Ex 15. $X \sim geo(p)$ $p \in [0,1]$

$P(X=n) = (1-p)^{n-1}p$ 4 2 6 5 8.

15.1. maximum de vraisemblance $\theta = p$.

$P(X=n|\theta) = (1-\theta)^{n-1}\theta$

$3+1+5+4+7$ . $\,=$

$L(x,\theta) = P(x|\theta) = \prod_{i=1}^{5} P(x_i|\theta) = (1-\theta)^{20}\theta^5$

$\lg L(x,\theta) = 20 \lg(1-\theta) + 5 \lg\theta$.

$\frac{\partial L(x,\theta)}{\partial\theta} = \frac{-20}{1-\theta} + \frac{5}{\theta}$

$\frac{\partial L(x,\theta)}{\partial\theta} = 0 \iff 20\theta = 5(1-\theta)$.

$\theta = 0.2$.

15.2. $\pi(\theta) \propto \theta^3(1-\theta)^4$

$\underset{\theta}{\text{argmax}}\ \pi(\theta|x) = \underset{\theta}{\text{argmax}}\ P(x|\theta)\pi(\theta)$.

$p(x|\theta)\pi(\theta) \propto \theta^5(1-\theta)^{20}\cdot\theta^3(1-\theta)^4 = (1-\theta)^{24}\theta^8$

$\lg \pi(\theta|x) = 24\lg(1-\theta) + 8\lg\theta + cste$.

$\frac{\partial \lg \pi(\theta|x)}{\partial\theta} = \frac{-24}{1-\theta} + \frac{8}{\theta}$

$\frac{\partial \lg \pi(\theta|\pi)}{\partial\theta} = 0. \iff 24\theta = 8(1-\theta) \iff \theta = 0.25$

Ex 16 $X \sim B(k,\theta)$ $k$: cste $\{x_1 \dots x_n\}$.

16.1. $P(X=m|\theta) = \binom{k}{m}\theta^m(1-\theta)^{k-m}$

$L(x,\theta) = P(x|\theta) = \prod_{i=1}^{n} P(x_i|\theta) = \prod_{i=1}^{n}\binom{k}{x_i}\theta^{x_i}(1-\theta)^{k-x_i}$

$\text{Log}\, L(x,\theta) = \sum_{i=1}^{n}\left(\lg\binom{k}{x_i} + x_i\lg\theta + (k-x_i)\lg(1-\theta)\right)$

$= \sum_{i=1}^{n}\lg\binom{k}{x_i} + (\lg\theta)(\sum_{i=1}^{n}x_i) + \lg(1-\theta)(nk - \sum_{i=1}^{n}x_i)$.

$\frac{\partial \lg L(x,\theta)}{\partial\theta} = \frac{\sum_{i=1}^{n}x_i}{\theta} - \frac{nk-\sum_{i=1}^{n}x_i}{1-\theta}$

$\frac{\partial \lg L(x,\theta)}{\partial\theta} = 0 \iff \sum_{i=1}^{n}x_i(1-\theta) = \theta(nk - \sum_{i=1}^{n}x_i)$.

$\sum_{i=1}^{n}x_i - \theta\sum_{i=1}^{n}x_i = n\theta k - \theta\sum_{i=1}^{n}x_i$

$\theta = \frac{\sum_{i=1}^{n}x_i}{nk}$

$\theta = \bar{x}/k$

16.2. $\theta \sim \pi(\theta) = Beta(\theta, a.b)$. $\pi(\theta) \propto \theta^a (1-\theta)^b$

$\theta_{MAP} = \arg\max_\theta \pi(\theta|x) = \arg_\theta P(x|\theta)\pi(\theta)$.

teste

$\log \pi(\theta|x) = \sum_{i=1}^n \log \binom{k}{x_i} + \log\theta\left(\sum_{i=1}^n x_i + a\right) + \log(1-\theta)\left(nk - \sum_{i=1}^n x_i + b\right)$

$$\frac{\partial \log \pi(\theta|x)}{\partial\theta} = \frac{\sum_{i=1}^n x_i + a}{\theta} - \frac{nk - \sum_{i=1}^n x_i + b}{1-\theta}$$

$$\frac{\partial \log \pi(\theta|\pi)}{\partial\theta} = 0. \iff (1-\theta)(\Sigma x_i + a) = \theta(nk - \Sigma x_i + b)$$

$$\Sigma x_i + a - \theta\Sigma x_i - \theta a = n\theta k - \theta\Sigma x_i + \theta b.$$

$$\Sigma x_i + a = \theta(nk + b + a)$$

$$\theta = \frac{\sum_{i=1}^n x_i + a}{nk + b + a}$$

Ex17. rouge (R). blenes (B) vert (V). jaune (J).

$X = \hat{A}$ $P(X=R) = P_R$ $P(X=B) = P_B$ $P(X=V) = P_V$. $P(X=J) = P_J$.

$P_R. P_B. P_V. P_J \geq 0$ et $P_R + P_B + P_V + P_J = 1$.

R R R R B B V V V J

Q17.1. $\theta = (P_R. P_B. P_V. P_J)$.

$L(X|\theta) = P_R^4 P_B^2 P_V^3 P_J$

$\log L(X.\theta) = 4\log P_R + 2\log P_B + 3\log P_V + \log(1 - P_R - P_B - P_V)$

$$\begin{cases} \dfrac{\partial \log L(X.\theta)}{\partial P_R} = \dfrac{4}{P_R} - \dfrac{1}{1-P_R-P_B-P_V} \\ \dfrac{\partial \log L(X.\theta)}{\partial P_B} = \dfrac{2}{P_B} - \dfrac{1}{1-P_R-P_B-P_V} \\ \dfrac{\partial \log L(X.\theta)}{\partial P_V} = \dfrac{3}{P_V} - \dfrac{1}{1-P_R-P_B-P_V} \end{cases}$$

$\nabla_\theta \log L(X.\theta) = 0$.

$$\Rightarrow \begin{cases} 1-P_R-P_B-P_V = \dfrac{P_R}{4} & \alpha. \\ 1-P_R-P_B-P_V = \dfrac{P_B}{2} & \beta \\ 1-P_R-P_B-P_V = \dfrac{P_V}{3} & \gamma. \end{cases}$$

$$\iff \begin{cases} P_B = 2 \cdot P_R/4 \\ P_V = 3/4\ P_R \\ P_R = 4 - 4P_R - 2P_R - 3P_R \end{cases} \Rightarrow \begin{cases} P_V = 3/10. \\ P_B = 1/5 \\ P_R = 2/5. \end{cases} P_J = 1/10.$$

$\pi(\theta) = \pi(p_R, p_B, p_V, p_J) \propto p_R^2 p_B p_V^3 p_J^2$.

$\pi(\theta|x) \propto P(x|\theta)\pi(\theta)$.

Ex 18. $P_\theta(\text{pile}) = \theta$.

$\tilde{\theta}$:

| $\theta$ | $\frac{1}{4}$ | $\frac{1}{3}$ | $\frac{1}{2}$ | $\frac{2}{3}$ | $\frac{3}{4}$ |
|---|---|---|---|---|---|
| $\pi(\theta)$ | 0.1 | 0.2 | 0.4 | 0.2 | 0.1. |

$X$ v.a. $\mathcal{X} = \{0, 1, 2, 3, 4, 5\}$.

Q18.1. $P(X = x \mid \tilde{\theta} = \theta) = \binom{5}{x}\theta^x (1-\theta)^{5-x}$

$P(X = 5-x \mid \tilde{\theta} = 1-\theta) = P(X = x \mid \tilde{\theta} = \theta)$.

| $x \backslash \theta$ | 1/4 | 1/3 | 1/2 | 2/3 | 3/4 |
|---|---|---|---|---|---|
| 0 | $\boxed{0.24}$ | 0.13 | 0.031 | 0.0041 | 0.0010. |
| 1 | $\boxed{0.40}$ | 0.37 | 0.16 | 0.041 | 0.015 |
| 2 | 0.26 | $\boxed{0.33}$ | 0.31 | 0.16. | 0.088 |
| 3 | | | | $\boxed{0.33}$ | |
| 4 | | | | | $\boxed{0.40}$ |
| 5 | 0.0010 | 0.0041 | 0.031 | 0.13 | $\boxed{0.24}$ |

Q18.2. $\pi(x, \theta) = P(X|\theta)\pi(\theta)$.

| $x \backslash \theta$ | 1/4 | 1/3 | 1/2 | 2/3 | 3/4 | $P(x)$. |
|---|---|---|---|---|---|---|
| 0 | 0.024 | 0.026. | 0.012. | 0.00082 | 0.001 | 0.063. |
| 1 | 0.040. | 0.066 | 0.062 | 0.0082 | 0.0015. | 0.18 |
| 2 | 0.026 | | | | | 0.26 |
| 3 | | | | | | 0.26 |
| 4 | | | | | | 0.18. |
| 5 | | | | | | 0.063 |
| $\pi(\theta)$. | 0.1. | 0.2 | 0.4 | 0.2 | 0.1 | 1 |

0.024 ← ×0.1.

18.3 $\pi(\theta|x) = \frac{\pi(\theta|x)}{P(x)}$.

用上一个表 $\boxed{\phantom{xx}}$ 除 $\underset{\boxed{\phantom{x}}}{P(x)}$

| $x \backslash \theta$ | 1/4 | 1/3 | 1/2 | 2/3 | 3/4 |
|---|---|---|---|---|---|
| 0 | | 0.42. | | | |
| 1 | | 0.37 | | | |
| 2 | | | 0.48. | | |
| 3 | | | 0.48. | | |
| 4 | | | | 0.37 | |
| 5 | | | | 0.42. | |

$$d(x) = \underset{\theta}{\text{Argmax}}\ \pi(\theta/x)$$

Q18.4. Q18.4.1. la proba d'erreur minimum $\pi(\theta/x)$.

| $x$ | 0 | 1 | 2 | 3 | 4. | 5. | |
|---|---|---|---|---|---|---|---|
| $d(x)$. | 1/3. | 1/3 | 1/2. | 1/2. | 2/3 | 2/3 | 最大值时. |

Q18.4.2.

| $x$ | 0 | 1 | 2 | 3 | 4 | 5. | |
|---|---|---|---|---|---|---|---|
| $d'(x)$. | 1/4 | 1/4 | 1/3. | 2/3. | 3/4 | 3/4 | (Q18.2 最大值). |

Q18.4.3 si $\pi(\theta)$ uniform

Ex19. $f(x) = \lambda e^{-\lambda x}\ \theta\ x>0$. 2 7 3 4 1 2 6 5 1 9.

Q19.1. durée de vie ~ loi exp. $\underline{\pi(x/\lambda) = \lambda e^{-\lambda x}}$

$$L(x,\lambda) = \prod_{i=1}^{10} \lambda e^{-\lambda x_i} = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$$

$$\lg L(x) = n\lg\lambda + (-\lambda \textstyle\sum_{i=1}^{n} x_i).$$

$$= n\lg\lambda - \lambda \sum_{i=1}^{n} x_i$$

$$\frac{d\lg L(x)}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i.$$

$$\frac{d\lg L(x)}{dx} = 0 \iff \lambda = \frac{n}{\sum x_i}$$

$$A.N. = \frac{10}{2+7+3+4+1+2+6+5+1+9}. = \frac{10}{40} = 0.25.$$

Q19.2. $g(x) = \frac{1}{\Gamma(5)} x^4 e^{-x}$

$$\lambda_{MAP} = \underset{\lambda}{\arg\max}\ \pi(\lambda/x) = \underset{\lambda}{\arg\max}\ \underline{\pi(x/\lambda)}\ \pi(\lambda).$$

$$\pi(x/\lambda)\ \pi(\lambda) \propto \lambda^n e^{-\lambda \sum x_i} \cdot \lambda^4 e^{-\lambda}$$

$$\log \pi(x|\lambda)\,\pi(\lambda) = n \lg \lambda - \lambda \Sigma x_i + 4 \lg \lambda - \lambda + cste$$

$$\frac{\partial \lg \pi(x|\lambda)\,\pi(\lambda)}{\partial \lambda} = \frac{4+n}{\lambda} - (\Sigma x_i + 1)$$

$$\lambda_{MAP} = \frac{4+n}{1+\Sigma x_i}$$

$\infty$