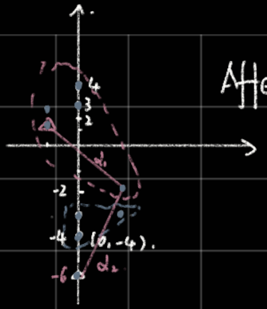
Ex22.

Q22.1.

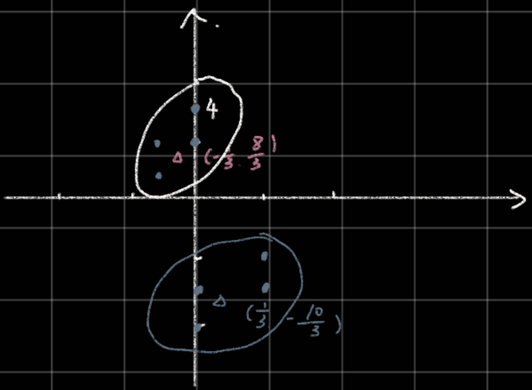


Affectation

$d_1^2 = (2^2+3^2) = 13$.

$d_2^2 = 4^2+1^2 = 17$.

Nouveau centroïdes : $(-\frac{1}{3}, \frac{8}{3})$. $(\frac{1}{3}, -\frac{10}{3})$.



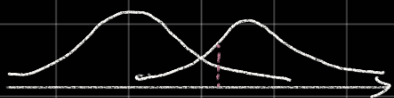
Nouveau centroïdes

$(-0.5, 2.5)$ $(0.5, -3)$.

ça ne bouge plus.

2). Max de vraisemblance pour effecter x

$$\arg\max_i \left\{ \frac{1}{\sqrt{(2\pi)^N|\Sigma|_i}} e^{-\frac{1}{2}(x-\mu_i)^T\Sigma_i^{-1}(x-\mu_i)}, i=1..2 \right\}$$



3). On recalcule μ_1. σ_1^2 et μ_2. σ_2^2

$$\mu_1^{(n+1)} = \frac{\sum_{i=1}^8 I(y_i^{(n)}=1)\, x_i}{\sum_{i=1}^8 I(y_i^{(n)}=1)}$$

$$\Sigma_1^{(n+1)} = \frac{\sum_{i=1}^8 I(y_i^{(n)}=1)\,(x_i-\mu_1^{(n+1)})\,(x_i-\mu_1)^T}{\sum_{i=1}^8 I(y_i^{(n)}=1)}$$

avec $I(b) \begin{cases} 0 & \text{si } b \text{ est faux.} \\ 1 & \text{sinon.} \end{cases}$

4). Resemble à EM.

mais affectation en dur avec I.

k-mean tombe plus facilement dans un minimum local.

Ex23. ① $\underset{\theta}{\operatorname{argmax}} \log L(X, \theta)$

$= \underset{\theta}{\operatorname{argmax}} \log \prod_{i=1}^{10} p(x_i | \theta)$

$= \underset{\theta}{\operatorname{argmax}} \sum_{i=1}^{10} \log p(x_i | \theta)$

$= \underset{\theta}{\operatorname{argmax}} \sum_{i=1}^{10} \log \left(\sum_{k=1}^{2} \pi_k \, p(x_i | \mu_k, \sigma_k^2) \right)$

$p(x_i|\theta) = \pi_1 P(x_i | \mu_1, \sigma_1^2) + \pi_2 P(x_i | \mu_2, \sigma_2^2)$

② Pourquoi difficile ? (dérivé difficile).

Difficile car on a un log de somme.

Plus facile si on savait quelle gaussienne a généré chaque x_i

Soit z_i cette variable caché

Le problème devient $\underset{\theta}{\operatorname{argmax}} \sum_{i=1}^{10} \log \left(\sum_{k=1}^{2} I(z_i = k) \pi_k P(x_i | \mu_k, \sigma_k^2) \right)$

$= \underset{\theta}{\operatorname{argmax}} \sum_{i=1}^{10} \sum_{k=1}^{2} I(z_i = k) (\log \pi_k + \log P(x_i | \mu_k, \sigma_k^2))$

$\log(1 \times \cdots + 0 \cdots)$

$= 1 \times \log \cdots + 0 \times \log \cdots$

③ $E : Q_i^{(t)}(k) = p(z_i = k | x_i, \theta^{(t)})$

$M : \underset{\theta}{\operatorname{argmax}} \sum_{i=1}^{10} \sum_{k=1}^{2} Q_i^{(t)}(k) [\log \pi_k + \log p(x_i | \mu_k, \sigma_k^2)]$

④

1	3	3	4	4	5	5	5	7	8

$\mu_1 \ \sigma_1^2$ $\qquad$ $\mu_2 \ \sigma_2^2$

$L(X, \theta) = \prod_i^n f(x_i) = \prod_i^n \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2} \frac{(x_i - \mu)^2}{\sigma^2} \right\}$

$\log L(X, \theta) = \sum_{i=1}^{n} -\frac{1}{2} \log 2\pi\sigma^2 + \sum_{i=1}^{n} -\frac{1}{2} \frac{(x_i - \mu)^2}{\sigma^2}$

- $\frac{\partial \log L(X, \theta)}{\partial \mu} = \sum_{i=1}^{n} -\frac{1}{2} \frac{2(x_i - \mu)^2(-1)}{\sigma^2}$

$\frac{\partial \log L(X, \theta)}{\partial \mu} = 0 \iff \sum_{i=1}^{n} (x_i - \mu) = 0.$

$\iff \mu = \frac{\sum_{i=1}^{n} x_i}{n}$

- $\frac{\partial \log L(X, \theta)}{\partial \sigma^2} = \sum_{i=1}^{n} \left(-\frac{1}{2}\right) \frac{2\pi}{2\pi\sigma^2} + \sum_{i=1}^{n} -\frac{1}{2} (x_i - \mu)^2 \left(-\frac{1}{(\sigma^2)^2}\right)$

$\frac{\partial \log L(X, \theta)}{\partial \sigma^2} = 0 \iff n\sigma^2 - \sum_{i=1}^{n} (x_i - \mu)^2 = 0.$

$\iff \sigma^2 = \frac{\sum_{i=1}^{n} (x_i - \mu)^2}{n}$

Applicat° numérique : $\pi_1 = \pi_2 = \frac{5}{10} = \frac{1}{2}.$

$\mu_1 = \frac{15}{5} = 3$

$\sigma_1^2 = \frac{1}{5}((-2)^2 + 0^2 + 0^2 + 1^2 + 1^2) = \frac{6}{5}$

$$\mu_1 = \frac{30}{5} = 6$$

$$\sigma_2^2 = \frac{1}{5}\left((-1)^2 + (-1)^2 + (-1)^2 + 1^2 + 2^2\right) = 8/5$$

⑤

i	1	2	3	4	5	6	7	8	9	10
x_i	1	3	3	4	4	5	5	5	7	8
$Q_i^1(y_1)$	·									
	+									
$Q_i^2(y_2)$	·									
	=									
	1									

calculer puis normaliser.

$$Q_i^1(y_1) = P(z_i = y_1 \mid x_i, \theta^{(0)}) \propto \frac{\pi_1}{\sqrt{2\pi}\,\sigma^2} \exp\left(-\frac{1}{2}\frac{(x-\mu_1)^2}{\sigma_1^2}\right)$$

$$Q_i^1(y_2) = P(z_i = y_2 \mid x_i, \theta^{(0)}) \propto \frac{\pi_2}{\sqrt{2\pi}\,\sigma^2} \exp\left(-\frac{1}{2}\frac{(x-\mu_2)^2}{\sigma^2}\right)$$

⑥ ⑤代入③中 M.

④ ''' ?

$$\operatorname{argmax} \sum_{i=1}^{10} \sum_{k=1}^{2} Q_i^{(1)}(k)\left[\log \pi_k + \log p(x_i \mid \mu_k, \sigma_k^2)\right]$$

et on $\frac{\partial}{\partial \pi_1}$ $\frac{\partial}{\partial \mu_1}$ $\frac{\partial}{\partial \sigma_1}$ et idm π_2. μ_2. σ_2.

Ex4. ①. On cherche à estimer $P(A.B)$.

$A \backslash B$	b_1	b_2
a_1	θ_{11}	θ_{12}
a_2	θ_{21}	θ_{22}

avec $\theta_{22} = 1-(\theta_{11}+\theta_{12}+\theta_{21})$

(2). $P(A.B)$ uniforme

$A \backslash B$	b_1	b_2
a_1	0.25	0.25
a_2	0.25	0.25

$P(A|B)$

$A \backslash B$	b_1	b_2
a_1	0.5	0.5
a_2	0.5	0.5

$P(B|A)$

$A \backslash B$	b_1	b_2
a_1	0.5	0.5
a_2	0.5	0.5

$$Q_i^{(t+1)}(x_i^k) = 1 \quad \text{pour } i = \{1, 2, 5, 6, 9, 10\}.$$

$$\begin{cases} Q_3^{(t+1)}(a_1) = P(A=a_1 \mid B=b_2) = 0.5 \\ Q_3^{(t+1)}(a_2) = P(A=a_2 \mid B=b_1) = 0.5 \end{cases}$$

Idem pr les autres : 0.5 partout.

$$\log L^{(t+1)}(X^\theta, \Theta) = \sum_{i=1}^{n} \sum_{x_i^k \in X^k} Q_i^{(t+1)}(x_i^{k}) \log \frac{P(x_i^\theta, x_i^k / \Theta)}{Q_i^{(t+1)}(x_i^k)}$$

$$= \underline{\log \theta_{11}} + \log \theta_{21} + \left(0.5 \log \frac{\theta_{12}}{0.5} + 0.5 \log \frac{\theta_{22}}{0.5}\right).$$

$$+ \left(0.5 \log \frac{\theta_{21}}{0.5} + 0.5 \log \frac{\theta_{22}}{0.5}\right).$$

$$+ \log \theta_{22} + \log \theta_{12} + \left(0.5 \log \frac{\theta_{11}}{0.5} + 0.5 \log \frac{\theta_{21}}{0.5}\right)$$

$$+ (0.5 \log \frac{\theta_{11}}{0.5} + 0.5 \log \frac{\theta_{12}}{0.5})$$

$$+ \log \theta_{12} + \log \theta_{21}.$$

$$= 2 \log \theta_{11} + 3 \log \theta_{12} + 3 \log \theta_{21} + 2 \log \theta_{22} - 4 \log 0.5$$

$$\frac{d L^{(t+1)}(X^0 . \Theta)}{d \theta_{11}} = 0 \iff \frac{2}{\theta_{11}} - \frac{2}{\theta_{22}} = 0.$$

$$\frac{dL}{d\theta_{12}} = 0 \iff \frac{3}{\theta_{12}} - \frac{2}{\theta_{22}} = 0$$

$$\frac{dL}{d\theta_{21}} = 0 \iff \frac{3}{\theta_{21}} - \frac{2}{\theta_{22}} = 0.$$

$\theta_{11} = 0.2$ $\theta_{12} = 0.3$ $\theta_{21} = 0.3$ $\theta_{22} = 0.2$.

$P(A, B) =$

	b_1	b_2
a_1	0.2	0.3
a_2	0.3	0.2.

$P(A|B)$

	b_1	b_2.
a_1	0.4	0.6
a_2	0.6	0.4.

$\frac{P(A, B)}{P(B)}$

$P(B|A)$

	b_1	b_2.
a_1	0.4	0.6
a_2	0.6	0.4.