## Unit 1 :- Linear Differential eq$^n$ with Constant Coefficients

The $n^{th}$ order Linear D.E with constant coefficients

$$a_0 \frac{d^ny}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = f(x)$$

Here $D = \frac{d}{dx}$, $Dy = \frac{dy}{dx}$, $D^2y = \frac{d^2y}{dx^2}$ ... $D^ny = \frac{d^ny}{dx^n}$

$$\therefore a_0 D^ny + a_1 D^{n-1}y + a_2 D^{n-2}y + \cdots + a_{n-1} Dy + a_n y = f(x)$$

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1}D + a_n)\,y = f(x) \quad -①$$

$\phi(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n$ → poly. in D

$\therefore$ eq$^n$ (1) ⇒ $\phi(D)\,y = f(x)$

Linear Differential eq$^n$ $\phi(D)y = 0$
Here $\phi(D) = 0$ is called auxiliary eq$^n$.

* Solution of $\phi(D)y = 0$ / Different cases depending upon the nature of roots of the auxiliary eq$^n$ $\phi(D) = 0$

**Case I :- Non repeated Real Roots :-**
If roots of $\phi(D) = 0$ be $m_1, m_2, m_3 \cdots m_n$ all are real & non repeated, then the sol$^n$ of $\phi(D)\,y = 0$ will be

$$\boxed{y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x} + \cdots + C_n e^{m_n x}}$$

**Case II :- Repeated Real Roots ;-** If $m_1 = m_2$

$$\boxed{y = (C_1 x + C_2)\, e^{m_1 x} + C_3 e^{m_3 x} + C_4 e^{m_4 x} + \cdots + C_n e^{m_n x}}$$

If $m_1 = m_2 = m_3$

$$\boxed{y = (C_1 x^2 + C_2 x + C_3) e^{m_1 x} + C_4 e^{m_4 x} + C_5 e^{m_5 x} + \dots + C_n e^{m_n x}}$$

Case III :- Complex non repeated roots:-

$m_1 = \alpha + i\beta$ , $m_2 = \alpha - i\beta$

$$\boxed{y = e^{\alpha x} [C_1 \cos\beta x + C_2 \sin\beta x]}$$

Case IV :- Complex repeated roots :-

$m_1 = \alpha \pm i\beta$, $m_2 = \alpha \pm i\beta$ occur twice

$$\boxed{y = e^{\alpha x} [(C_1 x + C_2) \cos\beta x + (C_3 x + C_4) \sin\beta x]}$$

Examples

1) The Soln of D.E $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$ is

$\Rightarrow$ $D^2y - 5Dy + 6y = 0$

$(D^2 - 5D + 6)\, y = 0$

$\therefore D^2 - 5D + 6 = 0$ → Auxiliary eqn

$(D-2)(D-3) = 0$

$D = 2,\ D = 3$ → directly by calci also.

non repeated real roots

$\therefore$ Use case - I

$\therefore y = C_1 e^{2x} + C_2 e^{3x}$

2) The soln of D.E. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 6y = 0$ is

$\Rightarrow$ $D^2y - 5Dy - 6y = 0$

$(D^2 - 5D - 6)\, y = 0$

$D^2 - 5D - 6 = 0$ → Auxiliary eqn

$(D-6)(D+1) = 0$

$D = 6,\ D = -1$ → Directly by calci

non repeated real roots by case I

$y = c_1 e^{6x} + c_2 e^{-x}$

(3) The soln of D.E. $\frac{d^2y}{dx^2} - 4y = 0$

⇒ $D^2 y - 4y = 0$

$(D^2 - 4)\, y = 0$

$D^2 - 4 = 0$ → Auxiliary eqn

$D^2 = 4$

$\therefore D = \pm 2$ → by calci also

$y = c_1 e^{2x} + c_2 e^{-2x}$ → by case I

(4) The soln of D.E $2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 10y = 0$

⇒ $2D^2 y - Dy - 10y = 0$

$(2D^2 - D - 10)\, y = 0$

$2D^2 - D - 10 = 0$ → Auxiliary eqn.

$2D^2 + 4D - 5D - 10 = 0$

$2D(D+2) - 5(D+2) = 0$

$(D+2)(2D-5) = 0$

$\therefore D + 2 = 0$ $\quad$ $2D - 5 = 0$

$\therefore D = -2$ $\quad$ $D = 5/2$ → by calci also

$y = c_1 e^{-2x} + c_2 e^{5/2 x}$ → case I

(5) The soln of D.E $2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 3y = 0$ is

⇒ $2D^2 y - 5Dy + 3y = 0$

$(2D^2-5D+3)\,y=0$

$2D^2-5D+3=0$ → Auxiliary eqⁿ

$2D^2-2D-3D+3=0$

$2D(D-1)-3(D-1)=0$

$(D-1)(2D-3)=0$

$D=1,\quad D=3/2$ → by calci also

$\therefore y=C_1e^{x}+C_2e^{3/2\,x}$ → case I

(6) The soln of D.E $\frac{d^2y}{dx^2}+\frac{2dy}{dx}+y=0$ is

⇒ $D^2+2D+1=0$ → Auxiliary eqⁿ

$(D+1)^2=0$

$(D+1)(D+1)=0$

$D=-1,\quad D=-1$ Repeated ~~roots~~ repeated roots

$y=(C_1x+C_2)e^{-x}$ → case II

(7) The soln of D.E $4\frac{d^2y}{dx^2}-4\frac{dy}{dx}+y=0$ is

⇒ $4D^2-4D+1=0$ → Auxiliary eqⁿ

$(2D-1)^2=0$

$\therefore D=\frac{1}{2},\quad D=\frac{1}{2}$ → Repeated roots

$\therefore y=(C_1x+C_2)e^{\frac{1}{2}x}$ → case II

(8) The soln of D.E $\frac{d^2y}{dx^2}+y=0$

$D^2+1=0$ → Auxiliary eqn

$D^2=-1$

$\therefore D=\pm i$ → by calci also

Compare with $D=\alpha\pm i\beta$ complex non repeated roots

$\therefore \alpha=0$ = Real part (without $i$)

$\beta=1$ = Imaginary Part (with $i$)

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Case III $\Rightarrow$ $y = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$

$\therefore y = e^{0x}(C_1 \cos x + C_2 \sin x)$

$= C_1 \cos x + C_2 \sin x$ $\quad (e^{0x} = e^0 = 1)$

(9) The Soln of D.E $\frac{d^2y}{dx^2} + 9y = 0$ is

$\Rightarrow$ $D^2 + 9 = 0$

$D^2 = -9$

$D = \pm 3i$ $\rightarrow$ Complex non repeated roots

$\alpha = 0, \ \beta = 3$

$\therefore y = e^{0x}(C_1 \cos 3x + C_2 \sin 3x)$ $\rightarrow$ Case III

$= C_1 \cos 3x + C_2 \sin 3x$ $\quad (e^{0x} = e^0 = 1)$

(10) The Soln of D.E $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$ is

$D^2 + D + 1 = 0$ $\rightarrow$ Auxiliary eqn.

$a = 1, \ b = 1, \ c = 1$

$D = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2}$

$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ $\quad (\sqrt{-1} = i)$ $\rightarrow$ Directly by calci also

$\longrightarrow$ Complex non repeated roots

$\therefore \alpha = -\frac{1}{2}, \ \beta = \frac{\sqrt{3}}{2}$

$\therefore y = e^{-\frac{1}{2}x}\left(C_1 \cos \frac{\sqrt{3}}{2} x + C_2 \sin \frac{\sqrt{3}}{2} x\right)$ $\rightarrow$ Case III

(11) The Soln of D.E $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$ is

$\Rightarrow$ $4D^2 + 4D + 5 = 0$ $\rightarrow$ Auxiliary eqn

$a = 4, \ b = 4, \ c = 5$

$D = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{16 - (4)(4)(5)}}{2(4)} = \frac{-4 \pm 8i}{8} = -\frac{4}{8} \pm \frac{8i}{8}$

$= -\frac{1}{2} \pm i$ $\rightarrow$ directly by calci also

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$\therefore \alpha = \frac{1}{2}$ , $\beta = 1$

$\therefore y = e^{-\frac{1}{2}x}[c_1 \cos x + c_2 \sin x]$ → by case III

12] The $\text{Sol}^n$ D.E $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0$ is

⇒ $D^3 + 6D^2 + 11D + 6 = 0$ → Auxiliary $\text{eq}^n$

| | | | | |
|---|---|---|---|---|
| $-1$ | 1 | 6 | 11 | 6 |
| | | $-1$ | $-5$ | $-6$ |
| | 1 | 5 | 6 | $\boxed{0}$ |

$\therefore D = -1$ , $D^2 + 5D + 6 = 0$

$(D+2)(D+3) = 0$

$D = -2$ , $D = -3$

$\therefore D = -1, -2, -3$ → Directly by calci also

$\therefore y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$ → Case I

13) The $\text{soln}$ of D.E $(D^3 - 4D)\,y = 0$

⇒ $D^3 - 4D = 0$ → Auxiliary $\text{eq}^n$

$D(D^2 - 4) = 0$

$D(D-2)(D+2) = 0$

$D = 0, 2, -2$ → by calci also

$y = c_1 e^{0x} + c_2 e^{2x} + c_3 e^{-2x}$ → case I

14] The $\text{soln}$ of D.E $\frac{d^3y}{dx^3} + y = 0$ is

⇒ $(D^3 + 1) = 0$ → Auxiliary $\text{eq}^n$.

$\underbrace{D = -1}_{\text{case I}}$ , $D = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ , $D = \frac{1}{2} - \frac{\sqrt{3}}{2}i$ → by calci

case III $(\alpha = \frac{1}{2}, \beta = \frac{\sqrt{3}}{2})$

$y = c_1 e^{-x} + e^{\frac{1}{2}x}[c_2 \cos \frac{\sqrt{3}}{2}x + c_3 \sin \frac{\sqrt{3}}{2}x]$

15) The soln of D.E $\frac{d^3y}{dx^3} + 3\frac{dy}{dx} = 0$ is

$\Rightarrow$ $D^3 + 3D = 0$ $\rightarrow$ Auxiliary eqn

$D(D^2+3) = 0$

$\therefore D = 0 \quad D^2 = -3$

$\therefore D = \pm\sqrt{3}\,i$

$\therefore \underset{\text{case I}}{D=0}, \quad D = \sqrt{3}\,i \quad D = -\sqrt{3}\,i$

Case III $(\alpha = 0, \beta = \sqrt{3})$

$\therefore y = C_1 e^{0x} + e^{0x}[C_2\cos\sqrt{3}x + C_3\sin\sqrt{3}x]$ $\quad [e^0 = 1]$

$= C_1 + C_2\cos\sqrt{3}x + C_3\sin\sqrt{3}x$

16) The soln of D.E. $(D^4+2D^2+1)y = 0$

$\Rightarrow$ $D^4 + 2D^2 + 1 = 0$ $\rightarrow$ Auxiliary eqn

put $D^2 = x$

$\therefore x^2 + 2x + 1 = 0$

$\therefore x = -1, -1$

$D^2 = -1, \quad D^2 = -1$

$D = \pm i \quad D = \pm i$ $\rightarrow$ Complex Repeated roots

$(\alpha = 0, \beta = 1)$

$\therefore y = e^{0x}[(C_1x+C_2)\cos x + (C_3x+C_4)\sin x]$ $\rightarrow$ case IV

$= (C_1x+C_2)\cos x + (C_3x+C_4)\sin x$

17) The soln of D.E. $\frac{d^6y}{dx^6} + 6\frac{d^4y}{dx^4} + 9\frac{d^2y}{dx^2} = 0$ is

$\Rightarrow$ $D^6 + 6D^4 + 9D^2 = 0$

put $D^2 = x$

$x^3 + 6x^2 + 9x = 0$

$x = 0, \quad x = -3, \quad x = -3$

$D^2 = 0 \quad D^2 = -3, \quad D^2 = -3$

$D = 0, 0$ (case II) $\quad \pm\sqrt{3}\,i \rightarrow$ case IV $\leftarrow \pm\sqrt{3}\,i$

$\therefore y = (C_1x+C_2)e^{0x} + e^{0x}[(C_3x+C_4)\cos\sqrt{3}x + (C_5x+C_6)\sin\sqrt{3}x]$

$\therefore y = (C_1x+C_2) + (C_3x+C_4)\cos\sqrt{3}x + (C_5x+C_6)\sin\sqrt{3}x$

The General sol$^n$ of the linear D.E $\phi(D)y = f(x)$

The general sol$^n$ of $\phi(D)y = f(x)$ is

$$y = y_c + y_p$$

$\therefore$ Complete sol$^n$ = C.F + P.I

= Complementary f$^n$ + Particular Integral

$y_c \rightarrow$ C.F i.e sol$^n$ of $\phi(D)y = 0$

$y_p \rightarrow y_p$ is f$^n$ of $x$ which satisfies the eq$^n$ $\phi(D)y = f(x)$

$$\phi(D)y_p = f(x)$$

$$\therefore \quad y_p = \frac{1}{\phi(D)} f(x) \rightarrow \text{P.I}$$

<u>Method I</u>:- shortcut method

<u>Case I</u>:- P.I when $f(x) = e^{ax}$, a is any constant

If $\phi(D)y = f(x) = e^{ax}$

put $\boxed{D = a}$

then $P.I = y_p = \frac{1}{\phi(a)} e^{ax}$ Provided $\phi(a) \neq 0$

Case of failure :- If $\phi(a) = 0$

then $y_p = x \frac{1}{\phi'(a)} e^{ax}$ provided $\phi'(a) \neq 0$

Case of failure :- If $\phi'(a) = 0$

then $y_p = x^2 \frac{1}{\phi''(a)} e^{ax}$ provided $\phi''(a) \neq 0$

(continue)

Imp Note :-

(i) If $f(x) = a^x$ then we use $a^x = e^{x \log a}$ by formula

Here put $\boxed{D = \log a}$

(ii) If $f(x) = a^{-x}$ then we use $a^{-x} = e^{x(-\log a)}$

put $\boxed{D = -\log a}$

(iii) $\dfrac{1}{(D-a)^r} e^{ax} = \dfrac{x^r}{r!} e^{ax}$

Examples

(i) Particular integral of D.E $\dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 6y = e^{2x}$ is

$\Rightarrow$ $(D^2 - 7D + 6)\, y = e^{2x}$

P.I $\rightarrow$ $y_p = \dfrac{1}{D^2 - 7D + 6} e^{2x}$ put $D = 2$

$= \dfrac{1}{2^2 - 7(2) + 6} e^{2x}$

$= \dfrac{e^{2x}}{-4}$

(ii) Particular integral of D.E $(D^2 - 5D + 6)\, y = 3e^{5x}$ is

$\Rightarrow$ P.I $y_p = \dfrac{1}{D^2 - 5D + 6} \cdot 3e^{5x}$

put $D = 5$

$= \dfrac{3 \cdot e^{5x}}{5^2 - 5(5) + 6}$

$= \dfrac{e^{5x}}{2}$

(iii) Particular integral of D.E $(D^2 - 9)\, y = e^{3x} + 1$ is

$\Rightarrow$ $y_p = \dfrac{1}{D^2 - 9} (e^{3x} + 1)$

$$= \frac{e^{3x}}{D^2-9} + \frac{1}{D^2-9}$$

Put $D=3$

$$= \frac{e^{3x}}{9-9} + \frac{e^{0x}}{D^2-9}$$

Case of failure — Put $D=0$

$$= \frac{x\,e^{3x}}{2D} + \frac{e^{0x}}{0^2-9}$$

Put $D=3$

$$= \frac{x \cdot e^{3x}}{2(3)} + \frac{1}{-9} \qquad (e^{0x} = e^0 = 1)$$

$$= \frac{x \cdot e^{3x}}{6} - \frac{1}{9}$$

iv) Particular integral D.E. $(D^2+4D+3)\,y = e^{-3x}$ is

P.I.

$$y_p = \frac{1}{D^2+4D+3}\, e^{-3x}$$

Put $D=-3$

$$= \frac{1}{(-3)^2+4(3)+3}\, e^{-3x}$$

$$= \frac{1}{0}\, e^{-3x} \rightarrow \text{case of failure}$$

$$= \frac{x \cdot e^{-3x}}{2D+4} \rightarrow \text{Process } \frac{x \cdot f(x)}{\phi'(D)}$$

Put $D=-3$

$$= \frac{x \cdot e^{-3x}}{2(-3)+4}$$

$$= \frac{x \cdot e^{-3x}}{-6+4}$$

$$= \frac{x \cdot e^{-3x}}{-2}$$

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v) P.I. of D.E. $(D-2)^3 y = e^{2x} + 3^x$ is

$\Rightarrow$ P.I $\longrightarrow$ $y_p = \dfrac{1}{(D-2)^3}(e^{2x}+3^x)$

$= \dfrac{1}{(D-2)^3} \cdot e^{2x} + \dfrac{1}{(D-2)^3} \cdot 3^x$

Put $D=2$ case of failure  Put $D = \log 3$

$= \dfrac{x \cdot e^{2x}}{3(D-2)^2} + \dfrac{1}{(\log 3 - 2)^3} 3^x$

Put $D=2$ case of failure

$= \dfrac{x^2 \cdot e^{2x}}{6(D-2)} + \dfrac{1}{(\log 3-2)^3} 3^x$

put $D=2$ again case of failure

$= \dfrac{x^2 \cdot e^{2x}}{6} + \dfrac{3^x}{(\log 3 - 2)^3}$

vi) P.I. of D.E $(D^5 - D)\, y = 12 e^x$ is

$\Rightarrow$ P.I $\longrightarrow$ $y_p = \dfrac{1}{D^5 - D} \cdot 12 e^x$

Put $D = 1$

$= \dfrac{1}{1^5 - 1} \cdot 12 e^x \rightarrow \dfrac{1}{0}$ case of failure

$= \dfrac{x}{5D^4 - D} \cdot 12 e^x$  Process $\dfrac{x \cdot f(x)}{\phi'(D)}$

Put $D = 1$

$= \dfrac{12 x \cdot e^x}{5(1)^4 - 1}$

$= \dfrac{12\, x \cdot e^x}{4}$

$= 3x \cdot e^x$

## Examples for theory

Q.1 Solve the following D.E.

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = e^{2x}$$

$\Rightarrow$ $D^2y - 7Dy + 6y = e^{2x}$

$(D^2 - 7D + 6)y = e^{2x}$

Consider

$D^2 - 7D + 6 = 0$ → Auxiliary eq$^n$

$D = 6, D = 1$ → by calci

$y_c = c_1e^{6x} + c_2e^{x}$ → (case I (non repeated Real roots))

For $y_p$ → $y_p = \frac{1}{D^2 - 7D + 6} \cdot e^{2x}$

Put $D = 2$

$= \frac{1}{2^2 - 7(2) + 6} \cdot e^{2x}$

$= \frac{e^{2x}}{-4}$

∴ Complete Sol$^n$ = $y = y_c + y_p$

$= c_1e^{6x} + c_2e^{x} - \frac{e^{2x}}{4}$

(2) $(D^3 - 5D^2 + 8D - 4)y = e^{2x} + 2e^{x} + 3e^{-x} + 2$

For $y_c$ →

$\Rightarrow$ $D^3 - 5D^2 + 8D - 4 = 0$ → Auxiliary

$D = 1, 2, 2$ → by calci

$y_c = c_1e^{x} + (c_2x + c_3)e^{2x}$ → non repeated case 1

Repeated real roots case II

For $y_p$ $y_p = \frac{1}{D^3 - 5D^2 + 8D - 4}(e^{2x} + 2e^{x} + 3e^{-x} + 2)$

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$$= \frac{e^{2x}}{D^3-5D^2+8D-4} + \frac{2e^{x}}{D^3-5D^2+8D-4} + \frac{3e^{-x}}{D^3-5D^2+8D-4} + \frac{2e^{0x}}{D^3-5D^2+8D-4}$$

Put $D=2$ case of failure

Put $D=1$ case of failure

Put $D=-1$

Put $D=0$

$$= \frac{x \cdot e^{2x}}{3D^2-10D+8} + \frac{2x \cdot e^{x}}{3D^2-10D+8} + \frac{3e^{-x}}{(-1)^3-5(-1)^2+8(-1)-4} + \frac{2e^{0x}}{0^3-5(0)+8(0)-4}$$

Put $D=2$

Put $D=1$

again case of failure

$$= \frac{x^2 \cdot e^{2x}}{6D-10} + \frac{2x\, e^{x}}{3(1)^2-10(1)+8} - \frac{3e^{-x}}{18} - \frac{2}{4}(1)$$

Put $D=2$

$$= \frac{x^2 \cdot e^{2x}}{6(2)-10} + \frac{2x\, e^{x}}{1} - \frac{e^{-x}}{6} - \frac{1}{2}$$

$$= \frac{x^2 \cdot e^{2x}}{2} + 2x e^{x} - \frac{e^{-x}}{6} - \frac{1}{2}$$

$\therefore y = y_c + y_p$

$$= c_1 e^{x} + (c_2 x + c_3)\, e^{2x} + \frac{x^2 e^{2x}}{2} + 2x e^{x} - \frac{e^{-x}}{6} - \frac{1}{2}$$

Case II] When $f(x) = \sin(ax+b)$ or $\cos(ax+b)$

$$\phi(D)\, y = f(x)$$

$$y = \frac{1}{\phi(D)} f(x)$$

Put $D^2 = -a^2$

if $D^2$ is not in $\phi(D)$ $f^n$ then adjust it by Simplification or Rationalisation

Case of failure

if $\phi(-a^2) = 0$

Then $y_p = x \cdot \frac{1}{\phi'(-a^2)} \cdot \sin(ax+b)$

if $\phi'(-a^2) = 0$

Then $y_p = x^2 \cdot \frac{1}{\phi''(-a^2)} \cdot \sin(ax+b)$

⋮ (continue)

Similarly for $\cos(ax+b)$

Examples (M.C.Q)

(1) Particular integral of D.E $(D^2-4D+4)y = \sin 2x$ is

$$y_p = \frac{1}{D^2-4D+4} \sin 2x$$

Put $D^2 = -a^2, = -2^2 = -4$

$$= \frac{1}{-4-4D+4} \cdot \sin 2x$$

$$= \frac{1}{-4D} \cdot \sin 2x$$

$$= -\frac{1}{4} \int \sin 2x \, dx \qquad \left| \; \frac{1}{D} = \int \leftarrow \text{Integration} \right.$$

$$= -\frac{1}{4}\left(-\frac{\cos 2x}{2}\right)$$

$$= \frac{\cos 2x}{8}$$

② Particular Integral of D.E $(D^3+D)y = \cos x$ is

$$\Rightarrow \quad y_p = \frac{1}{D^3+D}\cdot\cos x$$

put $D^2 = -a^2 = -1^2 = -1$

$$= \frac{1}{D[D^2+1]}\cdot\cos x$$

$$= \frac{1}{D[-1+1]}\cdot\cos x$$

case of failure

$$= x\,\frac{1}{3D^2+1}\cdot\cos x$$

$$= x\,\frac{1}{3(-1)+1}\cos x$$

$$= x\,\frac{1}{-2}\cos x$$

$$y_p = -\frac{x\cdot\cos x}{2}$$

③ Particular integral of D.E $(D^2+1)y = \sin x$ is

$$\Rightarrow \quad y_p = \frac{1}{D^2+1}\cdot\sin x$$

$D^2+1 = -1+1 = 0$

case of failure

$$= x\cdot\frac{1}{2D}\cdot\sin x$$

$$= \frac{x}{2}\int \sin x\,dx$$

$$= \frac{-x\cdot\cos x}{2}$$

④ Particular integral of D.E $(D^3+9D)y = \sin 3x$ is

$$y_p = \frac{1}{D^3+9D}\cdot\sin 3x \quad \text{. case of failure}$$

$$= x \cdot \frac{1}{3D^2+9} \cdot \sin 3x$$

$$= x \cdot \frac{1}{3(-9)+9} \sin 3x$$

$$= x \cdot \frac{\sin 3x}{-18}$$

(5) P.I. of $(D^4+10D^2+9)y = \sin 2x + \cos 4x$ is

=> $y_p = \frac{1}{D^4+10D^2+9}[\sin 2x + \cos 4x]$

$$= \frac{\sin 2x}{D^4+10D^2+9} + \frac{\cos 4x}{D^4+10D^2+9}$$

Put $D^2 = -2^2 = -4$ $\qquad D^2 = -4^2 = -16.$

$$= \frac{\sin 2x}{(D^2)^2+10D^2+9} + \frac{\cos 4x}{(D^2)^2+10D^2+9}$$

$$= \frac{\sin 2x}{(-4)^2+10(-4)+9} + \frac{\cos 4x}{(-16)^2+10(-16)+9}$$

$$= \frac{\sin 2x}{-15} + \frac{\cos 4x}{105}$$

(6) P.I. of D.E. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 10\sin x$ is

=> For $y_p$ $\quad (D^2-2D+5)y = 10\sin x$

$$y_p = \frac{1}{D^2-2D+5} 10\cdot\sin x \qquad \text{Put } D^2 = -1^2$$

$$= \frac{1}{-1-2D+5}\cdot 10\cdot\sin x = \frac{10\sin x}{4-2D} = \frac{10\sin x}{2(2-D)}$$

$$= \frac{5\cdot\sin x}{2-D}$$

$$= \frac{5\cdot\sin x}{2-D} \times \frac{2+D}{2+D} = \frac{5\cdot(2+D)\cdot\sin x}{4-D^2}$$

$$= \frac{5[2\cdot\sin x + D\sin x]}{4-(-1)^2} = \frac{\cancel{5}[2\sin x+\cos x]}{\cancel{5}} = 2\sin x + \cos x$$

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Case III :- P.I when $f(x) = \cosh(ax+b)$ or $\sinh(ax+b)$.

To find $y_p = \frac{1}{\phi(D^2)} \cosh(ax+b)$ or $\frac{1}{\phi(D^2)} \sinh(ax+b)$

Same process as previous But Here put $D^2 = a^2$

Examples (M.C.Q's)

① Particular integral of D.E. $\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = 2\cosh x$ is

$\Rightarrow$ $(D^3 - 4D)y = 2\cosh x$

$$y_p = \frac{1}{D^3 - 4D} 2\cosh x$$

$$= \frac{1}{D^2 \cdot D - 4D} \cdot 2\cosh x$$

Put $D^2 = 1$

$$= \frac{1}{D - 4D} 2 \cdot \cosh x$$

$$= \frac{1}{-3D} \cdot 2\cosh x$$

$$= -\frac{1}{3} \int 2 \cdot \cosh x$$

$$= -\frac{2}{3} \cdot \sinh x$$

② Particular integral of D.E $(D^2 + 6D - 9)y = \sinh 3x$ is

$\Rightarrow$ $y_p = \frac{1}{D^2 + 6D - 9} \cdot \sinh 3x$

Put $D^2 = 1$

$D^r = D^2 + 6D - 9$

$= \cancel{9} + 6D - \cancel{9}$

$$= \frac{1}{9 + 6D - 9} \sinh 3x$$

$$= \frac{1}{6D} \cdot \sinh 3x$$

$\frac{1}{D} = \int \sinh 3x \, dx = \frac{\cosh 3x}{3}$

$$= \frac{1}{6} \cdot \frac{\cosh 3x}{3}$$

$$= \frac{\cosh 3x}{18}$$

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Examples on (Complete $sol^n$) $y = y_c + y_p$

① $(D^3+D)y = \cos x$

Auxiliary $eq^n$ $D^3 + D = 0$

$D(D^2+1) = 0$

$D = 0, \quad D^2+1 = 0$

$D^2 = -1$

$\therefore D = \pm i$

$\therefore y_c = c_1 e^{0x} + e^{0x}[c_2 \cos x + c_3 \sin x]$

$= c_1 + c_2 \cos x + c_3 \sin x$

$y_p = \frac{1}{D^3+D} \cdot \cos x$ — put $D^2 = -1$, $D^3 = D^2 \cdot D = -D$

$= \frac{1}{-D+D} \cos x$ ⟶ case of failure.

$= \frac{x \cdot \cos x}{3D^2+1}$

$= \frac{x \cdot \cos x}{-3+1}$

$= \frac{x \cdot \cos x}{-2}$

$\therefore y = y_c + y_p = c_1 + c_2 \cos x + c_3 \sin x - \frac{x \cdot \cos x}{2}$

② Solve $(D^2+1) \cdot y = \sin x \cdot \sin 2x$.

$D^2+1 = 0, \quad D^2 = -1 \quad \therefore D = \pm i$

$\therefore y_c = e^{0x}(c_1 \cos x + c_2 \sin x) = c_1 \cos x + c_2 \sin x$.

$y_p = \frac{1}{D^2+1} \cdot [\sin 2x \cdot \sin x]$ ⟶ By. $\cos(A+B) - \cos(A-B) = -2\sin A \cdot \sin B$

$\frac{\cos(A-B) - \cos(A+B)}{2} = \sin A \cdot \sin B$.

$= \frac{1}{D^2+1}\left[\frac{\cos(x) - \cos(3x)}{2}\right]$

$= \frac{1}{2} \frac{1}{D^2+1} \cdot \cos x - \frac{1}{2} \frac{1}{D^2+1} \cdot \cos 3x$ — [$1^{st}$ term → case of failure, $2^{nd}$ term → Put $D^2 = -9$

$= \frac{x}{2} \frac{1}{2D} \cdot \cos x - \frac{1}{2} \frac{\cos 3x}{-9+1}$

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$= \frac{x}{4}\int \cos x\,dx + \frac{1}{16}\cos 3x$

$= \frac{x}{4}(\sin x) + \frac{1}{16}\cdot\cos 3x$

$\therefore y = y_c + y_p = C_1\cos x + C_2\cdot\sin x + \frac{x}{4}(\sin x) + \frac{1}{16}\cdot\cos 3x$

Note:- trigo $\times$ trigo $\xrightarrow{\text{convert into}}$ trigo $\pm$ trigo by trigo formulae
for hyperbolic trigo also.

Useful formulae for trigonometry

(1) $\sin^2 x = \frac{1-\cos 2x}{2}$

(2) $\cos^2 x = \frac{1+\cos 2x}{2}$

(3) $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$

(4) $\sin(A+B) - \sin(A-B) = 2\cos A\cdot\sin B$

(5) $\cos(A+B) + \cos(A-B) = 2\cos A\cos B$

(6) $\cos(A+B) - \cos(A-B) = -2\sin A\sin B$

Useful formulae for hyperbolic

$\sinh x = \frac{e^x - e^{-x}}{2}$ $\cosh x = \frac{e^x + e^{-x}}{2}$

$\sinh^2 x = \frac{\cosh 2x - 1}{2}$

$\cosh^2 x = \frac{\cosh 2x + 1}{2}$

$\sinh(A+B) + \sinh(A-B) = 2\sinh(A)\cosh(B)$

$\sinh(A+B) - \sinh(A-B) = 2\cosh(A)\cdot\sinh(B)$

$\cosh(A+B) + \cosh(A-B) = 2\cosh(A)\cosh(B)$

$\cosh(A+B) - \cosh(A-B) = 2\sinh(A)\cdot\sinh(B)$

$\sinh(x\pm y) = \sinh(x)\cdot\cosh(y) \pm \cosh(x)\sinh(y)$

$\cosh(x\pm y) = \cosh(x)\cdot\cosh(y) \pm \sinh(x)\sinh(y)$

$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

$\sin 2x = 2\sin x\cdot\cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

(3). $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = \sin^2 t$

(4) $(D^2 + 13D + 36)y = e^{-4x} + \sinh x$

(5) $(D^3 + 1)y = \sin(2x+3) + e^{-x} + 2^x$

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Case IV :- P.I when $f(x) = x^m$

If $y_p = \frac{1}{\phi(D)} x^m$

$= [\phi(D)]^{-1} x^m$ use above formulae.

$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \cdots$

$\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \cdots$

Process :-

① Always take constant term common from the $D^r$ & use above formula.

② If constant term is absent in $D^r$ then minimum power of D is taken common from $D^r$.

e.g. $\frac{1}{D^2+3D+3} x^m = \frac{1}{3\left[1 + \left(\frac{D^2+3D}{3}\right)\right]}$

$\frac{1}{D^2-2D-5} x^m = \frac{1}{-5\left[1 - \left[\frac{D^2-2D}{5}\right]\right]}$

$\frac{1}{D^3-3D^2+2D} x^m = \frac{1}{2D\left[1 + \left[\frac{D^2-3D}{2D}\right]\right]}$

Examples [M.C.Q's]

① Particular integral of D.E. $(D^3+3D^2-4)y = x^2$ is

$\Rightarrow y = \frac{1}{D^3+3D^2-4} x^2$

$= \frac{1}{-4\left[1 - \left(\frac{D^3+3D^2}{4}\right)\right]} x^2$

$= -\frac{1}{4}\left[1 + \frac{D^3+3D^2}{4} + \cdots\right] x^2$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

$Dx^2 = 2x$

$D^2 = 2$

$D^3 = 0$

neglecting $D^3, D^4 \cdots$

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$$= -\frac{1}{4}\left[x^2 + \frac{D^3x^2 + 3D^2 \cdot x^2}{4} + \cdots\right]$$

$$= -\frac{1}{4}\left[x^2 + \frac{0+3(2)}{4}\right]$$

$$= -\frac{1}{4}\left[x^2 + \frac{3}{2}\right]$$

$$= -\frac{1}{4}x^2 - \frac{3}{8}$$

(2) Particular integral of D.E. $(D^2-1)y = x^3$ is ✓

$\Rightarrow$

$$y_p = \frac{1}{D^2-1} \cdot x^3$$

$$= -\frac{1}{(1-D^2)} x^3$$

$$= -1\,[1-D^2]^{-1} \cdot x^3$$

$$= -1\,[1 + D^2 + \cancel{D^4} + \cdots] \cdot x^3$$

$$= -1\,[x^3 + D^2 \cdot x^3]$$

$$= -1\,[x^3 + 6x]$$

$$= -x^3 - 6x$$

$(1-x)^{-1} = 1 + x + x^2 + \cdots$

$Dx^3 = 3x^2$

$D^2 = 6x$

$D^3 = 6$

$D^4 = 0$

neglecting $D^4, D^5 \cdots$

(3) Particular integral of D.E. $(D^2-D+1)y = 3x^2-1$ is

$$y_p = \frac{1}{D^2-D+1}(3x^2-1)$$

$$= \frac{1}{[1+(D^2-D)]}(3x^2-1)$$

$$= [1 + (D^2-D)]^{-1}(3x^2-1)$$

$$= [1 - (D^2-D) + (D^2-D)^2 + \cdots](3x^2-1)$$

$$= 1(3x^2-1) - (D^2-D)(3x^2-1) + (\cancel{D^4} - 2\cancel{D^3} + D^2)(3x^2-1)$$

$$= (3x^2-1) - [6-6x] + 6$$

$$= 3x^2 - 1 - \cancel{6} + 6x + \cancel{6}$$

$$= 3x^2 + 6x - 1$$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$

$D(3x^2-1) = 6x$

$D^2 = 6$

$D^3 = 0$

neglecting $D^3, D^4 \cdots$

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(4) $(D^4+D^2+1)y = 53x^2+17$ is.

$\Rightarrow$ $y_p = \frac{1}{D^4+D^2+1}[53x^2+17]$

$= \frac{1}{[1+(D^4+D^2)]}[53x^2+17]$

$= [1+(D^4+D^2)]^{-1}[53x^2+17]$

$D(53x^2+17) = 106x$
$D^2 = 106$
$D^3 = 0$.

$= [1-(\cancel{D^4}+D^2)+\cancel{(D^4+D^2)^2}+\cdots][53x^2+17]$

$= \cancel{[x^2+17 - D^2(53x^2+17)]}$

$= [(53x^2+17) - D^2(53x^2+17)]$

$= 53x^2+17-106$

$= 53x^2-89$

✓ (5) Particular integral of D.E. $\frac{d^3y}{dx^3}+8y = x^4+2x+1$ is.

$\Rightarrow$ $y_p = \frac{1}{(D^3+8)} x^4+2x+1$.

$= \frac{1}{8[1+\frac{D^3}{8}]}(x^4+2x+1)$

$= 8[1+\frac{D^3}{8}]^{-1}(x^4+2x+1)$

$D(x^4+2x+1) = 4x^3+2$
$D^2 = 12x^2$
$D^3 = 24x$
$D^4 = 24$.

$= 8[1 - \frac{D^3}{8} + \cancel{\left(\frac{D^3}{8}\right)^2} + \cdots](x^4+2x+1)$

$= 8[(x^4+2x+1) - \frac{D^3}{8}(x^4+2x+1)]$

$= 8[x^4+2x+1 - \frac{24x}{8}]$

$= 8[x^4+2x+1-3x]$

$= 8[x^4-x+1]$

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⑥ Particular integral of D.E $(D^4+25)y = x^4+x^2+1$ is

$\Rightarrow$ $$y = \frac{1}{D^4+25}[x^4+x^2+1]$$

$$= \frac{1}{25\left[1+\frac{D^4}{25}\right]}[x^4+x^2+1]$$

$$= \frac{1}{25}\left[1+\frac{D^4}{25}\right]^{-1}[x^4+x^2+1]$$

$$= \frac{1}{25}\left[1-\frac{D^4}{25}+\cancel{\left(\frac{D^4}{25}\right)^2}-\cdots\right][x^4+x^2+1]$$

$$= \frac{1}{25}\left[x^4+x^2+1-\frac{24}{25}\right]$$

$$= \frac{1}{25}\left[x^4+x^2+\frac{1}{25}\right]$$

$D = 4x^3+2x$
$D^2 = 12x^2+2$
$D^3 = 24x$
$D^4 = 24$
$D^5 = 0$
neglecting $D^5, D^6$ ....

Examples on Complete solⁿ

11) Solve $(D^2-2D+5)y = 25x^2$

$\Rightarrow$ $y = y_c + y_p$

for $y_c \Rightarrow D^2-2D+5=0$.

$D = 1 \pm 2i$

$\therefore y_c = e^x(C_1\cos 2x + C_2 \sin 2x)$

For $y_p \Rightarrow$ $$y_p = \frac{1}{D^2-2D+5}25x^2$$

$$= \frac{1}{\cancel{5}\left[1+\frac{D^2-2D}{5}\right]}\cancel{25}^{5}x^2$$

$$= \left[1+\frac{D^2-2D}{5}\right]^{-1}\cdot 5x^2$$

$$= \left[1-\frac{D^2-2D}{5}+\left(\frac{D^2-2D}{5}\right)^2\right]5x^2$$

$$= 5x^2 - \left(\frac{D^2-2D}{5}\right)5x^2 + \left(\frac{D^4-4D^3+4D^2}{25}\right)5x^2$$

$$= 5x^2 - \frac{10-20x}{5} + \frac{40}{25} = 5x^2-2+4x+\frac{8}{5} = 5x^2+4x-\frac{2}{5}$$

$D = 10x$.
$D^2 = 10$
$D^3 = 0$.

$(D^2-2D)^2$
$= D^4-4D^3+4D^2$

$\therefore\ y = y_c + y_p$

$= e^{x}(C_1\cos 2x + C_2\sin 2x) + 5x^2 + 4x - \frac{2}{5}$

(2) Solve $(D^2+5D+4)y = x^2+7x+9$

for $y_c \Rightarrow \quad D^2+5D+4=0$

$D = -4, -1$

$\therefore y_c = C_1 e^{-4x} + C_2 e^{-x}$

For $y_p \quad y_p = \frac{1}{D^2+5D+4}\, x^2+7x+9$

$= \frac{1}{4\left[1+\frac{D^2+5D}{4}\right]}\, x^2+7x+9$

$= \frac{1}{4}\left[1+\left(\frac{D^2+5D}{4}\right)\right]^{-1}(x^2+7x+9)$

$= \frac{1}{4}\left[1 - \frac{D^2+5D}{4} + \left(\frac{D^2+5D}{4}\right)^2 - \cdots\right](x^2+7x+9)$

$= \frac{1}{4}\left[(x^2+7x+9) - \frac{(D^2+5D)}{4}(x^2+7x+9) + \frac{\cancel{D^4}+\cancel{10D^3}+25D^2}{16}(x^2+7x+9)\right]$

$D = 2x+7$

$D^2 = 2$

$D^3 = 0$

$= \frac{1}{4}\left[x^2+7x+9 - \frac{2+5(2x+7)}{4} + \frac{25(2)}{16}\right]$

$= \frac{1}{4}\left[x^2+7x+9 - \frac{2+10x+35}{4} + \frac{25}{8}\right]$

$= \frac{1}{4}\left[x^2+7x+9 - \frac{10x+37}{4} + \frac{25}{8}\right]$

$= \frac{1}{4}\left[x^2+7x+9 - \frac{10x}{4} - \frac{37}{4} + \frac{25}{8}\right]$

$= \frac{1}{4}\left[x^2 + \frac{9x}{2} + \frac{23}{8}\right]$

$\therefore y = y_c+y_p = C_1 e^{-4x} + C_2 e^{-x} + \frac{1}{4}\left[x^2+\frac{9x}{2}+\frac{23}{8}\right]$

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(3) $(D^2-3D+2)y = x^2+\sin x$

$\Rightarrow$ For $y_c \Rightarrow$ $D^2-3D+2=0$

$D=2, \quad D=1$

$y_c = C_1 e^{2x} + C_2 e^{x}$

For $y_p \Rightarrow$ $y_p = \dfrac{1}{D^2-3D+2}(x^2+\sin x)$

$= \dfrac{x^2}{D^2-3D+2} + \dfrac{\sin x}{D^2-3D+2}$

$= \dfrac{x^2}{2\left[1+\dfrac{D^2-3D}{2}\right]} + \dfrac{\sin x}{-1-3D+2}$

$D = 2x$
$D^2 = 2$
$D^3 = 0$

$= \dfrac{1}{2}\left[1+\dfrac{D^2-3D}{2}\right]^{-1} x^2 + \dfrac{\sin x}{1-3D}$

$= \dfrac{1}{2}\left[1 - \dfrac{D^2-3D}{2} + \left(\dfrac{D^2-3D}{2}\right)^2\right]x^2 + \dfrac{1}{1-3D} \times \dfrac{1+3D}{1+3D}\sin x$

$= \dfrac{1}{2}\left[x^2 - \dfrac{2-3(2x)}{2} + \dfrac{(\cancel{D^4}-\cancel{6D^3}+9D^2)x^2}{4}\right] + \dfrac{(1+3D)}{1-9D^2}\sin x$

$= \dfrac{1}{2}\left[x^2 - \dfrac{2-6x}{2} + \dfrac{9\times 2}{4}\right] + \dfrac{(1+3D)\cdot\sin x}{1+9}$

$= \dfrac{1}{2}\left[x^2 - \dfrac{2}{2} + \dfrac{6x}{2} + \dfrac{9}{2}\right] + \dfrac{(1+3D)\sin x}{10}$

$= \dfrac{1}{2}\left[x^2+3x+\dfrac{7}{2}\right] + \dfrac{\sin x + 3\cos x}{10}$

$= \dfrac{1}{2}\left[x^2+3x+\dfrac{7}{2}\right] + \dfrac{1}{10}\sin x + \dfrac{3}{10}\cos x$

$\therefore y = y_c + y_p$

$= C_1 e^{2x} + C_2 e^{x} + \dfrac{1}{2}\left[x^2+3x+\dfrac{7}{2}\right] + \dfrac{1}{10}\sin x + \dfrac{3}{10}\cos x$

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(4) Solve $(D^2-1)y = e^x + x^3$ ✓

$\Rightarrow$ For $y_c$ $\quad D^2-1=0$

$D = 1, -1$

$\therefore y_c = C_1 e^x + C_2 e^{-x}$

For $y_p$ $\quad y_p = \frac{1}{D^2-1}(e^x + x^3)$

$= \frac{e^x}{D^2-1} + \frac{x^3}{D^2-1}$

case of failure

$= \frac{x e^x}{2D} + \frac{1}{-[1-D^2]} x^3$

$= \frac{x e^x}{2} - (1-D^2)^{-1} \cdot x^3$

$D = 3x^2$
$D^2 = 6x$
$D^3 = 6$
$D^4 = 0$

$= \frac{x e^x}{2} - [1 + D^2 + \cancel{D^4} + \dots] x^3$

$= \frac{x e^x}{2} - [x^3 + 6x]$

$\therefore y = y_c + y_p = C_1 e^x + C_2 e^{-x} + \frac{x e^x}{2} - x^3 - 6x$

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Case V  P.I. when $f(x) = e^{ax} \cdot V$  V is any $f^n$ of $x$.

$$\frac{1}{\phi(D)} e^{ax} \cdot V = e^{ax} \frac{1}{\phi(D+a)} V$$

$\boxed{D \xrightarrow{\text{replace}} D+a}$

(1) Particular integral of D.E. $(D^2-4D+4)y = e^{2x} \cdot x^4$ is

$\Rightarrow$ $y_p = \frac{1}{D^2-4D+4} e^{2x} \cdot x^4$

$= e^{2x} \frac{1}{(D+2)^2-4(D+2)+4} x^4$ → Replace $D \to D+2$

$= e^{2x} \frac{1}{D^2+4D+\cancel{4}-\cancel{4D}-\cancel{8}+\cancel{4}} x^4$

$= e^{2x} \frac{1}{D^2} x^4$

$= e^{2x} \cdot \frac{x^6}{30}$

$\frac{1}{D^2} = \int\int$

$\int x^4 dx = \frac{x^5}{5}$

$\int \frac{x^5}{5} = \frac{1}{5} \frac{x^6}{6} = \frac{x^6}{30}$

(2) Particular integral of D.E. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{-x} \cos x$ is

$\Rightarrow$ $D^2y + 2Dy + y = e^{-x} \cos x$

$(D^2+2D+1)\, y = e^{-x} \cdot \cos x$

$y = \frac{1}{D^2+2D+1} e^{-x} \cos x$

$= e^{-x} \frac{1}{(D-1)^2 + 2(D-1)+1} \cos x$ → Replace $D \to D-1$

$= e^{-x} \frac{1}{D^2 - \cancel{2D} + \cancel{1} + \cancel{2D} - \cancel{2} + \cancel{1}} \cos x$

$= e^{-x} \cdot \frac{1}{D^2} \cos x$

$= -e^{-x} \cdot \cos x$

$\frac{1}{D^2} = \int\int$

$\int \cos x\, dx = \sin x$

$\int \sin x\, dx = -\cos x$

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3) P.I. of D.E. $(D^2+6D+9)y = e^{-3x} \cdot x^{-3}$ is

$\Rightarrow$

$$y = \frac{1}{D^2+6D+9} \cdot e^{-3x} \cdot x^{-3}$$

$$= e^{-3x} \cdot \frac{1}{D^2+6D+9} \cdot x^{-3}$$ Replace $D \rightarrow D-3$

$$= e^{-3x} \cdot \frac{1}{(D-3)^2+6(D-3)+9} x^{-3}$$

$$= e^{-3x} \cdot \frac{1}{D^2-6D+\cancel{9}+6\cancel{D}-\cancel{18}+\cancel{9}} x^{-3}$$

$$= e^{-3x} \cdot \frac{1}{D^2} x^{-3}$$

$$= e^{-3x} \cdot \frac{1}{2x}$$

$\int x^{-3} = \frac{x^{-2}}{-2}$

$\int \frac{x^{-2}}{-2} = -\frac{1}{2} \frac{x^{-1}}{-1}$

$= \frac{1}{2} \cdot \frac{1}{x}$

4) P.I. of D.E. $(D^2+2D+1)y = e^{-x}(1+x^2)$ is

$\Rightarrow$

$$y = \frac{1}{D^2+2D+1} \cdot e^{-x}(1+x^2)$$

$$= \frac{e^{-x}}{D^2+2D+1} + \frac{e^{-x} \cdot x^2}{D^2+2D+1}$$

case of failure

$$= \frac{x e^{-x}}{2D+2} + e^{-x} \cdot \frac{1}{(D-1)^2+2(D-1)+1} x^2$$ $\rightarrow$ Replace $D \rightarrow D-1$

case of failure

$$= \frac{x^2 \cdot e^{-x}}{2} + e^{-x} \cdot \frac{1}{D^2-\cancel{2D}+\cancel{1}+\cancel{2D}-\cancel{2}+\cancel{1}} x^2$$

$$= \frac{x^2 \cdot e^{-x}}{2} + e^{-x} * \frac{x^4}{12}$$

$$= e^{-x}\left[\frac{x^2}{2} + \frac{x^4}{12}\right]$$

$\int x^2 = \frac{x^3}{3}$

$\int \frac{x^3}{3} = \frac{1}{3} \cdot \frac{x^4}{4}$

$= \frac{x^4}{12}$

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5) Particular integral $(D-1)^3 y = e^x \sqrt{x}$ is

$$\Rightarrow y = \frac{1}{(D-1)^3} \cdot e^x \sqrt{x}$$

$$= e^x \cdot \frac{1}{(D+1-1)^3} \sqrt{x}$$

$$= e^x \cdot \frac{1}{D^3} \sqrt{x}$$

$$= e^x \cdot \frac{8}{105} \cdot x^{7/2}$$

Explaination

$D^3 = \int\int\int$ / $\because \sqrt{x} = x^{1/2}$

$$\int x^{\frac{1}{2}} dx = \frac{x^{3/2}}{\frac{3}{2}} = \frac{2}{3} x^{3/2}$$

$$\int \frac{2}{3} x^{3/2} dx = \frac{2}{3} \cdot \frac{x^{5/2}}{5/2} = \frac{4}{15} x^{5/2}$$

$$\int \frac{4}{15} x^{5/2} dx = \frac{4}{15} \cdot \frac{x^{7/2}}{7/2} = \frac{8}{105} x^{7/2}$$

Complete soln.

1) Solve $(D^2-4)\, y = e^{3x} \cdot x^2$

$$y_p = \frac{1}{D^2-4} \cdot e^{3x} \cdot x^2$$

$$= e^{3x} \cdot \frac{1}{D^2-4} \cdot x^2$$

$$= e^{3x} \cdot \frac{1}{(D+3)^2-4} \, x^2$$

Replace $D \to D+3$

$$= e^{3x} \frac{1}{D^2+6D+9-4} \, x^2$$

$$= e^{3x} \cdot \frac{1}{D^2+6D+5} \cdot x^2$$

$$= e^{3x} \cdot \frac{1}{5\left(1+\left(\frac{D^2+6D}{5}\right)\right)} \, x^2$$

Explaination

$D = 2x$

$D^2 = 2$

$D^3 = 0$.

neglecting $D^3, D^4, \ldots$

$$= \frac{e^{3x}}{5} \left[1 - \left(\frac{D^2+6D}{5}\right) + \left(\frac{D^2+6D}{5}\right)^2 + \cdots\right] \cdot x^2$$

$$= \frac{e^{3x}}{5} \left[1 - \frac{D^2+6D}{5} + \frac{\cancel{D^4}+12\cancel{D^3}+36D^2}{25} + \cdots\right] x^2$$

$$= \frac{e^{3x}}{5} \left[x^2 - \frac{2+6(2x)}{5} + \frac{36(2)}{25}\right]$$

$$= \frac{e^{3x}}{5} \left[x^2 - \frac{2}{5} - \frac{12x}{5} + \frac{72}{25}\right] = \frac{e^{3x}}{5} \left[\frac{25x^2 - 60x + 62}{25}\right]$$

$$\frac{e^{3x}}{5} \left[x^2 - \frac{12x}{5} + \frac{62}{25}\right] \longrightarrow = \frac{e^{3x}}{125} \left[25x^2 - 60x + 62\right]$$

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$y_c \Rightarrow \quad D^2-4=0 \quad D=2,-2$

$y_c = c_1 e^{2x} + c_2 e^{-2x}$

$\therefore y = y_c + y_p$

$= c_1 e^{2x} + c_2 e^{-2x} + \frac{e^{3x}}{125}\left[25x^2-60x+62\right]$

② Solve $(D^2+2D+1)y = \frac{e^{-x}}{x+2}$

$y_c \Rightarrow \quad D^2+2D+1=0$

$D=-1,-1$

$y_c = (c_1x+c_2)e^{-x}$

$y_p \Rightarrow \quad y = \frac{1}{D^2+2D+1}\cdot\frac{e^{-x}}{x+2}$

$= e^{-x}\frac{1}{(D-1)^2+2(D-1)+1}\cdot\frac{1}{x+2}$

$= e^{-x}\frac{1}{D^2-\cancel{2D}+\cancel{1}+\cancel{2D}-\cancel{2}+\cancel{1}}\cdot\frac{1}{x+2}$

$= e^{-x}\frac{1}{D^2}\cdot\frac{1}{x+2}$

$= e^{-x}\left[x\log(x+2) - x + 2\log(x+2)\right]$

$\frac{1}{D^2} = \int\int$

$\int\frac{1}{x+2}dx$

$= \log(x+2)$

$\int\log(x+2)\,dx$

$= \log(x+2)\cdot x - \int\frac{x}{x+2}dx$

$= \log(x+2)\,x - \int\frac{x+2-2}{x+2}dx$

$= x\log(x+2) - \int 1 - \frac{2}{x+2}dx$

$= x\log(x+2) - x + 2\log(x+2)$

③ Solve $(D^2+6D+9)y = \frac{1}{x^3}\cdot e^{-3x}$

$y_c \Rightarrow \quad D^2+6D+9=0$

$D=-3,-3$

$y_c = (c_1x+c_2)e^{-3x}$

$y_p \Rightarrow \quad y_p = \frac{1}{D^2+6D+9}\cdot e^{-3x}\cdot x^{3}$

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$$= e^{-3x} \frac{1}{D^2-6D+9+6D-18+9} \cdot x^{-3}$$

$$= e^{-3x} \cdot \frac{1}{D^2} \cdot x^{-3}$$

$$= e^{-3x} \cdot \frac{1}{2} \cdot \frac{1}{x}$$

$$= \frac{e^{-3x}}{2x}$$

$$\int x^{-3} dx = \frac{x^{-3+1}}{-3+1} = \frac{x^{-2}}{-2}$$

$$\int \frac{x^{-2}}{-2} dx = -\frac{1}{2} \cdot \frac{x^{-2+1}}{-2+1} = -\frac{1}{2} \frac{x^{-1}}{-1}$$

$$\therefore y = y_c + y_p = (C_1 x + C_2) \cdot e^{-3x} + \frac{e^{-3x}}{2x}$$

Case VI :- $\frac{1}{\phi(D)} \cdot xV = \left[x - \frac{\phi'(D)}{\phi(D)}\right] \cdot \frac{1}{\phi(D)} \cdot V$

Examples

① Solve $\frac{d^2y}{dx^2} + 4y = x \cdot \sin x$

$\Rightarrow$ $D^2 y + 4y = x \cdot \sin x$

$(D^2+4) y = x \cdot \sin x$

For $y_p$ $\quad y = \frac{1}{D^2+4} x \cdot \sin x$

$\phi(D) = D^2+4$

$\phi'(D) = 2D$

$$= \left[x - \frac{2D}{D^2+4}\right] \frac{1}{D^2+4} \sin x$$

$\rightarrow$ Case II Put $D^2 = -1^2 = -1$

$$= \left[x - \frac{2D}{D^2+4}\right] \frac{1}{-1+4} \sin x$$

$$= \frac{1}{3}\left[x \cdot \sin x - \frac{2D \cdot \sin x}{D^2+4}\right]$$

$$= \frac{1}{3}\left[x \cdot \sin x - \frac{2D \cdot \sin x}{-1+4}\right]$$

$$= \frac{1}{3}\left[x \cdot \sin x - \frac{2D \sin x}{3}\right]$$

$$= \frac{1}{3}\left[x \cdot \sin x - \frac{2}{3} \cdot \cos x\right]$$

$$= \frac{1}{3} x \cdot \sin x - \frac{2}{9} \cdot \cos x$$

(2) Particular integral is $(D-1)^2 y = e^x \cdot x \cdot \sin x$.

$$y = \frac{1}{(D-1)^2} \cdot e^x \cdot x \cdot \sin x$$

$$= e^x \cdot \frac{1}{(D+1-1)^2} x \cdot \sin x$$

$$= e^x \frac{1}{D^2} x \cdot \sin x$$

$$= e^x \left[ x - \frac{2D}{D^2} \right] \frac{1}{D^2} \cdot \sin x \quad \text{Put } D^2 = -1$$

$$= e^x \left[ x - \frac{2D}{D^2} \right] \frac{1}{-1} \cdot \sin x$$

$$= -e^x \left[ x \cdot \sin x - \frac{2}{D} \cdot \sin x \right]$$

$$= -e^x \left[ x \cdot \sin x + 2 \cdot \cos x \right]$$

Solve

(3) $(D^2+2D+1) y = x e^{-x} \cos x$

$D^2 + 2D + 1 = 0$ Auxiliary eqn

$(D+1)^2 = 0$

$D = -1, -1$

$y_c = (c_1 x + c_2) e^{-x}$

$y_p \rightarrow \quad y_p = \dfrac{1}{D^2+2D+1} x e^{-x} \cos x$

$$= e^{-x} \frac{1}{(D-1)^2 + 2(D-1) + 1} x \cdot \cos x \quad \rightarrow \text{Replace } D \rightarrow D-1$$

Use case V $e^{ax} \cdot V$

$$= e^{-x} \frac{1}{D^2} \cdot x \cdot \cos x$$

Use case VI $x \cdot f(x) = x \cdot v$.

$$= e^{-x} \left[ x - \frac{2D}{D^2} \right] \frac{1}{D^2} \cos x$$

Use case II put $D^2 = -1$

$$= e^{-x} \left[ x - \frac{2}{D} \right] \frac{1}{-1} \cdot \cos x$$

$$= -e^{-x} \left[ x \cos x - \frac{2}{D} \cdot \cos x \right]$$

$\therefore y_p = -e^{-x} \left[ x \cos x - 2 \sin x \right]$

$\therefore y = y_c + y_p$

$= (c_1 x + c_2) e^{-x}$

$+ (-e^{x}) \left[ x \cos x - 2 \sin x \right)$

Method 2 :-

Method of variation of parameter

When shortcut method fails to determine particular Integral then we have general method but this method is very tedious (complicated) for integration. So that time use method of variation of parameter. Each D.E of order 2 can be solved by variation parameter.

Procedure

To solve D.E $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = X$ — ①

Step 1 :- First find Complementary $f^n$.

Write in the form $y = Ay_1 + By_2$ — A & B are constants like $c_1$ & $c_2$

Step 2 :- Find Wronskian W

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$y_1'$, $y_2'$ derivative of $y_1$ & $y_2$

Step 3 :- Calculate $u = \int \frac{-y_2 X}{W} dx$

& $v = \int \frac{y_1 X}{W} dx$

X from ①

$y_1, y_2$ from step 1

W from step 2

Step 4 :- Particular Integral

$P.I = uy_1 + vy_2$

Step 5 :- Complete $sol^n$

$\boxed{y = y_c + y_p}$ → Imp

Hint for W (i) If $y_1 = \cos ax$, $y_2 = \sin ax$ then $W = a$

(ii) If $y_1 = xe^{ax}$ & $y_2 = e^{ax}$ then $W = -e^{2ax}$

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## Examples (Theory)

① Solve $(D^2+1)\,y = \operatorname{cosec} x$ by variation of parameter.

⇒ $D^2+1=0$ , $D=\pm i$ $\therefore \alpha=0,\ \beta=1$

$\therefore y_c = e^{0x}(c_1\cos x + c_2\sin x)$

$\therefore y_c = c_1\cos x + c_2\sin x$ compare with $y = Ay_1 + By_2$

$\therefore y_1 = \cos x$ $\quad y_2 = \sin x$

$y_1' = -\sin x$ $\quad y_2' = \cos x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$u = \int \frac{-y_2 X}{W}\,dx = \int \frac{-\sin x \cdot \operatorname{cosec} x}{1}\,dx$$

$$= \int -\sin x \cdot \frac{1}{\sin x}\,dx \qquad \left[\because \operatorname{cosec} x = \frac{1}{\sin x}\right.$$

$$= \int -1\,dx$$

$$u = -x$$

$$V = \int \frac{y_1 X}{W}\,dx = \int \frac{\cos x \cdot \operatorname{cosec} x}{1}\,dx$$

$$= \int \cos x \cdot \frac{1}{\sin x}\,dx \qquad \left[\because \frac{\cos x}{\sin x} = \cot x\right.$$

$$= \int \cot x\,dx$$

$$V = \log(\sin x)$$

$\therefore$ P.I. $= y = uy_1 + vy_2 = -x\cdot\cos x + \log(\sin x)\cdot\sin x$

Complete solⁿ $\therefore y = y_c + y_p$ (Put $y_c$ & $y_p$ here)

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② Solve by Method of variation of parameters

$$(D^2-6D+9)\,y = \frac{e^{3x}}{x^2}$$

$\Rightarrow$ $D^2-6D+9=0$ $\rightarrow$ Auxilliary eqn

$D = 3, 3$ $\rightarrow$ by calci

$\therefore y_c = (C_1x + C_2)\,e^{3x}$

$= C_1 x\cdot e^{3x} + C_2 e^{3x}$ Compare with $y = Ay_1 + By_2$

$\therefore y = x\cdot e^{3x}$ $\quad y_2 = e^{3x}$.

u.v Rule $\quad y_1' = x\cdot e^{3x}\cdot 3 + e^{3x}$ $\quad y_2' = e^{3x}\cdot 3$

$= (3x+1)\,e^{3x}$ $\quad = 3\cdot e^{3x}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x\cdot e^{3x} & e^{3x} \\ (3x+1)e^{3x} & 3\cdot e^{3x} \end{vmatrix}$$

$= 3x\cdot e^{3x}\cdot e^{3x} - e^{3x}(3x+1)e^{3x}$ $\quad (\because a^m a^n = a^{m+n})$

$= 3x\cdot e^{6x} - (3x+1)\,e^{6x}$

$= \cancel{3xe^{6x}} - \cancel{3xe^{6x}} - e^{6x}$

$= -e^{6x}$

$$u = \int \frac{-y_2 X}{W}\,dx = \int \frac{-e^{3x}\cdot \frac{e^{3x}}{x^2}}{-e^{6x}}\,dx$$

$$= \int \frac{e^{6x}/x^2}{e^{6x}}\,dx$$

$$= \int \frac{1}{x^2}\,dx$$

$$= -\frac{1}{x}$$

$\because \int x^{-2}\,dx$

$= \frac{x^{-2+1}}{-2+1}$

$= \frac{x^{-1}}{-1}$

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$$v = \int \frac{y_1 X}{W} dx = \int \frac{\cancel{x} e^{3x} (e^{3x}/x^2)}{-e^{6x}} dx$$

$$= \int -\frac{\cancel{e^{6x}}/x}{\cancel{e^{6x}}} dx$$

$$= \int -\frac{1}{x} dx$$

$$= -\log x$$

$$P.I = y_p = u y_1 + v y_2$$

$$= -\frac{1}{\cancel{x}} (\cancel{x} \cdot e^{3x}) + (-\log x) e^{3x}$$

$$= -e^{3x} [1 + \log x]$$

$\therefore$ Complete soln $\quad y = y_c + y_p$

$$= (C_1 x + C_2) e^{3x} - e^{3x} (1 + \log x)$$

③ Solve $(D^2 - 4D + 4) y = e^{2x} \cdot \sec^2 x$ by variation of parameters

$\Rightarrow$ $D^2 - 4D + 4 = 0 \quad \rightarrow$ Auxiliary eqn

$D = 2, 2 \quad \rightarrow$ by calci

$\therefore y = (C_1 x + C_2) e^{2x}$

$= C_1 x \cdot e^{2x} + C_2 e^{2x}$ $\quad$ compare with $y = A y_1 + B y_2$

$\therefore \quad y_1 = x \cdot e^{2x} \qquad y_2 = e^{2x}$

$y_1' = x \cdot e^{2x} \cdot 2 + e^{2x} \qquad y_2' = e^{2x} \cdot 2$

$= e^{2x} (2x + 1) \qquad = 2 \cdot e^{2x}$

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$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} xe^{2x} & e^{2x} \\ e^{2x}(2x+1) & 2\cdot e^{2x} \end{vmatrix}$$

$$= 2x(e^{2x})^2 - (e^{2x})^2(2x+1)$$

$$= \cancel{2x\,e^{4x}} - \cancel{2xe^{4x}} - e^{4x} \qquad (e^m)^n = e^{mn}$$

$$W = -e^{4x}$$

$$u = \int \frac{-y_2 X}{W}\,dx = \int -\frac{\cancel{e^{2x}}\cdot \cancel{e^{2x}}\sec^2 x \cdot dx}{-\cancel{e^{4x}}}$$

$$= \int \sec^2 x\,dx$$

$$\therefore u = \tan x.$$

$$v = \int \frac{y_1 X}{W}\,dx = \int \frac{x\cdot \cancel{e^{2x}}\cdot \cancel{e^{2x}}\sec^2 x}{-\cancel{e^{4x}}}\,dx.$$

$$= -\int x\cdot \sec^2 x\cdot dx \quad \rightarrow \quad \int u\cdot v \cdot \text{Rule} \qquad LIATE$$

$$u\int v\,dx - \int\left[\int v\,dx \cdot \frac{du}{dx}\right]dx$$

$$= -\left[x\int \sec^2 x\,dx - \int\left[\int \sec^2 x\,dx \cdot \frac{dx}{dx}\right]dx\right]$$

$$= -x\cdot\tan x - \int \tan x\,dx$$

$$= -[x\tan x - \log(\sec x)] = -x\tan x + \log(\sec x)$$

$$\therefore P.I. = y_p = uy_1 + vy_2 = \tan x(x\cdot e^{2x}) + [-x\cdot\tan x + \log(\sec x)]e^{2x}$$

$$= \cancel{x\cdot e^{2x}\tan x} - \cancel{x\cdot e^{2x}\tan x} + \log(\sec x)\cdot e^{2x}$$

$$y_p = P.I. = \log(\sec x)\cdot e^{2x}$$

Complete soln $\boxed{y = y_c + y_p}$

$$y = (c_1 x + c_2)e^{2x} + \log(\sec x)\cdot e^{2x}$$

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Examples for M.C.Q's.

① In solving D.E. $\frac{d^2y}{dx^2} + 4y = \sec 2x$ by method of variation of

parameters. Complementary fn $= c_1 \cos 2x + c_2 \sin 2x$

particular Integral $y_p = u\cos 2x + v\sin 2x$ then $u$ is

=> We know $u = \int \frac{-y_2 X}{W} dx$.

$y_c = c_1 \cos 2x + c_2 \sin 2x$ compare with $y = Ay_1 + By_2$

$\therefore y_1 = \cos 2x \qquad y_2 = \sin 2x$

$\therefore y_1' = -2\sin 2x \qquad y_2' = 2\cos 2x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2\cdot\cos^2 2x + 2\cdot\sin^2 2x = 2(\cos^2 2x + \sin^2 2x) = 2(1)$$

$W = 2$

$$\therefore u = -\int \frac{\sin 2x \cdot \sec 2x}{2} dx = -\int \frac{\sin 2x \cdot \frac{1}{\cos 2x}}{2} dx$$

$$= -\int \frac{\tan 2x}{2} dx$$

$$= -\frac{1}{2}\log(\sec x)$$

$$= \frac{1}{2}[\log(\sec x)]^{-1} \quad \text{--- log Rule } n\log m = \log m^n$$

$$= \frac{1}{2}\log\left(\frac{1}{\sec x}\right) \quad \text{---} \quad x^{-1} = \frac{1}{x}$$

$$= \frac{1}{2}\log(\cos x) \quad \text{---} \quad \sec x = \frac{1}{\cos x}$$

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(2) In solving D.E. $\frac{d^2y}{dx^2} + 9y = \frac{1}{1+\sin 3x}$ by method of variation of parameters Complementary fn = $C_1 \cos 3x + C_2 \sin 3x$
Particular integral = $u \cdot \cos 3x + v \sin 3x$ then $v$ is equal to

$\Rightarrow$ we know $v = \int \frac{y_1 X}{W} dx$

$C.F = C_1 \cos 3x + C_2 \sin 3x$ compare with $y = Ay_1 + By_2$

$\therefore y_1 = \cos 3x$ $\quad y_2 = \sin 3x$

$y_1' = -3 \cdot \sin 3x$ $\quad y_2' = 3 \cdot \cos 3x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 3x & \sin 3x \\ -3 \cdot \sin 3x & 3 \cdot \cos 3x \end{vmatrix}$$

$$= 3 \cdot \cos^2 3x + 3 \sin^2 3x = 3 (\cos^2 3x + \sin^2 3x)$$
$$= 3(1) = 3$$

$$\therefore v = \int \frac{\cos 3x \cdot \frac{1}{1+\sin 3x}}{3} dx$$

$$= \frac{1}{3} \int \frac{\cos 3x}{1+\sin 3x} dx \qquad --- \int \frac{f'(x)}{f(x)} dx = \log |f(x)| + c$$

$$= \frac{1}{3} \cdot \frac{1}{3} \int \frac{3 \cdot \cos 3x}{1+\sin 3x} dx \qquad --- \frac{d(1+\sin 3x)}{dx} = 0 + 3 \cdot \cos 3x = 3 \cos 3x$$

$$= \frac{1}{9} \log (1+\sin 3x)$$

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③ In Solving D.E. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \sin e^x$ by method of variation of parameters, complementary fn $= c_1 e^{-x} + c_2 e^{-2x}$, particular Integral $= u e^{-x} + v e^{-2x}$ then $u$ is equal to

$\Rightarrow$ $(D^2+3D+2)y = \sin e^x$.

Auxiliary eqn $D^2+3D+2 = 0$

Given C.F.

$y = c_1 e^{-x} + c_2 e^{-2x}$ Compare with $y = Ay_1 + By_2$

$\therefore y_1 = e^{-x}$ $\quad y_2 = e^{-2x}$

$y_1' = -e^{-x}$ $\quad y_2' = -2e^{-2x}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}$$

$= -2e^{-x}\cdot e^{-2x} - (-e^{-x})\cdot e^{-2x}$ $\quad \because a^m a^n = a^{m+n}$

$= -2e^{-3x} + e^{-3x}$

$W = -e^{-3x}$

$$u = \int \frac{-y_2 X}{W}\,dx = \int -\frac{e^{-2x}\cdot \sin e^x}{-e^{-3x}}\,dx$$ $\quad \frac{a^m}{a^n} = a^{m-n}$

$\therefore \frac{e^{-2x}}{e^{-3x}} = e^{-2x+3x}$

$= e^x$

$= \int e^x \cdot \sin e^x \cdot dx$

Put $e^x = t$.

$\therefore e^x dx = dt$.

$= \int \sin t \cdot dt$

$= -\cos t = -\cos e^x$

For theory

(4) Solve by variation of parameter

$$\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=e^{e^x}$$

$$(D^2+3D+2)y=e^{e^x}$$

$\Rightarrow$ For $y_c$

$D^2+3D+2=0$

$(D+2)(D+1)=0$

$\therefore D=-2,\ D=-1$

$\therefore y_c=C_1e^{-2x}+C_2e^{-x}$ Compare with $y=Ay_1+By_2$

$\therefore y_1=e^{-2x},\ y_2=e^{-x}$

$$\therefore W=\begin{vmatrix} y_1 & y_2\\ y_1' & y_2'\end{vmatrix}=\begin{vmatrix} e^{-2x} & e^{-x}\\ -2e^{-2x} & -e^{-x}\end{vmatrix}$$

$=-e^{-2x}\cdot e^{-x}-(-2)\cdot e^{-2x}\cdot e^{-x}$

$=-e^{-3x}+2e^{-3x}$

$=e^{-3x}$

$$u=\int\frac{-y_2X}{W}dx=\int\frac{-e^{-x}e^{e^x}}{e^{-3x}}dx=-\int e^{2x}\cdot e^{e^x}\cdot dx$$

$\frac{a^m}{a^n}=a^{m-n}$

$\therefore \frac{e^{-x}}{e^{-3x}}=e^{-x+3x}=e^{2x}$

Put $t=e^x$, $dt=e^x\cdot dx$

$=-\int e^x\cdot e^x\cdot e^{e^x}dx$

$=-\int t\cdot e^t\cdot dt$ by parts $\quad -\int uv\,dx=u\int v\,dx-\int\left[\int v\,dx\cdot\frac{du}{dx}\right]dx$

$=-\left[t\cdot\int e^t dt-\int\left[\int e^t dt\cdot\frac{dt}{dt}\right]dt\right]$

$=-\left[t\cdot e^t-\int e^t\cdot dt\right]$

$=-\left[te^t-e^t\right]$

$=-e^xe^{e^x}+e^{e^x}$

(B) $$v=\int\frac{y_1X}{W}dx=\int\frac{e^{-2x}\cdot e^{e^x}}{e^{-3x}}dx=\int e^x\cdot e^{e^x}dx$$

Put $t=e^x$, $dt=e^xdx$

$v=\int e^t dt=e^t=e^{e^x}$ $\Rightarrow y_p=uy_1+vy_2=(-e^xe^{e^x}+e^{e^x})e^{-2x}+e^{e^x}(e^{-x})$

$\therefore y=y_c+y_p=C_1e^{-2x}+C_2e^{-x}+e^{-2x}(-e^xe^{e^x})+e^{-x}\cdot e^{e^x}$

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For theory

⑤ $\frac{d^2y}{dx^2} + 4y = \tan 2x$ variation of parameter

$\Rightarrow$ $D^2+4 = 0$ $D^2 = -4$, $D = \pm 2i$

$\therefore y_c = e^{0x}(c_1 \cos 2x + c_2 \sin 2x)$

$= c_1 \cos 2x + c_2 \sin 2x$ compare with $y = Ay_1 + By_2$

$\therefore y_1 = \cos 2x$, $y_2 = \sin 2x$

$y_1' = -2\sin 2x$ $y_2' = 2\cos 2x$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$ (shortcut)

$u = \int \frac{-y_2 X}{W} dx = \int -\frac{\sin 2x \cdot \tan 2x}{2} dx = \frac{-1}{2}\int \sin 2x \cdot \frac{\sin 2x}{\cos 2x} dx$

$= -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x} dx = -\frac{1}{2}\int \frac{1-\cos^2 2x}{\cos 2x} dx$ $(\because \sin^2\theta + \cos^2\theta = 1)$

$= -\frac{1}{2}\int \frac{1}{\cos 2x} - \frac{\cos^2 2x}{\cos 2x} dx = -\frac{1}{2}\int \sec 2x - \cos 2x \, dx$

$= -\frac{1}{2} \frac{\log|\sec 2x + \tan 2x|}{2} + \frac{1}{2}\frac{\sin 2x}{2}$

$= -\frac{1}{4}\log|\sec 2x + \tan 2x| + \frac{1}{4}\sin 2x$

$v = \int \frac{y_1 X}{W} dx = \int \frac{\cos 2x \cdot \tan 2x}{2} dx = \frac{1}{2}\int \cancel{\cos 2x} \cdot \frac{\sin 2x}{\cancel{\cos 2x}} dx$

$= \frac{1}{2}\int \sin 2x \, dx = \frac{1}{2}\left(-\frac{\cos 2x}{2}\right) = \frac{-\cos 2x}{4}$

$\therefore y_p = uy_1 + vy_2$

$= \left[-\frac{1}{4}\log|\sec 2x + \tan 2x| + \frac{1}{4}\sin 2x\right]\cos 2x$

$+ \left(-\frac{\cos 2x}{4}\right)\sin 2x$

$\therefore y = y_c + y_p$

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III] General Method :-

Particular Integral

$$y_p = P.I = \frac{1}{D-m} f(x) = e^{mx} \int e^{-mx} f(x)\, dx$$

Similarly $$= \frac{1}{D+m} f(x) = e^{-mx} \int e^{mx} f(x)\, dx.$$

(1) ⇒ Solve $\frac{d^2y}{dx^2} + \frac{3dy}{dx} + 2y = e^{e^x}$

⇒ $(D^2+3D+2)\, y = e^{e^x}$

$\therefore\ D^2+3D+2=0$

$D = -1, -2$

$y_c = c_1 e^{-x} + c_2 e^{-2x}$

$$P.I = y_p = \frac{1}{(D+2)(D+1)} \cdot e^{e^x}$$

$$= \frac{1}{D+2}\left[\frac{1}{D+1} \cdot e^{e^x}\right]$$

$$= \frac{1}{D+2}\left[e^{-x} \int e^{x} . e^{e^x} dx\right]$$ | Put $e^x = t$, $e^x dx = dt$

$$= \frac{1}{D+2}\left[e^{-x} \int e^{t} dt\right]$$

$\int e^t dt = e^t$

$$= \frac{1}{D+2}\left[e^{-x} . e^{e^x}\right]$$

$$= e^{-2x} \int e^{+2x} . e^{-x} . e^{e^x} . dx$$

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$= e^{-2x} \int e^{x} . e^{e^{x}} dx$ put $e^{x} = t$

$= e^{-2x} \int e^{t} . dt$ $e^{x} dx = dt$

$P.I. = e^{-2x} . e^{t} = e^{-2x} . e^{e^{x}}$

$\therefore y = y_c + y_p$

$= C_1 e^{-x} + C_2 e^{-2x} + e^{-2x} . e^{e^{x}}$

(2) Solve $(D^2+3D+2) y = \sin e^{x}$

$D^2+3D+2 = 0$

$(D+2)(D+1) = 0$

$D = -2 , \quad D = -1$

$C.F = C_1 e^{-2x} + C_2 e^{-x}$.

$P I = \frac{1}{(D+2)(D+1)} \cdot \sin e^{x}$

$= \frac{1}{D+2} \cdot e^{-x} . \int e^{+x} \sin e^{x} . dx$ put $e^{x} = t$

$e^{x} dx = dt$

$= \frac{1}{D+2} \cdot e^{-x} \int \sin t \cdot dt$

$= \frac{1}{D+2} \cdot e^{-x} . (-\cos t)$

$= \frac{1}{D+2} \cdot e^{-x} (-\cos e^{x})$

$= -e^{-2x} \int e^{2x} . e^{-x} . \cos e^{x} dx$

$= - e^{-2x} \int e^{x} . \cos e^{x} dx$ put $e^{x} = t$

$e^{x} dx = dt$.

$= - e^{-2x} \int \cos t \, dt$

$= -e^{-2x} . \sin t$

$y_p = -e^{-2x} . \sin e^{x}$

$y = y_c + y_p = C_1 e^{-2x} + C_2 e^{-x} - e^{-2x} . \sin e^{x}$

## Cauchy's OR Euler's Homogenous Linear D.E.

Cauchy's D.E.

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = F(x)$$

It can be reduced D.E with constant coefficients by putting

$x = e^z$ or $z = \log x$

$$\therefore x \frac{dy}{dx} = Dy$$

$$x^2 \frac{d^2y}{dx^2} = D(D-1)y$$

$$x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

⋮

(1) Solve $x^3 \frac{d^2y}{dx^2} + 3x^2 \frac{dy}{dx} + xy = \sin(\log x)$

Divide by $x$ for standard form

$$x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + y = \frac{\sin(\log x)}{x}$$

$\therefore$ By putting $x = e^z \quad \therefore z = \log x$

$$x\frac{dy}{dx} = Dy, \quad x^2 \frac{d^2y}{dx^2} = D(D-1)y$$

$$\therefore D(D-1)y + 3Dy + y = \frac{\sin z}{e^z}$$

$$(D^2 - D + 3D + 1)y = e^{-z}\sin z$$

$$(D^2 + 2D + 1)y = e^{-z}\sin z$$

$$y = \frac{1}{(D+1)^2} e^{-z} \cdot \sin z$$

$$= e^{-z} \frac{1}{(D-1+1)^2} \cdot \sin z$$

$$= e^{-z} \frac{1}{D^2} \sin z$$

$$= e^{-z} \iint \sin z \, dz$$

$$= -e^{-z} \cdot \sin z$$

$$y_p = e^{-\log x} \cdot \sin(\log x)$$

For $y_c$ $\quad D^2 + 2D + 1 = 0 \quad \rightarrow$ Auxiliary eqn

$$D = -1, -1$$

$$\therefore y_c = (C_1 x + C_2) e^{-x}$$

$$\therefore y = y_c + y_p = (C_1 x + C_2) e^{-x} + e^{-\log x} \cdot \sin(\log x)$$

(2) Solve $\quad \dfrac{d^2 y}{dx^2} + \dfrac{1}{x} \dfrac{dy}{dx} = A + B \cdot \log x$

For standard form ~~div~~ multiply by $x^2$ on both sides

$$\therefore x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} = Ax^2 + Bx^2 \cdot \log x$$

By putting $\quad x = e^z, \quad z = \log x$

$$\therefore x \frac{dy}{dx} = Dy, \quad x^2 \frac{d^2 y}{dx^2} = D(D-1) y$$

$$D(D-1)y + Dy = A(e^z)^2 + B(e^z)^2 \cdot z$$

$$(D^2 - \cancel{D} + \cancel{D}) y = A e^{2z} + B e^{2z} \cdot z$$

$$D^2 y = A e^{2z} + B \cdot e^{2z} \cdot z$$

For $y_c$

$$D^2 = 0$$

$$\therefore D = 0, \ D = 0$$

$$\therefore y_c = (C_1 x + C_2) e^{0x}$$

$$= C_1 x + C_2$$

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For $y_p$

$$y = \frac{A \cdot e^{2z}}{D^2} + \frac{B \cdot e^{2z} \cdot z}{D^2}$$

Put $D = 2$     Use $x \cdot V$ case i.e. $\left[x - \frac{\phi'(D)}{\phi(D)}\right] \frac{1}{\phi(D)} \cdot V$

$$= \frac{A \cdot e^{2z}}{2^2} + B\left[z - \frac{2D}{D^2}\right] \frac{1}{D^2} \cdot e^{2z}$$

$$= \frac{A \cdot e^{2z}}{4} + B\left[z - \frac{2D}{D^2}\right] \frac{1}{2^2} \cdot e^{2z}$$

$$= \frac{A \cdot e^{2z}}{4} + B\left[z - \frac{2D}{D^2}\right] \frac{1}{4} \cdot e^{2z}$$

$$= \frac{A \cdot e^{2z}}{4} + \frac{B}{4}\left[z e^{2z} - \frac{2D}{D^{\cancel{2}}} e^{2z}\right]$$

$$= \frac{A \cdot e^{2z}}{4} + \frac{B}{4}\left[z \cdot e^{2z} - 2 \cdot \frac{1}{D} \cdot e^{2z}\right]$$

$$= \frac{A \cdot e^{2z}}{4} + \frac{B}{4}\left[z \cdot e^{2z} - \cancel{2} \cdot \frac{e^{2z}}{\cancel{2}}\right]$$

$$= \frac{A}{4} e^{2z} + \frac{B}{4}\left[z \cdot e^{2z} - e^{2z}\right]$$

$$y_p = \frac{A}{4} \cdot x^2 + \frac{B}{4} x^2 [\log x - 1]$$

$$\therefore y = y_c + y_p$$

$$= (c_1 x + c_2) + \frac{A}{4} x^2 + \frac{B}{4} x^2 [\log x - 1]$$

3) Solve $x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = x^5$ - (1)

⇒ Given eqn is in standard form

By putting $x = e^z$, $\therefore z = \log x$

$x\frac{dy}{dx} = Dy$, $x^2 \frac{d^2y}{dx^2} = D(D-1)y$

from (1)

$D(D-1)y - 4Dy + 6y = (e^z)^5$

$(D^2 - D - 4D + 6)y = e^{5z}$

$(D^2 - 5D + 6)y = e^{5z}$ - (2)

$D^2 - 5D + 6 = 0$ → Auxiliary eqn

$D = 3$, $D = 2$ → non repeated linear case I

$\therefore y_c = c_1 e^{3x} + c_2 e^{2x}$

$y_p$ → $y_p = \frac{1}{D^2 - 5D + 6} \cdot e^{5z}$ → from (2)

Put $D = 5$.

$= \frac{1}{5^2 - 5(5) + 6} \cdot e^{5z}$

$= \frac{e^{5z}}{6}$

$\therefore y = y_c + y_p$

$= c_1 e^{3x} + c_2 e^{2x} + \frac{e^{5z}}{6}$

$= c_1 e^{3x} + c_2 e^{2x} + e^{5\log x}$ Resubstitue in $x$ form

4) Solve $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 3y = \sin(\sqrt{3}\log x) + x^3$

⇒ Given eqn in standard form

Put $x = e^z$ $\therefore z = \log x$

$x\frac{dy}{dx} = Dy$, $x^2\frac{d^2y}{dx^2} = D(D-1)y$

$D(D-1)y - 3Dy + 3y = \sin(\sqrt{3}z) + (e^z)^3$

$(D^2 - D - 3D + 3)\,y = \sin\sqrt{3}z + e^{3z}$

$(D^2 - 4D + 3)\,y = \sin\sqrt{3}z + e^{3z}$

$D^2 - 4D + 3 = 0$ → Auxiliary eqn

$D = 3, \quad D = 1$

$\therefore\ y_c = C_1 e^{3x} + C_2 e^{x}$

$y_p \longrightarrow \quad y_p = \dfrac{1}{D^2 - 4D + 3}\left(\sin\sqrt{3}z + e^{3z}\right)$

$= \dfrac{1}{D^2 - 4D + 3}\cdot \sin\sqrt{3}z + \dfrac{1}{D^2 - 4D + 3}\, e^{3z}$

Put $D^2 = -a^2 = -(\sqrt{3})^2 = -3$, Put $D = 3$.

$= \dfrac{1}{-3 - 4D + 3}\sin\sqrt{3}z + \dfrac{1}{3^2 - 4(3) + 3}\, e^{3z}$

$= -\dfrac{1}{4D}\cdot \sin\sqrt{3}z + \dfrac{1}{0}\, e^{3z}$

case of failure

$= -\dfrac{1}{4}\int \sin\sqrt{3}z\, dz + \dfrac{z \cdot e^{3z}}{2D - 4}$

Put $D = 3$

$= -\dfrac{1}{4}\left(-\dfrac{\cos\sqrt{3}z}{\sqrt{3}}\right) + \dfrac{z \cdot e^{3z}}{2(3) - 4}$

$= \dfrac{\cos\sqrt{3}z}{4\sqrt{3}} + \dfrac{z \cdot e^{3z}}{2}$

## Legendre's Linear Equation

An equation of the type

$$a_0(ax+b)^n \frac{d^ny}{dx^n} + a_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_n y = F(x)$$

Such eqn can be reduced into linear D.E with constant coefficients by putting

$$ax+b = e^z \Rightarrow z = \log(ax+b)$$

$$(ax+b)\frac{dy}{dx} = a\,Dy$$

$$(ax+b)^2\frac{d^2y}{dx^2} = a^2 D(D-1)y$$

$$(ax+b)^3\frac{d^3y}{dx^3} = a^3 D(D-1)(D-2)y$$

$$\vdots$$

1. Solve $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin[\log(1+x)]$

$\Rightarrow$ Put $1+x = e^z \quad \therefore z = \log(1+x)$

$\therefore x = e^z - 1$

&

$$(1+x)\frac{dy}{dx} = Dy$$

$$(1+x)^2\frac{d^2y}{dx^2} = D(D-1)y$$

$\therefore$ eqn becomes $D(D-1)y + Dy + y = 2\sin(z)$

$$[D^2 - \cancel{D} + \cancel{D} + 1]\,y = 2\sin z$$

$$[D^2+1]\,y = 2\sin z$$

For $y_c$: $D^2+1 = 0$

$$D = \pm i$$

$$\therefore y_c = e^{0z}(c_1\cos z + c_2\sin z) = c_1\cos z + c_2\sin z$$

For $y_p$

$$y_p = \frac{1}{D^2+1} \cdot 2\sin z$$

$$= 2 \cdot z \cdot \frac{1}{2D} \cdot \sin z \quad \text{Case of failure}$$

$$= z \int \sin z \, dz$$

$$y_p = -z\cos z$$

$$\therefore y = y_c + y_p$$

$$= C_1 \cos z + C_2 \sin z - z\cos z$$

$$= C_1 \cos[\log(1+x)] + C_2 \sin[\log(1+x)] = -\log(1+x)\cos[\log(1+x)]$$

(2) Solve $(3x+2)^2 \frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1$

$\Rightarrow$ Put $3x+2 = e^z \quad \therefore x = \frac{e^z - 2}{3}$

$\therefore \quad z = \log(3x+2)$

$$(3x+2)\frac{dy}{dx} = 3Dy$$

$$(3x+2)^2 \frac{d^2y}{dx^2} = 3^2(D)(D-1)y = 9D(D-1)y$$

$$\therefore 9D(D-1)y + 3\cdot 3\cdot Dy - 36y = 3\left[\frac{e^z-2}{3}\right]^2 + 4\left[\frac{e^z-2}{3}\right] + 1$$

$$[9D^2 - \cancel{9D} + \cancel{9D} - 36]\,y = \frac{\cancel{3}\cdot[(e^z)^2 - 4e^z + 4]}{9} + \frac{4(e^z-2)}{3} + 1\times\frac{3}{3}$$

$$9[D^2-4]\,y = \frac{e^{2z} - \cancel{4e^z} + 4 + \cancel{4e^z} - 8 + 3}{3}$$

$$(D^2-4)y = \frac{e^{2z}-1}{9\times 3}$$

$$(D^2-4)y = \frac{e^{2z}-1}{27}$$

For $y_c$ $\quad D^2 - 4 = 0$

$D = \pm 2$

$\therefore y_c = C_1 e^{2x} + C_2 e^{-2x}$

For $y_p$ $\quad y = \frac{1}{D^2-4}\left[\frac{e^{2z}-1}{27}\right]$

$= \frac{1}{27}\left[\frac{1}{D^2-4} \cdot e^{2z} - \frac{1}{D^2-4} e^{0z}\right] \quad (\because e^0 = 1)$

case of failure

$= \frac{1}{27}\left[z \cdot \frac{1}{2D} \cdot e^{2z} - \frac{1}{0^2-4} \cdot e^{0z}\right]$

$= \frac{1}{27}\left[\frac{z}{2} \cdot \frac{e^{2z}}{2} + \frac{1}{4}\right] \quad \left(\frac{1}{D} = S\right)$

$= \frac{1}{27}\left[\frac{z \cdot e^{2z}}{4} + \frac{1}{4}\right]$

$\therefore y = y_c + y_p$

$= C_1 e^{2x} + C_2 e^{-2x} + \frac{1}{27}\left[\frac{z \cdot e^{2z}}{4} + \frac{1}{4}\right]$

# Simultaneous Linear D.E (only for M.C.Q's)

1. For the simultaneous linear D.E

$\frac{dx}{dt} + 2x - 3y = t, \quad \frac{dy}{dt} - 3x + 2y = e^{2t}$ soln of $x$ using

$D = \frac{d}{dx}$ is obtain from

$\Rightarrow$ Reduced form

$Dx + 2x - 3y = t \Rightarrow (D+2)x - 3y = t$ — ①

$Dy - 3x + 2y = e^{2t} \Rightarrow -3x + (D+2)y = e^{2t}$ — ②

We want soln of $x$ $\therefore$ cancel $y$ by making coefficients of $y$ same.

eqn(1) $\times (D+2)$ + eqn(2) $\times 3$

$$(D+2)^2 x - 3(D+2)y = (D+2)t$$
$$-9x + 3(D+2)y = 3e^{2t}$$

$$[(D+2)^2 x - 9x] = (D+2)t + 3e^{2t}$$
$$[D^2 + 4D + 4 - 9]x = D \cdot t + 2t + 3e^{2t}$$
$$[D^2 + 4D - 5]x = 1 + 2t + 3e^{2t}$$

② Previous example elimination of $x$ results

$\Rightarrow$ eqn(1) $\times 3$ + eqn(2) $\times (D+2)$

$$3\cancel{(D+2)x} - 9y = 3t$$
$$-3\cancel{(D+2)x} + (D+2)^2 y = (D+2)e^{2t}$$

$$[(D+2)^2 - 9]y = D \cdot e^{2t} + 2e^{2t} + 3t$$
$$[D^2 + 4D + 4 - 9]y = 2 \cdot e^{2t} + 2e^{2t} + 3t$$
$$[D^2 + 4D - 5]y = 4e^{2t} + 3t$$

③ For the simultaneous Linear DE $\frac{du}{dx} + v = \sin x$, $\frac{dv}{dx} + u = \cos x$ soln of u using $D \equiv \frac{d}{dx}$ is obtain from

⇒ Reduced form

$Du + v = \sin x$ — ①

$Dv + u = \cos x$ — ②

we want soln of u means cancel v by making coefficients v same

eqn (1) × D − eqn (2)

$D^2u + \cancel{Dv} = D \cdot \sin x$

$\underline{-}u \underline{+} \cancel{Dv} = \underline{-}\cos x$ [∵ $D\sin x = \cos x$]

$(D^2 - 1)u = \cos x - \cos x$

∴ $(D^2 - 1)u = 0$

④ For previous example eliminating u results

⇒ eqn (2) × D − eqn (1)

$\cancel{Du} + D^2v = D\cos x$

$\underline{-}\cancel{Du} \underline{\pm} v = \underline{-}\sin x$

$(D^2 - 1)v = -\sin x - \sin x$

$(D^2 - 1)v = -2\sin x$

## Symmetrical simultaneous DE (only for MCQ's)

* Eqn of the type $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ - (1)

$P, Q, R$ are $f^n s$ in $x$

Method of Multipliers

choose multipliers $l, m, n$ such that

$$\frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR} \quad \text{from (1)}$$

$lP + mQ + nR = 0 \rightarrow D^r$

then soln of D.E (1) is

$$\int l\,dx + \int m\,dy + \int n\,dz = c$$

(1) Soln of symmetric simultaneous DE $\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{1}$ is

(option in $xz$ & $yz$)

$\Rightarrow$ $\frac{dx}{1} = \frac{dz}{1}$ $\qquad \frac{dy}{1} = \frac{dz}{1}$

Integrating

$\int dx = \int dz$ $\qquad \int dy = \int dz$

$x = z + c_1$ $\qquad y = z + c_2$

$x - z = c_1$ $\qquad y - z = c_2$

(2) Soln of symmetric simultaneous DE $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$ is

$\Rightarrow$ $\frac{dx}{x} = \frac{dy}{y}$ $\qquad \frac{dy}{y} = \frac{dz}{z}$

$\log x = \log y + \log c_1$ $\qquad \log y = \log z + \log c_2$

$\log x - \log y = \log c_1$ $\qquad \log y - \log z = \log c_2$

$\log\left(\frac{x}{y}\right) = \log c$ $\qquad \log\left(\frac{y}{z}\right) = \log c_2$

$\frac{x}{y} = c_1 \Rightarrow x = c_1 y$ $\qquad \frac{y}{z} = c_2 \Rightarrow y = c_2 z$

③ Considering the first two ratio of the symmetrical simultaneous DE $\frac{dx}{y^2} = \frac{dy}{x^2} = \frac{dz}{x^2y^2z^2}$ one of the relation in the $sol^n$ is DE is

$$\Rightarrow \quad \frac{dx}{y^2} = \frac{dy}{x^2}$$

$x^2 dx = y^2 dy$ --- separating variables

$\int x^2 dx = \int y^2 dy$

$$\frac{x^3}{3} = \frac{y^3}{3} + c$$

$\cancel{x^2}x^3 = y^3 \cancel{+} 3c$ --- multiplying thr out by 3

$x^3 - y^3 = C$ $\rightarrow$ const of Integration

④ Considering the first two ratio of the symmetrical simultaneous DE $\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$, one of the relation in the $sol^n$ of DE is

$$\Rightarrow \quad \frac{dx}{y^2} = \frac{dy}{-xy}$$

$x\,dx = -\frac{y^2 dy}{y}$ separating variables

$x\,dx = -y\,dy$

$\int x\,dx = \int -y\,dy$

$$\frac{x^2}{2} = -\frac{y^2}{2} + c$$

$x^2 = -y^2 + 2c$ multiplying by 2

$x^2 + y^2 = C$ ... Constant of Integration

(5) Considering the first & third ratio of the symmetrical simultaneous DE $\frac{x\,dx}{y^3 z} = \frac{dy}{x^2 z} = \frac{dz}{y^3}$, one of the relation in the soln of D.E is

$\Rightarrow$

$$\frac{x\,dx}{\cancel{y^3} z} = \frac{dz}{\cancel{y^3}}$$

$$x\,dx = z\,dz$$

$$\int x\,dx = \int z\,dz$$

$$\frac{x^2}{2} = \frac{z^2}{2} + c$$

$$x^2 = z^2 + c$$

$$x^2 - z^2 = c$$

(6) Considering the second & third ratio of the symmetrical simultaneous DE $\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$, one of the relation in the soln of D.E is

$\Rightarrow$

$$\frac{dy}{\cancel{2}\cancel{x}y} = \frac{dz}{\cancel{2}\cancel{x}z}$$

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating

$$\log y = \log z + \log c$$

$$\log y - \log z = \log c$$

$$\log\left(\frac{y}{z}\right) = \log c$$

$$\frac{y}{z} = c$$

$$y = cz$$

(7) Using a set multiplier as 1, 1, 1 the $sol^n$ of DE

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y}$$

$\Rightarrow$ Here multipliers are given 1, 1, 1 $(l, m, n)$

$\therefore$ $sol^n$ of D.E is

$$\int l\,dx + \int m\,dy + \int n\,dz = c$$

$$\int 1\,dx + \int 1\,dy + \int 1\,dz = c$$

$$x + y + z = c$$

(8) Using a set multiplier as $x, y, z$ the $sol^n$ of DE

$$\frac{dx}{3z-4y} = \frac{dy}{4x-2z} = \frac{dz}{2y-3x} \text{ is}$$

$\Rightarrow$ multiplier's are $x, y, z$ $(l, m, n)$

$\therefore$ $sol^n$ of D.E is

$$\int l\,dx + \int m\,dy + \int n\,dz = c$$

$$\int x\,dx + \int y\,dy + \int z\,dz = c$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c$$

$$x^2 + y^2 + z^2 = c$$

(9) Using a set of multiplier as $1, y, z$ the $sol^n$ of DE

$$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{y+z} = \frac{dz}{y-z} \text{ is}$$

$\Rightarrow$ $\int 1\,dx + \int y\,dy + \int z\,dz = c$

$$x + \frac{y^2}{2} + \frac{z^2}{2} = c$$

$$2x + y^2 + z^2 = c$$

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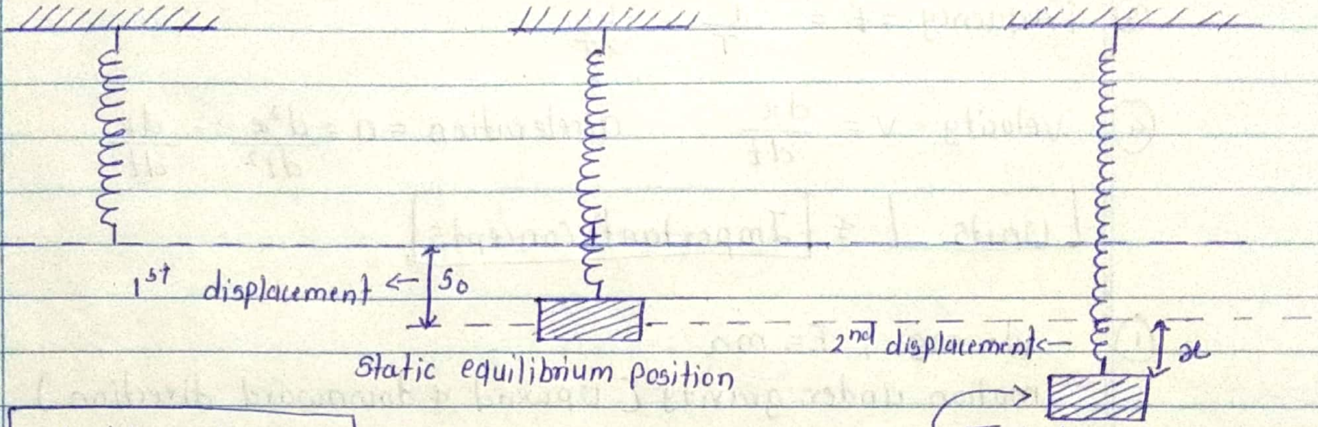
# Applications of Linear D.E

* Undamped system (only for Theory exam)

Very ideal system. Effect of external forces may be negligible




$t = 0$

$x$ = displacement

$\frac{dx}{dt} = V = 0$ (if not given)

① $\boxed{W = \text{Weight} = mg}$ - ①

$m$ = mass

$g$ = gravitational force

$= 9.8\ m/s^2$

$= 980\ cm/s^2$

② $\boxed{F = k\, S_0}$ - ②

$S_0$ = 1st displacement

$k$ = proportionality constant

By previous experiment (above) we get L.D.E.

$$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$$

--- find $k$ from ② & $m$ from ①

Putting $\omega^2 = \frac{k}{m}$ $\therefore \omega = \sqrt{\frac{k}{m}}$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

## Some Important formulae

1. Amplitude $= \sqrt{c_1^2 + c_2^2}$

2. Period $= T = \frac{2\pi}{\omega}$  $\because \omega = \sqrt{\frac{k}{m}}$  find k from (2) & m from (1)

3. Frequency $= f = \frac{1}{T} = \frac{\omega}{2\pi}$

4. velocity $= v = \frac{dx}{dt}$ , acceleration $= a = \frac{d^2x}{dt^2} = \frac{dv}{dt}$

## Units & Important concepts

1. $W = mg$ , $F = ma$
   motion under gravity ( upward & downward direction )
   $a = g$
   $\therefore \boxed{W = F}$  ( If no force is given )

2. If weight & pulled force are given
   then (a) W = weight  use to find m in eq$^n$(1)
   * unit of weight is Newton (N)
   (b) F = pulled force  use to find k in eq$^n$(2)
   * Unit of Force is also Newton (N)

3. $1\ m = 100\ cm$

4. For $c_1$ & $c_2$ find $x$ & $\frac{dx}{dt}$ eq$^n$.

## Examples

① A body of weight 2 N is suspended from a spring it 4 cm. If the weight is pulled down 8 cm below the equilibrium position & then released, find amplitude & period of motion.

⇒ Here Given Weight = Pulled force = 2 N

$\therefore W = 2N$

& $F = 2N$

Find m, from $W = mg$

$2 = m(9.8)$

$\therefore m = 2/9.8 = \frac{2}{9.8}$

Find k, from $F = kS_0$ $\quad S_0 = 1^{st}$ displacement = 4 cm only in m = 0.04 m

$2 = k(0.04)$

$\therefore k = \frac{2}{0.04}$

Now consider L.D.E.

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$\frac{d^2x}{dt^2} + \frac{\frac{\cancel{2}}{0.04}}{\frac{\cancel{2}}{9.8}} \cdot x = 0$$

$$\frac{d^2x}{dt^2} + \frac{9.8}{0.04}x = 0$$

$$\therefore \frac{d^2x}{dt^2} + 245x = 0$$

$$\therefore D^2x + 245x = 0$$

$$(D^2 + 245)x = 0$$

$\therefore D^2 + 245 = 0 \rightarrow$ Auxiliary eqn

$\therefore D = \pm\sqrt{245}\,i = \pm 7\sqrt{5}\,i$

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$\therefore x = e^{0t}(C_1 \cos 7\sqrt{5}t + C_2 \sin 7\sqrt{5}t)$

$x = C_1 \cos 7\sqrt{5}t + C_2 \sin 7\sqrt{5}t \quad -(1)$

Diff w.r.t. t:

$\frac{dx}{dt} = C_1(-\sin 7\sqrt{5}t) \cdot 7\sqrt{5} + C_2(\cos 7\sqrt{5}t) 7\sqrt{5} \quad -(2)$

Initially $t = 0$, $\frac{dx}{dt} = 0$ (not given)

& $x = 8\,cm = 0.08\,m$ (2nd displacement)

From eqn (1)

$0.08 = C_1 \cos 0 + C_2 \sin 0$ $\qquad \sin 0 = 0$

$\boxed{0.08 = C_1}$ $\qquad \cos 0 = 1$

From eqn (2)

$0 = C_1(-\sin 0) 7\sqrt{5} + C_2(\cos 0) 7\sqrt{5}$

$0 = 0 + C_2(7\sqrt{5})$

$\therefore \boxed{C_2 = 0}$

$\therefore$ From (1) $x = (0.08) \cos 7\sqrt{5}t + (0) \sin 7\sqrt{5}t$

$= 0.08 \cos(7\sqrt{5}t)$

* Amplitude $= \sqrt{C_1^2 + C_2^2} = \sqrt{(0.08)^2 + 0^2} = 0.08$

* Period $= \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{k}{m}}} = \frac{2\pi}{\sqrt{245}} = \frac{2\pi}{7\sqrt{5}}$

② A body of weight $W = 20\,N$ is hung from a spring a pull of 40 N will strech the spring to 10 cm. The body is pulled down to 20 cm below the static equilibrium position & then released. Find the displacement of the body from its equilibrium position in time 't' seconds. & the maximum velocity.

⇒ Given weight $= W = 20\,N$

Pulled force $= F = 40\,N$

Find m from $W = mg$

$$20 = m(9.8)$$

$$\therefore m = \frac{20}{9.8}$$

Find K from $F = K s_0$ $\qquad s_0 = 1^{st}$ displacement $= 10$ cm $= 0.1$ m

$$40 = K(0.1)$$

$$K = \frac{40}{0.1}$$

$$K = 400$$

Now consider L.D.E

$$\frac{d^2x}{dt^2} + \frac{K}{m}x = 0$$

$$\frac{d^2x}{dt^2} + \frac{400}{\left(\frac{20}{9.8}\right)}x = 0$$

$$\therefore \frac{d^2x}{dt^2} + 196x = 0$$

$$D^2x + 196x = 0$$

$$(D^2 + 196)x = 0$$

$$D^2 + 196 = 0 \rightarrow \text{Auxiliary eq}^n$$

$$\therefore D = \pm 14i$$

$$\therefore x = e^{0t}(C_1 \cos 14t + C_2 \sin 14t)$$

$$x = C_1 \cos 14t + C_2 \sin 14t \quad -①$$

Diff wrt $\cdot t$.

$$\frac{dx}{dt} = -C_1\,14 \cdot \sin 14t + C_2\,14 \cdot \cos 14t \quad -(2)$$

Initial conditions

$t = 0, \quad \frac{dx}{dt} = 0$ (not given)

& $x = 20\text{ cm} = 0.2\text{ m}$ ($2^{nd}$ displacement)

From eqn (1) $\quad 0.2 = C_1 \cos 0 + C_2 \sin 0$

$$\boxed{0.2 = C_1}$$

From eqn (2)

$$0 = -C_1\,14 \cdot \sin 0 + C_2\,14 \cdot \cos 0$$

$$0 = 0 + C_2(14)$$

$$\boxed{C_2 = 0}$$

* Displacement $= x = (0.2)\cos(14t) + (0)\cdot\sin(14t)$

$= (0.2)\cos 14t$

* velocity $= v = \frac{dx}{dt} = -(0.2)14\sin(14t) + (0)\,14\cdot\cos(14t)$

$= -(0.2)\,14 \cdot \sin(14t)$

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# Imp examples from University paper (for Theory only)

**Note** :- For Theory prepare shortcut method, variation of parameter method, Cauchy's and Legendre's L.D.E & Application of L.D.E

Examination :- December 2016 [8 marks]

Q.1 a. Solve any two

(i) $(D^3 - 7D - 6)y = e^{2x}(1+x)$

(ii) $(D^2+1)y = 3x - 8\cot x$ (by variation of parameter method)

(iii) $(2x+1)^2 \dfrac{d^2y}{dx^2} - 2(2x+1)\dfrac{dy}{dx} - 12y = 12x$

Examination :- November 2017 [8 M]

Q.1 a Solve any two

(i) $(D^2 - 4D + 3)y = x^3 e^{2x}$

(ii) $(D^2+4)y = \sec 2x$ (using method of variation of parameter)

(iii) $x^2 \dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 6y = x^5$

Examination :- May 2017 [8 M]

Q.1 a Solve any two of the following

(i) $(D^2 + 13D + 36)y = e^{-4x} + \sinh x$

(ii) $(D^2 - 2D + 2)y = e^x + \tan x$ (using variation of parameter)

(iii) $x^2 \dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 6y = x^5$

Examination :- December 2018 [8M]

Q.1 a Solve any two

(i) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = x^2 + x + 1$

(ii) $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 3y = \sin(\sqrt{3}\log x) + x^3$

(iii) $\dfrac{d^2y}{dx^2} + 4y = \tan 2x$ (variation of parameter)

Examination :- May 2018 [8M]

(i) $(D^2+2D+1)y = xe^{-x}.\cos x$

(ii) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$ (variation of parameter)

(iii) $(3x+2)^2\frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2+4x+1$

Examination :- May 2019 [8M]

(i) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = e^{-3x}\cos 4x + 6e^{2x}$

(ii) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 16y = x^2 + 2^{\log x} + 4\cosh(\log x)$

(iii) $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$ (variation of parameter)

Examination :- December 2016 [4m]

Q.2 a) A body weighing $W = 20$ N is hung from spring. A pull of 40 N will strech the spring 10 cm. The body is pulled down to 20 cm below the static equilibrium position and then released. Find the displacement of body, the maximum velocity and period.

Examination :- November 2017 [4m]

Q.2 a) A body of weight $W = 3$ N streches a spring of 15 cm. If the weight is pulled down 10 cm below the equilibrium position & given a downward velocity 60 cm/sec, determine the amplitude, period and frequency