Unit 2: Logic and Proofs

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Propositional Logic

What is a proposition?

- It is raining
- $1 + 1 = 2$
- every odd number is a prime
- $2^{67} - 1$ is a prime
- $(n + 1)(n - 1) = (n^2 - 1)$ for any integer n

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What is common between these statements?

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A proposition is a statement that is either true or false (but not both).

Definition: A ***proposition*** (denoted *p*, *q*, *r*, …) is simply:

- a *statement* (*i.e.*, a declarative sentence)
  - *with some definite meaning*, (not vague or ambiguous)
- having a *truth value* that's either *true* (**T**) or *false* (**F**)
  - it is **never** both, neither, or somewhere "in between!"
    - However, you might not *know* the actual truth value,
    - and, the truth value might *depend* on the situation or context.
- Later, we will study *probability theory*, in which we assign *degrees of certainty* ("between" **T** and **F**) to propositions.
  - But for now: think True/False only! (or in terms of **1** and **0**)

Examples of Propositions

- It is raining. (In a given situation)
- Beijing is the capital of China. (T)
- 2 + 2 = 5. (F)
- 1 + 2 = 3. (T)
- A fact-based declaration is a proposition, even if no one knows whether it is true
  - 11213 is prime.
  - There exists an odd perfect number.

- Give an example of a statement that is not a proposition.

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    - $x + 1 = 8$

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- propositions are statements that are either true or false.
- Just as we use variables $x; y; : : :$ for numbers, we will use variables $p; q; : : :$ for propositions.

"if it is raining, it will be wet" : $p \to q$

Examples of Non-Propositions

The following are **NOT** propositions:

- Who's there? (interrogative, question)
- Just do it! (imperative, command)
- La la la la la. (meaningless interjection)
- Yeah, I sorta dunno, whatever... (vague)
- 1 + 2 (expression with a non-true/false value)
- x + 2 = 5 (declaration about semantic tokens of non-constant value)

Truth Tables

- An *operator* or *connective* combines one or more *operand* expressions into a larger expression. (*e.g.*, “+” in numeric expressions.)
- ***Unary*** operators take *one* operand (*e.g.*, −3);
  Binary operators take *two* operands (*e.g.* 3 × 4).
- ***Propositional*** or ***Boolean operators*** operate on propositions (or their truth values) instead of on numbers.
- The ***Boolean domain*** is the set {T, F}. Either of its elements is called a ***Boolean value***.
  An n-tuple $(p_1,\ldots,p_n)$ of Boolean values is called a ***Boolean n-tuple***.
- An n-operand truth table is a table that assigns a Boolean value to the set of all Boolean n-tuples.

Some Popular Boolean Operators

Formal Name	**Nickname**	**Arity**	**Symbol**
Negation operator	NOT	Unary	$\neg$
Conjunction operator	AND	Binary	$\wedge$
Disjunction operator	OR	Binary	$\vee$
Exclusive-OR operator	XOR	Binary	$\oplus$
Implication operator	IMPLIES	Binary	$\rightarrow$
Biconditional operator	IFF	Binary	$\leftrightarrow$

The Negation Operator

- The unary ***negation** operator* "¬" (*NOT*) transforms a proposition into its logical *negation*.
- *E.g.* If *p* = "I have brown hair."
  then ¬*p* = "It is not the case that I have brown hair" or "I do **not** have brown hair."
- The *truth table* for NOT:

p	$\neg p$
T	F
F	T

Operand column

Result column

The Conjunction Operator

- The binary ***conjunction** operator* "$\wedge$" (*AND*) combines two propositions to form their logical *conjunction*.

- *E.g.* If p = "I will have salad for lunch." and
  q = "I will have steak for dinner."

  then, $p \wedge q$ = "I will have salad for lunch **and** I will have steak for dinner."

Conjunction Truth Table

Operand columns

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

- Note that a conjunction $p_1 \wedge p_2 \wedge \ldots \wedge p_n$ of n propositions will have 2^n rows in its truth table

The Disjunction Operator

- The binary ***disjunction*** *operator* "$\vee$" (*OR*) combines two propositions to form their logical *disjunction*.

- *E.g.* If p = "My car has a bad engine." and
  q = "My car has a bad carburetor."

  then, $p \vee q$ = "My car has a bad engine, **or** my car has a bad carburetor."

Meaning is like "and/or" in informal English.

Disjunction Truth Table

p	q	$p \vee q$
T	T	T
T	F	**T**
F	T	**T**
F	F	F

Note difference from AND

- Note that $p \vee q$ means that p is true, or q is true, **or both** are true!
- So, this operation is also called ***inclusive or***, because it **includes** the possibility that both p and q are true.

The Exclusive-Or Operator

- The binary ***exclusive-or*** *operator* "$\oplus$" (*XOR*) combines two propositions to form their logical "exclusive or"

- *E.g.* If p = "I will earn an A in this course." and
  q = "I will drop this course.", then

  $p \oplus q$ = "I will **either** earn an A in this course, **or** I will drop it (**but not both**!)"

Exclusive-Or Truth Table

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p	q	$p \oplus q$
T	T	**F**
T	F	T
F	T	T
F	F	F

Note difference from OR.

- Note that $p \oplus q$ means that p is true, or q is true, but **not both**!
- This operation is called ***exclusive or,*** because it **excludes** the possibility that both p and q are true.

Natural Language is ambiguous

- Note that the English "or" can be ambiguous regarding the "both" case!
- "Pat is a singer or Pat is a writer." - $\vee$
- "Pat is a man or Pat is a woman." - $\oplus$

p	q	p "or" q
T	T	?
T	F	T
F	T	T
F	F	F

- Need context to disambiguate the meaning!
- For this class, assume "or" means inclusive ($\vee$).

The Implication Operator

- The conditional statement (aka ***implication***) $p \rightarrow q$ states that p implies q.
- *I.e.*, If p is true, then q is true; but if p is not true, then q could be either true or false.
- *E.g.*, let p = "You study hard."
  q = "You will get a good grade."
  $p \rightarrow q$ = "If you study hard, then you will get a good grade." (else, it could go either way)
  - p: *hypothesis* or *antecedent* or *premise*
  - q: *conclusion* or *consequence*

Implication Truth Table

p	q	$p \to q$
T	T	T
T	F	**F**
F	T	T
F	F	T

The only False case!

- $p \to q$ is **false** only when p is true but q is **not** true.
- $p \to q$ does **not** require that p or q **are ever true**!
- *E.g.* "(1=0) $\to$ pigs can fly" is TRUE!

Examples of Implications

- "If this lecture ever ends, then the sun will rise tomorrow." True or False? $(T \rightarrow T)$
- "If 1+1=6, then Obama is president." True or False? $(F \rightarrow T)$
- "If the moon is made of green cheese, then I am richer than Bill Gates." True or False? $(F \rightarrow F)$
- "If Tuesday is a day of the week, then I am a penguin." True or False? $(T \rightarrow F)$

English Meaning Phrases of $p \rightarrow q$

"if p, then q"
"if p, q"
"p is sufficient for q"
"q if p"
"q when p"
"a necessary condition for p is q"
"q unless $\neg p$"
"p implies q"
"p only if q"
"a sufficient condition for q is p"
"q whenever p"
"q is necessary for p"
"q follows from p"

Understanding the Implication Operation

- An obligation or an contract
- Example:
  - *"If I am elected, then I will lower taxes."*
  - *"If you get 100% on the final, then you will get an A".*
- Home exercise
  - Equivalence of "*if p, then q*" and "*p, only if q*".
  - Equivalence of "*if p, then q*" and "*q unless* $\neg$p".

Exercise

22. Write each of these statements in the form "if p, then q" in English. [*Hint:* Refer to the list of common ways to express conditional statements provided in this section.]

a) It is necessary to wash the boss's car to get promoted.

b) Winds from the south imply a spring thaw.

c) A sufficient condition for the warranty to be good is that you bought the computer less than a year ago.

d) Willy gets caught whenever he cheats.

e) You can access the website only if you pay a subscription fee.

f) Getting elected follows from knowing the right people.

g) Carol gets seasick whenever she is on a boat.

Converse, Inverse and Contrapositive

- Some terminology, for an implication $p \rightarrow q$:
- Its ***converse*** is: $q \rightarrow p$.
- Its ***inverse*** is: $\neg p \rightarrow \neg q$.
- Its ***contrapositive***: $\neg q \rightarrow \neg p$.

p	q	$p \rightarrow q$	$q \rightarrow p$	$\neg p \rightarrow \neg q$	$\neg q \rightarrow \neg p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

- One of these three has the *same meaning* (same truth table) as $p \rightarrow q$. Can you figure out which? Contrapositive

Examples

- p: Today is Easter
  q: Tomorrow is Monday

- $p \to q$:
  If today is Easter then tomorrow is Monday.
- ***Converse***: $q \to p$
  If tomorrow is Monday then today is Easter.
- ***Inverse***: $\neg p \to \neg q$
  If today is not Easter then tomorrow is not Monday.
- ***Contrapositive***: $\neg q \to \neg p$
  If tomorrow is not Monday then today is not Easter.

The Biconditional Operator

- The ***biconditional*** statement $p \leftrightarrow q$ states that p ***if and only if*** (*iff*) q.
- p = "It is below freezing."

  q = "It is snowing."

  $p \leftrightarrow q$ = "It is below freezing if and only if it is snowing."

  or

  = "That it is below freezing is necessary and sufficient for it to be snowing"

Biconditional Truth Table

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- p is necessary and sufficient for q
- If p then q, and conversely
- p iff q

- $p \leftrightarrow q$ is equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$.
- $p \leftrightarrow q$ means that p and q have the **same** truth value.
- $p \leftrightarrow q$ does **not** imply that p and q are true.
- Note this truth table is the exact **opposite** of $\oplus$'s! Thus, $p \leftrightarrow q$ means $\neg(p \oplus q)$.

Combining Propositions

- Conjunction: $p \land q$, (read p and q), "discrete math is a required course and I am a computer science major".
- Disjunction: , $p \lor q$, (read p or q), "discrete math is a required course or I am a computer science major".
- Exclusive or: $p \oplus q$, "discrete math is a required course or I am a computer science major but not both".
- Implication: $p \rightarrow q$, "if discrete math is a required course then I am a computer science major".
- Biconditional: $p \leftrightarrow q$, "discrete math is a required course if and only if I am a computer science major".

Boolean Operations Summary

- We have seen 1 unary operator and 5 binary operators. What are they? Their truth tables are below.

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	F	T	T	F	T	T
T	F	F	F	T	T	F	F
F	T	T	F	T	T	T	F
F	F	T	F	F	F	T	T

- For an implication $p \rightarrow q$
- Its ***converse*** is: $q \rightarrow p$
- Its ***inverse*** is: $\neg p \rightarrow \neg q$
- Its ***contrapositive***: $\neg q \rightarrow \neg p$

Compound Propositions

- A ***propositional variable*** is a variable such as p, q, r (possibly subscripted, e.g. p_j) over the Boolean domain.
- An ***atomic proposition*** is either Boolean constant or a propositional variable: e.g. T, F, p
- A ***compound proposition*** is derived from atomic propositions by application of propositional operators: e.g. $\neg p$, $p \vee q$, $(p \vee \neg q) \to q$
- Precedence of logical operators: $\neg$, $\wedge$, $\vee$, $\to$, $\leftrightarrow$
- Precedence also can be indicated by parentheses.
  - e.g. $\neg p \wedge q$ means $(\neg p) \wedge q$, not $\neg(p \wedge q)$

An Exercise

- Any compound proposition can be evaluated by a truth table
- $(p \vee \neg q) \to q$

p	q	$\neg q$	$p \vee \neg q$	$(p \vee \neg q) \to q$
T	T	F	T	T
T	F	T	T	F
F	T	F	F	T
F	F	T	T	F

Translating English Statements

- Let p = "It rained last night",
  q = "The sprinklers came on last night,"
  r = "The lawn was wet this morning."

  Translate each of the following into English:

  $\neg p$ = "It didn't rain last night."

  $r \wedge \neg p$ = "The lawn was wet this morning, and it didn't rain last night."

  $\neg r \vee p \vee q$ = "The lawn wasn't wet this morning, or it rained last night, or the sprinklers came on last night."

Exercise

- Find the converse of the following statement.

  “Raining tomorrow is a sufficient condition for my not going to town.

Another Example

- Find the converse of the following statement.
  - "Raining tomorrow is a sufficient condition for my not going to town."
- **Step 1**: Assign propositional variables to component propositions.
  - p: It will rain tomorrow
  - q: I will not go to town
- **Step 2**: Symbolize the assertion: $p \rightarrow q$
- **Step 3**: Symbolize the converse: $q \rightarrow p$
- **Step 4**: Convert the symbols back into words.
  - "If I don't go to town then it will rain tomorrow" or
  - "Raining tomorrow is a *necessary condition* for my not going to town."

Logic and Bit Operations

- A ***bit*** is a binary (base 2) digit: 0 or 1.
- Bits may be used to represent truth values.
  - By convention:
    0 represents “False”; 1 represents “True”.
- A ***bit string of length n*** is an ordered sequence of $n \geq 0$ bits.
- By convention, bit strings are (sometimes) written left to right:
  - e.g. the “first” bit of the bit string “1001101010” is 1.
  - What is the length of the above bit string?

Bitwise Operations

- Boolean operations can be extended to operate on bit strings as well as single bits.

- Example:

01 1011 0110	
11 0001 1101	
11 1011 1111	Bit-wise OR
01 0001 0100	Bit-wise AND
10 1010 1011	Bit-wise XOR

Propositional Equivalences

- A ***tautology*** is a compound proposition that is **true** *no matter what* the truth values of its atomic propositions are!
  - *e.g.* $p \lor \neg p$ ("Today the sun will shine or today the sun will not shine.") [What is its truth table?]
- A ***contradiction*** is a compound proposition that is **false** no matter what!
  - *e.g.* $p \land \neg p$ ("Today is Wednesday and today is not Wednesday.") [Truth table?]
- A ***contingency*** is a compound proposition that is neither a tautology nor a contradiction.
  - *e.g.* $(p \lor q) \to \neg r$

Logical Equivalence

- Compound proposition p is ***logically equivalent*** to compound proposition q, written $p \equiv q$ or $p \Leftrightarrow q$, **iff** the compound proposition $p \leftrightarrow q$ is a tautology.
- Compound propositions p and q are logically equivalent to each other **iff** p and q contain the same truth values as each other in all corresponding rows of their truth tables.

Proving Equivalence Via Truth Tables

- Prove that $\neg(p \wedge q) \equiv \neg p \vee \neg q$. (De Morgan's law)

p	q	$p \wedge q$	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$\neg(p \wedge q)$
T	T	T	F	F	F	F
T	F	F	F	T	T	T
F	T	F	T	F	T	T
F	F	F	T	T	T	T

- Show that **Check out the solution in the textbook!**
  - $\neg(p \vee q) \equiv \neg p \wedge \neg q$ (De Morgan's law)
  - $p \rightarrow q \equiv \neg p \vee q$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ (distributive law)

Equivalence Laws

- These are similar to the arithmetic identities you may have learned in algebra, but for propositional equivalences instead.
- They provide a pattern or template that can be used to match part of a much more complicated proposition and to find an equivalence for it and possibly simplify it.

Equivalence Laws

- *Identity*: $p \wedge \mathbf{T} \equiv p$ $\quad p \vee \mathbf{F} \equiv p$
- *Domination*: $p \vee \mathbf{T} \equiv \mathbf{T}$ $\quad p \wedge \mathbf{F} \equiv \mathbf{F}$
- *Idempotent*: $p \vee p \equiv p$ $\quad p \wedge p \equiv p$
- *Double negation:* $\neg\neg p \equiv p$
- *Commutative:* $p \vee q \equiv q \vee p$ $\quad p \wedge q \equiv q \wedge p$
- *Associative:* $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

More Equivalence Laws

- *Distributive*: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
  $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- *De Morgan's*:
  $\neg(p \wedge q) \equiv \neg p \vee \neg q$
  $\neg(p \vee q) \equiv \neg p \wedge \neg q$
- *Absorption*
  $p \vee (p \wedge q) \equiv p$ $\quad$ $p \wedge (p \vee q) \equiv p$
- *Trivial tautology/contradiction*:
  $p \vee \neg p \equiv \mathbf{T}$ $\quad$ $p \wedge \neg p \equiv \mathbf{F}$

Defining Operators via Equivalences

Using equivalences, we can *define* operators in terms of other operators.

- Exclusive or: $p \oplus q \equiv (p \land \neg q) \lor (\neg p \land q)$
  $p \oplus q \equiv (p \lor q) \land \neg(p \land q)$
- Implies: $p \rightarrow q \equiv \neg p \lor q$
- Biconditional: $p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p)$
  $p \leftrightarrow q \equiv \neg(p \oplus q)$

This way we can “normalize” propositions

An Example Problem

- Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.

$\neg(p \rightarrow q)$ [Expand definition of $\rightarrow$]

$\equiv \neg(\neg p \vee q)$ [DeMorgan's Law]

$\equiv \neg(\neg p) \wedge \neg q$ [Double Negation]

$\equiv p \wedge \neg q$

Another Example

- Check using a symbolic derivation whether

$$\boxed{(p \wedge \neg q) \rightarrow (p \oplus r) \equiv \neg p \vee q \vee \neg r}$$

Solution …

$$(p \wedge \neg q) \rightarrow (p \oplus r) \quad \text{[Expand definition of } \rightarrow\text{]}$$

$$\equiv \neg(p \wedge \neg q) \vee (p \oplus r) \quad \text{[Expand definition of } \oplus\text{]}$$

$$\equiv \neg(p \wedge \neg q) \vee ((p \vee r) \wedge \neg(p \wedge r))$$

[DeMorgan's Law]

$$\equiv (\neg p \vee q) \vee ((p \vee r) \wedge \neg(p \wedge r))$$

cont.

$(\neg p \vee q) \vee ((p \vee r) \wedge \neg(p \wedge r))$ [$\vee$ Commutative]

$\equiv (q \vee \neg p) \vee ((p \vee r) \wedge \neg(p \wedge r))$ [$\vee$ Associative]

$\equiv q \vee (\neg p \vee ((p \vee r) \wedge \neg(p \wedge r)))$ [Distribute $\vee$ over $\wedge$]

$\equiv q \vee (((\neg p \vee (p \vee r)) \wedge (\neg p \vee \neg(p \wedge r))))$ [$\vee$ Assoc.]

$\equiv q \vee (((\neg p \vee p) \vee r) \wedge (\neg p \vee \neg(p \wedge r))))$ [Trivial taut.]

$\equiv q \vee ((\mathbf{T} \vee r) \wedge (\neg p \vee \neg(p \wedge r))))$ [Domination]

$\equiv q \vee (\mathbf{T} \wedge (\neg p \vee \neg(p \wedge r)))$ [Identity]

$\equiv q \vee (\neg p \vee \neg(p \wedge r))$

$$
\begin{aligned}
& q \vee (\neg p \vee \neg(p \wedge r)) && \text{[DeMorgan's Law]} \\
&\equiv q \vee (\neg p \vee (\neg p \vee \neg r)) && [\vee \text{ Associative}] \\
&\equiv q \vee ((\neg p \vee \neg p) \vee \neg r) && \text{[Idempotent]} \\
&\equiv q \vee (\neg p \vee \neg r) && \text{[Associative]} \\
&\equiv (q \vee \neg p) \vee \neg r && [\vee \text{ Commutative}] \\
&\equiv \neg p \vee q \vee \neg r \quad \blacksquare
\end{aligned}
$$

Review: Propositional Logic

- Atomic propositions: p, q, r, …
- Boolean operators: $\neg \ \wedge \ \vee \ \oplus \ \rightarrow \ \leftrightarrow$
- Compound propositions: $(p \wedge \neg q) \vee r$
- Equivalences: $p \wedge \neg q \Leftrightarrow \equiv \neg(p \rightarrow q)$
- Proving equivalences using:
  - Truth tables
  - Symbolic derivations (series of logical equivalences) $p \equiv q \equiv r \equiv \cdots$