Bandit Algorithms

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Outline

1. From Contextual to Linear Bandits
2. Stochastic Linear Bandits
3. Confidence Bounds for Least-Squares Estimators
4. Improved Regret for Fixed, Finite Action Sets

Outline

Stochastic Contextual Bandits

Set of contexts, $\mathcal{C}$, set of actions $[K]$; distributions $(P_{c,a})$.

Interaction

For rounds $t = 1, 2, 3, \ldots$:

1. Context $C_t \in \mathcal{C}$ is revealed to the learner.
2. Based on its past observations (including C_t), the learner chooses an action $A_t \in [K]$. The chosen action is sent to the environment.
3. The environment sends the reward $X_t \sim P_{C_t,A_t}$ to the learner.

Regret Definition

Definition: Expected reward for action a under context c:

$$r(c, a) = \int x \, P_{c,a}(dx) \, .$$

Regret:

$$R_n = \mathbb{E}\left[\sum_{t=1}^{n} \max_{a \in [K]} r(C_t, a) - \sum_{t=1}^{n} X_t\right] \, .$$

Poor Man's Contextual Bandit Algorithm

Assumption: C is finite.

Idea: Assign a bandit to each context.

Worst-case regret: $R_n = \Theta(\sqrt{nMK})$, where $M = |\mathrm{C}|$.

Problem: M (and K) can be very large.

How to save this? Assume structure.

Linear Models

Assumption:

$$r(c, a) = \langle \psi(c, a), \theta_* \rangle \,, \qquad \forall (c, a) \in \mathrm{C} \times [K] \,.$$

where $\psi : \mathrm{C} \times [K] \to \mathbb{R}^d$, $\theta_* \in \mathbb{R}^d$.

- ψ: **feature map**;
- $\mathcal{H}_\psi \doteq \mathrm{span}(\, \psi(c, k) \; : \; c \in \mathrm{C}, k \in [K] \,) \subset \mathbb{R}^d$: **feature space**;
- $\mathcal{S}_\psi \doteq \{r_\theta \; : \; r_\theta : \mathrm{C} \times [K] \to \mathbb{R}, \theta \in \mathbb{R}^d\}$, where
  $r_\theta(c, k) = \langle \psi(c, a), \theta \rangle$:
  space of linear reward functions.

Note: RKHS let us deal with $d = \infty$ (or d very large).

Outline

Choosing $\psi \Rightarrow$ betting on smoothness of r

Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$, $\|\cdot\|_*$ its dual (e.g., both 2-norms).
Hölder's inequality:

$$|r(c,a) - r(c',a')| \leq \|\theta_*\| \, \|\psi(c,a) - \psi(c',a')\|_* \, .$$

$r = \langle \psi, \theta_* \rangle \Rightarrow r$ is $(\rho, \|\theta_*\|)$-smooth:

$$|r(z) - r(z')| \leq \|\theta_*\| \, \rho(z,z')$$

where $z = (c,a), z' = (c',a')$, $\rho(z,z') = \|\psi(z) - \psi(z')\|_*$.

Choice of ψ + preference for small $\|\theta_*\| \Leftrightarrow$
preference for ρ-smooth r.

Influence of ψ is the largest when $d = +\infty$.

Sparsity

Concern: Is $r \in \mathcal{S}_\psi$ really true?

Thinking of free lunches: Can't we just add a lot of features to make sure $r \in \mathcal{S}_\psi$?

- Feature i unused: $\theta_{*,i} = 0$;
- Small $\|\theta_*\|_0 = \sum_{i=1}^{d} \mathbb{I}_{\{\theta_{*i} \neq 0\}}$;
- '0-norm'.

Sparsity

Outline

Features Are All What You Need

Given $C_1, A_1, \ldots, C_t, A_t$, the reward X_t in round t satisfies

$$\mathbb{E}\left[X_t | C_1, A_1, \ldots, C_t, A_t\right] = \langle \psi(C_t, A_t), \theta_* \rangle ,$$

for some known ψ and unknown θ_*.

$$\Leftrightarrow$$

At the beginning of any round t, observe action set $\mathcal{A}_t \subset \mathbb{R}^d$. If $A_t \in \mathcal{A}_t$,

$$\mathbb{E}\left[X_t | \mathcal{A}_1, A_1, \ldots, \mathcal{A}_t, A_t\right] = \langle A_t, \theta_* \rangle$$

with some unknown θ_*.

Why? Let $$\mathcal{A}_t = \{\psi(C_t, a) \,:\, a \in [K]\} .$$

Stochastic Linear Bandits

1. In round t, observe action set $\mathcal{A}_t \subset \mathbb{R}^d$.
2. The learner chooses $A_t \in \mathcal{A}_t$ and receives X_t, satisfying

$$\mathbb{E}\left[X_t \middle| \mathcal{A}_1, A_1, \dots, \mathcal{A}_t, A_t\right] = \langle A_t, \theta_* \rangle$$

with some unknown θ_*.

Goal: Keep regret

$$R_n = \mathbb{E}\left[\sum_{t=1}^{n} \max_{a \in \mathcal{A}_t} \langle a, \theta_* \rangle - X_t\right]$$

small.

Additional assumptions: *(i)* $\mathcal{A}_1, \dots, \mathcal{A}_n$ is any fixed sequence; *(ii)* $X_t - \langle A_t, \theta_* \rangle$ is light tailed, given $\mathcal{A}_1, A_1, \dots, \mathcal{A}_t, A_t$.

Finite-armed bandits

Case (a): $\mathcal{A}_t$ has always the same number of vectors in it:

"finite-armed stochastic contextual bandit".

Case (b): Also, $\mathcal{A}_t$ does not change, or $\mathcal{A}_t = \{a_1, \ldots, a_K\}$:

"finite-armed stochastic linear bandit".

Case (c): If the vectors in $\mathcal{A}_t$ are also orthogonal to each other:

"finite-armed stochastic bandit".

Difference between cases (c) and (b):

- Case (c): Learn about mean of arm $i \Leftrightarrow$ Choose action i;
- Case (b): Learn about mean of arm $i \Leftrightarrow$ Choose action j s.t. $\langle x_j, x_i \rangle \neq 0$.

Outline

Once an Optimist, Always an Optimist

Optimism in the Face of Uncertainty Principle:

"Choose the best action in the best environment amongst the plausible ones."

Environment $\Leftrightarrow \theta \in \mathbb{R}^d$.

Plausible environments:

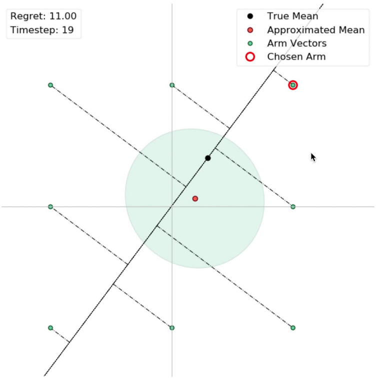
$\mathcal{C}_t \subset \mathbb{R}^d$ s.t. $\mathbb{P}\left(\theta_* \notin \mathcal{C}_t\right) \sim 1/t$.

Best environment:

$\tilde{\theta}_t = \mathrm{argmax}_{\theta \in \mathcal{C}_t} \max_{a \in \mathcal{A}} \langle a, \theta \rangle$.

Best action:

$\mathrm{argmax}_{a \in \mathcal{A}} \langle a, \hat{\theta}_t \rangle$.



Choosing the Confidence Set

Say, reward in round t is X_t, action in round t is $A_t \in \mathbb{R}^d$:

$$X_t = \langle A_t, \theta_* \rangle + \eta_t \,,$$

η_t is noise.
Regularized least-squares estimator: $\hat{\theta}_t = V_t^{-1} \sum_{s=1}^t A_s X_s$,

$$V_0 = \lambda I \,, \qquad V_t = V_0 + \sum_{s=1}^t A_s A_s^\top \,.$$

Choice of $\mathcal{C}_t$:

$$\mathcal{C}_t \subset \mathcal{E}_t \doteq \left\{ \theta \in \mathbb{R}^d \,:\, \|\theta - \hat{\theta}_{t-1}\|_{V_{t-1}}^2 \leq \beta_t \right\} \,.$$

Here, $(\beta_t)_t$ decreasing, $\beta_t \geq 1$, for A positive definite, $\|x\|_A^2 = x^\top A x$.

LinUCB

Choose $\mathcal{C}_t = \mathcal{E}_t$ with suitable $(\beta_t)_t$ and let

$$A_t = \operatorname*{argmax}_{a \in \mathcal{A}} \max_{\theta \in \mathcal{C}_t} \langle a, \theta \rangle \,.$$

Then,

$$A_t = \operatorname*{argmax}_{a} \langle a, \hat{\theta}_t \rangle + \sqrt{\beta_t}\, \|a\|_{V_{t-1}^{-1}} \,.$$

LinUCB (a.k.a. LinRel, OFUL, ConfEllips, ...)

Outline

Regret Analysis

Assumptions:

1. *Bounded scalar mean reward:* $|\langle a, \theta_* \rangle| \leq 1$ for any $a \in \cup_t \mathcal{A}_t$.
2. *Bounded actions:* for any $a \in \cup_t \mathcal{A}_t$, $\|a\|_2 \leq L$.
3. *Honest confidence intervals:* There exists a $\delta \in (0, 1)$ such that with probability $1 - \delta$, for all $t \in [n]$, $\theta_* \in \mathcal{C}_t$ where $\mathcal{C}_t$ satisfies $\mathcal{C}_t \subset \mathcal{E}_t$ with $(\beta_t)_t \geq 1$, decreasing as on the previous slide.

Theorem (LinUCB Regret)

Let the conditions listed above hold. Then with probability $1 - \delta$ *the regret of LinUCB satisfies*

$$\hat{R}_n^{pseudo} \leq \sqrt{8n\beta_n \log\left(\frac{\det V_n}{\det V_0}\right)} \leq \sqrt{8dn\beta_n \log\left(\frac{\mathrm{trace}(V_0) + nL^2}{d \det^{\frac{1}{d}}(V_0)}\right)}.$$

Proof

Assume $\theta_* \in \mathcal{C}_t$, $t \in [n]$. Let $A_t^* \doteq \mathrm{argmax}_{a \in \mathcal{A}_t} \langle a, \theta_* \rangle$, $r_t = \langle A_t^* - A_t, \theta_* \rangle$ and let $\tilde{\theta}_t \in \mathcal{C}_t$ s.t. $\langle A_t, \tilde{\theta}_t \rangle = \mathrm{UCB}_t(A_t)$.
From $\theta_* \in \mathcal{C}_t$ and the definition of LinUCB,

$$\langle A_t^*, \theta_* \rangle \le \mathrm{UCB}_t(A_t^*) \le \mathrm{UCB}_t(A_t) = \langle A_t, \tilde{\theta}_t \rangle \,.$$

Then,

$$r_t \le \langle A_t, \tilde{\theta}_t - \theta_* \rangle \le \|A_t\|_{V_{t-1}^{-1}} \|\tilde{\theta}_t - \theta_*\|_{V_{t-1}} \le 2 \|A_t\|_{V_{t-1}^{-1}} \beta_t \,.$$

From $\langle a, \theta_* \rangle \le 1$, $r_t \le 2$. This combined with $\beta_n \ge \max\{1, \beta_t\}$ gives

$$r_t \le 2 \wedge 2\sqrt{\beta_t} \|A_t\|_{V_{t-1}^{-1}} \le 2\sqrt{\beta_n}(1 \wedge \|A_t\|_{V_{t-1}^{-1}}) \,.$$

Jensen's inequality shows that

$$\hat{R}_n^{\text{pseudo}} = \sum_{t=1}^{n} r_t \le \sqrt{n \sum_{t=1}^{n} r_t^2} \le 2\sqrt{n\beta_n \sum_{t=1}^{n} (1 \wedge \|A_t\|_{V_{t-1}^{-1}}^2)} \,.$$

Lemma

Lemma

Let $x_1, \ldots, x_n \in \mathbb{R}^d$, $V_t = V_0 + \sum_{s=1}^t x_s x_s^\top$, $t \in [n]$, $v_0 = \mathrm{trace}(V_0)$ *and* $L \geq \max_t \|x_t\|_2$. *Then,*

$$\sum_{t=1}^n \left(1 \wedge \|x_t\|^2_{V_{t-1}^{-1}}\right) \leq 2 \log\left(\frac{\det V_n}{\det V_0}\right) \leq d \log\left(\frac{v_0 + nL^2}{d \det^{1/d}(V_0)}\right) .$$

Outline

LinUCB and Finite-Armed Bandits

Recall that if $\mathcal{A}_t = \{e_1, \ldots, e_d\}$, we get back the finite armed bandits.
LinUCB:

$$A_t = \operatorname*{argmax}_a \langle a, \hat{\theta}_t \rangle + \sqrt{\beta_t} \, \|a\|_{V_{t-1}^{-1}} \, .$$

If we set $\lambda = 0$, $\langle e_i, \hat{\theta}_t \rangle = \hat{\mu}_{i,t}$: empirical mean, $V_{t-1} = \mathrm{diag}(T_1(t-1), \ldots, T_K(t-1))$. Hence,

$$\sqrt{\beta_t} \, \|e_i\|_{V_{t-1}^{-1}} = \sqrt{\frac{\beta_t}{T_i(t-1)}} \, ,$$

and

$$A_t = \operatorname*{argmax}_a \langle a, \hat{\theta}_t \rangle + \sqrt{\frac{\beta_t}{T_i(t-1)}} \, ,$$

We recover UCB when $\beta_t = 2 \log(\cdot)$.

History

- Abe and Long (1999) introduced stochastic linear bandits into machine learning literature.
- Auer (2002) was the first to consider optimism for linear bandits (LinRel, SupLinRel). Main restriction: $|\mathcal{A}_t| < +\infty$.
- Confidence ellipsoids: Dani et al. (2008) (ConfidenceBall$_2$), Rusmevichientong and Tsitsiklis (2010) (Uncertainty Ellipsoid Policy), Abbasi-Yadkori et al. (2011) (OFUL).
- The name LinUCB comes from Chu et al. (2011).
- Alternative routes:
  - Explore then commit for action sets with smooth boundary. Abbasi-Yadkori et al. (2009); Abbasi-Yadkori (2009); Rusmevichientong and Tsitsiklis (2010).
  - Phased elimination
  - Thompson sampling

Extensions of Linear Bandits

- **Generalized linear model** (Filippi et al., 2009):

$$X_t = g^{-1}(\langle A_t, \theta_* \rangle + \eta)\,, \tag{1}$$

  where $g : \mathbb{R} \to \mathbb{R}$ is called the **link function**.
  Common choice: $g(p) = \log(p/(1-p))$ when
  $g^{-1}(x) = 1/(1+\exp(-x))$ (sigmoid).
- Spectral bandits: Spectral Eliminator Valko et al. (2014).
- Kernelised UCB: Valko et al. (2013).
- (Nonlinear) Structured bandits: $r : \mathcal{C} \times [K] \to [0,1]$ belongs to some known set (Anantharam et al., 1987; Russo and Roy, 2013; Lattimore and Munos, 2014).

Outline

Setting

1. *Subgaussian rewards:* The reward is $X_t = \langle A_t, \theta_* \rangle + \eta_t$, where η_t is conditionally 1-subgaussian ($\eta_t | \mathcal{F}_{t-1} \sim \mathrm{subG}(1)$):

$$\mathbb{E}[\exp(\lambda \eta_t) | \mathcal{F}_{t-1}] \leq \exp(\lambda^2/2) \qquad \text{almost surely for all } \lambda \in \mathbb{R}\,,$$

   where $\mathcal{F}_t = \sigma(A_1, \eta_1, \ldots, A_{t-1}, \eta_{t-1}, A_t)$.

2. *Bounded parameter vector:* $\|\theta_*\|_2 \leq S$ with $S > 0$ known.

Least Squares: Recap

Linear model:

$$X_t = \langle A_t, \theta_* \rangle + \eta_t \,.$$

Regularized squared loss:

$$L_t(\theta) = \sum_{s=1}^{t} (X_s - \langle A_s, \theta \rangle)^2 + \lambda \, \|\theta\|_2^2 \,,$$

Least squares estimate: $\hat{\theta}_t = \operatorname{argmin}_\theta L_t(\theta)$:

$$\hat{\theta}_t = V_t(\lambda)^{-1} \sum_{s=1}^{t} X_s A_s \quad \text{with } V_t(\lambda) = \lambda I + \sum_{s=1}^{t} A_s A_s^\top \,. \tag{2}$$

Abbreviation: $V_t = V_t(0)$.

Main difficulty

The actions $(A_s)_{s<t}$ are neither fixed nor independent, but are intricately correlated via the rewards $(X_s)_{s<t}$.

Outline

Fixed Design

1. *Nonsingular Grammian:* $\lambda = 0$ and V_t is invertible.
2. *Independent subgaussian noise:* $(\eta_s)_s$ are independent and 1-subgaussian.
3. *Fixed design:* $A_1, \ldots, A_t$ are deterministically chosen without the knowledge of $X_1, \ldots, X_t$.

Notation: $A_1, \ldots, A_t$ replaced by $a_1, \ldots, a_t$, so

$$V_t = \sum_{s=1}^{t} a_s a_s^\top \qquad \text{and} \qquad \hat{\theta}_t = V_t^{-1} \sum_{s=1}^{t} a_s X_s \,.$$

Note: $(a_s)_{s=1}^t$ must span $\mathbb{R}^d$ for $\exists V_t^{-1} \Rightarrow t \geq d$.

First Steps

Fix $x \in \mathbb{R}^d$. From $X_s = a_s^\top \theta_* + \eta_s$,

$$\hat{\theta}_t = V_t^{-1} \sum_{s=1}^{t} a_s X_s = \cancel{V_t^{-1}} \cancel{V_t}\, \theta_* + V_t^{-1} \sum_{s=1}^{t} a_s \eta_s \,,$$

so

$$\langle x, \hat{\theta}_t - \theta_* \rangle = \sum_{s=1}^{t} \langle x, V_t^{-1} a_s \rangle \, \eta_s \,.$$

Since $(\eta_s)_s$ are independent and 1-subgaussian: With probability $1 - \delta$,

$$\langle x, \hat{\theta}_t - \theta_* \rangle < \sqrt{2 \sum_{s=1}^{t} \langle x, V_t^{-1} a_s \rangle^2 \log\left(\frac{1}{\delta}\right)} = \sqrt{2 \|x\|^2_{V_t^{-1}} \log\left(\frac{1}{\delta}\right)} \,.$$

Bounding $\|\hat{\theta}_t - \theta^*\|_{V_t}$

For $x \in \mathbb{R}^d$ fixed, with probability $1 - \delta$,

$$\langle x, \hat{\theta}_t - \theta_* \rangle < \sqrt{2\|x\|^2_{V_t^{-1}} \log\left(\tfrac{1}{\delta}\right)}\,. \qquad (*)$$

We have $\|u\|^2_{V_t} = (V_t u)^\top u$.

Idea 1: Apply (*) to $X = V_t(\hat{\theta}_t - \theta_*)$..
Problem: (*) holds for non-random x only!

Idea 2: Take finite $\mathcal{C}_\varepsilon$ s.t. $X \approx_\varepsilon x$ for some $x \in \mathcal{C}_\varepsilon$. Make (*) hold for each $x \in \mathcal{C}_\varepsilon$, combine.
Refinement: Let $X = V_t^{1/2}(\hat{\theta}_t - \theta_*)/\|\hat{\theta}_t - \theta_*\|_{V_t}$.
Then $X \in S^{d-1} = \{x \in \mathbb{R}^d : \|x\|_2 = 1\}$ and we can choose a finite $\mathcal{C}_\varepsilon \subset S^{d-1}$ s.t. for any $u \in S^{d-1}$, $\exists x \in \mathcal{C}_\varepsilon$ with $\|u - x\| \le \varepsilon$.

Bounding $\|\hat{\theta}_t - \theta^*\|_{V_t}$: Part II.

Let

$$X = \frac{V_t^{1/2}(\hat{\theta}_t - \theta_*)}{\|\hat{\theta}_t - \theta_*\|_{V_t}}, \qquad X^* = \operatorname*{argmin}_{x \in \mathcal{C}_\varepsilon} \|x - X\| \ .$$

Then, with probability $1 - \delta$,

$$\begin{aligned}
\|\hat{\theta}_t - \theta_*\|_{V_t} &= \langle X, V_t^{1/2}(\hat{\theta}_t - \theta_*)\rangle \\
&= \langle X - X^*, V_t^{1/2}(\hat{\theta}_t - \theta_*)\rangle + \langle V_t^{1/2} X^*, \hat{\theta}_t - \theta_*\rangle \\
&\leq \varepsilon \|\hat{\theta}_t - \theta_*\|_{V_t} + \sqrt{2\cancel{\|V_t^{1/2} X^*\|^2_{V_t^{-1}}} \log\left(\frac{|\mathcal{C}_\varepsilon|}{\delta}\right)} \,.
\end{aligned}$$

or

$$\|\hat{\theta}_t - \theta_*\|_{V_t} \leq \frac{1}{1 - \varepsilon} \sqrt{2 \log\left(\frac{|\mathcal{C}_\varepsilon|}{\delta}\right)} \,.$$

We can choose $\mathcal{C}_\varepsilon$ so that $|\mathcal{C}_\varepsilon| \leq (5/\varepsilon)^d$:

$$\|\hat{\theta}_t - \theta_*\|_{V_t} < 2\sqrt{2\left(d \log(10) + \log\left(\frac{1}{\delta}\right)\right)} \,.$$

Outline

Bounding $\|\hat{\theta}_t - \theta^*\|_{V_t}$: Sequential Design

$X_s = \langle A_s, \theta_* \rangle + \eta_s$,
$\eta_s | \mathcal{F}_{s-1} \sim \mathrm{subG}(1)$ for $\mathcal{F}_s = \sigma(A_1, \eta_1, \ldots, A_s, \eta_s)$.

Previous bound exploited $A_1, \ldots, A_t$ fixed, non-random. Known as:

fixed design

When $A_1, \ldots, A_t$ is i.i.d., we have a

random design

Bandits: A_s is chosen based on $A_1, X_1, \ldots, A_{s-1}, X_{s-1}$!

sequential design

How to bound $\|\hat{\theta}_t - \theta^*\|_{V_t}$ in this case?

Bounding $\|\hat{\theta}_t - \theta^*\|_{V_t}$: Sequential Design

- Linearization trick
- Vector Chernoff
- Laplace method

A Start: Linearization & Vector Chernoff

Let $S_t = \sum_{s=1}^t \eta_s A_s$. "Linearization" of the quadratic:

$$\frac{1}{2}\|\hat{\theta}_t - \theta_*\|_{V_t}^2 = \frac{1}{2}\|S_t\|_{V_t^{-1}}^2 = \max_{x \in \mathbb{R}^d} \langle x, S_t \rangle - \frac{1}{2}\|x\|_{V_t}^2 \ .$$

Let

$$M_t(x) = \exp\left(\langle x, S_t \rangle - \frac{1}{2}\|x\|_{V_t}^2\right) \ .$$

One can show that $\mathbb{E}\left[M_t(x)\right] \leq 1$ for any $x \in \mathbb{R}^d$. Chernoff's method:

$$\begin{aligned}
\mathbb{P}\left(\frac{1}{2}\|\hat{\theta}_t - \theta_*\|_{V_t}^2 \geq u\right) &= \mathbb{P}\left(\exp(\max_x \log M_t(x)) \geq \exp(u)\right) \\
&\leq \mathbb{E}\left[\exp(\max_x \log M_t(x))\right] \exp(-u) = \mathbb{E}\left[\max_x M_t(x))\right] \exp(-u) \ .
\end{aligned}$$

Can we control $\mathbb{E}\left[\max_x M_t(x)\right]$?

Controlling $\mathbb{E}\left[\max_x M_t(x)\right]$: Covering Argument

Recall:

$$M_t(x) = \exp\left(\langle x, S_t\rangle - \frac{1}{2}\|x\|^2_{V_t}\right) .$$

Let $\mathcal{C}_\varepsilon \subset \mathbb{R}^d$ be finite, to be chosen later,

$$X = \operatorname*{argmax}_{x\in\mathbb{R}^d} M_t(x), \quad Y = \operatorname*{argmin}_{y\in\mathcal{C}_\varepsilon} \|X - y\| .$$

Then,

$$\max_{x\in\mathbb{R}^d} M_t(x) = M_t(X) = M_t(X) - M_t(Y) + M_t(Y) \leq \varepsilon + \sum_{y\in\mathcal{C}_\varepsilon} M_t(y) .$$

Challenge: ensure $M_t(X) - M_t(Y) \leq \varepsilon$!

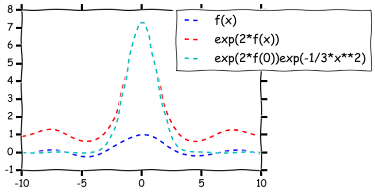
Laplace: One Step Back, Two Forward

The need to control $\mathbb{E}\left[\max_x M_t(x)\right]$ comes from the identity $\exp(\frac{1}{2}\|\hat{\theta}_t - \theta_*\|^2_{V_t}) = \max_x M_t(x)$.

Laplace: Integral of $\exp(sf(x))$ is dominated by $\exp(s \max_x f(x))$:

$$\int_a^b e^{sf(x)} dx \sim e^{sf(x_0)} \sqrt{\frac{2\pi}{s|f''(x_0)|}},$$

$$x_0 = \mathrm{argmax}_{x \in [a,b]} f(x) \in (a,b).$$



Idea: Replace $\max_x M_t(x)$ with $\int M_t(x)h(x)dx$ with h appropriate:

1. $\int M_t(x)h(x)dx \approx \max_x M_t(x)$ (in a way);
2. $\mathbb{E}\left[\int M_t(x)h(x)dx\right] = \int \mathbb{E}\left[M_t(x)\right] h(x)dx \leq 1$.

Choose $h(x)$ as density of $\mathcal{N}(0, H^{-1})$ for $H \succ 0$.

Step 2: Finishing

$$\int M_t(x)h(x)dx = \left(\frac{\det(H)}{\det(H+V)}\right)^{1/2} \exp\left(\frac{1}{2}\|S_t\|^2_{(H+V_t)^{-1}}\right) .$$

Choose $H = \lambda I$. Then, with probability $1 - e^{-u}$,

$$\tfrac{1}{2}\|S_t\|^2_{V_t^{-1}(\lambda)} < u + \frac{1}{2}\log\left(\frac{\det(V_t(\lambda))}{\lambda^d}\right) \qquad (**)$$

and from $\hat{\theta}_t - \theta_* = V_t^{-1}(\lambda)S_t - \lambda V_t^{-1}(\lambda)\theta_*$,

$$\begin{aligned}\|\hat{\theta}_t - \theta_*\|_{V_t(\lambda)} &\le \|V_t^{-1}(\lambda)S_t\|_{V_t(\lambda)} + \lambda\left\|V_t^{-1}(\lambda)\theta_*\right\|_{V_t(\lambda)} \\ &\le \|S_t\|_{V_t^{-1}(\lambda)} + \lambda^{1/2}\|\theta_*\| .\end{aligned}$$

Confidence Ellipsoid for Sequential Design

Assumptions: $\|\theta_*\| \leq S$, and let $(A_s)_s, (\eta_s)_s$ be so that for any $1 \leq s \leq t$, $\eta_s | \mathcal{F}_{s-1} \sim \mathrm{subG}(1)$, where $\mathcal{F}_s = \sigma(A_1, \eta_1, \ldots, A_{s-1}, \eta_{s-1}, A_s)$

Fix $\delta \in (0,1)$. Let

$$\beta_{t+1} = \sqrt{\lambda} S + \sqrt{2 \log\left(\frac{1}{\delta}\right) + \log\left(\frac{\det V_t(\lambda)}{\lambda^d}\right)}\,,$$

and

$$\mathcal{C}_{t+1} = \left\{\theta \in \mathbb{R}^d \,:\, \|\hat{\theta}_t - \theta_*\|_{V_t(\lambda)} \leq \beta_{t+1}\right\}.$$

Theorem

$\mathcal{C}_{t+1}$ is a confidence set for θ_* at level $1-\delta$:

$$\mathbb{P}\left(\theta_* \in \mathcal{C}_{t+1}\right) \geq 1 - \delta\,.$$

Note: β_{t+1} is a function of $(A_s)_{s \leq t}$.

Freedman's Stopping Trick

We want $\theta_* \in \mathcal{C}_t$ hold with probability $1 - \delta$, simultaneously for all $1 \leq t \leq n$.

Can we avoid the union bound over time?

Freedman's Stopping Trick: II

Let

$$\mathcal{E}_t = \left\{ \|\hat{\theta}_t - \theta_*\|_{V_{t-1}(\lambda)} \geq \sqrt{\lambda} S + \sqrt{2u + \log\left(\frac{\det V_{t-1}(\lambda)}{\lambda^d}\right)} \right\}, t \in [n] .$$

Define $\tau \in [n]$ as follows: τ to be the smallest round index $t \in [n]$ such that $\mathcal{E}_t$ holds, or n when none of $\mathcal{E}_1, \ldots, \mathcal{E}_n$ hold.

Note: Because $\hat{\theta}_t$ and $V_{t-1}(\lambda)$ is a function of $H_{t-1} = (A_1, \eta_1, \ldots, A_{t-1}, \eta_{t-1})$, whether $\mathcal{E}_t$ holds can be decided based on H_{t-1}.

$\Rightarrow \tau$ is a $(H_t)_t$ stopping time $\Rightarrow \mathbb{E}\left[M_\tau(x)\right] \leq 1$ and also $\int \mathbb{E}\left[M_\tau(x)\right] h(x) dx \leq 1$ and thus $\mathbb{P}\left(\cup_{t \in [n]} \mathcal{E}_t\right) \leq \mathbb{P}\left(\mathcal{E}_\tau\right) \leq e^{-u}$. $n \to \infty$.

Corollary

$\mathbb{P}\left(\exists t \geq 0 \text{ such that } \theta_* \notin C_{t+1}\right) \leq \delta.$

Historical Remarks

- Presentation mostly follows Abbasi-Yadkori et al. (2011).
- Auer (2002); Chu et al. (2011) avoided the need to construct ellipsoidal confidence sets
- Previous ellipsoidal constructions by Dani et al. (2008) and Rusmevichientong and Tsitsiklis (2010) used covering arguments.
- The improvement that results from using Laplace's method as compared to the previous ellipsoidal constructions that are based on covering arguments is quite enormous.
- Laplace's method is also called the "Method of Mixtures" (Peña et al., 2008); its use goes back to the work of Robbins and Siegmund in the 1970s (Robbins and Siegmund, 1970, 1971).
- Freedman's Stopping is by Freedman (1975).

Outline

Regret for LinUCB: Final Steps

Previously we have seen, for $\beta_t \geq 1$, nondecreasing, using LinUCB with $V_0 = \lambda I$, w.p. $1-\delta$,

$$\hat{R}_n^{\text{pseudo}} \leq \sqrt{8n\beta_n \log\left(\frac{\det V_n}{\det V_0}\right)} \leq \sqrt{8dn\beta_n \log\left(\frac{\lambda d + nL^2}{\lambda d}\right)}.$$

Now,

$$\beta_n = \sqrt{\lambda}S + \sqrt{2\log\left(\tfrac{1}{\delta}\right) + \log\left(\frac{\det V_{n-1}(\lambda)}{\lambda^d}\right)}$$
$$\leq \sqrt{\lambda}S + \sqrt{2\log\left(\tfrac{1}{\delta}\right) + d\log\left(\frac{\lambda d + nL^2}{\lambda d}\right)},$$

from $\|A_s\| \leq L$ and $\log\det V \leq d\log(\mathrm{trace}(V)/d)$.

$$\Rightarrow \hat{R}_n^{\text{pseudo}} \leq C_1 d\sqrt{n\log(n)} + C_2\sqrt{nd}\log(1/\delta) + C_3.$$

Summary

$$\hat{R}_n^{\text{pseudo}} \le C_1 d \sqrt{n \log(n)} + C_2 \sqrt{nd} \log(1/\delta) + C_3 .$$

- Optimism, confidence ellipsoid
- Getting the ellipsoid is tricky because the bandit algorithm makes $(A_s)_s$ and $(\eta_s)_s$ interdependent: “sequential design”.
- Hsu et al. (2012): Random design, fixed design
- Kernelization: One can directly kernelize the proof presented here. See Abbasi-Yadkori (2012). Gaussian Process Bandits effectively do the same (Srinivas et al., 2010).
- This presentation followed mostly Abbasi-Yadkori et al. (2011); Abbasi-Yadkori (2012).

Outline

Challenge!

The previous bound is $\tilde{O}(d\sqrt{n})$ even for $\mathcal{A} = \{e_1, \dots, e_d\}$ – finite-armed stochastic bandits.

Can we do better?

Setting

1. *Fixed finite action set:* The set of actions available in round t is $\mathcal{A} \subset \mathbb{R}^d$ and $|\mathcal{A}| = K$ for some natural number K.

2. *Subgaussian rewards:* The reward is $X_t = \langle \theta_*, A_t \rangle + \eta_t$ where η_t is conditionally 1-subgaussian: $\eta_t | \mathcal{F}_{t-1} \sim \mathrm{subG}(1)$, where $\mathcal{F}_t = \sigma(A_1, \eta_1, \ldots, A_{t-1}, \eta_t, A_t)$.

3. *Bounded mean rewards:* $\Delta_a = \max_{b \in \mathcal{A}} \langle \theta_*, b - a \rangle \leq 1$ for all $a \in \mathcal{A}$.

Key difference to previous setting:

Finite, fixed action set.

Avoiding Sequential Designs

Recall result for fixed design:

For $x \in \mathbb{R}^d$ fixed, with probability $1-\delta$,

$$\langle x, \hat{\theta}_t - \theta_* \rangle < \sqrt{2\|x\|^2_{V_t^{-1}} \log\left(\frac{1}{\delta}\right)} \,. \qquad (*)$$

Goal: Use this result! How?

Idea: Use a phased elimination algorithm!

- $\mathcal{A} = \mathcal{A}_1 \supset \mathcal{A}_2 \supset \mathcal{A}_3 \supset \ldots .$
- In phase ℓ, use actions in $\mathcal{A}_\ell$ to collect enough data to ensure that by the end of the phase, the data collected in the phase is sufficient to rule out all $\varepsilon_\ell \doteq 2^{-\ell}$-suboptimal actions.

Which action to use & how many times in phase ℓ?

How to Collect Data?

Recall: If $V_t = \sum_{s=1}^{t} a_s a_s^\top$, for any $x \in \mathbb{R}^d$.

$$\langle x, \hat{\theta}_t - \theta_* \rangle < \sqrt{2\|x\|^2_{V_t^{-1}} \log\left(\frac{1}{\delta}\right)}\,. \qquad (*)$$

If we need to know whether $\langle x, \hat{\theta}_t - \theta_* \rangle \leq 2^{-\ell}$, $x \in \mathcal{A}$, we better choose $a_1, a_2, \ldots, a_t$ so that

$$\max_{x \in \mathcal{A}} \|x\|^2_{V_t^{-1}} \qquad (*)$$

is minimized (and make t big enough).

$\Rightarrow$ Experimental design

Minimizing (*) is known as the G-optimal design problem.

Outline

G-optimal Design

Let $\pi : \mathcal{A} \to [0,1]$ be a distribution on $\mathcal{A}$: $\sum_{a\in\mathcal{A}} \pi(a) = 1$. Define

$$V(\pi) = \sum_{a\in\mathcal{A}} \pi(a) a a^\top \,, \qquad\qquad g(\pi) = \max_{a\in\mathcal{A}} \|a\|^2_{V(\pi)^{-1}} \,.$$

G-optimal design π^*:

$$g(\pi^*) = \min_{\pi} g(\pi) \,.$$

How to use this?

Using a Design π

Given a design π, for $a \in \mathrm{Supp}(\pi)$, set

$$n_a = \left\lceil \pi(a) \frac{g(\pi)}{\varepsilon^2} \log\left(\frac{1}{\delta}\right) \right\rceil .$$

Choose each action $a \in \mathrm{Supp}(\pi)$ exactly n_a times. Then:

$$V = \sum_{a \in \mathrm{Supp}(\pi)} n_a \, a a^\top \geq \frac{g(\pi)}{\varepsilon^2} \log\left(\frac{1}{\delta}\right) V(\pi) ,$$

and so for any $a \in \mathcal{A}$, w.p. $1 - \delta$,

$$\langle \hat{\theta} - \theta_*, a \rangle \leq \sqrt{\|a\|^2_{V^{-1}} \log\left(\frac{1}{\delta}\right)} \leq \varepsilon .$$

How big is n?

$$n = \sum_{a \in \mathrm{Supp}(\pi)} n_a = \sum_{a \in \mathrm{Supp}(\pi)} \left\lceil \pi(a) \frac{g(\pi)}{\varepsilon^2} \log\left(\frac{1}{\delta}\right) \right\rceil$$

$$\leq |\,\mathrm{Supp}(\pi)| + \frac{g(\pi)}{\varepsilon^2} \log\left(\frac{1}{\delta}\right) .$$

Bounding $g(\pi)$ and $|\operatorname{Supp}(\pi)|$

Theorem (Kiefer–Wolfowitz)

The following are equivalent:

1. π^* *is a minimizer of* g.
2. π^* *is a minimizer of* $f(\pi) = -\log\det V(\pi)$.
3. $g(\pi^*) = d$.

Note: Designs, minimizing f are known as D-optimal designs.

KW says that G-optimality is the same as D-optimality.

Combining this with John's Theorem for minimum-volume enclosing ellipsoids (John, 1948), we get $|\operatorname{Supp}(\pi)| \le d(d+3)/2$.

Outline

PEGOE Algorithm[1]

Input: $\mathcal{A} \subset \mathbb{R}^d$ and δ. Set $\mathcal{A}_1 = \mathcal{A}$, $\ell = 1$, $t = 1$.

1. Let $t_\ell = t$: current round. Find G-optimal design $\pi_\ell : \mathcal{A}_\ell \to [0, 1]$ that maximizes
$$\log \det V(\pi_\ell) \text{ subject to } \sum_{a \in \mathcal{A}_\ell} \pi_\ell(a) = 1$$
2. Let $\varepsilon_\ell = 2^{-\ell}$ and
$$N_\ell(a) = \left\lceil \frac{2\pi(a)}{\varepsilon_\ell^2} \log\left(\frac{K\ell(\ell+1)}{\delta}\right) \right\rceil \text{ and } N_\ell = \sum_{a \in \mathcal{A}_\ell} N_\ell(a)$$
3. Choose each action $a \in \mathcal{A}_\ell$ exactly $N_\ell(a)$ times
4. Calculate estimate: $\hat{\theta} = V_\ell^{-1} \sum_{t=t_\ell}^{t_\ell + N_\ell} A_t X_t$.
5. Eliminate poor arms:
$$\mathcal{A}_{\ell+1} = \left\{ a \in \mathcal{A}_\ell : \max_{b \in \mathcal{A}_\ell} \langle \hat{\theta}_\ell, b - a \rangle \geq 2\varepsilon_\ell \right\}.$$

[1] Phased Elimination with G-Optimal Exploration

The Regret of PEGOE

Theorem

With probability at least $1 - \delta$ *the pseudo-regret of PEGOE is at most:*

$$\hat{R}_n^{pseudo} \leq C\sqrt{nd\log\left(\frac{K\log(n)}{\delta}\right)},$$

where $C > 0$ *is a universal constant. If* $\delta = O(1/n)$, *then*

$$\mathbb{E}[R_n] \leq C\sqrt{nd\log(Kn)}$$

for appropriately chosen universal constant $C > 0$.

Summary and Historical Remarks

- Phased exploration allows one to use methods developed for fixed-design
- PEGOE: Exploration tuned to maximize information gain
- Finding an $(1+\varepsilon)$-optimal design is sufficient; convex problem
- This algorithm and analysis in this form is new.
- “Phased Elimination” is well known: Even-Dar et al. (2006) (pure exploration), Auer and Ortner (2010) (finite-armed bandits), Valko et al. (2014) (linear bandits, spanners instead of G-optimality).
- Finite, but changing action set: PEGOE cannot be applied! SupLinRel and SupLinUCB get the same bound (Auer, 2002; Chu et al., 2011). Sadly, these algorithms are very conservative..

Outline

General Setting

1. *(Sparse parameter)* There exist known constants M_0 and M_2 such that $\|\theta_*\|_0 \leq M_0$ and $\|\theta_*\|_2 \leq M_2$.
2. *(Bounded mean rewards):* $\langle a, \theta_* \rangle \leq 1$ for all $a \in \mathcal{A}_t$ and all rounds t.
3. *(Subgaussian noise):* The reward is $X_t = \langle A_t, \theta_* \rangle + \eta_t$ where $\eta_t | \mathcal{F}_{t-1} \sim \mathrm{subG}(1)$ for $\mathcal{F}_t = \sigma(A_1, \eta_1, \dots, A_t, \eta_t)$.

The Case of the Hypercube

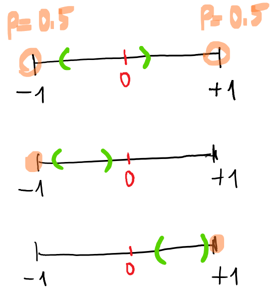
$$\mathcal{A} = [-1, 1]^d\,, \qquad \theta \doteq \theta_*\,, \qquad X_t = \langle A_t, \theta \rangle + \eta_t.$$

Assumptions:

1. *(Bounded mean rewards):* $\|\theta\|_1 \leq 1$, which ensures that $\langle a, \theta \rangle| \leq 1$ for all $a \in \mathcal{A}$.
2. *(Subgaussian noise):* The reward is $X_t = \langle A_t, \theta_* \rangle + \eta_t$ where $\eta_t | \mathcal{F}_{t-1} \sim \mathrm{subG}(1)$ for $\mathcal{F}_t = \sigma(A_1, \eta_1, \ldots, A_t, \eta_t)$.

Selective Explore-Then-Commit (SETC)

Recall: $\theta = \theta_*$.



For any $i \in [d]$ such that A_{ti} is randomized:

$$A_{ti}(A_t^\top \theta + \eta_t) = \theta_i + \underbrace{A_{ti} \sum_{j \neq i} A_{tj}\theta_j + A_{ti}\eta_t}_{\text{``noise''}} \; .$$

Regret of SETC

Theorem

There exists a universal constant $C > 0$ such that the regret of SETC satisfies:

$$R_n \leq 2 \|\theta\|_1 + C \sum_{i:\theta_i \neq 0} \frac{\log(n)}{|\theta_i|} \,.$$

Furthermore $R_n \leq C \|\theta\|_0 \sqrt{n \log(n)}$.

SETC adapts to $\|\theta\|_0$!

Outline

(General) LinUCB: Recap

GLinUCB

Choose $\mathcal{C}_t \subset \mathbb{R}^d$ and let

$$A_t = \operatorname*{argmax}_{a \in \mathcal{A}} \max_{\theta \in \mathcal{C}_t} \langle a, \theta \rangle \, .$$

Previous choice leads to regret $\tilde{O}(d\sqrt{n})$.

How to choose $\mathcal{C}_t$, knowing that $\|\theta_*\|_0 \leq p$,
so that the regret gets smaller?

Outline

Online Linear Regression (OLR)

Learner-environment interaction:

1. The environment chooses $X_t \in \mathbb{R}$ and $A_t \in \mathbb{R}^d$ in an arbitrary fashion.
2. The value of A_t is revealed to the learner (but not X_t).
3. The learner produces a real-valued prediction $\hat{X}_t$ in some way.
4. The environment reveals X_t to the learner and the loss is $(X_t - \hat{X}_t)^2$.

Goal: Compete with the total loss of the best linear predictors in some set $\Theta \subset \mathbb{R}^d$.

Regret against $\theta \in \Theta$:

$$\rho_n(\theta) = \sum_{t=1}^{n} (X_t - \hat{X}_t)^2 - \sum_{t=1}^{n} (X_t - \langle A_t, \theta \rangle)^2 .$$

From OLR to Confidence Sets

Let $\mathcal{L}$ be a learner that enjoys a regret guarantee $B_n = B_n(A_1, X_1, \ldots, A_n, X_n)$ relative to Θ: For any strategy of the environment,

$$\sup_{\theta \in \Theta} \rho_n(\theta) \leq B_n \,.$$

Combine

$$\rho_n(\theta) = \sum_{t=1}^{n} (X_t - \hat{X}_t)^2 - \sum_{t=1}^{n} (X_t - \langle A_t, \theta \rangle)^2 \,.$$

and $X_t = \langle A_t, \theta_* \rangle + \eta_t$ to get

$$\begin{aligned} Q_t \doteq \sum_{s=1}^{t} (\hat{X}_s - \langle A_s, \theta_* \rangle)^2 &= \rho_t(\theta_*) + 2 \sum_{s=1}^{t} \eta_s (\hat{X}_s - \langle A_s, \theta_* \rangle) \\ &\leq B_t + 2 \sum_{s=1}^{t} \eta_s (\hat{X}_s - \langle A_s, \theta_* \rangle) \,. \end{aligned}$$

From OLR to Confidence Sets: II.

$$Q_t \le B_t + 2Z_t\,, \qquad Z_t = \sum_{s=1}^{t} \eta_s(\hat{X}_s - \langle A_s, \theta_* \rangle)\,. \qquad (*)$$

Goal: Bound Z_t for $t \ge 0$.

$\hat{X}_t$, chosen by OLR learner $\mathcal{L}$, is $\mathcal{F}_{t-1}$-measurable,

$$(Z_t - Z_{t-1})|\mathcal{F}_{t-1} \sim \mathrm{subG}(\sigma_t)\,, \qquad \text{where } \sigma_t^2 = (\hat{X}_t - \langle A_t, \theta_* \rangle)^2\,.$$

Previous self-normalized bound (**): With probability $1-\delta$,

$$|Z_t| < \sqrt{(1+Q_t)\log\left(\frac{1+Q_t}{\delta^2}\right)}, t = 0, 1, \ldots\,.$$

Combining with (*), solve for Q_t:

$$Q_t \le \beta_t(\delta)\,, \quad \beta_t(\delta) = 1 + 2B_t + 32\log\left(\frac{\sqrt{8}+\sqrt{1+B_t}}{\delta}\right)\,.$$

OLR to Confidence Sets: III.

Theorem

Let $\delta \in (0,1)$ *and assume that* $\theta_* \in \Theta$ *and* $\sup_{\theta \in \Theta} \rho_t(\theta) \leq B_t$*. If*

$$\mathcal{C}_{t+1} = \left\{ \theta \in \mathbb{R}^d \,:\, \|\theta\|_2^2 + \sum_{s=1}^{t} (\hat{X}_s - \langle A_s, \theta \rangle)^2 \leq M_2^2 + \beta_t(\delta) \right\},$$

then $\mathbb{P}\left(\textit{exists } t \in \mathbb{N} \textit{ such that } \theta_* \notin \mathcal{C}_{t+1}\right) \leq \delta\,.$

Sparse LinUCB

Input OLR Learner $\mathcal{L}$, regret bound B_t, confidence parameter $\delta \in (0,1)$

for $t = 1, \dots, n$

  Receive action set $\mathcal{A}_t$

  Computer confidence set:

$$\mathcal{C}_t = \left\{ \theta \in \mathbb{R}^d \,:\, \|\theta\|_2^2 + \sum_{s=1}^{t-1} (\hat{X}_s - \langle A_s, \theta \rangle)^2 \leq M_2^2 + \beta_t(\delta) \right\}$$

  Calculate optimistic action

$$A_t = \operatorname*{argmax}_{a \in \mathcal{A}_t} \max_{\theta \in \mathcal{C}_t} \langle a, \theta \rangle$$

  Feed A_t to $\mathcal{L}$ and obtain prediction $\hat{X}_t$

  Play A_t and receive reward X_t

  Feed X_t to $\mathcal{L}$ as feedback

Regret of OLR-UCB

Theorem

With probability at least $1 - \delta$ the pseudo-regret of OLR-UCB satisfies

$$\hat{R}_n^{pseudo} \leq \sqrt{8dn\left(M_2^2 + \beta_{n-1}(\delta)\right)\log\left(1 + \frac{n}{d}\right)}.$$

The Regret of OLR-UCB(π)

Theorem (Sparse OLR Algorithm)

$\exists\pi$ for the learner such that for any $\theta \in \mathbb{R}^d$, the regret $\rho_n(\theta)$ of π against any strategic environment such that $\max_{t\in[n]} \|A_t\|_2 \le L$ and $\max_{t\in[n]} |X_t| \le X$ satisfies

$$\rho_n(\theta) \le cX^2 \|\theta\|_0 \left\{\log(e + n^{1/2}L) + C_n \log\left(1 + \frac{\|\theta\|_1}{\|\theta\|_0}\right)\right\} + (1 + X^2)C_n \,,$$

where $c > 0$ is some universal constant and $C_n = 2 + \log_2 \log(e + n^{1/2}L)$.

Corollary

The expected regret of OLR-UCB when using the strategy π from above satisfies

$$R_n = \tilde{O}(\sqrt{dpn})\,.$$

Outline

Summary

- OLR algorithm used inside OLR-UCB to construct center
- Regret guarantee of the OLR controls "width" of confidence ellipsoid
- Regret: $\tilde{O}(\sqrt{dpn})$, and p is known.
- Hypercube: $p\sqrt{n}$, p is unknown!
- In general, the regret can be as high as $\Omega(\sqrt{pdn})$ ($p = 1$: think of $\mathcal{A} = \{e_1, \ldots, e_d\}$)
- Under parameter noise ($X_t = \langle A_t, \theta_* + \eta_t \rangle$), for "rounded" action sets, $\tilde{O}(p\sqrt{n})$ is possible!
- Very much unlike in the "passive" case:

  Major conflict between exploration and exploitation!

Historical Notes

- Selective Explore-Then-Commit algorithm is due to (Lattimore et al., 2015).
- OLR-UCB is from Abbasi-Yadkori et al. (2012).
- The Sparse OLR algorithm is due to Gerchinovitz (2013).
- Rakhlin and Sridharan (2015) also discusses relationship between online learning regret bounds and self-normalized tail bounds of the type given here.

Outline

Minimax Lower Bound

Theorem

Let the action set be $\mathcal{A} = \{-1, 1\}^d$ *and* $\Theta = \{-n^{-1/2}, n^{-1/2}\}^d$. *Then for any policy* π *there exists a* $\theta \in \Theta$ *such that*

$$R_n^{\pi}(\mathcal{A}, \theta) \geq C\, d\sqrt{n}$$

for some universal constant $C > 0$.

Some Thoughts

- LinUCB with our confidence set construction is “nearly” worst-case optimal.
- The theorem is “new”, but the proof is standard; see (Shamir, 2015).
- Similar results for some other action sets: Rusmevichientong and Tsitsiklis (2010) (ℓ^2-ball), Dani et al. (2008) (products of 2D balls).
- Some action sets will have smaller minimax regret! Can you think of one?

Outline

Lower Bound

Setting:

1. Actions: $\mathcal{A} \subset \mathbb{R}^d$ finite, $K = |\mathcal{A}|$.
2. Reward is $X_t = \langle A_t, \theta \rangle + \eta_t$, where $\theta \in \mathbb{R}^d$ and η_t is a sequence of independent standard Gaussian variables.

Regret of policy π:

$$R_n^\pi(\mathcal{A}, \theta) = \mathbb{E}_{\theta,\pi}\left[\sum_{t=1}^n \Delta_{A_t}\right] , \qquad \Delta_a = \max_{a' \in \mathcal{A}} \langle a' - a, \theta \rangle ,$$

Recall: a policy π is consistent in some class of bandits $\mathcal{E}$ if the regret is subpolynomial for any bandit in that class:

$$R_n^\pi(\mathcal{A}, \theta) = o(n^p) \qquad \text{for all } p > 0 \text{ and } \theta \in \mathbb{R}^d .$$

Lower Bound: II

Theorem

Assume that $\mathcal{A} \subset \mathbb{R}^d$ *is finite and spans* $\mathbb{R}^d$ *and suppose* π *is consistent. Let* $\theta \in \mathbb{R}^d$ *be any parameter such that there is a unique optimal action and let* $\bar{G}_n = \mathbb{E}_{\theta,\pi}\left[\sum_{t=1}^n A_t A_t^\top\right]$ *be the expected Gram matrix . Then* $\liminf_{n\to\infty} \lambda_{\min}(\bar{G}_n)/\log(n) > 0$*. Furthermore, for any* $a \in \mathcal{A}$ *it holds that:*

$$\limsup_{n\to\infty} \log(n) \|a\|^2_{\bar{G}_n^{-1}} \leq \frac{\Delta_a^2}{2} .$$

Lower Bound: III

Corollary

Let $\mathcal{A} \subset \mathbb{R}^d$ be a finite set that spans $\mathbb{R}^d$ and $\theta \in \mathbb{R}^d$ be such that there is a unique optimal action. Then for any consistent policy π,

$$\liminf_{n \to \infty} \frac{R_n^\pi(\mathcal{A}, \theta)}{\log(n)} \geq c(\mathcal{A}, \theta)\,,$$

where $c(\mathcal{A}, \theta)$ is defined as

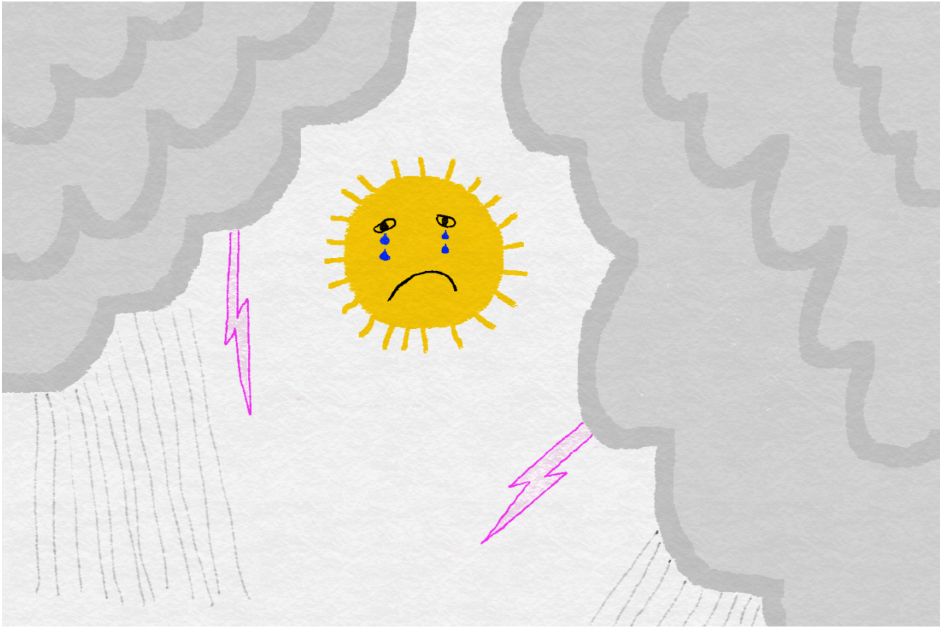
$$c(\mathcal{A}, \theta) = \inf_{\alpha \in [0,\infty)^{\mathcal{A}}} \sum_{a \in \mathcal{A}} \alpha(a) \Delta_a$$

$$\text{subject to } \|a\|^2_{H_\alpha^{-1}} \leq \frac{\Delta_a^2}{2} \text{ for all } a \in \mathcal{A} \text{ with } \Delta_a > 0\,,$$

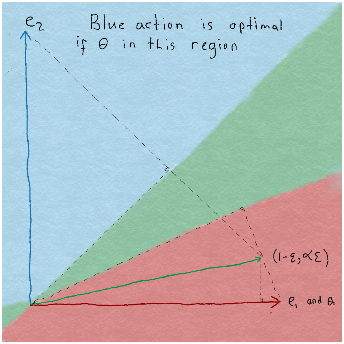
where $H = \sum_{a \in \mathcal{A}} \alpha(a) a a^\top$.

Outline

Poor Outlook for Optimism



Poor Outlook for Optimism



Actions:
$\mathcal{A} = \{a_1, a_2, a_3\}$, $a_1 = e_1$, $a_2 = e_2$, $a_3 = (1-\varepsilon, \gamma\varepsilon)$. $\varepsilon > 0$ small, $\gamma \geq 1$.

Let $\theta = (1, 0)$, so $a^* = a_1$.

Solving for the lower bound,
$\alpha(a_2) = 2\gamma^2$
and $\alpha(a_3) = 0$, $c(\mathcal{A}, \theta) = 2\gamma^2$ and

$$\liminf_{n\to\infty} \frac{R_n^\pi(\mathcal{A}, \theta)}{\log(n)} = 2\gamma^2 \,.$$

Moreover, for γ large, ε sufficiently small, π “optimistic”,

$$\limsup_{n\to\infty} \frac{R_n^\pi(\mathcal{A}, \theta)}{\log(n)} = \Omega(1/\varepsilon) \,,$$

Instance-Optimal Asymptotic Regret

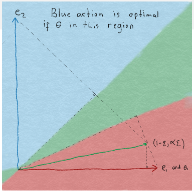
Theorem

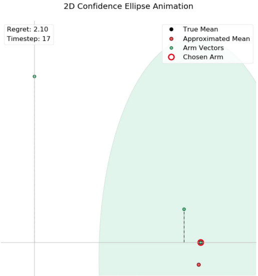
There exists a policy π that is consistent and satisfies

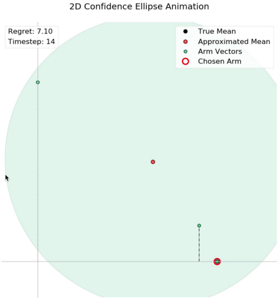
$$\limsup_{n\to\infty} \frac{R_n^\pi(\mathcal{A},\theta)}{\log(n)} = c(\mathcal{A},\theta)\,,$$

where $c(\mathcal{A},\theta)$ was defined in the lower bound.

Illustration: LinUCB







Summary

The instance-optimal regret of consistent algorithms is asymptotically $c(\mathcal{A}, \theta) \log(n)$.

Optimistic algorithms fail to achieve this: Their regret can be worse by an arbitrarily large constant factor.

Remember:

Finite-armed bandits

Case (a): $\mathcal{A}_t$ has always the same number of vectors in it:

"finite-armed stochastic contextual bandit".

Case (b): Also, $\mathcal{A}_t$ does not change, or $\mathcal{A}_t = \{a_1, \ldots, a_K\}$:

"finite-armed stochastic linear bandit".

Case (c): If the vectors in $\mathcal{A}_t$ are also orthogonal to each other:

"finite-armed stochastic bandit".

Difference between cases (c) and (b):

- Case (c): Learn about mean of arm $i \Leftrightarrow$ Choose action i;
- Case (b): Learn about mean of arm $i \Leftrightarrow$ Choose action j s.t. $\langle x_j, x_i \rangle \neq 0$.

Departing Thoughts

- These results are from Lattimore and Szepesvári (2016)
- The asymptotically optimal algorithm is given there (the algorithm solves for the optimal allocation, while monitoring whether things went wrong)
- Combes et al. (2017) refine the algorithm and generalize it to other settings.
- Soare et al. (2014), in best arm identification with linear payoff functions, gave essentially the same example that we use to argue for the large regret of optimistic algorithms.
- Open questions:
  - Simultaneously finite-time near-optimal and asymptotically optimal algorithm
  - Changing, or infinite action sets?

Summary

Summary of This Talk

- Contextual vs. linear bandits:

  Changing action sets can model contextual bandits

- Optimistic algorithms:
  - Optimism can achieve minimax optimality
  - Optimism can be expensive
  - Optimistic algorithms require a careful design of the underlying confidence sets

- Sparsity:

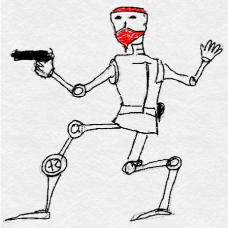
  Exploiting sparsity is sometimes at odds with the requirement to collect rewards

What's Next for Bandits?

- Today: Finite-armed and linear stochastic bandits.
- But bandits come in all forms and shapes!
  - Adversarial (finite, linear, . . .)
  - Combinatorial action sets: From shortest path to ranking
  - Continuous action sets, continuous time, delays
  - Resourceful, nonstationary, various structures (low-rank), . . .
- Nearby problems:
  - Reinforcement learning/Markov decision processes
  - Partial monitoring

Learning Material

- Bandit Visualizer:
  `https://github.com/alexrutar/banditvis`
- Online bandit simulator:
  `http://downloads.tor-lattimore.com/bandits/`
- Most of this tutorial (and more): `http://banditalgs.com`
  - Book to be published by early next year: Looking for reviewers!
  - Tor's lightweight C++ bandit library
- Sebastien Bubeck's tutorial
  - Blog post 1
  - Blog post 2
- Bubeck and Cesa-Bianchi's book;
  (Bubeck and Cesa-Bianchi, 2012)



`banditalgs.com`

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