Math 512B. Homework 5. Solutions

Problem 1. Determine whether each of the following series converges or does not converge.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\alpha}{n^2}$.

(ii) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{n}$

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{\log n}{n}$.

(iv) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$

(v) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\log n)^n}$.

Solution. (i) Converges by the comparison test: $\left|\dfrac{\cos n\alpha}{n^2}\right| \leq \dfrac{1}{n^2}$.

(ii) Converges by Leibniz's Theorem: it is an alternating series and $\log n/n \geq \log(n+1)/(n+1)$ for $n \geq 3$.

(iii) Diverges by the comparison test: $\dfrac{1}{n} \leq \dfrac{\log n}{n}$ for $n > 3$.

(iv) Converges by the comparison test: $0 \leq \dfrac{\sqrt{n+1}-\sqrt{n}}{n} = \dfrac{1}{n(\sqrt{n+1}+\sqrt{n})} \leq \dfrac{1}{2n^{3/2}}$.

(v) Converges by the comparison test: $\dfrac{1}{(\log n)^n} \leq \dfrac{1}{2^n}$ for $n \geq 9$.

Problem 2. (i) Suppose that f is nondecreasing on $[1,\infty)$. Prove that

$$f(1)+f(2)+\cdots+f(n-1) \leq \int_1^n f \leq f(2)+f(3)+\cdots+f(n).$$

(ii) Take $f = \log$ in (i) and prove that

$$e^{1-n}n^n \leq n! \leq e^{-n}(n+1)^{n+1}$$

(iii) Prove that

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}.$$

(iv) Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{a^n n!}{n^n}$ converges if $a < e$.

(v) Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{a^n n!}{n^n}$ does not converge if $a \geq e$.

Solution. (i) If f is nondecreasing, then $f(k) \leq \int_k^{k+1} f \leq f(k+1)$ for all $k \geq 1$.

(ii) The integral $\displaystyle\int_1^n \log = n\log n - n$. Because of the properties of log, $\log k! = \log k + \log(k-1) + \cdots + \log 1$, so (i) implies that $\log(n-1)! \leq \log n^n - n \leq \log n!$, since exp is increasing, $(n-1)! \leq n^n e^{-n} \leq n!$, which is equivalent to the stated inequality.

(iii) Obvious by virtue of (ii).

(iv) and (v) follow from the ratio test because $\dfrac{a^{n+1}(n+1)!/(n+1)^{n+1}}{a^n n!/n^n} = a\left(\dfrac{n}{n+1}\right)^n$ converges to a as $n \to \infty$.

Problem 3. (i) Let $a_n \geq 0$. Prove that if $\displaystyle\sum_{n=1}^{\infty} a_n$ does not converge, then $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ does not converge.

(ii) Let $a_n \geq 0$. Prove that if b_n is a bounded sequence and $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n b_n$ also converges.

(iii) Let $a_n \geq 0$. Prove that if $\displaystyle\limsup_{n\to\infty} \sqrt[n]{a_n} < 1$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges; and if $\displaystyle\limsup_{n\to\infty} \sqrt[n]{a_n} > 1$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ does not converge.

(iv) (Not required) Let $a_n \geq 0$. Prove that if a_n is decreasing and $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\lim_{n\to\infty} na_n = 0$.

(v) Suppose that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. Let $n_1 < n_2 < n_3 < \cdots$ be an increasing sequence of natural numbers. From the sequence a_n obtain a new sequence b_n by setting

$$\begin{aligned}
b_1 &= a_1 + a_2 + \cdots + a_{n_1} \\
b_2 &= a_{n_1+1} + a_{n_1+2} + \cdots + a_{n_2} \\
&\vdots \\
b_k &= a_{n_{k-1}+1} + a_{n_{k-1}+2} + \cdots + a_{n_k}
\end{aligned}$$

Prove that $\displaystyle\sum_{n=1}^{\infty} b_n$ also converges and $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$

Solution. (i) If $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges, then the sequence $\dfrac{a_n}{1+a_n}$ converges to 0 as $n \to \infty$. Because of this, there is N such that $a_n \leq 1$ for $n > N$, and so $a_n \leq 2\dfrac{a_n}{1+a_n}$ for all $n > N$. By the comparison test, the series $\displaystyle\sum_{n=N+1}^{\infty} a_n$ converges and thus the original series $\displaystyle\sum_{n=1}^{\infty}$ converges.

(ii) If $|b_n| \le M$, then $|a_n b_n| \le M a_n$, so $\sum_{n=1}^{\infty} |a_n b_n|$ converges by the comparison test.

Problem 4. Let $b_n \neq 0$. We say that the infinite product $\prod_{n=1}^{\infty} b_n$ converges if the sequence of partial products $p_n = \prod_{k=1}^{n} b_k$ converges and $\lim_{n\to\infty} p_n \neq 0$.

(i) Prove that if $\prod_{n=1}^{\infty} b_n$ converges, then $\lim_{n\to\infty} b_n = 1$.

(ii) Suppose that $b_n > 0$. Prove that $\prod_{n=1}^{\infty} b_n$ converges if and only if $\sum_{n=1}^{\infty} \log b_n$ converges.

(iii) (Not required) Suppose that $a_n \ge 0$. Prove that $\prod_{n=1}^{\infty}(1 + a_n)$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges.

(iv) (Not required) Prove that $\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^{\alpha}}\right)$ converges if and only if $\alpha > 1$.

(v) (Not required) Evaluate the infinite product $\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right)$

Solution. (i) and (ii) were done in class

Problem 5. Prove that each of the following series converge to the given limit.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!} = e - 1$

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n2^n} = \log 2$

(iii) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\pi^{2n}}{(2n)!} = 2$

(iv) (Not required) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

(v) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n+1}{n^2+n} = 1$

(vi) (Not required) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n} \frac{(2n)!}{2^{4n}(n!)^2} = \frac{2}{\sqrt{5}} - 1$.

Solution. (i) When studying the Taylor Reaminder for exp, we have shown that $\left| e - 1 - \sum_{n=1}^{N} \frac{1}{n!} \right| \le \frac{1}{(N+1)!}$, so the partial sums $s_N = \sum_{n=1}^{N} \frac{1}{n!}$ converge to $e - 1$

(ii) Argue as in (i) using the Taylor series for $\log(1-x)$ at $x = 1/2$.

(iii) (There was a typo.) Use the Taylor series for $\cos x$ at π