Sets

Section 2.1

Section Summary [1]

Definition of sets

Describing Sets

- Roster Method
- Set-Builder Notation

Some Important Sets in Mathematics

Empty Set and Universal Set

Subsets and Set Equality

Cardinality of Sets

Tuples

Cartesian Product

Introduction

Sets are one of the basic building blocks for the types of objects considered in discrete mathematics.

- Important for counting.
- Programming languages have set operations.

Set theory is an important branch of mathematics.

- Many different systems of axioms have been used to develop set theory.
- Here we are not concerned with a formal set of axioms for set theory. Instead, we will use what is called naïve set theory.

Sets

A *set* is an unordered collection of objects.

- the students in this class
- the chairs in this room

The objects in a set are called the *elements*, or *members* of the set. A set is said to *contain* its elements.

The notation $a \in A$ denotes that a is an element of the set A.

If a is not a member of A, write $a \notin A$

Describing a Set: Roster Method

$S = \{a, b, c, d\}$

Order not important

$S = \{a, b, c, d\} = \{b, c, a, d\}$

Each distinct object is either a member or not; listing more than once does not change the set.

$S = \{a, b, c, d\} = \{a, b, c, b, c, d\}$

Elipses (…) may be used to describe a set without listing all of the members when the pattern is clear.

$S = \{a, b, c, d, \ldots, z\}$

Roster Method

Set of all vowels in the English alphabet:

$V = \{a, e, i, o, u\}$

Set of all odd positive integers less than 10:

$O = \{1, 3, 5, 7, 9\}$

Set of all positive integers less than 100:

$S = \{1, 2, 3, \ldots, 99\}$

Set of all integers less than 0:

$S = \{\ldots, -3, -2, -1\}$

Some Important Sets

N = *natural numbers* = {1, 2, 3, …}

W = *whole numbers* = {0, 1, 2, 3, …}

Z = *integers* = {…, -3, -2, -1, 0, 1, 2, 3, …}

$\mathbf{Z}^+$ = *positive integers* = {1, 2, 3,…}

R = set of *real numbers*

$\mathbf{R}^+$ = set of *positive real numbers*

C = set of *complex numbers*

Q = set of *rational numbers*

Set-Builder Notation

Specify the property or properties that all members must satisfy:

$S = \{x \mid x \text{ is a positive integer less than } 100\}$

$O = \{x \mid x \text{ is an odd positive integer less than } 10\}$

$O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}$

A predicate may be used:

$S = \{x \mid P(x)\}$

Example: $S = \{x \mid \text{Prime}(x)\}$

Positive rational numbers:

$\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p,q\}$

Interval Notation

$$[a,b] = \{x \mid a \le x \le b\}$$
$$[a,b) = \{x \mid a \le x < b\}$$
$$(a,b] = \{x \mid a < x \le b\}$$
$$(a,b) = \{x \mid a < x < b\}$$

closed interval [a, b]

open interval (a, b)

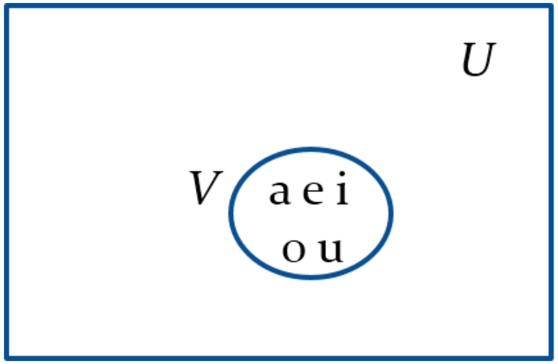
Universal Set and Empty Set

The *universal set U* is the set containing everything currently under consideration.

- Sometimes implicit
- Sometimes explicitly stated.
- Contents depend on the context.

The empty set is the set with no elements. Symbolized ∅, but { } also used.

Venn Diagram



John Venn (1834-1923)
Cambridge, UK

Russell's Paradox

Let S be the set of all sets which are not members of themselves. A paradox results from trying to answer the question "Is S a member of itself?"

Related Paradox:

- Henry is a barber who shaves all people who do not shave themselves. A paradox results from trying to answer the question "Does Henry shave himself?"

Some things to remember

Sets can be elements of sets.

$\{\{1, 2, 3\}, a, \{b, c\}\}$

$\{\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}\}$

The empty set is different from a set containing the empty set.

$\emptyset \neq \{\emptyset\}$

Set Equality

Definition: Two sets are *equal* if and only if they have the same elements.

- Therefore if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$
- We write *A* = *B* if *A* and *B* are equal sets.

$\{1, 3, 5\} = \{3, 5, 1\}$

$\{1, 5, 5, 5, 3, 3, 1\} = \{1, 3, 5\}$

Subsets

Definition: The set A is a *subset* of B, if and only if every element of A is also an element of B.

- The notation $A \subseteq B$ is used to indicate that A is a subset of the set B.
- $A \subseteq B$ holds if and only if $\forall x(x \in A \rightarrow x \in B)$ is true.
  1. Because $a \in \emptyset$ is always false, $\emptyset \subseteq S$ for every set S.
  2. Because $a \in S \rightarrow a \in S$, $S \subseteq S$ for every set S.

Showing a Set Is or Is Not a Subset of Another Set

Showing that *A* is a Subset of *B*: To show that $A \subseteq B$, show that if x belongs to A, then x also belongs to B.

Showing that *A* is not a Subset of *B*: To show that A is not a subset of B, $A \nsubseteq B$, find an element $x \in A$ with $x \notin B$ (such an x is a counterexample to the claim that $x \in A$ implies $x \in B$).

Examples:

1. The set of all computer science majors at your school is a subset of all students at your school.
2. The set of integers with squares less than 100 is not a subset of the set of nonnegative integers.

Another Look at Equality of Sets

Recall that two sets A and B are *equal*, denoted by $A = B$, iff

$$\forall x\left(x \in A \leftrightarrow x \in B\right)$$

Using logical equivalences we have that $A = B$ iff

$$\forall x\left[\left(x \in A \rightarrow x \in B\right) \wedge \left(x \in B \rightarrow x \in A\right)\right]$$

This is equivalent to

$$A \subseteq B \quad \text{and} \quad B \subseteq A$$

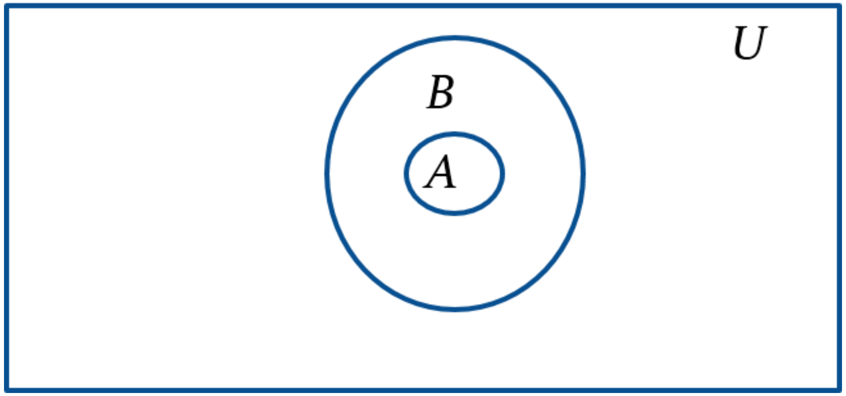
Proper Subsets

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B, denoted by $A \subset B$. If $A \subset B$, then

$$\forall x \wedge \left(x \in A \rightarrow x \in B\right) \wedge \exists x\left(x \in B \wedge x \notin A\right)$$

is true.

Venn Diagram



Set Cardinality

Definition: If there are exactly *n* distinct elements in *S* where *n* is a nonnegative integer, we say that *S* is *finite*. Otherwise it is *infinite*.

Definition: The *cardinality* of a finite set *A*, denoted by $|A|$, is the number of (distinct) elements of *A*.

Examples:

1. $|\emptyset| = 0$
2. Let S be the letters of the English alphabet. Then $|S| = 26$
3. $|\{1, 2, 3\}| = 3$
4. $|\{\emptyset\}| = 1$
5. The set of integers is infinite.

Power Sets

Definition: The set of all subsets of a set A, denoted $P(A)$, is called the *power set* of A.

Example: If $A = \{a, b\}$ then

$$P(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

If a set has n elements, then the cardinality of the power set is 2^n. (In Chapters 5 and 6, we will discuss different ways to show this.)

Tuples

The *ordered n-tuple* $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element and a_2 as its second element and so on until a_n as its last element.

Two n-tuples are equal if and only if their corresponding elements are equal.

2-tuples are called *ordered pairs*.

The ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.

Cartesian Product[1]

René Descartes
(1596-1650)

Definition: The *Cartesian Product* of two sets A and B, denoted by $A \times B$ is the set of ordered pairs (a, b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$$

Example:

$A = \{a,b\}$ $B = \{1,2,3\}$

$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

Definition: A subset R of the Cartesian product $A \times B$ is called a *relation* from the set A to the set B. (Relations will be covered in depth in Chapter 9.)

Cartesian Product 2

Definition: The Cartesian products of the sets $A_1, A_2, \ldots\ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots\ldots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where a_i belongs to A_i for $i = 1, \ldots, n$.

$$A_1 \times A_2 \times \cdots \times A_n =$$

$$\{(a_1, a_2 \cdots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Example: What is $A \times B \times C$ where $A = \{0, 1\}$, $B = \{1, 2\}$ and $C = \{0, 1, 2\}$

Solution: $A \times B \times C$ **=** {(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), (1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)}

Truth Sets of Quantifiers

Given a predicate *P* and a domain *D*, we define the *truth set* of *P* to be the set of elements in *D* for which *P*(*x*) is true. The truth set of *P*(x) is denoted by

$$\{x \in D \mid P(x)\}$$

Example: The truth set of *P*(*x*) where the domain is the integers and *P*(*x*) is "$|x| = 1$" is the set {-1, 1}

Set Operations

Section 2.2

Section Summary 2

Set Operations

- Union
- Intersection
- Complementation
- Difference

More on Set Cardinality

Set Identities

Proving Identities

Membership Tables

Boolean Algebra

Propositional calculus and set theory are both instances of an algebraic system called a *Boolean Algebra*. This is discussed in Chapter 12.

The operators in set theory are analogous to the corresponding operator in propositional calculus.

As always there must be a universal set U. All sets are assumed to be subsets of U.

Union

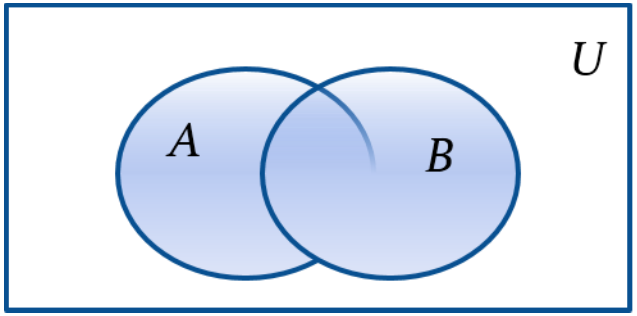
Definition: Let *A* and *B* be sets. The *union* of the sets *A* and *B*, denoted by *A* ∪ *B*, is the set:

$$\{x \mid x \in A \vee x \in B\}$$

Example: What is {1, 2, 3} ∪ {3, 4, 5}?

Solution: {1, 2, 3, 4, 5}

Venn Diagram for *A* ∪ *B*



Intersection

Definition: The *intersection* of sets *A* and *B*, denoted by $A \cap B$, is

$$\{x \mid x \in A \wedge x \in B\}$$

Note if the intersection is empty, then *A* and *B* are said to be *disjoint*.

Example: What is? $\{1, 2, 3\} \cap \{3, 4, 5\}$?

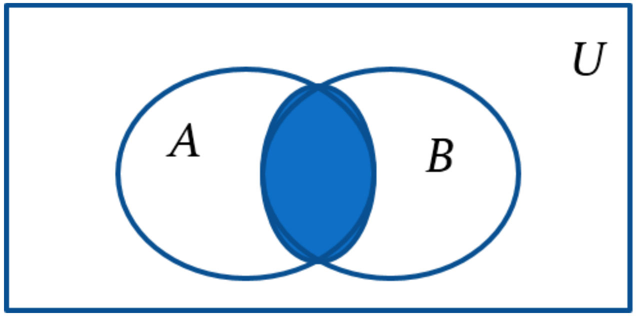
Solution: $\{3\}$

Example: What is?

$\{1, 2, 3\} \cap \{4, 5, 6\}$?

Solution: $\emptyset$

Venn Diagram for $A \cap B$



Complement

Definition: If A is a set, then the *complement* of the A (with respect to U), denoted by $\bar{A}$ is the set $U - A$

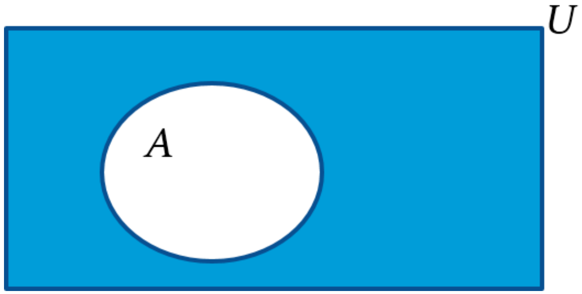
$$\overline{A} = \{x \mid x \in U \mid x \notin A\}$$

(The complement of A is sometimes denoted by A^c.)

Example: If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$

$$\text{Solution}: \{x \mid x \leq 70\}$$

Venn Diagram for Complement

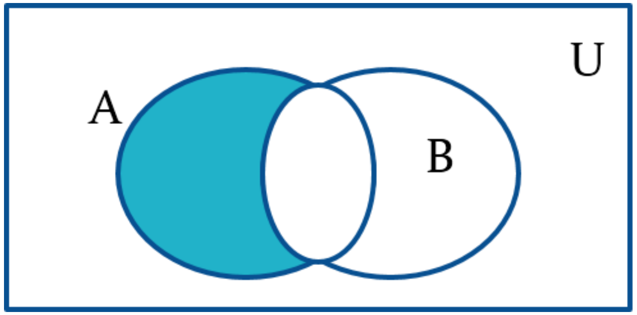


Difference

Definition: Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing the elements of A that are not in B. The difference of A and B is also called the complement of B with respect to A.

$$A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \overline{B}$$

Venn Diagram for $A - B$



The Cardinality of the Union of Two Sets

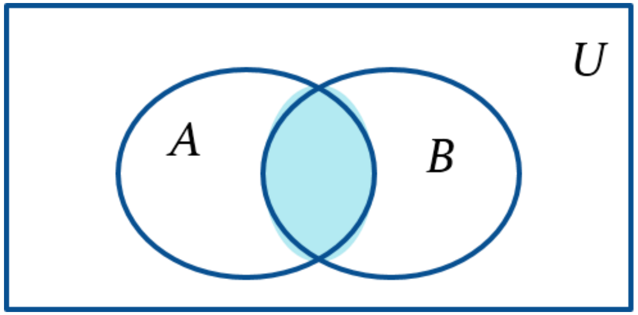
Inclusion-Exclusion

$|A \cup B| = |A| + |B| - |A \cap B|$

Example: Let A be the math majors in your class and B be the CS majors. To count the number of students who are either math majors or CS majors, add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.

We will return to this principle in Chapter 6 and Chapter 8 where we will derive a formula for the cardinality of the union of n sets, where n is a positive integer.

Venn Diagram for A, B, A ∩ B, A ∪ B



Review Questions

Example: $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7, 8\}$

1. $A \cup B$

   Solution: $\{1, 2, 3, 4, 5, 6, 7, 8\}$

2. $A \cap B$

   Solution: $\{4, 5\}$

3. $\bar{A}$

   Solution: $\{0, 6, 7, 8, 9, 10\}$

4. $\bar{B}$

   Solution: $\{0, 1, 2, 3, 9, 10\}$

5. $A - B$

   Solution: $\{1, 2, 3\}$

6. $B - A$

   Solution: $\{6, 7, 8\}$

Symmetric Difference (*optional*)

Definition: The *symmetric difference* of **A** and **B**, denoted by $A \oplus B$ is the set

$$(A-B)\cup(B-A)$$

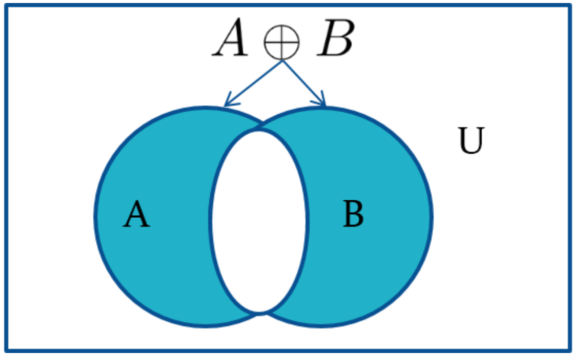
Example:

$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{1, 2, 3, 4, 5\}$ $B = \{4, 5, 6, 7, 8\}$

What is $A \oplus B$:

Solution: {1, 2, 3, 6, 7, 8}



Venn Diagram

Set Identities [1]

Identity laws

$$A \cup \varnothing = A \qquad A \cap U = A$$

Domination laws

$$A \cup U = U \qquad A \cap \varnothing = \varnothing$$

Idempotent laws

$$A \cup A = A \qquad A \cap A = A$$

Complementation law

$$\left(\overline{\overline{A}}\right) = A$$

Set Identities [2]

Commutative laws

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Set Identities 3

De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

Absorption laws

$$A \cup \left(A \cap B\right) = A \qquad A \cap \left(A \cup B\right) = A$$

Complement laws

$$A \cup \overline{A} = U \qquad A \cap \overline{A} = \varnothing$$

Proving Set Identities

Different ways to prove set identities:

1. Prove that each set (side of the identity) is a subset of the other.
2. Use set builder notation and propositional logic.
3. Membership Tables: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and a 0 to indicate that it is not

Proof of Second De Morgan Law 1

Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution: We prove this identity by showing that:

$$1)\ \overline{A \cap B} \subseteq \overline{A} \cup \overline{B} \qquad \text{and}$$

$$2)\ \overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$$

Proof of Second De Morgan Law 2

These steps show that: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	defn. of complement
$\neg\big((x \in A) \wedge (x \in B)\big)$	by defn. of intersection
$\neg(x \in A) \vee \neg(x \in B)$	1st De Morgan law for Prop Logic
$x \notin A \vee x \notin B$	defn. of negation
$x \in \overline{A} \vee x \in \overline{B}$	defn. of complement
$x \in \overline{A} \cup \overline{B}$	by defn. of union

Proof of Second De Morgan Law [3]

These steps show that: $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

$x \in \overline{A} \cup \overline{B}$	by assumption
$\left(x \in \overline{A}\right) \vee \left(x \in \overline{B}\right)$	by defn. of union
$\left(x \notin A\right) \vee \left(x \in \overline{B}\right)$	defn. of complement
$\neg\left(x \in A\right) \vee \neg\left(x \in B\right)$	defn. of negation
$\neg\left(\left(x \in A\right) \wedge \neg\left(x \in B\right)\right)$	1st De Morgan law for Prop Logic
$\neg\left(x \in A \cap B\right)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

Set-Builder Notation: Second De Morgan Law

$$
\begin{aligned}
\overline{A\cap B} &= x \in \overline{A\cap B} && \text{by defn. of complement}\\
&= \left\{x \mid \neg\left(x \in \left(A\cap B\right)\right)\right\} && \text{by defn. of does not belong symbol}\\
&= \left\{x \mid \neg\left(x \in A \wedge x \in B\right\}\right. && \text{by defn. of intersection}\\
&= \left\{x \mid \neg\left(x \in A\right) \vee \neg\left(x \in B\right)\right\} && \text{by 1st De Morgan law for Prop Logic}\\
&= \left\{x \mid x \notin A \vee x \notin B\right\} && \text{by defn. of not belong symbol}\\
&= \left\{x \mid x \in \overline{A} \vee x \in \overline{B}\right\} && \text{by defn. of complement}\\
&= \left\{x \mid x \in \overline{A} \cup \overline{B}\right\} && \text{by defn. of union}\\
&= \overline{A} \cup \overline{B} && \text{by meaning of notation}
\end{aligned}
$$

Membership Table

Example: Construct a membership table to show that the distributive law holds.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Solution:

A	***B***	***C***	***B* ∩ *C***	***A* ∪ (*B* ∩ *C*)**	***A* ∪ *B***	***A* ∪ *C***	**(*A* ∪ *B*) ∩ (*A* ∪ *C*)**
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

Generalized Unions and Intersections

Let $A_1, A_2, \ldots, A_n$ be an indexed collection of sets.

We define:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

These are well defined, since union and intersection are associative.

For $i = 1,2,\ldots$, let $A_i = \{i, i + 1, i + 2, \ldots\}$. Then,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{1, 2, 3, \ldots\}$$

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{n, n+1, n+2, \ldots\} = A_n$$

Functions

Section 2.3

Section Summary [3]

Definition of a Function

- Domain, Codomain
- Image, Preimage

Injection, Surjection, Bijection

Inverse Function

Function Composition

Graphing Functions

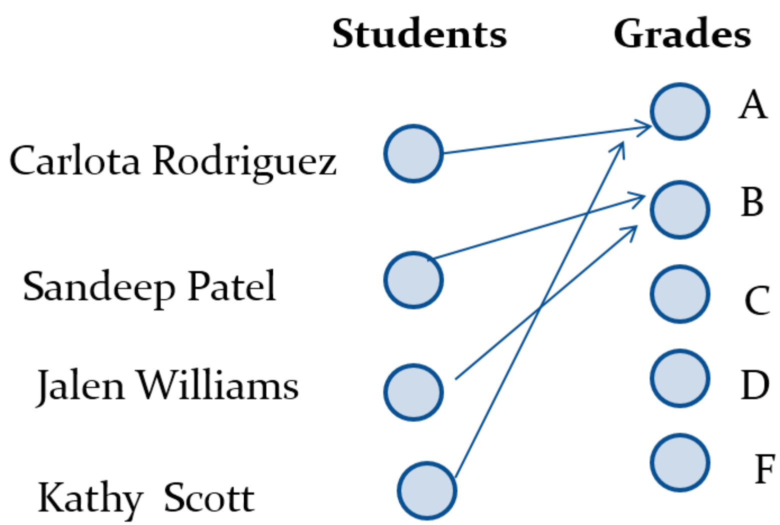
Floor, Ceiling, Factorial

Partial Functions (optional)

Functions [1]

Definition: Let A and B be nonempty sets. A *function* f from A to B, denoted $f: A \to B$ is an assignment of each element of A to exactly one element of B. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

- Functions are sometimes called *mappings* or *transformations*.



Functions[2]

A function f: $A \rightarrow B$ can also be defined as a subset of $A \times B$ (a relation). This subset is restricted to be a relation where no two elements of the relation have the same first element.

Specifically, a function *f* from *A* to *B* contains one, and only one ordered pair (*a, b*) for every element $a \in A$.

$$\forall x \left[x \in A \rightarrow \exists y \left[y \in B \wedge (x, y) \in f \right] \right]$$

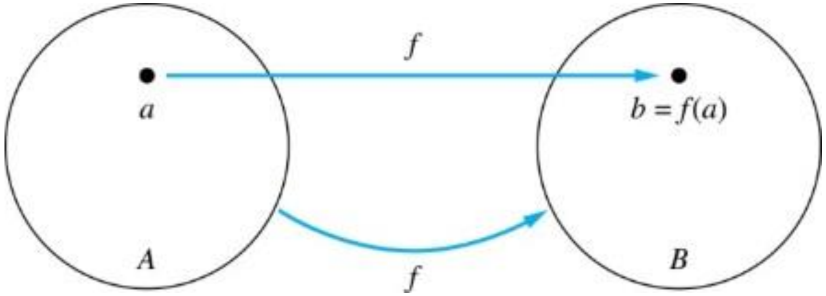
and

$$\forall x, y_1, y_2 \left[\left[(x, y_1) \in f \wedge (x, y_2) \in f \right] \rightarrow y_1 = y_2 \right]$$

Functions [3]

Given a function f: $A \rightarrow B$**:**

- We say f *maps* A to B *or* f is a *mapping* from A to B.
- A is called the *domain* of f.
- B is called the *codomain* of f.
- If $f(a) = b$,
  - then b is called the *image* of a under f.
  - a is called the *preimage* of b.



- The range of f is the set of all images of points in **A** under f. We denote it by $f(\boldsymbol{A})$.
- Two functions are *equal* when they have the same domain, the same codomain and map each element of the domain to the same element of the codomain.

Jump to long description

Representing Functions

Functions may be specified in different ways:

- An explicit statement of the assignment. Students and grades example.
- A formula.

$$f(x) = x + 1$$

- A computer program.
  - A Java program that when given an integer *n*, produces the *n*th Fibonacci Number (covered in the next section and also in Chapter 5).

Questions

$f(\mathrm{a})$ = ? z

The image of d is ? z

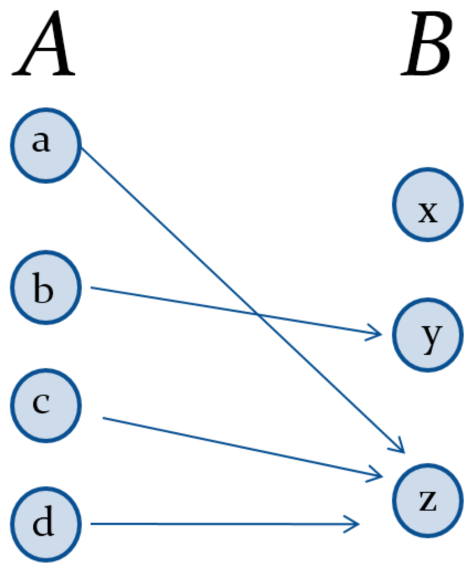
The domain of f is ? A

The codomain of f is ? B

The preimage of y is ? b

The range of f, $f(A)$ = ? {y, z}

The preimage(s) of z is (are) ? {a, c, d}



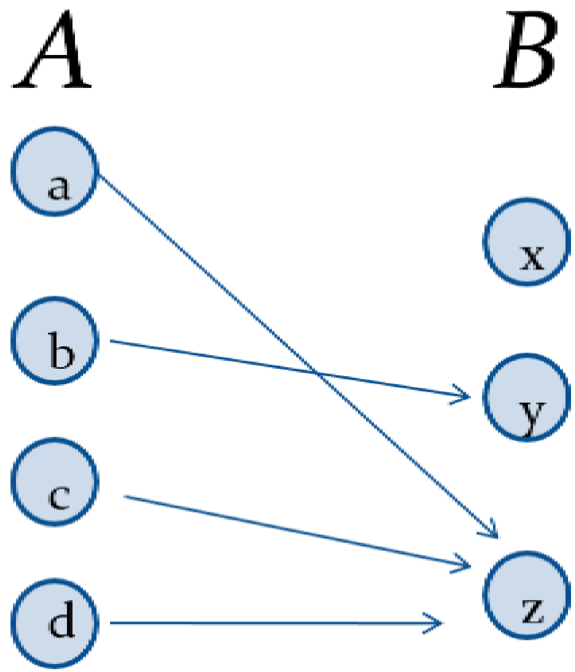
Question on Functions and Sets

If $f : A \rightarrow B$ and S is a subset of A, then

$$f(S) = \{ f(s) \mid s \in S \}$$

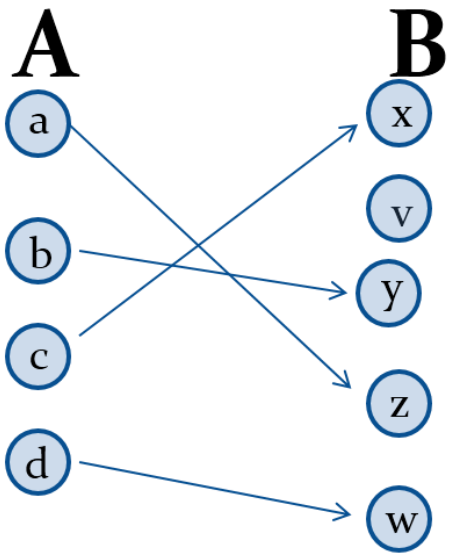
f {a,b,c,} is ? {y,z}

f {c,d} is ? {z}



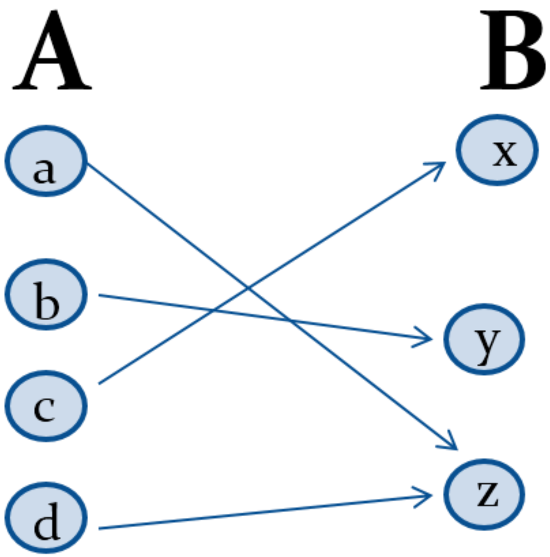
Injections

Definition: A function f is said to be *one-to-one*, or *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be an *injection* if it is one-to-one.



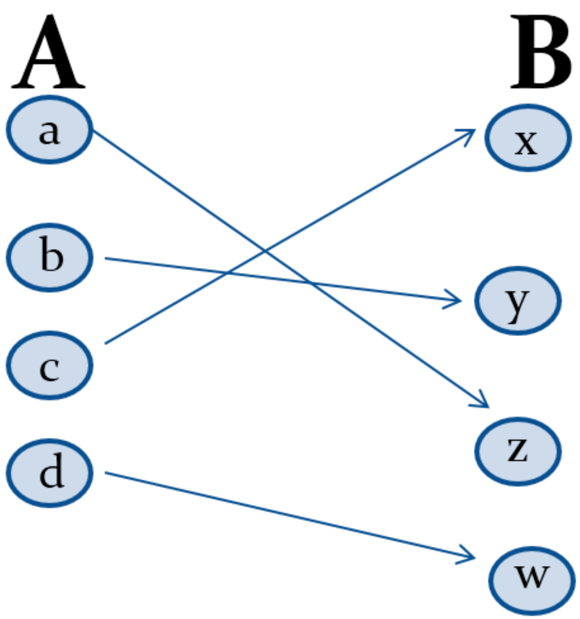
Surjections

Definition: A function *f* from *A* to *B* is called *onto,* or *surjective*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.
A function *f* is called a *surjection* if it is *onto*.



Bijections

Definition: A function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto (surjective and injective).



Showing that f is one-to-one or onto [1]

Suppose that $f : A \rightarrow B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Showing that *f* is one-to-one or onto 2

Example 1: Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

Solution: Yes, f is onto since all three elements of the codomain are images of elements in the domain. If the codomain were changed to $\{1, 2, 3, 4\}$, f would not be onto.

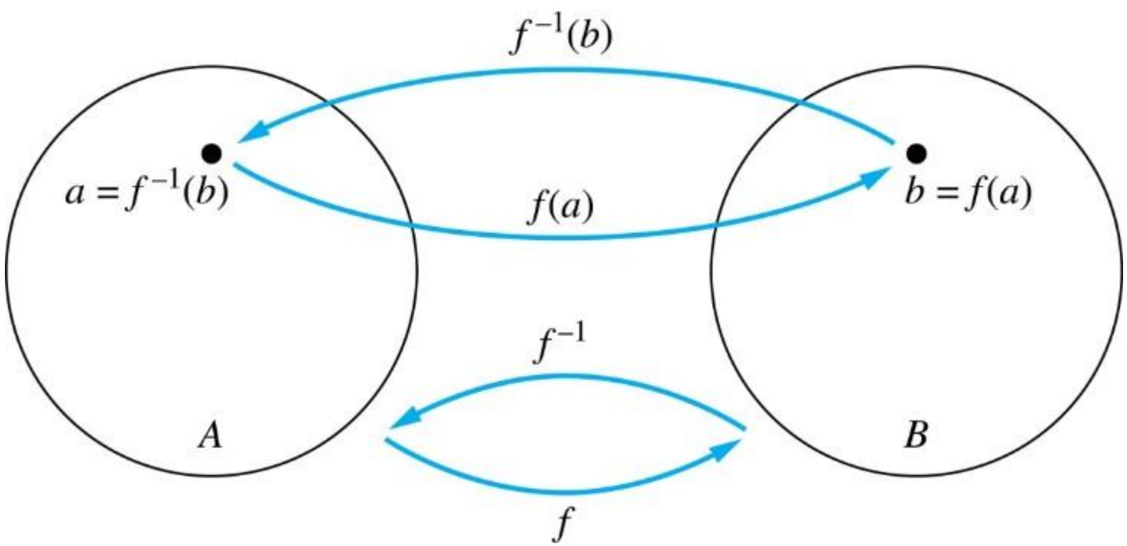
Example 2: Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

Solution: No, f is not onto because there is no integer x with $x^2 = -1$, for example.

Inverse Functions [1]

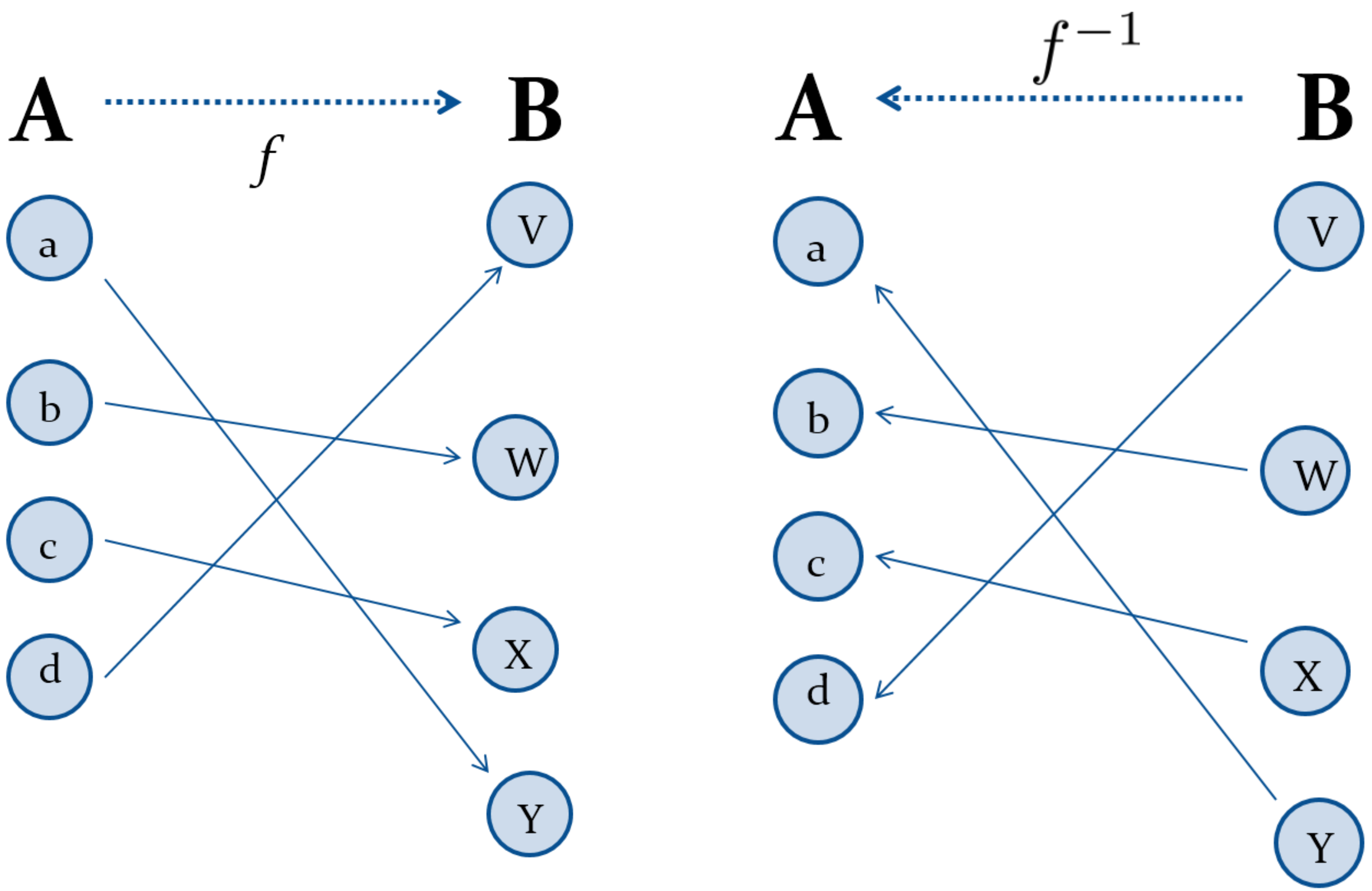
Definition: Let *f* be a bijection from *A* to *B*. Then the *inverse* of *f*, denoted f^{-1}, is the function from *B* to *A* defined as $f^{-1}(y) = x \text{ iff } f(x) = y$

No inverse exists unless *f* is a bijection. Why?



Jump to long description

Inverse Functions 2



Questions [1]

Example 1: Let f be the function from $\{a,b,c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible and if so what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Questions [2]

Example 2: Let f: **Z** $\rightarrow$ **Z** be such that $f(x) = x + 1$. Is f invertible, and if so, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence so $f^{-1}(y) = y - 1$.

Questions[3]

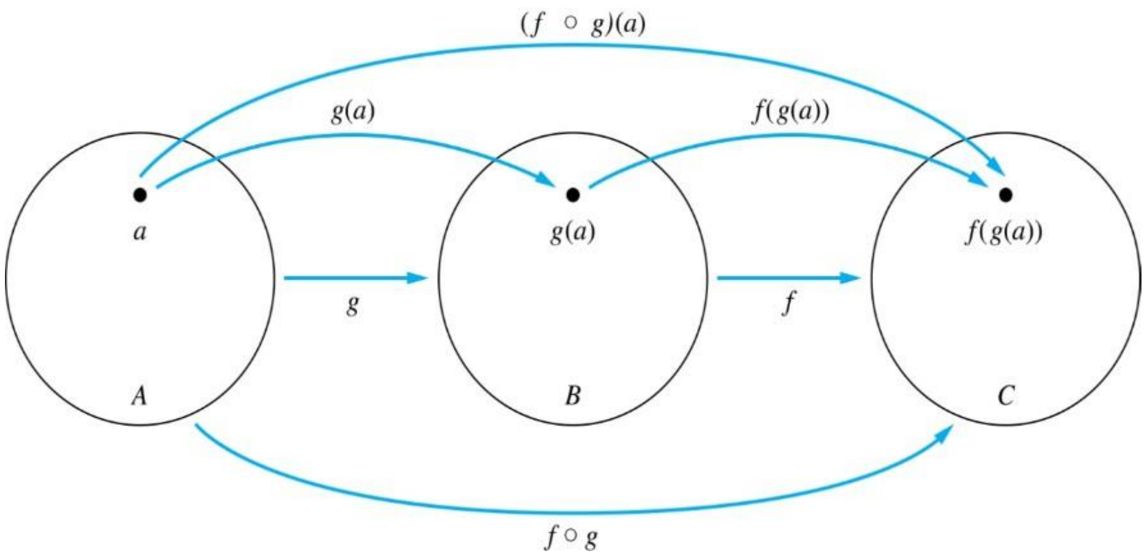
Example 3: Let *f*: **R** $\rightarrow$ **R** be such that $f(x) = x^2$
Is *f* invertible, and if so, what is its inverse?

Solution: The function *f* is not invertible because it is not one-to-one.

Composition [1]

Definition: Let *f*: *B*→*C*, *g*: *A*→*B*. The *composition of f with g*, denoted $f \circ g$ is the function from *A* to *C* defined by

$$f \circ g(x) = f(g(x))$$



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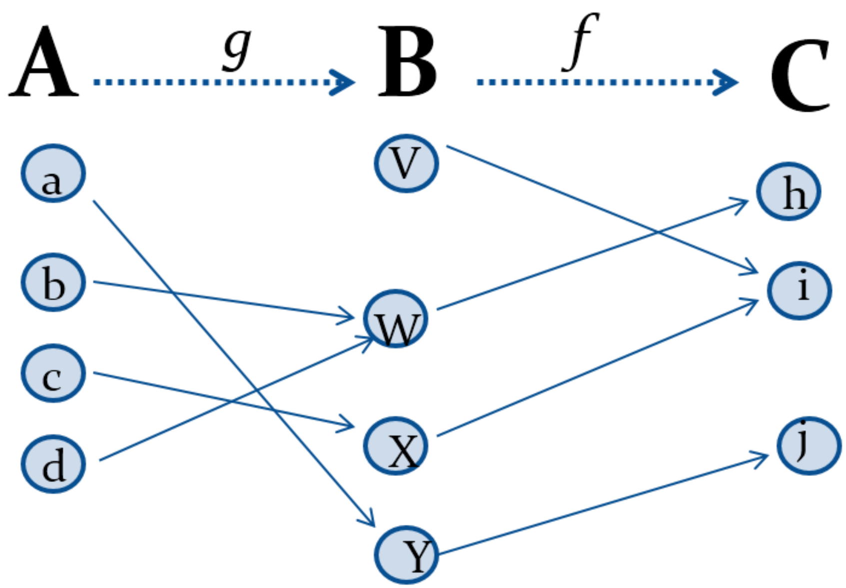
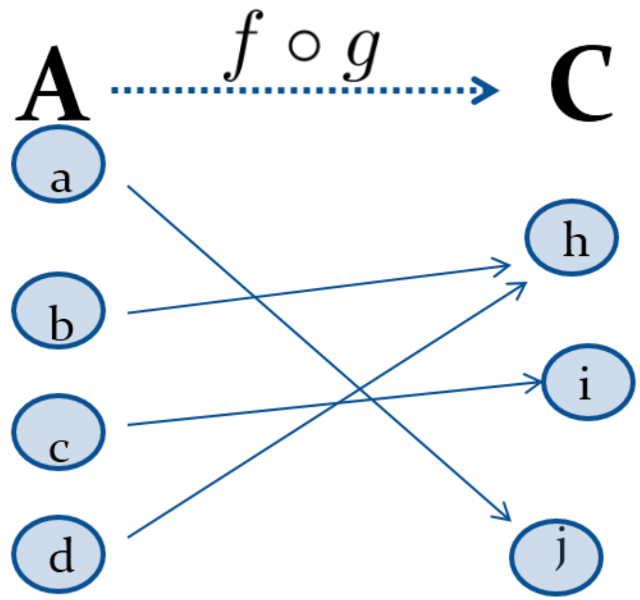
Composition 2





Composition[3]

Example 1: If

$$f(x) = x^2 \text{ and } g(x) = 2x + 1,$$

then

$$f(g(x)) = (2x+1)^2$$

and

$$g(f(x)) = 2x^2 + 1$$

Composition Questions[1]

Example 2: Let g be the function from the set $\{a, b, c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a, b, c\}$ to the set $\{1, 2, 3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

What is the composition of f and g, and what is the composition of g and f?

Solution: The composition $f \circ g$ is defined by

$$f \circ g(a) = f(g(a)) = f(b) = 2.$$

$$f \circ g(b) = f(g(b)) = f(c) = 1.$$

$$f \circ g(c) = f(g(c)) = f(a) = 3.$$

Note that $g \circ f$ is not defined, because the range of f is not a subset of the domain of g.

Composition Questions 2

Example 2: Let f and g be functions from the set of integers to the set of integers defined by

$$f(x) = 2x + 3 \quad \text{and} \quad g(x) = 3x + 2.$$

What is the composition of f and g, and also the composition of g and f?

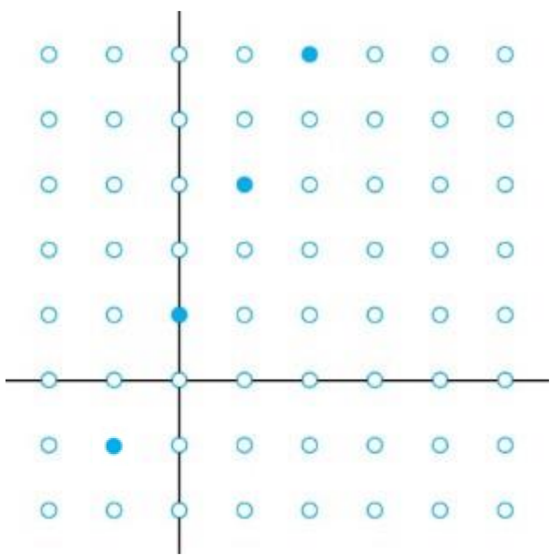
Solution:

$$f \circ g\ (x) = f\big(g(x)\big) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$

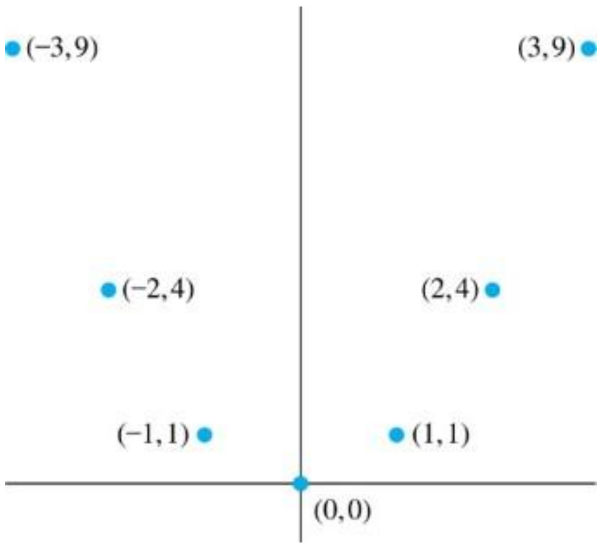
$$g \circ f\ (x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$$

Graphs of Functions

Let *f* be a function from the set *A* to the set *B*. The *graph* of the function *f* is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$.



Graph of *f*(*n*) = 2*n* + 1 from Z to Z



Graph of *f*(*x*) = x^2 from Z to Z

Jump to long description

Some Important Functions

The *floor* function, denoted

$$f(x) = \lfloor x \rfloor$$

is the largest integer less than or equal to *x*.

The *ceiling* function, denoted

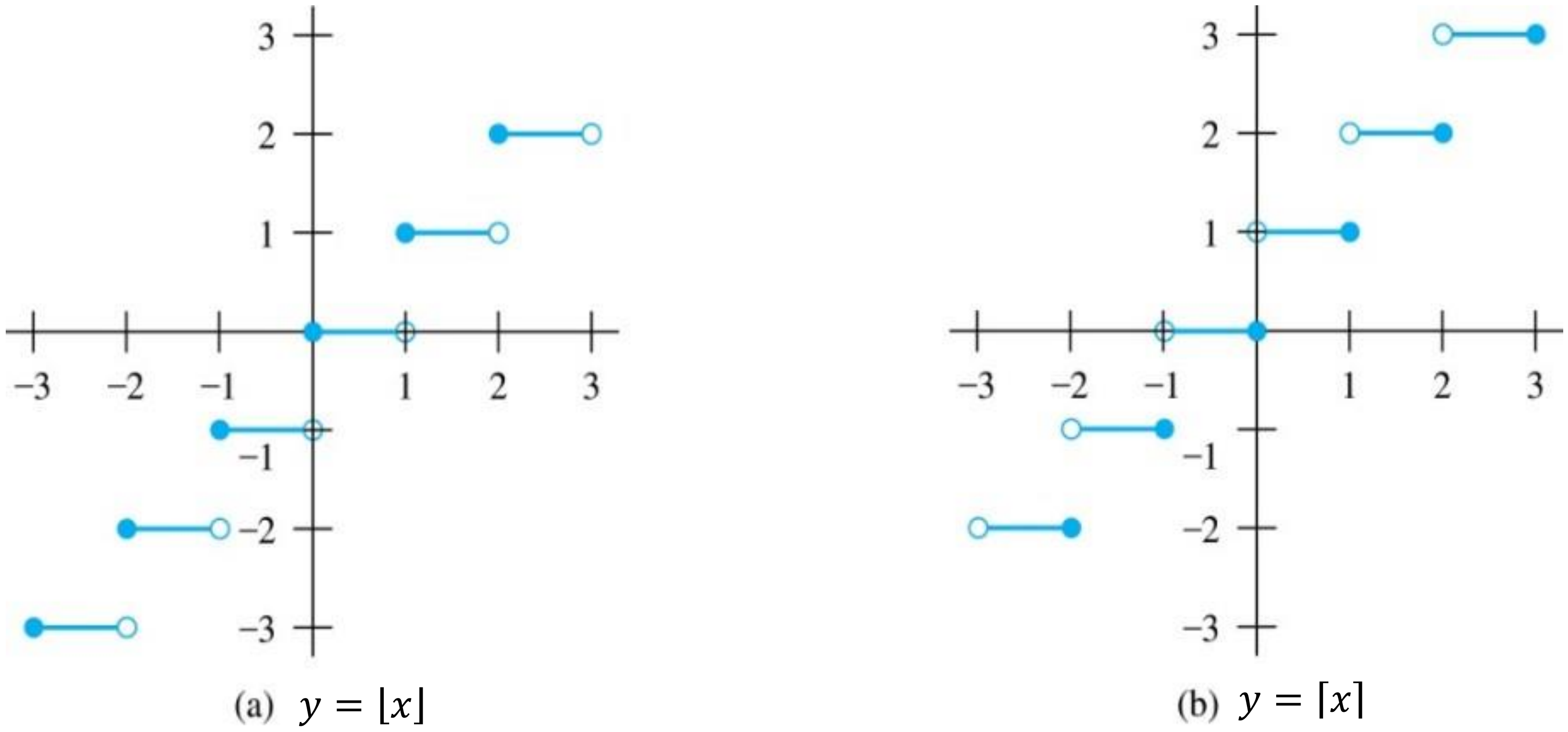
$$f(x) = \lceil x \rceil$$

is the smallest integer greater than or equal to *x*

Example: $\lceil 3.5 \rceil = 4$ $\lfloor 3.5 \rfloor = 3$

$\lceil -1.5 \rceil = -1$ $\lfloor -1.5 \rfloor = -2$

Floor and Ceiling Functions 1



Graph of (a) Floor and (b) Ceiling Functions

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Floor and Ceiling Functions 2

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(*n* is an integer, *x* is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n = x < n+1$
(1b)	$\lceil x \rceil = n$ if and only if $n-1 < x = n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x-1 < n = x$
(1d)	$\lceil x \rceil = n$ if and only if $x = n < x+1$
(2)	$x-1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x+1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x+n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x+n \rceil = \lceil x \rceil + n$

Proving Properties of Functions

Example: Prove that if x is a real number, then

$$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$$

Solution: Let $x = n + \varepsilon$, where n is an integer and $0 \le \varepsilon < 1$.

Case 1: $\varepsilon < ½$

- $2x = 2n + 2\varepsilon$ and $\lfloor 2x \rfloor = 2n$, since $0 \le 2\varepsilon < 1$.
- $\lfloor x + 1/2 \rfloor = n$, since $x + ½ = n + (1/2 + \varepsilon)$ and $0 \le ½ + \varepsilon < 1$.
- Hence, $\lfloor 2x \rfloor = 2n$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + n = 2n$.

Case 2: $\varepsilon \ge ½$

- $2x = 2n + 2\varepsilon = (2n + 1) + (2\varepsilon - 1)$ and $\lfloor 2x \rfloor = 2n + 1$, since $0 \le 2\varepsilon - 1 < 1$.
- $\lfloor x + 1/2 \rfloor = \lfloor n + (1/2 + \varepsilon) \rfloor = \lfloor n + 1 + (\varepsilon - 1/2) \rfloor = n + 1$ since $0 \le \varepsilon - 1/2 < 1$.
- Hence, $\lfloor 2x \rfloor = 2n + 1$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + (n + 1) = 2n + 1$.

Factorial Function

Definition: f: $\mathbf{N} \rightarrow \mathbf{Z}^+$ **,** denoted by $f(n) = n!$ is the product of the first n positive integers when n is a nonnegative integer.

$$f(n) = 1 \cdot 2 \cdots (n-1) \cdot n, \qquad f(0) = 0! = 1$$

Stirling's Formula:

$$n! \sim \sqrt{2\pi n}(n/e)^n$$

$$f(n) \sim g(n) \doteq lim_{n\to\infty} f(n)/g(n) = 1$$

Examples:

$$f(1) = 1! = 1$$

$$f(2) = 2! = 1 \cdot 2 = 2$$

$$f(6) = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$$

$$f(20) = 2,432,902,008,176,640,000.$$

Recurrence Relations

Definition: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.

The *initial conditions* for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

Questions about Recurrence Relations 1

Example 1: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1,2,3,4,\ldots.$ and suppose that $a_0 = 2$. What are a_1 , a_2 and a_3?

[Here $a_0 = 2$ is the initial condition.]

Solution: We see from the recurrence relation that

$$a_1 = a_0 + 3 = 2 + 3 = 5$$

$$a_2 = 5 + 3 = 8$$

$$a_3 = 8 + 3 = 11$$

Questions about Recurrence Relations 2

Example 2: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for n = 2,3,4,…. and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

[Here the initial conditions are $a_0 = 3$ and $a_1 = 5$.]

Solution: We see from the recurrence relation that

$$a_2 = a_1 - a_0 = 5 - 3 = 2$$

$$a_3 = a_2 - a_1 = 2 - 5 = -3$$

Fibonacci Sequence

Definition: Define the *Fibonacci sequence*, $f_0, f_1, f_2, \ldots,$ by:

- Initial Conditions: $f_0 = 0, f_1 = 1$
- Recurrence Relation: $f_n = f_{n-1} + f_{n-2}$

Example: Find f_2, f_3, f_4, f_5 and f_6.

Answer :

$$f_2 = f_1 + f_0 = 1 + 0 = 1,$$

$$f_3 = f_2 + f_1 = 1 + 1 = 2,$$

$$f_4 = f_3 + f_2 = 2 + 1 = 3,$$

$$f_5 = f_4 + f_3 = 3 + 2 = 5,$$

$$f_6 = f_5 + f_4 = 5 + 3 = 8.$$

Solving Recurrence Relations

Finding a formula for the *n*th term of the sequence generated by a recurrence relation is called *solving the recurrence relation*.

Such a formula is called a *closed formula*.

Various methods for solving recurrence relations will be covered in Chapter 8 where recurrence relations will be studied in greater depth.

Here we illustrate by example the method of iteration in which we need to guess the formula. The guess can be proved correct by the method of induction (Chapter 5).

Iterative Solution Example 1

Method 1: Working upward, forward substitution Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2,3,4,\ldots.$ and suppose that $a_1 = 2$.

$$a_2 = 2 + 3$$

$$a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$$

$$a_4 = (2 + 3 \cdot 2) + 3 = 2 + 3 \cdot 3$$

.

.

.

$$a_n = a_{n-1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n - 1)$$

Iterative Solution Example 2

Method 2: Working downward, backward substitution Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2,3,4,\ldots$. and suppose that $a_1 = 2$.

$$
\begin{aligned}
a_n &= a_{n-1} + 3 \\
&= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2 \\
&= (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3 \\
&\quad . \\
&\quad . \\
&\quad . \\
&= a_2 + 3(n-2) = (a_1 + 3) + 3(n-2) = 2 + 3(n-1)
\end{aligned}
$$

Financial Application [1]

Example: Suppose that a person deposits \$10,000.00 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after *n* years?

Let P_n denote the amount in the account after n years. P_n satisfies the following recurrence relation:

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)\, P_{n-1}$$

with the initial condition $P_0 = 10{,}000$

Financial Application[2]

$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)\, P_{n-1}$

with the initial condition $P_0 = 10{,}000$

Solution: Forward Substitution

$P_1 = (1.11)P_0$

$P_2 = (1.11)P_1 = (1.11)^2 P_0$

$P_3 = (1.11)P_2 = (1.11)^3 P_0$

$\vdots$

$P_n = (1.11)P_{n-1} = (1.11)^n \mathrm{P}_0 = (1.11)^n\, 10{,}000$

$P_n = (1.11)^n\, 10{,}000$ (Can prove by induction, covered in Ch. 5)

$P_{30} = (1.11)^{30}\, 10{,}000 = \$228{,}992.97$

Cardinality of Sets

Section 2.5

Section Summary[6]

Cardinality

Countable Sets

Computability

Cardinality[1]

Definition: The *cardinality* of a set A is equal to the cardinality of a set B, denoted

$$|A| = |B|,$$

if and only if there is a one-to-one correspondence (*i.e.*, a bijection) from A to B.

If there is a one-to-one function (*i.e.*, an injection) from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$.

When $|A| \leq |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and write $|A| < |B|$.

Cardinality 2

Definition: A set that is either finite or has the same cardinality as the set of positive integers ($\mathbf{Z}^+$) is called *countable*. A set that is not countable is *uncountable*.

The set of real numbers $\mathbf{R}$ is an uncountable set.

When an infinite set is countable (*countably infinite*) its cardinality is $\aleph_0$ (where $\aleph$ is aleph, the 1st letter of the Hebrew alphabet). We write $|S| = \aleph_0$ and say that S has cardinality "aleph null."

Showing that a Set is Countable

An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers).

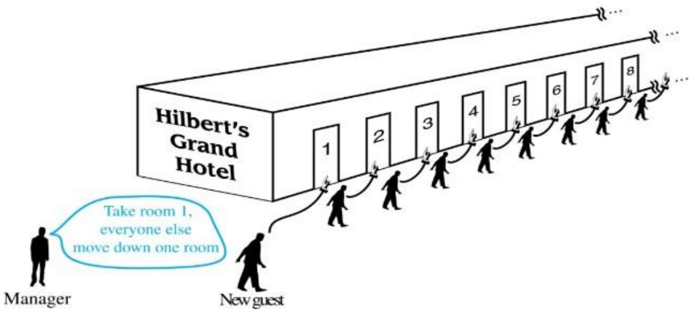
The reason for this is that a one-to-one correspondence f from the set of positive integers to a set S can be expressed in terms of a sequence $a_1, a_2, \ldots, a_n, \ldots$ where $a_1 = f(1)$, $a_2 = f(2), \ldots, a_n = f(n), \ldots$

Hilbert's Grand Hotel

David Hilbert

The Grand Hotel (example due to David Hilbert) has countably infinite number of rooms, each occupied by a guest. We can always accommodate a new guest at this hotel. How is this possible?

Explanation: Because the rooms of Grand Hotel are countable, we can list them as Room 1, Room 2, Room 3, and so on. When a new guest arrives, we move the guest in Room 1 to Room 2, the guest in Room 2 to Room 3, and in general the guest in Room n to Room $n + 1$, for all positive integers n. This frees up Room 1, which we assign to the new guest, and all the current guests still have rooms.



The hotel can also accommodate a countable number of new guests, all the guests on a countable number of buses where each bus contains a countable number of guests (see exercises).

Showing that a Set is Countable 1

Example 1: Show that the set of positive even integers *E* is countable set.

Solution: Let $f(x) = 2x$.

$$
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\
2 & 4 & 6 & 8 & 10 & 12
\end{array}
$$

Then f is a bijection from **N** to E since f is both one-to-one and onto. To show that it is one-to-one, suppose that $f(n) = f(m)$. Then $2n = 2m$, and so $n = m$. To see that it is onto, suppose that t is an even positive integer. Then $t = 2k$ for some positive integer k and $f(k) = t$.

Showing that a Set is Countable [2]

Example 2: Show that the set of integers **Z** is countable.

Solution: Can list in a sequence:

$0, 1, -1, 2, -2, 3, -3, \ldots\ldots\ldots$

Or can define a bijection from **N** to **Z**:

- When n is even: $f(n) = n/2$
- When n is odd: $f(n) = -(n-1)/2$

The Positive Rational Numbers are Countable[1]

Definition: A *rational number* can be expressed as the ratio of two integers p and q such that $q \neq 0$.

- ¾ is a rational number
- √2 is not a rational number.

Example 3: Show that the positive rational numbers are countable.

Solution: The positive rational numbers are countable since they can be arranged in a sequence:

$$r_1, r_2, r_3, \ldots$$

The next slide shows how this is done.

The Positive Rational Numbers are Countable 2

Constructing the List

First list p/q with $p + q = 2$.

Next list p/q with $p + q = 3$

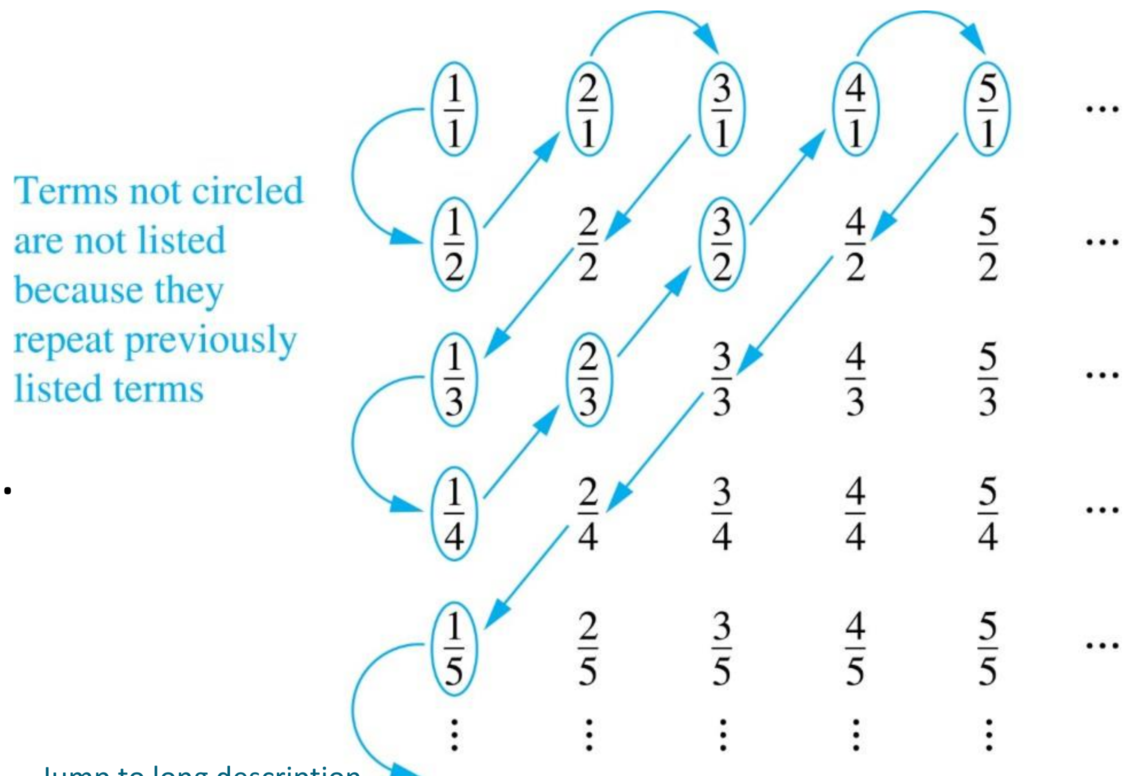
And so on.

1, ½, 2, 3, 1/3,1/4, 2/3, …

First row $q = 1$.

Second row $q = 2$.

etc.



Jump to long description

Strings 2

Example 4: Show that the set of finite strings *S* over a finite alphabet *A* is countably infinite.

Assume an alphabetical ordering of symbols in A

Solution: Show that the strings can be listed in a sequence. First list

1. All the strings of length 0 in alphabetical order.
2. Then all the strings of length 1 in lexicographic (as in a dictionary) order.
3. Then all the strings of length 2 in lexicographic order.
4. And so on.

This implies a bijection from **N** to *S* and hence it is a countably infinite set.

The set of all Java programs is countable.

Example 5: Show that the set of all Java programs is countable.

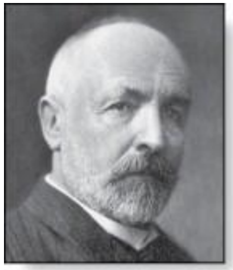
Solution: Let *S* be the set of strings constructed from the characters which can appear in a Java program. Use the ordering from the previous example. Take each string in turn:

- Feed the string into a Java compiler. (A Java compiler will determine if the input program is a syntactically correct Java program.)
- If the compiler says YES, this is a syntactically correct Java program, we add the program to the list.
- We move on to the next string.

In this way we construct an implied bijection from **N** to the set of Java programs. Hence, the set of Java programs is countable.

The Real Numbers are Uncountable

Georg Cantor
(1845-1918)



Example: Show that the set of real numbers is uncountable.

Solution: The method is called the Cantor diagnalization argument, and is a proof by contradiction.

1. Suppose **R** is countable. Then the real numbers between 0 and 1 are also countable (any subset of a countable set is countable - an exercise in the text).
2. The real numbers between 0 and 1 can be listed in order $r_1, r_2, r_3, \ldots$.
3. Let the decimal representation of this listing be
$$r_1 = 0.d_{11}d_{12}d_{13}d_{14}d_{15}d_{16}\ldots$$
$$r_2 = 0.d_{21}d_{22}d_{23}d_{24}d_{25}d_{26}\ldots$$
$$\vdots$$
4. Form a new real number with the decimal expansion $r = .r_1r_2r_3r_4\ldots$
 where $r_i = 3$ if $d_{ii} \neq 3$ and $r_i = 4$ if $d_{ii} = 3$
5. r is not equal to any of the $r_1, r_2, r_3, \ldots$ Because it differs from r_i in its ith position after the decimal point. Therefore there is a real number between 0 and 1 that is not on the list since every real number has a unique decimal expansion. Hence, all the real numbers between 0 and 1 cannot be listed, so the set of real numbers between 0 and 1 is uncountable.
6. Since a set with an uncountable subset is uncountable (an exercise), the set of real numbers is uncountable.

Computability (Optional)

Definition: We say that a function is **computable** if there is a computer program in some programming language that finds the values of this function. If a function is not computable we say it is **uncomputable**.

There are uncomputable functions. We have shown that the set of Java programs is countable. Exercise 38 in the text shows that there are uncountably many different functions from a particular countably infinite set (i.e., the positive integers) to itself. Therefore (Exercise 39) there must be uncomputable functions.