Contents

Math 350 - Advanced Calculus
Homework 8

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Problem 1 Let $\mathsf{C}[0,1]$ denote set of continuous functions on interval $[0,1]$. If $f \in \mathsf{C}[0,1]$, define

$$||f|| = \sup\{|f(x)| : 0 \le x \le 1\}$$

(i) Prove that $||\lambda f|| = |\lambda|||f||$, for every real number λ and every $f \in \mathsf{C}[0,1]$.

Proof. We claim that $|\lambda \cdot f(x)| = |\lambda| \cdot |f(x)|$ for all $x \in \mathbb{R}$ and for all $\lambda \in \mathbb{R}$. There are two cases to consider,

- If λ and $f(x)$ have the same sign, then $|\lambda \cdot f(x)| = \lambda \cdot f(x) = |\lambda| \cdot |f(x)|$
- If λ and $f(x)$ have opposite sign, then $|\lambda \cdot f(x)| = -(\lambda \cdot f(x))$ where $|\lambda| \cdot |f(x)| = -(\lambda \cdot f(x))$

Thus in both cases, $|\lambda \cdot f(x)| = |\lambda| \cdot |f(x)|$. Now go back to our problem, we have that

$$||f|| = \sup\{|f(x)| : 0 \le x \le 1\}$$

Hence

$$||\lambda \cdot f|| = \sup\{|\lambda \cdot f(x)| : 0 \le x \le 1\} = |\lambda| \cdot \sup\{|f(x)| : 0 \le x \le 1\} = |\lambda|||f||$$

□

(ii) Prove that $||f|| = 0$ if and only if $f(x) = 0$ for all $x \in [0,1]$.

Proof. We have $||f|| = 0 \Rightarrow \sup\{|f(x)| : 0 \le x \le 1\} = 0$. By definition of sup we have $|f(x)| \le 0$ for all x, and by definition of absolute value, we have $0 \le f(x)$. Combine these two inequalities, $0 \le |f(x)| \le 0$ implies $f(x) = 0$ for all $x \in [0,1]$. □

(iii) Prove that $||f + g|| \le ||f|| + ||g||$ for every $f, g \in \mathsf{C}[0,1]$.

Proof. We have $||f+g|| = \sup\{|(f+g)(x)| : 0 \le x \le 1\}$. By definition of sup, for all $\epsilon > 0$, there exists some $a \in [0,1]$ such that

$$||f+g|| - |f+g|(a) < \epsilon$$

which implies that

$$||f+g|| - (|f|(a) + |g|(a)) < \epsilon$$

Since $|f|(a) \le ||f||$ and $|g|(a) \le ||g||$, it follows that

$$||f+g|| - (||f|| + ||g||) < \epsilon$$

Since this is true for every $\epsilon > 0$, it follows that $||f+g|| \le ||f|| + ||g||$. □

(iv) Find $f, g \in \mathsf{C}[0,1]$ such that $||f+g|| \ne ||f|| + ||g||$. This follows from (iii).

Problem 2 Prove that if A and B are connected subsets of $\mathbb{R}$ and $A \cap B = \emptyset$, then $A \cup B$ is connected.

Proof. Let $S = A \cup B$ is disconnected since $A \cap B = \emptyset$, $A \neq \emptyset$ and $B \neq \emptyset$, where A, B are open. Suppose that A and B are open and connected, $A \cap B = \emptyset$ and $S = A \cup B$ is disconnected. Then $S \subset (U \cup V)$, where U, V are open in $\mathbb{R}$, $U \cap S \neq \emptyset$, $V \cap S \neq \emptyset$, and $U \cap V \cap S = \emptyset$. Since A is connected, and $A \subset U \cup V$, we must have $U \cap A = \emptyset$, or $V \cap B = \emptyset$. That is, either $A \subset U$ or $A \subset V$. Similarly, we must have either $B \subset U$, or $B \subset V$. In any case, we always have $(A \cap B) \subset (U \cap V)$. Since $A \cap B = \emptyset$, this contradicts the fact that $S \cap U \cap V = \emptyset$. Thus S is connected. $\square$

Problem 3 True of False: If f is continuous and S is connected, then $f^{-1}S$ is connected.

Problem 4 Prove that if f is continuous on an interval J and $f(x)$ is rational for every x in J, then f is constant on J.

Proof. Suppose that f is not constant, so for $a, b \in J$, $f(a) \neq f(b)$. Then by Intermediate Value Theorem, for any $c \in (f(a), f(b)$ there exists $x \in (a, b)$ such that $f(x) = c$. This contradicts that f takes on all rational numbers only because between 2 rational numbers $f(a), f(b)$, there are irrational numbers. $\square$

Problem 5 Suppose f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$.

(i) Prove that either $f(\frac{a+b}{2}) = 0$ or f has different signs at the end points $[a, (a+b)/2]$ or f has different signs at the end points of $[(a+b)/2, b]$. If $f(\frac{a+b}{2}) \neq 0$, let I_1 be one of the two intervals on which f has different signs at the endpoints. Now bisect I_1. Then either f is 0 at the midpoint, or f has opposite signs at the endpoints of one of the two intervals into which I_1 is bisected. Let I_2 be such an interval. Continue this way to define I_n for each natural number n (unless f is 0 at some midpoint).

Proof. If $f((a+b)/2) \neq 0$, then this number is either > 0 and f has different signs at the endpoints of $[a, (a+b)/2]$, or < 0 and f has different signs at the endpoints of $[(a+b)/2, b]$. $\square$

(ii) Prove that there is a point $x \in (a, b)$ where $f(x) = 0$.

Proof. Let c be in each interval I_n. If $f(c) < 0$, then there is some $\delta > 0$ such that $f(x) < 0$ for all $x \in [a, b]$ with $|x - c| < \delta$. Choose n with $\frac{1}{2^n} < \delta$. Since c is in I_n which has total length of $\frac{1}{2^n}$, it follows that all points x of I_n satisfy $|x - c| < \delta$. This contradicts the fact that f changes sign on I_n. Similarly, we cannot have $f(c) > 0$, so $f(c) = 0$. $\square$

(iii) Use the scheme described in (i) and (ii) to approximate the solution of $x^3 + 6x - 2 = 0$ with an error smaller than $\frac{1}{100}$.

Proof. If $f(x) = x^3 + 6x - 2$, then $f(0) = -2$ and $f(1/3) > 0$. Let $[a, b] = [0, 1/3]$. Since length $I_n = \frac{1}{3} \cdot 2^n$ and $3 \cdot 2^5 < 100 < 3 \cdot 2^6$, any of the endpoints of I_6 will approximate the solution of $f(x) = 0$ with an error smaller than $1/100$. $\square$

Problem 6 Find an integer n such that the polynomial equations $x^3 - x + 3 = 0$ has a solution between n and $n+1$.

Proof. By Polynomial Root Theorem, we know that $f(x) = x^3 - x + 3$ has a solution because $n = 3$ is odd. Apply Location of Root Theorem, we want to find n and $n+1$ such that $\text{sign}(f(n)) \neq \text{sign}(f(n+1))$, then the solution must be between $[n, n+1]$. Since the first term x^3 is much larger than other term, to make $f(x)$ negative, we obviously need a negative x, we could try
$f(-2) = (-2)^3 - (-2) + 3 = -8 + 4 + 3 = -1$
$f(-1) = (-1)^3 - (-1) + 3 = -1 + 1 + 3 = 3$
It follows that $f(-2) < 0 < f(-1)$, thus the solution is between $[-2, -1]$. $\square$

Problem 7 Prove that there is some number x such that $\sin(x) = x - 1$.

Proof. Let $f(x) = x1\sin(x)$. Then $f(0) = 1 < 0$ and $f(\pi/2) = \pi/2 > 0$, so there is $x \in (0, \pi/2)$ such that $f(x) = 0 \Leftrightarrow \sin(x) = x1$. $\square$

Problem 8

(a) Suppose that f is continuous on the interval $[0, 1]$ and that $0 \leq f(x) \leq 1$ for all $x \in [0, 1]$. Prove that $f(x) = x$ for some numbers $x \in [0, 1]$.

Proof. Recall,
Location of Roots Theorem. If f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$, then there is some $x \in [a, b]$ such that $f(x) = 0$.
Our goal is to apply this Theorem for $f(x) - x$. Let $g(x) = f(x) - x$, we have that $f(1) - 1 \leq 0$ and $f(0) - 0 \geq 0$. If the "=" occurs, then $f(1) - 1 = 0 \Rightarrow f(1) = 1$ and $f(0) - 0 = 0 \Rightarrow f(0) = 0$ which is what we want to show. Otherwise, we have $h(1) < 0$ and $h(0) > 0$ which implies there is some $x \in [0, 1]$ such that $h(x) = 0$ by Location of Roots Theorem. Hence, $f(x) - x = 0 \Leftrightarrow f(x) = x$ for some $x \in [0, 1]$. $\square$

(b) Let f be continuous and bounded above and below on $\mathbb{R}$. Prove that there is some number x such that $f(x) = x$.

Proof. Let m, n be lower bound and upper bound of f respectively, then $f(x) \geq m, f(x) \leq n$ for all $x \in \mathbb{R}$. If $f(x) = m$ or $f(x) = n$ for all x, then $f(x)$ is constant, so $f(x) = x$ is trivially true. If not, there is some x such that $m < f(x) < n$. Using the same idea from part (a), we consider

$$g(x) = f(x) - x$$

We have that $g(m) = f(m) - m > 0$ and $g(n) = f(n) - n < 0$ so apply Location of Roots Theorem, we have there is some $x \in \mathbb{R}$ such that $g(x) = 0 \Leftrightarrow f(x) = x$ which was what we want to show.

$\square$

Problem 9 One morning, exactly at sunrise, a Buddhist monk began to climb a tall mountain. The narrow path, no more than a foot or two wide, spiraled around the mountain to a glittering temple at the summit The monk ascended the path at varying rates of speed stopping along the way to rest and to eat the dried fruit he carried with him. He reached the temple shortly before sunset. After several days of fasting and meditation he began his journey back along the same path, starting at sunrise and again walking at variable speeds with many pauses along the way. His average speed descending was, of course, greater than his average climbing speed. Prove that there is a spot along the path that the monk will occupy on both trips at precisely the same time of day. (Martin Gardner, in *My Best Mathematical Puzzles* Dover 1994.)

Problem 10 A function f defined on an interval I has the Intermediate Value Property on I if for any two numbers $a < b$ in I and every y strictly between $f(a)$ and $f(b)$, there is $c \in (a, b)$ such that $f(c) = y$.

(i) Prove that the function f given by $f(x) = \sin(1/x)$ if $x \neq 0$ and $f(0) = 0$ has the Intermediate Value Property on the interval $[0, B]$ for any $B > 0$.

Proof. If $0 < a < b$ are two numbers in $[0, B]$, then we apply the Intermediate Value Theorem to $f(x) = \sin(1/x)$ on $[a, b]$ because f is continuous on $[a, b]$. However, if $0 = a < b$ and x is strictly between $0 = f(0)$ and $f(b)$, let n be a natural number such that $\frac{2}{b} < (2n + 1) \cdot \pi$ so that the interval

$$I = \left[\frac{2}{2n+3} \cdot \pi, \frac{2}{2n+1} \cdot \pi\right]$$

is contained in $[0, b]$. The function $f(x) = \sin(1/x)$ is continuous on I and takes on the values 1 and 1 at the endpoints of I. Since $1 \leq f(x) \leq 1$, the Intermediate Value Theorem applied to f on I implies that given any y such that $1 < y < 1$, there is c in I such that $f(c) = y$. In particular, if y is strictly between $f(a)$ and $f(b)$, then $1 < y < 1$ also, and c in I satisfies $0 = a < c < b$, as desired. ☐

(ii) Prove that if f is non-decreasing on the interval I and has the Intermediate Value Property on I, then f is continuous on I. (Adopting the terminology of the textbooks, f is said to be increasing on I if $f(x) < f(y)$ whenever $x < y$ in I; it is said to be non-decreasing if $f(x) \leq f(y)$ whenever $x < y$.

Proof. Suppose that there is a in I where f fails to be continuous. Then there is a sequence (x_n) in I such that $x_n \to a$ but $f(x_n)$ does not converge to $f(a)$. We may assume, by taking a subsequence if necessary, that x_n increases (or decreases) to a. Then $f(x_n)$ is non decreasing and bounded above by $f(a)$, thus it converges to a number p with $p < f(a)$. Let q be a number such that $p < q < f(a)$. For each x_n we have $f(x_n) \leq p < q$, so the Intermediate Value Property of f on the interval $[x_n, a]$ implies the existence of y_n in (x_n, a) such that $f(y_n) = q$. The sequence (y_n) converges to a and $f(y_n) = q$ for all n. Since x_n also converges to a, given n there is m such that $y_n < x_m$, but $f(y_n) = q > p \geq f(x_m)$, contradicting that f is non decreasing.

☐

Problem 11 Let $f(x) = x^2$ if x is rational, and $f(x) = 0$ if x is not rational. Prove that f is differentiable at 0.

$$f(x) = \begin{cases} x^2 & \text{, if } x \text{ is rational} \\ 0 & \text{, if } x \text{ is not rational} \end{cases}$$

Proof. To prove that f is differentiable at 0, we want to show that:

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h} \text{ exists}$$

Since 0 is both rational and irrational we have that $f(0) = 0$, it reduces to prove $\lim_{h \to 0} \frac{f(h)}{h}$ exists. On the other hand, we have

$$\frac{f(x)}{x} = \begin{cases} \frac{x^2}{x} = x & \text{, if } x \text{ is rational} \\ 0 & \text{, if } x \text{ is not rational} \end{cases}$$

So in either case, as $x \to 0$, $\frac{f(x)}{x} = 0$, in other words $\lim_{h \to 0} \frac{f(h)}{h} = 0$, so $f'(0)$ exists or f is differentiable at 0.

☐

Problem 12 Let f be a function such that $|f(x)| \leq x^2$ for all x. Prove that f is differentiable at 0 and find $f'(0)$.

Proof. Since $|f(x)| \leq x^2$ for all x, we have $0 \leq |f(x)| \leq 0^2 = 0$ which implies $f(0) = 0$. Also since $\left|\frac{f(h)}{h}\right| \leq \frac{h^2}{|h|}$, it follows that $-h \leq \frac{f(h) - f(0}{h} \leq -h \Rightarrow \lim_{h \to 0} \frac{f(h)}{h} = 0$, i.e, $f'(0) = 0$. □

Problem 13 Find $f'(x)$ for $-1 < x < 1$ if $f(x) = \sqrt{1 - x^2}$.

Proof. Apply Chain rule, we have

$$f'(x) = \frac{1}{2}(1 - x^2)^{1/2-1} \cdot (-2x) = \frac{-x}{\sqrt{1 - x^2}}$$

□

Problem 14 If f is differentiable three times and $f'(x) \neq 0$, the Schwartz derivative of f at x is defined to be

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

(i) Show that

$$S(f \circ g) = [Sf \circ g] \cdot (g')^2 + Sg$$

Consider

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$
$$(f \circ g)''(x) = f''(g(x)) \cdot g'(x)^2 + f'(g(x)) \cdot g''(x)$$

$$\begin{aligned}(f \circ g)'''(x) &= [f'''(g(x)) \cdot g'(x)^3 + 2f''(g(x)) \cdot g'(x)g''(x)] + [f''(g(x)) \cdot g'(x)g''(x) + f'(g(x)) \cdot g'''(x)] \\ &= f'''(g(x)) \cdot g'(x)^3 + 3f''(g(x)) \cdot g'(x)g''(x) + f'(g(x))g'''(x)\end{aligned}$$

Plug it back into the the Schwartz derivative, we have

$$\begin{aligned}Sf(x) &= \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 \\ &= \frac{(f \circ g)'''}{(f \circ g)'} - \frac{3}{2}\left(\frac{(f \circ g)''}{(f \circ g)'}\right)^2 \\ &= \frac{(f''' \circ g)g'^2}{f' \circ g} + \frac{3(f'' \circ g)g''}{f' \circ g} + \frac{g'''}{g'} - \frac{3}{2}\left(\frac{(f'' \circ g) \cdot g'}{f' \circ g} + \frac{g''}{g'}\right)^2 \\ &= \frac{(f''' \circ g)g'^2}{f' \circ g} + \frac{3(f'' \circ g)g''}{f' \circ g} + \frac{g'''}{g'} - \frac{3}{2}\left(\frac{(f'' \circ g) \cdot g'}{f' \circ g}\right)^2 - 3\frac{(f'' \circ g)g''}{f' \circ g} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2 \\ &= \left[\frac{f'''}{f'} \circ g - \frac{3}{2}\frac{f'' \circ g}{f' \circ g}\right] \cdot g'^2 + \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2 \\ &= [Sf \circ g] \cdot g'^2 + Sg\end{aligned}$$

(ii) Show that if $f(x) = \frac{ax + b}{cx + d}$ with $ad - bc \neq 0$, then $Sf = 0$, and consequently $S(f \circ g) = Sg$.

Proof. Consider

$$f'(x) = \frac{a(cx+d) - c(ax+b)}{(cx+d)^2} = \frac{ad-bc}{(cx+d)^2}$$
$$f''(x) = \frac{-2c(ad-bc)}{(cx+d)^3}$$
$$f'''(x) = \frac{6c^2(ad-bc)}{(cx+d)^4}$$

Hence,

$$Sf = \frac{6c^2}{(cx+d)^2} - \frac{3}{2}\left(\frac{-2c}{cx+d}\right)^2 = 0$$

□

Problem 15 A function f is Lipschitz of order α at x is, there is a constant C such that

$$|f(x) - f(y)| \leq C|x-y|^\alpha$$

for all y in an interval around x. Then function f is Lipschitz of order α on an interval if (1) holds for all x and y in the interval.

(a) If f is Lipschitz of order $\alpha > 0$ at x, then f is continuous at x.

Proof. Since $|f(x) - f(x+h)| \leq C|h|^\alpha$ it follows that $\lim_{h\to 0} f(x+h) = f(x)$. □

(b) If f is Lipschitz of order $\alpha > 0$ on an interval, then f is uniformly continuous on this interval.

Proof. Given $\epsilon > 0$ choose $\delta = \left(\frac{\epsilon}{C}\right)^{1/\alpha}$ so that $\delta^\alpha = \frac{\epsilon}{C}$. Then for all x and y in the interval with $|x-y| < \delta$ we have

$$|f(x) - f(y)| \leq C|x-y|^\alpha < C \cdot \frac{\epsilon}{C} = \epsilon$$

□

(c) If f is differentiable at x, then f is Lipschitz of order 1 at x.

Proof. If f is differentiable at x, then

$$\lim_{y\to x} \frac{f(y)-f(x)}{y-x} = f'(x)$$

so for all y in some interval around x we have that

$$\left|\frac{f(y)-f(x)}{y-x} - f'(x)\right| < 1$$

Hence,

$$\left|\frac{f(y)-f(x)}{y-x}\right| < 1 + |f'(x)|$$

or

$$|f(y) - f(x)| \leq (1 + |f'(x)|)|y-x|$$

so we can choose $C = 1 + |f'(x)|$ or we $C = \epsilon + |f'(x)|$ for any $\epsilon > 0$. The converse is not true though, for example $f(x) = x$. □

(d) If f is Lipschitz of order $\alpha > 1$, then f is differentiable at x and $f'(x) = 0$.

Proof. We have that

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}$$

Now consider

$$\left| \frac{f(y) - f(x)}{y - x} \right| \le |x - y|^{n-1}$$

and $\lim_{y \to x} |x - y|^{n-1} = 0$ since $n - 1 > 0$. Consequently $f'(x) = 0$ for all x and so f is constant.

□

Problem 16 Let $a > 0$. Show that the maximum value of

$$f(x) = \frac{1}{1 + |x|} + \frac{1}{1 + |x - a|}$$

is $\dfrac{2 + a}{1 + a}$. Consider the interval $(-\infty, 0)$, $(0, a)$ and (a, ∞) separately.

Proof.

$$f(x) = \begin{cases} \dfrac{1}{1 - x} + \dfrac{1}{1 + a - x} & , x < 0 \\ \dfrac{1}{1 + x} + \dfrac{1}{1 + a - x} & , 0 < x < a \\ \dfrac{1}{1 + x} + \dfrac{1}{1 + x - a} & , a < x \end{cases}$$

So the derivative is

$$f'(x) = \begin{cases} \dfrac{1}{(1 - x)^2} + \dfrac{1}{(1 + a - x)^2} & , x < 0 \\ \dfrac{-1}{(1 + x)^2} + \dfrac{1}{(1 + a - x)^2} & , 0 < x < a \\ \dfrac{-1}{(1 + x)^2} - \dfrac{1}{(1 + x - a)^2} & , a < x \end{cases}$$

Thus f is increasing on $(-\infty, 0]$ and decreasing on $[a, \infty)$ so the maximum of f on $[0, a]$ is the maximum on $\mathbb{R}$. If $f'(x) = 0$ for $x \in (0, a)$ then

$$(1 + x)^2 - (1 + a - x)^2 = 0$$

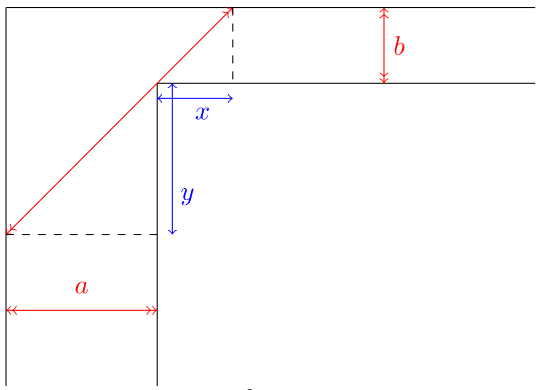
whose only solution is $x = \dfrac{a}{2}$. Since

$$f(\frac{a}{2}) = \frac{4}{2 + a} < \frac{2 + a}{1 + a} = f(0) = f(a)$$

The maximum value is $\dfrac{2 + a}{1 + a}$. □

Problem 17 Find the length of the longest ladder that can be moved around a right-angle corner from a corridor of width a to a corridor of width b.

Proof.



We note that the ratio $\frac{b}{x} = \frac{y}{a}$, so the length of ladder is

$$L = \sqrt{b^2 + x^2} + \sqrt{a^2 + \frac{a^2 b^2}{x^2}} = \sqrt{b^2 + x^2} + \frac{a}{x}\sqrt{x^2 + b^2} = \left(1 + \frac{a}{x}\right)\sqrt{x^2 + b^2}$$

The maximum length of the ladder is the minimum length of this length of the dashed line. Set the derivative of the above equation to 0 and solve for x,

$$\begin{aligned} & \frac{-a}{x^2}\sqrt{x^2 + b^2} + \left(1 + \frac{a}{x}\right)\frac{x}{\sqrt{x^2 + b^2}} = 0 \\ \Leftrightarrow \quad & \left[\frac{-a}{x^2}(x^2 + b^2) + x + a\right] \cdot \frac{1}{\sqrt{x^2 + b^2}} \\ \Leftrightarrow \quad & ax^2 + ab^2 = x^3 + ax^2 \\ \Leftrightarrow \quad & x = a^{1/3} \cdot b^{2/3} \end{aligned}$$

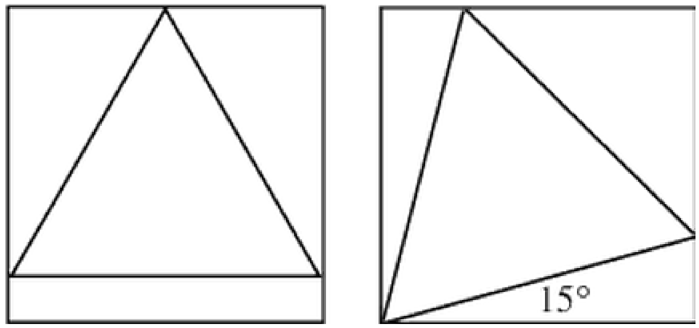
Therefore the length is

$$\left(1 + \frac{a^{2/3}}{b^{2/3}}\right) \cdot \sqrt{a^{2/3} b^{4/3} + b^2} = (b^{2/3} + a^{2/3}) \cdot \sqrt{\frac{a^{2/3} b^{4/3} + b^2}{b^{4/3}}} = (b^{2/3} + a^{2/3})^{3/2}$$

□

Problem 17 Find the equilateral triangle of maximum area that can be inscribed in a unit square.

Proof. From MathWorld



The smallest equilateral triangle which can be inscribed in a unit square (left figure) has side length and area

$$s = 1 \tag{18}$$
$$A = \frac{1}{4}\sqrt{3} \approx 0.4330. \tag{19}$$

The largest equilateral triangle which can be inscribed (right figure) is oriented at an angle of $15°$ and has side length and area

$$s = \sec(15°) = \sqrt{6} - \sqrt{2} \tag{20}$$
$$A = 2\sqrt{3} - 3 \approx 0.4641 \tag{21}$$

(Madachy 1979).

□

Problem 18 Suppose that f and g are two differentiable functions which satisfy $fg' - f'g = 0$. Prove that if $f(a) = 0$ and $g(a) \neq 0$, then $f(x) = 0$ for all x in an interval around a.

Proof. On any interval where $h(x) = \dfrac{f(x)}{g(x)}$ is defined, it is differentiable and by hypothesis,

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} = 0$$

so f/g is constant on that interval. So if $f(a) = 0$, then $f(x) = 0$ in an interval around a which $g \neq 0$. □

Problem 19

(a) Give an example of a function f for which $\lim_{x\to\infty} f(x)$ exists, but $\lim_{x\to\infty} f'(x)$ does not exist.

Proof. An example is $f(x) = \dfrac{\sin(x^2)}{x}$, then $\lim_{x\to\infty} f(x) = 0$, but

$$f'(x) = \frac{2x^2\sin(x^2) - \sin(x^2)}{x^2} = 2\sin(x^2) - \frac{\sin(x^2)}{x^2}$$

so $\lim_{x\to\infty} f'(x)$ does not exist. □

(b) Prove that if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} f'(x)$ both exist, then $\lim_{x\to\infty} f'(x) = 0$.

Proof. Let $L = \lim_{x\to\infty} f'(x)$. If $L < 0$ then there would be some N such that $|f'(x) - L| < \dfrac{|L|}{2}$ for $x > N$. This would imply that $f'(x) > \dfrac{|L|}{2}$. But that would imply, by the Mean Value Theorem that

$$f(x) > f(N) + \frac{(x-N)|L|}{2}, \quad \text{for } x > N$$

which would mean that $\lim_{x\to\infty} f(x)$ does not exists. Similarly, $\lim_{x\to\infty} f'(x)$ cannot be < 0. □

(c) Prove that if $\lim_{x\to\infty} f(x)$ exists and $\lim_{x\to\infty} f''(x)$ exists, then $\lim_{x\to\infty} f''(x) = 0$.

Proof. Let $I = \lim_{x\to\infty} f''(x)$. If $L > 0$ then as in part (a), we have $\lim_{x\to\infty} f'(x) = \infty$. Apply the Mean Value Theorem shows that $\lim_{x\to\infty} f(x) = \infty$ contradicting the hypothesis. Similarly, $\lim_{x\to\infty} f''(x)$ cannot be < 0. □

Problem 20 Suppose that $f'(x) \geq M > 0$ for all $x \in [0,1]$. Show that there is an interval of length $\dfrac{1}{4}$ on which $|f| \geq \dfrac{M}{4}$.

Proof. Note that f is increasing. If $f(1/2) \geq 0$, then $f(3/4) \geq M/4$, so certainly $f \geq M/4$ on the interval $[3/4, 1]$. On the other hand, if $f(1/2) \leq 0$ then $f(1/4) \leq -M/4$ so $f \leq -M/4$ on the interval $[0, 1/4]$. □

Problem 21

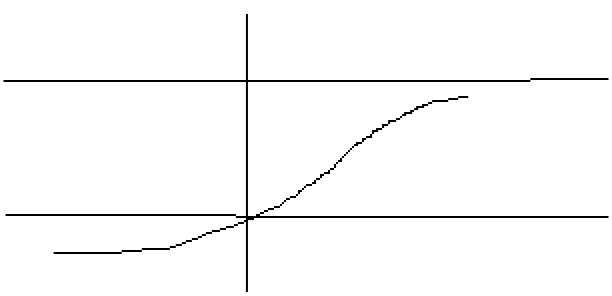
(a) Suppose that $f'(x) > g'(x)$ for all x and that $f(a) = g(a)$. Show that $f(x) > g(x)$ for $x > a$ and $f(x) < g(x)$ for $x < a$.

Proof. Apply the Mean Value Theorem to $f - g$. If $x > a$, then

$$\frac{f(x) - g(x)}{x - a} = \frac{f(x) - g(x) - [f(a) - g(a)]}{x - a} = f'(y) - g'(y) > 0 \text{ for some } y \in (a, x)$$

Since $x - a > 0$, it follows that $f(x) - g(x) > 0$. Similarly, if $x - a < 0$, then $f(x) < g(x)$. □

(b) Show an example that these conclusion do not follow without the hypothesis $f(a) = g(a)$.



(c) Suppose that $f(a) = g(a)$, and that $f'(x) \geq g'(x)$ for all x, and that $f'(x_0) > g'(x_0)$ for some $x_0 > a$. Show that $f(x) > g(x)$ for all $x \geq x_0$.

Proof. An argument similar to that for part (a) shows that $f(x) \geq g(x)$ for all $x > a$. More generally, $(f-g)(y) \geq (f-g)(x)$ for $a < x < y$. If we had $0 = f(x_0) - g(x_0) = (f-g)(x_0)$, then since $(f-g)(a) = 0$, we could have $(f-g)$ constant on $[a, x]$. But then we could not have $(f-g)'(x_0) > 0$. So we have $(f-g)(x_0) > 0$ which then implies that $(f-g)(x) > 0$ for all $x > x_0$. □

Problem 22 Let $a \neq 0$ and n be even. Prove that the polynomial equation $x^n + a^n = (x+a)^n$ has exactly one (real) solution.

Proof. Note first that if n is even, then $n1 > 1$ is odd and the function $g(x) = nx^{n-1}$ is strictly increasing (in particular, one-one). Let $f(x) = (x+a)^n - x^n - a^n$. If $f(x_0) = 0$ for some $x_0 \neq 0$, then $f'(c) = 0$ for some c between 0 and x_0, by Rolles Theorem. But $f'(x) = n(x+a)^{n-1} - nx^{n-1}$, so that $f'(c) = 0$ implies that $g(c+a) = g(c)$, contradicting that g is one-one. □

Problem 23

(a) Give an example of a continuous function on $(0, 1)$ which is bounded but attains neither a maximum nor a minimum value.

Proof. The function $f(x) = x$ for $x \in (0, 1)$ has $\sup\{f(x) : 0 < x < 1\} = 1$ and $\inf\{f(x) : 0 < x < 1\} = 0$, but $0 < f(x) < 1$ for all $x \in (0, 1)$. □

(b) Suppose that f is continuous on $\mathbb{R}$ and that for any number M there exists $\delta > 0$ such that $f(x) > M$ if $|x| > \delta$. Prove that f attains a min.

Proof. Let $M = f(0)$ and let δ be such that $f(x) > f(0)$ for $|x| > \delta$. The function f is continuous on the compact interval $[\delta, \delta]$, so f attains a min value in that interval; that is there is $y \in [-\delta, \delta]$ such that $f(x) \geq f(y)$ for all $x \in [-\delta, \delta]$. In particular $f(0) \geq f(y)$ and thus $f(x) \geq f(0) \geq f(y)$ for all x such that $|x| > \delta$. □

Problem 24 Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $(1, \infty)$.

Proof. If $x, y > 1$, then $\dfrac{1}{\sqrt{x}+\sqrt{y}} < \dfrac{1}{1+1} = \dfrac{1}{2}$. Rationalizing yields,

$$\begin{aligned} |\sqrt{x} - \sqrt{y}| &= \frac{|\sqrt{x}-\sqrt{y}|(\sqrt{x}+\sqrt{y})}{\sqrt{x}+\sqrt{y}} \\ &= \frac{|x-y|}{\sqrt{x}+\sqrt{y}} < \frac{1}{2}|x-y| \end{aligned}$$

Therefore given $\epsilon > 0$, take $\delta = 2\epsilon$, then if $|x - y| < \delta$, then $|\sqrt{x} - \sqrt{y}| < \dfrac{1}{2} \cdot 2\epsilon = \epsilon$. □

Problem 25 Suppose that f were continuous on $[a, b]$ but not bounded on $[a, b]$, then f would be unbounded on either $[a, (a+b)/2]$ or $[(a+b)/2, b]$.

Proof. Let c be in each I_n. Since f is continuous at c, there is $\delta > 0$ such that f is bounded on the set of all points in $[0, 1]$ satisfying $|x - c| < \delta$. Choose n with $\dfrac{1}{2^n} < \delta$. Since c is in I_n, all points x of I_n satisfy $|x - c| < \delta$. This contradicts the fact that f is not bounded on I_n. □

Problem 26 Determine (either prove or give a counterexample) whether the following statements are true: (a) The union of two uncountable sets is uncountable. (b) The intersection of two uncountable sets is uncountable.

Proof. (a) True. If $A \cup B$ is countable, then there is a one-one mapping $f : A \cup B \to N$. The composite $f \circ i : A \to A \cup B \to N$ is one-one, and hence A is countable. (b) False. $A = (\infty, 1]$ and $B = [1, \infty)$ are uncountable but $A \cap B = \{1\}$ is countable. □

Problem 27 Prove that $a_n = \dfrac{2n-1}{n+3}$ converges to $L = 2$.

Proof. Given $\epsilon > 0$, let $N = \max(2, 7/\epsilon - 2)$. If $n > N$, then

$$\left|\frac{2n-1}{n+3} - 2\right| = \frac{7}{N+3} < \epsilon$$

$\square$

Problem 28 Suppose that a and b are two consecutive roots of the polynomial function f, but that a, b are not double roots, we can write $f(x) = (x-a)(x-b)g(x)$ where $g(a) \neq 0, g(b) \neq 0$.

(a) Prove that $g(a), g(b)$ have the same sign.

Proof. If $g(a)$ and $g(b)$ have opposite sign, then by Intermediate Value Theorem, there is $c \in (a, b)$ such that $g(c) = 0 \Rightarrow f(c) = 0$, contradicting a, b are consecutive roots of f. $\square$

(b) Prove that there is some number x with $a < x < b$ and $f'(x) = 0$.

Proof. Because a and b are roots of f, $f(a) = f(b) = 0$. Moreover, f is continuous on $[a, b]$ and differentiable on (a, b) because it is a polynomial. Thus by Rolle's Theorem, we have that $f'(x) = 0$ for some $x \in (a, b)$. Note that the derivative $f'(x) = (x-a)g(x) + (x-b)g(x) + (x-a)(x-b)g'(x)$, so that $f'(a) = (a-b)g(a) \neq 0$ and $f'(b) = (b-a)g(b) \neq 0$. $\square$

Problem 29 Let $f(x) = |x|^3$. Find $f'(x), f''(x)$, and find all numbers x for which $f'''(x)$ exists.

Proof. We have

$$f(x) = \begin{cases} x^3 & , \text{ if } x \geq 0 \\ -x^3 & , \text{ if } x < 0 \end{cases}$$

Hence

$$f'(x) = \begin{cases} 3x^2 & , \text{ if } x \geq 0 \\ -3x^2 & , \text{ if } x < 0 \end{cases}$$

For $x = 0$, we find

$$\lim_{h \to 0} \frac{f(0+h) - f(0}{h} = \lim_{h \to 0} \frac{|h|^3}{h} = \lim_{h \to 0} h \cdot |h| = 0$$

and we obtain that $f'(x) = 3x|x|$. Similarly, we have

$$f''(x) = \begin{cases} 6x & , \text{ if } x \geq 0 \\ -6x & , \text{ if } x < 0 \end{cases}$$

For $x = 0$, we have

$$f''(0) = \lim_{h \to 0} \frac{f'(h) - f'(0)}{h} = \lim_{h \to 0} \frac{3h|h|}{h} = 0$$

and we obtain $f''(x) = 6|x|$ for all x. It follows by similar arguments that $f'''(x) = 6\dfrac{|x|}{x}$ if $x \neq 0$, and that $f'''(0)$ does not exist. $\square$