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Unit - 6

Numerical Method

Interpolation, Numerical differentiation & Integration D. E.

Interpolation

- **Lagrange's Interpolation** (x_i are not equally spaced). difference 'h' is different
- **Netwton's formulae**. (x_i's are equally sapaced) difference is same (h)

* **Lagrange's Interpolation:**

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\cdots(x_0-x_n)} \times y_0$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)} \times y_1$$

$$\vdots$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1})} \times y_n$$

* **Netwon's Forward Difference operator:-**

$$\boxed{\Delta f(x) = f(x+h) - f(x)}$$

Foward difference operator

$$\boxed{p = \frac{x - x_0}{h}}$$

$x \rightarrow$ given value

$x_0 \rightarrow$ First value of x.

$h \rightarrow$ difference x_i's

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$$y = f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$ (5)

$$+ \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \ldots$$ (6)

Note :- For n degree polynomial

$\Delta^{n+1} = 0$

$\Delta^n = \text{constant}$

* Newton's Backward difference operator

$$\boxed{\nabla f(x) = \cancel{f(x+h)}\ f(x) - f(x-h)}$$

Here, $\boxed{p = \frac{x - x_n}{h}}$

x = given value
x_n = last value of x_i
h = difference betn x_i's

$$y = f(x) = y_n + p\nabla y_n + \frac{p(p-1)}{2!}\nabla^2 y_n + \frac{p(p-1)(p-2)}{3!}\nabla^3 y_n$$

$+ \ldots$

- Finite differences & differences operators :-

1. $\Delta f(x) = f(x+h) - f(x)$ — forward operator
2. $\nabla f(x) = f(x) - f(x+h)$ — backward operator
3. $Ef(x) = f(x+h)$ → shift operator

$E^2 f(x) = f(x+2h)$

$\vdots$

$E^n f(x) = f(x+nh)$

4. $E^{-1} f(x) = f(x-h)$ → Inverse shif operator

$E^{-2} f(x) = f(x-2h)$

$\vdots$

$E^{-n} f(x) = f(x-nh)$

(5) $\delta f(x) = f(x+h/2) - f(x-h/2)$ -- central difference operator.

(6) $\mu f(x) = \frac{1}{2}\left[f(x+h/2) + f(x-h/2)\right]$ → Average operator

- Relations betn operators :-

(1) $\Delta = E - 1$
(2) $\nabla = 1 - E^{-1}$
(3) $\delta = E^{1/2}\nabla$
(4) $\delta = E^{-1/2}\Delta$
(5) $\Delta\nabla = \nabla\Delta = \delta^2$

- Numerical Differentiation :-

Newton's Forwards difference formula

$$y = f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

$$\therefore y = f(x) = y_0 + p\Delta y_0 + \left(\frac{p^2-p}{2}\right)\Delta^2 y_0 + \left(\frac{p^3-3p^2+2p}{6}\right)\Delta^3 y_0$$

$$+ \left(\frac{p^4-6p^3+11p^2-6p}{24}\right)\Delta^4 y_0 + \ldots \quad -(1)$$

Here, $p = \frac{x-x_0}{h} = \frac{x}{h} - \frac{x_0}{h}$

$$\boxed{\therefore \frac{dp}{dx} = \frac{1}{h}}$$

w.r.t. $\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx}$ ← [at $x = x_0$] (Take eqn (1))

$\therefore \frac{dy}{dp} =$ [when missing value not in table]

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$\therefore y' = f'(x) = \frac{dy}{dx} = \frac{dy}{dp} \times \frac{dp}{dx}$

$$\therefore \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2p-1}{2}\right)\Delta^2 y_0 + \left(\frac{3p^2-6p+2}{6}\right)\Delta^3 y_0 + \left(\frac{4p^3-18p^2+22p-6}{24}\right)\Delta^4 y_0 + \cdots\right]$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + \left(\frac{6p-6}{6}\right)\Delta^3 y_0 + \left(\frac{12p-36p+22}{24}\right)\Delta^4 y_0 + \cdots\right]$$

missing value not in table $(x \neq x_0)$

Similarly for backward difference,

$$\therefore y = f(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

$\therefore$ ~~$y' = f'(x) = y_n + p\nabla y_n +$~~

$$\therefore y' = f'(x) = \frac{1}{h}\left[\nabla y_n + \left(\frac{2p+1}{2}\right)\nabla^2 y_n + \left(\frac{3p^2-6p+2}{6}\right)\nabla^3 y_n + \cdots\right]$$

$$\therefore y'' = f''(x) = \frac{1}{h^2}\left[\nabla^2 y_n + \left(\frac{6p+6}{6}\right)\nabla^3 y_n + \left(\frac{12p^2+36p+22}{24}\right)\nabla^4 y_n + \cdots\right]$$

[put, $x \neq x_n$ when missing value is not in table]

$$\therefore \frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right]$$

$(x = x_n)$ so, $p = 0$ (missing value is in table)

Backward difference

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$$\therefore \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \ldots\right]$$

at $x = x_0$ so $p = 0$.

$$\therefore \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\right]$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 + \ldots\right]$$

Forward difference

- **Numerical Integration :-**

① Trapezoidal Rule :-

$$\int_{a=x_0}^{b=x_n} y\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \ldots + y_{n-1})\right]$$

$$= \frac{h}{2}\left[\left(\text{sum of 1st \& last ordinates}\right) + 2\left(\text{sum of remaining ordinates}\right)\right]$$

② Simpson's $\frac{1}{3}$rd Rule :-

$$\int_{a=x_0}^{b=x_n} y\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \ldots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \ldots + y_{n-2})\right]$$

$$= \frac{h}{3}\left[\left(\text{sum of 1st \& last}\right) + 4(\text{odd ordinates}) + 2(\text{even ordinates})\right]$$

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(9) Simpson's $(3/8)^{th}$ Rule :-

$$\int_a^b y\,dx = \left(\frac{3}{8}\right) h \left[(y_0 - y_n) + 3(y_1 + y_2 + y_3 + \dots) + 2(y_3 + y_6 + y_9 \dots) \right]$$

multiple of 3

$$\boxed{h = \frac{b-a}{n}} = \frac{\text{upper limit} - \text{lower limit}}{\text{no. of equal parts of interval}}$$

h = difference of x_i's

$y_0 \quad y_1 \quad y_2 \quad y_3 \quad y_4 \quad y_5 \quad y_6$

Interval = a 1 2 3 4 5 6 b

$n = 6$ cause it divided into 6 equal parts

& ordinates are $7 \Rightarrow y_0, y_1, y_2, y_3, y_4, y_5, y_6$

- Numerical soln of ordinary Differential equations

- Euler's Method :-

consider the differential equations

$$\frac{dy}{dx} = f(x, y) \qquad \text{Intial values } \boxed{y(x_0) = y_0}$$

$x_1 = x_0 + h$ $\qquad y_1 = y$ at x_1, $y(x_1)$

$x_2 = x_1 + h$ $\qquad y_2 = y$ at x_2 $y(x_2)$

$x_3 = x_2 + h$ $\qquad y_3 = y$ at x_3 $y(x_3)$

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$\boxed{x_n = x_{n-1} + h}$ $\qquad \boxed{y_n = y \text{ at } x_n = y(x_n)}$

Process :- Consider, D.E. $\frac{dy}{dx} = f(x, y)$

$$y(x_1) = y_1 = y_0 + h f(x_0, y_0)$$
$$y(x_2) = y_2 = y_1 + h f(x_1, y_1)$$
$$y(x_3) = y_3 = y_2 + h f(x_2, y_2)$$
$$\vdots$$
$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)}$$

- **Modified Euler's Method** :-

Process :- y_1 is computed by Euler's formula,

$$y_1 = y_0 + h f(x_0, y_0)$$

then, find, $f(x_1, y_1)$.

- The first modification of y_1,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$

- The second modification of y_1,

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

- The Third modification of y_1,

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

And so on $\rightarrow$ nth

- **Range kutta 4th order method :-**

Process :-

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf(x_0 + h/2,\ y_0 + k_1/2)$$

$$k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right).$$

$$k_4 = hf(x_0 + h,\ k_3).$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$\boxed{y_1 = y_0 + k}$$