Relations

Chapter 9

Chapter Summary

Relations and Their Properties

n-ary Relations and Their Applications (*not currently included in overheads*)

Representing Relations

Closures of Relations

Equivalence Relations

Partial Orderings

Relations and Their Properties

Section 9.1

Section Summary

Relations and Functions

Properties of Relations

- Reflexive Relations
- Symmetric and Antisymmetric Relations
- Transitive Relations

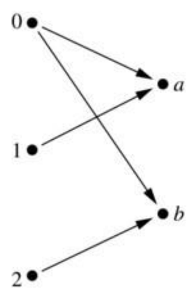
Combining Relations

Binary Relations

Definition: A *binary relation R* from a set *A* to a set *B* is a subset $R \subseteq A \times B$.

Example:

- Let *A* = {0,1,2} and *B* = {*a*,*b*}
- {(0, *a*), (0, *b*), (1,*a*) , (2, *b*)} is a relation from *A* to *B*.
- We can represent relations from a set *A* to a set *B* graphically or using a table:



R	a	b
0	×	×
1	×	
2		×

Relations are more general than functions. A function is a relation where exactly one element of *B* is related to each element of *A*.

Jump to long description

Binary Relations on a Set[1]

Definition: A binary relation R *on a set* A is a subset of $A \times A$ or a relation from A to A.

Example:

- Suppose that $A = \{a,b,c\}$. Then $R = \{(a,a),(a,b), (a,c)\}$ is a relation on A.
- Let $A = \{1, 2, 3, 4\}$. The ordered pairs in the relation R $= \{(a,b) \mid a \text{ divides } b\}$ are

  (1,1), (1, 2), (1,3), (1, 4), (2, 2), (2, 4), (3, 3), and (4, 4).

Binary Relations on a Set [2]

Question: How many relations are there on a set A?

Solution: Because a relation on A is the same thing as a subset of $A \times A$, we count the subsets of $A \times A$. Since $A \times A$ has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are $2^{|A|^2}$ subsets of $A \times A$. Therefore, there are $2^{|A|^2}$ relations on a set A.

Binary Relations on a Set 3

Example: Consider these relations on the set of integers:

$$R_1 = \{(a,b) \mid a \le b\},$$
$$R_2 = \{(a,b) \mid a > b\},$$
$$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\},$$
$$R_4 = \{(a,b) \mid a = b\},$$
$$R_5 = \{(a,b) \mid a = b + 1\},$$
$$R_6 = \{(a,b) \mid a + b \le 3\}.$$

Note that these relations are on an infinite set and each of these relations is an infinite set.

Which of these relations contain each of the pairs

(1,1), (1, 2), (2, 1), (1, −1), and (2, 2)?

Solution: Checking the conditions that define each relation, we see that the pair (1,1) is in R_1, R_3, R_4 , and R_6: (1,2) is in R_1 and R_6: (2,1) is in R_2, R_5, and R_6: (1, −1) is in R_2, R_3, and R_6 : (2,2) is in R_1, R_3, and R_4.

Reflexive Relations

Definition: *R* is *reflexive* iff (*a*,*a*) ∊ *R* for every element *a* ∊ A. Written symbolically, R is reflexive if and only if

$$\forall x \left[\mathrm{x} \in U \rightarrow (x,x) \in R \right]$$

Example: The following relations on the integers are reflexive:

$$R_1 = \left\{ (a,b) \middle| a \leq b \right\},$$
$$R_3 = \left\{ (a,b) \middle| a = b \text{ or } a = -b \right\},$$
$$R_4 = \left\{ (a,b) \middle| a = b \right\}.$$

The following relations are not reflexive:

$$R_2 = \left\{ (a,b) \middle| a > b \right\} \quad (\text{note that } 3 \ngtr 3),$$
$$R_5 = \left\{ (a,b) \middle| a = b + 1 \right\} \quad (\text{note that } 3 \neq 3 + 1),$$
$$R_6 = \left\{ (a,b) \middle| a + b \leq 3 \right\} \quad (\text{note that } 4 + 4 \nleq 3).$$

If *A* = ∅ then the empty relation is reflexive vacuously. That is the empty relation on an empty set is reflexive!

Symmetric Relations

Definition: R is *symmetric* iff $(b,a) \in R$ whenever $(a,b) \in R$ for all $a,b \in A$. Written symbolically, R is symmetric if and only if

$$\forall x \forall y \left[(x,y) \in R \rightarrow (y,x) \in R \right]$$

Example: The following relations on the integers are symmetric:

$$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\},$$

$$R_4 = \{(a,b) \mid a = b\},$$

$$R_6 = \{(a,b) \mid a + b \leq 3\}.$$

The following are not symmetric:

$$R_1 = \{(a,b) \mid a \leq b\} \quad (\text{note that } 3 \leq 4, \text{ but } 4 \not\leq 3),$$

$$R_2 = \{(a,b) \mid a > b\} \quad (\text{note that } 4 \leq 3, \text{ but } 3 \not> 4),$$

$$R_5 = \{(a,b) \mid a = b + 1\} \quad (\text{note that } 4 = 3 + 1, \text{ but } 3 \neq 4 + 1).$$

Antisymmetric Relations

Definition: A relation R on a set A such that for all $a,b \in A$ if $(a,b) \in R$ and $(b,a) \in R$, then $a = b$ is called *antisymmetric*. Written symbolically, R is antisymmetric if and only if

$$\forall x \forall y \left[(x,y) \in R \wedge (y,x) \in R \rightarrow x = y \right]$$

Example: The following relations on the integers are antisymmetric:

$$R_1 = \{(a,b) | a \leq b\},$$

For any integer, if a $a \leq b$ and $b \leq a$, then $a = b$.

$$R_2 = \{(a,b) | a > b\},$$
$$R_4 = \{(a,b) | a = b\},$$
$$R_5 = \{(a,b) | a = b + 1\}.$$

The following relations are notantisymmetric:

$$R_3 = \{(a,b) | a = b \text{ or } a = -b\}$$
$$\left(\text{note that both } (1,-1) \text{ and } (-1,1) \text{ belongs to } R_3\right),$$
$$R_6 = \{(a,b) | a + b \leq 3\} \ \left(\text{note that both } (1,2) \text{ and } (2,1) \text{ belongs to } R_6\right).$$

Transitive Relations

Definition: A relation R on a set A is called transitive if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a,b,c \in A$. Written symbolically, R is transitive if and only if

$$\forall x \forall y \forall z \left[(x,y) \in R \wedge (y,z) \in R \rightarrow (x,z) \in R \right]$$

Example: The following relations on the integers are transitive:

$R_1 = \{(a,b) \mid a \le b\}$, ← For every integer, $a \le b$ and $b \le c$, then $a \le c$.

$R_2 = \{(a,b) \mid a > b\}$,

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,

$R_4 = \{(a,b) \mid a = b\}$.

The following are not transitive:

$R_5 = \{(a,b) \mid a = b+1\}$ (note that both $(3,2)$ and $(4,3)$ belongs to R_5, but not (4,2)),

$R_6 = \{(a,b) \mid a+b \le 3\}$ (note that both $(2,1)$ and $(1,2)$ belongs to R_6, but not $(2,2)$).

Combining Relations

Given two relations R_1 and R_2, we can combine them using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and $R_2 - R_1$.

Example: Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relations $R_1 = \{(1,1),(2,2),(3,3)\}$ and $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$ can be combined using basic set operations to form new relations:

$$R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2)(3,3)\}$$

$$R_1 \cap R_2 = \{(1,1)\} \qquad R_1 - R_2 = \{(2,2),(3,3)\}$$

$$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$$

Composition

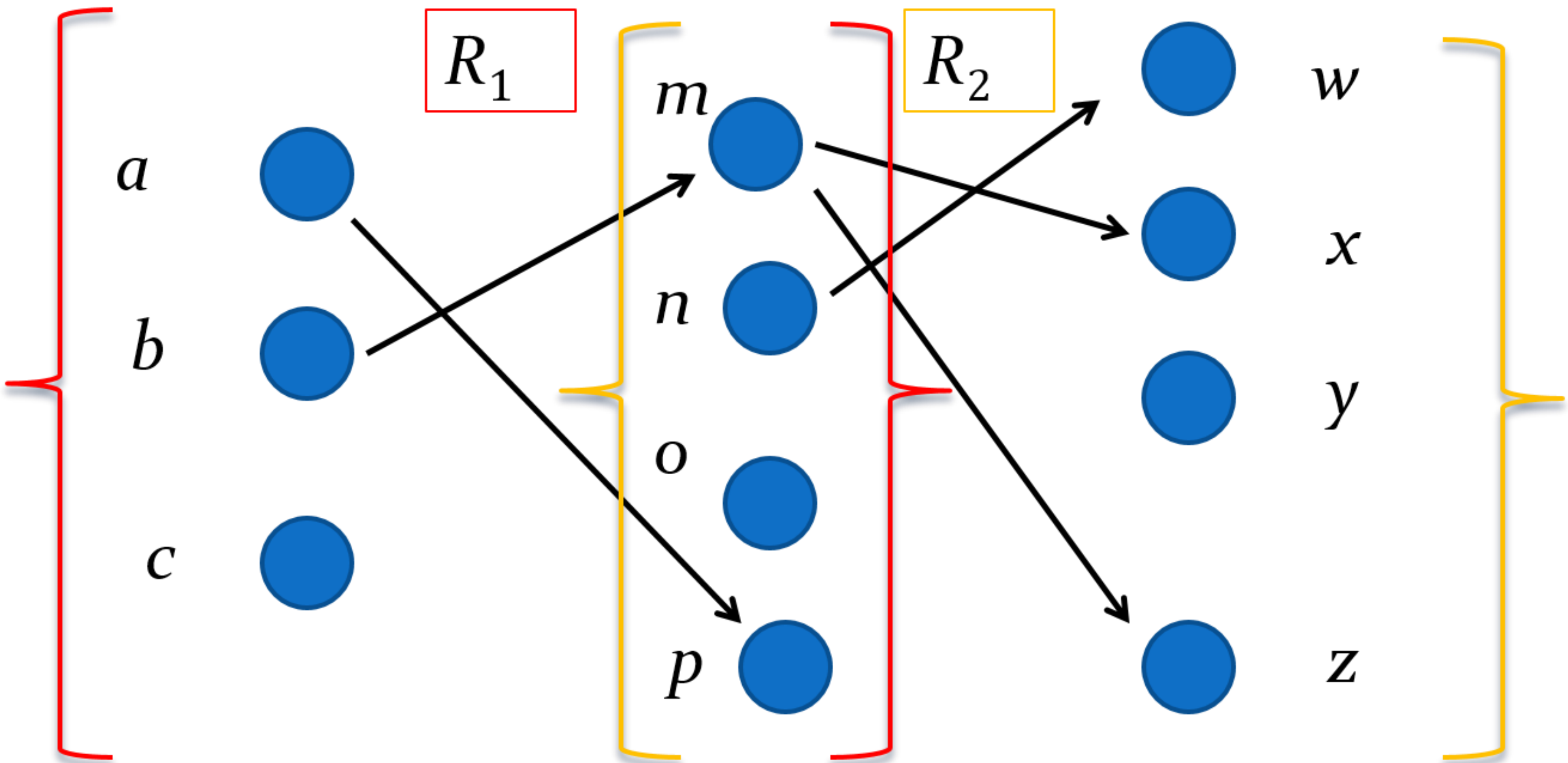
Definition: Suppose

- R_1 is a relation from a set A to a set B.
- R_2 is a relation from B to a set C.

Then the *composition* (or *composite*) of R_2 with R_1, is a relation from A to C where

- if (x,y) is a member of R_1 and (y,z) is a member of R_2, then (x,z) is a member of $R_2 \circ R_1$.

Representing the Composition of Relations



$$R_2 \circ R_1 = \{(b, x), (b, z)\}$$

Powers of a Relation

Definition: Let R be a binary relation on A. Then the powers R^n of the relation R can be defined inductively by:

- Basis Step: $R^1 = R$
- Inductive Step: $R^{n+1} = R^n \circ R$

(*see the slides for Section* 9.3 *for further insights*)

The powers of a transitive relation are subsets of the relation. This is established by the following theorem:

Theorem 1: The relation R on a set A is transitive iff $R^n \subseteq R$ for $n = 1,2,3 \ldots.$

(*see the text for a proof via mathematical induction*)

Representing Relations

Section 9.3

Section Summary

Representing Relations using Matrices

Representing Relations using Digraphs

Representing Relations Using Matrices

A relation between finite sets can be represented using a zero-one matrix.

Suppose R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$.

- The elements of the two sets can be listed in any particular arbitrary order. When $A = B$, we use the same ordering.

The relation R is represented by the matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R, \\ 0 \text{ if } (a_i, b_j) \notin R. \end{cases}$$

The matrix representing R has a 1 as its (i,j) entry when a_i is related to b_j and a 0 if a_i is not related to b_j.

Examples of Representing Relations Using Matrices [1]

Example 1: Suppose that $A = \{1,2,3\}$ and $B = \{1,2\}$. Let R be the relation from A to B containing (a,b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R (assuming the ordering of elements is the same as the increasing numerical order)?

Solution: Because $R = \{(2,1), (3,1),(3,2)\}$, the matrix is

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Examples of Representing Relations Using Matrices [2]

Example 2: Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

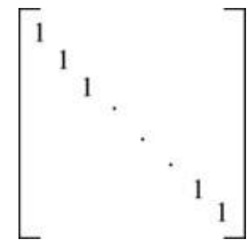
$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

Solution: Because R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$, it follows that:

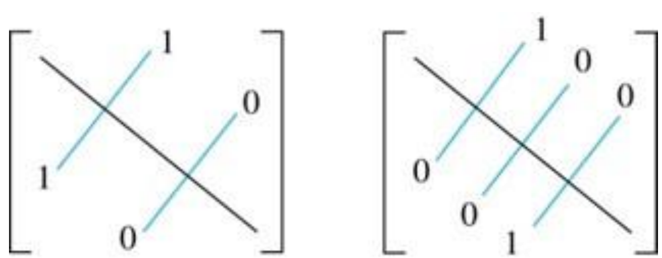
$$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}.$$

Matrices of Relations on Sets

If R is a reflexive relation, all the elements on the main diagonal of M_R are equal to 1.



R is a symmetric relation, if and only if $m_{ij} = 1$ whenever $m_{ji} = 1$. R is an antisymmetric relation, if and only if $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.



(a) Symmetric (b) Antisymmetric

Jump to long description

Example of a Relation on a Set

Example 3: Suppose that the relation *R* on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Is *R* reflexive, symmetric, and/or antisymmetric?

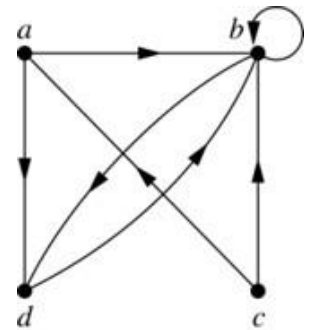
Solution: Because all the diagonal elements are equal to 1, *R* is reflexive. Because M_R is symmetric, *R* is symmetric and not antisymmetric because both $m_{1,2}$ and $m_{2,1}$ are 1.

Representing Relations Using Digraphs

Definition: A *directed graph*, or *digraph*, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex a is called the *initial vertex* of the edge (a,b), and the vertex b is called the *terminal vertex* of this edge.

- An edge of the form (a,a) is called a *loop*.

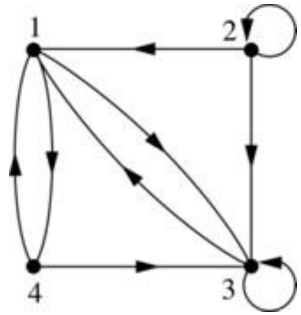
Example 7: A drawing of the directed graph with vertices a, b, c, and d, and edges (a, b), (a, d), (b, b), (b, d), $(c,$ a$)$, (c, b), and (d, b) is shown here.



Jump to long description

Examples of Digraphs Representing Relations

Example 8: What are the ordered pairs in the relation represented by this directed graph?



Solution: The ordered pairs in the relation are $(1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3),$ $(4, 1)$, and $(4, 3)$

Jump to long description

Determining which Properties a Relation has from its Digraph

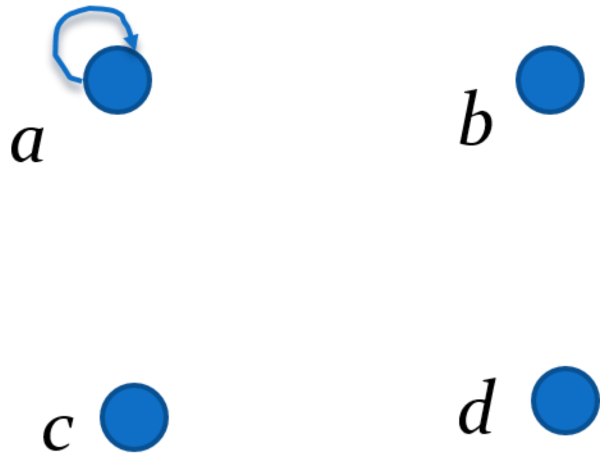
Reflexivity: A loop must be present at all vertices in the graph.

Symmetry: If (x,y) is an edge, then so is (y,x).

Antisymmetry: If (x,y) with $x \neq y$ is an edge, then (y,x) is not an edge.

Transitivity: If (x,y) and (y,z) are edges, then so is (x,z).

Determining which Properties a Relation has from its Digraph – Example 1



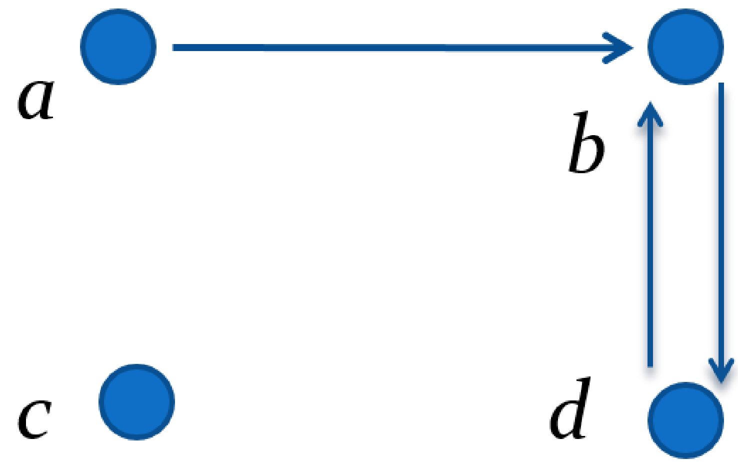
Reflexive? No, not every vertex has a loop

Symmetric? Yes (trivially), there is no edge from one vertex to another

Antisymmetric? Yes (trivially), there is no edge from one vertex to another

Transitive? Yes, (trivially) since there is no edge from one vertex to another

Determining which Properties a Relation has from its Digraph – Example 2



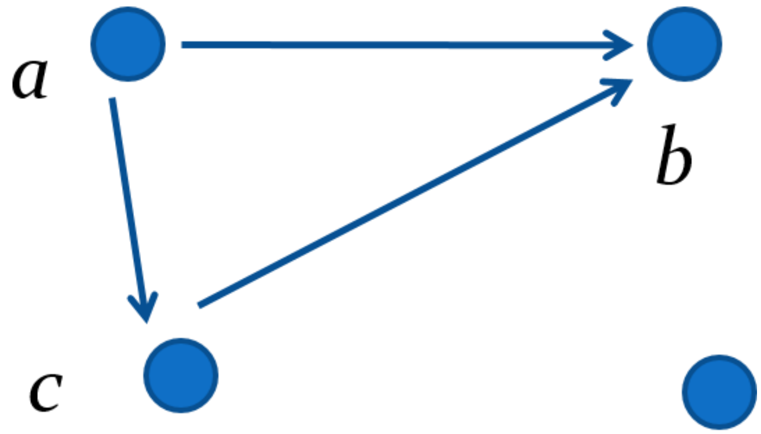
Reflexive? No, there are no loops

Symmetric? No, there is an edge from *a* to *b*, but not from *b* to *a*

Antisymmetric? No, there is an edge from *d* to *b* and *b* to *d*

Transitive? No, there are edges from *a* to *c* and from *c* to *b*, but there is no edge from *a* to *d*

Determining which Properties a Relation has from its Digraph – Example 3



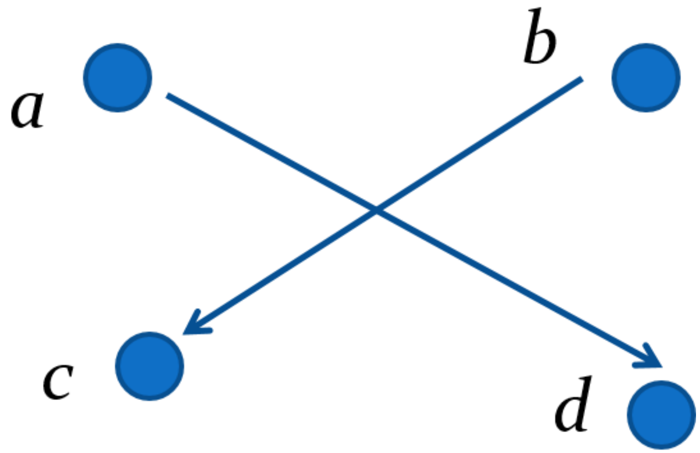
Reflexive? No, there are no loops

Symmetric? No, for example, there is no edge from *c* to *a*

Antisymmetric? Yes, whenever there is an edge from one vertex to another, there is not one going back

Transitive? Yes.

Determining which Properties a Relation has from its Digraph – Example 4



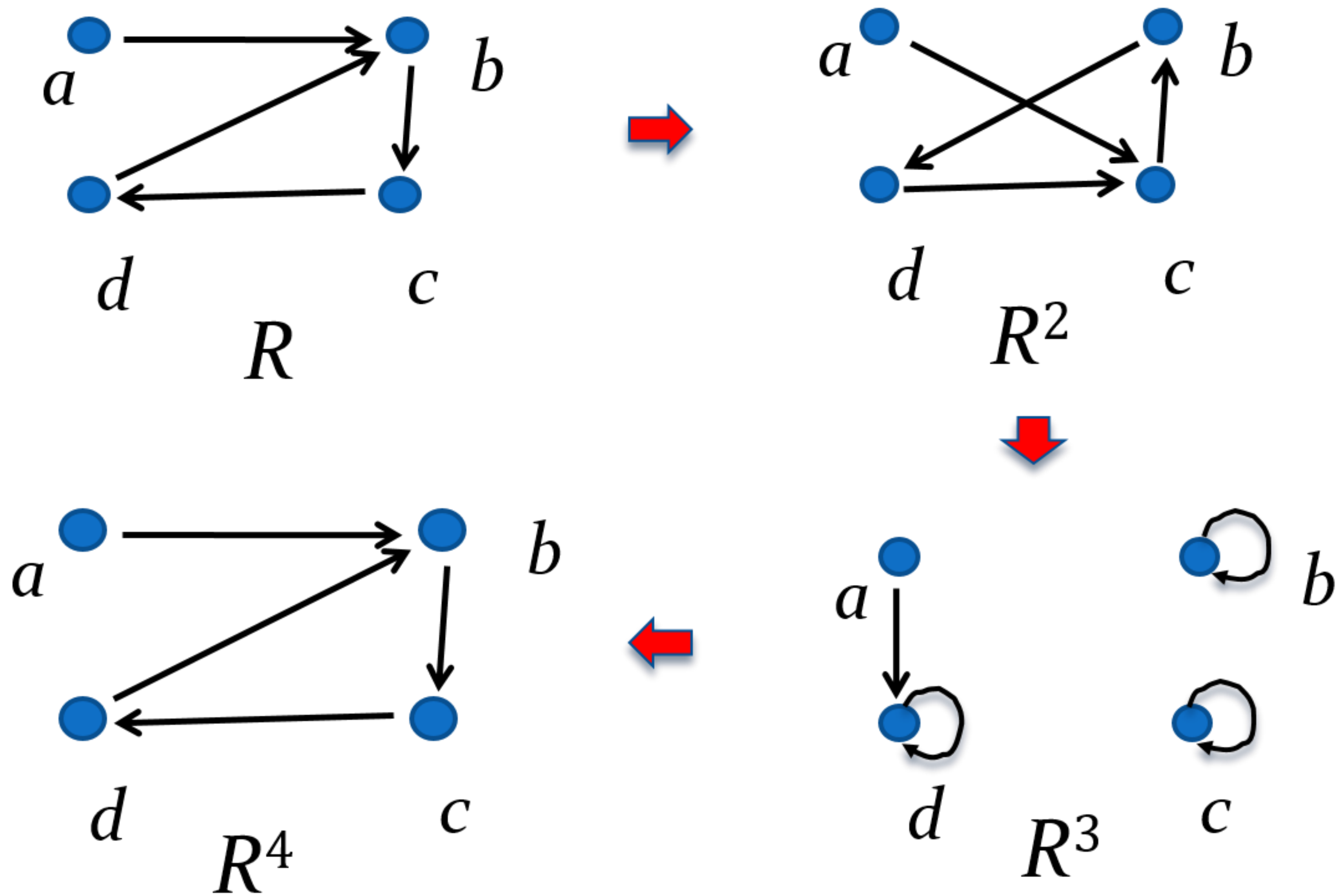
Reflexive? No, there are no loops

Symmetric? No, for example, there is no edge from *d* to *a*

Antisymmetric? Yes, whenever there is an edge from one vertex to another, there is not one going back

Transitive? Yes (trivially), there are no two edges where the first edge ends at the vertex where the second edge begins

Example of the Powers of a Relation



The pair (x,y) is in R^n if there is a path of length n from x to y in R (following the direction of the arrows).

Closure of Relations

Section 9.4

Section Summary

Reflexive Closure

Symmetric Closure

Transitive Closure

Closure of Relations

A relation R may or may not have some property P, such as reflexivity, symmetry, or transitivity.

If there is a relation *S* with property *P* containing *R* such that *S* is a subset of every relation with property *P* containing *R*, then *S* is called the closure of *R* with respect to *P*.

Closure of Relations

Example Consider relation R = {(1, 1), (1, 2), (2, 1), (3, 2)} on the set A = {1, 2, 3}. What are the reflexive, symmetric, and transitive closures of R?

Solution How to produce a reflexive relation containing R that is as small as possible? Add (2, 2) and (3, 3).

All reflexive relations containing R will also contain these pairs, so the reflexive closure of R is

$$R \cup \{(2,2),(3,3)\}$$

Similarly, the symmetric closure is

$$R \cup \{(2,3)\}$$

The transitive closure is

$$R \cup \{(2,2),(3,1)\}$$

Closure of Relations

Example Consider relation R = {(1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3, 2)} on the set A = {1, 2, 3}. What are the reflexive, symmetric, and transitive closures of R?

Solution The reflexive closure of R is

$$R \cup \{(3, 3)\}$$

Similarly, the symmetric closure is

$$R \cup \{(2, 1), (1, 3)\}$$

The transitive closure is

$$R \cup \{(2, 1), (3, 3)\}$$

Closure of Relations

Example What are the reflexive, symmetric, and transitive closures of relation $R = \{(a, b) \mid a < b\}$ on the of integers?

Solution The reflexive closure of R is

$$R \cup \{(a, a) \mid a \in \mathbf{Z}\} = \{(a, b) \mid a \leq b\}$$

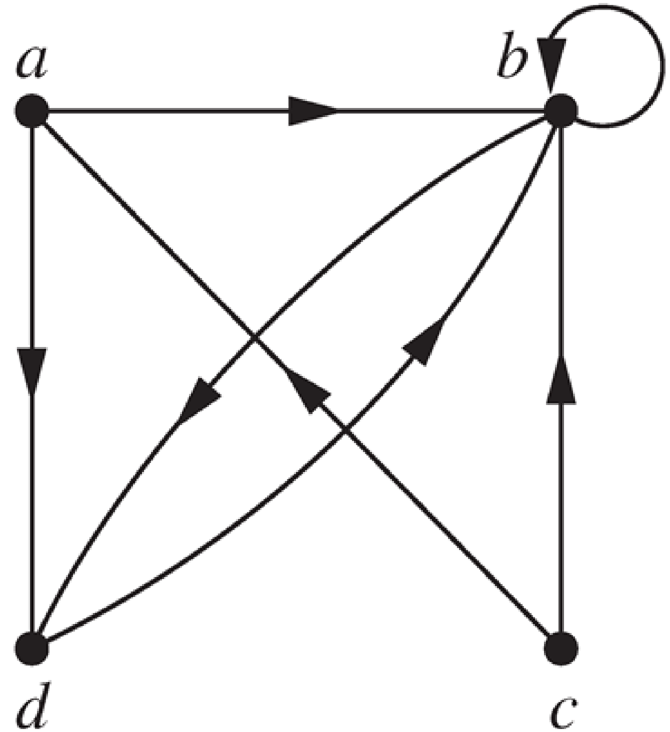
The symmetric closure is

$$R \cup \{(a, b) \mid a > b)\} = \{(a, b) \mid a \neq b\}$$

R is already transitive

Closure of Relations

Example What are the reflexive, symmetric, and transitive closures of the relation depicted below?



Closure of Relations

Example Consider relation R = {(1, 3), (1, 4), (2, 1), (3, 2)} on the set A = {1, 2, 3, 4}. What is the transitive closure of R?

Solution The transitive closure is

$$R \cup \{(1,2),(2,3),(2,4),(3,1)\}$$

PROBLEM: still not transitive – need (3, 4) also

The transitive closure is therefore more complicated to compute.

Closure of Relations

Example Find the transitive closure of the relation R = {(1, 3), (1, 4), (2, 1), (3, 2)} on the set A = {1, 2, 3, 4}.

R can be represented by the following matrix M_R:

$$M_R = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Closure of Relations

$$M_R = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_R^{[2]} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_R^{[3]} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_R^{[4]} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} \vee M_R^{[4]} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Closure of Relations

Solution The transitive closure of the relation $R = \{(1, 3), (1, 4), (2, 1), (3, 2)\}$ on the set $A = \{1, 2, 3, 4\}$ is given by the relation:

$$\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4)\}$$

Equivalence Relations

Section 9.5

Section Summary

Equivalence Relations

Equivalence Classes

Equivalence Classes and Partitions

Equivalence Relations

Definition 1: A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Definition 2: Two elements a, and b that are related by an equivalence relation are called *equivalent.* The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Strings

Example: Suppose that R is the relation on the set of strings of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Is R an equivalence relation?

Solution: Show that all of the properties of an equivalence relation hold.

- *Reflexivity*: Because $l(a) = l(a)$, it follows that aRa for all strings a.
- *Symmetry*: Suppose that aRb. Since $l(a) = l(b)$, $l(b) = l(a)$ also holds and bRa.
- *Transitivity*: Suppose that aRb and bRc. Since $l(a) = l(b)$,and $l(b) = l(c)$, $l(a) = l(a)$ also holds and aRc.

Congruence Modulo *m*

Example: Let m be an integer with $m > 1$. Show that the relation
$R = \{(a,b) \mid a \equiv b \pmod{m}\}$
is an equivalence relation on the set of integers.

Solution: Recall that $a \equiv b \pmod{m}$ if and only if m divides $a - b$.

- *Reflexivity*: $a \equiv a \pmod{m}$ since $a - a = 0$ is divisible by m since $0 = 0 \cdot m$.
- *Symmetry*: Suppose that $a \equiv b \pmod{m}$. Then $a - b$ is divisible by m, and so $a - b = km$, where k is an integer. It follows that $b - a = (-k)\, m$, so $b \equiv a \pmod{m}$.
- *Transitivity*: Suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. Then m divides both $a - b$ and $b - c$. Hence, there are integers k and l with $a - b = km$ and $b - c = lm$. We obtain by adding the equations: $a - c = (a - b) + (b - c) = km + lm = (k + l)\, m.$

$$a - c = (a - b) + (b - c) = km + lm = (k + l)m.$$

Therefore, $a \equiv c \pmod{m}$.

Divides

Example: Show that the "divides" relation on the set of positive integers is not an equivalence relation.

Solution: The properties of reflexivity and transitivity do hold, but the relation is not transitive. Hence, "divides" is not an equivalence relation.

- *Reflexivity*: $a \mid a$ for all a.
- *Not Symmetric*: For example, $2 \mid 4$, but $4 \nmid 2$. Hence, the relation is not symmetric.
- *Transitivity*: Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c. Therefore, the relation is transitive.

Equivalence Classes

Definition 3: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a. The equivalence class of a with respect to R is denoted by $[a]_R$.

When only one relation is under consideration, we can write $[a]$, without the subscript R, for this equivalence class.

Note that $[a]_R = \{s \mid (a,s) \in R\}$.

If $b \in [a]_R$, then b is called a representative of this equivalence class. Any element of a class can be used as a representative of the class.

The equivalence classes of the relation congruence modulo m are called the *congruence classes modulo m*. The congruence class of an integer a modulo m is denoted by $[a]_m$, so $[a]_m = \{\ldots,\ a-2m,\ a-m,\ a+2m,\ a+2m,\ \ldots\}$.

For example, $[0]_4 = \{\ldots,\ -8,\ -4, 0, 4, 8, \ldots\}$ $\quad [1]_4 = \{\ldots,\ -7,\ -3, 1, 5, 9, \ldots\}$

$[2]_4 = \{\ldots,\ -6,\ -2, 2, 6, 10, \ldots\}$ $\quad [3]_4 = \{\ldots,\ -5,\ -1, 3, 7, 11, \ldots\}$

Equivalence Classes and Partitions

Theorem 1: let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

(i) aRb

(ii) $[a] = [b]$

(iii) $[a] \cap [b] \neq \emptyset$

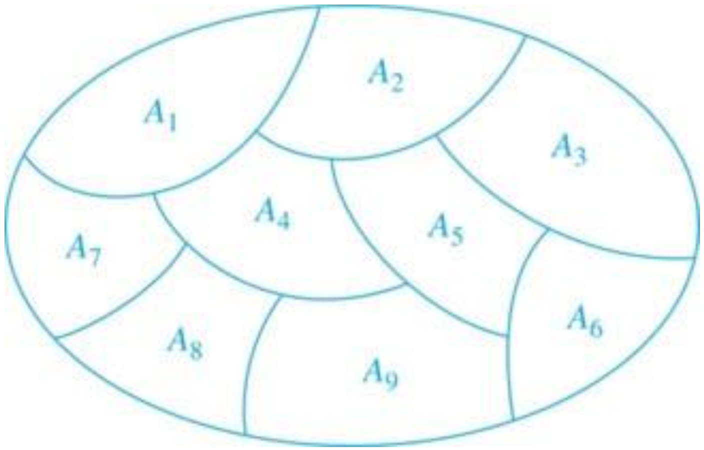
Proof: We show that (i) implies (ii). Assume that aRb. Now suppose that $c \in [a]$. Then aRc. Because aRb and R is symmetric, bRa. Because R is transitive and bRa and aRc, it follows that bRc. Hence, $c \in [b]$. Therefore, $[a] \subseteq [b]$. A similar argument (omitted here) shows that $[b] \subseteq [a]$. Since $[a] \subseteq [b]$ and $[b] \subseteq [a]$, we have shown that $[a] = [b]$.

(*see text for proof that* (ii) *implies* (iii) *and* (iii) *implies* (i))

Partition of a Set

Definition: A *partition* of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i, where $i \in I$ (where I is an index set), forms a partition of S if and only if

- $A_i \neq \emptyset$ for $i \in I$,
- $A_i \cap A_j = \emptyset$ when $i \neq j$,
- and $\bigcup_{i \in I} A_i = S$.



A Partition of a Set

An Equivalence Relation Partitions a Set[1]

Let *R* be an equivalence relation on a set *A*. The union of all the equivalence classes of *R* is all of *A*, since an element *a* of *A* is in its own equivalence class $[a]_R$. In other words,

$$\bigcup_{a \in A} [a]_R = A.$$

From Theorem 1, it follows that these equivalence classes are either equal or disjoint, so $[a]_R \cap [b]_R = \varnothing$ when $[a]_R \neq [b]_R$.

Therefore, the equivalence classes form a partition of *A*, because they split *A* into disjoint subsets.

An Equivalence Relation Partitions a Set 2

Theorem 2: Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

Proof: We have already shown the first part of the theorem.

For the second part, assume that $\{A_i \mid i \in I\}$ is a partition of S. Let R be the relation on S consisting of the pairs (x, y) where x and y belong to the same subset A_i in the partition. We must show that R satisfies the properties of an equivalence relation.

- *Reflexivity*: For every $a \in S$, $(a,a) \in R$, because a is in the same subset as itself.
- *Symmetry*: If $(a,b) \in R$, then b and a are in the same subset of the partition, so $(b,a) \in R$.
- *Transitivity*: If $(a,b) \in R$ and $(b,c) \in R$, then a and b are in the same subset of the partition, as are b and c. Since the subsets are disjoint and b belongs to both, the two subsets of the partition must be identical. Therefore, $(a,c) \in R$ since a and c belong to the same subset of the partition.

Partial Orderings

Section 9.6

Section Summary

Partial Orderings and Partially-ordered Sets

Lexicographic Orderings

Hasse Diagrams

Lattices (*not currently in overheads*)

Topological Sorting (*not currently in overheads*)

Partial Orderings 1

Definition 1: A relation R on a set S is called a *partial ordering,* or *partial order,* if it is reflexive, antisymmetric, and transitive. A set together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R). Members of S are called *elements* of the poset.

Partial Orderings 2

Example 1: Show that the "greater than or equal" relation ($\geq$) is a partial ordering on the set of integers.

- *Reflexivity*: $a \geq a$ for every integer a.
- *Antisymmetry*: If $a \geq b$ and $b \geq a$, then $a = b$.
- *Transitivity*: If $a \geq b$ and $b \geq c$, then $a \geq c$.

These properties all follow from the order axioms for the integers. (*See Appendix* 1).

Partial Orderings 3

Example 2: Show that the divisibility relation (|) is a partial ordering on the set of integers.

- *Reflexivity*: $a \mid a$ for all integers a. (*see Example* 9 *in Section* 9.1)
- *Antisymmetry*: If a and b are positive integers with $a \mid b$ and $b \mid a$, then $a = b$. (*see Example* 12 *in Section* 9.1)
- *Transitivity*: Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c. Therefore, the relation is transitive.

$(\boldsymbol{Z}^{+}, \mid)$ is a poset.

Partial Orderings 4

Example 3: Show that the inclusion relation ($\subseteq$) is a partial ordering on the power set of a set S.

- *Reflexivity*: $A \subseteq A$ whenever A is a subset of S.
- *Antisymmetry*: If A and B are positive integers with $A \subseteq B$ and $B \subseteq A$, then $A = B$.
- *Transitivity*: If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

The properties all follow from the definition of set inclusion.

Comparability

Definition 2: The elements a and b of a poset $(S, \preccurlyeq)$ are *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When a and b are elements of S so that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, then a and b are called *incomparable*.

The symbol $\preccurlyeq$ is used to denote the relation in any poset.

Definition 3: If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a *totally ordered* or *linearly ordered set*, and $\preccurlyeq$ is called a *total order* or a *linear order.* A totally ordered set is also called a *chain.*

Definition 4: $(S, \preccurlyeq)$ is a well-ordered set if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of S has a least element.

Lexicographic Order

Definition: Given two posets $(A_1,\preccurlyeq_1)$ and $(A_2,\preccurlyeq_2)$, the *lexicographic ordering* on $A_1 \times A_2$ is defined by specifying that (a_1, a_2) is less than (b_1,b_2), that is,

$(a_1, a_2) \prec (b_1,b_2)$,

either if $a_1 \prec_1 b_1$ or if $a_1 = b_1$ and $a_2 \prec_2 b_2$.

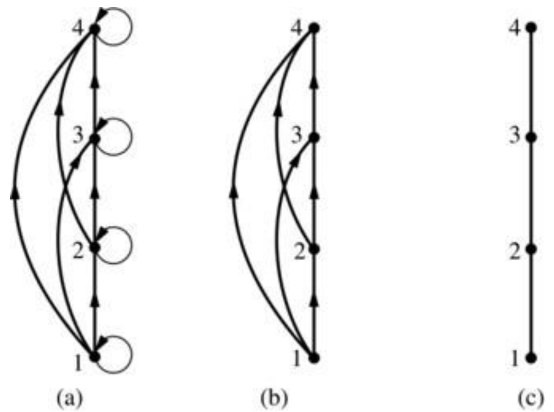
This definition can be easily extended to a lexicographic ordering on strings (*see text*).

Example: Consider strings of lowercase English letters. A lexicographic ordering can be defined using the ordering of the letters in the alphabet. This is the same ordering as that used in dictionaries.

- *discreet* $\prec$ *discrete*, because these strings differ in the seventh position and $e \prec t$.
- *discreet* $\prec$ *discreetness*, because the first eight letters agree, but the second string is longer.

Hasse Diagrams

Definition: A *Hasse diagram* is a visual representation of a partial ordering that leaves out edges that must be present because of the reflexive and transitive properties.



A partial ordering is shown in (a) of the figure above. The loops due to the reflexive property are deleted in (b). The edges that must be present due to the transitive property are deleted in (c). The Hasse diagram for the partial ordering (a) is depicted in (c).

Jump to long description

Procedure for Constructing a Hasse Diagram

To represent a finite poset $(S,\preccurlyeq)$ using a Hasse diagram, start with the directed graph of the relation:

- Remove the loops (a, a) present at every vertex due to the reflexive property.
- Remove all edges (x, y) for which there is an element $z \in S$ such that $x \prec z$ and $z \prec y$. These are the edges that must be present due to the transitive property.
- Arrange each edge so that its initial vertex is below the terminal vertex. Remove all the arrows, because all edges point upwards toward their terminal vertex.

Procedure for Constructing a Hasse Diagram

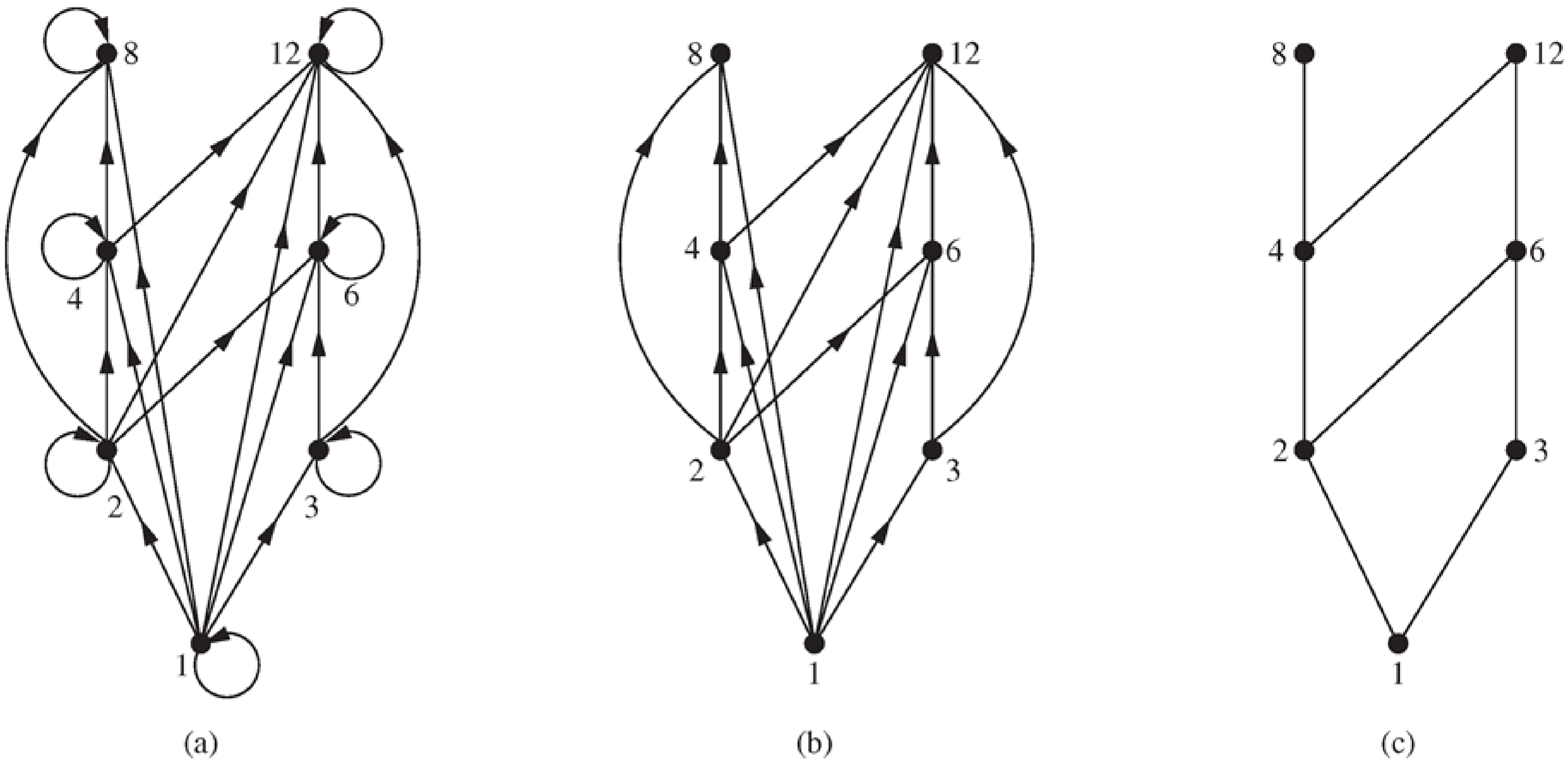


FIGURE 3 **Constructing the Hasse Diagram of ({1, 2, 3, 4, 6, 8, 12}, |).**

Maximal and Minimal Elements

Elements in posets have an ordering that allow maximal and minimal elements to be identified.

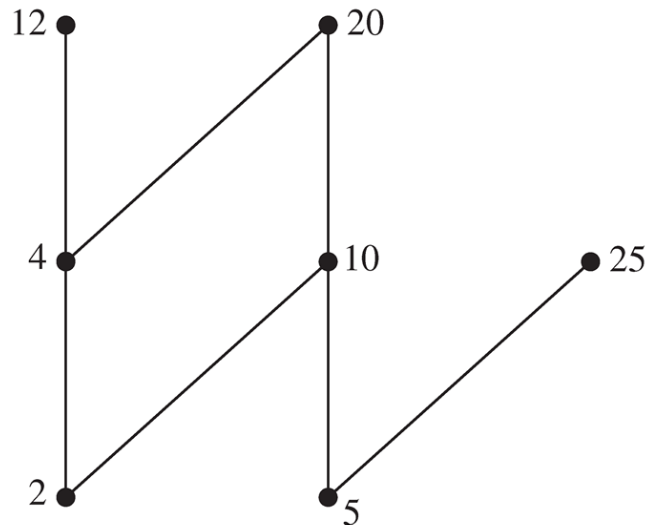
- a is maximal in the poset $(S, \preccurlyeq)$ if there is no $b \in S$ such that $a \prec b$
- minimal elements are identified similarly

Easy to identify maximal and minimal elements in a Hasse diagram

Maximal and Minimal Elements

Example: Which elements of the poset ({2, 4, 5, 10, 12, 20, 25}, |) are maximal, and which are minimal?

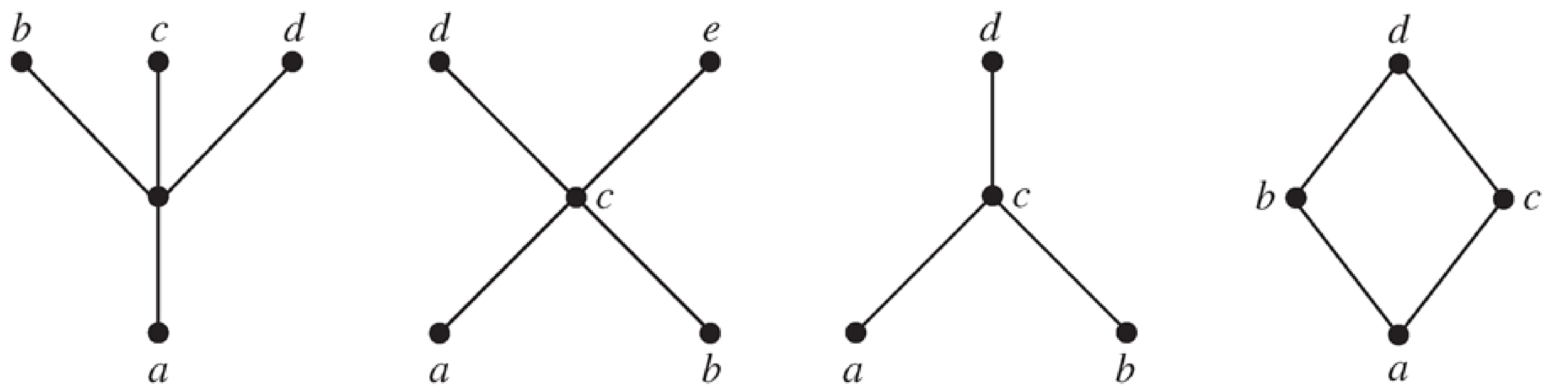
Solution: the Hasse diagram shows maximal elements 12, 20, and 25, and minimal elements 2 and 5



Greatest and Least Elements

Some posets have a unique greatest element that is greater than every other element, and/or a least element that is less than every element

Easy to identify greatest and least elements in a Hasse diagram



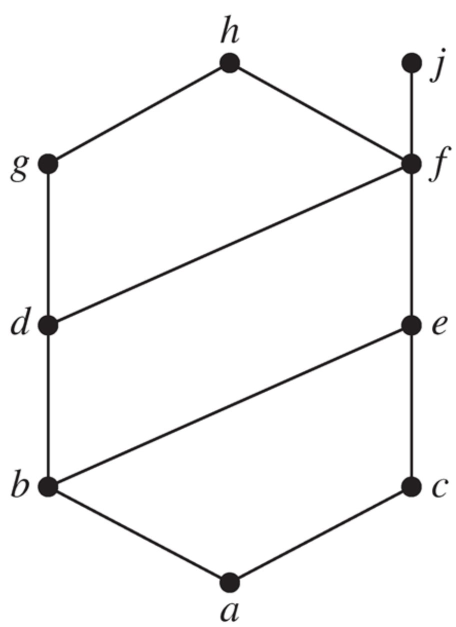
(a) (b) (c) (d)

Upper and Lower Bounds

- Sometimes it is possible to find an element that is greater than or equal to all the elements in a subset A of a poset $(S, \preccurlyeq)$. If u is an element of S such that $a \preccurlyeq u$ for all elements $a \in A$, u is an upper bound of A
- Likewise, if $l \preccurlyeq a$ for all $a \in A$, then l is a lower bound of A.

Upper and Lower Bounds

- **Example**: Find the lower and upper bounds of the subsets {*a*, *b*, *c*}, {*j*, *h*}, and {*a*, *c*, *d*, *f*} in the poset with the Hasse diagram shown below.
- **Solution**:



	upper	lower
{*a*, *b*, *c*}	*e*, *f*, *j*, *h*	*a*
{*j*, *h*}	none	*a*, *b*, *c*, *d*, *e*, *f*
{*a*, *c*, *d*, *f*}	*f*, *h*, *j*	*a*

{*b*, *d*, *g*} least upper bound: *g*

{*b*, *d*, *g*} greatest lower bound: *b*

Appendix of Image Long Descriptions

Binary Relations - Appendix

In the left part of the picture, there are 5 points. Points 0, 1 , and 2 are placed one above the other forming a column. Points a and b are located aside forming another column. Arrows point from 0 to A, from 0 to B, from 1 to A, and from 2 to B. In the right part of the picture there is a table with two columns labeled as A and B and three rows labeled as 0,1 and 2. Some elements of the intersection of rows and columns are marked with a cross, the others are empty. Both elements of the first row are marked with a cross. Only the first element of the second line is marked with a cross, the second one is empty. The first element of the second line is empty, and the second one is marked with a cross.

Jump to the image

Matrices of Relations on Sets - Appendix

There is a square matrix, only the main diagonal is shown. All the elements of this diagonal are 1.

There are two square matrices A and B. In the matrix A, the elements that are symmetric with respect to the main diagonal are equal; in matrix B, these elements are not equal.

Jump to the image

Representing Relations Using Digraphs - Appendix

Points A, B, C, and D are vertices of the graph. Arrows point from A to B, from A to D, from B to D. From C to A, from C to B and from D to B. The arrow pointing from B to D does not coincide with the arrow pointing from D to B. There is a circle that starts and ends at vertex B. This circle is a loop.

Jump to the image

Examples of Digraphs Representing Relations - Appendix

Points 1, 2, 3, and 4 are vertices of the graph. Arrows point from 1 to 4, from 1 to 3, from 2 to 1, from 2 to 3, from 3 to 1, from 4 to 1, from 4 to 3. The arrow pointing from 1 to 3 does not coincide with the arrow pointing from 3 to 1. The arrow pointing from 1 to 4 does not coincide with the arrow pointing from 4 to 1. There are loops at vertices 2 and 3.

Jump to the image

Hasse Diagrams - Appendix

Case A. Points 1, 2, 3, and 4 located from the bottom to the top on the same line are vertices of the graph. Arrows point from 1 to 2, from 1 to 3, from 1 to 4, from 2 to 3, from 2 to 4, and from 3 to 4. There are loops at each vertex. Case B. The same graph is shown, but there are no loops at the vertices. Case C. The same graph is shown, but there are only edges connecting the vertices consequently present.

Jump to the image