# Math 512B. Homework 4. Solutions

**Problem 1.** (i) Prove that the remainder $R_{2n+1,0,\arctan}(x)$ of degree $2n+1$ of the function arctan satisfies

$$|R_{2n+1,0,\arctan}(x)| \leq \frac{1}{2n+3}$$

for $|x| \leq 1$.

(ii) Use the relation $\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$ (valid for $xy \neq 1$) to show that

$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3},$$

and that

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}.$$

(This last identity and its use for estimating $\pi$ are due to J. Machin.)

(iii) Use the above and the appropriate Taylor polynomial for arctan to show that $\pi = 3.14159\cdots$. That is, compute the first 5 decimal digits of $\pi$ exactly. (Be careful with many little errors; they tend to add up to make a big one.)

**Solution.**(i) We have shown that $R_{2n+1,0,\arctan}(x) = \int_0^x \frac{(-1)^{n+1}t^{2n+2}}{1+t^2}\,dt$, so

$$\left|\int_0^x \frac{(-1)^{n+1}t^{2n+2}}{1+t^2}\,dt\right| \leq \left|\int_0^x t^{2n+2}\right| = \frac{|x|^{2n+3}}{2n+3} \leq \frac{1}{2n+3}$$

if $|x| \leq 1$.

(ii) $\arctan\frac{1}{2} + \arctan\frac{1}{3} = \arctan\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\frac{1}{3}} = \arctan 1 = \frac{\pi}{4}$. Similarly, $\arctan\frac{1}{5} + \arctan\frac{1}{5} = \arctan\frac{5}{12}$, $4\arctan\frac{1}{5} = \arctan\frac{120}{119}$, and finally

$$4\arctan\frac{1}{5} - \arctan\frac{1}{239} = \arctan 1 = \frac{\pi}{4}.$$

(iii) To compute $\pi$ with an error $< 1/10^6$, we should compute $\pi/4$ with an error $< 1/4\cdot 10^6$. Since we need 5 values of arctan, we shoudl compute $\arctan\frac{1}{5}$ and $\arctan\frac{1}{239}$ each with an error $< 1/20\cdot 10^6 = 1/2\cdot 10^7$. Since the remainder $|R_{2n+1,0}(x)| \leq |x|^{2n+3}/2n+3$, we need to find $n$ such that

$$\frac{(1/5)^{2n+3}}{2n+3} < \frac{1}{2\cdot 10^7} \quad \text{and} \quad \frac{(1/239)^{2n+3}}{(2n+3)} < \frac{1}{2\cdot 10^7}.$$

It suffices to take $n = 5$ and $n = 0$, respectively.

**Problem 2.** For any real number $\alpha$ and integer $n \geq 0$, define the binomial coefficient $\binom{\alpha}{n}$ by

$$\binom{\alpha}{n} = \begin{cases} 1, & n = 0 \\ \dfrac{\alpha\cdot(\alpha-1)\cdot\ldots\cdot(\alpha-n+1)}{1\cdot 2\cdot\ldots\cdot n} & n > 0. \end{cases}$$

(i) (Not required) Let $f(x) = (1+x)^m$, where $m$ is a non-negative integer. What is the Taylor polynomial of degree $n$ for $f$ at 0?

(ii) Let $f(x) = (1+x)^\alpha$, where $\alpha$ is any real number, not necessarily a natural number. Prove that the Taylor polynomial of degree $n$ for $f$ at 0 is

$$P_{n,0,f}(x) = \sum_{k=0} \binom{\alpha}{k} x^k.$$

(iii) Find the integral form of the remainder $R_{n,0,f}(x)$ of degree $n$ for $f(x) = (1+x)^\alpha$ at 0, and prove that if $|x| < 1$, then

$$\lim_{n\to\infty} R_{n,0,f}(x) = 0.$$

The Taylor polynomials of $f(x) = (1+x)^\alpha$ can be used to approximate radicals quite fast and accurately. For example, if your want to find the first few decimals of $\sqrt[q]{p}$ (where $p$ is a positive number). However if you simply write $p^{1/q} = (1+x)^\alpha$ with $x = p-1$ and $\alpha = 1/q$, you may have $x > 1$ and in order to obtain a good approximation to $\sqrt[q]{p}$ you may be required to compute the remainder of very high degree for $f$ at 0.

(iii) To experience what I mean above, try it with $\sqrt{11} = (1+10)^{1/2}$. How large must $n$ be so that the remainder $R_{n,0}(10) < 1/10^2$?

There is a trick for approximating $\sqrt[q]{p}$ using Taylor polynomials which will speed up your calculations. For that you pick a number $d$ for which $\sqrt[q]{d}$ is known and such that $0 < p/d < 2$. Then write

$$\frac{p}{d} = 1 + x$$

with $|x| < 1$ and obtain

$$p^{1/q} = d^{1/q}(1+x)^{1/q}.$$

(iv) Use the above for approximating $\sqrt{11}$ with an error $< 1/10^7$. (There are many choices for $d$, choose wisely!)

**Solution.**(ii) The $n$th derivative of $f(x) = (1+x)^\alpha$ is easy to compute:

$$f^{(n)}(x) = \alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n+1)(1+x)^{\alpha-n}$$

. Thus the Taylor polynomial of degree $n$ for $f$ at 0 is

$$P_{n,0,f}(x) = 1 + \frac{\alpha}{1}x + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n.$$

(iii) The integral form of the remainder is $R_{n,0,f}(x) = \int_0^x (-1)^{n+1}\binom{\alpha}{n}(1+t)^{\alpha-n-1}(x-t)^n dt$. If $|x| < 1$ and $\alpha < n+1$, then

$$|R_{n,0,f}(x)| \leq (n+1)|x|^{n+1},$$

so $R_{n,0,f}(x) \to 0$ as $n \to \infty$ for $|x| < 1$.

**Problem 3** (Based on Giesy, *Mathematics Magazine*, 45 (1972), pp 148–149.). Let $f_n$ be the function defined by

$$f_n(x) = \frac{1}{2} + \cos 2x + \cos 4x + \cdots + \cos 2nx$$

(i) Show that

$$f_n(x) = \frac{\sin[(2n+1)x]}{2\sin x}$$

if $x$ is not an integral multiple of $\pi$. (Hint. Use the trigonometric identity $\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha\sin\beta$.)

(ii) Let $E_n$ be the number defined by

$$E_n = \int_0^{\pi/2} 2x f_n(x)\,dx$$

Prove that

$$E_n = \frac{\pi^2}{8} + \sum_{k=1}^{n} \frac{(-1)^k - 1}{2k^2}$$

In particular, for odd indexes:

$$\frac{\pi^2}{8} = E_{2n-1} + \sum_{k=1}^{n} \frac{1}{(2k-1)^2}.$$

(iii) Prove that

$$E_{2n-1} = \frac{1}{4n-1}\left[1 + \int_0^{\pi/2} u'(x)\cos(4n-1)x\,dx\right]$$

Hint. Let $u(x) = x/\sin x$ if $0 < x \le \pi/2$ and $u(0) = 0$, and let $v(x) = \sin(4n-1)x$; show that $u'$ and $v'$ are continuous on $[0, \pi/2]$, and the apply integration by parts to the integral used to define $E_{2n-1}$ in (ii).

(iv) Prove that $0 \le u(x) \le \pi/2$ for $x$ in $[0, \pi/2]$.

(v) Prove that $\lim_{n\to\infty} E_{2n-1} = 0$.

**Problem 4.** Let $f(x) = \log(1+x)$.

(i) Find the integral expression for the remainder $R_{n,0}(x)$ of degree $n$ for $f$ at 0.

(ii) Prove that if $0 < x \le 1$, then

$$|R_{n,0}(x)| < \frac{1}{n+1}.$$

(iii) Prove that if $-1 < x \le 0$, then

$$|R_{n,0}(x)| < \frac{1}{(1+x)(n+1)}.$$

**Solution.**(i) If $f(x) = \log(1+x)$, then $f^{(n+1)}(t) = \dfrac{(-1)^n n!}{(1+t)^{n+1}}$. Thus, the integral expression for the remainder is

$$R_{n,0}(x) = \begin{cases} \displaystyle\int_0^x \frac{(-1)^n}{(1+t)^{n+1}}(x-t)^n\,dt, & x \ge 0 \\ \displaystyle\int_x^0 \frac{(-1)^n}{(1+t)^{n+1}}(t-x)^n\,dt & x < 0 \end{cases}$$

There is second integral expression for the remainder $R_{n,0,f}$ which is obtained as follows. First write

$$\frac{1}{1+t} = 1 - t + t^2 - \cdots + (-1)^{n-1}t^{n-1} + \frac{(-1)^n t^n}{1+t}$$

so that

$$\begin{aligned} \log(1+x) &= \int_0^x \frac{1}{1+t}dt \\ &= x - \frac{x^2}{2} + \cdots + (-1)^{n-1}\frac{x^n}{n} \\ &\quad + \int_0^x \frac{(-1)^n t^n}{1+t}dt \end{aligned}$$

and thus

$$R_{n,0,f}(x) = \int_0^x \frac{(-1)^n t^n}{1+t}dt.$$

(ii) If $0 < x \le 1$, using the first integral expression in (i) for the remainder $R_{n,0,f}$ we obtain

$$|R_{n,0,f}(x)| \le \int_0^x (x-t)^n dt = \frac{1}{n+1}.$$

(iii) If $-1 < x \le 0$, then $0 \le 1+x \le 1+t$. By using the second expression for the remainder $R_{n,0,f}$ in (i), we obtain

$$\begin{aligned} |R_{n,0,f}(x)| &= \left|\int_0^x \frac{(-1)^n t^n}{1+t}dt\right| \\ &\le \frac{|x|^{n+1}}{(1+x)(n+1)} \le \frac{1}{(1+x)(n+1)}. \end{aligned}$$