Contents

Math 350 - Advanced Calculus
Homework 2

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December 11, 2012

Problem 1 Let I be a set and for each $i \in I$, let X_i be another set. We may speak of I as being an indexing set, whose elements, i, are indexes used to specify the sets X_i which we direct our attention. The set of all sets X_i as i ranges over I is denoted by $\{X_i \mid i \in I\}$, or $\{X_i\}_{i\in I}$ If all X_i are subsets of a set S, let $\bigcap_{i\in I} X_i$ and $\bigcup_{i\in I} X_i$ be the subsets of S given by:

$$\bigcup_{i\in I} X_i = \{x \in S \mid x \in X_i \text{ for some } i \in I\}$$

$$\bigcap_{i\in I} X_i = \{x \in S \mid x \in X_i \text{ for all } i \in I\}$$

Prove the following:

(i) $S \setminus \bigcup_{i\in I} X_i = \bigcap_{i\in I}(S \setminus X_i)$

Proof. Since all X_i are subsets of S, we can write it as,

$$\left(\bigcup_{i\in I} X_i\right)^c = \bigcap_{i\in I} X_i^c$$

Suppose that $x \in \left(\bigcup_{i\in I} X_i\right)^c \implies x \notin \bigcup_{i\in I} X_i \implies x \notin X_1$ and $x \notin X_2 \ldots$ and $x \notin X_i, i \in I \implies x \in X_1^c$ and $x \in X_2^c \ldots$ and $x \in X_i^c, i \in I$. In other words, $x \in \bigcap_{i\in I} X_i^c (1)$

To go the other way, suppose that $x \in \bigcap_{i\in I} X_i^c \implies (x \in X_1^c) \cap (x \in X_2^c) \cap \ldots \cap (x \in X_i^c), i \in I \implies (x \notin X_1) \cup (x \notin X_2) \cup \ldots \cup (x \notin X_i), i \in I \implies x \notin \bigcup_{i\in I} X_i \implies x \in \left(\bigcup_{i\in I} X_i\right)^c (2)$

From (1) and (2) we can conclude that

$$\left(\bigcup_{i\in I} X_i\right)^c = \bigcap_{i\in I} X_i^c$$

$\square$

(ii) $S \setminus \bigcap_{i\in I} X_i = \bigcup_{i\in I}(S \setminus X_i)$

Proof. Similarly, we can rewrite it as,

$$\left(\bigcap_{i\in I} X_i\right)^c = \bigcup_{i\in I} X_i^c$$

Suppose that $x \in \left(\bigcap_{i\in I} X_i\right)^c \implies x \notin \bigcap_{i\in I} X_i \implies x \in \bigcup_{i\in I} X_i^c$ (1)

To go the other way, suppose that $x \in \bigcup_{i\in I} X_i^c \implies (x \in X_1^c) \cup (x \in X_2^c) \cup \ldots \cup (x \in X_i^c) \implies (x \notin X_1) \cup (x \notin X_2) \cup \ldots \cup (x \notin X_i) \implies x \notin \bigcap_{i\in I} X_i \implies x \in \left(\bigcap_{i\in I} X_i\right)^c$ (2)

From (1) and (2) we can conclude that

$$\left(\bigcap_{i\in I} X_i\right)^c = \bigcup_{i\in I} X_i^c$$

□

(iii) $\bigcup_{i\in I} X_i \cap \bigcup_{j\in J} Y_j = \bigcup_{(i,j)\in I\times J} (X_i \cap Y_j)$

Proof. Suppose that $x \in \left(\bigcup_{i\in I} X_i \cap \bigcup_{j\in J} Y_j\right) \implies x \in \Big((X_1 \cup X_2 \cup \ldots \cup X_i) \cap (Y_1 \cup Y_2 \cup \ldots \cup Y_i)\Big) \implies x \in \Big([X_1 \cap (Y_1 \cup Y_2 \cup \ldots \cup Y_i)] \cup [X_2 \cap (Y_1 \cup Y_2 \cup \ldots \cup Y_i)] \ldots [X_i \cap (Y_1 \cup Y_2 \cup \ldots \cup Y_i)]\Big) \implies x \in \bigcup_{(i,j)\in I\times J} (X_i \cap Y_j)$ (1)

To go the other way around, suppose that $x \in \bigcup_{(i,j)\in I\times J} (X_i \cap Y_j) \implies x \in \Big([X_1 \cap (Y_1 \cup Y_2 \cup \ldots \cup Y_i)] \cup [X_2 \cap (Y_1 \cup Y_2 \cup \ldots \cup Y_i)] \ldots [X_i \cap (Y_1 \cup Y_2 \cup \ldots \cup Y_i)]\Big) \implies x \in \left(\bigcup_{i\in I} X_i \cap \bigcup_{j\in J} Y_j\right)$ (2)

From (1) and (2) we can conclude that

$$\bigcup_{i\in I} X_i \cap \bigcup_{j\in J} Y_j = \bigcup_{(i,j)\in I\times J} (X_i \cap Y_j)$$

□

(iv) $\bigcap_{i\in I} X_i \cup \bigcap_{j\in J} Y_j = \bigcap_{(i,j)\in I\times J} (X_i \cup Y_j)$

Proof. Similarly, suppose that $x \in \left(\bigcap_{i\in I} X_i \cup \bigcap_{j\in J} Y_j\right) \implies x \in \Big((X_1 \cap X_2 \cap \ldots \cap X_I) \cup (Y_1 \cap Y_2 \cap \ldots \cap Y_i)\Big) \implies x \in \Big([X_1 \cup (Y_1 \cap Y_2 \cap \ldots \cap Y_i)] \cap [X_2 \cup (Y_1 \cap Y_2 \cap \ldots \cap Y_i)] \cap \ldots [X_i \cup (Y_1 \cap Y_2 \cap \ldots \cap Y_i)]\Big) \implies x \in \left(\bigcap_{(i,j)\in I\times J} (X_i \cup Y_j)\right)$ (1)

To go the other way, suppose that $x \in \left(\bigcap_{(i,j)\in I\times J} (X_i \cup Y_j)\right) \implies x \in \Big([X_1 \cup (Y_1 \cap Y_2 \cap \ldots \cap Y_i)] \cap [X_2 \cup (Y_1 \cap Y_2 \cap \ldots \cap Y_i)] \cap \ldots [X_i \cup (Y_1 \cap Y_2 \cap \ldots \cap Y_i)]\Big) \implies x \in \Big((X_1 \cap X_2 \cap \ldots \cap X_I) \cup (Y_1 \cap Y_2 \cap \ldots \cap Y_i)\Big) \implies x \in \left(\bigcap_{i\in I} X_i \cup \bigcap_{j\in J} Y_j\right)$ (2) From (1) and (2) we can conclude that

$$\bigcap_{i\in I} X_i \cup \bigcap_{j\in J} Y_j = \bigcap_{(i,j)\in I\times J} (X_i \cup Y_j)$$

□

Problem 2 Prove that if a set A is enumerable (at most countable), then either $A = \emptyset$ or there is a surjective mapping $f : \mathbf{N} \to A$.

Proof. First we recall 4 properties of set:

1. Two sets are equipotent if there is a bijection between them. To be equipotent is an equivalence relation among sets.
2. A set is finite if it's equipotent to one of the finite sets $\mathbf{0}, \mathbf{1}, \ldots$.
3. A set is countable if it is equipotent to the set $\mathbf{N}$ for all natural numbers.
4. A set is enumerable if it is either finite or countable.

Essentially, we know that A is either finite or countable, so we have two cases:

- Assume that A is finite then A is equipotent to one of the sets $\mathbf{0}, \mathbf{1}, \ldots$. Next let's assume that A is equipotent to $\mathbf{0} = \emptyset$, thus A must equal to $\emptyset$.
- Assume that A is countable, then A is equipotent to $\mathbf{N}$ which implies there is a bijection between A and $\mathbf{N}$. Thus there is a surjective mapping $f : \mathbf{N} \to A$

$\therefore A = \emptyset$ or there is a surjective mapping $f : \mathbf{N} \to A$. $\square$

Problem 3

(i) Prove that if $\{X_i \mid i \in I\}$ is a set of sets where the indexing set I is countable and each X_i is countable, then $\bigcup_{i \in I} X_i$ is countable.

To be precise, first we recall the definition of countable set. What does it mean to count the number of elements in the set? What we really do we count is to assign each element of the set a unique natural number, either starting from 0 or 1. and proceeding upward. For example, count the set of lower-case letters in alphabet:

$$S = \{a, b, c, \ldots, x, y, z\}$$

$$\begin{array}{ccccccc} a & b & c & \ldots & x & y & z \\ \updownarrow & \updownarrow & \updownarrow & \ldots & \updownarrow & \updownarrow & \updownarrow \\ 1 & 2 & 3 & \ldots & 24 & 25 & 26 \end{array}$$

In fact, when we count we are using a function from our set S to a subset of the natural number $\mathbf{N}$ that is both one-to-one and onto.

- A function is *one-to-one* if two distinct inputs always have two distinct outputs.
- A function is *onto* if every element in the range is the output of one or more elements in the domain.
- *bijection* = *one-to-one* + *onto*.

Proof. First we have that the indexing set I is countable, so there is a mapping from $\mathbf{N}$ to I, symbolically,

$$f : \mathbf{N} \to I$$

so we can label each element $x \in I$ with the natural number $1, 2, 3, \ldots$ Let denote all countable sets X_i as:

$$X_1, X_2, X_3, X_4 \ldots$$

What we want to show is the union of these sets is also countable.

$$X = \bigcup_{i=1}^{\infty} X_i = X_1 \cup X_2 \cup X_3 \cup \ldots$$

Suppose X_i has elements:

$$a_{i1}, a_{i2}, a_{i3} \ldots$$

Next we put these elements of the union of these countable set into a 2-dimension infinite array:

$$\begin{array}{ccccc}
a_{11} & a_{12} & a_{13} & a_{14} & \ldots \\
a_{21} & a_{22} & a_{23} & a_{24} & \ldots \\
a_{31} & a_{32} & a_{33} & a_{34} & \ldots \\
a_{41} & a_{42} & a_{43} & a_{44} & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{array}$$

Now to show that this is a countable set, we need to create bijection mapping between these elements with the natural number sets $\mathbf{N}$ using the method as described in lecture:

$$\begin{tikzcd}
a_{11} \arrow[r] & a_{12} \arrow[dl] & a_{13} \arrow[r] & a_{14} \arrow[dl] \\
a_{21} \arrow[d] & a_{22} \arrow[ur] & a_{23} \arrow[dl] & a_{24} \\
a_{31} \arrow[ur] & a_{32} \arrow[dl] & a_{33} & a_{34} \\
a_{41} \arrow[r] & \cdots & &
\end{tikzcd}$$

The bijection mapping above indeed map each element in the union to the set of natural number $\mathbf{N}$

$$\begin{array}{cccccccc}
a_{11} & a_{12} & a_{21} & a_{31} & a_{22} & a_{13} & a_{14} & \ldots \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
1 & 2 & 3 & 3 & 4 & 5 & 6 & \ldots
\end{array}$$

Therefore X is countable. $\square$

(ii) Prove that $P(\mathbf{N})$, the set of all finite subsets of $\mathbf{N}$ is countable.

Proof. Let $S = P(\mathbf{N})$ be the set of all finite subset of $\mathbf{N}$. In addition, we have that if a set is finite, then it is countable. Now let S_k be the set of subsets of $\mathbf{N}$ that consists of k elements, we can reorder these k elements in increasing order as follows:

$$x_0 < x_1 < x_2 < \ldots x_k$$

Let define the sum and sum square of all elements in S_k as:

$$s = \sum_{i=0}^{k} x_i \text{ and } sq = \sum_{i=0}^{k} x_i^2$$

It's easy to see that a triplet of (s, sq, k) is unique. Since S_k is finite, we can list all the finite subsets of $\mathbf{N}$ as: $\square$

Problem 4 Prove the following:

(i) $a \cdot c^{-1} = (a \cdot b) \cdot (b \cdot c)^{-1}$ if $b, c \neq 0$

(ii) $a \cdot b^{-1} + c \cdot d^{-1} = (a \cdot d + b \cdot c) \cdot (b \cdot d)^{-1}$ if $b, d \neq 0$

(iii) $(a \cdot b)^{-1} = a^{-1} \cdot b^{-1}$ if $a, b \neq 0$

(iv) $(a \cdot b^{-1}) \cdot (c \cdot d^{-1}) = (ac) \cdot (db)^{-1}$ if $b, d \neq 0$

(v) $(a \cdot b^{-1}) \cdot (c \cdot d^{-1})^{-1} = (a \cdot d) \cdot (b \cdot c)^{-1}$ if $b, c, d \neq 0$

(vi) If $b, d \neq 0$, then $a \cdot b^{-1} = c \cdot d^{-1}$ if and only if $a \cdot d = b \cdot c$

Proof.

(i) Multiply both sides by $(b \cdot c)$ to obtain,

$$\begin{aligned} & a \cdot c^{-1}(b \cdot c) = (a \cdot b) \cdot (b \cdot c)^{-1} \cdot (b \cdot c) \\ \Leftrightarrow\quad & a \cdot c^{-1}(c \cdot b) = (a \cdot b) \cdot 1 \text{ (multiplicative inverse, associative)} \\ \Leftrightarrow\quad & a \cdot (c^{-1} \cdot c) \cdot b = a \cdot b \text{ (associative, neutral element)} \\ \Leftrightarrow\quad & (a \cdot 1) \cdot b = a \cdot b \text{ (multiplicative inverse)} \\ \Leftrightarrow\quad & a \cdot b = a \cdot b \text{ (neutral element)} \end{aligned}$$

(ii) From (i) we have

$$\begin{aligned} & a \cdot b^{-1} + c \cdot d^{-1} = (a \cdot d + b \cdot c) \cdot (b \cdot d)^{-1} \\ \Leftrightarrow\quad & (a \cdot d) \cdot (d \cdot b)^{-1} + (c \cdot b) \cdot (b \cdot d)^{-1} = (a \cdot d + b \cdot c) \cdot (b \cdot d)^{-1} \text{ (from (i))} \\ \Leftrightarrow\quad & (a \cdot d + b \cdot c) \cdot (b \cdot b)^{-1} = (a \cdot d + b \cdot c) \cdot (b \cdot d)^{-1} \text{ (associative, distributive)} \end{aligned}$$

(iii) Multiply both side by $(a \cdot b)$ to obtain

$$\begin{aligned} & (a \cdot b)^{-1} = a^{-1} \cdot b^{-1} \\ \Leftrightarrow\quad & (a \cdot b)^{-1} \cdot (a \cdot b) = a^{-1} \cdot b^{-1} \cdot (a \cdot b) \\ \Leftrightarrow\quad & 1 = (a^{-1} \cdot a) \cdot (b^{-1} \cdot b) \text{ (multiplicative inverse, associative)} \\ \Leftrightarrow\quad & 1 = 1 \cdot 1 \text{ (multiplicative inverse)} \\ \Leftrightarrow\quad & 1 = 1 \text{ (neutral element)} \end{aligned}$$

(iv) We have

$$\begin{aligned} & (a \cdot b^{-1}) \cdot (c \cdot d^{-1}) = (ac) \cdot (db)^{-1} \\ \Leftrightarrow\quad & (ac) \cdot (b^{-1} \cdot d^{-1}) = (ac) \cdot (db)^{-1} \text{ (associative)} \\ \Leftrightarrow\quad & (ac) \cdot (bd)^{-1} = (ac) \cdot (db)^{-1} \text{ (from (iii))} \end{aligned}$$

(v) First we prove that $(d^{-1})^{-1} = d$. Multiply both sides by d^{-1} to obtain:

$$\begin{aligned} & (d^{-1})^{-1} \cdot d^{-1} = d \cdot d^{-1} \\ \Leftrightarrow \quad & 1 = 1 \text{ (multiplicative inverse)} \end{aligned}$$

Now we have,

$$\begin{aligned} & (a \cdot b^{-1}) \cdot (c \cdot d^{-1})^{-1} = (a \cdot d) \cdot (b \cdot c)^{-1} \\ \Leftrightarrow \quad & (a \cdot b^{-1}) \cdot (c^{-1} \cdot (d^{-1})^{-1}) = (a \cdot d) \cdot (b \cdot c)^{-1} \text{ (from (iii))} \\ \Leftrightarrow \quad & (a \cdot b^{-1}) \cdot (c^{-1} \cdot d) = (a \cdot d) \cdot (b \cdot c)^{-1} \text{ (from proof above))} \\ \Leftrightarrow \quad & (a \cdot d) \cdot (b^{-1} \cdot c^{-1}) = (a \cdot d) \cdot (b \cdot c)^{-1} \text{ (associative))} \\ \Leftrightarrow \quad & (a \cdot d) \cdot (b \cdot c)^{-1} = (a \cdot d) \cdot (b \cdot c)^{-1} \text{ (from (iii))} \end{aligned}$$

(vi) We have,
$\Rightarrow$: If $a \cdot b^{-1} = c \cdot d^{-1}$ where $b, d \neq 0$, then there exists multiplicative inverse of b and d. Next we multiply both sides by bd to obtain: $a \cdot b^{-1} \cdot (bd) = c \cdot d^{-1}(bd) \Leftrightarrow a \cdot d = c \cdot b$
$\Leftarrow$: If $a \cdot d = b \cdot c$ where $b, d \neq 0$, then there exists a multiplicative of b and d. Again, we multiply both sides of this equality by $(bd)^{-1}$ to obtain, $a \cdot d \cdot d^{-1} \cdot b^{-1} = b \cdot c \cdot b^{-1} \cdot d^{-1}$. Using associative and multiplicative inverse property we reduce it to: $a \cdot b^{-1} = c \cdot d^{-1}$

□

Problem 5 Find all the numbers x such that:

(i) $x^2 - 3x - 16 < 2$

(ii) $\dfrac{1}{x+1} + \dfrac{1}{x+2} < 0$

(iii) $\dfrac{x-1}{x+1} > 0$

Proof.

(i)

$$\begin{aligned} & x^2 - 3x - 16 < 2 \\ \Leftrightarrow \quad & x^2 - 3x - 18 < 0 \\ \Leftrightarrow \quad & (x+3)(x-6) < 0 \end{aligned}$$

We have two cases:

$$\begin{cases} x + 3 < 0 \text{ and } x - 6 > 0 \rightarrow \text{ no solution} \\ x + 3 > 0 \text{ and } x - 6 < 0 \rightarrow x \in (-3, 6) \end{cases}$$

$$\therefore x \in (-3, 6)$$

(ii)

$$\begin{aligned} & \frac{1}{x+1} + \frac{1}{x+2} < 0 \\ \Leftrightarrow \quad & \frac{(x+2) + (x+1)}{(x+1)(x+2)} < 0 \\ \Leftrightarrow \quad & \frac{2x+3}{(x+1)(x+2)} < 0 \end{aligned}$$

We have

x		-2		-3/2		-1	
y	$-$	und.	$+$	0	$-$	und.	$+$

$$\therefore x \in (-\infty, -2) \cup (-3/2, -1)$$

(iii) We have

x		-1		1	
y	$+$	und.	$-$	0	$+$

$$\therefore x \in (-\infty, -1) \cup (0, \infty)$$

□

Problem 6 Prove the following and write ab for $a \cdot b$:

(i) If $a < b$ and $c \leq d$ then $a + c < b + d$

(ii) If $a < b$ then $-b < -a$

(iii) If $a < b$ and $c < d$ then $a - d < b - c$

(iv) If $a < b$ and $0 < c$, then $ac < bc$

(v) If $a > 1$ then $a^2 > a$

(vi) If $0 < a < 1$ then $a^2 < a$

(vii) If $0 \leq a < b$ and $0 \leq c < d$ then $ac < bd$

(viii) If $0 \leq a < b$ then $a^2 < b^2$

(ix) If $a \geq 0, b \geq 0$, and $a^2 > b^2$ then $a > b$

Proof.

(i) Consider $a < b$

$$\begin{aligned} & a < b \\ \Leftrightarrow \quad & a + c < b + c \\ \Leftrightarrow \quad & a + c < b + d \text{ (since } c \leq d) \end{aligned}$$

(ii) Consider $a < b$

$$\begin{aligned} & a < b \\ \Leftrightarrow \quad & a \cdot -1 < b \cdot -1 \\ \Leftrightarrow \quad & -b < -a \end{aligned}$$

(iii) Consider $c < d$

$$\begin{aligned} & c < d \\ \Leftrightarrow \quad & -d < -c \\ \Leftrightarrow \quad & -d + a < -c + a \\ \Leftrightarrow \quad & -d + a < -c + a < -c + b \\ \Leftrightarrow \quad & a - d < b - c \end{aligned}$$

(iv) Consider $a < b \to b - a \in \mathbf{F}_+$. Since $\mathbf{F}_+ \cdot \mathbf{F}_+ \subset \mathbf{F}_+$, we have $(b-a) \cdot c \in \mathbf{F}_+ \Leftrightarrow bc - ac \in \mathbf{F}_+ \Leftrightarrow bc > ac$ by definition of "$<$".

(v) Consider If $a > 1$

$$\begin{aligned} & a \cdot a > 1 \cdot a \\ \Leftrightarrow \quad & a^2 > a \end{aligned}$$

(vi) Consider $0 < a < 1$ then $a^2 < a$

$$\begin{aligned} & a < 1 \\ \Leftrightarrow \quad & a \cdot a < 1 \cdot a \\ \Leftrightarrow \quad & a^2 < a \end{aligned}$$

(vii) Consider $a < b$

$$\begin{aligned} & a < b \\ \Leftrightarrow \quad & a \cdot c < b \cdot c \\ \Leftrightarrow \quad & a \cdot c < b \cdot c < b \cdot d \\ \Leftrightarrow \quad & a \cdot c < b \cdot d \end{aligned}$$

(viii) Consider $a < b$

$$\begin{aligned} & a < b \\ \Leftrightarrow \quad & a \cdot a < b \cdot a \\ \Leftrightarrow \quad & a \cdot a < b \cdot a < b \cdot b \\ \Leftrightarrow \quad & a^2 < b^2 \end{aligned}$$

(ix) Consider $a^2 > b^2$

$$\begin{aligned} & a^2 > b^2 \\ \Leftrightarrow \quad & a^2 - b^2 > 0 \\ \Leftrightarrow \quad & (a-b)(a+b) > 0 \\ \Leftrightarrow \quad & (a-b) > 0 \\ \Leftrightarrow \quad & a > b \end{aligned}$$

$\square$

Problem 7 A *Peano* system is a triple (P, z, σ) where P is a set ($\neq \emptyset$), $z \in P$, and $\sigma : P \to P$ is a function such that:

(a) $\sigma(p) \neq z$ for every $p \in P$

(b) σ is one to one

(c) If $B \subset P$ satisfies $z \in B$ and $\sigma(b) \in B$ whenever $b \in B$ then $B = P$.
For a *Peano* system (P, z, σ) prove the following:

(i) $P = \{z\} \cup \sigma(P)$

(ii) For any $p \in P$, either $p = z$ or there is exactly one number $n = 1, 2, 3 \ldots$ such that $p = \overbrace{\sigma(\sigma(\sigma(\ldots(\sigma(z))))))}_{\sigma \text{ applied } n \text{ times}}$

(iii) If (Q, w, τ) is any other *Peano* system, then there is a bijection $f : P \to Q$ such that $f(z) = w$ and $\tau(f(p)) = f(\sigma(p))$ for any $p \in P$

Proof.

(i) $P = \{z\} \cup \sigma(P)$ Since $z \in P$ and $\sigma(p) \neq z$ for every $p \in P$, we have that $\sigma(P) = P \setminus \sigma(p)$ where $\sigma(p) = z$. Thus $(P \setminus \{z\}) \cup \{z\} = P$

(ii) Since $z \in P$, saying either $p \in P$ is equal to z is trivial. Now suppose that $z \neq p$, we need to show that there is exactly one number $n = 1, 2, 3 \ldots$ such that $p = \overbrace{\sigma(\sigma(\sigma(\ldots(\sigma(z))))))}_{\sigma \text{ applied } n \text{ times}}$

Let's define:

$$\sigma(z) = x, \text{ for some } x \text{ such that } x \notin P$$

Next, we have

$$\begin{aligned} \sigma(z) &= x \\ \sigma(\sigma(z)) &= x \end{aligned}$$

$\square$