac{3}{4}$. Suppose that the return on some security is $K_1^{\omega_1} = -2\%$ in the first scenario and $K_1^{\omega_2} = 8\%$ in the second scenario. If the return on another security is $K_2^{\omega_1} = -4\%$ in the first scenario, find the return $K_2^{\omega_2}$ in the other scenario such that the two securities have the same risk.

Example 3.4

Let the return on an investment be $K = 3\%$ or -1%, both with probability 0.5. Then the risk is

$$\sigma_K = 0.02.$$

Now suppose that the return on another investment is double that on the first investment, being equal to $2K = 6\%$ or -2%, also with probability 0.5 each. Then the risk of the second investment will be

$$\sigma_{2K} = 0.04.$$

The risk as measured by the standard deviation is doubled.

This illustrates the following general rule:

$$\sigma_{aK} = |a|\,\sigma_K,$$
$$\mathrm{Var}(aK) = a^2\mathrm{Var}(K),$$

for any real number a.

3.2 Two Securities

We begin a detailed discussion of the relationship between risk and expected return in the simple situation of a portfolio with just two risky securities.

Example 3.5

Suppose that the prices of two stocks behave as follows:

Scenario	Probability	Return K_1	Return K_2
ω_1	0.5	10%	−5%
ω_2	0.5	−5%	10%

If we split our money equally between these two stocks, then we shall earn 5% in each scenario (losing 5% on one stock, but gaining 10% on the other). Even though an investment in either stock separately involves risk, we have reduced the overall risk to nil by splitting the investment between the two stocks. This is a simple example of diversification, which is particularly effective here because the returns have perfect negative correlation.

We introduce weights to describe the allocation of funds between the securities as a convenient alternative to specifying portfolios in terms of the number of shares of each security. The *weights* are defined by

$$w_1 = \frac{x_1 S_1(0)}{V(0)}, \quad w_2 = \frac{x_2 S_2(0)}{V(0)},$$

where x_k denotes the number of shares of kind $k = 1, 2$ in the portfolio. This means that w_k is the percentage of the initial value of the portfolio invested in security k. Observe that the weights always add up to 100%,

$$w_1 + w_2 = \frac{x_1 S_1(0) + x_2 S_2(0)}{V(0)} = \frac{V(0)}{V(0)} = 1. \tag{3.1}$$

If short selling is allowed, then one of the weights may be negative and the other one greater than 100%.

Example 3.6

Suppose that the prices of two kinds of stock are $S_1(0) = 30$ and $S_2(0) = 40$ dollars. We prepare a portfolio worth $V(0) = 1,000$ dollars by purchasing $x_1 = 20$ shares of stock 1 and $x_2 = 10$ shares of stock 2. The allocation of funds between the two securities is

$$w_1 = \frac{30 \times 20}{1,000} = 60\%, \quad w_2 = \frac{10 \times 40}{1,000} = 40\%.$$

These are the weights in the portfolio. If the stock prices change to $S_1(T) = 35$ and $S_2(T) = 39$ dollars, then the portfolio will be worth $V(T) = 20 \times 35 + 10 \times 39 = 1,090$ dollars. Observe that this amount is no longer split between the two securities as 60% to 40%, but as follows:

$$\frac{20 \times 35}{1,090} \cong 64.22\%, \quad \frac{10 \times 39}{1,090} \cong 35.78\%,$$

even though the actual number of shares of each stock in the portfolio remains unchanged.

Remark 3.7

In reality the number of shares has to be an integer, which places a constraint on possible weights. To simplify matters, as in Chapter 1 we shall assume divisibility of assets. This means that the weights can be any real numbers that add up to one. Since not all real markets allow short selling, sometimes we need to distinguish a special case when the weights are non-negative.

Example 3.8

Suppose that a portfolio worth $V(0) = 1,000$ dollars is constructed by taking a long position in stock 1 and a short position in stock 2 from Example 3.6 with

weights $w_1 = 120\%$ and $w_2 = -20\%$. The portfolio will consist of

$$x_1 = w_1 \frac{V(0)}{S_1(0)} = 120\% \times \frac{1,000}{30} = 40,$$
$$x_2 = w_2 \frac{V(0)}{S_2(0)} = -20\% \times \frac{1,000}{40} = -5$$

shares of type 1 and 2. If the stock prices change as in Example 3.6, then this portfolio will be worth

$$V(T) = x_1 S_1(T) + x_2 S_2(T) = V(0)\left(w_1 \frac{S_1(T)}{S_1(0)} + w_2 \frac{S_2(T)}{S_2(0)}\right)$$
$$= 1,000\left(120\% \times \frac{35}{30} - 20\% \times \frac{39}{40}\right) = 1,205$$

dollars, benefiting from both the rise in the price of stock 1 and the fall of stock 2. However, a small investor might face some restrictions on short selling. For example, it may be necessary to provide a cash collateral equal to 50% of the amount raised by shorting stock 2. The investor would then have to forego the interest that could otherwise have been earned on the collateral, which will need to be subtracted from the final value $V(T)$ of the portfolio.

Exercise 3.4

Compute the value $V(T)$ of a portfolio worth initially $V(0) = 100$ dollars that consists of two securities with weights $w_1 = 25\%$ and $w_2 = 75\%$, given that the security prices are $S_1(0) = 45$ and $S_2(0) = 33$ dollars initially, changing to $S_1(T) = 48$ and $S_2(T) = 32$ dollars.

We can see in Example 3.8 and Exercise 3.4 that $V(T)/V(0)$ depends on the security prices only through the ratios $S_1(T)/S_1(0) = 1 + K_1$ and $S_2(T)/S_2(0) = 1 + K_2$, and on the positions in the securities only through the weights w_1, w_2. This indicates that the return on the portfolio should depend only on the weights w_1, w_2 and the returns K_1, K_2 on the two securities.

Proposition 3.9

The return K_V on a portfolio consisting of two securities is the weighted average

$$K_V = w_1 K_1 + w_2 K_2, \tag{3.2}$$

where w_1 and w_2 are the weights and K_1 and K_2 are the returns on the two components.

Proof

Suppose that the portfolio consists of x_1 shares of security 1 and x_2 shares of security 2. Then the initial and final values of the portfolio are

$$\begin{aligned} V(0) &= x_1 S_1(0) + x_2 S_2(0), \\ V(T) &= x_1 S_1(0)(1+K_1) + x_2 S_2(0)(1+K_2) \\ &= V(0)\,(w_1(1+K_1) + w_2(1+K_2)) \\ &= V(0)(1 + w_1 K_1 + w_2 K_2). \end{aligned}$$

As a result, the return on the portfolio is

$$K_V = \frac{V(T) - V(0)}{V(0)} = w_1 K_1 + w_2 K_2.$$

□

Exercise 3.5

Find the return on a portfolio consisting of two kinds of stock with weights $w_1 = 30\%$ and $w_2 = 70\%$ if the returns on the components are as follows:

Scenario	Return K_1	Return K_2
ω_1	12%	−4%
ω_2	−10%	7%

3.2.1 Risk and Expected Return on a Portfolio

The expected return on a portfolio consisting of two securities can be expressed in terms of the weights and the expected returns on the components as

$$\mathbb{E}(K_V) = w_1 \mathbb{E}(K_1) + w_2 \mathbb{E}(K_2). \tag{3.3}$$

This follows at once from (3.2) by the additivity of mathematical expectation.

Example 3.10

Consider three scenarios with the probabilities given below (a trinomial model). Let the returns on two different stocks in these scenarios be as follows:

Scenario	Probability	Return K_1	Return K_2
ω_1 (recession)	0.2	−10%	−30%
ω_2 (stagnation)	0.5	0%	20%
ω_3 (boom)	0.3	10%	50%

The expected returns are

$$\begin{aligned} \mathbb{E}(K_1) &= -0.2 \times 10\% + 0.5 \times \ \ 0\% + 0.3 \times 10\% = \ \ 1\%, \\ \mathbb{E}(K_2) &= -0.2 \times 30\% + 0.5 \times 20\% + 0.3 \times 50\% = 19\%. \end{aligned}$$

Suppose that $w_1 = 60\%$ of available funds is invested in stock 1 and 40% in stock 2. The expected return on such a portfolio will be

$$\begin{aligned} \mathbb{E}(K_V) &= w_1 \mathbb{E}(K_1) + w_2 \mathbb{E}(K_2) \\ &= 0.6 \times 1\% + 0.4 \times 19\% = 8.2\%. \end{aligned}$$

Exercise 3.6

Compute the weights in a portfolio consisting of two kinds of stock if the expected return on the portfolio is to be $\mathbb{E}(K_V) = 10\%$, given the following information on the returns on stock 1 and 2:

Scenario	Probability	Return K_1	Return K_2
ω_1	0.1	−10%	10%
ω_2	0.3	0%	−5%
ω_3	0.6	15%	20%

To compute the variance of K_V we need to know not only the variances of the returns K_1 and K_2 on the components in the portfolio, but also the covariance between the two returns.

Theorem 3.11

The variance of the return on a portfolio is given by

$$\mathrm{Var}(K_V) = w_1^2 \mathrm{Var}(K_1) + w_2^2 \mathrm{Var}(K_2) + 2 w_1 w_2 \mathrm{Cov}(K_1, K_2). \tag{3.4}$$

Proof

Substituting $K_V = w_1 K_1 + w_2 K_2$ and collecting the terms with w_1^2, w_2^2 and $w_1 w_2$, we compute

$$\begin{aligned} \mathrm{Var}(K_V) &= \mathbb{E}(K_V^2) - \mathbb{E}(K_V)^2 \\ &= w_1^2[\mathbb{E}(K_1^2) - \mathbb{E}(K_1)^2] + w_2^2[\mathbb{E}(K_2^2) - \mathbb{E}(K_2)^2] \\ &\quad + 2 w_1 w_2 [\mathbb{E}(K_1 K_2) - \mathbb{E}(K_1)\mathbb{E}(K_2)] \\ &= w_1^2 \mathrm{Var}(K_1) + w_2^2 \mathrm{Var}(K_2) + 2 w_1 w_2 \mathrm{Cov}(K_1, K_2). \end{aligned}$$

□

To avoid clutter, we introduce the following notation:

$$\begin{aligned}
\mu_V &= \mathbb{E}(K_V), & \sigma_V &= \sqrt{\operatorname{Var}(K_V)},\\
\mu_1 &= \mathbb{E}(K_1), & \sigma_1 &= \sqrt{\operatorname{Var}(K_1)},\\
\mu_2 &= \mathbb{E}(K_2), & \sigma_2 &= \sqrt{\operatorname{Var}(K_2)}.
\end{aligned}$$
$$c_{12} = \operatorname{Cov}(K_1, K_2).$$

Formulae (3.3) and (3.4) can be written as

$$\mu_V = w_1\mu_1 + w_2\mu_2, \tag{3.5}$$
$$\sigma_V^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2c_{12}. \tag{3.6}$$

We shall also use the correlation coefficient

$$\rho_{12} = \frac{c_{12}}{\sigma_1\sigma_2}. \tag{3.7}$$

Note that the correlation coefficient is undefined when $\sigma_1\sigma_2 = 0$. This condition means that at least one of the assets is risk free.

Remark 3.12

For risky securities the returns K_1 and K_2 are non-constant random variables. Because of this $\sigma_1\sigma_2 > 0$ and ρ_{12} is well defined.

Example 3.13

We use the following data:

Scenario	Probability	Return K_1	Return K_2
ω_1	0.2	-10%	5%
ω_2	0.4	0%	30%
ω_3	0.4	20%	-5%

We want to compare the risk of a portfolio such that $w_1 = 40\%$ and $w_2 = 60\%$ with the risk of each of its components. Direct computations give

$$\sigma_1^2 \cong 0.0144, \quad \sigma_2^2 \cong 0.0254, \quad \rho_{12} \cong -0.6065.$$

By (3.6)

$$\begin{aligned}
\sigma_V^2 &\cong (0.4)^2 \times 0.0144 + (0.6)^2 \times 0.0254\\
&\quad +2 \times 0.4 \times 0.6 \times (-0.6065) \times \sqrt{0.0144} \times \sqrt{0.0254}\\
&\cong 0.00588.
\end{aligned}$$

Observe that the variance σ_V^2 is smaller than both σ_1^2 and σ_2^2.

Example 3.14

Consider another portfolio with weights $w_1 = 10\%$ and $w_2 = 90\%$, all other things being the same as in Example 3.13. Then

$$\begin{aligned}
\sigma_V^2 &\cong (0.1)^2 \times 0.0144 + (0.9)^2 \times 0.0254\\
&\quad +2 \times 0.1 \times 0.9 \times (-0.6065) \times \sqrt{0.0144} \times \sqrt{0.0254}\\
&\cong 0.01863,
\end{aligned}$$

which is between σ_1^2 and σ_2^2.

Proposition 3.15

The variance σ_V^2 of a portfolio cannot exceed the greater of the variances σ_1^2 and σ_2^2 of the components,

$$\sigma_V^2 \le \max\{\sigma_1^2, \sigma_2^2\},$$

if short sales are not allowed.

Proof

Let us assume that $\sigma_1^2 \le \sigma_2^2$. If short sales are not allowed, then $w_1, w_2 \ge 0$ and

$$w_1\sigma_1 + w_2\sigma_2 \le (w_1 + w_2)\sigma_2 = \sigma_2.$$

Since the correlation coefficient satisfies $-1 \le \rho_{12} \le 1$, it follows that

$$\begin{aligned}
\sigma_V^2 &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2\\
&\le w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\\
&= (w_1\sigma_1 + w_2\sigma_2)^2 \le \sigma_2^2.
\end{aligned}$$

If $\sigma_1^2 \ge \sigma_2^2$, the proof is analogous. □

Example 3.16

Now consider a portfolio with weights $w_1 = -50\%$ and $w_2 = 150\%$ (allowing short sales of security 1), all the remaining data being the same as in Example 3.13. The variance of this portfolio is

$$\begin{aligned}
\sigma_V^2 &\cong (-0.5)^2 \times 0.0144 + (1.5)^2 \times 0.0254\\
&\quad +2 \times (-0.5) \times 1.5 \times (-0.6065) \times \sqrt{0.0144} \times \sqrt{0.0254}\\
&\cong 0.2795,
\end{aligned}$$

which is greater than both σ_1^2 and σ_2^2.

Exercise 3.7

Using the data in Example 3.13, find the weights in a portfolio with expected return $\mu_V = 50\%$ and compute the risk σ_V of this portfolio.

3.2.2 Feasible Set

The collection of all portfolios that can be manufactured by investing in two given assets is called the ***feasible*** (or ***attainable***) ***set***. Each portfolio can be represented by a point with coordinates σ_V and μ_V in the σ, μ plane.

We want to investigate the shape of the feasible set in this plane, which will allow valuable insights. The set consists of all points with coordinates

$$\begin{aligned}\mu_V &= w_1\mu_1 + w_2\mu_2,\\ \sigma_V^2 &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2c_{12},\end{aligned}$$

where $w_1, w_2 \in \mathbb{R}$ and

$$1 = w_1 + w_2.$$

Portfolios in the feasible set can be parameterised by one of the weights. Here we shall use $s = w_1$ as a parameter. Then $1 - s = w_2$, and the above expressions for σ_V^2 and μ_V can be written as

$$\mu_V = s\mu_1 + (1-s)\mu_2, \tag{3.8}$$

$$\sigma_V^2 = s^2\sigma_1^2 + (1-s)^2\sigma_2^2 + 2s(1-s)c_{12}, \tag{3.9}$$

where $s \in \mathbb{R}$.

We begin with finding the portfolio with the smallest variance (that is, smallest risk) among all feasible portfolios.

Proposition 3.17

If $\rho_{12} < 1$ or $\sigma_1 \neq \sigma_2$, then σ_V^2 as a function of s attains its minimum value at

$$s_0 = \frac{\sigma_2^2 - c_{12}}{\sigma_1^2 + \sigma_2^2 - 2c_{12}}. \tag{3.10}$$

The corresponding values of the expected return μ_V and variance σ_V^2 are

$$\mu_0 = \frac{\mu_1\sigma_2^2 + \mu_2\sigma_1^2 - (\mu_1 + \mu_2)c_{12}}{\sigma_1^2 + \sigma_2^2 - 2c_{12}}, \tag{3.11}$$

$$\sigma_0^2 = \frac{\sigma_1^2\sigma_2^2 - c_{12}}{\sigma_1^2 + \sigma_2^2 - 2c_{12}}. \tag{3.12}$$

If $\rho_{12} = 1$ and $\sigma_1 = \sigma_2$, then all feasible portfolios have the same variance equal to $\sigma_1^2 = \sigma_2^2$.

Proof

To find the value of s for which σ_V^2 in (3.9) attains a minimum we differentiate σ_V^2 with respect to s and equate the derivative to zero. This gives an equation for s,

$$\frac{d(\sigma_V^2)}{ds} = 2s\left(\sigma_1^2 + \sigma_2^2 - 2c_{12}\right) - 2\left(\sigma_2^2 - c_{12}\right) = 0,$$

which has a solution s_0 given by (3.10) as long as the denominator in (3.10) is non-zero. This is guaranteed by the condition that $\rho_{12} < 1$ or $\sigma_1 \neq \sigma_2$. Indeed, if $\rho_{12} < 1$, then

$$\sigma_1^2 + \sigma_2^2 - 2c_{12} > \sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2 = (\sigma_1 - \sigma_2)^2 \geq 0,$$

and if $\sigma_1 \neq \sigma_2$, then

$$\sigma_1^2 + \sigma_2^2 - 2c_{12} \geq \sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2 = (\sigma_1 - \sigma_2)^2 > 0.$$

For the second derivative this implies that

$$\frac{d^2(\sigma_V^2)}{ds^2} = 2\left(\sigma_1^2 + \sigma_2^2 - 2c_{12}\right) > 0.$$

We can conclude that σ_V^2 attains its minimum at s_0. The expressions for μ_0 and σ_0^2 follow by substituting s_0 for s in (3.8) and (3.9). If $\rho_{12} = 1$ and $\sigma_1 = \sigma_2$, then $c_{12} = \sigma_1\sigma_2$, and for any $s \in \mathbb{R}$

$$\sigma_V^2 = s^2\sigma_1^2 + (1-s)^2\sigma_2^2 + 2s(1-s)c_{12} = (s\sigma_1 + (1-s)\sigma_2)^2 = \sigma_1^2 = \sigma_2^2.$$

□

The curve described by the parametric equations (3.8), (3.9), which represents the feasible set on the σ, μ plane, turns out to be a branch of a hyperbola. This is shown in Figure 3.1 along with the asymptotes of the hyperbola. The bold segment of the hyperbola corresponds to portfolios without short selling.

Proposition 3.18

Assume that $-1 < \rho_{12} < 1$ and $\mu_1 \neq \mu_2$. Then for each portfolio V in the feasible set, $x = \sigma_V$ and $y = \mu_V$ satisfy the equation of a hyperbola

$$x^2 - A^2(y - \mu_0)^2 = \sigma_0^2 \tag{3.13}$$

with μ_0 and $\sigma_0^2 > 0$ given in Proposition 3.17 and with

$$A^2 = \frac{\sigma_1^2 + \sigma_2^2 - 2c_{12}}{(\mu_1 - \mu_2)^2} > 0.$$

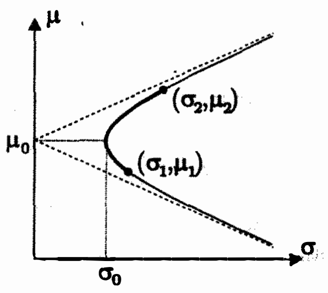

Figure 3.1 Hyperbola representing feasible portfolios on the σ, μ plane

The two asymptotes of the hyperbola are

$$y = \mu_0 \pm \frac{1}{A}x.$$

Proof

From (3.8) we have $s = \frac{\mu_V - \mu_2}{\mu_1 - \mu_2}$. On substituting this expression for s into (3.9), it follows by simple (if slightly tedious) transformations that $x = \sigma_V$ and $y = \mu_V$ satisfy (3.13). Since $-1 < \rho_{12} < 1$ and $\mu_1 \neq \mu_2$, it follows that $A^2 > 0$ and $\sigma_0^2 > 0$, so that (3.13) indeed describes a hyperbola. The equation for the asymptotes follows directly from (3.13). □

The correlation coefficient always satisfies $-1 \leq \rho_{12} \leq 1$. The next proposition is concerned with the two special cases when ρ_{12} assumes one of the extreme values 1 or -1. This means perfect positive or negative correlation between the securities in the portfolio. In these cases the feasible set will have a particularly simple shape, see Figure 3.2.

Proposition 3.19

Suppose that $\rho_{12} = 1$ and $\sigma_1 \neq \sigma_2$. Then $\sigma_V = 0$ if and only if

$$w_1 = -\frac{\sigma_2}{\sigma_1 - \sigma_2}, \quad w_2 = \frac{\sigma_1}{\sigma_1 - \sigma_2}. \tag{3.14}$$

This involves short selling, since either w_1 or w_2 is negative.

Suppose that $\rho_{12} = -1$. Then $\sigma_V = 0$ if and only if

$$w_1 = \frac{\sigma_2}{\sigma_1 + \sigma_2}, \quad w_2 = \frac{\sigma_1}{\sigma_1 + \sigma_2}. \tag{3.15}$$

No short selling is necessary, since both w_1 and w_2 are positive.

Proof

Let $\rho_{12} = 1$ and $\sigma_1 \neq \sigma_2$. Then (3.6) takes the form

$$\sigma_V^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2 = (w_1\sigma_1 + w_2\sigma_2)^2$$

and $\sigma_V^2 = 0$ if and only if $w_1\sigma_1 + w_2\sigma_2 = 0$. This is equivalent to (3.14) because $w_1 + w_2 = 1$.

Now let $\rho_{12} = -1$. Then (3.6) becomes

$$\sigma_V^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 - 2w_1w_2\sigma_1\sigma_2 = (w_1\sigma_1 - w_2\sigma_2)^2$$

and $\sigma_V^2 = 0$ if and only if $w_1\sigma_1 - w_2\sigma_2 = 0$. The last equality is equivalent to (3.15) because $w_1 + w_2 = 1$. (We know that $\sigma_1 + \sigma_2 > 0$ because both securities are risky.) □

Figure 3.2 shows two typical lines representing portfolios with $\rho_{12} = -1$ (left) and $\rho_{12} = 1$ (right). The bold segments correspond to portfolios without short selling.

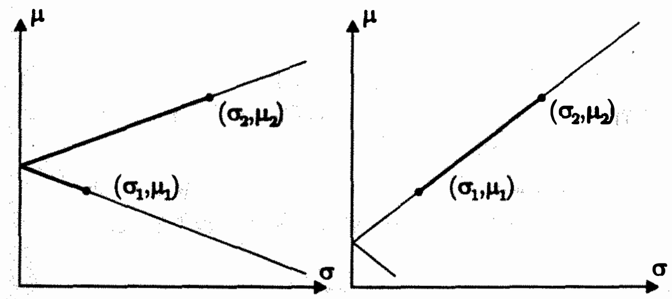

Figure 3.2 Typical portfolio lines with $\rho_{12} = -1$ and 1

Suppose that $\rho_{12} = -1$. It follows from the proof of Proposition 3.19 that $\sigma_V = |w_1\sigma_1 - w_2\sigma_2|$. In addition, $\mu_V = w_1\mu_1 + w_2\mu_2$ by (3.5) and $w_1 + w_2 = 1$ by (3.1). We can choose $s = w_1$ as a parameter. Then $1 - s = w_2$ and

$$\sigma_V = |s\sigma_1 - (1-s)\sigma_2|,$$
$$\mu_V = s\mu_1 + (1-s)\mu_2.$$

These parametric equations describe the line in Figure 3.2 with a broken segment between (σ_1, μ_1) and (σ_2, μ_2). As s decreases, the point (σ_V, μ_V) moves along the line in the direction from (σ_1, μ_1) to (σ_2, μ_2).

If $\rho_{12} = 1$, then $\sigma_V = |w_1\sigma_1 + w_2\sigma_2|$. We choose $s = w_1$ as a parameter once again, and obtain the parametric equations

$$\sigma_V = |s\sigma_1 + (1-s)\sigma_2|,$$
$$\mu_V = s\mu_1 + (1-s)\mu_2$$

of the line in Figure 3.2 with a straight segment between (σ_1, μ_1) and (σ_2, μ_2).

Exercise 3.8

Suppose that there are just two scenarios ω_1 and ω_2 and consider two risky securities with returns K_1 and K_2. Show that $K_1 = aK_2 + b$ for some numbers $a \neq 0$ and b, and deduce that $\rho_{12} = 1$ or -1.

Figure 3.3 illustrates the following corollary, in which we study the shape of the feasible set depending on the correlation coefficient ρ_{12}.

Corollary 3.20

Suppose that $\sigma_1 \leq \sigma_2$. The following five cases are possible:

1) If $\rho_{12} = 1$, then there is a feasible portfolio V with short selling such that $\sigma_V = 0$ (line 1 in Figure 3.3) whenever $\sigma_1 < \sigma_2$. Each portfolio V in the feasible set has the same σ_V whenever $\sigma_1 = \sigma_2$.
2) If $\frac{\sigma_1}{\sigma_2} < \rho_{12} < 1$, then there is a feasible portfolio V with short selling such that $\sigma_V < \sigma_1$, but for each portfolio without short selling $\sigma_V \geq \sigma_1$ (line 2 in Figure 3.3).
3) If $\rho_{12} = \frac{\sigma_1}{\sigma_2}$, then $\sigma_V \geq \sigma_1$ for each feasible portfolio V (line 3 in Figure 3.3).
4) If $-1 < \rho_{12} < \frac{\sigma_1}{\sigma_2}$, then there is a feasible portfolio V without short selling such that $\sigma_V < \sigma_1$ (line 4 in Figure 3.3).
5) If $\rho_{12} = -1$, then there is a feasible portfolio V without short selling such that $\sigma_V = 0$ (line 5 in Figure 3.3).

Proof

1) The case when $\rho_{12} = 1$ is covered in Proposition 3.19.
2) If $\frac{\sigma_1}{\sigma_2} < \rho_{12} < 1$, then $s_0 > 1$. In this case the portfolio V with minimum variance that corresponds to the parameter s_0 involves short selling of security 2 and satisfies $\sigma_V < \sigma_1$. For $s \geq s_0$ the variance σ_V is an increasing function of s, which means that $\sigma_V \geq \sigma_1$ for every portfolio without short selling.

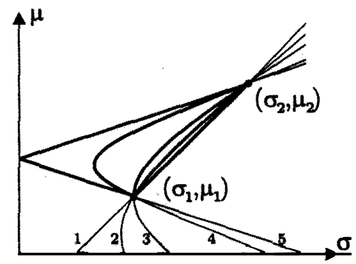

Figure 3.3 Portfolio lines for various values of ρ_{12}

3) If $\rho_{12} = \frac{\sigma_1}{\sigma_2}$, then $s_0 = 1$. As a result, $\sigma_V \geq \sigma_1$ for every portfolio V because σ_1^2 is the minimum variance.
4) If $-1 < \rho_{12} < \frac{\sigma_1}{\sigma_2}$, then $0 < s_0 < 1$, which means that the portfolio V with minimum variance, which corresponds to s_0, involves no short selling and satisfies $\sigma_V < \sigma_1$.
5) The case when $\rho_{12} = -1$ has been covered in Proposition 3.19.

□

The above corollary is important because it shows when it is possible to construct a portfolio with risk lower than that of either of its components. In cases 4) and 5) this is possible without short selling. In cases 1) and 2) this is also possible, but only if short selling is allowed. In case 3) it is impossible to construct such a portfolio.

Example 3.21

Suppose that

$$\sigma_1^2 = 0.0041, \quad \sigma_2^2 = 0.0121, \quad \rho_{12} = 0.9796.$$

Clearly, $\sigma_1 < \sigma_2$ and $\frac{\sigma_1}{\sigma_2} < \rho_{12} < 1$, so this is case 2) in Corollary 3.20. Our task will be to find the portfolio with minimum risk with and without short selling. Using Proposition 3.17, we compute

$$s_0 \cong 2.1663.$$

It follows that in the portfolio with minimum risk the weights should be $w_1 \cong 2.1663$ and $w_2 \cong -1.1663$ if short selling is allowed. Without short selling $w_1 = 1$ and $w_2 = 0$.

Exercise 3.9

Compute the weights in the portfolio with minimum risk for the data in Example 3.13. Does this portfolio involve short selling?

We conclude this section with a brief discussion of portfolios in which one of the securities is risk free. The variance of the risky security is positive, whereas that of the risk-free component is zero.

Proposition 3.22

The standard deviation σ_V of a portfolio consisting of a risky security with expected return μ_1 and standard deviation $\sigma_1 > 0$, and a risk-free security with return R and standard deviation zero depends on the weight w_1 of the risky security as follows:

$$\sigma_V = |w_1|\sigma_1.$$

Proof

Let $\sigma_1 > 0$ and $\sigma_2 = 0$. Then (3.6) reduces to $\sigma_V^2 = w_1^2\sigma_1^2$, and the formula for σ_V follows by taking the square root. □

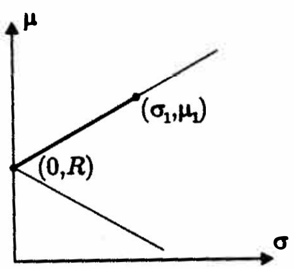

Figure 3.4 Portfolio line for one risky and one risk-free security

The line on the σ, μ plane representing portfolios constructed from one risky and one risk-free security is shown in Figure 3.4. As before, the bold line segment corresponds to portfolios without short selling.

3.3 Several Securities

3.3.1 Risk and Expected Return on a Portfolio

A portfolio constructed from n different securities can be described in terms of their weights

$$w_i = \frac{x_i S_i(0)}{V(0)}, \quad i = 1, \ldots, n,$$

where x_i is the number of shares of type i in the portfolio, $S_i(0)$ is the initial price of security i, and $V(0)$ is the amount initially invested in the portfolio. It will prove convenient to arrange the weights into a one-row matrix

$$\mathbf{w} = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix}.$$

Just like for two securities, the weights add up to one, which can be written in matrix form as

$$1 = \mathbf{w}\mathbf{u}^T, \tag{3.16}$$

where

$$\mathbf{u} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}$$

is a one-row matrix with all n entries equal to 1, $\mathbf{u}^T$ is a one-column matrix, the transpose of $\mathbf{u}$, and the usual matrix multiplication rules apply. The *feasible* (or *attainable*) *set* consists of all portfolios with weights $\mathbf{w}$ satisfying (3.16), called the *feasible* (or *attainable*) *portfolios*.

Suppose that the returns on the securities are $K_1, \ldots, K_n$. The expected returns $\mu_i = \mathbb{E}(K_i)$ for $i = 1, \ldots, n$ will also be arranged into a one-row matrix

$$\mathbf{m} = \begin{bmatrix} \mu_1 & \mu_2 & \cdots & \mu_n \end{bmatrix}.$$

The covariances between returns will be denoted by $c_{ij} = \mathrm{Cov}(K_i, K_j)$. They are the entries of the $n \times n$ *covariance matrix*

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}.$$

The diagonal elements of $\mathbf{C}$ are simply the variances of returns, $c_{ii} = \sigma_i = \mathrm{Var}(K_i)$.

The covariance matrix is symmetric and non-negative definite, see Appendix 10.4. In what follows, we shall assume that $\det \mathbf{C} \neq 0$, which implies that $\mathbf{C}$ has an inverse $\mathbf{C}^{-1}$.

Formula (3.2) for the return on a portfolio extends to the case of n securities:

$$K_V = w_1 K_1 + \cdots + w_n K_n.$$

Proposition 3.23

The expected return $\mu_V = \mathbb{E}(K_V)$ and variance $\sigma_V^2 = \mathrm{Var}(K_V)$ of the return $K_V = w_1 K_1 + \cdots + w_n K_n$ on a portfolio V with weights $\mathbf{w} = [w_1, \ldots, w_n]$ are given by

$$\mu_V = \mathbf{w}\mathbf{m}^{\mathrm{T}}, \tag{3.17}$$
$$\sigma_V^2 = \mathbf{w}\mathbf{C}\mathbf{w}^{\mathrm{T}}. \tag{3.18}$$

Proof

The formula for μ_V follows by the linearity of expectation,

$$\mu_V = \mathbb{E}(K_V) = \mathbb{E}\left(\sum_{i=1}^{n} w_i K_i\right) = \sum_{i=1}^{n} w_i \mu_i = \mathbf{w}\mathbf{m}^{\mathrm{T}}.$$

For σ_V^2 we use the linearity of covariance with respect to each of its arguments,

$$\begin{aligned}\sigma_V^2 &= \mathrm{Var}(K_V) = \mathrm{Var}\left(\sum_{i=1}^{n} w_i K_i\right)\\ &= \mathrm{Cov}\left(\sum_{i=1}^{n} w_i K_i, \sum_{j=1}^{n} w_j K_j\right) = \sum_{i,j=1}^{n} w_i w_j c_{ij}\\ &= \mathbf{w}\mathbf{C}\mathbf{w}^{\mathrm{T}}.\end{aligned}$$

□

Exercise 3.10

Compute the expected return μ_V and standard deviation σ_V of a portfolio consisting of three securities with weights $w_1 = 40\%$, $w_2 = -20\%$, $w_3 = 80\%$, given that the securities have expected returns $\mu_1 = 8\%$, $\mu_2 = 10\%$, $\mu_3 = 6\%$, standard deviations $\sigma_1 = 0.15$, $\sigma_2 = 0.05$, $\sigma_3 = 0.12$ and correlations $\rho_{12} = 0.3$, $\rho_{23} = 0.0$, $\rho_{31} = -0.2$.

3.3.2 Minimum Variance Portfolio

In Proposition 3.17 we found the portfolio with the smallest variance (that is, smallest risk) among all feasible portfolios constructed from two risky assets. Here we shall extend this result to the case of n risky assets.

The portfolio with the smallest variance among all feasible portfolios will be called the *minimum variance portfolio* (MVP). To find this portfolio we need

to minimise the variance $\sigma_V^2 = \mathbf{w}\mathbf{C}\mathbf{w}^{\mathrm{T}}$ over all weights $\mathbf{w}$. Because the weights must add up to 1 this leads to a constrained minimum problem:

$$\min \mathbf{w}\mathbf{C}\mathbf{w}^{\mathrm{T}},$$

where the minimum is taken over all vectors $\mathbf{w} \in \mathbb{R}$ that satisfy the condition

$$\mathbf{w}\mathbf{u}^{\mathrm{T}} = 1.$$

To compute this constrained minimum we can use the method of Lagrange multipliers, see Appendix 10.1.2.

Proposition 3.24

Assume that $\det \mathbf{C} \neq 0$. Then the minimum variance portfolio has weights

$$\mathbf{w}_{\mathrm{MVP}} = \frac{\mathbf{u}\mathbf{C}^{-1}}{\mathbf{u}\mathbf{C}^{-1}\mathbf{u}^{\mathrm{T}}}. \tag{3.19}$$

Proof

We need to find the minimum of $\mathbf{w}\mathbf{C}\mathbf{w}^{\mathrm{T}}$ subject to the constraint $\mathbf{w}\mathbf{u}^{\mathrm{T}} = 1$. According to the method of Lagrange multipliers, we put

$$F(\mathbf{w}, \lambda) = \mathbf{w}\mathbf{C}\mathbf{w}^{\mathrm{T}} - \lambda(\mathbf{w}\mathbf{u}^{\mathrm{T}} - 1).$$

The first order necessary condition gives $2\mathbf{w}\mathbf{C} - \lambda\mathbf{u} = 0$, hence

$$\mathbf{w} = \frac{\lambda}{2}\mathbf{u}\mathbf{C}^{-1}.$$

Substituting this into the constraint, we obtain $\frac{\lambda}{2}\mathbf{u}\mathbf{C}^{-1}\mathbf{u}^{\mathrm{T}} = 1$. The properties of the covariance matrix guarantee that $\mathbf{u}\mathbf{C}^{-1}\mathbf{u}^{\mathrm{T}} \neq 0$, see Appendix 10.4. Therefore, we can compute the Lagrange multiplier λ and substitute the result into the expression for $\mathbf{w}$ to obtain the formula for $\mathbf{w}_{\mathrm{MVP}}$.

We have verified a necessary condition for a minimum. Observe that $\mathbf{w}\mathbf{C}\mathbf{w}^{\mathrm{T}}$ is a quadratic function of the weights, bounded below by 0. It must therefore have a minimum, which completes the proof. □

Exercise 3.11

Among all feasible portfolios constructed using three securities with expected returns $\mu_1 = 0.20$, $\mu_2 = 0.13$, $\mu_3 = 0.17$, standard deviations of returns $\sigma_1 = 0.25$, $\sigma_2 = 0.28$, $\sigma_3 = 0.20$, and correlations between returns $\rho_{12} = 0.30$, $\rho_{23} = 0.00$, $\rho_{31} = 0.15$, find the minimum variance portfolio. What are the weights in this portfolio? Also compute the expected return and standard deviation of this portfolio.

3.3.3 Efficient Frontier

Given the choice between two securities a rational investor will, if possible, choose that with the higher expected return and lower standard deviation, that is, lower risk. This motivates the following definition.

Definition 3.25

We say that a security with expected return μ_1 and standard deviation σ_1 *dominates* another security with expected return μ_2 and standard deviation σ_2 whenever

$$\mu_1 \geq \mu_2 \quad \text{and} \quad \sigma_1 \leq \sigma_2.$$

This definition readily extends to portfolios, which can be considered as securities in their own right.

The ordering is only partial. For example, for two securities with $\mu_1 = 10\%, \sigma_1 = 20\%$ and $\mu_2 = 15\%, \sigma_2 = 22\%$, neither of them dominates the other.

Definition 3.26

A portfolio is called *efficient* if there is no other portfolio, except itself, that dominates it. The subset of efficient portfolios among all feasible portfolios is called the *efficient frontier*.

We assume that every rational investor prefers dominating portfolios to dominated ones. However, different investors may select different portfolios on the efficient frontier, depending on their individual preferences. Given two efficient portfolios with $\mu_1 \leq \mu_2$ and $\sigma_1 \leq \sigma_2$, a cautious person may prefer that with lower risk σ_1 and lower expected return μ_1, while others may choose a portfolio with higher risk σ_2, regarding the higher expected return μ_2 as compensation for increased risk.

To find the efficient frontier we have to recognise and eliminate the dominated portfolios. To this end, for any $\mu \in \mathbf{R}$ consider all feasible portfolios whose expected return is μ. The set of such portfolios is called an *isoexpected line*. All portfolios on an isoexpected line are dominated by that with the smallest variance among them.

Definition 3.27

The family of portfolios V, parameterised by $\mu \in \mathbf{R}$, such that $\mu_V = \mu$ and $\sigma_V^2 \leq \sigma_{V'}^2$ for each portfolio V' with $\mu_{V'} = \mu$ is called the *minimum variance line* (MVL).

To compute the weights of the portfolio on the minimum variance line for any $\mu \in \mathbf{R}$ we shall solve the constrained minimum problem

$$\min \mathbf{w}\mathbf{C}\mathbf{w}^{\mathrm{T}},$$

where the minimum is taken over all vectors $\mathbf{w} \in \mathbf{R}^n$ such that

$$\mathbf{w}\mathbf{m}^{\mathrm{T}} = \mu, \quad \mathbf{w}\mathbf{u}^{\mathrm{T}} = 1.$$

We introduce Lagrange multipliers λ_1, λ_2 and minimise the function

$$G(\mathbf{w}, \lambda_1, \lambda_2) = \mathbf{w}\mathbf{C}\mathbf{w}^{\mathrm{T}} - \lambda_1(\mathbf{w}\mathbf{m}^{\mathrm{T}} - \mu) - \lambda_2(\mathbf{w}\mathbf{u}^{\mathrm{T}} - 1).$$

The gradient of G with respect to $\mathbf{w}$ and the partial derivatives with respect to λ_1, λ_2 give necessary conditions for a minimum, see Appendix 10.1.2:

$$\begin{aligned} 2\mathbf{w}\mathbf{C} - \lambda_1\mathbf{m} - \lambda_2\mathbf{u} &= \mathbf{0}, \\ \mathbf{w}\mathbf{m}^{\mathrm{T}} - \mu &= 0, \\ \mathbf{w}\mathbf{u}^{\mathrm{T}} - 1 &= 0. \end{aligned}$$

The first of these is a system of n equations, $\mathbf{0}$ representing a vector of zeros. Assuming invertibility of $\mathbf{C}$, we solve this system for $\mathbf{w}$

$$2\mathbf{w} = \lambda_1\mathbf{m}\mathbf{C}^{-1} + \lambda_2\mathbf{u}\mathbf{C}^{-1}. \tag{3.20}$$

Multiplying the last equality on the right by $\mathbf{m}^{\mathrm{T}}$ and, respectively, by $\mathbf{u}^{\mathrm{T}}$, we obtain a system of linear equations for the Lagrange multipliers:

$$\begin{aligned} \lambda_1\mathbf{m}\mathbf{C}^{-1}\mathbf{m}^{\mathrm{T}} + \lambda_2\mathbf{u}\mathbf{C}^{-1}\mathbf{m}^{\mathrm{T}} &= 2\mu, \\ \lambda_1\mathbf{m}\mathbf{C}^{-1}\mathbf{u}^{\mathrm{T}} + \lambda_2\mathbf{u}\mathbf{C}^{-1}\mathbf{u}^{\mathrm{T}} &= 2. \end{aligned}$$

The matrix of coefficients of this system will be denoted by

$$M = \begin{bmatrix} \mathbf{m}\mathbf{C}^{-1}\mathbf{m}^{\mathrm{T}} & \mathbf{u}\mathbf{C}^{-1}\mathbf{m}^{\mathrm{T}} \\ \mathbf{m}\mathbf{C}^{-1}\mathbf{u}^{\mathrm{T}} & \mathbf{u}\mathbf{C}^{-1}\mathbf{u}^{\mathrm{T}} \end{bmatrix}.$$

If the vectors $\mathbf{u}, \mathbf{m}$ are linearly independent, then M is an invertible matrix, see Appendix 10.4. We can solve the system of linear equations for λ_1 and λ_2, obtaining a linear expression in μ:

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = 2M^{-1}\begin{bmatrix} \mu \\ 1 \end{bmatrix}. \tag{3.21}$$

Inserting this into (3.20), we get

$$\mathbf{w} = \mu\mathbf{a} + \mathbf{b} \tag{3.22}$$

for some vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. The important point here is that the vectors $\mathbf{a}, \mathbf{b}$ are the same for each portfolio on the minimum variance line. They can be expressed in terms of $\mathbf{C}$, $\mathbf{m}$ and $\mathbf{u}$.

We can summarise the above results as a theorem.

Theorem 3.28

Assume that $\det \mathbf{C} \neq 0$ and $\mathbf{m}, \mathbf{u}$ are linearly independent vectors. Then a vector of weights $\mathbf{w}$ represents a portfolio V on the minimum variance line if and only if $\mathbf{w}$ is of the form (3.20) with λ_1, λ_2 given by (3.21) and with $\mu = \mu_V$.

Proof

The above argument shows that (3.20) with (3.21) is a necessary condition for minimising the risk at a given level μ of expected return. Sufficiency follows from the fact that the minimisation problem has a unique solution for each μ since the function minimised is quadratic in $\mathbf{w}$ and bounded below by 0. □

Example 3.29

Consider three securities with expected returns, standard deviations of returns and correlations between returns

$$\begin{aligned} \mu_1 &= 0.10, & \sigma_1 &= 0.28, & \rho_{12} &= \rho_{21} = -0.10, \\ \mu_2 &= 0.15, & \sigma_2 &= 0.24, & \rho_{23} &= \rho_{32} = 0.20, \\ \mu_3 &= 0.20, & \sigma_3 &= 0.25, & \rho_{31} &= \rho_{13} = 0.25. \end{aligned}$$

We arrange the μ_i's into a one-row matrix $\mathbf{m}$ and 1's into a one-row matrix $\mathbf{u}$,

$$\mathbf{m} = \begin{bmatrix} 0.10 & 0.15 & 0.20 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

Next we compute the entries $c_{ij} = \rho_{ij}\sigma_i\sigma_j$ of the covariance matrix $\mathbf{C}$, and find the inverse matrix $\mathbf{C}^{-1}$,

$$\mathbf{C} \cong \begin{bmatrix} 0.0784 & -0.0067 & 0.0175 \\ -0.0067 & 0.0576 & 0.0120 \\ 0.0175 & 0.0120 & 0.0625 \end{bmatrix}, \quad \mathbf{C}^{-1} \cong \begin{bmatrix} 13.954 & 2.544 & -4.396 \\ 2.544 & 18.548 & -4.274 \\ -4.396 & -4.274 & 18.051 \end{bmatrix}.$$

This gives

$$\begin{aligned} \mathbf{u}\mathbf{C}^{-1} &\cong \begin{bmatrix} 12.102 & 16.818 & 9.382 \end{bmatrix}, \\ \mathbf{m}\mathbf{C}^{-1} &\cong \begin{bmatrix} 0.898 & 2.182 & 2.530 \end{bmatrix}, \end{aligned}$$

and

$$M = \begin{bmatrix} \mathbf{m}\mathbf{C}^{-1}\mathbf{m}^T & \mathbf{u}\mathbf{C}^{-1}\mathbf{m}^T \\ \mathbf{m}\mathbf{C}^{-1}\mathbf{u}^T & \mathbf{u}\mathbf{C}^{-1}\mathbf{u}^T \end{bmatrix} \cong \begin{bmatrix} 0.923 & 5.609 \\ 5.609 & 38.302 \end{bmatrix},$$

$$M^{-1} \cong \begin{bmatrix} 9.842 & -1.441 \\ -1.441 & 0.237 \end{bmatrix}.$$

The weights in the minimum variance portfolio can be found using (3.19):

$$\mathbf{w} = \frac{\mathbf{u}\mathbf{C}^{-1}}{\mathbf{u}\mathbf{C}^{-1}\mathbf{u}^T} \cong \begin{bmatrix} 0.316 & 0.439 & 0.245 \end{bmatrix}.$$

The expected return and standard deviation of this portfolio are

$$\mu_V = \mathbf{m}\mathbf{w}^T \cong 0.146, \quad \sigma_V = \sqrt{\mathbf{w}\mathbf{C}\mathbf{w}^T} \cong 0.162.$$

To obtain the weights $\mathbf{w}$ of the portfolio on the minimum variance line with expected return μ_V we apply formula (3.22):

$$\mathbf{w} \cong \mu_V \mathbf{a} + \mathbf{b},$$

where

$$\mathbf{a} = \begin{bmatrix} -8.614 & -2.769 & 11.384 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1.578 & 0.845 & -1.422 \end{bmatrix}.$$

The standard deviation of this portfolio is

$$\sigma_V = \sqrt{\mathbf{w}\mathbf{C}\mathbf{w}^T} \cong \sqrt{0.237 - 2.885\mu_V + 9.850\mu_V^2}.$$

Exercise 3.12

Among all feasible portfolios with expected return $\mu_V = 20\%$ constructed using the three securities in Exercise 3.11 find the portfolio with the smallest variance. Compute the weights and the standard deviation of this portfolio.

The following result is an important consequence of the linear dependence (3.22) of the weights of portfolios belonging to the minimum variance line on the expected return.

Theorem 3.30 (Two-Fund Theorem)

Under the assumptions of Theorem 3.28, let w_1, w_2 be the weights of any two portfolios V_1, V_2 on the minimum variance line with different expected returns

$\mu_{V_1} \neq \mu_{V_2}$. Then each portfolio V on the minimum variance line can be obtained as a linear combination of these two, that is,

$$\mathbf{w} = \alpha\mathbf{w}_1 + (1-\alpha)\mathbf{w}_2$$

for some $\alpha \in \mathbb{R}$.

Proof

We find α so that

$$\mu_V = \alpha\mu_{V_1} + (1-\alpha)\mu_{V_2}.$$

This is possible since the returns are different:

$$\alpha = \frac{\mu_V - \mu_{V_2}}{\mu_{V_1} - \mu_{V_2}}.$$

Since the two portfolios V_1, V_2 belong to the minimum variance line, they satisfy

$$\mathbf{w}_1 = \mu_{V_1}\mathbf{a} + \mathbf{b}, \quad \mathbf{w}_2 = \mu_{V_2}\mathbf{a} + \mathbf{b}.$$

As a result, because V also belongs to the minimum variance line

$$\alpha\mathbf{w}_1 + (1-\alpha)\mathbf{w}_2 = (\alpha\mu_{V_1} + (1-\alpha)\mu_{V_2})\mathbf{a} + \mathbf{b} = \mu_V\mathbf{a} + \mathbf{b} = \mathbf{w}.$$

□

The practical significance of the two-fund theorem stems from the fact that any portfolio on the minimum variance line can be realised by splitting available wealth between just two different investment funds (two different portfolios) instead of investing in individual assets. Trading in the units of only two investment funds can significantly reduce the costs and simplify the procedures as compared with simultaneous transactions in n risky assets.

Another consequence of the two-fund theorem is that the minimum variance line can be viewed as if it were a two-asset line constructed from the portfolios V_1 and V_2 (regarded for this purpose as if they were individual risky assets). From the results of Section 3.2 we can therefore conclude that the minimum variance line is a hyperbola of the kind described in Proposition 3.18. We can therefore conclude that the efficient frontier consists of portfolios with weights of the form $\mathbf{w} = \mu\mathbf{a} + \mathbf{b}$ for μ greater than or equal to expected return on the minimal variance portfolio.

Example 3.31

There are two convenient ways to visualise all portfolios that can be constructed from the three securities in Example 3.29. One is presented in Figure 3.5. Here

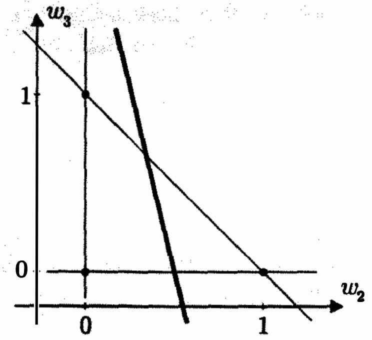

Figure 3.5 Feasible portfolios on the w_2, w_3 plane

two of the three weights, namely w_2 and w_3, are used as parameters. The remaining weight is given by $w_1 = 1 - w_2 - w_3$. (Of course any other two weights can also be used as parameters.) Each point on the w_2, w_3 plane represents a different portfolio. The vertices of the triangle represent the portfolios consisting of only one of the three securities. For example, the vertex with coordinates $(1,0)$ corresponds to weights $w_1 = 0$, $w_2 = 1$ and $w_3 = 0$, that is, represents a portfolio with all money invested in security 2. The straight lines through the vertices correspond to portfolios consisting of two securities only. For example, the line through $(1,0)$ and $(0,1)$ corresponds to portfolios containing securities 2 and 3 only. Points inside the triangle, including the boundaries, correspond to portfolios without short selling. For example, $(\frac{2}{5}, \frac{1}{2})$ represents a portfolio with 10% of the initial funds invested in security 1, 40% in security 2, and 50% in security 3. Points outside the triangle correspond to portfolios with one or two of the three securities shorted. The minimum variance line is a straight line because of the linear dependence of the weights on the expected return. It is represented by the bold line in Figure 3.5.

Figure 3.6 shows another way to visualise feasible portfolios by plotting the expected return of a portfolio against the standard deviation. This is sometimes called the risk–expected return graph. The three points indicated in this picture correspond to portfolios consisting of only one of the three securities. For instance, the portfolio with all funds invested in security 2 is represented by the point $(0.24, 0.15)$. The hyperbolas passing through a pair of these three points correspond to portfolios consisting of just two securities. These are the two-security lines studied in detail in Section 3.2. For example, all portfolios containing securities 2 and 3 only lie on the line through $(0.24, 0.15)$ and $(0.25, 0.20)$. The three points and the lines passing through them correspond to the vertices of the triangle and the straight lines passing through them in Figure 3.5. The shaded area (both dark and light), including the boundary, represents portfolios that can be constructed from the three securities, that is, all feasible portfolios. The boundary, shown as a bold line, is the minimum variance line. The shape of it is known as the *Markowitz bullet*. The darker part of the shaded area corresponds to the interior of the triangle in Figure 3.5, that is, it represents portfolios without short selling.

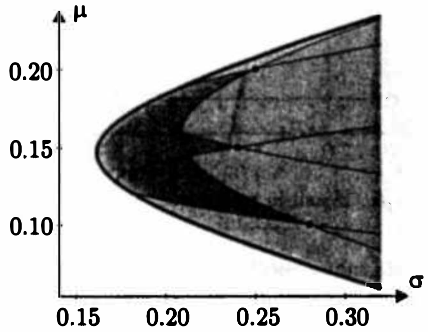

Figure 3.6 Feasible portfolios on the σ, μ plane

It is instructive to imagine how the whole w_2, w_3 plane in Figure 3.5 is mapped onto the shaded area representing all feasible portfolios in Figure 3.6. Namely, the w_2, w_3 plane is folded along the minimum variance line, being simultaneously warped and stretched to attain the shape of the Markowitz bullet. This means, in particular, that pairs of points on opposite sides of the minimum variance line on the w_2, w_3 plane are mapped into single points on the σ, μ plane. In other words, each point inside the shaded area in Figure 3.6 corresponds to two different portfolios. However, each point on the minimum variance line corresponds to a single portfolio.

Example 3.32

For the same three securities as in Examples 3.29 and 3.31, Figure 3.7 shows what happens if no short selling is allowed. All portfolios without short selling are represented by the interior and boundary of the triangle on the w_1, w_2 plane and by the shaded area with boundary on the σ, μ plane. The minimum variance line without short selling is shown as a bold line in both plots. For comparison, the minimum variance line with short selling is shown as a broken line.

Exercise 3.13

For portfolios constructed with and without short selling from the three securities in Exercise 3.11 compute the minimum variance line parameterised by the expected return and sketch it a) on the w_2, w_3 plane and b) on the σ, μ plane. Also sketch the set of all feasible portfolios with and without short selling.

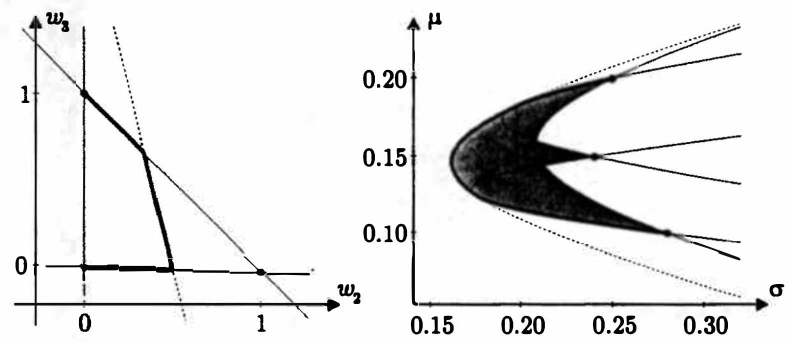

Figure 3.7 Portfolios without short selling

3.3.4 Market Portfolio

From now on we shall assume that a risk-free security is available in addition to n risky securities. The return on the risk-free security will be denoted by R. The standard deviation of the return is of course zero for the risk-free security.

Consider a portfolio consisting of the risk-free security and a specified risky security (or a portfolio of risky securities) V with expected return μ_V and standard deviation $\sigma_V > 0$. By Proposition 3.22 all such portfolios form a line consisting of two rectilinear half-lines meeting at the point $(0, R)$ on the vertical axis on the σ, μ plane, represented by the broken line in Figure 3.8. By taking portfolios containing the risk-free security and a risky security V with σ_V, μ_V anywhere in the feasible set on the σ, μ plane, represented by the shaded area in Figure 3.8, we can construct any portfolio between the two solid half-lines. The area between these two half-lines serves as a new set of admissible portfolios, which can now include the risk-free security.

The efficient frontier of this new feasible set is the bold half-line passing through $(0, R)$ and tangent to the hyperbola representing the minimum variance line constructed from risky securities. Every rational investor who respects the dominance relation between portfolios will select his or her portfolio on this half-line, called the *capital market line* (CML). This argument works as long as the risk-free return R is not too high, namely R is less than the expected return on the minimum variance portfolio (this is because the asymptotes of the hyperbola intersect on the μ axis, see Proposition 3.18).

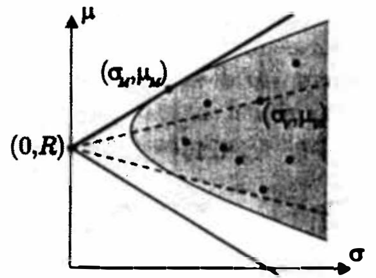

Figure 3.8 Efficient frontier for portfolios with a risk-free security

The portfolio M corresponding to the tangency point (σ_M, μ_M) plays a special role. It is called the *market portfolio*. In order to compute the weights in the market portfolio we observe that the straight line passing though $(0, R)$ and (σ_M, μ_M) has the steepest slope (highest gradient) among all lines through $(0, R)$ and (σ_V, μ_V) for any portfolio V in the feasible set constructed from risky assets. The slope of such a line can be written as

$$\frac{\mu_V - R}{\sigma_V} = \frac{\mathbf{w}\mathbf{m}^{\mathrm{T}} - R}{\sqrt{\mathbf{w}\mathbf{C}\mathbf{w}^{\mathrm{T}}}},$$

where $\mathbf{w}$ are the weights in portfolio V. We shall therefore find

$$\max \frac{\mathbf{w}\mathbf{m}^{\mathrm{T}} - R}{\sqrt{\mathbf{w}\mathbf{C}\mathbf{w}^{\mathrm{T}}}}$$

over all vectors $\mathbf{w} \in \mathbf{R}^n$ subject to the usual constraint

$$\mathbf{w}\mathbf{u}^{\mathrm{T}} = 1.$$

To solve this maximisation problem we form the Lagrange function

$$F(\mathbf{w}, \lambda) = \frac{\mathbf{w}\mathbf{m}^{\mathrm{T}} - R}{\sqrt{\mathbf{w}\mathbf{C}\mathbf{w}^{\mathrm{T}}}} - \lambda(\mathbf{w}\mathbf{u}^{\mathrm{T}} - 1),$$

and equate its gradient to zero:

$$\nabla_{\mathbf{w}} F(\mathbf{w}, \lambda) = \frac{\sqrt{\mathbf{w}\mathbf{C}\mathbf{w}^{\mathrm{T}}}\,\mathbf{m} - \frac{\mathbf{w}\mathbf{m}^{\mathrm{T}} - R}{\sqrt{\mathbf{w}\mathbf{C}\mathbf{w}^{\mathrm{T}}}}\mathbf{w}\mathbf{C}}{\mathbf{w}\mathbf{C}\mathbf{w}^{\mathrm{T}}} - \lambda\mathbf{u} = 0.$$

This can be written as

$$\frac{\mu_V - R}{\sigma_V^2}\mathbf{w}\mathbf{C} = \mathbf{m} - \lambda\sigma_V\mathbf{u}.$$

Multiplying by $\mathbf{w}^{\mathrm{T}}$ on the right we get

$$\frac{\mu_V - R}{\sigma_V^2}\mathbf{w}\mathbf{C}\mathbf{w}^{\mathrm{T}} = \mu_V - \lambda\sigma_V$$

so

$$\lambda = \frac{R}{\sigma_V}.$$

Therefore we have the equation

$$\gamma\mathbf{w}\mathbf{C} = \mathbf{m} - R\mathbf{u} \tag{3.23}$$

where $\gamma = \frac{\mu_V - R}{\sigma_V^2}$, so that

$$\gamma\mathbf{w} = (\mathbf{m} - R\mathbf{u})\mathbf{C}^{-1}.$$

The difficulty in solving this for $\mathbf{w}$ lies in the fact that γ depends on $\mathbf{w}$. To compute γ we multiply by $\mathbf{u}^{\mathrm{T}}$ on the right and use the constraint once again, which gives

$$\gamma = (\mathbf{m} - R\mathbf{u})\mathbf{C}^{-1}\mathbf{u}^{\mathrm{T}}.$$

This leads to the following result.

Theorem 3.33

If $\det \mathbf{C} \neq 0$ and the risk-free rate R is lower than the expected return μ_{MVP} of the minimum variance portfolio, then the market portfolio M exists and its weights are given by

$$\mathbf{w}_M = \frac{(\mathbf{m} - R\mathbf{u})\mathbf{C}^{-1}}{(\mathbf{m} - R\mathbf{u})\mathbf{C}^{-1}\mathbf{u}^{\mathrm{T}}}.$$

Proof

It only remains to make sure that $\gamma \neq 0$. By (3.19) the expected return of the minimum variance portfolio can be written as

$$\mu_{\mathrm{MVP}} = \mathbf{w}_{\mathrm{MVP}}\mathbf{m}^{\mathrm{T}} = \frac{\mathbf{u}\mathbf{C}^{-1}\mathbf{m}^{\mathrm{T}}}{\mathbf{u}\mathbf{C}^{-1}\mathbf{u}^{\mathrm{T}}}.$$

Since, by assumption, $R < \mu_{\mathrm{MVP}}$ and $\mathbf{u}\mathbf{C}^{-1}\mathbf{u}^{\mathrm{T}} > 0$, it follows that

$$\gamma = \mathbf{u}\mathbf{C}^{-1}\mathbf{m}^{\mathrm{T}} - R\mathbf{u}\mathbf{C}^{-1}\mathbf{u}^{\mathrm{T}} > 0.$$

□

Exercise 3.14

Suppose that the risk-free return is $R = 5\%$. Compute the weights in the market portfolio constructed from the three securities in Exercise 3.10. Also compute the expected return and standard deviation of the market portfolio.

Exercise 3.15

In a market consisting of the three securities in Exercise 3.11, consider the portfolio V on the efficient frontier with expected return $\mu_V = 21\%$. Compute the values of γ and R such that the weights $\mathbf{w}$ in this portfolio satisfy $\gamma\mathbf{w}\mathbf{C} = \mathbf{m} - R\mathbf{u}$.

The capital market line (CML), which joins the risk-free security represented by $(0, R)$ and the market portfolio with coordinates (σ_M, μ_M), satisfies the equation

$$\mu = R + \frac{\mu_M - R}{\sigma_M}\sigma. \tag{3.24}$$

For a portfolio on the CML with risk σ the term $\frac{\mu_M - R}{\sigma_M}\sigma$ is called the *risk premium*. It can be viewed as additional return above the risk-free return R which compensates for exposure to risk.

If each investor uses the same values of the model parameters (the expected returns on the basic assets and the entries of the covariance matrix) and chooses a portfolio that is not dominated by another one, then each investor will construct their portfolio on the CML. In effect, everyone will combine an investment in the risk-free security with just one risky portfolio, namely the market portfolio M. Consequently, the relative proportions of risky assets held in the portfolio will be the same for all investors. This can only happen if the weights in the market portfolio are proportional to the total value (called *market capitalisation*) of each of the risky assets available in the market.

In practice, the market portfolio can be represented by a market index. An empirical test of the theory is therefore to see if the market index lies on the efficient frontier and determines the tangent line when combined with the risk-free asset.

3.4 Capital Asset Pricing Model (CAPM)

Paradoxically, in a model where decisions are based on risk expressed as standard deviation, if we look at a portfolio of assets to assess its risk, the standard

deviations of individual assets may turn out not as relevant as the covariances between them. This can be illustrated by the following example.

Example 3.34

Suppose that the weights in a portfolio are of the form $w_j = \frac{1}{n}$ where n is the number of risky assets. We shall investigate the risk of this portfolio depending on n. Assume that the variances of all assets traded in the market are uniformly bounded, $\sigma_j^2 \leq L$ for each j. Then

$$\begin{aligned}\sigma_V^2 &= \sum_{j,k} w_j w_k c_{jk} \\ &= \sum_j w_j^2\sigma_j^2 + \sum_{j\neq k} w_j w_k c_{jk} \leq \frac{1}{n}L + \frac{1}{n^2}\sum_{j\neq k} c_{jk}.\end{aligned}$$

Assume further that the off-diagonal elements of the covariance matrix are all equal, $c_{jk} = c > 0$ for each $j \neq k$. Then

$$\sigma_V^2 \leq \frac{L}{n} + \frac{1}{n^2}n(n-1)c.$$

The upper bound converges to c as $n \to \infty$. The risk of a portfolio containing many such assets is determined by the covariances. The variances of the individual assets become irrelevant for large n.

This example motivates the following distinction between two kinds of risk: ***diversifiable***, or ***specific risk***, which can be reduced to zero by expanding the portfolio, and ***undiversifiable***, ***systematic***, or ***market risk***, which cannot be avoided because the securities are to some extent linked to the market.

As we know, the CML is tangent to the efficient frontier at the point (σ_M, μ_M) representing the market portfolio M with weights $\mathbf{w}_M$. Consider any other portfolio (or an individual asset) V, represented by a point (σ_V, μ_V) on the σ, μ plane. Now consider all portfolios constructed from M and V. They form a hyperbola, which we claim to be tangent to the CML at (σ_M, μ_M). If this hyperbola were to intersect the CML, this would contradict the fact that the slope of CML is maximal, see Figure 3.9.

We shall compute the slope of the tangent line to the hyperbola at M, and then use the fact that the slope of the CML must be the same. Denote the weights of V and M in a portfolio P on the hyperbola by $(x, 1-x)$. The risk and return of P are

$$\begin{aligned}\sigma_P &= (x^2\sigma_V^2 + (1-x)^2\sigma_M^2 + 2x(1-x)\mathrm{Cov}(K_V, K_M))^{\frac{1}{2}}, \\ \mu_P &= x\mu_V + (1-x)\mu_M.\end{aligned}$$

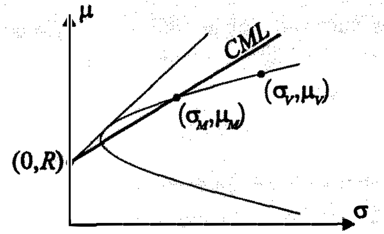

Figure 3.9 The CML cannot both have the steepest slope and intersect the two-asset line through M and V

We compute the derivatives with respect to x at $x = 0$ to get

$$\left.\frac{\partial\sigma_P}{\partial x}\right|_{x=0} = \frac{\mathrm{Cov}(K_V, K_M) - \sigma_M^2}{\sigma_M},$$
$$\left.\frac{\partial\mu_P}{\partial x}\right|_{x=0} = \mu_V - \mu_M,$$

The slope of the tangent line is the ratio of these derivatives, which we equate to the slope of the CML:

$$\frac{\mu_V - \mu_M}{\frac{\mathrm{Cov}(K_V,K_M)-\sigma_M^2}{\sigma_M}} = \frac{\mu_M - R}{\sigma_M}.$$

Solving for μ_V, we get

$$\mu_V = R + \frac{\mathrm{Cov}(K_V, K_M)}{\sigma_M^2}(\mu_M - R).$$

Definition 3.35

We call

$$\beta_V = \frac{\mathrm{Cov}(K_V, K_M)}{\sigma_M^2}$$

the *beta factor* of the portfolio V.

We have proved the following theorem.

Theorem 3.36 (CAPM)

Suppose that the risk-free rate R is lower than the expected return μ_{MVP} of the minimum variance portfolio (so that the market portfolio M exists). Then

the expected return μ_V on any feasible portfolio V is given by

$$\mu_V = R + \beta_V(\mu_M - R). \tag{3.25}$$

The term $\beta_V(\mu_M - R)$ is called the *risk premium*. It represents the additional return required by an investor who faces risk due to the link between the portfolio and the market. This is similar to the risk premium in (3.24). However, (3.24) applies only to portfolios on the capital market line, whereas (3.25) is much more general in that it applies to all feasible portfolios and individual securities. Of course, for a portfolio V on the capital market line the risk premium in (3.25) and that in (3.24) are the same:

$$\beta_V(\mu_M - R) = \frac{\mu_M - R}{\sigma_M}\sigma_V.$$

Exercise 3.16

Show that the beta factor β_V of a portfolio consisting of n securities with weights $w_1, \ldots, w_n$ is given by $\beta_V = w_1\beta_1 + \cdots + w_n\beta_n$, where $\beta_1, \ldots, \beta_n$ are the beta factors of the securities.

The CAPM formula (3.25) is concerned with expected returns. Our next step is to consider the returns themselves. Suppose that we want to approximate the return K_V on a feasible portfolio V by a linear function $\beta K_M + \alpha$ of the return K_M on the market portfolio M. The error of this approximation

$$\varepsilon = K_V - (\beta K_M + \alpha)$$

is called the *residual error*. The best approximation in the sense of minimising the variance $\mathrm{Var}(\varepsilon)$, known as *linear regression*, is achieved for

$$\beta = \beta_V, \tag{3.26}$$
$$\alpha = \mu_V - \beta_V\mu_M = R(1 - \beta_V), \tag{3.27}$$

see Appendix 10.5. This provides an alternative interpretation of β_V as the gradient of the regression line, which minimises the variance of the residual error.

We shall denote the residual error by ε_V when β, α are given by (3.26) and (3.27). This means that

$$\begin{aligned} \varepsilon_V &= (K_V - \mu_V) - \beta_V(K_M - \mu_M) \\ &= (K_V - R) - \beta_V(K_M - R). \end{aligned} \tag{3.28}$$

The second equality follows from the CAPM formula (3.25).

Exercise 3.17

Verify that

$$\mathbb{E}(\varepsilon_V) = 0,$$
$$\mathrm{Cov}(K_M, \varepsilon_V) = 0.$$

Formula (3.28) can be written as $K_V - \mu_V = \varepsilon_V + \beta_V(K_M - \mu_M)$. Applying variance to both sides of this equality and using the results of Exercise 3.17, we obtain

$$\sigma_V^2 = \mathrm{Var}(\varepsilon_V) + \beta_V^2\sigma_M^2.$$

This sheds more light on the distinction between diversifiable and systematic risk. The left-hand side is directly related to the risk of the portfolio V. The second term on the right represents the systematic risk that is due to the link between the portfolio and the market and is measured by β_V. The first term corresponds to the diversifiable part of the risk. In particular, if $V = M$, then $\mathrm{Var}(\varepsilon_M) = 0$. There is no diversifiable risk if we invest in the market portfolio (or in a portfolio on the CML, which is a combination of the market portfolio and the risk-free asset).

Remark 3.37

Drawing on the results established in Section 3.3.4, we can give an alternative proof of the CAPM formula. Consider an arbitrary portfolio V with weights $\mathbf{w}$. The weights $\mathbf{w}_M$ in the market portfolio satisfy

$$\gamma\mathbf{w}_M\mathbf{C} = \mathbf{m} - R\mathbf{u}$$

for some number $\gamma > 0$, see (3.23). The beta factor of portfolio V with weights $\mathbf{w}$ can, therefore, be written as

$$\beta_V = \frac{\mathrm{Cov}(K_V, K_M)}{\sigma_M^2} = \frac{\gamma\mathbf{w}\mathbf{C}\mathbf{w}_M^{\mathrm{T}}}{\gamma\mathbf{w}_M\mathbf{C}\mathbf{w}_M^{\mathrm{T}}} = \frac{\mathbf{w}\frac{1}{\gamma}(\mathbf{m} - R\mathbf{u})^{\mathrm{T}}}{\mathbf{w}_M\frac{1}{\gamma}(\mathbf{m} - R\mathbf{u})^{\mathrm{T}}} = \frac{\mu_V - R}{\mu_M - R}.$$

Solving this for μ_V, we obtain the CAPM formula (3.25) once again.

According to (3.25), the expected return on a portfolio is an affine function of the beta factor. The graph of this function in the β, μ plane is called the *security market line* (SML). This straight line is shown in Figure 3.10, in which the CML is also plotted for comparison.

The CAPM describes a state of equilibrium in the market. Everyone is holding a portfolio of risky securities with the same weights as the market

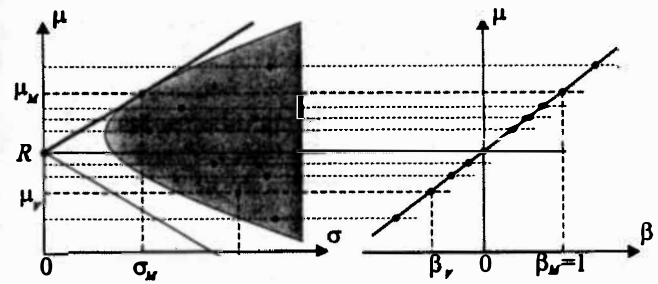

Figure 3.10 Capital market line and security market line

portfolio. Any trades that may be executed by investors will only affect their split of funds between the risk-free security and the market portfolio. As a result, the demand and supply of all securities will be balanced.

The market will remain in equilibrium as long as the estimates of expected returns and beta factors satisfy (3.25). Any new information about a particular security V can affect the expected return μ_V or the beta factor β_V, so that the CAPM formula (3.25) may no longer be valid. Suppose, for example, that

$$\mu_V > R + \beta_V(\mu_M - R).$$

In this case investors will want to increase their relative position in this security, which offers higher expected return than that required as compensation for systematic risk. Demand will exceed supply, the price of the security will begin to rise and the expected return will decline. On the other hand, if the reverse inequality

$$\mu_V < R + \beta_V(\mu_M - R)$$

holds, then investors will want to sell or even short sell the security, causing the price to fall because of the excess supply, so that the expected return will increase. In both cases we should observe price adjustments restoring the CAPM formula and the state of equilibrium.

Case 3: Discussion

Let us set similar targets for the pension scheme and make similar assumptions as in Case 2. When you retire after 40 years you want to receive a pension for 20 years. Your salary is expected to grow at 2% annually, and you want the pension payments to start at 50% of your final salary and to grow at the same 2% rate. The interest rate offered by the risk-free bank account is 5%.

Moreover, you consider risk of up to 10% (as measured by standard deviation) to be reasonable, and select three stocks A, B, C to be included in your portfolio in addition to the bank account. You estimate the expected returns on the stocks to be, respectively, 8%, 10% and 14% per annum, and the standard deviations of the annual returns to be 15%, 22% and 26%. The estimates for the correlation coefficients are 0.5 between stocks A, B, -0.3 between A, C, and 0.3 between B, C.

The three chosen stocks constitute a mini-market, to which we can apply the theory developed in this chapter. This requires selecting a suitable portfolio on the capital market line. To this end, using the formula in Theorem 3.33, we can compute the weights $(0.6773, -0.1518, 0.4745)$ in the market portfolio M.

However, as a small investor, you are unlikely to be able to do short selling and will need to restrict your portfolios to those with non-negative weights only. We no longer have an analytic formula for the weights, and need to resort to a numerical solution. Using Solver in Excel to maximise the gradient of the straight line through the risk-free security and feasible portfolios without short selling, we find the weights in the market portfolio M' with no short selling to be $(0.5646, 0.0000, 0.4354)$. Portfolio M' has expected return 10.61% (as compared to 10.54% with short selling) and standard deviation 11.93% (11.52% with short selling).

We can now combine M' with the risk-free investment into a bank account at 5% to construct a portfolio P on the capital market line. The choice of how to split your investment between M' and the bank account will depend on your attitude to risk. To construct a portfolio P with 10% risk, you will need to invest 16.19% of your funds in the bank account and the rest in portfolio M'. The expected return on P will then be 9.70%. Going back to Case 2, we can see that at this level of return, the premium you need to pay into the pension scheme will be just 2.20% of your salary, as compared to 10.05% when investing into the bank account only.

If risk at 10% is considered too high for a pension scheme, it can be reduced by increasing the proportion of funds invested in the bank account. This is shown in Table 3.1.

risk	0.00%	2.50%	5.00%	7.50%	10.00%
expected return	5.00%	6.18%	7.35%	8.53%	9.70%
proportion of salary paid	10.05%	6.94%	4.76%	3.24%	2.20%
of which invested risk free	100.00%	79.05%	58.09%	37.14%	16.19%

Table 3.1 Pension schemes based on different attitudes to risk