UNIT I : LINEAR DIFFERENTIAL EQUATIONS and APPLICATIONS

Recall: LDE of order 1 and degree 1 is of the form, $\frac{dy}{dx} + Py = Q$.

$\therefore$ $I.F. = e^{\int P\,dx}$ and sol^n is given by,

$$y(I.F.) = \int Q.(I.F.)\,dx + c$$

i.e, $y\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$

i.e., $\boxed{y = e^{-\int P\,dx}\left\{\int Q \cdot e^{\int P\,dx}\,dx\right\} + c \cdot e^{-\int P\,dx}}$

e.g. 1) Solve, $\frac{dy}{dx} + 2y = 0$

$\rightarrow$ Here $P = 2$ and $Q = 0$

$\therefore$ sol^n is, $y = e^{-\int 2\,dx}\{0\} + c \cdot e^{-\int 2\,dx}$

i.e., $y = 0 + c.e^{-2x}$, i.e., $\boxed{y = ce^{-2x}}$

2) Solve, $\frac{dy}{dx} - 3y = 0$

$\rightarrow$ Here $P = -3$ and $Q = 0$.

$\therefore$ sol^n is, $y = e^{-\int(-3)\,dx}\{0\} + c \cdot e^{-\int(-3)\,dx}$

$= 0 + c \cdot e^{3x}$

i.e., $\boxed{y = c \cdot e^{3x}}$

We will denote $\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2$, ...

$\left(\frac{d^2y}{dx^2} \neq \left(\frac{dy}{dx}\right)^2 \right)$

Ex. 1) $Dy + 2y = 0$, i.e., $\underbrace{(D+2)}_{\Phi(D)}y = 0$

i.e., $\Phi(D) \cdot y = 0$, where $\Phi(D) = D+2$ is polynomial in D.

Auxiliary eq^n is, $\Phi(m) = 0$

i.e., $m+2=0$, i.e., $m=-2$

i.e., $y = c\,e^{mx} = c \cdot e^{-2x}$

2) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$ — ①

→ Let $\frac{d}{dx} = D$ and $\frac{d^2}{dx^2} = D^2$

∴ eq^n ① becomes,

$D^2y - 3Dy + 2y = 0$

i.e., $(D^2 - 3D + 2)y = 0$, i.e., $\Phi(D) = D^2 - 3D + 2$

∴ Auxiliary eq^n is, $\Phi(m) = 0$

i.e., $m^2 - 3m + 2 = 0$, i.e., $(m-2)(m-1) = 0$

i.e., $m = 2, 1$

∴ sol^n is, $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$$\boxed{y = c_1 e^{2x} + c_2 e^{x}}$$

Some standard factorization :

1) $m^2 - a^2 = (m-a)(m+a)$ $[\because a^2 - b^2 = (a-b)(a+b)]$

2) $m^2 + a^2 = m^2 - (-a^2) = m^2 - (i^2 a^2)$ $(\because i = \sqrt{-1})$

i.e, $m^2 + a^2 = m^2 - (ia)^2 = (m - ia)(m + ia)$

e.g. i) $m^2 + 1 = (m-i)(m+i)$

ii) $m^2 + 9 = (m - 3i)(m + 3i)$

3) The roots of $am^2 + bm + c = 0$ are,

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

e.g. i) $m^2 + m + 1 = 0$

i.e, $m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}\, i}{2}$

i.e, $m = \frac{-1 \pm \sqrt{3}\, i}{2} = \alpha \pm i\beta$

i.e, $\alpha = \frac{-1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$

ii) $m^2 + 1 = 0 \Rightarrow m = \pm i \Rightarrow \alpha = 0, \beta = 1$

4) $m^3 - a^3 = (m-a)(m^2 + am + a^2)$

5) $m^3 + a^3 = (m+a)(m^2 - am + a^2)$

6) $m^3 + 3m^2 + 3m + 1 = (m+1)^3$

7) $m^3 - 3m^2 + 3m - 1 = (m-1)^3$

8) Synthetic division:

e.g. i) $m^3 - 7m - 6 = 0$, observe that one root is $m_1 = -1$

$$\begin{array}{c|cccc} -1 & 1 & 0 & -7 & -6 \\ & & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array}$$

i.e., $m^2 - m - 6 = 0 \Rightarrow (m-3)(m+2) = 0$

i.e., $m = 3, -2$

all roots are, $m = -1, -2, 3$

9) $m^4 - a^4 = (m^2)^2 - (a^2)^2 = (m^2 - a^2)(m^2 + a^2)$

$= (m-a)(m+a)(m-ia)(m+ia)$

10) $m^4 + a^4 = \underline{m^4 + 2a^2m^2 + a^4} - 2a^2m^2$

$= (m^2 + a^2)^2 - (\sqrt{2}\,am)^2$

$= (m^2 + a^2 - \sqrt{2}\,am)(m^2 + a^2 + \sqrt{2}\,am)$

$= (m^2 - \sqrt{2}\,am + a^2)(m^2 + \sqrt{2}\,am + a^2)$

e.g. $m^4 + 16 = m^4 + 8m^2 + 16 - 8m^2$

$= (m^2 + 4)^2 - (\sqrt{8}\,m)^2$

$= (m^2 + 4)^2 - (2\sqrt{2}\,m)^2$

$= (m^2 + 4 - 2\sqrt{2}\,m)(m^2 + 4 + 2\sqrt{2}\,m)$

$= (m^2 - 2\sqrt{2}\,m + 4)(m^2 + 2\sqrt{2}\,m + 4)$

11) $m^4 + 8m^2 + 16 = 0$, i.e., $(m^2)^2 + 8m^2 + 16 = 0$

Put $m^2 = t$ gives, $t^2 + 8t + 16 = 0$

i.e., $(t+4)(t+4) = 0$

i.e., $t = -4$ and $t = -4$

i.e., $m^2 = -4$ and $m^2 = -4$

i.e., $m = \pm\sqrt{-4}$ and $m = \pm\sqrt{-4}$

i.e., $m = \pm 2i$ and $m = \pm 2i$

The n^{th} order LDE with constant coefficients:

The general form of n^{th} order LDE with constant coefficients is,

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = z(x), \quad \text{—①}$$

where $a_0, a_1, a_2, \ldots$ are all constants.

Let $\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2$, ..., $\frac{d^n}{dx^n} = D^2$,

equation ① becomes,

$$a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \cdots + a_{n-1} Dy + a_n y = z(x)$$

i.e., $(a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} + a_n) y = z(x)$

i.e., $\Phi(D) y = z(x)$, —②

where $\Phi(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$, which is a polynomial in D.

The general solution is given by,

$$y = y_c + y_p \quad \text{--- (3)}$$

where, i) y_c is known as complementary function (C.F.) or complementary solution and it is a solution of $\Phi(D) y = 0$. (i.e., $f(x) = 0$)

ii) y_p is known as particular integral (P.I.) or particular solution and it satisfies the equation, $\Phi(D) y = f(x)$.

i.e., $\Phi(D)\, y_p = f(x)$

operate $\frac{1}{\Phi(D)}$ on left side of above eq^n we get,

$$\frac{1}{\Phi(D)} \left[\Phi(D)\, y_p \right] = \frac{1}{\Phi(D)} f(x)$$

i.e., $\boxed{y_p = \frac{1}{\Phi(D)} f(x)}$

Note: 1) If $f(x) = 0$, then equation (2) is known as homogeneous LDE with constant coefficients; otherwise known as non-homogeneous LDE with constant coefficients.

2) If $f(x) = 0$, then $y_p = \frac{1}{\Phi(D)}(0) = 0$

i.e., General solution of homogeneous LDE $\Phi(D)y = 0$ is given by,

$$y = y_c + y_p = y_c + 0 = y_c \text{ , i.e., } \boxed{y = y_c}$$

Methods to find y_c (complimentary function):

Consider Auxiliary Equation (A.E.) of the LDE $\Phi(D) y = f(x)$ as, $\Phi(m) = 0$.

Suppose $m_1, m_2, m_3, \cdots, m_n$ are the roots of $\Phi(m) = 0$.

Case i): If the roots of A.E. are real and distinct, then

$$\boxed{y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}}$$

Case ii): If the roots of A.E. are real but not distinct (some roots may be repeated; suppose $m_1 = m_2$), then

$$y_c = (c_1 x + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

If $m_1 = m_2 = m_3$, then

$$y_c = (c_1 x^2 + c_2 x + c_3) e^{m_1 x} + c_4 e^{m_4 x} + \cdots + c_n e^{m_n x}$$

If $m_1 = m_2 = m_3 = \cdots = m_n$, then

$$\boxed{y_c = (c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_{n-1} x + c_n) e^{m_1 x}}$$

Case iii) : If the roots of A.E. are complex and distinct.

suppose $\alpha \pm i\beta$ and $c \pm id$ are four distinct complex roots, then

$$\boxed{y_c = e^{\alpha x}\left[c_1 \cos(\beta x) + c_2 \sin(\beta x)\right] + e^{cx}\left[c_3 \cos(dx) + c_4 \sin(dx)\right]}$$

Case iv): If the roots of A.E. are complex but not distinct (some roots may be repeated)

suppose $\alpha \pm i\beta$ is repeated twice, then

$$y_c = e^{\alpha x}\left[(c_1 x + c_2)\cos(\beta x) + (c_3 x + c_4)\sin(\beta x)\right]$$

lly, If $\alpha \pm i\beta$ is repeated three times, then

$$y_c = e^{\alpha x}\left[(c_1 x^2 + c_2 x + c_3)\cos(\beta x) + (c_4 x^2 + c_5 x + c_6)\sin(\beta x)\right]$$

e.g. 1) suppose roots of A.E. are $-1, \frac{1}{2}, 5$

then $y_c = c_1 e^{(-1)x} + c_2 e^{(\frac{1}{2})x} + c_3 e^{(5)x}$

i.e., $y_c = c_1 e^{-x} + c_2 e^{x/2} + c_3 e^{5x}$

2) suppose roots of A.E. are $-1, -1, -1, -1, 2, 3$

then $y_c = (c_1 x^3 + c_2 x^2 + c_3 x + c_4) e^{(-1)x} + c_5 e^{(2)x} + c_6 e^{(3)x}$

i.e., $y_c = (c_1 x^3 + c_2 x^2 + c_3 x + c_4) e^{-x} + c_5 e^{2x} + c_6 e^{3x}$

3) suppose roots of A.E. are $\frac{3}{2}, \frac{-5}{4}, 1\pm i, \frac{-1}{2}\pm\frac{\sqrt{3}}{2}i$

then $y_c = c_1 e^{\frac{3}{2}x} + c_2 e^{\frac{-5}{4}x} + e^{(1)x}[c_3\cos(1\cdot x) + c_4\sin(1\cdot x)]$
$+ e^{(\frac{-1}{2})x}[c_5\cos(\frac{\sqrt{3}}{2}x) + c_6\sin(\frac{\sqrt{3}}{2}x)]$

i.e., $y_c = c_1 e^{3x/2} + c_2 e^{-5x/4} + e^{x}[c_3\cos(x) + c_4\sin(x)]$
$+ e^{-x/2}[c_5\cos(\frac{\sqrt{3}x}{2}) + c_6\sin(\frac{\sqrt{3}x}{2})]$

4) suppose roots of A.E. are $1\pm i, \frac{-1}{2}\pm\frac{\sqrt{3}}{2}i, \frac{-1}{2}\pm\frac{\sqrt{3}}{2}i, \frac{-1}{2}\pm\frac{\sqrt{3}}{2}i$, then

$y_c = e^{x}[c_1\cos(x) + c_2\sin(x)]$
$+ e^{-x/2}[(c_3x^2 + c_4x + c_5)\cos(\frac{\sqrt{3}x}{2}) + (c_6x^2 + c_7x + c_8)\sin(\frac{\sqrt{3}x}{2})]$

Note: 1) If order of LDE is 'n', then complimentary function (C.F.) y_c must contain n arbitrary constants.

2) particular integral (P.I.) does not contain any arbitrary constants.

Examples: (All examples below are homogeneous LDE, hence it's general solⁿ is, $y = y_c$)

1) Solve, $4\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + y = 0$ —— ①

→ Let $\frac{d}{dx} = D$ and $\frac{d^2}{dx^2} = D^2$, hence eqⁿ ① reduces to,

$4D^2y - 8Dy + y = 0$, i.e., $(4D^2 - 8D + 1)y = 0$

i.e., $\Phi(D) = 4D^2 - 8D + 1$

Now, Auxiliary equation (A.E.) is, $\Phi(m) = 0$

i.e., $4m^2 - 8m + 1 = 0$

i.e., $m = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(4)(1)}}{2(4)}$

$= \frac{8 \pm \sqrt{64 - 16}}{8}$

$= \frac{8 \pm \sqrt{48}}{8} = \frac{8 \pm 4\sqrt{3}}{8} = \frac{8}{8} \pm \frac{4}{8}\sqrt{3}$

i.e., $m = 1 \pm \frac{\sqrt{3}}{2}$, i.e., $m_1 = 1 + \frac{\sqrt{3}}{2}$ & $m_2 = 1 - \frac{\sqrt{3}}{2}$

i.e., roots are real and distinct, hence

$y_c = c_1 e^{\left(1 + \frac{\sqrt{3}}{2}\right)x} + c_2 e^{\left(1 - \frac{\sqrt{3}}{2}\right)x}$

Thus, general solⁿ is, $\boxed{y = c_1 e^{\left(1 + \frac{\sqrt{3}}{2}\right)x} + c_2 e^{\left(1 - \frac{\sqrt{3}}{2}\right)x}}$

2) Solve, $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$. —— ①

→ Let, $\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2$ and $\frac{d^3}{dx^3} = D^3$, hence eqⁿ ① reduces to

$D^3y - 6D^2y + 11Dy - 6y = 0$

i.e., $(D^3 - 6D^2 + 11D - 6)y = 0$, i.e., $\Phi(D) = D^3 - 6D^2 + 11D - 6$

i.e., Auxiliary eqⁿ (A.E.) is, $\Phi(m) = 0$

i.e., $m^3 - 6m^2 + 11m - 6 = 0$

We will find roots by using synthetic division.

Note that $m = 1$ is one root. ($\because$ $m=1$ satisfies A.E.)

1	1	−6	11	−6
		1	−5	6
	1	−5	6	$\boxed{0}$

i.e., factor polynomial is, $m^2 - 5m + 6 = 0$

i.e., $(m-2)(m-3) = 0$

i.e., $m = 2,\ m = 3$

i.e., all roots of A.E. are $m = 1, 2, 3$

i.e., $y_c = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$

Thus, the general solⁿ is, $\boxed{y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}}$

3) Solve, $(D^2 + 2D + 5)y = 0$

→ Here, $\Phi(D) = D^2 + 2D + 5$

$\therefore$ A.E. is, $\Phi(m) = 0$, i.e., $m^2 + 2m + 5 = 0$

i.e., $m = \dfrac{-2 \pm \sqrt{4-20}}{2} = \dfrac{-2 \pm \sqrt{-16}}{2} = \dfrac{-2 \pm 4i}{2} = -1 \pm 2i$

$= \alpha \pm i\beta$

i.e., $\alpha = -1$ & $\beta = 2$

i.e., $y_c = e^{\alpha x}[c_1 \cos(\beta x) + c_2 \sin(\beta x)] = e^{(-1)x}[c_1 \cos(2x) + c_2 \sin(2x)]$

Thus, the general solution is, $\boxed{y = e^{-x}[c_1 \cos(2x) + c_2 \sin(2x)]}$

4) Solve, $(D^4 - 5D^2 + 12D + 28)y = 0$

→ Here, $\Phi(D) = D^4 - 5D^2 + 12D + 28$.

∴ A.E. is, $\Phi(m) = 0$, i.e., $m^4 - 5m^2 + 12m + 28 = 0$

We will find roots by using synthetic division.

Note that $m = -2$ is one root.

-2	1	0	-5	12	28
		-2	4	2	-28
-2	1	-2	-1	14	$\boxed{0}$
		-2	8	-14	
	1	-4	7	$\boxed{0}$	

(∵ $m = -2$ is again root of $m^3 - 2m^2 - m + 14 = 0$)

i.e., factor polynomial is, $m^2 - 4m + 7 = 0$

i.e., $m = \dfrac{4 \pm \sqrt{16 - 28}}{2} = \dfrac{4 \pm \sqrt{-12}}{2} = \dfrac{4 \pm 2\sqrt{3}i}{2} = 2 \pm \sqrt{3}i$

i.e., all roots of A.E. are, $-2, -2, 2 \pm \sqrt{3}i$

∴ $y_c = (c_1 x + c_2)e^{-2x} + e^{2x}[c_3 \cos(\sqrt{3}x) + c_4 \sin(\sqrt{3}x)]$

Thus, the general solution is,

$$\boxed{y = (c_1 x + c_2)e^{-2x} + e^{2x}[c_3 \cos(\sqrt{3}x) + c_4 \sin(\sqrt{3}x)]}$$

5) Solve, $(D^6 + 6D^4 + 9D^2)y = 0$

$\longrightarrow$ Here, $\Phi(D) = D^6 + 6D^4 + 9D^2$.

$\therefore$ A.E. is, $\Phi(m) = 0$, i.e., $m^6 + 6m^4 + 9m^2 = 0$

i.e., $(m^2)^3 + 6(m^2)^2 + 9m^2 = 0$

put $m^2 = t$ gives $t^3 + 6t^2 + 9t = 0$

i.e., $t(t^2 + 6t + 9) = 0$, i.e., $t(t+3)(t+3) = 0$

i.e., $t = 0$, $t+3 = 0$, $t + 3 = 0$

i.e., $t = 0$, $t = -3$, $t = -3$

i.e., $m^2 = 0$, $m^2 = -3$, $m^2 = -3$

i.e., $m = 0, 0$, $m = \pm\sqrt{3}i$, $m = \pm\sqrt{3}i$

i.e., all roots of A.E. are, $0, 0, \pm\sqrt{3}i, \pm\sqrt{3}i$

$\therefore$ $y_c = (c_1x + c_2)e^{(0).x} + e^{(0).x}\left[(c_3x + c_4)\cos(\sqrt{3}x) + (c_5x + c_6)\sin(\sqrt{3}x)\right]$

$(\because \alpha = 0, \beta = \sqrt{3})$

i.e., $y_c = (c_1x + c_2)(1) + (1)\left[(c_3x + c_4)\cos(\sqrt{3}x) + (c_5x + c_6)\sin(\sqrt{3}x)\right]$

Thus, the general solution is, $(\because e^0 = 1)$

$$\boxed{y = c_1x + c_2 + (c_3x + c_4)\cos(\sqrt{3}x) + (c_5x + c_6)\sin(\sqrt{3}x)}$$

6) solve, $\frac{d^2x}{dt^2} + 4x = 0$. ——— ①

→ Let

$\frac{d}{dt} = D$ and $\frac{d^2}{dt^2} = D^2$, hence eqⁿ ① reduces to

$D^2x + 4x = 0$, i.e., $(D^2+4)x = 0$, i.e., $\Phi(D) = D^2+4$

∴ A.E. is, $\Phi(m) = 0$, i.e., $m^2+4 = 0$, i.e., $m^2 = -4$

i.e., $m = \pm\sqrt{-4} = \pm 2i = \alpha \pm i\beta \Rightarrow \alpha = 0,\ \beta = 2$

∴

$x_c = e^{\alpha t}\left[c_1 \cos(\beta t) + c_2 \sin(\beta t)\right]$ (∵ here dependent and independent variables are x & t instead of y & x resp.)

i.e., $x_c = e^{(0)t}\left[c_1 \cos(2t) + c_2 \sin(2t)\right]$

$= (1)\left[c_1 \cos(2t) + c_2 \sin(2t)\right]$

Thus, the general solution is,

$$\boxed{x = c_1 \cos(2t) + c_2 \sin(2t)}$$

Home work questions:

1) solve, $2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 10y = 0$

2) solve, $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$, 3) solve, $\frac{d^3y}{dx^3} + y = 0$

4) solve, $\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 8\frac{dy}{dx} - 4y = 0$

5) solve, $\frac{d^4y}{dx^4} - y = 0$, 6) solve, $(D^2+9)^2 y = 0$.