Unit-III

Interpolation, Numerical Differentiation and Integration, D.E.

Interpolation

- Lagrange's Interpolation (x_i's are not equally spaced)
- Newton's Formulae (x_i's are equally spaced)
  - Newton's forward [If missing value is at beginning of table]
  - Newton's Backward [If missing value at end of table]

Lagrange's Interpolation:-

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\cdots(x_0-x_n)} \times y_0$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)} \times y_1$$

$$+ \quad \vdots$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1})} \times y_n$$

Q.1 Find Lagrange's interpolating polynomial passing through the set of points:

x	0	1	3
y	3	4	12

$\Rightarrow$

	x_0	x_1	x_2
x	0	1	3
y	3	4	12
	y_0	y_1	y_2

$$y = f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \times y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \times y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \times y_2$$

$$= \frac{(x-1)(x-3)}{(0-1)(0-3)} \times 3 + \frac{(x-0)(x-3)}{(1-0)(1-3)} \times 4 + \frac{(x-0)(x-1)}{(3-0)(3-1)} \times 12$$

$$= \frac{(x-1)(x-3)}{-1 \times -3} \times 3 + \frac{x(x-3)}{(1)(-2)} \times 4 + \frac{x(x-1)}{3 \times 2} \times 12$$

$$= \frac{(x-1)(x-3)}{\cancel{3}} \times \cancel{3} + \frac{x(x-3)}{-2} \times 4 + \frac{x(x-1)}{6} \times 12$$

$$= (x-1)(x-3) - 2x(x-3) + 2x(x-1)$$

$$= x^2 - 3\cancel{x} - \cancel{x} + 3 - 2x^2 + 6\cancel{x} + 2x^2 - 2\cancel{x}$$

$$= x^2 + 3$$

Q2 Find Lagrange's interpolation polynomial passing through points

x	0	1	2	5
y	2	3	12	147

$\Rightarrow$

	x_0	x_1	x_2	x_3
x	0	1	2	5
y	2	3	12	147
	y_0	y_1	y_2	y_3

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times y_3$$

$$= \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)} \times 2 + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)} \times 3$$

$$+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)} \times 12 + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)} \times 147$$

$$= \frac{(x-1)(x-2)(x-5)}{-1 \times -\cancel{2} \times -5} \times \cancel{2} + \frac{x(x-2)(x-5)}{(1)(-1)(-4)} \cdot 3$$

$$+ \frac{x(x-1)(x-5)}{(\cancel{2})(1)(-\cancel{3})} \times \cancel{12}^{2} + \frac{x(x-\cancel{1})(x-2)}{(5)(4)(\cancel{3})} \times \cancel{147}^{49}$$

$$= \frac{(x-1)(x^2-5x-2x+10)}{5} + \frac{3x(x^2-5x-2x+10)}{4} - 2x(x^2-5x-x+5)$$

$$+ \frac{49x(x^2-3x+2)}{20}$$

$$= \frac{(x-1)(x^2-7x+10)}{5} + \frac{3x(x^2-7x+10)}{4} - 2x(x^2-6x+5)$$

$$+ \frac{49x(x^2-3x+2)}{20}$$

$$= \frac{4(x^3-8x^2+17x-10)}{20} + \frac{5(3x^3-21x^2+30x)}{20} - \frac{20(2x^3-12x^2+10x)}{20}$$

$$+ \frac{49x^3-147x^2+98x}{20}$$ (making D^r same)

$$= \frac{4x^3-32x^2+68x-40+15x^3-105x^2+150x-40x^3+240x^2-200x+49x^3-147x^2+98x}{20}$$

$$= \frac{28x^3-44x^2+116x-40}{20}$$

$$y = 1.4x^3 - 2.2x^2 + 5.8x - 2$$

③ Given $y_0 = 1.23$, $y_2 = 3.78$, $y_3 = 6.9$, $y_4 = 10.9$ Find y_1

x	0	2	3	4
y	1.23	3.78	6.9	10.9

$\Rightarrow$

	x_0	x_1	x_2	x_3
x	0	2	3	4
y	1.23	3.78	6.9	10.9
	y_0	y_1	y_2	y_3

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_3$$

$$= \frac{(1-2)(1-3)(1-4)}{(0-2)(0-3)(0-4)} \times 1.23 + \frac{(1-0)(1-3)(1-4)}{(2-0)(2-3)(2-4)} \times 3.78$$

$$+ \frac{(1-0)(1-2)(1-4)}{(3-0)(3-2)(3-4)} \times 6.9 + \frac{(1-0)(1-2)(1-3)}{(4-0)(4-2)(4-3)} \times 10.9$$

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$= \frac{-1\times-2\times-3}{-2\times-3\times-4} \times 1.23 + \frac{1\times-2\times-3}{2\times-1\times-2} \times 3.78 + \frac{1\times-1\times-3}{3\times1\times-1} \times 6.9 + \frac{1\times-1\times-2}{4\times2\times1} \times 10.9$

$= 0.3075 + 5.67 - 6.9 + 2.725$

$= 1.8025$

Q.4 Find lagrange's interpolating polynomial passing through set of points

x	0	1	2
y	4	3	6

- Use it to find y at $x = 1.5$, $\frac{dy}{dx}$ at $x = 0.5$ and find $\int_0^3 y\,dx$.

$\Rightarrow$

$$y = f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \times y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \times y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \times y_2$$

$$= \frac{(x-1)(x-2)}{(0-1)(0-2)} \times 4 + \frac{(x-0)(x-2)}{(1-0)(1-2)} \times 3 + \frac{(x-0)(x-1)}{(2-0)(2-1)} \times 6.$$

$$= \frac{x^2-2x-x+2}{-1\times-2} \times 4 + \frac{x^2-2x}{1\times-1} \times 3 + \frac{x^2-x}{2\times1} \times 6$$

$$= 2(x^2-3x+2) - 3(x^2-2x) + 3(x^2-x)$$

$$= 2x^2-6x+4 - 3x^2 + 6x + 3x^2 - 3x$$

$$y = 2x^2 - 3x + 4$$

① y at $x = 1.5$

$\Rightarrow y = 2(1.5)^2 - 3(1.5) + 4 = 4$

② $y = 2x^2 - 3x + 4$

$\frac{dy}{dx} = 4x - 3$

$\left(\frac{dy}{dx}\right)_{x=0.5} = 4(0.5) - 3 = -1$

③ $\int_0^3 y\,dx = \int_0^3 2x^2 - 3x + 4\,dx = \left[\frac{2x^3}{3} - \frac{3x^2}{2} + 4x\right]_0^3$

$$= \left[\frac{2(3)^3}{3} - \frac{3(3)^2}{2} + 4(3)\right] - [0]$$

$= 16.5$

⑤ Given $\log 2 = 0.3010$, $\log 3 = 0.4771$, $\log 5 = 0.6990$, $\log 7 = 0.8451$.
Find $\log 47$ using lagrange's interpolation formula. ~~De~~

⇒

	x_0	x_1	x_2	x_3
x	2	3	5	7
$y = \log x$	0.3010	0.4771	0.6990	0.8451
	y_0	y_1	y_2	y_3

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times y_3$$

$$= \frac{(47-3)(47-5)(47-7)}{(2-3)(2-5)(2-7)} \times 0.3010 + \frac{(47-2)(47-5)(47-7)}{(3-2)(3-5)(3-7)} \times 0.4771$$
$$+ \frac{(47-2)(47-3)(47-7)}{(5-2)(5-3)(5-7)} \times 0.6990 + \frac{(47-2)(47-3)(47-5)}{(7-2)(7-3)(7-5)} \times 0.8451$$

$$= 168.8299$$

⑥ Given $(1.0)^3 = 1.000$, $(1.2)^3 = 1.728$, $(1.3)^3 = 2.197$ and $(1.5)^3 = 3.375$
Using lagrange's interpolation formula evaluate $(1.07)^3$

⇒ Given data

	x_0	x_1	x_2	x_3
x	1.0	1.2	1.3	1.5
y	1.000	1.728	2.197	3.375
	y_0	y_1	y_2	y_3

By formula

$$y = 0.4285666 + 1.126944 - 1.432810 + 0.23546$$
$$= 0.3582$$

⑦ Given

x	1	1.2	1.3	1.4
$\sqrt{x}$	1	1.095	1.140	1.183

Find $\sqrt{1.1}$ using lagrange's interpolation. Determine.

$$y = \frac{(1.1-1.2)(1.1-1.3)(1.1-1.4)}{(1-1.2)(1-1.3)(1-1.4)} \times 1 + \frac{(1.1-1)(1.1-1.3)(1.1-1.4)}{(1.2-1)(1.2-1.3)(1.2-1.4)} \times 1.095$$

$$+ \frac{(1.1-1)(1.1-1.2)(1.1-1.4)}{(1.3-1)(1.3-1.2)(1.3-1.4)} \times 1.140 + \frac{(1.1-1)(1.1-1.2)(1.1-1.3)}{(1.4-1)(1.4-1.2)(1.4-1.3)} \times 1.183$$

$= 1.0483$

Newton's Forward Difference operator:

$$\Delta f(x) = f(x+h) - f(x).$$

Newton's forward interpolation formula.

$$y = f(x) = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0$$

$$+ \frac{P(P-1)(P-2)(P-3)}{4!}\Delta^4 y_0$$

where $p = \frac{x - x_0}{h}$ where $x \rightarrow$ given value

$x_0 \rightarrow$ First value of x

$h \rightarrow$ Difference in x_i's.

Note: For n degree polynomial

$\Delta^{n+1} = 0$

$\Delta^{n} =$ constant.

Q.1 Evaluate $f(15)$ by Newton's forward interpolation formula given

x	10	20	30	40	50
$y=f(x)$	46	66	81	93	101

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x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0 10	y_0 46				
		Δy_0 20			
20	66		$\Delta^2 y_0$ -5		
		15		$\Delta^3 y_0$ 2	
30	81		-3		$\Delta^4 y_0$ -3
		12		-1	
40	93		-4		
		8			
50	101				

$$P = \frac{x-x_0}{h} = \frac{15-10}{10} = \frac{5}{10} = 0.5$$

Newton's forward interpolation formula.

$$y = f(x) = f(15) = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0$$

$$+ \frac{P(P-1)(P-2)(P-3)}{4!}\Delta^4 y_0$$

$$= 46 + (0.5)\times 20 + \frac{(0.5)(0.5-1)}{2!}(-5) + \frac{(0.5)(0.5-1)(0.5-2)}{3!}2$$

$$+ \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{4!}(-3)$$

$$= 46 + 10 + 0.625 + 0.125 + 0.1172$$

$$= 56.8672$$

(2) Find a cubic polynomial in x which takes on the values $-3, 3, 11, 27, 57$ and 107 when $x = 0, 1, 2, 3, 4$ and 5.

$\Rightarrow$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0 0	y_0 -3				
		Δy_0 6			
1	3		$\Delta^2 y_0$ 2		
		8		$\Delta^3 y_0$ 6	
2	11		8		$\Delta^4 y_0$ 0
		16		6	
3	27		14		0
		30		6	
4	57		20		
		50			
5	107				

$$P = \frac{x-x_0}{h} = \frac{x-0}{1} = x$$

$$y = f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$+ \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$= (-3) + x6 + \frac{x(x-1)}{2}(2) + \frac{x(x-1)(x-2)}{6} \times 6 + 0$$

$$= -3 + 6x + x^2 - x + x[x^2 - 2x - x + 2]$$

$$= -3 + 6x + x^2 - x + x[x^2 - 3x + 2]$$

$$= -3 + 6x + x^2 - x + x^3 - 3x^2 + 2x$$

$$= x^3 - 2x^2 + 7x - 3$$

③ Find a polynomial passing thr the points (0,1), (1,1), (2,7), (3,25), (4,61), (5,121) using Newton's interpolation formula & hence find the value of the polynomial at $x = 0.5$.

⇒

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1	0	6	6	0
1	1	6	12	6	0
2	7	18	18	6	
3	25	36	24		
4	61	60			
5	121				

$$p = \frac{x - x_0}{h} = \frac{x - 0}{1} = x.$$

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$= 1 + x(0) + \frac{x(x-1)}{2}.(6) + \frac{x(x-1)(x-2)}{6} 6$$

$$= 1 + 3x(x-1) + (x^2 - x)(x-2)$$

$$= 1 + \cancel{3x^2} - \cancel{3x} + x^3 - \cancel{2x^2} - \cancel{x^2} + \cancel{2x}.$$

$$f(x) = x^3 - x + 1.$$

$f(x)$ at $x = 0.5$.

$$f(0.5) = (0.5)^3 - 0.5 + 1$$

$$= 0.625$$

(4) Find y for $x=5$ Using Newton's forward interpolation formula for the following data.

x:	4	6	8	10
y:	1	3	8	16

⇒

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
4	1			
		2		
6	3		3	
		5		0
8	8		3	
		8		
10	16			

$$p = \frac{x-x_0}{h} = \frac{x-4}{2} = \frac{5-4}{2} = \frac{1}{2} = 0.5$$

$$y = f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0$$

$$= 1 + (0.5)(2) + \frac{(0.5)(0.5-1)}{2}(3)$$

$$= 1 + 1 - 0.375$$

$$= 1.625$$

Newton's Backward difference operator

$$\nabla f(x) = f(x) - f(x-h)$$

Newton's Backward interpolation formula.

$$y = f(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots$$

where $p = \dfrac{x-x_n}{h}$

Q.1 Apply Newton's backward difference formula to the data below to find population in 1925.

Year x.	1891	1901	1911	1921	1931
Population (y) in thousands	46	66	81	93	101

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

$x_n = 1931$, $y_n = 101$, $\nabla y_n = 8$, $\nabla^2 y_n = -4$, $\nabla^3 y_n = -1$, $\nabla^4 y_n = -3$

$$p = \frac{x - x_n}{h} = \frac{1925-1931}{10} = -0.6$$

$$y = f(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

$$= 101 + (-0.6)8 + \frac{(-0.6)(-0.6+1)}{2}(-4) + \frac{(-0.6)(-0.6+1)(-0.6+2)}{6}(-1) + \frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{24}(-3)$$

$$= 101 - 4.8 + 0.48 + 0.056 + 0.1008$$

$$= 96.8368 \approx 97$$

② Find $f(1.45)$ by Newton's backward interpolation formula.

x	1	1.1	1.2	1.3	1.4	1.5
$y = f(x)$	2	2.1	2.3	2.7	3.5	4.5

$\Rightarrow$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
1	2					
		0.1				
1.1	2.1		0.1			
		0.2		0.1		
1.2	2.3		0.2		0.1	
		0.4		0.2		-0.5
1.3	2.7		0.4		-0.4	
		0.8		-0.2		
1.4	3.5		0.2			
		1				
1.5	4.5					

$x_n = 1.5$, $y_n = 4.5$, $\nabla y_n = 1$, $\nabla^2 y_n = 0.2$, $\nabla^3 y_n = -0.2$, $\nabla^4 y_n = -0.4$, $\nabla^5 y_n = -0.5$

$$p = \frac{x - x_n}{h} = \frac{1.45 - 1.5}{0.1} = -0.5$$

$$y = f(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \frac{p(p+1)(p+2)(p+3)(p+4)}{5!}\nabla^5 y_n$$

$$= 4.5 + (-0.5)(1) + \frac{(-0.5)(-0.5+1)}{2}(0.2) + \frac{(-0.5)(-0.5+1)(-0.5+2)}{6}(-0.2)$$

$$+ \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)}{24}(-0.4) + \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)(-0.5+4)(-0.5+5)}{120} \times (-0.5)$$

$$= 4.5 - 0.5 + (-0.025) + 0.0125 + 0.015625 + 0.01367$$

$$= 4.0167$$

Q.3. For the following table estimate $f(7.5)$ by using Newton's interpolation formula.

x	1	2	3	4	5	6	7	8
$f(x)$	1	8	27	64	125	216	343	512

⇒ Here $x = 7.5$ is at end of the table.

∴ Newton's backward interpolation formula

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1	1				
		7			
2	8		12		
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		0
		91		6	
6	216		36		0
		127		6	
7	343		42		
		169			
8	512				

$$p = \frac{x - x_n}{h} = \frac{7.5 - 8}{1} = -0.5.$$

$$y = f(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

$$f(7.5) = 512 + (-0.5)(169) + \frac{(-0.5)(-0.5+1)}{2}(42) + \frac{(-0.5)(-0.5+1)(-0.5+2)}{6}6$$

$$= [512 - 84.5 - 5.25 - 0.375]$$

$$= 421.875$$

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Finite Differences and Difference operators.

① $\Delta f(x) = f(x+h) - f(x)$ → Forward difference operator

② $\nabla f(x) = f(x) - f(x-h)$ → Backward difference operator

③ $E f(x) = f(x+h)$ → Shift operator

$E^2 f(x) = f(x+2h)$

$\vdots$

$E^n f(x) = f(x+nh)$

④ $E^{-1} f(x) = f(x-h)$ → Inverse shift operator

$E^{-2} f(x) = f(x-2h)$

$\vdots$

$E^{-n} f(x) = f(x-nh)$

⑤ $\delta f(x) = f(x+\frac{h}{2}) - f(x-\frac{h}{2})$ → Central difference operator

⑥ $\mu f(x) = \frac{1}{2}\left[f(x+\frac{h}{2}) + f(x-\frac{h}{2})\right]$ → Average operator.

Relation between operators

① $\Delta = E-1$ ② $\nabla = 1-E^{-1}$ ③ $\delta = E^{1/2}\nabla$ ④ $\delta = E^{-1/2}\Delta$ ⑤ $\Delta\nabla = \nabla\Delta = \delta^2$

→ ① $\Delta f(x) = f(x+h) - f(x)$

$= Ef(x) - f(x)$

$= (E-1) f(x)$

$\therefore \Delta = E-1$

② $\nabla f(x) = f(x) - f(x-h)$

$= f(x) - E^{-1} f(x)$

$= [1-E^{-1}] f(x)$

$\nabla = 1-E^{-1}$

③ $\delta f(x) = f(x+\frac{h}{2}) - f(x-\frac{h}{2})$

$= E^{1/2} f(x) - E^{1/2} f(x-h)$

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$= E^{1/2}[f(x) - f(x-h)]$

$= E^{1/2}\,\nabla f(x)$

$\therefore \delta = E^{1/2}\nabla$.

(4) $\delta f(x) = f(x+\frac{h}{2}) - f(x-\frac{h}{2})$

$= E^{-1/2} f(x+h) - E^{-1/2} f(x)$

$= E^{-1/2}[f(x+h) - f(x)]$

$= E^{-1/2}\Delta f(x)$

$\therefore \delta = E^{-1/2}\Delta$

(5) Show that $\Delta\nabla = \nabla\Delta = \delta^2$

$\Rightarrow$ $\Delta\nabla f(x) = \Delta[f(x) - f(x-h)]$

$= \Delta f(x) - \Delta f(x-h)$

$= [f(x+h) - f(x)] - [f(x) - f(x-h)]$

$= f(x+h) - f(x) - f(x) + f(x-h)$

$= f(x+h) - 2f(x) + f(x-h)$ — (1).

$\nabla\Delta f(x) = \nabla[\Delta f(x)]$

$= \nabla[f(x+h) - f(x)]$

$= \nabla f(x+h) - \nabla f(x)$

$= [f(x+h) - f(x)] - [f(x) - f(x-h)]$

$= f(x+h) - f(x) - f(x) + f(x-h)$

$= f(x+h) - 2f(x) + f(x-h)$ — (2)

$\delta^2 f(x) = \delta[\delta f(x)] = \delta\left[f(x+\frac{h}{2}) - f(x-\frac{h}{2})\right]$

$= \delta\left[f(x+\frac{h}{2})\right] - \delta\left[f(x-\frac{h}{2})\right]$

$= \left[f(x+\frac{h}{2}+\frac{h}{2}) - f(x+\frac{h}{2}-\frac{h}{2})\right] - \left[f(x-\frac{h}{2}+\frac{h}{2}) - f(x-\frac{h}{2}-\frac{h}{2})\right]$

$= f(x+h) - f(x) - f(x) + f(x-h)$

$= f(x+h) - 2f(x) + f(x-h)$ — ③

from ①, ② & ③ $\Delta\nabla = \nabla\Delta = \delta^2$

Examples (MCQ's)

① If $f(x) = x^2+1$, $h=2$ $Ef(x)$ is.

⇒ $Ef(x) = f(x+h)$

$= f(x+2)$ $\because h=2$

$= (x+2)^2+1 = x^2+4x+4+1 = x^2+4x+5$

② If $f(x) = x^3$, $h=1$ $Ef(x)$ is given by

⇒ $Ef(x) = f(x+h) = f(x+1)$

$f(x+1) = (x+1)^3$ $\because (a+b)^3 = a^3+3a^2b+3ab^2+b^3$

$= x^3+3x^2+3x+1$

③ For $f(x) = x^2$, $h=1$ first forward difference of $f(x)$ is

⇒ First forward difference $= \Delta$

$\therefore \Delta f(x) = f(x+h) - f(x)$

$= f(x+1) - f(x)$ $\because h=1$

$= (x+1)^2 - x^2$ --- value of f^n

$= \cancel{x^2}+2x+1-\cancel{x^2}$

$= 2x+1$

④ If $f(x) = x^2-2$, $h=1$ first backward difference $\nabla f(x)$ is

⇒ First backward difference $= \nabla$

$\nabla f(x) = f(x) - f(x-h)$

$= f(x) - f(x-1)$

$= [x^2-2] - [(x-1)^2-2]$

$= x^2-2 - [x^2-2x+1-2]$

$= x^2-2 - [x^2-2x-1]$

$= \cancel{x^2}-2-\cancel{x^2}+2x+1$

$\nabla f(x) = 2x-1$

⑤ If $f(x)=x^2, h=1$, $\delta f(x)$ is given by

$\Rightarrow$ $\delta f(x) = f(x+\frac{h}{2}) - f(x-\frac{h}{2})$

$= f(x+\frac{1}{2}) - f(x-\frac{1}{2}) \quad \because h=1 \quad - (1)$

$f(x+\frac{1}{2}) = (x+\frac{1}{2})^2 = x^2+2x\cdot\frac{1}{2}+(\frac{1}{2})^2$

$= x^2+x+\frac{1}{4} \quad - (2)$

$f(x-\frac{1}{2}) = (x-\frac{1}{2})^2 = x^2-2x\cdot\frac{1}{2}+(\frac{1}{2})^2$

$= x^2-x+\frac{1}{4} \quad - (3).$

$\therefore$ from (1), (2) & (3).

$\delta\cdot f(x) = f(x+\frac{1}{2}) - f(x-\frac{1}{2})$

$= [x^2+x+\frac{1}{4}] - [x^2-x+\frac{1}{4}]$

$= \cancel{x^2}+x+\cancel{\frac{1}{4}}-\cancel{x^2}+x-\cancel{\frac{1}{4}}$

$\to = 2x$

⑥ If $f(x)=x^2, h=1$ $\mu f(x)$ is given by

$\Rightarrow$ $\mu f(x) = \frac{1}{2}[f(x+\frac{h}{2})+f(x-\frac{h}{2})]$

$= \frac{1}{2}[f(x+\frac{1}{2})+f(x-\frac{1}{2})] \quad - (1)$

$f(x+\frac{1}{2}) = (x+\frac{1}{2})^2 = x^2+2\cdot x\cdot\frac{1}{2}+(\frac{1}{2})^2 = x^2+x+\frac{1}{4} \quad - (2)$

$f(x-\frac{1}{2}) = (x-\frac{1}{2})^2 = x^2-2x\cdot\frac{1}{2}+(\frac{1}{2})^2 = x^2-x+\frac{1}{4} \quad - (3)$

$\therefore$ From (1), (2) and (3).

$\mu f(x) = \frac{1}{2}[f(x+\frac{h}{2})+f(x-\frac{h}{2})]$

$= \frac{1}{2}[f(x+\frac{1}{2})+f(x-\frac{1}{2})]$

$= \frac{1}{2}[x^2+\cancel{x}+\frac{1}{4}+x^2-\cancel{x}+\frac{1}{4}]$

$= \frac{1}{2}[2x^2+2\cdot\frac{1}{4}]$

$= \frac{2}{2}[x^2+\frac{1}{4}] = x^2+\frac{1}{4}$

⑦ For $f(x)=x^2$, $h=2$ second forward difference $\Delta^2 f(x)$ is

$\Rightarrow$ $\Delta^2 f(x) = \Delta\{\Delta f(x)\}$

$= \Delta[f(x+h)-f(x)]$

$= \Delta[f(x+h)-f(x)]$

$= \Delta f(x+h) - \Delta f(x)$

$= [f(x+h+h)-f(x+h)] - [f(x+h)-f(x)]$

$= f(x+2h) - f(x+h) - f(x+h) + f(x)$

$= f(x+2h) - 2f(x+h) + f(x)$

$= f(x+2(2)) - 2f(x+2) + f(x)$ ... $h=2$

$= f(x+4) - 2f(x+2) + f(x)$

$= (x+4)^2 - 2(x+2)^2 + x^2$ ... value of f^n

$= x^2 + 8x + 16 - 2[x^2+4x+4] + x^2$

$= \cancel{x^2} + \cancel{8x} + 16 - \cancel{2x^2} - \cancel{8x} - 8 + \cancel{x^2}$

$\therefore \Delta^2 f(x) = 8$

⑧ For $f(x)=x^2$, $h=1$ $\Delta\nabla f(x)$ is

$\Rightarrow$ $\Delta\nabla f(x) = \Delta[\nabla f(x)] = \Delta[f(x)-f(x-h)] = \Delta f(x) - \Delta f(x-h)$

$= [f(x+h) - f(x)] - [f(x-\cancel{h}+\cancel{h}) - f(x-h)]$

$= f(x+h) - f(x) - f(x) + f(x-h)$

$= f(x+h) - 2f(x) + f(x-h)$

$= f(x+1) - 2f(x) + f(x-1)$ ... $h=1$

$= (x+1)^2 - 2x^2 + (x-1)^2$ ... value of f^n

$= \cancel{x^2} + \cancel{2x} + 1 - \cancel{2x^2} + \cancel{x^2} - \cancel{2x} + 1$

$= 2$

⑨ For $f(x)=x^2$, $h=1$ $\delta^2 f(x)$ is

$\Rightarrow$ $\delta^2 f(x) = f(x+h) - 2f(x) + f(x-h)$ [It is proved in relation part]

then solve as previous

Numerical Differentiation

Newton's forward difference formula

$$y = f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$+ \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots$$

$$= y_0 + p\Delta y_0 + \left(\frac{p^2-p}{2}\right)\Delta^2 y_0 + \left(\frac{p^3-3p^2+2p}{6}\right)\Delta^3 y_0$$

$$+ \left(\frac{p^4-6p^3+11p^2-6p}{24}\right)\Delta^4 y_0 + \cdots$$

$$p = \frac{x-x_0}{h} \qquad \frac{dp}{dx} = \frac{1}{h}$$

$$= \frac{x}{h} - \frac{x_0}{h}$$

$$f'(x) = \frac{dy}{dx} = \frac{dy}{dp} \times \frac{dp}{dx}$$

at $x = x_0$ (not in table)

$$\therefore \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2p-1}{2}\right)\Delta^2 y_0 + \left(\frac{3p^2-6p+2}{6}\right)\Delta^3 y_0 + \left(\frac{4p^3-18p^2+22p-6}{24}\right)\Delta^4 y_0 + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + \left(\frac{6p-6}{6}\right)\Delta^3 y_0 + \left(\frac{12p-36p+22}{24}\right)\Delta^4 y_0 + \cdots\right]$$

then when $x = x_0$ (value $\overset{\text{missing}}{}$ in table) then $p = 0$

$$\left(\frac{dy}{dx}\right)_{\text{at } x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{\text{at } x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 + \cdots\right]$$

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Similarly for Newton's Backward interpolation formula.

$$y = f(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

at $x = x_n$ [missing value not in table]

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \left(\frac{2p+1}{2}\right)\nabla^2 y_n + \left(\frac{3p^2+6p+2}{6}\right)\nabla^3 y_n + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \left(\frac{6p+6}{6}\right)\nabla^3 y_n + \left(\frac{12p^2+36p+22}{24}\right)\nabla^4 y_n + \cdots\right]$$

at $x = x_n$ [missing value in the table]

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \cdots\right]$$

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① The first & second derivative value of interpolating fn $y = f(x)$ at $x = 2$ using the given data.

x	1.0	1.2	1.4	1.6	1.8	2.0
$f(x) = y$	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891

⇒ Backward formula because $x=2$ is at the end of table.

$h = 0.2$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
1.0	2.7183					
		0.6018				
1.2	3.3201		0.1333			
		0.7351		0.0294		
1.4	4.0552		0.1627		0.0067	
		0.8978		0.0361		0.0013
1.6	4.9530		0.1988		0.008	
		1.0966		0.0441		
1.8	6.0496		0.2429			
		1.3395				
2.0	7.3891					

$x = x_n = 2 =$ missing value in the table.

$$\therefore \frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n\right]$$

$$= \frac{1}{0.2}\left[1.3395 + \frac{1}{2}(0.2429) + \frac{1}{3}(0.0441) + \frac{1}{4}(0.008) + \frac{1}{5}(0.0013)\right]$$

$$= 7.38825 \text{ or } (07.38955)$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n\right]$$

$$= \frac{1}{(0.2)^2}\left[0.2429 + 0.0441 + \frac{11}{12}(0.008) + \frac{5}{6}(0.0013)\right]$$

$$= 7.3854$$

② Find $y'(0)$ and $y''(0)$ from the following table

x	0	1	2	3	4
y	1	3	15	43	93

=> Forward differentiation formula. $x=0$ at the begining

$h=1$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		2			
1	3		10		
		12		6	
2	15		16		0
		28		6	
3	43		22		
		50			
4	93				

Here $x = x_0 \quad \therefore p = 0$

y' at 0 $= f'(x) = \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0\right]$

$y'(0) = \frac{1}{1}\left[2 - \frac{1}{2}(10) + \frac{1}{3}(6) - \frac{1}{4}(0)\right]$

$= -1$

y'' at 0 $= f''(x) = \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0\right]$

$= \frac{1}{1^2}\left[10 - 6 + 0\right]$

$= 4$

③ Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1.2$

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y	2.72	3.32	4.06	4.96	6.05	7.39	9.02

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⇒ Here $x = x_0 = 1.2$ at begining of table.

$h = 0.2$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1.0	2.72	0.60	0.14	0.02	0.01	0.02	-0.07
1.2	3.32	0.74	0.16	0.03	0.03	-0.05	
1.4	4.06	0.90	0.19	0.06	-0.02		
1.6	4.96	1.09	0.25	0.04			
1.8	6.05	1.34	0.29				
2.0	7.39	1.63					
2.2	9.02						

By Newton's forward difference formula for differentiation

$$y' = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0\right]$$

$$= \frac{1}{0.2}\left[0.74 - \frac{1}{2}(0.16) + \frac{1}{3}(0.03) - \frac{1}{4}(0.03) + \frac{1}{5}(-0.05)\right]$$

$$= \frac{1}{0.2}\left[0.74 - 0.08 + 0.01 - 0.0075 - 0.01\right]$$

$$= 3.2625$$

$$y'' = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0\right]$$

$$= \frac{1}{(0.2)^2}\left[0.16 - 0.03 + \frac{11}{12}(0.03) - \frac{5}{6}(-0.05)\right]$$

$$= 4.98$$

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(4) Find $\frac{dy}{dx}$ at $x = 1.2$ for the following data:

x	1.1	1.3	1.5	1.7	1.9
$f(x)$	0.21	0.69	1.25	1.89	2.61

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1.1	0.21			
		0.48		
1.3	0.69		0.08	
		0.56		0
1.5	1.25		0.08	
		0.64		0
1.7	1.89		0.08	
		0.72		
1.9	2.61			

$h = 0.2$, $x = 1.2$, $x_0 = 1.1$

$$p = \frac{x - x_0}{h} = \frac{1.2 - 1.1}{0.2} = 0.5$$

$$y' = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2p-1}{2}\right)\Delta^2 y_0 + \frac{3p^2 - 6p + 2}{6}\Delta^3 y_0 + \ldots\right]$$

$$= \frac{1}{0.2}\left[0.48 + \left(\frac{2(0.5)-1}{2}\right)(0.08)\right]$$

$y'(1.2) = 2.4$

Numerical Integration

① Trapezoidal Rule:

$$\int_{a=x_0}^{b=x_n} y\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})\right]$$

$$= \frac{h}{2}\left[\left(\text{sum of first \& last ordinates}\right)+2\left(\text{sum of remaining ordinates}\right)\right]$$

② Simpson's $\frac{1}{3}$rd Rule:

$$\int_{a=x_0}^{b=x_n} y\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+y_5+\cdots+y_{n-1}) + 2(y_2+y_4+y_6+\cdots+y_{n-2})\right]$$

$$= \frac{h}{3}\left[\left(\text{sum of first \& last ordinates}\right)+4\left[\text{odd ordinates}\right]+2\left[\text{even ordinates}\right]\right]$$

③ Simpson's $\frac{3}{8}$th Rule:

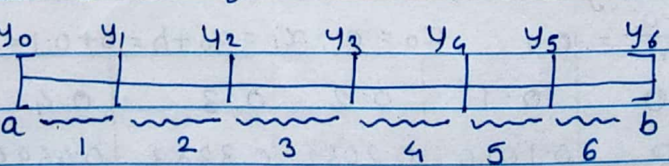
$$\int_{a}^{b} y\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\cdots) + 2(y_3+y_6+y_9+\cdots)\right]$$

multiple of 3.

$$h = \frac{b-a}{n} = \frac{\text{Upper limit} - \text{lower limit}}{\text{no. of equal parts of interval}}$$

h = difference in x_is.

Interval =



Here $n = 6$ = Interval divide into 6 equal parts

ordinates = 7 i.e. $y_0, y_1, y_2, y_3, y_4, y_5, y_6$.

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Examples

Trapezoidal Rule.

① Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by Trapezoidal rule taking $n=4$. Hence obtain approximate value of π.

$\Rightarrow$ $I = \int_0^1 \frac{dx}{1+x^2}$, $h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25 = \frac{1}{4}$

	x_0	x_1	x_2	x_3	x_4
x	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$y = \frac{1}{1+x^2}$	1	$\frac{16}{17}$	$\frac{4}{5}$	$\frac{16}{25}$	$\frac{1}{2}$
	y_0	y_1	y_2	y_3	y_4

$$I = \int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_4) + 2(y_1+y_2+y_3)\right]$$

$$= \frac{\frac{1}{4}}{2}\left[\left(1+\frac{1}{2}\right) + 2\left(\frac{16}{17}+\frac{4}{5}+\frac{16}{25}\right)\right]$$

$$= \frac{1}{8}\left[\frac{3}{2} + 2\left(\frac{16}{17}+\frac{4}{5}+\frac{16}{25}\right)\right]$$

$$= 0.7828 \quad -①$$

For Exact value

$$\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_0^1 = \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4} \quad -②$$

from ① & ②

$$\frac{\pi}{4} = 0.7828 \quad (\pi = 3.1312)$$

$$\therefore \pi = 3.1312$$

② Use Trapezoidal rule to numerically evaluate $\int_0^1 x \cdot e^{x^2} dx$ by taking $h = 0.1$, compare with exact value

$\Rightarrow$ $h = \frac{b-a}{n} = 0.1$ $\quad x_0 = 0,\ x_1 = x_0 + h = 0 + 0.1 = 0.1,\ x_2 = x_1 + h = 0.2 \ldots$

x	0	0.1	0.2	0.3	0.4	0.5	0.6
y	0 (y_0)	0.1010 (y_1)	0.2082 (y_2)	0.3283 (y_3)	0.4694 (y_4)	0.6420 (y_5)	0.8599 (y_6)
	0.7	0.8	0.9	1.0			
	1.1426 (y_7)	1.5172 (y_8)	2.0231 (y_9)	2.7183 (y_{10})			

$$I = \frac{h}{2}\left[(y_0 + y_{10}) + 2(y_1 + y_2 + y_3 + \cdots + y_9)\right]$$

$$= \frac{0.1}{2}\left[(0 + 2.7183) + 2(y_1 + y_2 + y_3 + \cdots + y_9)\right]$$

$$= 0.865085 \quad -(1)$$

For exact value.

$$I = \int_0^1 x \cdot e^{x^2}\, dx.$$

Put $t = x^2$ — $x = 0$, $t = 0$; $x = 1$, $t = 1^2 = 1$

$$dt = 2x\, dx$$

$$\therefore \frac{dt}{2} = x\, dx.$$

$$I = \int_0^1 e^t \cdot \frac{dt}{2} = \left[\frac{1}{2} \cdot e^t\right]_0^1 = \frac{1}{2}e^1 - \frac{1}{2}e^0$$

$$= \frac{1}{2}[e - 1]$$

$$= 0.8591409 \quad -(2)$$

(3) Use Trapezoidal Rule to evaluate the value of $\int_0^2 \frac{x}{\sqrt{2+x^2}}\, dx$ by taking $h = 0.5$.

$\Rightarrow$ $x_0 = 0$, $x_1 = x_0 + h = 0 + 0.5 = 0.5$, $x_2 = x_1 + h = 0.5 + 0.5 = 1$, ...

x	0 (x_0)	0.5 (x_1)	1 (x_2)	1.5 (x_3)	2 (x_4)
y	0 y_0	0.3333 y_1	0.5773 y_2	0.7276 y_3	0.8165 y_4

$$I = \int_0^2 \frac{x}{\sqrt{2+x^2}}\, dx. = \frac{h}{2}\left[(y_0 + y_4) + 2(y_1 + y_2 + y_3)\right]$$

$$= \frac{0.5}{2}\left[(0 + 0.8165) + 2(0.3333 + 0.5773 + 0.7276)\right]$$

$$= 1.023225$$

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4) Using Trapezoidal rule, Calculate the approximate value of $\int_0^4 \sqrt{x}\,dx$ given by.

x	0	1	2	3	4
$\sqrt{x}$	0	1	1.4142	1.7321	2
	y_0	y_1	y_2	y_3	y_4

$\Rightarrow$ Table given $\int_0^4 \sqrt{x}\,dx = \frac{h}{2}\left[(y_0+y_4)+2(y_1+y_2+y_3)\right]$

$$= \frac{1}{2}\left[(0+2)+2(1+1.4142+1.7321)\right]$$

$$= 5.1463$$

$\left[h \text{ is } x_1 - x_0 = 1-0 = 1\right]$

(5) Using Trapezoidal rule, evaluate $\int_2^7 \frac{1}{x}\,dx$, dividing into 5 sub intervals.

$\Rightarrow h = \frac{b-a}{n} = \frac{7-2}{5} = \frac{5}{5} = 1$ $\quad$ ∴ Ans $\int_2^7 \frac{1}{x}\,dx = 1.2714$

Simpson's $\frac{1}{3}$rd Rule

(1) Evaluate $\int_1^2 \frac{dx}{x}$ using Simpson's $\left(\frac{1}{3}\right)^{rd}$ rule, taking $h = 0.25$

$\Rightarrow$ $y = f(x) = \frac{1}{x}$. $\quad$ $h = \frac{b-a}{n} = 0.25$ (Given).

$x_0 = 1$, $x_1 = x_0 + h = 1 + 0.25 = 1.25$, $x_2 = x_1 + h = 1.25 + 0.25 = 1.5$ ----

	(x_0)	(x_1)	(x_2)	(x_3)	(x_4)
x	1	1.25	1.5	1.75	2
y	1	0.8	0.6667	0.5714	0.5
	y_0	y_1	y_2	y_3	y_4

$\Rightarrow$ Using Simpson's $(\frac{1}{3})^{rd}$ rule

$$\int_1^2 \frac{dx}{x} = \frac{h}{3}\left[(y_0+y_4)+4(y_1+y_3)+2(y_2)\right]$$

$$= \frac{0.25}{3}\left[(1+0.5)+4(0.8+0.5714)+2(0.6667)\right]$$

$$= 0.69325$$

2) Stating the formula for Simpson's $\frac{1}{3}^{rd}$ rule, evaluate $\int_1^{1.04} f(x)\,dx$ from the following data:

	x_0	x_1	x_2	x_3	x_4
x :	1	1.01	1.02	1.03	1.04
$f(x)$:	3.953	4.066	4.182	4.300	4.421
	y_0	y_1	y_2	y_3	y_4

$\Rightarrow$ Here $h = x_1 - x_0 = 1.01 - 1 = 0.01$

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_4)+4(y_1+y_3)+2(y_2)\right]$$

$$= \frac{0.01}{3}\left[(3.953+4.421)+4(4.066+4.300)+2(4.182)\right]$$

$$= 0.16734$$

3) Use Simpson's $(\frac{1}{3})^{rd}$ rule to obtain $\int_0^{\pi/2} \frac{\sin x}{x}\,dx$, by dividing interval into four parts.

$\Rightarrow$ Here $n=4$ $\therefore$ $h = \frac{b-a}{n} = \frac{\frac{\pi}{2}-0}{4} = \frac{\pi}{8}$

$y = \frac{\sin x}{x}$, $x_0 = 0$, $x_1 = x_0 + h = 0 + \frac{\pi}{8} = \frac{\pi}{8}$ . . . .

	x_0	x_1	x_2	x_3	x_4
x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
y	1	0.9745	0.9003	0.7842	0.6366
	y_0	y_1	y_2	y_3	y_4

$\because y = \lim_{x\to 0} \frac{\sin x}{x} = 1$

$\boxed{\text{Note}}$:- For trigonometry adjust radian measure in calci

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_4)+4(y_1+y_3)+2y_2\right]$$

$$= \frac{\frac{\pi}{8}}{3}\left[(1+0.6366)+4(0.9745+0.7842)+2(0.9003)\right]$$

$$= \frac{\pi}{24}\left[1.6366+4(0.9745+0.7842)+2(0.9003)\right]$$

$$= 1.37078$$

④ Determine the value of integral $\int^{\pi/2} \cos x\,dx$ using Simpson's $\frac{1}{3}$ rd rule taking 4 sub intervals

⇒ $n=4$ $\therefore h = \frac{b-a}{n} = \frac{\frac{\pi}{2}-0}{4} = \frac{\pi}{8}$

$\therefore x_0 = 0$, $x_1 = x_1 + h = 0 + \frac{\pi}{8} = \frac{\pi}{8}$, $x_2 = x_1 + h = \frac{\pi}{8}+\frac{\pi}{8} = \frac{2\pi}{8} = \frac{\pi}{4}$ - - - -

	x_0	x_1	x_2	x_3	x_4
x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
y	1	0.9238	0.7071	0.3826	0
	y_0	y_1	y_2	y_3	y_4

$\boxed{\text{Note}}$:- For trigo adjust radian measure in calci

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_4)+4(y_1+y_3)+2y_2\right]$$

$$= 1.000047$$

⑤ Calculate $\int_4^{5.2} \log_e x\,dx$ [taking $h=0.2$] Using Simpson's $\frac{1}{3}$ rd Rule Compare the value with exact value.

⇒ $y = \log_e x$, $x_0 = 4$, $h = 0.2$

x	4	4.2	4.4	4.6	4.8	5	5.2
y	1.3862	1.4350	1.4816	1.5260	1.5686	1.6094	1.6486
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Note :- If $\log x$ or $\log_e x$ then press ln in calci.
If $\log_{10} x$ then press log in calci

$$\int_a^b f(x)dx = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{0.2}{3}\left[(1.3862+1.6486)+4(1.4350+1.5260+1.6094) + 2[1.4816+1.5686)\right]$$

$$= 1.8278.$$

Exact value.

$$\int_4^{5.2} \log_e x \, dx = \log x \cdot x - \int_4^{5.2} x \cdot \frac{1}{x} \cdot dx$$

| Suv Rule.
$\int \underset{u}{\log x} \cdot \underset{v}{1}\, dx$

$$= \log x \cdot x - \int 1\, dx$$

$$= \left[x \log x - x\right]_4^{5.2} = [5.2 \log 5.2 - 5.2] - [4 \log 4 - 4]$$

$$= 1.8278$$

Simpson's $\frac{3}{8}$ th Rule

(1) Evaluate $\int_0^6 \frac{1}{1+x}\, dx$, using simpson's $\frac{3}{8}$ th Rule dividing (or taking 7 ordinates) the interval into 6 parts, compare the numerical value with exact value.

$\Rightarrow$ $h = \frac{b-a}{n} = \frac{6-0}{6} = 1.$

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
x	0	1	2	3	4	5	6
y	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3\right]$$

$$= \frac{3(1)}{8}\left[\left(1+\frac{1}{7}\right)+3\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{6}\right)+2\left(\frac{1}{4}\right)\right]$$

$$= 1.9661$$

② Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using $\frac{3}{8}$ th rule taking $h = \frac{1}{6}$.

$\Rightarrow$

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
x	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
y	1	$\frac{36}{37}$	$\frac{9}{10}$	$\frac{4}{5}$	$\frac{9}{13}$	$\frac{36}{61}$	$\frac{1}{2}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0+y_6) + 3(y_1+y_2+y_4+y_5) + 2y_3\right]$$

$$= \frac{3\left(\frac{1}{6}\right)}{8}\left[\left(1+\frac{1}{2}\right) + 3\left(\frac{36}{37}+\frac{9}{10}+\frac{9}{13}+\frac{36}{61}\right) + 2\left(\frac{4}{5}\right)\right]$$

$$= 0.7860$$

③ Evaluate $\int_0^1 \sqrt{\sin x + \cos x}\,dx$ by using Simpson's $\frac{3}{8}$ th rule.

taking seven ordinates.

$\Rightarrow$ seven ordinates 1 2 3 4 5 6 $n = 6$ equal parts.

y_0 y_1 y_2 y_3 y_4 y_5 y_6 = 7 ordinates

$\therefore n = 6 \quad \therefore h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
x	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
y	1	1.0733	1.1279	1.1650	1.1850	1.1886	1.1755
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0+y_6) + 3(y_1+y_2+y_4+y_5) + 2(y_3)\right]$$

$$= \frac{3\left(\frac{1}{6}\right)}{8}\left[(1+1.1755) + 3(1.0733+1.1279+1.1850+1.1886) + 2(1.1755)\right]$$

$= 1.13934$

(4) Compute the value of definite integral : $\int_0^6 \frac{1}{1+x}\,dx$

Using simpson's $(\frac{3}{8})^{th}$ rule, dividing the interval into 6 parts.

$\Rightarrow$ $h = \frac{b-a}{n} = \frac{6-0}{6} = 1$

x	0	1	2	3	4	5	6
y	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)\right]$$

$$= \frac{3(1)}{8}\left[\left(1+\frac{1}{7}\right)+3\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{6}\right)+2\left(\frac{1}{4}\right)\right]$$

$$= 1.9661$$

Numerical Soln of Ordinary Differential Equations

Euler's Method :-

Consider the differential equation

$\frac{dy}{dx} = f(x, y)$ Initial value $y(x_0) = y_0$

$x_1 = x_0 + h$ $y_1 = y$ at $x_1 = y(x_1)$

$x_2 = x_1 + h$ $y_2 = y$ at $x_2 = y(x_2)$

$x_3 = x_2 + h$ $y_3 = y$ at $x_3 = y(x_3)$

$\vdots$ $\vdots$

$x_n = x_{n-1} + h$ $y_n = y$ at $x_n = y(x_n)$

Process :-

Consider D.E. $\frac{dy}{dx} = f(x, y)$

$$y(x_1) = y_1 = y_0 + h f(x_0, y_0)$$
$$y(x_2) = y_2 = y_1 + h f(x_1, y_1)$$
$$y_3 = y_2 + h f(x_2, y_2)$$
$$\vdots$$
$$y_{n+1} = y_n + h f(x_n, y_n)$$

Examples :-

① Solve by Euler's method, the equation $\frac{dy}{dx} = x + y$, $y(0) = 0$, choose $h = 0.2$ and compute $y(0.4)$ and $y(0.6)$

$\Rightarrow$ Given $\frac{dy}{dx} = f(x, y) = x + y$

Initially $y(x_0) = y_0$

$y(0) = 0$ --- (Given)

$\therefore$ $x_0 = 0$, $y_0 = 0$

$\therefore x_1 = x_0 + h = 0 + 0.2 = 0.2$

$x_2 = x_1 + h = 0.2 + 0.2 = 0.4$

$x_3 = x_2 + h = 0.4 + 0.2 = 0.6$

find $y(0.4)$ i.e $y(x_2) = y_2$

& $y(0.6)$ i.e $y(x_3) = y_3$

$\Rightarrow$ $y_1 = y_0 + hf(x_0, y_0)$

$= 0 + (0.2)[0+0]$ --- $f(x,y) = x + y$

$= 0$

$y_2 = y_1 + hf(x_1, y_1)$

$= 0 + (0.2) f[0.2, 0]$

$= 0.2\ [0.2 + 0]$ --- $f(x,y) = x + y$

$= 0.04$

$y_3 = y_2 + hf(x_2, y_2)$

$= 0.04 + (0.2)\ f(0.4, 0.04)$

$= 0.04 + (0.2)[0.4 + 0.04]$ --- $f(x,y) = x + y$

$= 0.1280$

② Use Euler's Method to solve the eqn $\frac{dy}{dx} = x^2 + y$, subject to the Conditions $x = 0, y = 1$ and tabulate the solution for $x = 0\ (0.1)\ 0.5$

$\Rightarrow$ $\frac{dy}{dx} = f(x, y) = x^2 + y$

Initially $x_0 = 0$, $y_0 = 1$ (Given)

$x_0 = 0$, $x_1 = x_0 + h = 0 + 0.1 = 0.1$ [Given $x = 0\ \underbrace{(0.1)}_{h}\ 0.5$]

$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

$x_3 = x_2 + h = 0.2 + 0.1 = 0.3$

$x_4 = x_3 + h = 0.3 + 0.1 = 0.4$

$x_5 = x_4 + h = 0.4 + 0.1 = 0.5$

To find

$y(0.1)$, $y(0.2)$, $y(0.3)$, $y(0.4)$, $y(0.5)$

$y(x_1)$, $y(x_2)$, $y(x_3)$, $y(x_4)$, $y(x_5)$

$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$

y_1 y_2 y_3 y_4 y_5

$$
\begin{aligned}
y_1 &= y_0 + h f(x_0, y_0) \\
&= 1 + (0.1)\, f(0, 1) \\
&= 1 + (0.1)\,[0^2 + 1] \quad \cdots f(x,y) = x^2 + y \\
&= 1.1
\end{aligned}
$$

$$
\begin{aligned}
y_2 &= y_1 + h f(x_1, y_1) \\
&= 1.1 + (0.1)\, f(0.1, 1.1) \\
&= 1.1 + (0.1)\,[(0.1)^2 + 1.1] \quad \cdots f(x,y) = x^2 + y \\
&= 1.211
\end{aligned}
$$

$$
\begin{aligned}
y_3 &= y_2 + h f(x_2, y_2) \\
&= 1.211 + (0.1)\, f(0.2, 1.211) \\
&= 1.211 + (0.1)\,[0.2^2 + 1.211] \quad \cdots f(x,y) = x^2 + y \\
&= 1.3361
\end{aligned}
$$

$$
\begin{aligned}
y_4 &= y_3 + h f(x_3, y_3) \\
&= 1.3361 + (0.1)\, f(0.3, 1.3361) \\
&= 1.3361 + (0.1)\,[0.3^2 + 1.3361] \quad \cdots f(x,y) = x^2 + y \\
&= 1.4787
\end{aligned}
$$

$$
\begin{aligned}
y_5 &= y_4 + h f(x_4, y_4) \\
&= 1.4787 + (0.1)\,[0.4^2 + 1.4787] \\
&= 1.6426
\end{aligned}
$$

③ Using Euler's Method solve for y at $x=0.1$ from

$\frac{dy}{dx} = x+y+xy$ $\quad y(0)=1$, taking step size $h=0.025$

$\Rightarrow \frac{dy}{dx} = f(x,y) = x+y+xy$

Initially $x_0 = 0,\ y_0 = 1$ Given $y(x_0) = y_0$

$y(0) = 1$

$\therefore x_1 = x_0 + h = 0 + 0.025 = 0.025$

$x_2 = x_1 + h = 0.025 + 0.025 = 0.05$

$x_3 = x_2 + h = 0.05 + 0.025 = 0.075$

$x_4 = x_3 + h = 0.075 + 0.025 = 0.1$

To find y at $x=0.1$ i.e. y at $x_4 \Rightarrow y(x_4) = y_4$

$y_1 = y_0 + h f(x_0, y_0)$

$= 1 + (0.025)\, f(0,1)$

$= 1 + (0.025)[0+1+(0)(1)]$ $\quad$ --- $f(x,y) = x+y+xy$

$= 1.025$

$y_2 = y_1 + h f(x_1, y_1)$

$= 1.025 + (0.025)\, f(0.025, 1.025)$

$= 1.025 + (0.025)[0.025 + 1.025 + (0.025)(1.025)]$

$= 1.0519$

$y_3 = y_2 + h f(x_2, y_2)$

$= 1.0519 + (0.025)\, f(0.05, 1.0519)$

$= 1.0519 + (0.025)[0.05 + 1.0519 + (0.05)(1.0519)]$

$= 1.0808$

$y_4 = y_3 + h f(x_3, y_3)$

$= 1.0808 + (0.025)[0.075 + 1.0808 + (0.075)(1.0808)]$

$= 1.1117$

$y_5 = y_4 + h f(x_4, y_4) = 1.1117 (0.025)[0.1 + 1.1117 + (0.1)(1.1117)]$

$= 1.1448$

(4) Use Euler's method to solve the eqn $\frac{dy}{dx} = -y$, $y(0) = 1$

and tabulate the soln for $x = 0(0.01)0.04$

$\Rightarrow$

$$\frac{dy}{dx} = -y = f(x, y)$$

Given $y(0) = 1$

$y(x_0) = y_0$

$\therefore x_0 = 0, \quad y_0 = 1$ Initially

$x_1 = x_0 + h = 0 + 0.01 = 0.01$ $\qquad x = 0\underbrace{(0.01)}_{h}0.04$

$x_2 = x_1 + h = 0.01 + 0.01 = 0.02$

$x_3 = x_2 + h = 0.02 + 0.01 = 0.03$

$x_4 = x_3 + h = 0.03 + 0.01 = 0.04$

We have to find y_1, y_2, y_3 & y_4

$$\begin{aligned} y_1 &= y_0 + h f(x_0, y_0) \\ &= 1 + (0.01) f(0, 1) \\ &= 1 + (0.01)[-1] = 0.99 \qquad \text{--- } f(x,y) = -y \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + h f(x_1, y_1) \\ &= 0.99 + (0.01) f(0.01, 0.99) \\ &= 0.99 + (0.01)[-0.99] \\ &= 0.9801 \end{aligned}$$

$$\begin{aligned} y_3 &= y_2 + h f(x_2, y_2) \\ &= 0.9801 + (0.01) f(0.02, 0.9801) \\ &= 0.9801 + (0.01)[-0.9801] \\ &= 0.9703 \end{aligned}$$

$$\begin{aligned} y_4 &= y_3 + h f(x_3, y_3) \\ &= 0.9703 + (0.01)[-0.9703] \\ &= 0.9606 \end{aligned}$$

Modified Euler's Method

Process :-

y_1 is computed by using Euler's formula

$$y_1 = y_0 + h f(x_0, y_0)$$

then find $f(x_1, y_1)$

The first modification of y_1 is obtained by formula

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$$

Second modification is given by

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

Third modification is given by

$$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

$\vdots$

n^{th} modification

$$y_1^{(n)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(n-1)})]$$

In this way, we can tabulate (x, y) for the desired interval $x_0 < x < x_n$

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① Solve the equation $\frac{dy}{dx} = 1 + xy$, $x_0 = 0$, $y_0 = 1$ to find y at $x = 0.1$ and $x = 0.2$ using modified Euler's method taking $h = 0.1$. Correct upto four decimal places.

$\Rightarrow$

$$y_1 = y_0 + h f(x_0, y_0)$$
$$= 1 + (0.1) f(0, 1)$$
$$= 1 + (0.1)[1 + (0)(1)]$$
$$= 1.1$$

First modification

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$
$$= y_0 + \frac{0.1}{2}\left[f(0,1) + f(0.1, 1.1)\right]$$
$$= 1 + \frac{0.1}{2}\left[(1+0) + (1 + (0.1)(1.1))\right]$$
$$= 1.1055$$

Second modification

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$
$$= 1 + \frac{0.1}{2}\left[f(0,1) + f(0.1, 1.1055)\right]$$
$$= 1 + \frac{0.1}{2}\left[(1) + [1 + (0.1)(1.1055)]\right]$$
$$= 1.1055275$$

Accuracy is upto 4th decimal.

$\therefore$ $y_1 = 1.1055$. y at $x = 0.1$

For y at $x = 0.2 = x_2$.

$$y_2 = y_1 + h f(x_1, y_1)$$
$$= 1.1055 + 0.1\, f(0.1, 1.1055)$$
$$= 1.1055 + (0.1)[1 + (0.1)(1.1055)]$$
$$= 1.216555$$

First modification

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2)]$$

$$= 1.1055 + \frac{0.1}{2}[f(0.1, 1.1055) + f(0.2, 1.216555)]$$

$$= 1.1055 + \frac{0.1}{2}[[1 + (0.1)(1.1055)] + [1 + (0.2)(1.216555)]]$$

$$= 1.2232$$

Second modification

$$y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})]$$

$$= 1.1055 + \frac{0.1}{2}[[1 + (0.1)(1.1055)] + [1 + (0.2)(1.2232)]]$$

$$= 1.2233$$

Third modification

$$y_2^{(3)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(2)})]$$

$$= 1.1055 + \frac{0.1}{2}[(1 + (0.1)(1.1055)) + (1 + (0.2)(1.2233))]$$

$$= 1.2233$$

Correct $y = 1.2233$

(2) Using modified Euler's method solve the equation $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$

$y(1) = 1$ to find y at $x = 1.2$ correct to three decimal places taking $h = 0.1$

$\Rightarrow$ $x_0 = 1$, $x_1 = x_0 + h = 1 + 0.1 = 1.1$, $x_2 = x_1 + h = 1.1 + 0.1 = 1.2$

$\therefore$ we have to find y at $x = 1.2$ i.e. y_2

$$\frac{dy}{dx} = \frac{1}{x^2} - \frac{y}{x} = f(x, y)$$

$$y_1 = y_0 + hf(x_0, y_0)$$
$$= 1 + 0.1\, f(1,1)$$
$$= 1 + (0.1)\left[\frac{1}{1^2} - \frac{1}{1}\right] = 1$$

First modification:

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]$$
$$= 1 + \frac{0.1}{2}\left[f(1,1) + f(1.1, 1)\right]$$
$$= 1 + \frac{0.1}{2}\left[0 + \left(\frac{1}{1.1^2} - \frac{1}{1.1}\right)\right]$$
$$= 0.996$$

Second modification:

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$
$$= 1 + \frac{0.1}{2}\left[f(1,1) + f(1.1, 0.996)\right]$$
$$= 1 + \frac{0.1}{2}\left[0 + \left(\frac{1}{1.1^2} - \frac{0.996}{1.1}\right)\right]$$
$$= 0.996$$

Correct upto three decimal places $y_1 = 0.996$

$$y_2 = y_1 + hf(x_1, y_1) = 0.996 + 0.1\, f(1.1, 0.996)$$
$$= 0.996 + 0.1\left[\frac{1}{1.1^2} - \frac{0.996}{1.1}\right]$$
$$= 0.988$$

First modification $y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2)\right]$

$$= 0.996 + \frac{0.1}{2}\left[-0.079 + \left[\frac{1}{1.2^2} - \frac{0.988}{1.2}\right]\right]$$
$$= 0.986$$

Second modification $y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$

$$= 0.996 + \frac{0.1}{2}\left[-0.079 + f(1.2, 0.986)\right]$$
$$= 0.996 + \frac{0.1}{2}\left[-0.079 + \left[\frac{1}{1.2^2} - \frac{0.986}{1.2}\right]\right]$$
$$= 0.986$$

$\therefore$ value of y correct upto three decimal $= 0.986$

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Runge kutta 4^{th} order Method:

Process :-

$$K_1 = hf(x_0, y_0)$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$

$$K_4 = hf(x_0 + h, y_0 + k_3)$$

$$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$y_1 = y_0 + K.$$

Examples:-

(1) Solve the following differential eqn $\frac{dy}{dx} = x - 2y$ using Runge-kutta fourth order method, given that $y = 1$ when $x = 0$ and find y at $x = 0.1$

→ $f(x, y) = x - 2y$

Initially $x_0 = 0$, $y_0 = 1$ find y at $x = 0.1 = x_1$ (y_1).

$\therefore h = x_1 - x_0 = 0.1 - 0 = 0.1$

$$K_1 = hf(x_0, y_0)$$
$$= 0.1\, f(0, 1)$$
$$= 0.1\,[0 - 2(1)]$$
$$= -0.2$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$
$$= 0.1\, f(0.05, 0.9)$$
$$= 0.1\,[0.05 - 2(0.9)]$$
$$= -0.175$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$
$$= 0.1\, f(0.05, 0.9125)$$
$$= 0.1\,[0.05 - 2(0.9125)]$$
$$= -0.1775$$

$$K_4 = hf(x_0 + h, y_0 + k_3)$$
$$= 0.1\, f(0.1, 0.8225)$$
$$= 0.1\,[0.1 - 2(0.8225)]$$
$$= -0.1545$$

$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = -0.3566$ | $y_1 = y_0 + K = 0.6434$

2) Using fourth order Runge-Kutta method, solve the D.E:

$\frac{dy}{dx} = \sqrt{x+y}$ with $y(0)=1$, and find $y(0.2)$ taking $h=0.2$

$\Rightarrow$ Initially $y(x_0) = y_0$

$y(0) = 1$

$\therefore x_0 = 0, \; y_0 = 1$

$x_1 = x_0 + h = 0 + 0.2 = 0.2$ i.e. to find $y_1 = 0.2$

$$\frac{dy}{dx} = f(x,y) = \sqrt{x+y}$$

$$K_1 = h f(x_0, y_0)$$
$$= 0.2 f(0,1)$$
$$= 0.2 \sqrt{0+1}$$
$$= 0.2$$

$$K_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$
$$= 0.2 f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right)$$
$$= 0.2 f(0.1, 1.1)$$
$$= 0.2 \sqrt{0.1 + 1.1}$$
$$= 0.2191$$

$$K_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$
$$= 0.2 f(0.1, 1.1096)$$
$$= 0.2 \sqrt{0.1 + 1.1096}$$
$$= 0.22$$

$$K_4 = h f(x_0 + h, y_0 + K_3)$$
$$= 0.2 f(0.2, 1.22)$$
$$= 0.2 \sqrt{0.2 + 1.22}$$
$$= 0.2383$$

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$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.2 + 2(0.2191) + 2(0.22) + 0.2383)$$

$$= 0.2194$$

$$y_1 = y_0 + k$$
$$= 1 + 0.2194$$
$$= 1.2194$$

2) Given: $\frac{dy}{dx} = 3x + \frac{y}{2}$, $y(0) = 1$, $h = 0.1$. Evaluate $y(0.1)$ by using Runge-kutta method of fourth order.

⇒

Initially $y(0) = 1$

$y(x_0) = y_0$

$\therefore x_0 = 0, \quad y_0 = 1$

$h = 0.1 \qquad x_1 = x_0 + h = 0 + 0.1 = 0.1$

$\therefore$ to find $y(0.1)$ i.e. y_1.

$$k_1 = h f(x_0, y_0)$$
$$= (0.1) f(0, 1)$$
$$= (0.1)\left[3(0) + \frac{1}{2}\right]$$
$$= 0.05$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$= (0.1) f(0.05, 1.025)$$
$$= (0.1)\left[3(0.05) + \frac{1.025}{2}\right]$$

$$= 0.0663$$

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$$k_3 = h f\left(x_0+\frac{h}{2},\ y_0+\frac{k_2}{2}\right)$$

$$= (0.1) f(0.05,\ 1.0332) \qquad ; \quad 3x+\frac{y}{2}$$

$$= (0.1)\left[3(0.05)+\frac{1.0332}{2}\right]$$

$$= 0.0667$$

$$k_4 = h f(x_0+h,\ y_0+k_3)$$

$$= (0.1) f(0.1,\ 1.0667)$$

$$= (0.1)\left[3(0.1)+\frac{1.0667}{2}\right]$$

$$= 0.0833$$

$$K = \frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

$$= 0.0666$$

$$y_1 = y_0 + K$$

$$= 1 + 0.0666$$

$$= 1.0666$$

3) Using fourth order Runge-Kutta method, solve the D.E. $\frac{dy}{dx} = x+y+xy$ with $y(0)=1$ to get $y(0.1)$ taking $h=0.1$

$\Rightarrow$ Initially $y(0)=1$

$y(x_0) = y_0$

$\therefore x_0 = 0$ & $y_0 = 1$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

to find y at 0.1 i.e. y_1

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$k_1 = h f(x_0, y_0)$
$= (0.1) f(0, 1)$
$= (0.1) [0 + 1 + (0)(1)]$
$= 0.1$

$k_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$
$= (0.1) f(0.05, 1.05)$
$= (0.1) [0.05 + 1.05 + (0.05)(1.05)]$
$= (0.1) [0.05 + 1.05 + 0.0525]$
$= 0.1153$

$k_3 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$
$= (0.1) f(0.05, 1.0577)$
$= (0.1) [0.05 + 1.0577 + (0.05)(1.0577)]$
$= 0.1161$

$k_4 = h f(x_0 + h, y_0 + k_3)$
$= (0.1) f(0.1, 1.1161)$
$= (0.1) [0.1 + 1.1161 + (0.1)(1.1161)]$
$= 0.1328$

$K = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 0.1159$

$y_1 = y_0 + K = 1 + 0.1159 = 1.1159$

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4) Solve the differential eqn $\frac{dy}{dx} = 1+xy$, $y(0) = 2$ to get $y(0.2)$ by using Runge Kutta method of fourth order. $(h = 0.2)$

$\Rightarrow$ Initially $y(0) = 2$

$y(x_0) = y_0$

$\therefore x_0 = 0, \quad y_0 = 2, \quad x_1 = x_0 + h = 0 + 0.2 = 0.2$

to find $y(0.2) = y(x_1) = y_1$

$$\frac{dy}{dx} = f(x, y) = 1 + xy$$

$$K_1 = hf(x_0, y_0)$$
$$= (0.2) f(0, 2)$$
$$= (0.2)[1 + (0)(2)]$$
$$= 0.2$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$
$$= (0.2) f(0.1, 2.1)$$
$$= (0.2)[1 + (0.1)(2.1)]$$
$$= 0.242$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$
$$= (0.2) f(0.1, 2.121)$$
$$= (0.2)[1 + (0.1)(2.121)]$$
$$= 0.2424$$

$$K_4 = hf(x_0 + h, y_0 + k_3)$$
$$= (0.2) f(0.2, 2.2424)$$
$$= (0.2)[1 + (0.2)(2.2424)]$$
$$= 0.2897$$

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$$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= \frac{1}{6}\left(0.2 + 2(0.242) + 2(0.2424) + 0.2897\right)$$

$$= 0.2431.$$

$$y_1 = y_0 + K$$

$$= 2.2431$$

5) Given $\frac{dy}{dx} = y - x$ with $y(0) = 2$ find $y(0.1)$, & $y(0.2)$.

⇒ Initially $y(0) = 2$

$y(x_0) = y_0$

$y_0 = 2$ $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$

$\therefore h = 0.1$.

to find y_1 & y_2.

$$\frac{dy}{dx} = f(x, y) = y - x.$$

For y_1

$$K_1 = h f(x_0, y_0)$$

$$= (0.1) f(0, 2)$$

$$= (0.1)[2 - 0]$$

$$= 0.2$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$

$$= (0.1) f(0.05, 2.1).$$

$$= (0.1)[2.1 - 0.05]$$

$$= 0.205$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$

$$= (0.1) f(0.05, 2.1025)$$

$$= 0.2053$$

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$k_4 = hf(x_0+h,\ y_0+k_3)$
$= (0.1)\, f(0.1,\ 2.2053)$
$= (0.1)\,[2.2053 - 0.1]$
$= 0.2105$

$k = \frac{1}{6}\,[\,k_1 + 2k_2 + 2k_3 + k_4\,]$

$= 0.2052.$

$y_1 = y_0 + k$
$= 2 + 0.2052$
$= 2.2052$

$(x_1, y_1) = (0.1,\ 2.2052)$, $h = 0.1$

For y_2

$k_1 = hf(x_1, y_1)$
$= (0.1)\, f(0.1,\ 2.2052)$
$= (0.1)\,[2.2052 - 0.1]$
$= 0.2105$

$k_2 = hf\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_1}{2}\right)$
$= (0.1)\, f(0.15,\ 2.3105)$
$= (0.1)\,[2.3105 - 0.15]$
$= 0.2161$

$k_3 = hf\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_2}{2}\right)$
$= (0.1)\, f(0.15,\ 2.3133)$
$= (0.1)\,[2.3133 - 0.15]$
$= 0.2163$

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$$K_4 = hf(x_1 + h, y_1 + K_3)$$
$$= (0.1)\, f(0.2, 2.4215) = 0.2222$$

$$K = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$
$$= 0.2163.$$

$$y_2 = y_1 + K$$
$$= 2.4215$$