CHAPTER 14

Ordered Sets and Lattices

14.1 INTRODUCTION

Order and precedence relationships appear in many different places in mathematics and computer science. This chapter makes these notions precise. We also define a lattice, which is a special kind of an ordered set.

14.2 ORDERED SETS

Suppose R is a relation on a set S satisfying the following three properties:

[$\mathbf{O}_1$] (Reflexive) For any $a \in S$, we have aRa.

[$\mathbf{O}_2$] (Antisymmetric) If aRb and bRa, then $a = b$.

[$\mathbf{O}_3$] (Transitive) If aRb and bRc, then aRc.

Then R is called a *partial order* or, simply an *order* relation, and R is said to define a *partial ordering* of S. The set S with the partial order is called a *partially ordered set* or, simply, an *ordered set* or *poset*. We write (S, R) when we want to specify the relation R.

The most familiar order relation, called the *usual order*, is the relation $\leq$ (read "less than or equal") on the positive integers $\mathbf{N}$ or, more generally, on any subset of the real numbers $\mathbf{R}$. For this reason, a partial order relation is usually denoted by $\precsim$; and

$$a \precsim b$$

is read "a precedes b." In this case we also write:

$a \prec b$ means $a \precsim b$ and $a \neq b$; read "a strictly precedes b."
$b \succsim a$ means $a \precsim b$; read "b succeeds a."
$b \succ a$ means $a \prec b$; read "b strictly succeeds a."
$\not\precsim, \not\prec, \not\succsim$, and $\not\succ$ are self-explanatory.

When there is no ambiguity, the symbols $\leq$, $<$, $>$, and $\geq$ are frequently used instead of $\precsim$, $\prec$, $\succ$, and $\succsim$, respectively.

EXAMPLE 14.1

(a) Let S be any collection of sets. The relation $\subseteq$ of set inclusion is a partial ordering of S. Specifically, $A \subseteq A$ for any set A; if $A \subseteq B$ and $B \subseteq A$ then $A = B$; and if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

(b) Consider the set **N** of positive integers. We say "a divides b," written $a \mid b$, if there exists an integer c such that $ac = b$. For example, $2 \mid 4$, $3 \mid 12$, $7 \mid 21$, and so on. This relation of divisibility is a partial ordering of **N**.

(c) The relation "|" of divisibility is not an ordering of the set **Z** of integers. Specifically, the relation is not antisymmetric. For instance, $2 \mid -2$ and $-2 \mid 2$, but $2 \neq -2$.

(d) Consider the set **Z** of integers. Define aRb if there is a positive integer r such that $b = a^r$. For instance, $2\ R\ 8$ since $8 = 2^3$. Then R is a partial ordering of **Z**.

Dual Order

Let $\precsim$ be any partial ordering of a set S. The relation $\succsim$, that is, a succeeds b, is also a partial ordering of S; it is called the *dual order*. Observe that $a \precsim b$ if and only if $b \succsim a$; hence the dual order $\succsim$ is the inverse of the relation $\precsim$, that is, $\succsim = \precsim^{-1}$.

Ordered Subsets

Let A be a subset of an ordered set S, and suppose $a, b \in A$. Define $a \precsim b$ as elements of A whenever $a \precsim b$ as elements of S. This defines a partial ordering of A called the *induced order* on A. The subset A with the induced order is called an *ordered subset* of S. Unless otherwise stated or implied, any subset of an ordered set S will be treated as an ordered subset of S.

Quasi-order

Suppose $\prec$ is a relation on a set S satisfying the following two properties:

[$\mathbf{Q}_1$] (Irreflexive) For any $a \in A$, we have $a \nprec a$.

[$\mathbf{Q}_2$] (Transitive) If $a \prec b$, and $b \prec c$, then $a \prec c$.

Then $\prec$ is called a *quasi-order* on S.

There is a close relationship between partial orders and quasi-orders. Specifically, if $\precsim$ is a partial order on a set S and we define $a \prec b$ to mean $a \precsim b$ but $a \neq b$, then $\prec$ is a quasi-order on S. Conversely, if $\prec$ is a quasi-order on a set S and we define $a \precsim b$ to mean $a \prec b$ or $a = b$, then $\precsim$ is a partial order on S. This allows us to switch back and forth between a partial order and its corresponding quasi-orders using whichever is more convenient.

Comparability, Linearly Ordered Sets

Suppose a and b are elements in a partially ordered set S. We say a and b are *comparable* if

$$a \precsim b \quad \text{or} \quad b \precsim a$$

that is, if one of them precedes the other. Thus a and b are *noncomparable*, written

$$a \parallel b$$

if neither $a \precsim b$ nor $b \precsim a$.

The word "partial" is used in defining a partially ordered set S since some of the elements of S need not be comparable. Suppose, on the other hand, that every pair of elements of S are comparable. Then S is said to be *totally ordered* or *linearly ordered*, and S is called a *chain*. Although an ordered set S may not be linearly ordered, it is still possible for a subset A of S to be linearly ordered. Clearly, every subset of a linearly ordered set S must also be linearly ordered.

EXAMPLE 14.2

(a) Consider the set **N** of positive integers ordered by divisibility. Then 21 and 7 are comparable since $7 \mid 21$. On the other hand, 3 and 5 are noncomparable since neither $3 \mid 5$ nor $5 \mid 3$. Thus **N** is not linearly ordered by divisibility. Observe that $A = \{2, 6, 12, 36\}$ is a linearly ordered subset of **N** since $2 \mid 6$, $6 \mid 12$ and $12 \mid 36$.

(b) The set **N** of positive integers with the usual order $\leq$ (less than or equal) is linearly ordered and hence every ordered subset of **N** is also linearly ordered.

(c) The power set $P(A)$ of a set A with two or more elements is not linearly ordered by set inclusion. For instance, suppose a and b belong to A. Then $\{a\}$ and $\{b\}$ are noncomparable. Observe that the empty set $\varnothing$, $\{a\}$, and A do form a linearly ordered subset of $P(A)$ since $\varnothing \subseteq \{a\} \subseteq A$. Similarly, $\varnothing$, $\{b\}$, and A form a linearly ordered subset of $P(A)$.

Product Sets and Order

There are a number of ways to define an order relation on the Cartesian product of given ordered sets. Two of these ways follow:

(*a*) ***Product Order*:** Suppose S and T are ordered sets. Then the following is an order relation on the product set $S \times T$, called the *product order*:

$$(a, b) \precsim (a', b') \quad \text{if} \quad a \leq a' \text{ and } b \leq b'$$

(*b*) ***Lexicographical Order*:** Suppose S and T are linearly ordered sets. Then the following is an order relation on the product set $S \times T$, called the *lexicographical* or *dictionary order*:

$$(a, b) \prec (a', b') \quad \text{if} \quad a < b \quad \text{or if} \quad a = a' \text{ and } b < b'$$

This order can be extended to $S_1 \times S_2 \times \cdots \times S_n$ as follows:

$$(a_1, a_2, \ldots, a_n) \prec (a_1', a_2', \ldots, a_n') \quad \text{if} \quad a_i = a_i' \text{ for } i = 1, 2, \ldots, k-1 \text{ and } a_k < a_k'$$

Note that the lexicographical order is also linear.

Kleene Closure and Order

Let A be a (nonempty) linearly ordered alphabet. Recall that A^*, called the Kleene closure of A, consists of all words w on A, and $|w|$ denotes the length of w. Then the following are two order relations on A^*.

(*a*) ***Alphabetical (Lexicographical) Order*:** The reader is no doubt familiar with the usual alphabetical ordering of A^*. That is:

(i) $\lambda < w$, where λ is the empty word and w is any nonempty word.

(ii) Suppose $u = au'$ and $v = bv'$ are distinct nonempty words where $a, b \in A$ and $u', v' \in A^*$. Then

$$u \prec v \quad \text{if } a < b \quad \text{or} \quad \text{if } a = b \text{ but } u' \prec v'$$

(*b*) ***Short-lex Order*:** Here A^* is ordered first by length, and then alphabetically. That is, for any distinct words u, v in A^*,

$$u \prec v \quad \text{if } |u| < |v| \quad \text{or} \quad \text{if } |u| = |v| \text{ but } u \text{ precedes } v \text{ alphabetically}$$

For example, "to" precedes "and" since $|\text{to}| = 2$ but $|\text{and}| = 3$. However, "an" precedes "to" since they have the same length, but "an" precedes "to" alphabetically. This order is also called the *free semigroup order*.

14.3 HASSE DIAGRAMS OF PARTIALLY ORDERED SETS

Let S be a partially ordered set, and suppose a, b belong to S. We say that a is an *immediate predecessor* of b, or that b is an *immediate successor* of a, or that b is a *cover* of a, written

$$a \ll b$$

if $a < b$ but no element in S lies between a and b, that is, there exists no element c in S such that $a < c < b$.

Suppose S is a finite partially ordered set. Then the order on S is completely known once we know all pairs a, b in S such that $a \ll b$, that is, once we know the relation $\ll$ on S. This follows from the fact that $x < y$ if and only if $x \ll y$ or there exist elements $a_1, a_2, \ldots, a_m$ in S such that

$$x \ll a_1 \ll a_2 \ll \cdots \ll a_m \ll y$$

The *Hasse diagram* of a finite partially ordered set S is the directed graph whose vertices are the elements of S and there is a directed edge from a to b whenever $a \ll b$ in S. (Instead of drawing an arrow from a to b, we sometimes place b higher than a and draw a line between them. It is then understood that movement upwards indicates succession.) In the diagram thus created, there is a directed edge from vertex x to vertex y if and only if $x \ll y$. Also, there can be no (directed) cycles in the diagram of S since the order relation is antisymmetric.

The Hasse diagram of a poset S is a picture of S; hence it is very useful in describing types of elements in S. Sometimes we define a partially ordered set by simply presenting its Hasse diagram. We note that the Hasse diagram of a poset S need not be connected.

Remark: The Hasse diagram of a finite poset S turns out to be a directed cycle-free graph (DAG) studied in Section 9.9. The investigation here is independent of the previous investigation. Here we mainly think of order in terms of "less than" or "greater than" rather than in terms of directed adjacency relations. Accordingly, there will be some overlap in the content.

EXAMPLE 14.3

(a) Let $A = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$ be ordered by the relation "x divides y." The diagram of A is given in Fig. 14-1(a). (Unlike rooted trees, the direction of a line in the diagram of a poset is always upward.)

(b) Let $B = \{a, b, c, d, e\}$. The diagram in Fig. 14-1(b) defines a partial order on B in the natural way. That is, $d \le b$, $d \le a$, $e \le c$ and so on.

(c) The diagram of a finite linearly ordered set, i.e., a finite chain, consists simply of one path. For example, Fig. 14-1(c) shows the diagram of a chain with five elements.

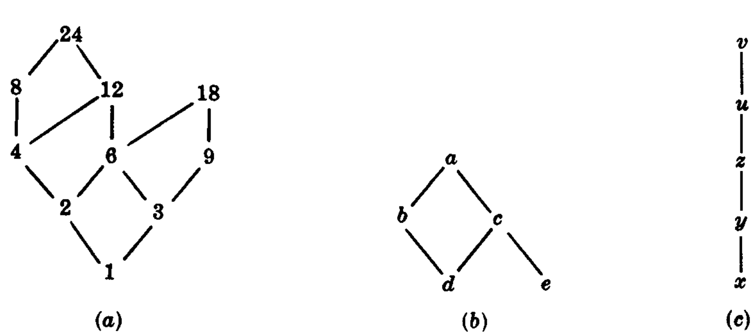


Fig. 14-1

EXAMPLE 14.4 A *partition* of a positive integer m is a set of positive integers whose sum is m. For instance, there are seven partitions of $m = 5$ as follows:

$$5, \quad 3-2, \quad 2-2-1, \quad 1-1-1-1-1, \quad 4-1, \quad 3-1-1, \quad 2-1-1-1$$

We order the partitions of an integer m as follows. A partition P_1 precedes a partition P_2 if the integers in P_1 can be added to obtain the integers in P_2 or, equivalently, if the integers in P_2 can be further subdivided to obtain the integers in P_1. For example,

$$2-2-1 \quad \text{precedes} \quad 3-2$$

since $2+1=3$. On the other hand, $3-1-1$ and $2-2-1$ are noncomparable.

Figure 14-2 gives the Hasse diagram of the partitions of $m = 5$.

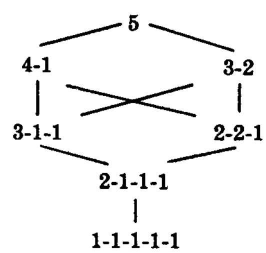


Fig. 14-2

Minimal and Maximal, and First and Last Elements

Let S be a partially ordered set. An element a in S is called a *minimal* element if no other element of S strictly precedes (is less than) a. Similarly, an element b in S is called a *maximal* element if no element of S strictly succeeds (is larger than) b. Geometrically speaking, a is a minimal element if no edge enters a (from below), and b is a maximal element if no edge leaves b (in the upward direction). We note that S can have more than one minimal and more than one maximal element.

If S is infinite, then S may have no minimal and no maximal element. For instance, the set **Z** of integers with the usual order $\leq$ has no minimal and no maximal element. On the other hand, if S is finite, then S must have at least one minimal element and at least one maximal element.

An element a is S is called a *first* element if for every element x in S,

$$a \precsim x$$

that is, if a precedes every other element in S. Similarly, an element b in S is called a *last* element if for every element y in S,

$$y \precsim b$$

that is, if b succeeds every other element in S. We note that S can have at most one first element, which must be a minimal element, and S can have at most one last element, which must be a maximal element. Generally speaking, S may have neither a first nor a last element, even when S is finite.

EXAMPLE 14.5

(a) Consider the three partially ordered sets in Example 14-3 whose Hasse diagrams appear in Fig. 14-1.

(i) A has two maximal elements 18 and 24 and neither is a last element. A has only one minimal element, 1, which is also a first element.

(ii) B has two minimal elements, d and e, and neither is a first element. B has only one maximal element, a, which is also a last element.

(iii) The chain has one minimal element, x, which is a first element, and one maximal element, v, which is a last element.

(b) Let A be any nonempty set and let $P(A)$ be the power set of A ordered by set inclusion. Then the empty set $\varnothing$ is a first element of $P(A)$ since, for any set X, we have $\varnothing \subseteq X$. Moreover, A is a last element of $P(A)$ since every element Y of $P(A)$ is, by definition, a subset of A, that is, $Y \subseteq A$.

14.4 CONSISTENT ENUMERATION

Suppose S is a finite partially ordered set. Frequently we want to assign positive integers to the elements of S in such a way that the order is preserved. That is, we seek a function $f\colon S \to \mathbf{N}$ so that if $a \prec b$ then $f(a) < f(b)$. Such a function is called a *consistent enumeration* of S. The fact that this can always be done is the content of the following theorem.

Theorem 14.1: There exists a consistent enumeration for any finite poset A.

We prove this theorem in Problem 14.8. In fact, we prove that if S has n elements then there exists a consistent enumeration $f\colon S \to \{1, 2, \ldots, n\}$.

We emphasize that such an enumeration need not be unique. For example, the following are two such enumerations for the poset in Fig. 14-1(b):

(i) $f(d) = 1,\ f(e) = 2,\ f(b) = 3,\ f(c) = 4,\ f(a) = 5.$

(ii) $g(e) = 1,\ g(d) = 2,\ g(c) = 3,\ g(b) = 4,\ g(a) = 5.$

However the chain in Fig. 14-1(c) admits only one consistent enumeration if we map the set into $\{1, 2, 3, 4, 5\}$. Specifically, we must assign:

$$h(x) = 1, \quad h(y) = 2, \quad h(z) = 3, \quad h(u) = 4, \quad h(v) = 5$$

14.5 SUPREMUM AND INFIMUM

Let A be a subset of a partially ordered set S. An element M in S is called an *upper bound* of A if M succeeds every element of A, i.e., if, for every x in A, we have

$$x \precsim M$$

If an upper bound of A precedes every other upper bound of A, then it is called the *supremum* of A and is denoted by

$$\sup(A)$$

We also write $\sup(a_1, \ldots, a_n)$ instead of $\sup(A)$ if A consists of the elements $a_1, \ldots, a_n$. We emphasize that there can be at most one $\sup(A)$; however, $\sup(A)$ may not exist.

Analogously, an element m in a poset S is called a *lower bound* of a subset A of S if m precedes every element of A, i.e., if, for every y in A, we have

$$m \precsim y$$

If a lower bound of A succeeds every other lower bound of A, then it is called the *infimum* of A and is denoted by

$$\inf(A), \quad \text{or} \quad \inf(a_1, \ldots, a_n)$$

if A consists of the elements $a_1, \ldots, a_n$. There can be at most one $\inf(A)$ although $\inf(A)$ may not exist.

Some texts use the term *least upper bound* instead of supremum and then write $\mathrm{lub}(A)$ instead of $\sup(A)$, and use the term *greatest lower bound* instead of infimum and write $\mathrm{glb}(A)$ instead of $\inf(A)$.

If A has an upper bound we say A is *bounded above*, and if A has a lower bound we say A is *bounded below*. In particular, A is *bounded* if A has an upper and lower bound.

EXAMPLE 14.6

(a) Let $S = \{a, b, c, d, e, f\}$ be ordered as pictured in Fig. 14-3(a), and let $A = \{b, c, d\}$. The upper bounds of A are e and f since only e and f succeed every element in A. The lower bounds of A are a and b since only a and b precede every element of A. Note that e and f are noncomparable; hence $\sup(A)$ does not exist. However, b also succeeds a, hence $\inf(A) = b$.

(b) Let $S = \{1, 2, 3, \ldots, 8\}$ be ordered as pictured in Fig. 14-3(b), and let $A = \{4, 5, 7\}$. The upper bounds of A are 1, 2, and 3, and the only lower bound is 8. Note that 7 is not a lower bound since 7 does not precede 4. Here $\sup(A) = 3$ since 3 precedes the other upper bounds 1 and 2. Note that $\inf(A) = 8$ since 8 is the only lower bound.

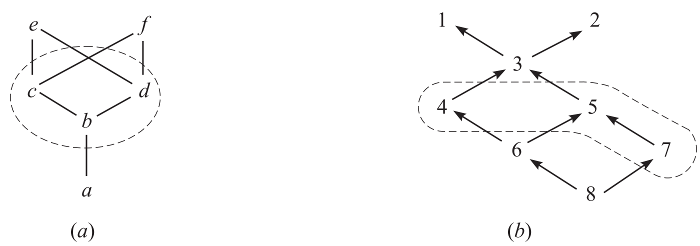


Fig. 14-3

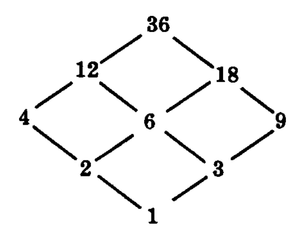


Fig. 14-4

Generally speaking, $\sup(a, b)$ and $\inf(a, b)$ need not exist for every pair of elements a and b in a poset S. We now give two examples of partially ordered sets where $\sup(a, b)$ and $\inf(a, b)$ do exist for every a, b in the set.

EXAMPLE 14.7

(a) Let the set $\mathbf{N}$ of positive integers be ordered by divisibility. The *greatest common divisor* of a and b in $\mathbf{N}$, denoted by

$$\gcd(a, b)$$

is the largest integer which divides a and b. The *least common multiple* of a and b, denoted by

$$\operatorname{lcm}(a, b)$$

is the smallest integer divisible by both a and b.

An important theorem in number theory says that every common divisor of a and b divides $\gcd(a, b)$. One can also prove that $\operatorname{lcm}(a, b)$ divides every multiple of a and b. Thus

$$\gcd(a, b) = \inf(a, b) \quad \text{and} \quad \operatorname{lcm}(a, b) = \sup(a, b)$$

In other words, $\inf(a, b)$ and $\sup(a, b)$ do exist for any pair of elements of $\mathbf{N}$ ordered by divisibility.

(b) For any positive integer m, we will let $\mathbf{D}_m$ denote the set of divisors of m ordered by divisibility. The Hasse diagram of

$$\mathbf{D}_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$$

appears in Fig. 14-4. Again, $\inf(a, b) = \gcd(a, b)$ and $\sup(a, b) = \operatorname{lcm}(a, b)$ exist for any pair a, b in $\mathbf{D}_m$.

14.6 ISOMORPHIC (SIMILAR) ORDERED SETS

Suppose X and Y are partially ordered sets. A one-to-one (injective) function $f\colon X \to Y$ is called a *similarity mapping* from X into Y if f preserves the order relation, that is, if the following two conditions hold for any pair a and a' in X:

(1) If $a \precsim a'$ then $f(a) \precsim f(a')$.

(2) If $a \parallel a'$ (noncomparable), then $f(a) \parallel f(a')$.

Accordingly, if A and B are linearly ordered, then only (1) is needed for f to be a similarity mapping.

Two ordered sets X and Y are said to be *isomorphic* or *similar*, written

$$X \simeq Y$$

if there exists a one-to-one correspondence (bijective mapping) $f\colon X \to Y$ which preserves the order relations, i.e., which is a similarity mapping.

EXAMPLE 14.8 Suppose $X = \{1, 2, 6, 8, 12\}$ is ordered by divisibility and suppose $Y = \{a, b, c, d, e\}$ is isomorphic to X; say, the following function f is a similarity mapping from X onto Y:

$$f = \{(1, e), (2, d), (6, b), (8, c), (12, a)\}$$

Draw the Hasse diagram of Y.

The similarity mapping preserves the order of the initial set X and is one-to-one and onto. Thus the mapping can be viewed simply as a relabeling of the vertices in the Hasse diagram of the initial set X. The Hasse diagrams for both X and Y appear in Fig. 14-5.

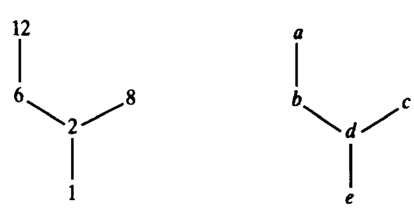


Fig. 14-5

14.7 WELL-ORDERED SETS

We begin with a definition.

Definition 14.1: An ordered set S is said to be *well-ordered* if every subset of S has a first element.

The classical example of a well-ordered set is the set $\mathbf{N}$ of positive integers with the usual order $\leq$. The following facts follow from the definition.

(1) A well-ordered set is linearly ordered. For if $a, b, \in S$, then $\{a, b\}$ has a first element; hence a and b are comparable.
(2) Every subset of a well-ordered set is well-ordered.
(3) If X is well-ordered and Y is isomorphic to X, then Y is well-ordered.
(4) All finite linearly ordered sets with the same number n of elements are well-ordered and are all isomorphic to each other. In fact, they are all isomorphic to $\{1, 2, \ldots, n\}$ with the usual order $\leq$.

(5) Every element $a \in S$, other than a last element, has an immediate successor. For, let $M(a)$ denote the set of elements which strictly succeed a. Then the first element of $M(a)$ is the immediate successor of a.

EXAMPLE 14.9

(a) The set $\mathbf{Z}$ of integers with the usual order $\leq$ is linearly ordered and every element has an immediate successor and an immediate predecessor, but $\mathbf{Z}$ is not well-ordered. For example, $\mathbf{Z}$ itself has no first element. However, any subset of $\mathbf{Z}$ which is bounded from below is well-ordered.

(b) The set $\mathbf{Q}$ of rational numbers with the usual order $\leq$ is linearly ordered, but no element in $\mathbf{Q}$ has an immediate successor or an immediate predecessor. For if $a, b \in Q$, say $a < b$, then $(a + b)/2 \in \mathbf{Q}$ and

$$a < \frac{a+b}{2} < b$$

(c) Consider the disjoint well-ordered sets

$$A = \{1, 3, 5, \ldots, \} \quad \text{and} \quad B = \{2, 4, 6, \ldots\}$$

Then the following ordered set

$$S = \{A; B\} = \{1, 3, 5, \ldots; 2, 4, 6, \ldots\}$$

is well-ordered. Note that, besides the first element 1, the element 2 does not have an immediate predecessor.

Notation: Here and subsequently, if $A, B, \ldots$ are disjoint ordered sets, then $\{A; B; \ldots\}$ means the set $A \cup B \cup \ldots$ ordered positionwise from left to right; that is, the elements in the same set keep their order, and any element in a set on the left precedes any element in a set on its right. Thus every element in A precedes every element in B, and so on.

Transfinite Induction

First we restate the principle of mathematical induction. (See Section 1.8 and 11.3.)

Principle of Mathematical Induction: Let A be a subset of the set $\mathbf{N}$ of positive integers with the following two properties:

(i) $1 \in A$.
(ii) If $k \in A$, then $k + 1 \in A$.

Then $A = \mathbf{N}$.

The above principle is one of Peano's axioms for the natural numbers (positive integers) $\mathbf{N}$. There is another form which is sometimes more convenient to use. Namely:

Principle of Mathematical Induction (Second Form): Let A be a subset of $\mathbf{N}$ with the following two properties:

(i) $1 \in A$.
(ii) If j belongs to A for $1 \leq j < k$, then $k \in A$.

Then $A = \mathbf{N}$.

The second form of induction is equivalent to the fact that $\mathbf{N}$ is well-ordered (Theorem 11.6). In fact, there is a somewhat similar statement which is true for every well-ordered set.

Principle of Transfinite Induction: Let A be a subset of a well-ordered set S with the following two properties:

(i) $a_0 \in A$.
(ii) If $s(a) \subseteq A$, then $a \in A$.

Then $A = S$.

Here a_0 is the first element of S, and $s(a)$, called the *initial segment* of a, is defined to be the set of all elements of S which strictly precede a.

Axiom of Choice, Well-Ordering Theorem

Let $\{A_i \mid i \in I\}$ be a collection of nonempty disjoint sets. We assume every $A_i \subseteq X$. A function $f\colon \{A_i\} \to X$ is called a *choice function* if $f(A_i) = a_i \in A_i$. In other words, f "chooses" a point $a_i \in A_i$ for each set A_i.

The axiom of choice lies at the foundations of mathematics and, in particular, the theory of sets. This "innocent looking" axiom, which follows, has as a consequence some of the most powerful and important results in mathematics.

Axiom of Choice: There exists a choice function for any nonempty collection of nonempty disjoint sets.

One of the consequences of the axiom of choice is the following theorem, which is attributed to Zermelo.

Well-Ordering Theorem: Every set S can be well-ordered.

The proof of this theorem lies beyond the scope of this text. Moreover, since all of our structures are finite or countable, we will not need to use this theorem. Ordinary mathematical induction suffices.

14.8 LATTICES

There are two ways to define a lattice L. One way is to define L in terms of a partially ordered set. Specifically, a lattice L may be defined as a partially ordered set in which $\inf(a, b)$ and $\sup(a, b)$ exist for any pair of elements $a, b \in L$. Another way is to define a lattice L axiomatically. This we do below.

Axioms Defining a Lattice

Let L be a nonempty set closed under two binary operations called *meet* and *join*, denoted respectively by $\wedge$ and $\vee$. Then L is called *lattice* if the following axioms hold where a, b, c are elements in L:

[$\mathbf{L}_1$] Commutative law:
(1a) $a \wedge b = b \wedge a$ (1b) $a \vee b = b \vee a$

[$\mathbf{L}_2$] Associative law:
(2a) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (2b) $(a \vee b) \vee c = a \vee (b \vee c)$

[$\mathbf{L}_3$] Absorption law:
(3a) $a \wedge (a \vee b) = a$ (3b) $a \vee (a \wedge b) = a$

We will sometimes denote the lattice by $(L, \wedge, \vee)$ when we want to show which operations are involved.

Duality and the Idempotent Law

The *dual* of any statement in a lattice $(L, \wedge, \vee)$ is defined to be the statement that is obtained by interchanging $\wedge$ and $\vee$. For example, the dual of

$$a \wedge (b \vee a) = a \vee a \quad \text{is} \quad a \vee (b \wedge a) = a \wedge a$$

Notice that the dual of each axiom of a lattice is also an axiom. Accordingly, the principle of duality holds; that is:

Theorem 14.2 (Principle of Duality): The dual of any theorem in a lattice is also a theorem.

This follows from the fact that the dual theorem can be proven by using the dual of each step of the proof of the original theorem.

An important property of lattices follows directly from the absorption laws.

Theorem 14.3 (Idempotent Law): (i) $a \wedge a = a$; (ii) $a \vee a = a$.

The proof of (i) requires only two lines:

$$\begin{aligned} a \wedge a &= a \wedge (a \vee (a \wedge b)) && \text{(using (3}b\text{))} \\ &= a && \text{(using (3}a\text{))} \end{aligned}$$

The proof of (ii) follows from the above principle of duality (or can be proved in a similar manner).

Lattices and Order

Given a lattice L, we can define a partial order on L as follows:

$$a \precsim b \quad \text{if} \quad a \wedge b = a$$

Analogously, we could define

$$a \precsim b \quad \text{if} \quad a \vee b = b$$

We state these results in a theorem.

Theorem 14.4: Let L be a lattice. Then:

(i) $a \wedge b = a$ if and only if $a \vee b = b$.

(ii) The relation $a \precsim b$ (defined by $a \wedge b = a$ or $a \vee b = b$) is a partial order on L.

Now that we have a partial order on any lattice L, we can picture L by a diagram as was done for partially ordered sets in general.

EXAMPLE 14.10 Let C be a collection of sets closed under intersection and union. Then $(C, \cap, \cup)$ is a lattice. In this lattice, the partial order relation is the same as the set inclusion relation. Figure 14-6 shows the diagram of the lattice L of all subsets of $\{a, b, c\}$.

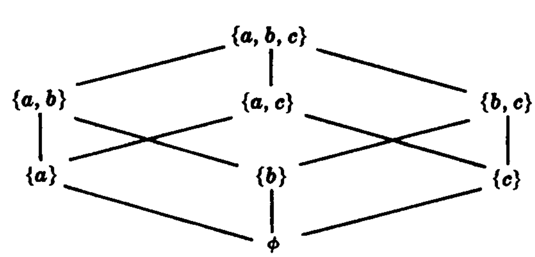


Fig. 14-6

We have shown how to define a partial order on a lattice L. The next theorem tells us when we can define a lattice on a partially ordered set P such that the lattice will give back the original order on P.

Theorem 14.5: Let P be a poset such that the $\inf(a, b)$ and $\sup(a, b)$ exist for any a, b in P. Letting

$$a \wedge b = \inf(a, b) \quad \text{and} \quad a \vee b = \sup(a, b)$$

we have that $(P, \wedge, \vee)$ is a lattice. Furthermore, the partial order on P induced by the lattice is the same as the original partial order on P.

The converse of the above theorem is also true. That is, let L be a lattice and let $\precsim$ be the induced partial order on L. Then $\inf(a, b)$ and $\sup(a, b)$ exist for any pair a, b in L and the lattice obtained from the poset $(L, \precsim)$ is the original lattice. Accordingly, we have the following:

Alternate Definition: A lattice is a partially ordered set in which

$$a \wedge b = \inf(a, b) \quad \text{and} \quad a \vee b = \sup(a, b)$$

exist for any pair of elements a and b.

We note first that any linearly ordered set is a lattice since $\inf(a, b) = a$ and $\sup(a, b) = b$ whenever $a \precsim b$. By Example 14.7, the positive integers $\mathbf{N}$ and the set $\mathbf{D}_m$ of divisors of m are lattices under the relation of divisibility.

Sublattices, Isomorphic Lattices

Suppose M is a nonempty subset of a lattice L. We say M is *a sublattice* of L if M itself is a lattice (with respect to the operations of L). We note that M is a sublattice of L if and only if M is closed under the operations of $\wedge$ and $\vee$ of L. For example, the set $\mathbf{D}_m$ of divisors of m is a sublattice of the positive integers $\mathbf{N}$ under divisibility.

Two lattices L and L' are said to be *isomorphic* if there is a one-to-one correspondence $f\colon L \to L'$ such that

$$f(a \wedge b) = f(a) \wedge f(b) \quad \text{and} \quad f(a \vee b) = f(a) \vee f(b)$$

for any elements a, b in L.

14.9 BOUNDED LATTICES

A lattice L is said to have a *lower bound* 0 if for any element x in L we have $0 \precsim x$. Analogously, L is said to have an *upper bound* I if for any x in L we have $x \precsim I$. We say L is *bounded* if L has both a lower bound 0 and an upper bound I. In such a lattice we have the identities

$$a \vee I = I, \quad a \wedge I = a, \quad a \vee 0 = a, \quad a \wedge 0 = 0$$

for any element a in L.

The nonnegative integers with the usual ordering,

$$0 < 1 < 2 < 3 < 4 < \cdots$$

have 0 as a lower bound but have no upper bound. On the other hand, the lattice $P(U)$ of all subsets of any universal set $\mathbf{U}$ is a bounded lattice with $\mathbf{U}$ as an upper bound and the empty set $\varnothing$ as a lower bound.

Suppose $L = \{a_1, a_2, \ldots, a_n\}$ is a finite lattice. Then

$$a_1 \vee a_2 \vee \cdots \vee a_n \quad \text{and} \quad a_1 \wedge a_2 \wedge \cdots \wedge a_n$$

are upper and lower bounds for L, respectively. Thus we have

Theorem 14.6: Every finite lattice L is bounded.

14.10 DISTRIBUTIVE LATTICES

A lattice L is said to be *distributive* if for any elements a, b, c in L we have the following:

[$\mathbf{L}_4$] Distributive law:

$$(4a)\ a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \qquad (4b)\ a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

Otherwise, L is said to be *nondistributive*. We note that by the principle of duality the condition (4a) holds if and only if (4b) holds.

Figure 14-7(a) is a nondistributive lattice since

$$a \vee (b \wedge c) = a \vee 0 = a \quad \text{but} \quad (a \vee b) \wedge (a \vee c) = I \wedge c = c$$

Figure 14-7(b) is also a nondistributive lattice. In fact, we have the following characterization of such lattices.

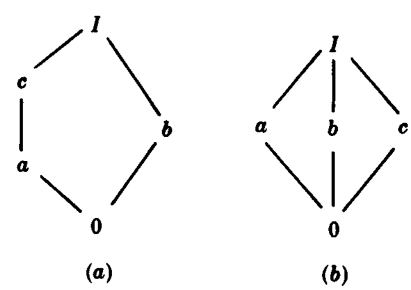


Fig. 14-7

Theorem 14.7: A lattice L is nondistributive if and only if it contains a sublattice isomorphic to Fig. 14-7(a) or to Fig. 14.7(b).

The proof of this theorem lies beyond the scope of this text.

Join Irreducible Elements, Atoms

Let L be a lattice with a lower bound 0. An element a in L is said to be *join irreducible* if $a = x \vee y$ implies $a = x$ or $a = y$. (Prime numbers under multiplication have this property, i.e., if $p = ab$ then $p = a$ or $p = b$ where p is prime.) Clearly 0 is join irreducible. If a has at least two immediate predecessors, say, b_1 and b_2 as in Fig. 14-8(a), then $a = b_1 \vee b_2$, and so a is not join irreducible. On the other hand, if a has a unique immediate predecessor c, then $a \neq \sup(b_1, b_2) = b_1 \vee b_2$ for any other elements b_1 and b_2 because c would lie between the b's and a as in Fig. 14-8(b). In other words, $a \neq 0$, is join irreducible if and only if a has a unique immediate predecessor. Those elements which immediately succeed 0, called *atoms*, are join irreducible. However, lattices can have other join irreducible elements. For example, the element c in Fig. 14-7(a) is not an atom but is join irreducible since a is its only immediate predecessor.

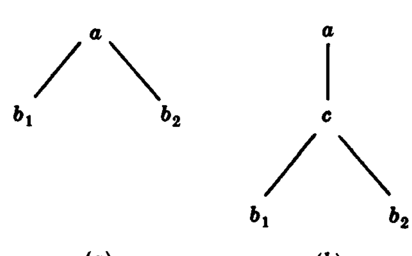


Fig. 14-8

If an element a in a finite lattice L is not join irreducible, then we can write $a = b_1 \vee b_2$. Then we can write b_1 and b_2 as the join of other elements if they are not join irreducible; and so on. Since L is finite we finally have

$$a = d_1 \vee d_2 \vee \cdots \vee d_n$$

where the d's are join irreducible. If d_i precedes d_j then $d_i \vee d_j = d_j$; so we can delete the d_i from the expression. In other words, we can assume that the d's are *irredundant*, i.e., no d precedes any other d. We emphasize that such an expression need not be unique, e.g., $I = a \vee b$ and $I = b \vee c$ in both lattices in Fig. 14-7. We now state the main theorem of this section (proved in Problem 14.28.)

Theorem 14.8: Let L be a finite distributive lattice. Then every a in L can be written uniquely (except for order) as the join of irredundant join irreducible elements.

Actually this theorem can be generalized to lattices with *finite length*, i.e., where all linearly ordered subsets are finite. (Problem 14.30 gives an infinite lattice with finite length.)

14.11 COMPLEMENTS, COMPLEMENTED LATTICES

Let L be a bounded lattice with lower bound 0 and upper bound I. Let a be an element of L. An element x in L is called a *complement* of a if

$$a \vee x = I \quad \text{and} \quad a \wedge x = 0$$

Complements need not exist and need not be unique. For example, the elements a and c are both complements of b in Fig. 14-7(a). Also, the elements y, z, and u in the chain in Fig. 14-1 have no complements. We have the following result.

Theorem 14.9: Let L be a bounded distributive lattice. Then complements are unique if they exist.

Proof: Suppose x and y are complements of any element a in L. Then

$$a \vee x = I, \quad a \vee y = I, \quad a \wedge x = 0, \quad a \wedge y = 0$$

Using distributivity,

$$x = x \vee 0 = x \vee (a \wedge y) = (x \vee a) \wedge (x \vee y) = I \wedge (x \vee y) = x \vee y$$

Similarly,

$$y = y \vee 0 = y \vee (a \wedge x) = (y \vee a) \wedge (y \vee x) = I \wedge (y \vee x) = y \vee x$$

Thus

$$x = x \vee y = y \vee x = y$$

and the theorem is proved.

Complemented Lattices

A lattice L is said to be *complemented* if L is bounded and every element in L has a complement. Figure 14-7(b) shows a complemented lattice where complements are not unique. On the other hand, the lattice $P(\mathbf{U})$ of all subsets of a universal set $\mathbf{U}$ is complemented, and each subset A of $\mathbf{U}$ has the unique complement $A^c = \mathbf{U} \backslash A$.

Theorem 14.10: Let L be a complemented lattice with unique complements. Then the join irreducible elements of L, other than 0, are its atoms.

Combining this theorem and Theorems 14.8 and 14.9, we get an important result.

Theorem 14.11: Let L be a finite complemented distributive lattice. Then every element a in L is the join of a unique set of atoms.

Remark: Some texts define a lattice L to be complemented if each a in L has a unique complement. Theorem 14.10 is then stated differently.

Solved Problems

ORDERED SETS AND SUBSETS

14.1. Let $\mathbf{N} = \{1, 2, 3, \ldots\}$ be ordered by divisibility. State whether each of the following subsets of $\mathbf{N}$ are linearly (totally) ordered.

(a) $\{24, 2, 6\}$; (c) $\mathbf{N} = \{1, 2, 3 \ldots\}$; ($e$) $\{7\}$;
(b) $\{3, 15, 5\}$; (d) $\{2, 8, 32, 4\}$; (f) $\{15, 5, 30\}$.

(a) Since 2 divides 6 which divides 24, the set is linearly ordered.

(b) Since 3 and 5 are not comparable, the set is not linearly ordered.

(c) Since 2 and 3 are not comparable, the set is not linearly ordered.

(d) This set is linearly ordered since $2 \prec 4 \prec 8 \prec 32$.

(e) Any set consisting of one element is linearly ordered.

(f) Since 5 divides 15 which divides 30, the set is linearly ordered.

14.2. Let $A = \{1, 2, 3, 4, 5\}$ be ordered by the Hasse diagram in Fig. 14-9(a).

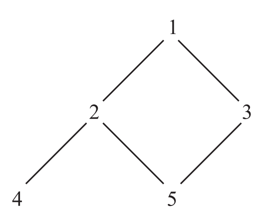
(a) Insert the correct symbol, $\prec$, $\succ$, or $\|$ (not comparable), between each pair of elements:

(i) 1 ___ 5; (ii) 2 ___ 3; (iii) 4 ___ 1; (iv) 3 ___ 4.

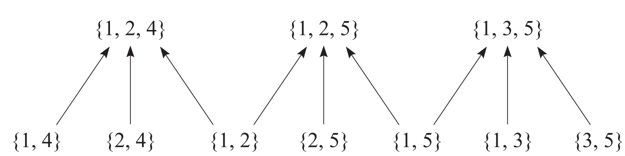
(b) Find all minimal and maximal elements of A.

(c) Does A have a first element or a last element?

(d) Let $L(A)$ denote the collection of all linearly ordered subsets of A with 2 or more elements, and let $L(A)$ be ordered by set inclusion. Draw the Hasse diagram of $L(A)$.



(a)



(b)

Fig. 14-9

(a) (i) Since there is a "path" (edges slanting upward) from 5 to 3 to 1, 5 precedes 1; hence $1 \succ 5$.

(ii) There is no path from 2 to 3, or vice versa; hence $2 \| 3$.

(iii) There is a path from 4 to 2 to 1; hence $4 \prec 1$.

(iv) Neither $3 \prec 4$ nor $4 \prec 3$; hence $3 \| 4$.

(b) No element strictly precedes 4 or 5, so 4 and 5 are minimal elements of A. No element strictly succeeds 1, so 1 is a maximal element of A.

(c) A has no first element. Although 4 and 5 are minimal elements of A, neither precedes the other. However, 1 is a last element of A since 1 succeeds every element of A.

(d) The elements of $L(A)$ are as follows:

$$\{1, 2, 4\}, \quad \{1, 2, 5\}, \quad \{1, 3, 5\}, \quad \{1, 2\}, \quad \{1, 4\}, \quad \{1, 3), \quad \{1, 5\}, \quad \{2, 4\}, \quad \{2, 5\}, \quad \{3, 5\}$$

(Note $\{2, 5\}$ and $\{3, 4\}$ are not linearly ordered.) The diagram of $L(A)$ appears in Fig. 14-9(b).

14.3. Prerequisites in college is a familiar partial ordering of available classes. We write $A \prec B$ if course A is a prerequisite for course B. Let C be the ordered set consisting of the mathematics courses and their prerequisites appearing in Fig. 14-10(a).

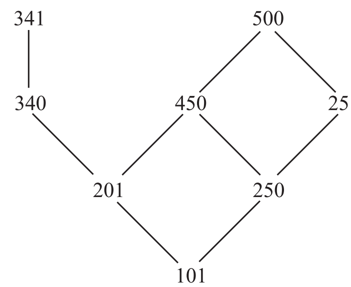
(a) Draw the Hasse diagram for the partial ordering C of these classes.

(b) Find all minimal and maximal elements of C.

(c) Does C have a first element or a last element?

Class	**Prerequisites**
Math 101	None
Math 201	Math 101
Math 250	Math 101
Math 251	Math 250
Math 340	Math 201
Math 341	Math 340
Math 450	Math 201, Math 250
Math 500	Math 450, Math 251

(a)



(b)

Fig. 14-10

(a) Math 101 must be on the bottom of the diagram since it is the only course with no prerequisites. Since Math 201 and Math 250 only require Math 101, we have Math 101 $\ll$ Math 201and Math 101 $\ll$ Math 250; hence draw a line slanting upward from Math 101 to Math 201 and one from Math 101 to Math 250. Continuing this process, we obtain the Hasse diagram in Fig. 14-10(b).

(b) No element strictly precedes Math 101 so Math 101 is a minimal element of C. No element strictly succeeds Math 341 or Math 500, so each is a maximal element of C.

(c) Math 101 is a first element of C since it precedes every other element of C. However, C has no last element. Although Math 341 and Math 500 are maximal elements, neither is a last element since neither precedes the other.

PRODUCT SETS AND ORDER

14.4. Suppose $\mathbf{N}^2 = \mathbf{N} \times \mathbf{N}$ is given the product order (Section 14.2) where $\mathbf{N}$ has the usual order $\leq$.

Insert the correct symbol, $\prec$, $\succ$, or $\parallel$ (not comparable), between each of the following pairs of elements of $\mathbf{N} \times \mathbf{N}$:

(a) (5, 7) ___ (7, 1); (c) (5, 5) ___ (4, 8); (e) (7, 9) ___ (4, 1);
(b) (4, 6) ___ (4, 2); (d) (1, 3) ___ (1, 7); (f) (7, 9) ___ (8, 2).

Here $(a, b) \prec (a', b')$ if $a < a'$ and $b \leq b'$ or if $a \leq a'$ and $b < b'$. Thus:

(a) $\parallel$ since $5 < 7$ but $7 > 1$. (c) $\parallel$ since $5 > 4$ and $5 < 8$. (e) $\succ$ since $7 > 4$ and $9 > 1$.
(b) $\succ$ since $4 = 4$ and $6 > 2$. (d) $\prec$ since $1 = 1$ and $3 < 7$. (f) $\parallel$ since $7 < 8$ and $9 > 2$.

14.5. Repeat Problem 14.4 using the lexicographical ordering of $\mathbf{N}^2 = \mathbf{N} \times \mathbf{N}$.

Here $(a, b) \prec (a',\ b')$ if $a < a'$ or if $a = a'$ but $b < b'$. Thus:

(a) $\prec$ since $5 < 7$. (c) $\succ$ since $5 > 4$. (e) $\succ$ since $7 > 4$.
(b) $\succ$ since $4 = 4$ and $6 > 2$. (d) $\prec$ since $1 = 1$ but $3 < 7$. (f) $\prec$ since $7 < 8$.

14.6. Consider the English alphabet $\mathbf{A} = \{a, b, c, \ldots, y, z\}$ with the usual (alphabetical) order. (Recall that $\mathbf{A}^*$ consisting of all words in $\mathbf{A}$.) Consider the following list of words in $\mathbf{A}^*$:

went, forget, to, medicine, me, toast, melt, for, we, arm

(a) Sort the list of words using the short-lex (free semigroup) order.

(b) Sort the list of words using the usual (alphabetical) order of $\mathbf{A}^*$.

(a) First order the elements by length and then order them lexicographically (alphabetically):

me, to, we, arm, for, melt, went, toast, forget, medicine

(b) The usual (alphabetical) ordering yields:

arm, for, forget, me, medicine, melt, to, toast, we, went

CONSISTENT ENUMERATIONS

14.7. Suppose a student wants to take all eight mathematics courses in Problem 14.3, but only one per semester.

(a) Which choice or choices does she have for her first and for her last (eighth) semester?

(b) Suppose she wants to take Math 250 in her first year (first or second semester) and Math 340 in her senior year (seventh or eighth semester). Find all the ways that she can take the eight courses.

(a) By Fig. 14-10, Math 101 is the only minimal element and hence must be taken in the first semester, and Math 341 and 500 are the maximal elements and hence one of them must be taken in the last semester.

(b) Math 250 is not a minimal element and hence must be taken in the second semester, and Math 340 is not a maximal element so it must be taken in the seventh semester and Math 341 in the eighth semester. Also Math 500 must be taken in the sixth semester. The following gives the three possible ways to take the eight courses:

$$101, 250, 251, 201, 450, 500, 340, 341, \quad 101, 250, 201, 251, 450, 500, 340, 341,$$
$$101, 250, 201, 450, 251, 500, 340, 341$$

14.8. Prove Theorem 14.1: Suppose S is a finite poset with n elements. Then there exists a consistent enumeration $f\colon S \to \{1, 2, \ldots, n\}$.

The proof is by induction on the number n of elements in S. Suppose $n = 1$, say $S = \{s\}$. Then $f(s) = 1$ is a consistent enumeration of S. Now suppose $n > 1$ and the theorem holds for posets with fewer than n elements. Let $a \in S$ be a minimal element. (Such an element a exists since S is finite.) Let $T = S\backslash\{a\}$. Then T is a finite poset with $n - 1$ elements and hence, by induction, T admits a consistent enumeration; say $g\colon T \to \{1, 2, \ldots, n - 1\}$. Define $f\colon S \to \{1, 2, \ldots, n\}$ by:

$$f(x) = \begin{cases} 1, & \text{if } x = a \\ g(x) + 1 & \text{if } x \neq a \end{cases}$$

Then f is the required consistent enumeration.

UPPER AND LOWER BOUNDS, SUPREMUM AND INFIMUM

14.9. Let $S = \{a, b, c, d, e, f, g\}$ be ordered as in Fig. 14-11(a), and let $X = \{c, d, e\}$.

(a) Find the upper and lower bounds of X.

(b) Identify $\sup(X)$, the supremum of X, and $\inf(X)$, the infimum of X, if either exists.

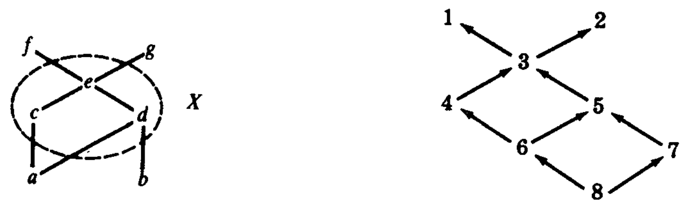
(a) The elements e, f, and g succeed every element of X; hence e, f, and g are the upper bounds of X. The element a precedes every element of X; hence a is the lower bound of X. Note that b is not a lower bound since b does not precede c; in fact, b and c are not comparable.

(b) Since e precedes both f and g, we have $e = \sup(X)$. Likewise, since a precedes (trivially) every lower bound of X, we have $a = \inf(X)$. Note that $\sup(X)$ belongs to X but $\inf(X)$ does not belong to X.

14.10. Let $S = \{1, 2, 3, \ldots, 8\}$ be ordered as in Fig. 14-11(b), and let $A = (2, 3, 6\}$.
(a) Find the upper and lower bounds of A. (b) Identify $\sup(A)$ and $\inf(A)$ if either exists.

(a) The upper bound is 2, and the lower bounds are 6 and 8.

(b) Here $\sup(A) = 2$ and $\inf(A) = 6$.



(a) (b)

Fig. 14-11

14.11. Repeat Problem 14.10 for the subset $B = \{1, 2, 5\}$.

(a) There is no upper bound for B since no element succeeds both 1 and 2. The lower bounds are 6, 7, 8.

(b) Trivially, $\sup(A)$ does not exist since there are no upper bounds. Although A has three lower bounds, $\inf(A)$ does not exist since no lower bound succeeds both 6 and 7.

14.12. Consider the set $\mathbf{Q}$ of rational numbers with the usual order $\leq$. Consider the subset D of $\mathbf{Q}$ defined by

$$D = \{x \mid x \in \mathbf{Q} \text{ and } 8 < x^3 < 15\}$$

(a) Is D bounded above or below? (b) Does $\sup(D)$ or $\inf(D)$ exist?

(a) The subset D is bounded both above and below. For example, 1 is a lower bound and 100 an upper bound.

(b) We claim that $\sup(D)$ does not exist. Suppose, on the contrary, $\sup(D) = x$. Since $\sqrt[3]{15}$ is irrational, $x > \sqrt[3]{15}$. However, there exists a rational number y such that $\sqrt[3]{15} < y < x$. Thus y is also an upper bound for D. This contradicts the assumption that $x = \sup(D)$. On the other hand, $\inf(D)$ does exist. Specifically, $\inf(D) = 2$.

ISOMORPHIC (SIMILAR) SETS, SIMILARITY MAPPINGS

14.13. Suppose a poset A is isomorphic (similar) to a poset B and $f: A \to B$ is a similarity mapping. Are the following statements true or false?

(a) An element $a \in A$ is a first (last, minimal, or maximal) element of A if and only if $f(a)$ is a first (last, minimal, or maximal) element of B.

(b) An element $a \in A$ immediately precedes an element $a' \in A$, that is, $a \ll a'$, if and only if $f(a) \ll f(a')$.

(c) An element $a \in A$ has r immediate successors in A if and only if $f(a)$ has r immediate successors in B.

All the statements are true; the order structure of A is the same as the order structure of B.

14.14. Let $S = \{a, b, c, d, e\}$ be the ordered set in Fig. 14-12(a). Suppose $A = \{1, 2, 3, 4, 5\}$ is isomorphic to S. Draw the Hasse diagram of A if the following is a similarity mapping from S to A:

$$f = \{(a, 1), (b, 3), (c, 5), (d, 2), (e, 4)\}$$

The similarity mapping f preserves the order structure of S and hence f may be viewed simply as a relabeling of the vertices in the diagram of S. Thus Fig. 14-12(b) shows the Hasse diagram of A.

14.15. Let $A = \{1, 2, 3, 4, 5\}$ is ordered as in Fig. 14-12(b). Find the number n of similarity mappings $f: A \to A$.

Since 1 is the only minimal element of A and 4 is the only maximal element, we must have $f(1) = 1$ and $f(4) = 4$. Also, $f(3) = 3$ since 3 is the only immediate successor of 1. On the other hand, there are two possibilities for $f(2)$ and $f(5)$, that is, we can have $f(2) = 2$ and $f(5) = 5$, or $f(2) = 5$ and $f(5) = 2$. Accordingly, $n = 2$.

14.16. Give an example of a finite nonlinearly ordered set $X = (A, R)$ which is isomorphic to $Y = (A, R^{-1})$, the set A with the inverse order.

Let R be the partial ordering of $A = \{a, b, c, d, e\}$ pictured in Fig. 14-13(a).

Then Fig. 14-13(b) shows A with the inverse order R. (The diagram of R is simply turned upside down to obtain R^{-1}.) Notice that the two diagrams are identical except for the labeling. Thus X is isomorphic to Y.

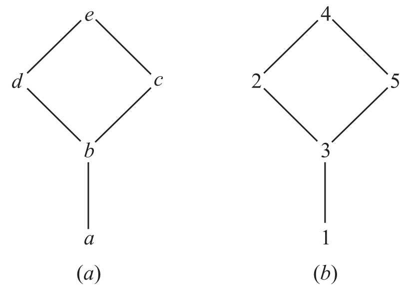


Fig. 14-12

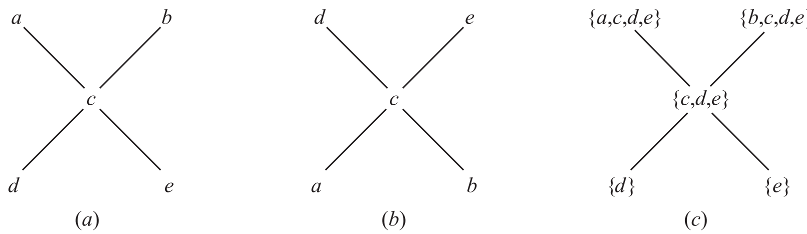


Fig. 14-13

14.17. Let A be an ordered set and, for each $a \in A$, let $p(a)$ denote the set of predecessors of a:

$$p(a) = \{x \mid x \precsim a\}$$

(called the *predecessor set* of a). Let $p(A)$ denote the collection of all predecessor sets of the elements in A ordered by set inclusion.

(a) Show that A and $p(A)$ are isomorphic by showing that the map $f: A \to p(A)$, defined by $f(a) = p(a)$, is a similarity mapping of A onto $p(A)$.

(b) Find the Hasse diagram of $p(A)$ for the set A in Fig. 14-13(a).

(a) First we show that f preserves the order relation of A. Suppose $a \precsim b$. Let $x \in p(a)$. Then $x \precsim a$, and hence $a \precsim b$; so $x \in p(b)$. Thus $p(a) \subseteq p(b)$. Suppose $a \parallel b$ (noncomparable). Then $a \in p(a)$ but $a \notin p(b)$; hence $p(a) \nsubseteq p(b)$. Similarly, $b \in p(b)$ but $b \notin p(a)$; hence $p(b) \nsubseteq p(a)$. Therefore, $p(a) \parallel p(b)$. Thus f preserves order.

We now need only show that f is one-to-one and onto. Suppose $y \in p(A)$. Then $y = p(a)$ for some $a \in A$. Thus $f(a) = p(a) = y$ so f is onto $p(A)$. Suppose $a \neq b$. Then $a \prec b$, $b \prec a$ or $a \parallel b$. In the first and third cases, $b \in p(b)$ but $b \notin p(a)$, and in the second case $a \in p(a)$ but $a \notin p(b)$. Accordingly, in all three cases, we have $p(a) \neq p(b)$. Therefore f is one-to-one.

Consequently, f is a similarity mapping of A onto $p(A)$ and so $A \simeq p(A)$.

(b) The elements of $p(A)$ follow:

$$p(a) = \{a, c, d, e\}, \quad p(b) = \{b, c, d, e\}, \quad p(c) = \{c, d, e\}, \quad p(d) = \{d\}, \quad p(e) = \{e\}$$

Figure 14-13(c) gives the diagram of $p(A)$ ordered by set inclusion. Observe that the diagrams in Fig. 14-13(a) and (c) are identical except for the labeling of the vertices.

WELL-ORDERED SETS

14.18. Prove the Principle of Transfinite Induction: Let A be a subset of a well-ordered set S with the following two properties: (i) $a_0 \in A$. (ii) If $s(a) \subseteq A$ then $a \in A$. Then $A = S$.

(Here a_0 is the first element of A, and $s(a)$ is the initial segment of a, i.e., the set of all elements strictly preceding a.)
Suppose $A \neq S$. Let $B = S \backslash A$. Then $B \neq \emptyset$. Since S is well-ordered, B has a first element b_0. Each element $x \in s(b_0)$

precedes b_0 and hence does not belong to B. Thus every $x \in s(b_0)$ belongs to A; hence $s(b_0) \subseteq A$. By (ii), $b_0 \in A$. This contradicts the assumption that $b_0 \in S \backslash A$. Thus the original assumption that $A \neq S$ is not true. Therefore $A = S$.

14.19. Let S be a well-ordered set with first element a_0. Define *a limit element* of S.

An element $b \in$ S is a limit element if $b \neq a_0$ and b has no immediate predecessor.

14.20. Consider the set $\mathbf{N} = \{1, 2, 3, \ldots\}$ of positive integers. Every number in $\mathbf{N}$ can be written uniquely as a product of a nonnegative power of 2 times an odd number. Suppose $a, a' \in \mathbf{N}$ and

$$a = 2^r(2s+1) \quad \text{and} \quad a' = 2^{r'}(2s'+1)$$

where r, r' and s, s' are nonnegative integers. Define:

$$a \prec a' \quad \text{if } r < r' \quad \text{or} \quad \text{if } r = r' \text{ but } s < s'.$$

(a) Insert the correct symbol, $\prec$ or $\succ$, between each pair of numbers:

(i) 5 ___ 14; (ii) 6 ___ 9; (iii) 3 ___ 20; (iv) 14 ___ 21

(b) Let $S = (\mathbf{N}, \prec)$. Show that S is well-ordered.

(c) Does S have any limit elements?

(a) The elements of $\mathbf{N}$ can be listed as in Fig. 14-14. The first row consists of the odd numbers, the second row of 2 times the odd numbers, the third row of $2^2 = 4$ times the odd numbers, and so on. Then $a \prec a'$ if a is in higher row then, a' or if a and a' are in the same row but a comes before a' in the row. Accordingly:

(i) $5 \prec 14$; (ii) $6 \succ 9$; (iii) $3 \succ 20$; (iv) $14 \succ 20$.

r \ s	0	1	2	3	4	5	6	7	
0	1	3	5	7	9	11	13	15	...
1	2	6	10	14	18	22	26	30	...
2	4	12	20	28	36	44	52	60	...
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Fig. 14-14

(b) Let A be a subset of S. The rows are well-ordered. Let r_0 denote the minimum row of elements in A. In r_0 there may be many elements of A. The columns are well-ordered, so let s_0 denote the minimum column of the elements of A in row r_0. Then $x = (r_0, s_0)$ is the first element of A. Thus S is well-ordered.

(c) As indicated by Fig. 14-14, every power of 2, that is, 1, 2, 4, 8, ..., has no immediate predecessor. Thus each, other than 1, is a limit element of S.

14.21. Let S be a well-ordered set. Let $f: S \to S$ be a similarity mapping of S into S. Prove that, for every $a \in S$, we have $a \precsim f(a)$.

Let $D = \{x \mid f(x) \prec x\}$. If D is empty, then the statement is true. Suppose $D \neq \varnothing$. Since D is well-ordered, D has a first element, say d_0. Since $d_0 \in D$, we have $f(d_0) \prec d_0$. Since f is a similarity mapping:

$$f(d_0) \prec d_0 \quad \text{implies} \quad f(f(d_0)) \prec f(d_0)$$

Thus $f(d_0)$ also belongs to D. But $f(d_0) \prec d_0$ and $f(d_0) \in D$ contradicts the fact that d_0 is the first element of D. Hence the original assumption that $D \neq \varnothing$ leads to a contradiction. Therefore D is empty and the statement is true.

14.22. Let A be a well-ordered set. Let $s(A)$ denote the collection of all initial segments $s(a)$ of elements $a \in A$ ordered by set inclusion. Prove A is isomorphic to $s(A)$ by showing that the map $f: A \to s(A)$, defined by $f(a) = s(a)$, is a similarity mapping of A onto $s(A)$. (Compare with Problem 14.17.)

First we show that f is one-to-one and onto. Suppose $y \in s(A)$. Then $y = s(a)$ for some $a \in A$. Thus $f(a) = s(a) = y$, so f is onto $s(A)$. Suppose $x \neq y$. Then one precedes the other, say, $x \prec y$. Then $x \in s(y)$. But $x \notin s(x)$. Thus $s(x) \neq s(y)$. Therefore, f is also one-to-one.

We now need only show that f preserves order, that is,

$$x \precsim y \quad \text{if and only if} \quad s(x) \subseteq s(y)$$

Suppose $x \precsim y$. If $a \in s(x)$, then $a \prec x$ and hence $a \prec y$; thus $a \in s(y)$. Thus $s(x) \subseteq s(y)$. On the other hand, suppose $x \not\precsim y$, that is, $x \succ y$. Then $y \in s(x)$. But $y \notin s(y)$; hence $s(x) \not\subseteq s(y)$. In other words, $x \precsim y$ if and only if $s(x) \subseteq s(y)$. Accordingly, f is a similarity mapping of A onto $S(A)$, and so $A \cong s(A)$.

LATTICES

14.23. Write the dual of each statement:

(a) $(a \wedge b) \vee c = (b \vee c) \wedge (c \vee a)$; ($b$) $(a \wedge b) \vee a = a \wedge (b \vee a)$.

Replace $\vee$ by $\wedge$ and replace $\wedge$ by $\vee$ in each statement to obtain the dual statement:

(a) $(a \vee b) \wedge c = (b \wedge c) \vee (c \wedge a)$; ($b$) $(a \vee b) \wedge a = a \vee (b \wedge a)$

14.24. Prove Theorem 14.4: Let L be a lattice. Then:

(i) $a \wedge b = a$ if and only if $a \vee b = b$.

(ii) The relation $a \precsim b$ (defined by $a \wedge b = a$ or $a \vee b = b$) is a partial order on L.

(i) Suppose $a \wedge b = a$. Using the absorption law in the first step we have:

$$b = b \vee (b \wedge a) = b \vee (a \wedge b) = b \vee a = a \vee b$$

Now suppose $a \vee b = b$. Again using the absorption law in the first step we have:

$$a = a \wedge (a \vee b) = a \wedge b$$

Thus $a \wedge b = a$ if and only if $a \vee b = b$.

(ii) For any a in L, we have $a \wedge a = a$ by idempotency. Hence $a \precsim a$, and so $\precsim$ is reflexive.

Suppose $a \precsim b$ and $b \precsim a$. Then $a \wedge b = a$ and $b \wedge a = b$. Therefore, $a = a \wedge b = b \wedge a = b$, and so $\precsim$ is antisymmetric.

Lastly, suppose $a \precsim b$ and $b \precsim c$. Then $a \wedge b = a$ and $b \wedge c = b$. Thus

$$a \wedge c = (a \wedge b) \wedge c = a \wedge (b \wedge c) = a \wedge b = a$$

Therefore $a \precsim c$, and so $\precsim$ is transitive. Accordingly, $\precsim$ is a partial order on L.

14.25. Which of the partially ordered sets in Fig. 14-15 are lattices?

A partially ordered set is a lattice if and only if $\sup(x, y)$ and $\inf(x, y)$ exist for each pair x, y in the set. Only (c) is not a lattice since $\{a, b\}$ has three upper bounds, c, d and I, and no one of them precedes the other two, that is, $\sup(a, b)$ does not exist.

14.26. Consider the lattice L in Fig. 14-15(a).

(a) Which nonzero elements are join irreducible?

(b) Which elements are atoms?

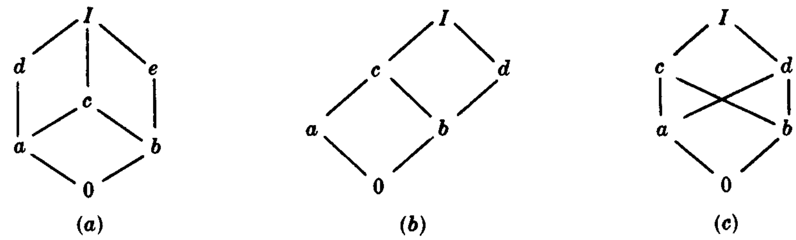


Fig. 14-15

(c) Which of the following are sublattices of L:

$$L_1 = \{0, a, b, I\}, \quad L_2 = \{0, a, e, I\}, \quad L_3 = \{a, c, d, I\}, \quad L_4 = \{0, c, d, I\}$$

(d) Is L distributive?

(e) Find complements, if they exist, for the elements, a, b and c.

(f) Is L a complemented lattice?

(a) Those nonzero elements with a unique immediate predecessor are join irreducible. Hence a, b, d, and e are join irreducible.

(b) Those elements which immediately succeed 0 are atoms, hence a and b are the atoms.

(c) A subset L' is a sublattice if it is closed under $\wedge$ and $\vee$. L_1 is not a sublattice since $a \vee b = c$, which does not belong to L_1. The set L_4 is not a sublattice since $c \wedge d = a$ does not belong to L_4. The other two sets, L_2 and L_3, are sublattices.

(d) L is not distributive since $M = \{0, a, d, e, I\}$ is a sublattice which is isomorphic to the nondistributive lattice in Fig. 14-7(a).

(e) We have $a \wedge e = 0$ and $a \vee e = I$, so a and e are complements. Similarly, b and d are complements. However, c has no complement.

(f) L is not a complemented lattice since c has no complement.

14.27. Consider the lattice M in Fig. 14-15(b).

(a) Find the nonzero join irreducible elements and atoms of M.

(b) Is M (i) distributive? (ii) complemented?

(a) The nonzero elements with a unique predecessor are a, b, and d, and of these three only a and b are atoms since their unique predecessor is 0.

(b) (i) M is distributive since M does not have a sublattice isomorphic to one of the lattices in Fig. 14-7. (ii) M is not complemented since b has no complement. Note a is the only solution to $b \wedge x = 0$ but $b \wedge a = c \neq I$.

14.28. Prove Theorem 14.8: Let L be a finite distributive lattice. Then every $a \in L$ can be written uniquely (except for order) as the join of irredundant join irreducible elements.

Since L is finite we can write a as the join of irredundant join irreducible elements as we discussed in Section 14.9. Thus we need to prove uniqueness. Suppose

$$a = b_1 \vee b_2 \vee \cdots \vee b_r = c_1 \vee c_2 \vee \cdots \vee c_s$$

where the b's are irredundant and join irreducible and the c's are irredundant and irreducible. For any given i we have

$$b_i \precsim (b_1 \vee b_2 \vee \cdots \vee b_r) = (c_1 \vee c_2 \vee \cdots \vee c_s)$$

Hence

$$b_i = b_i \wedge (c_1 \vee c_2 \vee \cdots \vee c_s) = (b_i \wedge c_1) \vee (b_i \wedge c_2) \vee \cdots \vee (b_i \wedge c_s)$$

Since b_i is join irreducible, there exists a j such that $b_i = b_i \wedge c_j$, and so $b_i \precsim c_j$. By a similar argument, for c_j there exists a b_k such that $c_j \precsim b_k$. Therefore

$$b_i \precsim c_j \precsim b_k$$

which gives $b_i = c_j = b_k$ since the b's are irredundant. Accordingly, the b's and c's may be paired off. Thus the representation for a is unique except for order.

14.29. Prove Theorem 14.10: Let L be a complemented lattice with unique complements. Then the join irreducible elements of L, other than 0, are its atoms.

Suppose a is join irreducible and a is not an atom. Then a has a unique immediate predecessor $b \neq 0$. Let b' be the complement of b. Since $b \neq 0$ we have $b' \neq I$. If a precedes b', then $b \precsim a \precsim b'$, and so $b \wedge b' = b'$, which is impossible since $b \wedge b' = I$. Thus a does not precede b', and so $a \wedge b'$ must strictly precede a. Since b is the unique immediate predecessor of a, we also have that $a \wedge b'$ precedes b as in Fig. 14-16(a). But $a \wedge b'$ precedes b'. Hence

$$a \wedge b' \precsim \inf(b, b') = b \wedge b' = 0$$

Thus $a \wedge b' = 0$. Since $a \vee b = a$, we also have that

$$a \vee b' = (a \vee b) \vee b' = a \vee (b \vee b') = a \vee I = I$$

Therefore b' is a complement of a. Since complements are unique, $a = b$. This contradicts the assumption that b is an immediate predecessor of a. Thus the only join irreducible elements of L are its atoms.

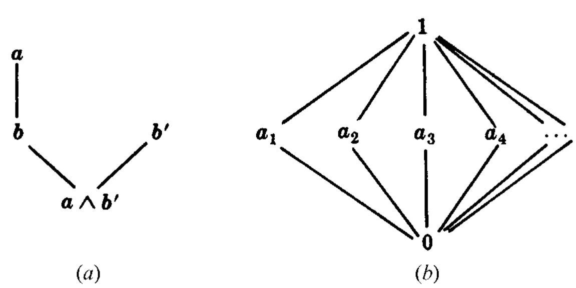


Fig. 14-16

14.30. Give an example of an infinite lattice L with finite length.

Let $L = \{0, 1, a_1, a_2, a_3, \ldots\}$ and let L be ordered as in Fig. 14-16(b). Accordingly, for each $n \in \mathbf{N}$, we have $0 < a_n < 1$. Then L has finite length since L has no infinite linearly ordered subset.

Supplementary Problems

ORDERED SETS AND SUBSETS

14.31. Let $A = \{1, 2, 3, 4, 5, 6\}$ be ordered as in Fig. 14-17(a).

(a) Find all minimal and maximal elements of A.

(b) Does A have a first or last element?

(c) Find all linearly ordered subsets of A, each of which contains at least three elements.

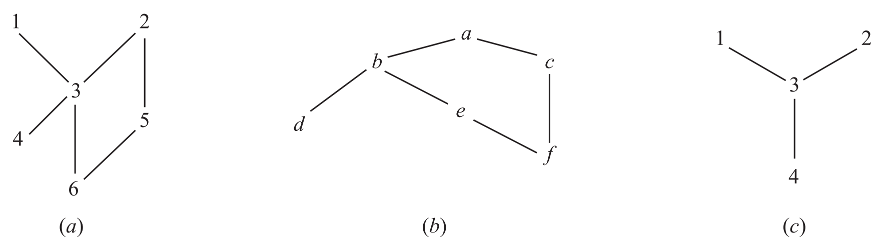


Fig. 14-17

14.32. Let $B = \{a, b, c, d, e, f\}$ be ordered as in Fig. 14-17(b).

(a) Find all minimal and maximal elements of B.

(b) Does B have a first or last element?

(c) List two and find the number of consistent enumerations of B into the set $\{1, 2, 3, 4, 5, 6\}$.

14.33. Let $C = \{1, 2, 3, 4\}$ be ordered as in Fig. 14-17(c). Let $L(C)$ denote the collection of all nonempty linearly ordered subsets of C ordered by set inclusion. Draw a diagram of $L(C)$.

14.34. Draw the diagrams of the partitions of m (see Example 14.4) where: (a) $m = 4$; (b) $m = 6$.

14.35. Let $\mathbf{D}_m$ denote the positive divisors of m ordered by divisibility. Draw the Hasse diagrams of: (a) $\mathbf{D}_{12}$; (b) $\mathbf{D}_{15}$; (c) $\mathbf{D}_{16}$; (d) $\mathbf{D}_{17}$.

14.36. Let $S = \{a, b, c, d, e, f\}$ be a poset. Suppose there are exactly six pairs of elements where the first immediately precedes the second as follows:

$$f \ll a, \quad f \ll d, \quad e \ll b, \quad c \ll f, \quad e \ll c, \quad b \ll f$$

(a) Find all minimal and maximal elements of S.

(b) Does S have any first or last element?

(c) Find all pairs of elements, if any, which are noncomparable.

14.37. State whether each of the following is true or false and, if it is false, give a counterexample.

(a) If a poset S has only one maximal element a, then a is a last element.

(b) If a finite poset S has only one maximal element a, then a is a last element.

(c) If a linearly ordered set S has only one maximal element a, then a is a last element.

14.38. Let $S = \{a, b, c, d, e\}$ be ordered as in Fig. 14-18(a).

(a) Find all minimal and maximal elements of S.

(b) Does S have any first or last element?

(c) Find all subsets of S in which c is a minimal element.

(d) Find all subsets of S in which c is a first element.

(e) List all linearly ordered subsets with three or more elements

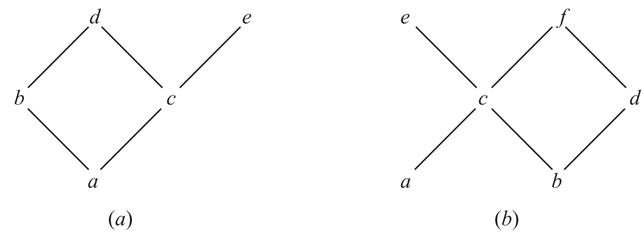


Fig. 14-18

14.39. Let $S = \{a, b, c, d, e, f\}$ be ordered as in Fig. 14-18(b).

(a) Find all minimal and maximal elements of S.

(b) Does S have any first or last element?

(c) List all linearly ordered subsets with three or more elements.

14.40. Let $S = \{a, b, c, d, e, f, g\}$ be ordered as in Fig. 14-11(a). Find the number n of linearly ordered subsets of S with: (a) four elements; (b) five elements.

14.41. Let $S = \{1, 2, \ldots, 7, 8\}$ be ordered as in Fig. 14-11(b). Find the number n of linearly ordered subsets of S with: (a) five elements; (b) six elements.

CONSISTENT ENUMERATIONS

14.42. Let $S = \{a, b, c, d, e\}$ be ordered as in Fig. 14-18(a). List all consistent enumerations of S into $\{1, 2, 3, 4, 5\}$.

14.43. Let $S = \{a, b, c, d, e, f\}$ be ordered as in Fig. 14-18(b). Find the number n of consistent enumerations of S into $\{1, 2, 3, 4, 5, 6\}$.

14.44. Suppose the following are three consistent enumerations of an ordered set $A = \{a, b, c, d\}$:

$$[(a, 1), (b, 2), (c, 3), (d, 4)], \quad [(a, 1), (b, 3), (c, 2), (d, 4)], \quad [(a, 1), (b, 4), (c, 2), (d, 3)]$$

Assuming the Hasse diagram D of A is connected, draw D.

ORDER AND PRODUCT SETS

14.45. Let $M = \{2, 3, 4, \ldots\}$ and let $M^2 = M \times M$ be ordered as follows:

$$(a, b) \prec (c, d) \quad \text{if} \quad a \mid c \text{ and } b < d$$

Find all minimal and maximal elements of $M \times M$.

14.46. Consider the English alphabet $A = \{a, b, c, \ldots, y, z\}$ with the usual (alphabetical) order. Recall A^* consists of all words in A. Let L consist of the following list of elements in A^*:

gone, or, arm, go, an, about, gate, one, at, occur

(a) Sort L according to the short-lex order, i.e., first by length and then alphabetically.

(b) Sort L alphabetically.

14.47. Consider the ordered sets A and B appearing in Fig. 14-17(a) and (b), respectively. Suppose $S = A \times B$ is given the product order. Insert the correct symbol, $\prec$, $\succ$ or $\|$, between each pair of elements of S:

(a) $(4, b)$___$(2, e)$; (b) $(3, a)$___$(6, f)$; (c) $(5, d)$___$(1, a)$; (d) $(6, e)$___$(2, b)$.

14.48. Suppose $\mathbf{N} = \{1, 2, 3, \ldots\}$ and $\mathbf{A} = \{a, b, c, \ldots, y, z\}$ are given the usual orders, and $S = \mathbf{N} \times \mathbf{A}$ is ordered lexicographically. Sort the following elements of S:

$$(2, z), \quad (1, c), \quad (2, c), \quad (1, y), \quad (4, b), \quad (4, z), \quad (3, b), \quad (2, a)$$

UPPER AND LOWER BOUNDS, SUPREMUM AND INFIMUM

14.49. Let $S = \{a, b, c, d, e, f, g\}$ be ordered as in Fig. 14-11(a). Let $A = \{a, c, d\}$.

(a) Find the set of upper bounds of A. (c) Does $\sup(A)$ exist?

(b) Find the set of lower bounds of A. (d) Does $\inf(A)$ exist?

14.50. Repeat Problem 14.49 for subset $B = \{b, c, \text{e}\}$ of S.

14.51. Let $S = \{1, 2, \ldots, 7, 8\}$ be ordered as in Fig. 14-11(b). Consider the subset $A = \{3, 6, 7\}$ of S.

(a) Find the set of upper bounds of A. (c) Does $\sup(A)$ exist?

(b) Find the set of lower bounds of A. (d) Does $\inf(A)$ exist?

14.52. Repeat Problem 14.51 for the subset $B = \{1, 2, 4, 7\}$ of S.

14.53. Consider the rational numbers $\mathbf{Q}$ with the usual order $\leq$. Let $A = \{x \mid x \in \mathbf{Q} \text{ and } 5 < x^3 < 27\}$.

(a) Is A bounded above or below?

(b) Does $\sup(A)$ or $\inf(A)$ exist?

14.54. Consider the real numbers $\mathbf{R}$ with the usual order $\leq$. Let $A = \{x \mid x \in \mathbf{Q} \text{ and } 5 < x^3 < 27\}$.

(a) Is A bounded above or below? (b) Does $\sup(A)$ or $\inf(A)$ exist?

ISOMORPHIC (SIMILAR) SETS, SIMILARITY MAPPINGS

14.55. Find the number of non-isomorphic posets with three elements a, b, c, and draw their diagrams.

14.56. Find the number of connected non-isomorphic posets with four elements a, b, c, d, and draw their diagrams.

14.57. Find the number of similarity mapings $f\colon S \to S$ where S is the ordered set in:

(a) Fig. 14-17(a); (b) Fig. 14-17(b); (c) Fig. 14-17(c).

14.58. Show that the isomorphism relation $A \cong B$ for ordered sets is an equivalence relation, that is:

(a) $A \cong A$ for any ordered set A. (b) If $A \cong B$, then $B \cong A$. (c) If $A \cong B$ and $B \cong C$, then $A \cong C$.

WELL-ORDERED SETS

14.59. Let the union S of sets $A = \{a_1, a_2, a_3, \ldots\}$, $B = \{b_1, b_2, b_3, \ldots\}$, $C = \{c_1, c_2, c_3, \ldots\}$ be ordered by:

$$S = \{A; B; C\} = \{a_1, a_2, \ldots, b_1, b_2, \ldots, c_1, c_2, \ldots\}$$

(a) Show that S is well-ordered.

(b) Find all limit elements of S.

(c) Show that S is not isomorphic to $\mathbf{N} = \{1, 2, \ldots\}$ with the usual order $\leq$.

14.60. Let $A = \{a, b, c\}$ be linearly ordered by $a < b < c$, and let $\mathbf{N}$ have the usual order $\leq$.

(a) Show that $S = \{A; \mathbf{N}\}$ is isomorphic to $\mathbf{N}$.

(b) Show that $S' = \{\mathbf{N}; A\}$ is not isomorphic to $\mathbf{N}$.

14.61. Suppose A is a well-ordered set under the relation $\precsim$, and suppose A is also well-ordered under the inverse relation $\succsim$. Describe A.

14.62. Suppose A and B are well-ordered isomorphic sets. Show that there is only one similarity mapping $f\colon A \to B$.

14.63. Let S be a well-ordered set. For any $a \in S$, the set $s(a) = \{x \mid x \prec a\}$ is called an *initial segment* of a. Show that S cannot be isomorphic to one of its *initial segments*. (*Hint*: Use Problem 14.21.)

14.64. Suppose $s(a)$ and $s(b)$ are distinct initial segments of a well-ordered set S. Show that $s(a)$ and $s(b)$ cannot be isomorphic. (*Hint*: Use Problem 14.63.)

LATTICES

14.65. Consider the lattice L in Fig. 14-19(a).

(a) Find all sublattices with five elements.

(b) Find all join-irreducible elements and atoms.

(c) Find complements of a and b, if they exist.

(d) Is L distributive? Complemented?

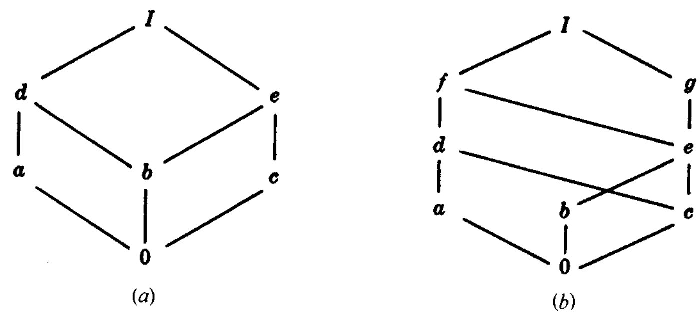


Fig. 14-19

14.66. Consider the lattice M in Fig. 14-19(b).

(a) Find all join-irreducible elements.

(b) Find the atoms.

(c) Find complements of a and b, if they exist.

(d) Express each x in M as the join of irredundant join-irreducible elements.

(e) Is M distributive? Complemented?

14.67. Consider the bounded lattice L in Fig. 14-20(a).

(a) Find the complements, if they exist, of e and f.

(b) Express I in an irredundant join-irreducible decomposition in as many ways as possible.

(c) Is L distributive?

(d) Describe the isomorphisms of L with itself.

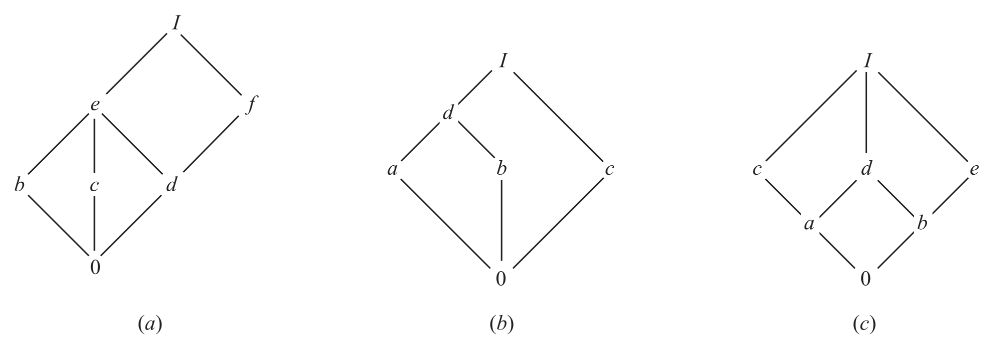


Fig. 14-20

14.68. Consider the bounded lattice L in Fig. 14-20(b).

(a) Find the complements, if they exist, of a and c.

(b) Express I in an irredundant join-irreducible decomposition in as many ways as possible.

(c) Is L distributive?

(d) Describe the isomorphisms of L with itself.

14.69. Consider the bounded lattice L in Fig. 14-20(c).

(a) Find the complements, if they exist, of a and c.

(b) Express I in an irredundant join-irreducible decomposition in as many ways as possible.

(c) Is L distributive?

(d) Describe the isomorphisms of L with itself.

14.70. Consider the lattice $\mathbf{D}_{60} = \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$, the divisors of 60 ordered by divisibility.

(a) Draw the diagram of $\mathbf{D}_{60}$.

(b) Which elements are join-irreducible and which are atoms?

(c) Find complements of 2 and 10, if they exist.

(d) Express each number x as the join of a minimum number of irredundant join irreducible elements.

14.71. Consider the lattice $\mathbf{N}$ of positive integers ordered by divisibility.

(a) Which elements are join-irreducible ?

(b) Which elements are atoms?

14.72. Show that the following "weak" distributive laws hold for any lattice L:

(a) $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$; (b) $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$.

14.73. Let $S = \{1, 2, 3, 4\}$. We use the notation $[12, 3, 4] \equiv [\{1, 2\}, \{3\}, \{4\}]$. Three partitions of S follow:

$$P_1 = [12, 3, 4], \quad P_2 = [12, 34], \quad P_3 = [13, 2, 4]$$

(a) Find the other twelve partitions of S.

(b) Let L be the collection of the 12 partitions of S ordered by *refinement*, that is, $P_i \prec P_j$ if each cell of P_i is a subset of a cell of P_j. For example $P_1 \prec P_2$, but P_2 and P_3 are noncomparable. Show that L is a bounded lattice and draw its diagram.

14.74. An element a in a lattice L is said to be meet-irreducible if $a = x \wedge y$ implies $a = x$ or $a = y$. Find all meet-irreducible elements in: (a) Fig. 14-19(a); (b) Fig. 14-19(b); (c) $\mathbf{D}_{60}$ (see Problem 14.70.)

14.75. A lattice M is said to be *modular* if whenever $a \leq c$ we have the law

$$a \vee (b \wedge c) = (a \vee b) \wedge c$$

(a) Prove that every distributive lattice is modular.

(b) Verify that the non-distributive lattice in Fig. 14-7(b) is modular; hence the converse of (a) is not true.

(c) Show that the nondistributive lattice in Fig. 14-7(a) is non-modular. (In fact, one can prove that every non-modular lattice contains a sublattice isomorphic to Fig. 14-7(a).)

14.76. Let R be a ring. Let L be the collection of all ideals of R. Prove that L is a bounded lattice where, for any ideals J and K of R, we define: $J \vee K = J + K$ and $J \wedge K = J \cap K$.

Answers to Supplementary Problems

14.31. (a) Minimal, 4 and 6; maximal, 1 and 2. (b) First, none; last, none, (c) $\{1, 3, 4\}$, $\{1, 3, 6\}$, $\{2, 3, 4\}$, $\{2, 3, 6\}$, $\{2, 5, 6\}$.

14.32. (a) Minimal, d and f; maximal, a. (b) First, none; last, a. (c) There are eleven: *dfebca*, *dfecba*, *dfceba*, *fdebca*, *fdecba*, *fdceba*, *fedbca*, *fedcba*, *fcdeba*, *fecdba*, *fcedba*.

14.33. See Fig. 14-21.

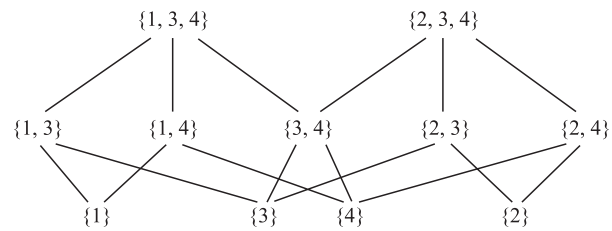


Fig. 14-21

14.34. See Fig. 14-22.

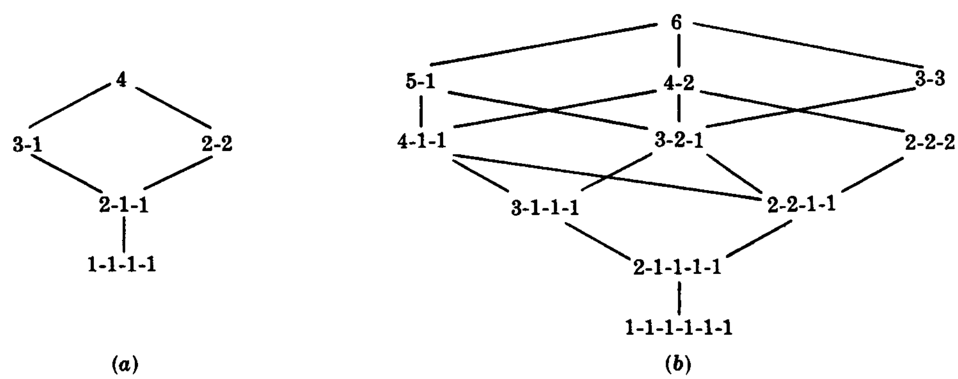


Fig. 14-22

14.35. See Fig. 14-23.

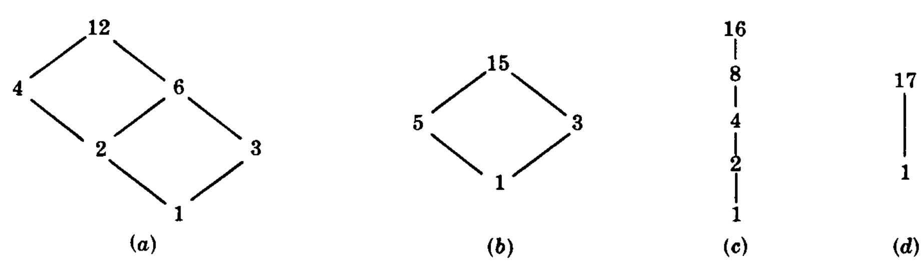


Fig. 14-23

14.36. *Hint*: Draw the diagram of S.

(a) Minimal, e; maximal, a, d.

(b) First, e; Last, none.

(c) $\{a, d\}$, $\{b, c\}$.

14.37. (a) False. Example: $\mathbf{N} \cup \{a\}$ where $1 \ll a$, and $\mathbf{N}$ ordered by $\leq$. (b) True. (c) True.

14.38. (a) Minimal, a; maximal, d and e. (b) First, a; last, none. (c) Any subset which contains c and omits a; that is: c, cb, cd, ce, cbd, cbe, cde, $cbde$. (d) c, cd, ce, cde. (e) abd, acd, ace.

14.39. (a) Minimal, a and b; maximal, e and f. (b) First, none; last, none. (c) ace, acf, bce, bcf, bdf.

14.40. (a) Four. (b) None.

14.41. (a) Six. (b) None.

14.42. $abcde$, $abced$, $acbde$, $acbed$, $acebd$.

14.43. Eleven.

14.44. $a \ll b$, $a \ll c$, $c \ll d$.

14.45. Minimal, $(p, 2)$ where p is a prime. Maximal, none.

14.46. (a) an, at, go, or, arm, one, gate, gone, about, occur. (b) an, about, arm, at, gate, go, gone, occur, one, or.

14.47. (a) $\|$; (b) $>$; (c) $\|$; (d) $<$.

14.48. $1c$, $1y$, $2a$, $2c$, $2z$, $3b$, $4b$, $4z$

14.49. (a) e, f, g; (b) a; (c) $\sup(A) = e$; (d) $\inf(A) = a$.

14.50. (a) e, f, g; (b) none; (c) $\sup(B) = e$; (d) none.

14.51. (a) 1, 2, 3; (b) 8; (c) $\sup(A) = 3$; (d) $\inf(A) = 8$.

14.52. (a) None; (b) 8; (c) none; (d) $\inf(B) = 8$.

14.53. (a) Both; (b) $\sup(A) = 3$; $\inf(A)$ does not exist.

14.54. (a) Both; (b) $\sup(A) = 3$; $\inf(A) = \sqrt[3]{5}$

14.55. Four: (1) a, b, c; (2) $a, b \ll c$; (3) $a \ll b, a \ll c$. (4) $a \ll b \ll c$.

14.56. Four: See Fig. 14-24.

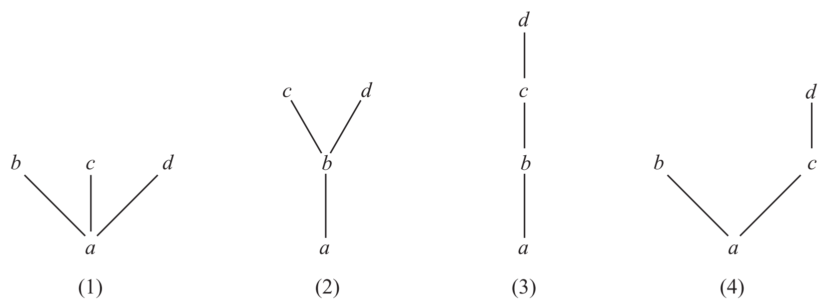


Fig. 14-24

14.57. (a) One: Identity mapping; (b) one; (c) two.

14.59. (b) b_1, c_1; (c) **N** has no limit points.

14.60. (a) Define $f : S \to \mathbf{N}$ by $f(a) = 1$, $f(b) = 2$, $f(3) = 3$, $f(n) = n + 3$.
(b) The element a is a limit point of S', but **N** has no limit points.

14.61. A is a finite linearly ordered set.

14.65. (a) Six: $0abdI$, $0acdI$, $0adeI$, $0bceI$, $0aceI$, $0cdeI$; (b) (i) a, b, c, 0; (ii) a, b, c. (c) c and e are complements of a. b has no complement. (d) No. No.

14.66. (a) a, b, c, g, 0. (b) a, b, c. (c) a has g; b has none. (d) $I = a \vee g$, $f = a \vee b$, $e = b \vee c$, $d = a \vee c$. Other elements are join-irreducible. (e) No. No.

14.67. (a) e has none; f has b and c. (b) $I = c \vee f = b \vee f = b \vee d \vee f$. (c) No, since decompositions are not unique. (d) Two: 0, d, e, f, I must be mapped into themselves. Then $F = 1_L$, identity map on L, or $F = \{(b, c), (c, b)\}$.

14.68. (a) a has c; c has a and b. (b) $I = a \vee c = b \vee c$. (c) No. (d) Two: 0, c, d, I must be mapped into themselves. Then $f = 1_L$ or $f = \{(a, b), (b, a)\}$.

14.69. (a) a has e, c has b and e. (b) $I = a \vee e = b \vee c = c \vee e$. (c) No. (d) Two: 0, d, I are mapped into themselves. Then $f = 1_L$ or $f = \{(a, b), (b, a), (c, d), (d, c)\}$.

14.70. (a) See Fig. 14-25. (b) 1, 2, 3, 4, 5. The atoms are 2, 3 and 5. (c) 2 has none, 10 has none. (d) $60 = 4 \vee 3 \vee 5$; $30 = 2 \vee 3 \vee 5$; $20 = 4 \vee 5$; $15 = 3 \vee 5$; $12 = 3 \vee 4$; $10 = 2 \vee 5$; $6 = 2 \vee 3$.

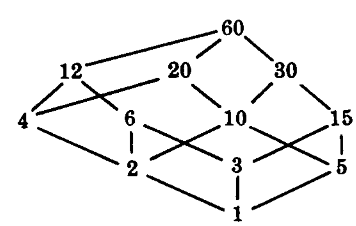


Fig. 14-25

14.73. (a) [1, 2, 3, 4], [14, 2, 3], [13, 24], [14, 23], [123, 4], [124, 3], [134, 2], [234, 1], [1234], [23, 1, 4] [24, 1, 3], [34, 1, 2]. (b) See Fig. 14-26.

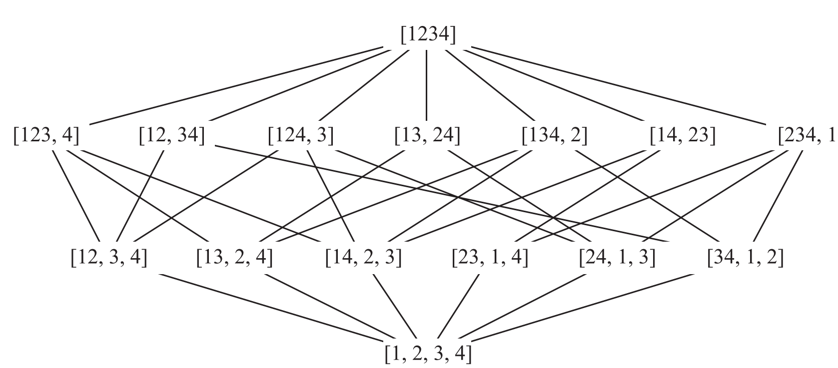


Fig. 14-26

14.74. Geometrically, an element $a \neq I$ is meet-irreducible if and only if a has only one immediate successor. (a) a, c, d, e, I; (b) a, b, d, f, g, I; (c) 4, 6, 12, 15, 60.

14.75. (a) If $a \leq c$ then $a \vee c = c$. Hence $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) = (a \vee b) \wedge c$; (b) Here $a \leq c$. But $a \vee (b \wedge c) = a \vee 0 = a$ and $(a \vee b) \wedge c = I \wedge c = c$; hence $a \vee (b \wedge c) \neq (a \vee b) \wedge c$.