Contents

Math 350 - Advanced Calculus
Homework 10

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Problem 1

(a) Prove that if f and g are continuous on $[a, b]$ if $m \le f(x) \le M$ for all x in $[a, b]$ and if g is non-negative on $[a, b]$ then

$$m \cdot \int_a^b g \le \int_a^b f \cdot g \le M \cdot \int_a^b g$$

Proof. Since $m \le f(x) \le M$ and $g(x)$ is non-negative on $[a, b]$, we have

$$mg(x) \le f(x) \cdot g(x) \le Mg(x)$$

Thus

$$m \int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le M \int_a^b g(x)dx$$

□

(b) Use part (a) to prove that

$$\frac{1}{7\sqrt{2}} \le \int_0^1 \frac{x^6}{\sqrt{1+x^2}}dx \le \frac{1}{7}$$

Proof. Let

$$f(x) = \frac{1}{\sqrt{1+x^2}}$$
$$g(x) = x^6$$

Also

$$f'(x) = \frac{-1}{2} \cdot (1+x^2)^{-1/2-1} \cdot 2x = \frac{-x}{(1+x^2)^{3/2}} < 0 \text{ for all } x \in [0, 1]$$

So f is decreasing on $[0, 1]$ and it attains min and max at two end points, $\frac{1}{\sqrt{2}} \le f(x) \le 1$. On the other hand,

$$\int_0^1 g(x) = \int_0^1 x^6 = \frac{x^7}{7}\Big|_0^1 = \frac{1}{7}$$

Apply part (a), it follows that

$$\frac{1}{7\sqrt{2}} \le \int_0^1 \frac{x^6}{\sqrt{1+x^2}}dx \le \frac{1}{7}$$

□

Problem 2 Prove that

$$\frac{3}{8} \leq \int_0^{1/2} \sqrt{\frac{1-x}{1+x}} dx \leq \frac{\sqrt{3}}{4}$$

Proof. We want to apply the Theorem from Problem 1., we need to find f and g that disguised in the form of $\sqrt{\dfrac{1-x}{1+x}}$. The most obvious choice is to complete the square, we have

$$\begin{aligned}\sqrt{\frac{1-x}{1+x}} &= \sqrt{\frac{1-x}{1+x}} \cdot \frac{\sqrt{1-x}}{\sqrt{1-x}} \\ &= \frac{1-x}{\sqrt{1-x^2}}\end{aligned}$$

Let

$$\begin{aligned}f(x) &= \frac{1}{\sqrt{1-x^2}} \\ g(x) &= 1-x\end{aligned}$$

Then $g(x) > 0$ for all $x \in [0, 1/2]$. And

$$\int_0^{1/2} g(x) = \left(x - \frac{x^2}{2}\right)\Big|_0^{1/2} = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$$

On the other hand,

$$f'(x) = \frac{-x}{(1-x^2)^{3/2}} \geq 0 \text{ for all } x \in [0, 1/2]$$

So f is increasing so it attains min at $x = 0$ and max at $x = \dfrac{1}{2}$.

$$\begin{aligned}m &= f(0) = \frac{1}{1-0^2} = 1 \\ M &= f(1/2) = \frac{1}{1-\frac{1}{4}} = \frac{2}{\sqrt{3}}\end{aligned}$$

Hence,

$$1 \leq f(x) \leq \frac{2}{\sqrt{3}}$$

combine with the integral of $g(x)$, we obtain

$$\begin{aligned}1 \cdot \frac{3}{8} &\leq \int_1^{1/2} \sqrt{\frac{1-x}{1+x}} \leq \frac{3}{8} \cdot \frac{2}{\sqrt{3}} \\ \Leftrightarrow \frac{3}{8} &\leq \int_1^{1/2} \sqrt{\frac{1-x}{1+x}} \leq \frac{\sqrt{3}}{4}\end{aligned}$$

□

Problem 3 Let f be integrable on $[a, b]$, let c be in (a, b) and let

$$F(x) = \int_a^x f \quad , a \leq x \leq b$$

For each of the following statements, give either a proof or a counterexample

(a) If f is differentiable at c, then F is differentiable at c.

Proof. Recall the Fundamental Theorem of Calculus,

Theorem . If f is integrable on $[a, b]$ and define F on $[a, b]$ by $F(x) = \int_a^x f$. If f is continuous at c in $[a, b]$, then F is differentiable at c and $F'(c) = f(c)$.

From theorem we see that this is true because f is differentiable at c which implies f is continuous at c. □

(b) If f is differentiable at c, then F' is continuous at c.

Proof. This is also true because $F'(c) = f(c)$ where f is continuous at c, so is $F'(c)$. □

(c) If f' is continuous at c, then F' is continuous at c.

Proof. If f' is continuous at c, then first, f has to be differentiable at c so f is continuous at c which implies F' is continuous at c. □

Problem 4 Around 1671, Newton discovered the approximation rule for the integral of a continuous function f,

$$\int_a^b f \approx \frac{b-a}{8}(f_0 + 3f_1 + 3f_2 + f_3)$$

where

$$f_i = f(a + \frac{(b-a)i}{3}) \text{ for } i = 0, 1, 2, 3$$

Prove that this Newton approximation gives exact value if $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ is a polynomial function of degree ≤ 3.

Proof. Integrate $f(x)$ by regular method we obtain,

$$\begin{aligned}
\int_a^b f(x) &= \int_a^b a_0 + a_1x + a_2x^2 + a_3x^3 \\
&= (a_0x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \frac{a_3}{4}x^4)\Big|_a^b \\
&= a_0(b-a) + \frac{a_1}{2}(b^2 - a^2) + \frac{a_2}{3}(b^3 - a^3) + \frac{a_3}{4}(b^4 - a^4)
\end{aligned}$$

To use Newton method, first we need to compute f_0, f_1, f_2, f_3

$$\begin{aligned}
f_0 &= f(a + 0) = f(a) \\
f_1 &= f(a + (b-a)/3) \\
f_2 &= f(a + 2(b-a)/3) \\
f_3 &= f(a + 3(b-a)/3) = f(b)
\end{aligned}$$

Substitute into the original expression to obtain,

$$\begin{aligned}
\int_a^b f &\approx \frac{b-a}{8} \cdot (f_0 + 3f_1 + 3f_2 + f_3) \\
&= \frac{b-a}{8} \cdot (f(a) + 3f(a + (b-a)/3) + 3f(a + 2(b-a)/3) + f(b)) \\
&= \frac{b-a}{8} \cdot \Big[(a_0 + a_1a + a_2a^2 + a_3a^3) \\
&+ 3(a_0 + a_1(a + (b-a)/3) + a_2(a + (b-a)/3)^2 + a_3(a + (b-a)/3)^3) \\
&+ 3(a_0 + a_1(a + 2(b-a)/3) + a_2(a + 2(b-a)/3)^2 + a_3(a + 2(b-a)/3)^3) \\
&+ (a_0 + a_1b + a_2b^2 + a_3b^3)\Big]
\end{aligned}$$

To avoid lengthy expression, we manipulate each coefficient a_0, a_1, a_2, a_3 one by one,
For a_0,

$$a_0 + 3a_0 + 3a_0 + a_0 = 8a_0$$

For a_1,

$$\begin{aligned}
&a_1\left(a + 3a + 3 \cdot \frac{b-a}{3} + 3a + 3 \cdot \frac{2(b-a)}{3} + b\right) \\
&= a_1(7a + 3(b-a) + b) \\
&= a_1(4a + 4b)
\end{aligned}$$

For a_2,

$$\begin{aligned}
&a_2\left[a^2 + 3 \cdot \left(a + \frac{b-a}{3}\right)^2 + 3 \cdot \left(a + \frac{2(b-a)}{3}\right)^2 + b^2\right] \\
&= a_2\left(\frac{8a^2}{3} + \frac{8ab}{3} + \frac{8b^2}{3}\right)
\end{aligned}$$

For a_3,

$$\begin{aligned}
&a_3\left[a^3 + 3 \cdot \left(a + \frac{b-a}{3}\right)^3 + 3 \cdot \left(a + \frac{2(b-a)}{3}\right)^3 + b^3\right] \\
&= a_3(2a^3 + 2a^2b + 2ab^2 + 2b^3)
\end{aligned}$$

Put altogether, we have

$$\begin{aligned}
\int_a^b f &\approx \frac{b-a}{8} \cdot (f_0 + 3f_1 + 3f_2 + f_3) \\
&= \frac{b-a}{8}\left[8a_0 + a_1(4a+4b) + a_2\left(\frac{8a^2}{3} + \frac{8ab}{3} + \frac{8b^2}{3}\right) + a_3(2a^3 + 2a^2b + 2ab^2 + 2b^3)\right] \\
&= a_0(b-a) + \frac{a_1}{2}(b^2 - a^2) + \frac{a_2}{3}(b^3 - a^3) + \frac{a_3}{4}(b^4 - a^4)
\end{aligned}$$

□

Problem 5 Let

$$f(x) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(a) Prove that f is integrable on $[0, 1]$.

Proof. Fix $\epsilon > 0$, choose a partition. Let $P = \{t_0, t_1, t_2, \ldots, t_n\}$ be a partition of $[0, 1]$ such that $t_i - t_{i-1} < \epsilon$. Also, let

$$\begin{aligned}
m_i &= \inf\{f(x) : t_{i-1} \leq x \leq t_i\} \\
M_i &= \sup\{f(x) : t_{i-1} \leq x \leq t_i\}
\end{aligned}$$

Note that $m_1 = \{0, t_1\} = 0$ because $f(x) = 0$ only if $x = 0$, where $m_i = 1$ for all $1 < i \leq n$. But $M_i = 1$ for all $i, 1 \leq i \leq n$ because $f(x) = 1$ for all $x \neq 0$. Hence,

$$L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1}) < 0 + (n-1) \cdot \epsilon$$
$$U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}) < n \cdot \epsilon$$

It follows that

$$U(f, P) - L(f, P) < (n - n + 1)\epsilon = \epsilon$$

Since ϵ is arbitrarily chosen, f is integrable on $[0, 1]$. □

(b) Compute $\int_0^1 f$.

Proof. It's obvious that $\int_0^1 f = 1$. □

Problem 6 Let $f(x) = x^2$, and let $a < b$.

1. Prove that f is integrable on $[a, b]$ by finding, for any $\epsilon > 0$, a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \epsilon$$

Proof. The tricky part in this problem is $f(x) = x^2$, so we need to consider two cases: $0 < a < b$ ($a < b < 0$) and $a < 0 < b$ since the $\inf\{f(x)\}$ and $\sup\{f(x)\}$ will be different for each of these cases.

Case 1: $0 < a < b$,
Fix $\epsilon > 0$. Let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a, b]$ where each interval is of equal length $\dfrac{b-a}{n}$.

$$\begin{aligned} L(f, P) &= \sum_{i=1}^{n} m_i(t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} t_{i-1}^2(t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} \left(\frac{(i-1)(b-a)}{n}\right)^2 \cdot \frac{b-a}{n} \\ &= \left(\frac{b-a}{n}\right)^3 \sum_{i=1}^{n} (i-1)^2 \\ &= \left(\frac{b-a}{n}\right)^3 \cdot \frac{n(n-1)(2n-1)}{6} \end{aligned}$$

Similarly,

$$\begin{aligned}
U(f,P) &= \sum_{i=1}^{n} m_i(t_i - t_{i-1}) \\
&= \sum_{i=1}^{n} t_{i-1}^2(t_i - t_{i-1}) \\
&= \sum_{i=1}^{n} \left(\frac{i(b-a)}{n}\right)^2 \cdot \frac{b-a}{n} \\
&= \left(\frac{b-a}{n}\right)^3 \sum_{i=1}^{n} i^2 \\
&= \left(\frac{b-a}{n}\right)^3 \cdot \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

Then

$$\begin{aligned}
U(f,P) - L(f,P) &= \left(\frac{b-a}{n}\right)^3 \cdot \frac{n(n+1)(2n+1)}{6} - \left(\frac{b-a}{n}\right)^3 \cdot \frac{n(n-1)(2n-1)}{6} \\
&= \left(\frac{b-a}{n}\right)^3 \cdot \frac{6n^2}{6} \\
&= (b-a)^3 \cdot \frac{n^2}{n^3} \\
&= (b-a)^3 \cdot \frac{1}{n}
\end{aligned}$$

But $\lim_{n\to\infty} (b-a)^3 \cdot \frac{1}{n} = 0 < \epsilon$ Thus for sufficiently large n we have $U(f,P) - L(f,P) < \epsilon$.
Case 2: $a < 0 < b$, Fix $\epsilon > 0$, but this time we consider divide $[a,b]$ into two partitions $P_1 = \{a, 0\}$ and $P_2 = \{0, b\}$ then $P = P_1 \cup P_2$ where each subinterval length of P_1, P_2 is defined as $\frac{a}{n/2}, \frac{b}{n/2}$ respectively.

$$\begin{aligned}
L(f,P) &= \sum_{i=1}^{n} m_i(t_i - t_{i-1}) \\
&= \sum_{i=1}^{n} t_{i-1}^2(t_i - t_{i-1}) \\
&= \sum_{i=1}^{n} \left(\frac{(i-1)(b-a)}{n}\right)^2 \cdot \frac{b-a}{n} \\
&= \left(\frac{b-a}{n}\right)^3 \sum_{i=1}^{n} (i-1)^2 \\
&= \left(\frac{b-a}{n}\right)^3 \cdot \frac{n(n-1)(2n-1)}{6}
\end{aligned}$$

Similarly,

$$U(f,P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1})$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} t_{i-1}^2 (t_i - t_{i-1}) \\
&= \sum_{i=1}^{n} \left(\frac{i(b-a)}{n} \right)^2 \cdot \frac{b-a}{n} \\
&= \left(\frac{b-a}{n} \right)^3 \sum_{i=1}^{n} i^2 \\
&= \left(\frac{b-a}{n} \right)^3 \cdot \frac{n(n+1)(2n+1)}{6}
\end{aligned}
$$

□

2. Use your work on part (a) to compute $\int_a^b f$.

   Proof.

$$\int_a^b x^2 dx = \frac{b^3}{3} - \frac{a^3}{3}$$

□

Problem 7 If $f(x) = \int_0^x \sqrt{t + t^6} dt$. Find $f'(3)$.

Proof. From Fundamental Theorem of Calculus, we have that

$$f'(x) = \sqrt{x + x^6}$$

Hence,

$$f'(3) = \sqrt{3 + 3^6}$$

□

Problem 8 Let $f(x) = \int_1^x (1 + \sin(\sin(t))) dt$. Compute $f'(x)$ and prove that f is increasing.

Proof. From Fundamental Theorem of Calculus, we have that

$$f'(x) = (1 + \sin(\sin(x)))$$

To prove that f is increasing, note that

$$
\begin{aligned}
&-1 &&\leq \sin(x) &&\leq 1 \\
\Leftrightarrow\ &\sin(-1) &&\leq \sin(\sin(x)) &&\leq \sin(1) \\
\Leftrightarrow\ &\sin(-1) + 1 &&\leq 1 + \sin(\sin(x)) &&\leq \sin(1) + 1
\end{aligned}
$$

So $f'(x) > 0$ for all x which implies $f(x)$ is increasing.

□

Problem 9 Find the derivatives of the following functions.

(a) $F(x) = \displaystyle\int_x^b \frac{1}{1+t^2+\sin^2(t)}dt$

Proof. $F'(x) = \dfrac{1}{1+x^2+\sin^2(x)}$ □

(b) $F(x) = \displaystyle\int_a^b \frac{x}{1+t^2+\sin^2(t)}dt$

Proof. $F'(x) = \dfrac{1}{1+x^2+\sin^2(x)}$ □

Problem 10 Prove that

$$\int_0^x \frac{1}{1+t^2}dt = c + \int_{1/x}^0 \frac{1}{1+t^2}dt$$

for some constant c.

Proof. The equation is equivalent to

$$\begin{aligned}\int_0^x \frac{1}{1+t^2}dt - \int_{1/x}^0 \frac{1}{1+t^2}dt = c\\ \int_0^x \frac{1}{1+t^2}dt + \int_0^{1/x} \frac{1}{1+t^2}dt = c\end{aligned}$$

Let $F(x) = \displaystyle\int_0^x \frac{1}{1+t^2}dt + \int_0^{1/x} \frac{1}{1+t^2}dt$ We want to show that $F'(x) = 0$ so that $F(x) = c$. By the First Fundamental Theorem of Calculus, we have

$$\begin{aligned}\int_0^x \frac{1}{1+t^2}dt - \int_{1/x}^0 \frac{1}{1+t^2}dt &= \frac{1}{1+x^2} + \frac{1}{1+(1/x)^2}\cdot\frac{-1}{x^2}\\ &= \frac{1}{1+x^2} - \frac{1}{x^2}\cdot\frac{x^2}{x^2+1}\\ &= 0\end{aligned}$$

□

Problem 11 Prove that if h is continuous, f and g are differentiable and

$$F(x) = \int_{f(x)}^{g(x)} h$$

then $F'(x) = h(g(x))\cdot g'(x) - h(f(x))\cdot f'(x)$.

Proof. We have

$$\begin{aligned}
F(x) &= \int_{f(x)}^{g(x)} h \\
&= \int_{f(x)}^{0} h + \int_{0}^{g(x)} h \\
&= \int_{0}^{g(x)} h - \int_{0}^{f(x)} h \\
&= h(g(x)) \cdot g'(x) - h(f(x)) \cdot f'(x)
\end{aligned}$$

□

Problem 12 Find all the continuous functions f satisfying

$$\int_0^x f = (f(x))^2 + C$$

for some constant C.

Proof. By the First Fundamental Theorem of Calculus, we have if

$$F(x) = \int_a^x f$$

and if f is continuous then $F'(x) = f(x)$.
Let $F(x) = (f(x))^2 + C \Rightarrow F'(x) = 2f(x)f'(x)$, it follows that if f is continuous, then

$$F'(x) = f(x) \Leftrightarrow 2f(x)f'(x) = f(x) \Leftrightarrow f'(x) = \frac{1}{2}$$

Integrate $f'(x)$ we obtain

$$f(x) = \int f'(x)dx = \int \frac{1}{2}dx = \frac{1}{2}x + k$$

for some $k \in \mathbb{R}$. To solve for k, we substitute $f(x)$ into the original equation, we have

$$\begin{aligned}
& \int_0^x \left(\frac{1}{2}t + k\right)dt = \left(\frac{x}{2} + k\right)^2 + C \\
\Leftrightarrow \quad & \left(\frac{1}{2} \cdot \frac{t^2}{2} + kt\right)\Bigg|_0^x = \frac{x^2}{4} + xk + k^2 + C \\
\Leftrightarrow \quad & \frac{x^2}{4} + kx = \frac{x^2}{4} + kx + k^2 + C \\
\Leftrightarrow \quad & k^2 = -C \\
\Leftarrow \quad & k = \sqrt{-C}
\end{aligned}$$

Therefore

$$f(x) = \frac{x}{2} + \sqrt{-C} \text{ for } C \leq 0$$

□

Problem 13 Prove that if f and g have continuous derivatives on $[a, b]$, then

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g$$

Proof. By Algebra Integral Theorem, we have

$$\int_a^b (fg' + f'g) = \int_a^b fg' + \int_a^b f'g$$

Moreover from Chain Rule, we know that

$$(f \cdot g)' = f'g + fg'$$

Apply the Second Fundamental Theorem of Calculus to $(f \cdot g)$, to obtain

$$\int_a^b (fg' + f'g) = (f \cdot g)(b) - (f \cdot g)(a) = f(b)g(b) - f(a)g(a)$$

which implies

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g$$

□

Problem 14 Let $f(x) = \log|x|$ for $x \neq 0$. Prove that $f'(x) = \dfrac{1}{x}$ for $x \neq 0$.

Proof. Using the definition of $\log(x)$, we have

$$\int_1^x \frac{1}{t} dt = \log(x)$$

So there are two cases,
If $x > 0$, then $f'(x) = [\log(x)]' = \dfrac{1}{x}$.
If $x < 0$, then $f'(x) = [\log(-x)]' = \dfrac{1}{-x} \cdot -1 = \dfrac{1}{x}$. □

Problem 15 Let e be the number such that $\log(e) = 1$. Prove that $\dfrac{5}{2} < e < 3$.

Proof. Consider the definition of $\log(x)$,

$$\log(x) = \int_1^x \frac{1}{t} dt$$

and using a constant lower and upper bound for $1/t$ on the interval $[1, x]$ it follows that

$$1 - \frac{1}{x} \leq \log(x) \leq x - 1$$

for all $x > 0$. Taking inverse functions this becomes

$$1 + x \leq e^x \leq \frac{1}{1 - x}$$

for all $x < 1$.
Choose $n \geq 1$, substitute $x \leftarrow x/n$ and raise to the power n to get

$$\left(1 + \frac{x}{n}\right)^n \leq e^{\frac{x}{n} n} = e^x \leq \left(1 - \frac{x}{n}\right)^{-n}$$

for all $x < n$. For $x = 1$ and $n = 6$ this becomes

$$\frac{5}{2} < \left(1 + \frac{1}{6}\right)^6 \leq e \leq \left(1 - \frac{1}{6}\right)^{-6} < 3.$$

□

Problem 16

(a) Prove that $\dfrac{\log(x)}{x} \leq \dfrac{1}{\sqrt{x}} \displaystyle\int_1^x \frac{1}{t^{3/2}} dt$ for all $x \geq 1$.

(b) Prove that $\displaystyle\lim_{x\to\infty} \frac{\log(x)}{x} = 0$.

Proof. From the definition of $\log(x)$,

$$\log(x) = \int_1^x \frac{1}{t} dt$$

Since 1 is the $\sup\{f(t) : m_{i-1} \leq t \leq m_i\}$, it follows that

$$\int_1^x \frac{1}{t} dt \leq U(f, P) < x - 1$$

So

$$\log(x) < x - 1 < x \Rightarrow \frac{\log(x)}{x} < 1$$

On the other hand, we have

$$\log(x) \leq \sqrt{x} \text{ because } \int_1^x \frac{1}{t} dt < \int_1^x \frac{1}{\sqrt{t}} dt$$

Or another way to prove this fact is to consider $\log(x) < x$ for all $x > 0 \Rightarrow \log(x) = 2\log\sqrt{x} < 2\sqrt{x}$.

Our goal is to apply Squeeze's Theorem (Sandwich's Lemma) to $\displaystyle\lim_{x\to\infty} \frac{\log(x)}{x}$.
In fact,

$$\frac{1}{x} \leq \frac{\log(x)}{x} \leq \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$$

where

$$\lim_{x\to\infty} \frac{1}{x} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$$

Therefore,

$$\lim_{x\to\infty} \frac{\log(x)}{x} = 0$$

□

(c) Prove that $\displaystyle\lim_{x\to\infty} \frac{\log(x)}{x^n} = 0$ for any $n > 0$.

Proof. Consider

$$\lim_{x\to\infty} \left[\frac{\log(x)}{x} \cdot \frac{1}{x^{n-1}}\right]$$

From part (b), we know that $\dfrac{\log(x)}{x} < 1$ and as $x \to \infty$, it's obvious that $\dfrac{1}{x^{n-1}} \to 0$. So

$$\lim_{x\to\infty} \frac{\log(x)}{x^n} = 0$$

□

Problem 17 Prove that if f is differentiable and $f'(x) = f(x)$ for all real number x, then there is a number c such that $f(x) = c \cdot \exp(x)$ for all x.

Let $g(x) = \dfrac{f(x)}{e^x}$, $e^x \neq 0 \ \forall x \in \mathbb{R}$. Then

$$g'(x) = \frac{e^x f'(x) - f(x)e^x}{(e^x)^2} = \frac{e^x[f(x) - f(x)]}{e^{2x}} = 0$$

So

$$g(x) = \int g'(x)dx = \int 0dx = c$$

Problem 18 Let $f(x) = \displaystyle\int_0^x f$, then prove that $f(x) = 0$ for all x.

Proof. Since $f(x)$ is defined by $f(x) = \displaystyle\int_0^x f$, f is continuous on $[a, b]$. Moreover, by the First Fundamental Theorem of Calculus, we have

$$f(x) = f'(x)$$

So $f(x) = ce^x$ by Problem 17, and integrate $f(x)$ over $[0, x]$ yields

$$\int_0^x ce^t dt = ce^x - ce^0 = ce^x - c = ce^x$$

which implies

$$c(e^x - 1) = ce^x \Leftrightarrow c = 0$$

Thus $f(x) = 0 \cdot e^x = 0$. □

Problem 19 Prove that

$$\lim_{x\to\infty} \frac{x^n}{\exp(x)} = 0$$

for any $n > 0$.

Proof. Consider the definition of $\log(x)$,

$$\log(x) = \int_1^x \frac{1}{t} dt$$

and using a constant lower and upper bound for $1/t$ on the interval $[1, x]$ it follows that

$$\log(x) \leq x - 1 < x \text{ for all } x > 0$$

Raise to the power e of both sides, the inequality becomes

$$x < \exp(x) \Rightarrow \frac{x}{\exp(x)} < 1 \text{ for all } x > 0$$

Next consider

$$\begin{aligned}
\lim_{x\to\infty} \frac{x}{\exp(x)} &= \lim_{x\to\infty} \frac{\frac{x}{2} \cdot 2}{\exp(x/2) \cdot \exp(x/2)} \\
&= \lim_{x\to\infty} \left[\frac{x/2}{\exp(x/2)}\right] \cdot \frac{1}{\exp(x/2)} \\
&= 0
\end{aligned}$$

because $\lim\limits_{x\to\infty}\dfrac{1}{\exp(x/2)}=0$ and $\dfrac{x/2}{\exp(x/2)}<1$. Now write $\dfrac{x^n}{\exp(x)}$ as

$$\frac{(x/n)^n\cdot n^n}{\exp(x/n)^n}=\left[\frac{x/n}{\exp(x/n)}\right]^n\cdot n^n$$

Since n^n is just a constant, Algebra Limit Theorem allows us to write

$$\lim_{x\to\infty}\left[\frac{x/n}{\exp(x/n)}\right]^n\cdot n^n=n^n\cdot\lim_{x\to\infty}\left[\frac{x/n}{\exp(x/n)}\right]^n=n^n\cdot 0=0$$

□

Problem 20 Let $f(x)=\dfrac{\exp(x)}{x^n}$ for $x>0$.

(a) Find the minimum value of $f(x)$ for $x>0$, and conclude that $f(x)>\dfrac{\exp(n)}{n^n}$ for all $x>n$.

(b) Using the expression for $f'(x)$ found in (a), prove that $f'(x)>\dfrac{\exp(n+1)}{(n+1)^{n+1}}$, for $x>n+1$.

Problem 21 Let $f(x)=\dfrac{1}{\sqrt{1+x^2}}$ and let $F(x)=\displaystyle\int_0^x f$.

(a) Prove that F is uniformly continuous on $\mathbb{R}$.

Proof. Consider

$$\int_0^x\frac{1}{\sqrt{1+t^2}}dt$$

First we integrate $f(x)$ by substitution. Let $t=\tan(u)$, so $dt=\sec^2(u)du$. Substitute into $F(x)$ to obtain,

$$\begin{aligned}
F(x)&=\int_0^{\tan(u)}f\\
&=\int_0^x\frac{\sec^2(u)}{\sqrt{1+\tan^2(u)}}\\
&=\int_0^x\sec(u)du\\
&=\log(\sec(u)+\tan(u))\Big|_0^x\\
&=\log(t+\sqrt{1+t^2})\Big|_0^x\\
&=\log(x+\sqrt{1+x^2})-\log(0+\sqrt{1+0^2})\\
&=\log(x+\sqrt{1+x^2})
\end{aligned}$$

Since $\sqrt{1+x^2}>0$ for all $x\in\mathbb{R}$, we have that $f(x)$ is continuous on $\mathbb{R}$. By Fundamental Theorem of Calculus, $F(x)$ is differentiable on $\mathbb{R}$. By Mean Value Theorem, there is a number $c\in\mathbb{R}$ such that

$$F'(c)=\frac{f(x)-f(y)}{x-y}$$

but

$$F'(x)=f(x)=\frac{1}{\sqrt{1+x^2}}>0\quad\forall x\in\mathbb{R}$$

which implies $F'(c) > 0$. And this satisfies Lipschitz condition because

$$|f(x) - f(y)| \leq F'(c)|x - y|$$

where $F'(c) > 0$. As we shown in class if $F(x)$ is a Lipschitz function then f is uniformly continuous. If we haven't proved it, then the proof should be straightforward as follows,
Given $\epsilon > 0$, choose $\delta = \dfrac{\epsilon}{K}$. If $x, y \in \mathrm{dom}(f)$ satisfy $|x - y| < \delta$, then

$$|f(x) - f(y)| < K \cdot \frac{\epsilon}{K} = \epsilon$$

In fact, compute the integral is redundant in this case. □

(b) Prove that $F(-x) = -F(x)$.

Proof. To show that $F(-x) = -F(x)$ is the same as $F(-x) + F(x) = 0$. So we have

$$\begin{aligned} F(-x) + F(x) &= \log(-x + \sqrt{1 + x^2}) + \log(x + \sqrt{1 + x^2}) \\ &= \log((-x + \sqrt{1 + x^2}) \cdot (x + \sqrt{1 + x^2}) \\ &= \log(x^2 + 1 - x^2) \\ &= \log(1) = 0 \end{aligned}$$

Another way to prove is consider the definition

$$F(x) = \int_0^x \frac{1}{\sqrt{1 + x^2}} \cdot dx$$

Then

$$\begin{aligned} F(-x) + F(x) &= \int_0^{-x} \frac{1}{\sqrt{1 + x^2}} + \int_0^x \frac{1}{\sqrt{1 + x^2}} \\ &= -\int_0^x \frac{1}{\sqrt{1 + x^2}} + \int_0^x \frac{1}{\sqrt{1 + x^2}} \\ &= 0 \end{aligned}$$

which implies $F(-x) = F(x)$. □

(c) Prove that F is increasing on $\mathbb{R}$.

Proof. Follows from (a) since $F'(x) = f(x) = \dfrac{1}{\sqrt{1 + x^2}} > 0$ for all $x \in \mathbb{R}$. □

(d) Prove that $F(x) \geq \log(\sqrt{x})$ for all $x \geq 1$.

Proof. For $x \geq 1$, we have $x + \sqrt{1 + x^2} \geq \sqrt{x}$, it follows that $F(x) = \log(x + \sqrt{1 + x^2}) \geq \log(\sqrt{x})$ because $\log(x)$ is a decreasing function ($\log(x)' = \dfrac{1}{x} > 0$ for $x \geq 1$).

To prove it using the first Fundamental Theorem of Calculus, we note that

$$G(x) = \log(\sqrt{x}) = \int_1^{\sqrt{x}} \frac{1}{t} dt$$

So

$$G'(x) = \frac{1}{\sqrt{x}} \cdot (\sqrt{x})' = \frac{1}{\sqrt{x} \cdot \sqrt{x}} \cdot \frac{1}{2} = \frac{1}{2x}$$

where

$$F'(x) = f(x) = \frac{1}{\sqrt{x^2+1}}$$

For all $x \geq 1$, we have that $F'(x) \geq G'(x)$ since $\sqrt{x^2+1} \leq 2x \Leftrightarrow x^2+1 \leq 4x^2$. Thus $F(x) \geq G(x)$.

$\square$

(e) Prove that F takes on all real numbers: if y is any number, there is a number x such that $F(x) = y$.

Proof. Since $\sqrt{x^2+1} > |x| \Rightarrow x+\sqrt{x^2+1} > 0$, so that $F(x) = \log(x+\sqrt{x^2+1})$ is defined on $\mathbb{R}$. If $F(x) = y$ then

$$\begin{aligned}
& y = \log(x+\sqrt{x^2+1}) \\
\Leftrightarrow\quad & e^y = x+\sqrt{x^2+1} \\
\Leftrightarrow\quad & e^y - x = \sqrt{x^2+1} \\
\Leftrightarrow\quad & (e^y-x)^2 = x^2+1 \\
\Leftrightarrow\quad & e^{2y} - 2e^y x + x^2 = x^2+1 \\
\Leftrightarrow\quad & e^{2y} - 1 = 2e^y x \\
\Leftrightarrow\quad & x = \frac{e^{2y}-1}{2e^y}
\end{aligned}$$

$\square$

To prove it without using $\log(x+\sqrt{x^2+1})$, notice that the derivative of $F(x)$ is $f(x) = \dfrac{1}{\sqrt{1+x^2}} > 0$ for all $x \geq 1$, so it is increasing on $[1,\infty)$ which implies $F(x)$ is one to one. Moreover, f is continuous so F is differentiable which is continuous. Hence $F(x)$ is also onto or if y is any number, there is a number x such that $F(x) = y$.

Problem 22 Let F be the function constructed in Problem 21. Let $S(x)$ be defined by $S(x) = y$ if and only if $F(y) = x$ (that is, S is the inverse function of F).

(a) Prove that S is differentiable.

Proof. From Problem 21, we know that $F(x)$ is differentiable because $f(x)$ is continuous, and $F(x)$ is also one-one. Since $S(x)$ is the inverse of $F(x)$, by Inverse Differentiation Theorem (proved in class), we have $S(x)$ is also differentiable. $\square$

(b) Prove that $S'(x) = \sqrt{1+S^2(x)}$ for all numbers x.

Proof. Since $S(x)$ is the inverse of $F(x)$, we have $F(S(x)) = x$. Take the derivative of from both side we have $[F(S(x))]' = 1 \Rightarrow F'(S(x)) \cdot S'(x) = 1 \Rightarrow S'(x) = \dfrac{1}{F'(S(x))}$. Moreover since $F'(x) = f(x)$, we have

$$S'(x) = \frac{1}{1/\sqrt{1+S^2(x)}} = \sqrt{1+S^2(x)}$$

$\square$

(c) Prove that $S''(x) = S(x)$.

Proof. We have

$$\begin{aligned}
S''(x) &= (\sqrt{1+S^2(x)})' \\
&= \frac{1}{2}(1+S^2(x))^{-1/2} \cdot 2S(x)S'(x) \\
&= \frac{S(x)S'(x)}{\sqrt{1+S^2(x)}} \\
&= \frac{S(x) \cdot \sqrt{1+S^2(x)}}{\sqrt{1+S^2(x)}} \\
&= S(x)
\end{aligned}$$

□

Problem 23 Let S be the function constructed in Problem 22 and let $C(x) = S'(x)$. Prove that $S(x) + C(x) = \exp(x)$ for all x.

Proof. Let $g(x) = S(x) + C(x) = S(x) + S'(x)$, we have

$$g'(x) = S'(x) + S''(x) = S'(x) + S(x) = g(x)$$

By Problem 17, it follows that $g(x) = \exp(x)$. □

Problem 24 If f has n derivatives at a, then

$$P_{n,a} = f(a) + \frac{1}{1!}f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \ldots + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

is called the Taylor polynomial of degree n for f at a.
Find the Taylor polynomial of degree n for $f(x) = \log(1+x)$ at $a = 0$.

Proof. We have

$$f^{(k)} = \log^{(k)}(1+x)$$

so

$$f^{(k)}(0) = \log^{(k)}(1) = (-1)^{k-1}(k-1)!$$

Therefore the Taylor polynomial of degree n for f at 0 is

$$P_{n,0}(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{(-1)^{n-1}x^n}{n}$$

□