Algebraic System

14 October 2020 09:06

N-ary operation → $x_1 + x_2$

binary operation

Special elements →

↳ identity → may or may not exist → Unique for whole set (if exist)

→ zero

→ inverse → 10 → may or may not have inverse.

→ that element is called invertible

→ Cancellable

If inverse exist, then it would be Unique for each element.

$a * a^{-1} = e$ → a^{-1} is Unique

Properties — A, C, D, C } w.r.t.

Identity → for whole set (Unique)

inverse → would be Unique for each element if exist.

Algebraic System

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→ Def^n

↳ Examples and general properties. → Special algebraic System

→ homomorphism
→ isomorphism
→ Congruence relation
→ direct product
→ Sub-algebra.

} general terms.

→ App^n:-

n-ary. $n = 0, 1, 2, \ldots$ }

Def^n:→ n-ary operation for $n = 1, 2, \ldots 3$ on a set X $X^n \to n$

$n = 1 \to$ unary.
$n = 2$ binary.

$n = 0$ → special element
distinguish element:
↳ identity
→ zero
→ inverse

0-ary operation

+ 1e122

1e425 →

Algebraic System :→ A System Consisting of a set and one or more n-ary operations on the set is called Algebraic System.

−1, −2, 0, 1, 2, 3, 4, → Z →

+, ×, −

$\langle S, f_1, f_2 \cdots \rangle$ S → non-empty set

$f_1, f_2 \cdots$ → operations on set S.

algebraic Structure → also

$\langle I, +, \times \rangle$

I + → Addition

× → Multiplication

$\oplus$ → Symbol $\langle X, * \rangle$, $\langle Y, \oplus \rangle$

+ →

$*, +, \times, \oplus, \Delta$ etc. → Binary operation

Example.
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(1) $\underline{I} \to$ Set of Integers.

$\langle I, +, \times \rangle$ $+ \to$ Addition
$\times \to$ Multiplication

$\langle I, +, \times \rangle \approx \langle I, +, \times, 0, 1 \rangle$

distinguish element of $\underline{I}$.

Properties: $+ \to$ Addition

(A-1) for $a, b, c \in I$ $(a+b)+c = a+(b+c) \to$ Associative

(A-2) for $a, b \in I$ $a+b = b+a$ Commutative

(A-3) Identity element $0 \in \underline{I}$, for any $a \in \underline{I}$, $a+0 = 0+a = \underline{a}$ w.r.t. Addition
Is Unique

(A-4) for each $a \in \underline{I}$, there exist $a^{-1} \in I$ s.t. $a + a^{-1} = 0$
$a^{-1} = -a \to$ Unique for each $\bar{a}$

$\times \to$ Multiplication

(M-1) $\to$ Associative

(M-2) — Commutative

(M-3) — $1 \in \underline{I}$, $1 \times a = a \times 1 = \underline{a}$.

(D) $\to$ $\times$ distributes over $+$
$a \times (b+c) = (a \times b) + (a \times c)$.

(C) Cancellation property
for $a, b, c \in \underline{I}$, $a \neq 0$
$a \times b = a \times c \Rightarrow b = c$

$S \to$ non-empty set $\quad P(S) \to$ Power set

$A, B \in P(S)$, $\quad +$, $\times$ on $P(S)$

$A + B = (A - B) \cup (B - A) = (A \cap B^c) \cup (B \cap A^c)$

$A \times B = A \cap B$

$\langle P(S), +, \times \rangle \to$ it satisfies all the properties except (C)

	Identity
$+$	ϕ
$\times$	S

	Inverse
$+$	

$A \to$ $\quad A + a^{-1} = e$

ϕ

$A + A^{-1} = \phi$ $\quad A + A = \phi$

$A^{-1} = A$

$E \to$ Universal set

$P(E) \to$ Power set

$+ \to$ Union $\quad A + B = A \cup B$

$* \to$ Intersection $\quad A * B = A \cap B$

	Identity	(C)
$+$	$\phi \to$ any inverse X	
$*$	E	

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$B = \{0, 1\}$.

$\langle B, +, \times \rangle$

$\langle I, +, \times \rangle$

+	0	1
0	0	1
1	1	0

$\times$	0	1
0	0	0
1	0	1

$+ \to$

$\times$ —

$\to$ A-1, A-2, A-3, A-4

M-1, M-2, M-3, D, C

	identity
+	0
$\times$	1

$\langle I, +, \times \rangle$

I

$a \in X \to *$ $\quad x, y \in X$.

$e * a = a * e = a$

$0 * 0 = 0 * 0 = 0$

$0 * 1 = 1 * 0 = 1$

$0 \to$ identity element

$(a * x = a * y) \vee (x * a = y * a) \Rightarrow (x = y)$

If $*$ is associative, $a \in X$ is invertible, then a is Cancellable

Stacks

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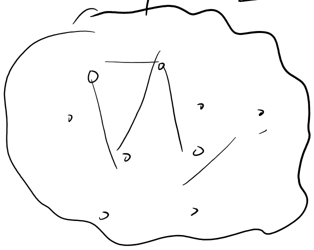
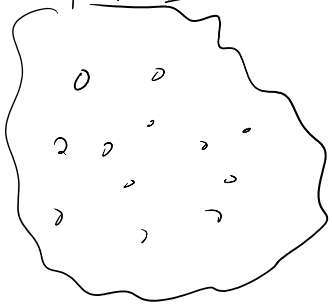
$\langle B, +, X \rangle$ $\langle I, +, X \rangle$ → set of op. $+, X$

$\langle B, +, \circ \rangle$ ↔ $\langle I, +, X \rangle$

properties → axioms   properties → behaviour



X



Set and the operations are just the symbol or more precisely

abstraction

$1+1=2$
$1+3=4$
$1+5=6$

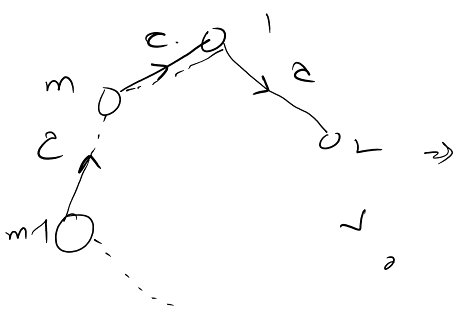
$\langle B, +, X \rangle$ share some of the features of abstract algebra

System of Integers.

Let $M = \{1, 2, \ldots m\}$ — $\langle M, \sigma \rangle$

$\sigma \to$ Unary operation

$$\sigma(j) = \begin{cases} j+1, & j \neq m \\ 1, & j = m \end{cases}$$



$f_4 \circ f_2(a) = f_4(f_2(a)) = f_4(a) = b$

$f_4 \circ f_2(b) = f_4(f_2(b)) = f_4(a) = b$

identity $\to$ $a * a^{-1} = e$

?? invertible or not ??

f_1 and f_4 are invertible elements

— f_2 and $f_3 \to$ X

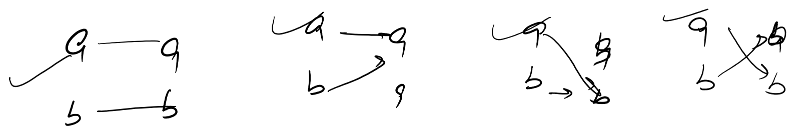
Let $X = \{a, b\}$ and

$S \to$ denotes the set of all the mappings $X \to X$

$S = \langle f_1, f_2, f_3, f_4 \rangle$

$f_1(a) = a$ $\quad f_2(a) = a$ $\quad f_3(a) = b$ $\quad f_4(a) = b$

$f_1(b) = b$ $\quad f_2(b) = a$ $\quad f_3(b) = b$ $\quad f_4(b) = a$



$\langle S, \circ \rangle$ $\circ \to$ left composition of function

$\circ$	f_1	f_2	f_3	f_4
f_1	f_1	f_2	f_3	f_4
f_2	f_2	f_2	f_2	f_2
f_3	f_3	f_3	f_3	f_3
f_4	f_4	f_3	f_2	f_1

$f_1 \to$ identity element

$f_1 \circ f_1 = f_1 \Rightarrow f_1^{-1} = f_1$

$f_4 \circ f_4 = f_1$ $\quad f_4^{-1} = f_4$

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$X \to \{1,2,3,4\}$ $f: X \to X$

$f = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 3,4\rangle, \langle 4,1\rangle\}$

f^0: identity mapping $f^0 = \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 4,4\rangle\}$

$f^1 = f$, $f^1 = \{\langle 1,2\rangle, \langle 2,3\rangle, \langle 3,4\rangle, \langle 4,1\rangle\}$

$f^2 = \{\langle 1,3\rangle, \langle 2,4\rangle, \langle 3,1\rangle, \langle 4,2\rangle\}$

$f^3 = \{\langle 1,4\rangle, \langle 2,1\rangle, \langle 3,2\rangle, \langle 4,3\rangle\}$

$f^4 \to \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 4,4\rangle\} = \underline{f^0}$

$F = \{f^0, f^1, f^2, f^3\}$

$f_0^{-1} \to f_0$ $f_2^{-1} = \underline{f_2}$

$f_1^{-1} = f_3$

$f_3^{-1} = f_1$

$\circ$	f_0	f_1	f_2	f_3
f_0	f_0	f_1	f_2	f_3
f_1	f_1	f_2	f_3	f_0
f_2	f_2	f_3	f_0	f_1
f_3	f_3	f_0	f_1	f_2

$f_0 \to$ identity element

$f_0 \circ f_0 = f_0$ $f_2 \circ f_2 = f_0$

$f_1 \circ f_3 = f_0$ $f_3 \circ f_1 = \underline{f_0}$

An Equivalence Relation Called "Congruence modulo m" on the set of integers

$Z \to$ Set of Integers

$R \to$ "Congruence modulo m"

$m = 4.$

$[1] \to$

$R = \{ \langle x, y \rangle \mid x \in Z \wedge y \in Z \wedge (x-y) \text{ is divisible by } m \}.$

$Z_4 \to$ Set of all equivalence classes.

$Z_4 = [0]\ [1]\ [2]\ [3]$

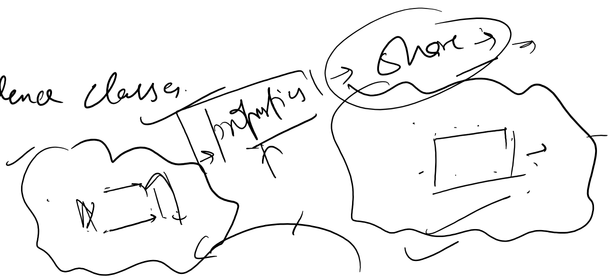
$m = 3.$

→ Equivalence classes.

$+_4 \to [i] +_4 [j] = [(i+j) \pmod 4]$

$[0].$

for $i, j = 0, 1, 2, 3.$



$+_4$	[0]	[1]	[2]	[3]
[0]	[0]	[1]	[2]	[3]
[1]	[1]	[2]	[3]	[0]
[2]	[2]	[3]	[0]	[1]
[3]	[3]	[0]	[1]	[2]

$\langle F, \circ \rangle \quad \langle Z, +_4 \rangle$

Set is different

$\circ \to +_4 \to$

w.r.t. $\circ$

abstract algebra.

$\langle F, \circ \rangle \longrightarrow \langle Z, +_4 \rangle$

You can define a mapping b/w two algebraic structures.

→ homomorphism
→ isomorphism

general form

Asso
C

→ Certain set of properties.

Semigroup → App^n. → Unit-4
Monoid →
Group
Abelian group

Un-3