Math 512A. Homework 7 Solutions

Problem 1. Let $f : E \subset \mathbf{R} \to \mathbf{R}$ be uniformly continuous. Prove that if (x_n) is a Cauchy sequence in E, then $(f(x_n))$ is also a Cauchy sequence. Show by counterexample that *uniformly* is necessary.

Solution. If f is uniformly continuous on E, then given $\varepsilon > 0$ there is $\delta > 0$ such that if x, y are in E and $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. Let (x_n) be a Cauchy sequence in E. Then given $\delta > 0$ there is N such that if $p, q > N$, then $|x_p - x_q| < \delta$, and thus $|f(x_p) - f(x_q)| < \varepsilon$, implying that $(f(x_n))$ is a Cauchy sequence.

A counterexample was described in class. Let $E = \{1, 1/2, 1/3, \cdots\}$ and $f(1/n) = 1$ is n is odd, $f(1/n) = -1$ if n is even. Then f is continuous but not uniformly continuous. The sequence $(x_n) = (1/n)$ in E is Cauchy but the sequence $(f(x_n)) = (1, -1, 1, -1, \cdots)$ is not Cauchy. □

Problem 2. (i) Prove that if f and g are uniformly continuous on E, then so is $f + g$.

(ii) Prove that if f and g are uniformly continuous and bounded on E, then fg is uniformly continuous on E.

(iii) Show that the conclusion in (ii) above does not hold if one of them is not bounded.

Solution. (i) Given $\varepsilon > 0$ there are $\delta_1 > 0$ and $\delta_2 > 0$ such that for any x, y in E, if $|x-y| < \delta_1$, then $|f(x)-f(y)| < \varepsilon/2$ and if $|x - y| < \delta_2$, then $|g(x) - g(y)| < \varepsilon/2$. Therefore, if $|x - y| < min\{\delta_1, \delta_2\}$, then

$$|(f + g)(x) - (f + g)(y)| \leq |f(x) - f(y)| + |g(x) - g(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

as desired.

(ii) Because f and g are bounded, there is $M > 0$ be such that $|f(x)| < M$ and $|g(x)| < M$ for all x in E. Because f and g are uniformly continuous on E, given $\varepsilon > 0$, there is $\delta > 0$ such that if x and y are in E and $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon/2M$ and $|g(x) - g(y)| < \varepsilon/2M$. Then

$$\begin{aligned}
|f(x)g(x) - f(y)g(y)| &= |f(x)g(x) - f(y)g(x) + f(y)g(x) - f(y)g(y)| && \text{(Add and subtract } f(y)g(x)\text{)}\\
&\leq |f(x)g(x) - f(y)g(x)| + |f(y)g(x) - f(y)g(y)| && \text{(Triangle Inequality)}\\
&\leq |g(x)||f(x) - f(y)| + |f(y)||g(x) - g(y)| && \text{(Extract common factors)}\\
&\leq M|f(x) - f(y)| + M|g(x) - g(y)| && (f \text{ and } g \text{ bounded by } M)\\
&< M\frac{\varepsilon}{2M} + M\frac{\varepsilon}{2M} && \text{(Uniform continuity)}\\
&= \varepsilon
\end{aligned}$$

(iii) Let $f(x) = x$ and $g(x) = \sin x$. Both f and g are uniformly continuous on $\mathbf{R}$ (you should prove that g is uniformly continuous) but the product $f \cdot g$ is not uniformly continuous on $\mathbf{R}$. To see this, let $\varepsilon = 1/2$. If n is a natural number, then

$$f(2n\pi + 1/2n\pi)g(2n\pi + 1/2n\pi) - f(2n\pi)g(2n\pi) = (2n\pi + 1/2n\pi)\sin(1/2n\pi) > \frac{\sin(1/2n\pi)}{1/2n\pi}$$

Since $\sin x/x \to 1$ as $x \to 0$, given $\delta > 0$ we can find a natural number n such that $1/2n\pi < \delta$ and $\dfrac{\sin(1/2n\pi)}{1/2n\pi} > 1/2$, proving that $f \cdot g$ is not uniformly continuous.

Note however that $f \cdot g$ takes Cauchy sequences to Cauchy sequences (Proof?). □

Problem 3. Let A and B be two nonempty sets of real numbers and suppose that $x \leq y$ for all x in A and all y in B.

(i) Prove that $\sup A \leq y$ for all y in B.

(ii) Prove that $\sup A \leq \inf B$.

Note. The supremum of a set A, $\sup A$, was defined in Homework 6. The infimum of a set B is $\inf B = -\sup(-B)$, where $-B$ is the set of all numbers x such that $-x$ is in B.

Solution. (i) If y is in B, then $x \le y$ for all x in A, so any y in B is an upper bound for A, thus $\sup A \le y$. (Review Homework 6.)

(ii) Part (i) shows that $\sup A$ is a lower bound for B, so $\sup A \le \inf B$. $\square$

Problem 4. (i) Consider a sequence of closed intervals $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$, $\ldots$. Suppose that $a_n \le a_{n+1}$ and $b_{n+1} \le b_n$ for all n. Prove that there is a point x which is in every I_n.

(ii) Prove that if $\operatorname{length} I_n \to 0$, then the point x in (i) is unique.

(iii) Show that this conclusion in Part (i) is false if we consider open intervals instead of closed intervals. Is it true if we consider open and bounded intervals?

Solution. (i) The sequence of left endpoints (a_n) is non-decreasing and bounded above (by b_1), so it converges to $a = \sup\{a_n \mid n \text{ in } \mathbf{N}\}$. Similarly the sequence of right end points (b_n) converges to $b = \inf\{b_n \mid n \text{ in } \mathbf{N}\}$. Since $a_n \le b_m$ (because $a_n \le a_{n+m} \le b_{n+m} \le b_m$), we have $a \le b$ (cf. previous problem). If x is any number such that $a \le x \le b$, then $a_n \le x \le b_n$ for all n, which means that x is in I_n for all n.

(ii) If $\operatorname{length} I_n = b_n - a_n \to 0$, then $a = b$.

(iii) $I_n = (0, 1/n)$. $\square$

Problem 5. Suppose f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$.

(i) Prove that either $f((a + b)/2) = 0$, or f has different signs at the end points $[a, (a + b)/2]$, or f has different signs at the end points of $[(a + b)/2, b]$.

If $f((a+b)/2) \ne 0$, let I_1 be one of the two intervals on which f has different signs at the endpoints. Now bisect I_1. Then either f is 0 at the midpoint, of f has opposite signs at the endpoints of one of the two intervals into which I_1 was bisected. Let I_2 be such an interval. Continue in this way to define I_n for each natural number n (unless f is 0 at some midpoint).

(ii) Prove that there is a point x in (a, b) where $f(x) = 0$.

(iii) Use the scheme described in (i) and (ii) to approximate the solution of $x^3 + 6x - 2 = 0$ with an error smaller than 1/100. (Calculators not allowed.)

Solution. (i) If $f((a + b)/2) \ne 0$, then this number is either > 0 (and f has different signs at the endpoints of $[a, (a + b)/2]$), or < 0 (and f has different signs at the endpoints of $[(a + b)/2, b]$).

(ii) Let x be in each I_n. If $f(x) < 0$, then there is some $\delta > 0$ such that $f(y) < 0$ for all y in $[a, b]$ with $|x-y| < \delta$. Let n be such that $(b - a)/2^n < \delta$. Since $\operatorname{length} I_n = (b - a)/2^n$, all the points y in I_n satisfy $|x - y| \le 1/2^n < \delta$, hence $f(y) < 0$ for all y in I_n, contradicting that f has opposite signs on the endpoints of I_n. Similarly we cannot have $f(x) > 0$, thus $f(x) = 0$.

(iii) If $f(x) = x^3 + 6x - 2$, then $f(0) = -2$ and $f(1/3) > 0$. Let $[a, b] = [0, 1/3]$. Since $\operatorname{length} I_n = 1/3 \cdot 2^n$ and $3 \cdot 2^5 < 100 < 3 \cdot 2^6$, any of the endpoints of I_6 will approximate the solution of $f(x) = 0$ with an error smaller than 1/100. $\square$

Problem 6 (Not required)**.** Let A and B be two nonempty sets of real numbers which are bounded above, and let $A + B$ denote the set of all real numbers of the form $x + y$ with x in A and y in B. Prove that $\sup(A + B) = \sup A + \sup B$.

Hint. The inequality $\sup(A + B) \le \sup A + \sup B$ should be easy. To prove that $\sup A + \sup B \le \sup(A + B)$, it suffices to prove that $\sup A + \sup B \le \sup(A + B) + \varepsilon$ for all $\varepsilon > 0$. For this, begin by choosing x in A and y in B with $\sup A - x < \varepsilon/2$ and $\sup B - y < \varepsilon/2$.