Important Theorems

Introduction to Analysis

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Chapter 5. Limit

$$f(x) = \begin{cases} 0 & , x \text{ irrational} \\ 1 & , x \text{ rational} \end{cases}$$

No matter what a is, f does not approach any number L near a.

$$f(x) = \begin{cases} x & , x \text{ rational} \\ 0 & , x \text{ irrational} \end{cases}$$

This function approaches 0 at 0 but does not approach any number $a \neq 0$.
Definition. The function f approaches the limit L near a means: for every $\epsilon > 0$, there is some $\delta > 0$, for all x if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.
If it is NOT true: there is some $\epsilon > 0$ such that for every $\delta > 0$ there is some x which satisfies $0 < |x - a| < \delta$ but not $|f(x) - L| < \epsilon$.

Chapter 6. Continuous Functions

Definition. The function f is continuous at a if

$$\lim_{x \to a} f(x) = f(a)$$

$$g(x) = \begin{cases} x \sin(1/x) & , x \neq 0 \\ a & , x = 0 \end{cases}$$

$g(x)$ is continuous at 0.

$$f(x) = \begin{cases} \sin(1/x) & , x \neq 0 \\ a & , x = 0 \end{cases}$$

$f(x)$ is not continuous at 0 no matter what a is because $\lim_{x \to 0} f(x)$ does not exists.

$$h(x) = \begin{cases} x & , x \text{ rational} \\ 0 & , x \text{ irrational} \end{cases}$$

$h(x)$ is not continuous at a if $a \neq 0$ since $\lim\limits_{x \to a} f(x)$ does not exist.

Theorem 1. If f and g are continuous at a, then

(1) $f + g$ is continuous at a.

(2) $f \cdot g$ is continuous at a.

(3) if $g(a) \neq 0$, then $\dfrac{1}{g}$ is continuous a.

Theorem 2. If g is continuous at a, and f is continuous at $g(a)$, then $f \circ g$ is continuous at a. (Notice that f is required to be continuous at $g(a)$, not a)

Theorem 3. Suppose f is continuous at a, and $f(a) > 0$. Then $f(x) > 0$ for all x in some interval containing a; more precisely, there is a number $\delta > 0$ such that $f(x) > 0$ for all x satisfying $|x - a| < \delta$. Similarly, $f(a) < 0$, then there is a number $\delta > 0$ such that $f(x) < 0$ for all x satisfying $|x - a| < \delta$.

Chapter 7. Three Hard Theorems

Theorem 1. If f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$, then there is some $x \in [a, b]$ such that $f(x) = 0$.

Theorem 2. If f is continuous on $[a, b]$ and f is bounded above on $[a, b]$ that is there is some numbers N such that $f(x) \leq N$ for all $x \in [a, b]$.

Theorem 3. If f is continuous on $[a, b]$ then there is some number $y \in [a, b]$ such that $f(y) \geq f(x)$ for all $x \in [a, b]$.

Theorem 4. If f is continuous on $[a, b]$ and $f(a) < c < f(b)$ (or $f(a) > c > f(b)$ then there is some x in $[a, b]$ such that $f(x) = c$.

Theorem 5. If f is continuous on $[a, b]$ and $f(a) > c > f(b)$ (or $f(a) > c > f(b)$ then there is some x in $[a, b]$ such that $f(x) = c$.

Theorem 6. If f is continuous on $[a, b]$, then f is bounded below on $[a, b]$, that is, there is some number N such that $f(x) \geq N$ for all $x \in [a, b]$.

Theorem 7. If f is continuous on $[a, b]$, then there is some y in $[a, b]$ such that $f(y) \leq f(x)$ for all x in $[a, b]$.

Theorem 8. Every positive number has a square root. In other words, if $\alpha > 0$, then there is some number x such that $x^2 = \alpha$.

Theorem 9. If n is odd, then any equation

$$x^n + a_{n-1}x^{n-1} + \ldots + a_0 = 0$$

has a root.

Theorem 10. If n is even and $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$, then there is a number y such that $f(y) \leq f(x)$ for all x.

Theorem 11. Consider the equation

$$x_n + a_{n-1}x^{n-1} + \ldots + a_0 = c \quad (*)$$

and suppose n is even. Then there is a number m such that $(*)$ has a solution for $c \geq m$ and has no solution for $c < m$.

Chapter 8. Least Upper Bounds

Theorem 1. If f is continuous at a, then there is a number $\delta > 0$ such that f is bounded above on the interval $(a - \delta, a + \delta)$.

Theorem 3. For any $\epsilon > 0$, there is natural number n with $\frac{1}{n} < \epsilon$.

Chapter 9. Derivative

Definition. The function f is differentiable at a if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists.

Theorem 1. If f is differentiable at a, then f is continuous at a.

Proof.

$$\begin{aligned} \lim_{h \to 0} f(a+h) - f(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \cdot h \\ &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \cdot \lim_{h \to 0} h \\ &= f'(a) \cdot 0 = 0 \end{aligned}$$

□

Chapter 10. Differentiation

Theorem 1. If f is constant function, $f(x) = c$, then $f'(a) = 0$ for all numbers a.

Theorem 2. If f is identity function, $f(x) = x$, then $f'(a) = 1$ for all numbers a.

Theorem 3. If f and g are differentiable at a, then $f+g$ is also differentiable at a and

$$(f+g)'(a) = f'(a) + g'(a)$$

Theorem 4. If f and g are differentiable at a, then $f \cdot g$ is also differentiable at a, and

$$(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$$

Theorem 5. If $g(x) = cf(x)$ and f is differentiable at a, then g is differentiable at a, and

$$g'(a) = c \cdot f'(a)$$

Theorem 6. If $f(x) = x^n$ for some natural number n, then

$$f'(a) = na^{n-1} \text{ for all } a$$

Theorem 7. If g is differentiable at a, and $g(a) \neq 0$, then $1/g$ is differentiable at a and

$$\left(\frac{1}{g}\right)'(a) = \frac{-g'(a)}{[g(a)]^2}$$

Theorem 8. If f and g are differentiable at a and $g(a) \neq 0$, then f/g is differentiable at a and

$$\left(\frac{f}{g}\right)'(a) = \frac{g(a) \cdot f'(a) - f(a) \cdot g'(a)}{[g(a)]^2}$$

Theorem 9. If g is differentiable at a, and f is differentiable at $g(a)$, then $f \circ g$ is differentiable at a, and

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$$

Chapter 11. Significance of The Derivative

Theorem 1. Let f be any function defined on (a, b). If x is a maximum (or a minimum) point for f on (a, b) and f is differentiable at x, then $f'(x) = 0$.

Theorem 2. If x is a local maximum or minimum for f on (a, b) and f is differentiable at x, then $f'(x) = 0$.

Definition. A **critical point** of a function f is a number x such that

$$f'(x) = 0$$

The number $f(x)$ itself is called a critical value of f.

Note: In order to locate the maximum and minimum of f three kinds of points must be considered:

1. The critical points of f in $[a, b]$.
2. The end points a and b.
3. Points x in $[a, b]$ such that f is NOT differentiable at x.

Intermediate Value Theorem. If f is continuous on $[a, b]$ and c is any number between $f(a)$ and $f(b)$, then there is at least one x in $[a, b]$ such that $f(x) = c$.

Theorem 3. (Rolle's Theorem) If f is continuous on $[a, b]$ and differentiable on (a, b) and $f(a) = f(b)$, then there is a number x in (a, b) such that $f'(x) = 0$.

Theorem 4. (The Mean Value Theorem) If f is continuous on $[a, b]$ and differentiable (a, b) then there is a number x in (a, b) such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

Corollary 1. If f is defined on interval and $f'(x) = 0$ for all x in the interval, then f is constant on the interval.

Corollary 2. If f and g are defined on the same interval, and $f'(x) = g'(x)$ for all x in the interval, then there is some number c such that $f = g + c$.

Corollary 3. If $f'(x) > 0$ for all x in an interval, then f is increasing on the interval; if $f'(x) < 0$ for all x in the interval, then f is decreasing on the interval.

Theorem 5. Suppose $f'(a) = 0$. If $f''(a) > 0$, then f has a local minimum at a; if $f''(a) < 0$, then f has a local maximum at a.

Theorem 6. Suppose $f''(a)$ exists. If f has a local minimum at a, then $f''(a) \geq 0$; if f has a local maximum at a, then $f''(a) \leq 0$.

Theorem 7. Suppose that f is continuous at a, and that $f'(x)$ exists for all x in some interval contain a, except perhaps for $x = a$. Suppose, moreover, that $\lim_{x \to a} f'(x)$ exists. Then $f'(a)$ also exists, and

$$f'(a) = \lim_{x \to a} f'(x)$$

Theorem 8. (The Cauchy Mean Value Theorem) If f and g are continuous on $[a, b]$ and differentiable on (a, b), then there is a number x in (a, b) such that

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)$$

Theorem 9. (L'Hopital Rule) Suppose that

$$\lim_{x \to a} f(x) = 0 \text{ and } \lim_{x \to a} g(x) = 0$$

and suppose that $\lim_{x \to a} f'(x)/g'(x)$ exists. Then $\lim_{x \to a} f(x)/g(x)$ exists and

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Chapter 12. Inverse Function

Definition. A function is **one-one** if $f(a) \neq f(b)$ whenever $a \neq b$.

Definition. For any function f, the inverse of f, denoted by $\mathbf{f^{-1}}$, is the set of all pairs (a, b) for which the pair (b, a) is in f.

Theorem 1. f^{-1} is a function if and only f is one to one.

Theorem 2. If f is continuous and one-one on an interval, then f is either increasing or decreasing on that interval.

Theorem 3. If f is continuous and one-one on an interval, then f^{-1} is also continuous.

Theorem 4. If f is continuous one-one function defined on an interval and $f'(f^{-1}(a)) = 0$, then f^{-1} is *not* differentiable at a.

Theorem 5. Let f be a continuous one-one function defined on an interval, and suppose that f is differentiable at $f^{-1}(b)$, withe derivative $f'(f^{-1}(b)) \neq 0$. Then f^{-1} is differentiable at b, and

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

Chapter 13. Integrals

Definition. Let $a < b$. A **partition** of the interval $[a, b]$ is a finite collection of points on $[a, b]$, one of which is a and one of which is b.

Definition. Suppose f is bounded on $[a, b]$ and $P = \{t_0, t_1, \ldots, t_n\}$ is a partition of $[a, b]$. Let

$$\begin{aligned} m_i &= \inf\{f(x) : t_{i-1} \leq x \leq t_i\} \\ M_i &= \sup\{f(x) : t_{i-1} \leq x \leq t_i\} \end{aligned}$$

The lower sum of f for P denoted by

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1})$$

The upper sum of f for P denoted by

$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1})$$

Lemma. If Q contains P, ($P \subseteq Q$), then

$$L(f, P) \leq L(f, Q)$$

$$U(f, P) \geq U(f, Q)$$

Theorem 1. Let P_1 and P_2 be partitions of $[a, b]$ and let f be a function which is bounded on $[a, b]$.
Then $L(f, P_1) \leq U(f, P_2)$.

Definition. A function f which is bounded on $[a, b]$ is **integrable** on $[a, b]$ if P is a partition of $[a, b]$ and

$$\sup\{L(f, P)\} = \inf\{U(f, P)\}$$

In this case, this common **number** is called the **integral** of f on $[a, b]$ and is denoted by

$$\int_a^b f$$

Theorem 2. If f is bounded on $[a, b]$, then f is integrable on $[a, b]$ if and only if for every $\epsilon > 0$, there is a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \epsilon$$

Theorem 3. If f is continuous on $[a, b]$ then f is integrable on $[a, b]$.

Theorem 4. Let $a < c < b$. If f is integrable on $[a, b]$, then f is integrable on $[a, c]$ and on $[c, b]$. Conversely, if f is integrable on $[a, c]$ and on $[c, b]$ then f is integrable on $[a, b]$. Finally if f is integrable on $[a, b]$ then

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Theorem 5. If f and g are integrable on $[a, b]$, then $f + g$ is integrable on $[a, b]$ and

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g$$

Theorem 6. If f is integrable on $[a, b]$, then for any number c, the function cf is integrable on $[a, b]$ and

$$\int_a^b cf = c \cdot \int_a^b f$$

Theorem 7. Suppose f is integrable on $[a, b]$ and that

$$m \leq f(x) \leq M \text{ for all } x \in [a, b]$$

Then

$$m(b - a) \leq \int_a^b \leq M(b - a)$$

Theorem 8. If f is integrable on $[a, b]$ and F is defined on $[a, b]$ by

$$F(x) = \int_a^x f$$

then F is continuous on $[a, b]$.

Chapter 14. The Fundamental Theorem of Calculus

Theorem 1. If f is integrable on $[a, b]$ and define F on $[a, b]$ by

$$F(x) = \int_a^x f$$

If f is continuous at c in $[a, b]$, then F is differentiable at c, and

$$F'(c) = f(c)$$

Corollary. If f is continuous on $[a, b]$ and $f = g'$ for some function g, then

$$\int_a^b f = g(b) - g(a)$$

Theorem 2. If f is integrable on $[a, b]$ and $f = g'$ for some function g, then

$$\int_a^b f = g(b) - g(a)$$

Chapter 15. The Trigonometric Functions

Definition. $$\boxed{\pi = 2 \cdot \int_{-1}^{1} \sqrt{1-x^2}\,dx}$$

Definition. If $-1 \le x \le 1$ then
$$A(x) = \frac{x\sqrt{1-x^2}}{2} + \int_x^1 \sqrt{1-t^2}\,dt$$

Definition. If $0 \le x \le \pi$, then $\cos(x)$ is the unique number in $[-1,1]$ such that
$$A(\cos(x)) = \frac{x}{2}$$
and
$$\sin(x) = \sqrt{1-\cos(x)^2}$$

Theorem 1. If $0 < x < \pi$, then
$$\cos'(x) = -\sin(x)$$
$$\sin'(x) = \cos(x)$$

Theorem 2. If $x \ne k\pi + \frac{\pi}{2}$, then
$$\sec'(x) = \sec(x)\tan(x)$$
$$\tan'(x) = \sec(x)^2$$
If $x \ne k\pi$, then
$$\csc'(x) = -\csc(x)\cot(x)$$
$$\cot'(x) = -\csc(x)^2$$

Theorem 3. If $-1 < x < 1$, then
$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$$
$$\arccos'(x) = \frac{-1}{\sqrt{1-x^2}}$$
Moreover, for all x we have
$$\arctan'(x) = \frac{1}{1+x^2}$$

Lemma. Suppose f has a second derivative everywhere and that
$$\begin{aligned} f''+f &= 0 \\ f(0) &= 0 \\ f'(0) &= 0 \end{aligned}$$
Then $f = 0$.

Theorem 4. Suppose f has a second derivative everywhere and that
$$\begin{aligned} f''+f &= 0 \\ f(0) &= a \\ f'(0) &= b \end{aligned}$$

Then $f = b \cdot \sin + a \cdot \cos$

Theorem 5. If x and y are any two numbers, then

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

Chapter 20. Approximation By Polynomial Functions

Theorem 1. Suppose that f is a function for which

$$f'(a), f''(a), \ldots, f^{(n)}(a)$$

all exist. Let

$$a_k = \frac{f^{(k)}(a)}{k!} \ , 0 \leq k \leq n$$

and define

$$P_{n,a}(x) = a_0 + a_1(x-a) + \ldots + a_n(x-a)^n$$

Then

$$\lim_{x \to a} \frac{f(x) - P_{n,a}(x)}{(x-a)^n} = 0$$

Theorem 2. Suppose that

$$f'(a) = \ldots = f^{(n-1)}(a) = 0$$
$$f^{(n)}(a) \neq 0$$

(1) If n is even and $f^{(n)}(a) > 0$, then f has a local minimum at a.

(2) If n is even and $f^{(n)}(a) < 0$, then f has a local maximum at a.

(3) If n is odd, then f has neither a local maximum nor a local minimum at a.

Definition. Two functions f and g are equal up to order n at a if

$$\lim_{x \to a} \frac{f(x) - g(x)}{(x-a)^n} = 0$$

Theorem 3. Let P and Q are two polynomials in $(x-a)$, of degree $\leq n$, and suppose that P and Q are equal up to order n at a. Then $P = Q$.

Corollary. Let f be n times differentiable at a, and suppose that P is a polynomial in $(x-a)$ of degree $\leq n$, which equals f up to order n at a. Then $P = P_{n,a,f}$.

Lemma. Suppose that the function R is $(n+1)$ times differentiable on $[a,b]$, and

$$R^{(k)} = 0 \text{ for } k = 0, 1, 2, \ldots n$$

Then for any x in $(a,b]$ we have

$$\frac{R(x)}{(x-a)^{n+1}} = \frac{R^{(n+1)}(t)}{(n+1)!} \text{ for some } t \text{ in } (a,x)$$

Chapter 22. Infinite Sequence

Definition. An **infinite sequence** of real numbers is a function whose domain is $\mathbb{N}$.

Definition. A sequence (a_n) converges to L if for every $\epsilon > 0$, there is a natural number N such that, for all natural number n,

$$\text{if } n > N \text{ then } |a_n - L| < \epsilon$$

Theorem 1. Let f be a function defined in an open interval containing c, except perhaps at c itself, with

$$\lim_{x \to c} f(x) = L$$

Suppose (a_n) is a sequence such that

(1) Each a_n is in the domain of f

(2) Each $a_n \neq c$

(3) $\lim_{n \to \infty} a_n = c$

Then the sequence $(f(a_n))$ satisfies

$$\lim_{n \to \infty} f(a_n) = L$$

Conversely if this is true for every sequence (a_n) satisfying the above conditions, then

$$\lim_{x \to c} f(x) = L$$

Theorem 2. If (a_n) is nondecreasing and bounded above, then (a_n) converges. Similar statement is true if (a_n) is nonincreasing and bounded below.

Lemma. Any sequence (a_n) contains a subsequence which is either nondecreasing or nonincreasing.

Definition. A sequence (a_n) is a Cauchy Sequence if for every $\epsilon > 0$ there is a natural number N such that for all m, n,

$$\text{if } m, n > N \text{ then } |a_n - a_m| < \epsilon$$

Theorem 3. A sequence (a_n) converges if and only if it is a Cauchy Sequence.

Monotone Subsequence Theorem If $X = (x_n)$ is a sequence of real numbers then there is a monotone subsequence.

Bolzano-Weierstrass Theorem A bounded sequence of real numbers has a convergent subsequence.

Theorem Let $X = (x_n)$ be a bounded sequence of real numbers and let $x \in \mathbb{R}$ have the property that every convergent subsequence of X converges to x, then the sequence X converges to x.

Chapter 18. The Logarithm and Exponential Functions

Definition.

$$\log(x) = \int_1^x \frac{1}{t} dt$$

Theorem 1. If $x, y > 0$, then $\log(xy) = \log(x) + \log(y)$.

Corollary 1. If n is natural number and $x > 0$, then

$$\log(x^n) = n \log(x)$$

Corollary 2. If $x, y > 0$, then

$$\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$$

Definition. The exponential function, exp is defined as $\log^{-1}$.

Theorem 2. For all number x,

$$\exp'(x) = \exp(x)$$

Theorem 3. If x and y are any two numbers, then

$$\exp(x+y) = \exp(x) \cdot \exp(y)$$

Definition. $\mathbf{e} = \exp(1)$

Definition. If $a > 0$, for any real number x,

$$a^x = e^{x \log(a)}$$

Theorem 4. If $a > 0$, then

(1) $(a^b)^c = a^{bc}$ for b, c

(2) $a^1 = a$ and $a^{x+y} = a^x \cdot a^y$ for x, y.

Theorem 5. If f is differentiable and

$$f'(x) = f(x) \text{ for all } x$$

then there is a number c such that

$$f(x) = ce^x \text{ for all } x$$

Proof. Let $g(x) = \dfrac{f(x)}{e^x}$. This is permissible since $e^x \neq 0$ for all x. Then

$$g'(x) = \frac{e^x f'(x) - f(x)e^x}{(e^x)^2} = \frac{e^x[f(x) - f(x)]}{e^{2x}} = 0$$

Therefore there is a number c such that

$$g(x) = \frac{f(x)}{e^x} = c \text{ for all } x$$

□

Theorem 6. For any natural number n,

$$\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$$

Proof. The proof consists of several steps:

(1) $e^x > x$ for all x, and consequently $\lim_{x\to\infty} e^x = \infty$ (this may be considered to be the case $n = 0$). To prove this statement (which is clear for $x \leq 0$) it suffices to show that

$$x > \log(x) \text{ for all } x > 0$$

If $x < 1$, this is clearly true, since $\log(x) < 0$. If $x > 1$, then $x - 1$ is an upper sum for $f(t) = \frac{1}{t}$ on $[1, x]$ so $\log(x) < x - 1 < x$.

(2) $\lim_{x\to\infty} \frac{e^x}{x} = \infty$
To prove this, note that

$$\frac{e^x}{x} = \frac{e^{x/2} \cdot e^{x/2}}{\frac{x}{2} \cdot 2} = \frac{1}{2}\left(\frac{e^{x/2}}{x/2}\right) \cdot e^{x/2}$$

By (1), the expression in parentheses is greater than 1, and $\lim_{x\to\infty} e^{x/2} = \infty$; this shows that $\lim_{x\to\infty} \frac{e^x}{x} = \infty$.

(3) $\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$
Note that

$$\frac{e^x}{x^n} = \frac{(e^{x/n})^n}{(x/n)^n \cdot n^n} = \frac{1}{n^n} \cdot \left(\frac{e^{x/n}}{x/n}\right)^n$$

The expression in parentheses becomes arbitrarily large, by (2), so the nth power certainly becomes arbitrarily large.

$\square$