Contents

Math 350 - Advanced Calculus
Homework 9

Chan Nguyen

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Problem 1 Let f be a function that has n derivatives at a. The Taylor polynomial of degree n for f at a is the polynomial $P_{n,a,f}$ in $(x-a)$ given by

$$P_{n,a,f}(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \ldots + a_n(x-a)^n$$

with coefficients $a_k = \dfrac{f^{(k)}(a)}{k!}$.

(a) Let $f : (0,\infty) \to \mathbb{R}$ be a function such that $f'(x) = \dfrac{1}{x}$ for all $x > 0$ and $f(1) = 0$. Find the Taylor polynomial of degree n for f at $a = 1$.

Proof. We have

$$\begin{aligned}
f^{(0)}(x) &= f(x) \\
f^{(1)}(x) &= \frac{1}{x} \\
f^{(2)}(x) &= \frac{-1}{x^2} \\
f^{(3)}(x) &= \frac{-1 \cdot -2}{x^3} \\
f^{(4)}(x) &= \frac{-1 \cdot -2 \cdot -3}{x^4} \\
f^{(5)}(x) &= \frac{-1 \cdot -2 \cdot -3 \cdot -4}{x^5} \\
\ldots &= \ldots \\
f^{(k)}(x) &= (-1)^{k+1} \cdot \frac{(k-1)!}{x^k}
\end{aligned}$$

So

$$a_k = (-1)^{k+1} \cdot \frac{(k-1)!}{a^k}$$

Thus the Taylor polynomial of degree n for f at $a = 1$ is

$$\begin{aligned}
P_{n,a,f} &= f(1) + (-1)^{1+1}\frac{(1-1)!}{1^1} \cdot (x-1) + \ldots + (-1)^{n-1}\frac{(n-1)!}{1^n} \cdot (x-1)^n \\
&= (x-1) - (x-1)^2 + 2! \cdot (x-1)^3 + \ldots + (-1)^{n+1}\frac{(n-1)!}{1^n} \cdot (x-1)^n
\end{aligned}$$

□

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be a function such that $g'(x) = \dfrac{1}{\sqrt{1+x^2}}$ for all $x > 0$ and $g(0) = 0$. Find the Taylor polynomial of degree n for g at $a = 0$.

Proof. Similarly,

$$
\begin{aligned}
g^{(0)}(x) &= g(x) \\
g^{(1)}(x) &= \frac{1}{(1+x^2)^{1/2}} \\
g^{(2)}(x) &= \frac{-x}{(1+x^2)^{3/2}} \\
g^{(3)}(x) &= \frac{2x^2-1}{(x^2+1)^{5/2}} \\
g^{(4)}(x) &= \frac{-6x^3+9x}{(x^2+1)^{7/2}} \\
g^{(5)}(x) &= \frac{24x^4-72x^2+9}{(x^2+1)^{9/2}} \\
g^{(6)}(x) &= \frac{-120x^5+600x^3-225x}{(x^2+1)^{11/2}} \\
g^{(7)}(x) &= \frac{720x^6-5400x^4+4050x^2-225}{(x^2+1)^{13/2}} \\
g^{(8)}(x) &= \frac{-5040x^7+52920x^5-66150x^3+11025x}{(x^2+1)^{15/2}} \\
g^{(9)}(x) &= \frac{40320x^8-564480x^5-1058400x^3+352800x^2+11025}{(x^2+1)^{17/2}}
\end{aligned}
$$

It seems like the general for the nth derivative of $\dfrac{1}{\sqrt{1+x^2}}$ is a little overkill. Note that we only need to find the nth derivative at $a = 0$, i.e. $g^{(k)}(0)$, but if we look at the pattern of the first 9 derivatives, we see that every even term is actually zeroed out.

$$
\begin{aligned}
g^{(0)}(0) &= 0 \\
g^{(1)}(0) &= \frac{1}{(1+x^2)^{1/2}} = 1 \\
g^{(2)}(0) &= 0 \\
g^{(3)}(0) &= \frac{2x^2-1}{(x^2+1)^{5/2}} = -1 \\
g^{(4)}(0) &= 0 \\
g^{(5)}(0) &= \frac{24x^4-72x^2+9}{(x^2+1)^{9/2}} = 9 \\
g^{(6)}(0) &= 0 \\
g^{(7)}(0) &= \frac{720x^6-5400x^4+4050x^2-225}{(x^2+1)^{13/2}} = -225 \\
g^{(8)}(0) &= 0 \\
g^{(9)}(x) &= \frac{40320x^8-564480x^5-1058400x^3+352800x^2+11025}{(x^2+1)^{17/2}} = 11025
\end{aligned}
$$

Now the remain task is to find out what is the general pattern of the sequence $1, 1, 225, 11025$ since the alternating sign can be handle easily by adding $(-1)^n$. By factoring out each of this number, we see that

$$\begin{aligned} 1 &= 1 \\ 1 &= 1 \\ 9 &= 3^2 \\ 225 &= 3^2 \cdot 5^2 \\ 11025 &= 3^2 \cdot 5^2 \cdot 7^2 \end{aligned}$$

which is the product of odd square, so we have

$$\left[\prod_{i=1}^{n}(2i-1)\right]^2$$

To find a more compact formula for this expression, we complete the factorial by write it as

$$\begin{aligned} \prod_{i=1}^{n}(2i-1) &= \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdots (2n-1) \cdot 2n}{2 \cdot 4 \cdot 6 \cdots 2n} \\ &= \frac{(2n)!}{2^n(1 \cdot 2 \cdot 3 \cdots n)} \\ &= \frac{(2n)!}{2^n n!} \end{aligned}$$

Now try out some values for k, we have $k = 7$, yields 225 which is

$$\left[\frac{(2 \cdot 3)!}{3! \cdot 2^3}\right]^2 = \left[\frac{(7-1)!}{((7-1)/2)! \cdot 2^{(7-1)/2}}\right]^2$$

Generally,

$$g^{(k)}(0) = (-1)^{\lfloor \frac{k}{2} \rfloor} \frac{(k-1)!}{[(k-1)/2]! \cdot 2^{\frac{k-1}{2}}}$$

So the Taylor polynomial of degree n for g at $a = 0$ is

$$P_{n,0,g} = \frac{1}{1!} \cdot x + \frac{-1}{3!} \cdot x^3 + \frac{9}{5!} \cdot x^5 + \frac{-225}{7!} \cdot x^7 + \ldots + \frac{(-1)^{\lfloor \frac{n}{2} \rfloor} \frac{(n-1)!}{[(n-1)/2]! \cdot 2^{\frac{n-1}{2}}}}{n!} \cdot x^n$$

□

Problem 2 Let f have n derivatives at a. Prove that

$$\lim_{x \to a} \frac{f(x) - P_{n,f,a}(x)}{(x-a)^n} = 0$$

Proof. Recall

$$P_{n,a,f}(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \ldots + a_n(x-a)^n$$

with coefficients $a_k = \dfrac{f^{(k)}(a)}{k!}$.

Let $g(x) = P_{n,a,f}(x)$, we notice that

$$\begin{aligned} g^{(1)}(a) &= 1! \cdot a_1 \\ g^{(2)}(a) &= 2! \cdot a_2 \\ \ldots &= \ldots \\ g^{(k)}(a) &= k! \cdot a_k \end{aligned}$$

Hence, $g^{(k)}(a) = k! \cdot \dfrac{f^{(k)}(a)}{k!} = f^{(k)}(a)$. Then

$$\lim_{x\to a}[f^{(k)}(x) - g^{(k)}(x)] = 0$$

for all $0 \leq k \leq n$.

...not done! $\square$

Problem 3 Two functions f and g are equal up to order n at a if

$$\lim_{x\to a}\frac{f(x)-g(x)}{(x-a)^n} = 0$$

(a) Let $P(x) = p_0 + p_1(x-a) + p_2(x-a)^2 + \ldots + p_M(x-a)^M$ and $Q(x) = q_0 + q_1(x-a) + \ldots q_N(x-a)^N$ be two polynomials in $(x-a)$ of degrees $M, N \leq n$. Prove that if P and Q are up to order n at a then $P = Q$.

Proof. Without loss of generality, assume that $N \leq M$. Since P and Q are up to order n at a, we have that

$$\lim_{x\to a}\frac{P(x)-Q(x)}{(x-a)^n} = 0$$

Consider

$P(x) - Q(x) = (p_0 - q_0) + (p_1 - q_1)(x-a) + (p_2-q_2)(x-a)^2 + \ldots + (p_N - q_N)(x-a)^N + \ldots p_M(x-a)^M$

What we want to show is that $P(x) - Q(x) = 0 \Rightarrow P(x) = Q(x)$. On the other hand, we claim that

$$\lim_{x\to a}\frac{f(x)-g(x)}{(x-a)^n} \Rightarrow \lim_{x\to a}\frac{f(x)-g(x)}{(x-a)^k}, 0 \leq k \leq n \tag{1}$$

Let $h(x) = f(x) - g(x)$, since $\lim_{x\to a}\dfrac{h(x)}{(x-a)^k} = 0$, by definition of limit, given $\epsilon > 0$, there exists $\delta > 0$ such that if $|x-a| < \delta$ then $\left|\dfrac{h(x)}{(x-a)^n} - 0\right| < \epsilon$. So we can choose $\gamma = \min(\delta, 1)$, then if $|x-a| < \gamma$ then,

$$\left|\frac{h(x)}{(x-a)^n}\cdot(x-a)^m - 0\right| = \left|\frac{h(x)}{(x-a)^n}\right|\cdot|x-a|^m < \epsilon\cdot 1 = \epsilon$$

where $m = n-k$. So (1) holds for any $k, 0 \leq k \leq n$. Inductively choose $k = 0, 1, \ldots n$, we have

- $k = 0$:
  $\lim_{x\to a}(p_0 - q_0) + (p_1 - q_1)(x-a) + (p_2 - q_2)(x-a)^2 + \ldots + (p_N - q_N)(x-a)^N + \ldots p_M(x-a)^M = 0$
  $$\Rightarrow p_0 - q_0 = 0 \Rightarrow p_0 = q_0$$
- $k = 1$:
  $\lim_{x\to a}(p_1 - q_1) + (p_2 - q_2)(x-a) + \ldots + (p_N - q_N)(x-a)^{N-1} + \ldots p_M(x-a)^{M-1} = 0$
  $$\Rightarrow p_1 - q_1 = 0 \Rightarrow p_1 = q_1$$

- $k = \ldots$
- $k = n \lim_{x \to a} p_n(x-a)^n = 0$

$$\Rightarrow p_n = 0$$

Thus it must he case that $M = N$ and $p_k - q_k = 0$ for all $0 \le k \le N = n$, in other words $P(x) = Q(x)$.

$\square$

(b) Let f be n times differentiable at a and suppose that P is a polynomial $(x-a)$ of degree $\le n$ that equals f up to order n at a. Prove that $P = P_{n,f,a}$, the Taylor polynomial of degree n for f at a.

Proof. We have

$$P_{n,a,f}(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \ldots + a_n(x-a)^n \text{ with } a_k = \frac{f^{(k)}(a)}{k!}$$

and

$$P_{n,a}(x) = a_0' + a_1'(x-a) + a_2'(x-a)^2 + \ldots a_n'(x-a)^n$$

Using the same technique from part (a), we have

$$\begin{aligned} a_0 &= a_0' \\ a_1 &= a_1' \\ a_2 &= a_2' \\ \ldots &= \ldots \\ a_n &= a_n' \end{aligned}$$

It follows that $P_{n,a} = P_{n,f,a}$. This is actually the result of part (a) since $P_{n,f,a}$ and $P_{n,a}$ are the same as P, Q. $\square$

Problem 4 Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f'(x) = f(x)$ for all x and $f(0) = 1$. Prove that, for all n and all x,

$$|f(x) - P_{n,0,f}(x)| \le \frac{F_x|x|^{n+1}}{(n+1)!}$$

where $F_x = \sup\{|f(y)| : |y| \le |x|\}$.

Proof. First we will show that the two conditions $f'(x) = f(x)$ for all x and $f(0) = 1$ implies that $f(x) = e^x$. The second part is just the result of the remainder term of Taylor polynomial of e^x. First note that $f'(x) = f(x)$ implies $f^{(k)}$ exists and $f^{(k)} = f(x)$ because $f''(x) = (f'(x))' = f'(x)$ and so on. Write $f(x)$ in Taylor polynomial form $P_{n,f,a}$,

$$\begin{aligned} P_{n,f,a} &= f(0) + \frac{f(a)}{1!}(x-a) + \frac{f(a)}{2!}(x-a)^2 + \ldots + \frac{f(a)}{n!}(x-a)^n \\ &= 1 + \frac{f(a)}{1!}(x-a) + \frac{f(a)}{2!}(x-a)^2 + \ldots + \frac{f(a)}{n!}(x-a)^n \end{aligned}$$

At $a = 0$, we have

$$P_{n,f,0} = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}$$

On the other hand, we know that from lecture notes:

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + R_{n,x} = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \frac{f^{n+1}(c)}{(n+1)!} \cdot x^{n+1}$$

Then

$$f(x) - P_{n,f,0} = e^x - P_{n,f,0} = R_{n,x} = \frac{f^{n+1}(c)}{(n+1)!} \cdot x^{n+1} = \frac{f(c)}{(n+1)!} x^{n+1}$$

for some c between x and 0.
Moreover, since $F_x = \sup\{|f(y)| : |y| \le |x|\}$, by definition of supremum, $F_x \le f(c)$ for all c between x and 0. Thus

$$|f(x) - P_{n,f,0}| = |R_{n,x}| = \left| \frac{f(c)}{(n+1)!} x^{n+1} \right| \le \frac{F_x |x|^{n+1}}{(n+1)!}$$

□

Problem 5 Given a function f and two integers $M \ge 0$ and $N \ge 0$, the Pade approximant of order $[M/N]$ of f (at 0) is the rational function

$$R(x) = \frac{a_0 + a_1 x + a_2 x^2 + \ldots + a_M x^M}{1 + b_1 x + b_2 x^2 + \ldots + b_N x^N}$$

that agrees with $f(x)$ to the highest possible order which amounts to

$$\begin{aligned} R(0) &= f(0) \\ R'(0) &= f'(0) \\ R''(0) &= f''(0) \\ \ldots &= \ldots \\ R^{(M+N)}(0) &= f^{(M+N)}(0) \end{aligned}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f'(x) = f(x)$ for all x and $f(0) = 1$. Find the Pade approximant of order $[2/2]$ for f.

Proof. The Pade approximant of order $[2/2]$ is given by

$$R(x) = \frac{a_0 + a_1 x + a_2 x^2}{1 + b_1 x + b_2 x^2}$$

For $M = 2, N = 2$, we want to satisfy

$$\frac{a_0 + a_1 x + a_2 x^2}{1 + b_1 x + b_2 x^2} = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4$$

where

$$f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

Multiplying up the denominator and then equating coefficients up to x^4, we have that

$$\begin{aligned} a_0 + a_1 x + a_2 x^2 = & \quad (1 + b_1 x + b_2 x^2) \cdot (c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4) \\ = & \quad c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \\ & \quad b_1 c_0 x + b_1 c_1 x^2 + b_1 c_2 x^3 + b_1 c_3 x^4 + b_1 c_4 x^5 + \\ & \quad b_2 c_0 x^2 + b_2 c_1 x^3 + b_2 c_2 x^4 + b_2 c_3 x^5 + b_2 c_4 x^6 \end{aligned}$$

which yields a system of equations

$$\begin{aligned} a_0 &= c_0 = 1 \\ a_1 &= c_1 + b_1 c_0 \\ a_2 &= c_2 + c_1 b_1 + c_0 b_2 \\ 0 &= c_3 + b_1 c_2 + b_2 c_1 \\ 0 &= c_4 + b_1 c_3 + b_2 c_2 \end{aligned}$$

where $c_0 = 1, \quad c_1 = 1, \quad c_2 = \frac{1}{2}, \quad c_3 = \frac{1}{6}, \quad c_4 = \frac{1}{24}$. Solving this system of equations we have

$$\begin{aligned} a_0 = 1, a_1 = \frac{1}{2}, a_2 = \frac{1}{12} \\ b_0 = 1, b_1 = \frac{-1}{2}, b_2 = \frac{1}{12} \end{aligned}$$

Therefore, the Pade approximant of order $[2/2]$ is

$$\frac{1 + \frac{x}{2} + \frac{x^2}{12}}{1 - \frac{x}{2} + \frac{x^2}{12}} = \frac{12 + 6x + x^2}{12 - 6x + x^2}$$

□

Problem 6 Let f and g be integrable on $[a, b]$ and let c be a constant.

(a) Prove that $c \cdot f$ is integrable on $[a, b]$ on $\int_a^b c\dot{f} = c \cdot \int_a^b f$.

Proof. Since f is integrable on $[a, b]$, for every $\epsilon > 0$, there exists a partition P in $[a, b]$ such that

$$U(f, P) - L(f, P) < \epsilon$$

Let

$$\begin{aligned} m_i &= \inf\{f(x) : t_{i-1} \le x \le t_i\} \\ M_i &= \sup\{f(x) : t_{i-1} \le x \le t_i\} \\ m_i' &= \inf\{c \cdot f(x) : t_{i-1} \le x \le t_i\} \\ M_i' &= \sup\{c \cdot f(x) : t_{i-1} \le x \le t_i\} \end{aligned}$$

Since $c > 0$ is just a constant, we have $m_i' = c \cdot m_i$ and $M_i' = c \cdot M_i$. Let Choose $\epsilon = \frac{\epsilon}{c}$ for P, we have

$$U(cf, P) - L(cf, P) = c[U(f, P) - L(f, P)] < \frac{\epsilon}{c} \cdot c = \epsilon$$

□

(b) Prove that $f + g$ is integrable on $[a, b]$ and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.

Proof. Let

$$\begin{aligned} m_i &= \inf\{f(x) : t_{i-1} \le x \le t_i\} \\ M_i &= \sup\{f(x) : t_{i-1} \le x \le t_i\} \\ n_i &= \inf\{g(x) : t_{i-1} \le x \le t_i\} \\ N_i &= \sup\{g(x) : t_{i-1} \le x \le t_i\} \\ s_i &= \inf\{(f + g)(x) : t_{i-1} \le x \le t_i\} \\ S_i &= \sup\{(f + g)(x) : t_{i-1} \le x \le t_i\} \end{aligned}$$

We claim that $m_i + n_i \le s_i$ since $m_i \le f(x)$ and $n_i \le g(x)$ for all $x \in [t_{i-1}, t_i]$, so $m_i + n_i \le f(x) + g(x)$ which implies $m_i + n_i$ is a lower bound of $f(x) + g(x) = (f+g)(x)$, where s_i is the greatest as such, thus $m_i + n_i \le s_i$.
Similarly, we also claim that $M_i + N_i \ge S_i$ since $M_i \ge f(x)$ and $N_i \ge g(x)$ for all $x \in [t_{i-1}, t_i]$ so $M_i + N_i \ge (f+g)(x)$ which implies $M_i + N_i$ is an upper bound of $(f+g)(x)$, where S_i is the least as such, thus $M_i + N_i \ge S_i$.
Thus,

$$U(f+g, P) \le U(f, P) + U(g, P)$$
$$L(f+g, P) \ge L(f, P) + L(g, P)$$

On the other hand, since f, g are both integrable on $[a, b]$, for all $\epsilon > 0$, there exists P_1, P_2, such that

$$U(f, P_1) - L(f, P_1) < \frac{\epsilon}{2}$$
$$U(g, P_2) - U(g, P_2) < \frac{\epsilon}{2}$$

Let $P = P_1 \cup P_2$, so that P contains both P_1, P_2 then by Lemma, we have

$$L(f, P_1) \le L(f, P) \text{ and } L(f, P_2) \le L(f, P)$$
$$U(f, P_1) \ge L(f, P) \text{ and } U(f, P_2) \ge U(f, P)$$

Hence,

$$U(f+g, P) - U(f+g, P) < U(f, P) - L(f, P) + U(f, P) - L(f, P) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

Problem 7 Prove that if f is integrable on $[a, b]$, then $|f|$ is integrable on $[a, b]$ and $\left| \int_a^b f \right| \le \int_a^b |f|$

Proof. By definition of absolute, we have

$$-|f| \le f \le |f|$$

On the other hand,

$$L(f, P) \le f \le U(f, P)$$

where

$$L(f, P) = \sum_{i=1}^{n} m_i (t_i - t_{i-1})$$
$$U(f, P) = \sum_{i=1}^{n} M_i (t_i - t_{i-1})$$

Since $t_i - t_{i-1}$ is the length of a subinterval, $t_i - t_{i-1} \ge 0$, so the $L(f, P)$ and $U(f, P)$ depend only on m_i and M_i, where

$$m_i = \inf\{f(x) : t_{i-1} \le x \le t_i\}$$
$$M_i = \sup\{f(x) : t_{i-1} \le x \le t_i\}$$

Let

$$m_i' = \inf\{|f|(x) : t_{i-1} \le x \le t_i\}$$
$$M_i' = \sup\{|f|(x) : t_{i-1} \le x \le t_i\}$$

It's easy to see that $M_i \le M_i'$ and $m_i \le m_i'$. Thus

$$-\int_a^b |f| \le \int_a^b f \le \int_a^b |f|$$

or

$$\left| \int_a^b f \right| \le \int_a^b |f|$$

□

Problem 8 Evaluate without doing any computations

(a) $\displaystyle\int_{-1}^{1} x^3\sqrt{1-x^2}dx$

Proof.

$$\begin{aligned}\int_{-1}^{1} x^3\sqrt{1-x^2} &= \int_{-1}^{0} x^3\sqrt{1-x^2} + \int_{0}^{1} x^3\sqrt{1-x^2} \\ &= -\int_{0}^{1} x^3\sqrt{1-x^2} + \int_{0}^{1} x^3\sqrt{1-x^2} = 0\end{aligned}$$

□

(b) $\displaystyle\int_{-1}^{1} (x^5+1)\sqrt{1-x^2}dx$

Proof.

$$\begin{aligned}\int_{-1}^{1} (x^5+1)\sqrt{1-x^2} &= \int_{-1}^{0} (x^5+1)\sqrt{1-x^2} + \int_{0}^{1} (x^5+1)\sqrt{1-x^2} \\ &= -\int_{0}^{1} (x^5+1)\sqrt{1-x^2} + \int_{0}^{1} (x^5+1)\sqrt{1-x^2} = 0\end{aligned}$$

□

Problem 9 Prove that if $f(x) = x^3$, then $\displaystyle\int_a^b f = \frac{b^4-a^4}{4}$, by considering upper and lower sums for partitions $[a, b]$ into n equal subintervals, using the formula: $1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$ for the sum of the cubes of the first n positive integers.

Proof. We have

$$\int_a^b x^3 = \int_0^b x^3 - \int_0^a x^3$$

Consider a partition P in $[0, a]$, and divide this partition into n equal subintervals of length $\dfrac{a}{n}$. So we have

$$\begin{aligned}t_0 &= 0 \\ t_1 &= \frac{a}{n} \\ t_2 &= a + \frac{2a}{n} \\ t_3 &= a + \frac{3a}{n} \\ \ldots &= \ldots \\ t_i &= a + \frac{ia}{n}\end{aligned}$$

Let

$$\begin{aligned}m_i &= \inf\{x^3 : x \in [t_{i-1}, t_i]\} \\ M_i &= \sup\{x^3 : x \in [t_{i-1}, t_i]\}\end{aligned}$$

For any subinterval $[t_{i-1}, t_i]$, we have $m_i = t_{i-1}^3$ and $M_i = t_i^3$ because $t_{i-1} < t_i$ and x^3 is increasing ($f'(x) = 3x^2 \geq 0, \forall x$). Thus

$$\begin{aligned}
M(f,P) &= \sum_{i=1}^{n} M_i(t_i - t_{i-1}) \\
&= \sum_{i=1}^{n} \left[\frac{ia}{n}\right]^3 \cdot \frac{a}{n} \\
&= \frac{a^4}{n^4} \sum_{i=1}^{n} i^3 \\
&= \frac{a^4}{n^4} \cdot \left[\frac{n(n+1)}{2}\right]^2 \\
&= \frac{a^4}{4} \cdot \frac{n^4 + 2n^3 + n^2}{n^4}
\end{aligned}$$

Since $\lim_{n\to\infty} \frac{n^4 + 2n^3 + n^2}{n^4} = 1$, for sufficiently large n, we have

$$M(f,P) = \frac{a^4}{n^4}$$

Similarly, for the lower bound

$$\begin{aligned}
L(f,P) &= \sum_{i=1}^{n} m_i(t_i - t_{i-1}) \\
&= \sum_{i=1}^{n} \left[\frac{(i-1)a}{n}\right]^3 \cdot \frac{a}{n} \\
&= \sum_{i=0}^{n-1} \left[\frac{(i-1)a}{n}\right]^3 \cdot \frac{a}{n} \\
&= \frac{a^4}{n^4} \sum_{i=0}^{n-1} i^3 \\
&= \frac{a^4}{n^4} \cdot \left[\frac{(n-1)n}{2}\right]^2 \\
&= \frac{a^4}{n^4} \cdot \frac{n^4 - 2n^3 + n^2}{4} \\
&= \frac{a^4}{4} \cdot \frac{n^4 - 2n^3 + n^2}{n^4}
\end{aligned}$$

Again, we have that $\lim_{n\to\infty} \frac{n^4 - 2n^3 + n^2}{n^4} = 1$, so for sufficiently large n, we have $L(f,P) = \frac{a^4}{n^4}$. Hence,

$$\frac{a^4}{4} \leq \int_0^a f \leq \frac{a^4}{4}$$

which implies $\int_0^a f = \frac{a^4}{4}$. Similarly, we have $\int_0^b f = \frac{b^4}{4}$. It follows that,

$$\int_a^b x^3 = \int_0^b x^3 - \int_0^a x^3 = \frac{b^4}{4} - \frac{a^4}{4} = \frac{b^4 - a^4}{4}$$

$\square$

Problem 10 Decide which of the following functions are integrable on $[0,2]$ and calculate the integral sum if you can.

(a) $f(x) = \begin{cases} x + [x] & , x \text{ rational} \\ 0 & , x \text{ irrational} \end{cases}$

Proof. We claim that $f(x)$ is not integrable, in order to prove this fact, for all partition P of $[0,2]$ such that given $\epsilon > 0$, then $U(f,P) - L(f,P) > \epsilon$.
Let $P = \{t_0, t_1, \ldots, t_n\}$ be any partition of $[0,2]$, then we have at least one point $t_i \geq 1$ which implies there is some $M_i = \sup\{f(x) : t_{i-1} \leq x \leq t_i\} \geq 1$ because $[x]$ is the integer part that is less than equal to x, plus x in $[0,2]$, so it must be ≥ 1 for some interval $[t_i, t_j]$ if x is rational. Hence, $U(f,P) \geq 1$ because

$$U(f,P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1})$$

where $M_i \geq 0$, and there some i such that $M_i \geq 1$.
On the other hand, $L(f,P)$ is always 0 because $x \in [0,2]$ and $f(x) = x + [x]$. Thus

$$U(f,P) - L(f,P) \geq 1 \text{ for all } P \text{ of } [0,2]$$

Any $\epsilon > 1$ would suffice. □

(b) f is the function shown in Figure 1 (set $f(0) = 0$).

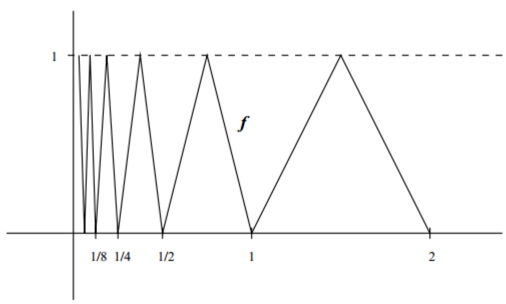

FIGURE 1

Proof. First we note that

- f is bounded above by 1 in $[0,2]$.
- f is discontinuous at $x = 0$.

As we can see each area of the triangle can be computed by height $h = 1$, a and base

$$\begin{aligned} b_0 &= 2 - 1 \\ b_1 &= 1 - \frac{1}{2} \\ b_2 &= \frac{1}{2} - \frac{1}{4} \\ \ldots &= \ldots \\ b_3 &= \frac{1}{2^n} - \frac{1}{2^{n+1}} \\ \ldots &= \ldots \end{aligned}$$

So total area of all triangles is

$$
\begin{aligned}
A &= \frac{1}{2}\cdot 1\cdot(2-1)+\sum_{i=0}^{\infty}\frac{1}{2}b_i\cdot h_i \\
&= \frac{1}{2}+\sum_{i=0}^{\infty}\left(\frac{1}{2^i}-\frac{1}{2^{i+1}}\right) \\
&= \frac{1}{2}+\sum_{i=0}^{\infty}\frac{2-1}{2^{i+1}} \\
&= \frac{1}{2}+\sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^i \\
&= \frac{1}{2}+\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots\right) \\
&= \frac{1}{2}\cdot\left(1+\frac{1}{2}+\frac{1}{4}+\dots\right)
\end{aligned}
$$

Recall the Geometric sum for $\frac{1}{2}<1$, we have

$$\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^i=\frac{1}{1-\frac{1}{2}}=\frac{1}{1/2}=2$$

Thus the area is $\frac{1}{2}\cdot 2=1$.
To prove that f is integrable formally, we have to consider an $\epsilon>0$. Notice that, although f is not continuous at 0, it's in fact continuous at $[\epsilon,2]$ for $0<\epsilon<2$. But continuity implies that f is integrable, so let P_i be the partition of of the i triangle in the picture, i.e.

$$
\begin{aligned}
P_1 &= \{2,1\} \\
P_2 &= \{1,\frac{1}{2}\} \\
P_3 &= \{\frac{1}{2},\frac{1}{4}\} \\
\dots &= \dots \\
P_n &= \{\frac{1}{2^n},\frac{1}{2^{n+1}}\} \\
\dots &= \dots
\end{aligned}
$$

For each of these intervals, we have

$$U(f,P_i)-L(f,P_i)<\frac{\epsilon}{2^i}$$

Now we choose $P=\bigcup_{i=1}^{\infty}P_i$, then

$$
\begin{aligned}
U(f,P)-L(f,P) &= \sum_{i=0}^{n}U(f,P_i)-L(f,P_i) \\
&< \sum_{i=1}^{n}\frac{\epsilon}{2^i}=\epsilon\cdot\sum_{i=1}^{n}\frac{1}{2^i}=\epsilon\cdot\left(\sum_{i=0}^{n}\frac{1}{2^i}-1\right) \\
&= \epsilon\left(\frac{1}{1-1/2}-1\right)=\epsilon(2-1)=\epsilon
\end{aligned}
$$

Since $\epsilon > 0$ is arbitrarily chosen, f is integrable on $[0, 2]$ and

$$\int_0^2 f = 1$$

□

Problem 11

(a) Prove that f is integrable on $[a, b]$ and $f(x) \geq 0$ for all x in $[a, b]$ then $\int_a^b f \geq 0$.

Proof. We have

$$L(f, P) \leq \int_a^b \leq U(f, P)$$

where

$$L(f, P) = \sum_i^n m_i(t_i - t_{i-1})$$

Notice that in this sum, the length of a subinterval is always positive, $t_i - t_{i-1} > 0$ because $t_i > t_{i-1}$. Moreover $m_i = \inf\{f(x) : t_{i-1} \leq x \leq t_i\}$ also ≥ 0 because $f(x) \geq 0$. Thus it follows from (2), that $\int_a^b f \geq 0$. □

(b) Prove that f and g are integrable on $[a, b]$ and $f(x) \geq g(x)$ for all x in $[a, b]$ then $\int_a^b f \geq \int_a^b g$.

Proof. Let $h(x) = f(x) - g(x)$, then from (a) and Problem 6, we have that $\int_a^b h \geq 0 \Rightarrow \int_a^b f - \int_a^b g \geq 0 \Rightarrow \int_a^b f \geq \int_a^b g$. □

Problem 12

(a) Give an example of a function f which is integrable on $[a, b]$ satisfies $f(x) \geq 0$ for all x, and $f(x) > 0$ for some x, and yet $\int_a^b f = 0$.

Proof. Let

$$f(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases} \quad \text{on } [0, 1]$$

The first two conditions satisfied because $f(x) \geq 0$ for all x and $f(1) = 1 > 0$.
To see why $\int_0^1 f = 0$, consider a partition P of $[0, 1]$, we have that the lower sum over P is,

$$L(f, P) = \sum_{i=1}^n m_i(t_i - t_{i-1})$$

since f is either 0 or 1 on $[0, 1]$, so $\inf\{f(x) : 0 \leq x \leq 1\} = 0$ which implies $\sum_{i=1}^n m_i(t_i - t_{i-1}) = 0$ for all $i, 1 \leq i \leq n$. On the other hand, by definition of Integral, we have

$$\int_0^1 f(x) = \sup\{L(f, P)\}$$

Thus

$$\int_0^1 f(x) = 0$$

$\square$

(b) Suppose that $f(x) \geq 0$ for all x in $[a, b]$ and f is continuous at x_0 in $[a, b]$ and $f(x_0) > 0$. Prove that $\displaystyle\int_a^b f > 0$. (Hint: It suffices to find partition P for which the lower sum $L(f, P) > 0$).

Proof. Since f is continuous at x_0, for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow f(x) - f(x_0) < \epsilon \text{ for all } x \in [a, b]$$

This implies that

$$-\epsilon < f(x) - f(x_0) < \epsilon$$

Let $\epsilon = f(x_0) > 0$ since from the hypothesis, $f(x_0) > 0$, we have

$$-f(x_0) < f(x) - f(x_0) < f(x_0) \Leftrightarrow 0 < f(x) < 2f(x_0)$$

So we can choose a partition P such that $|t_i - t_{i-1}| < \delta$, then $\inf\{f(x) : t_{i-1} \leq x \leq t_i\} > 0$.
Moreover by Integral Theorem, we have that

$$L(f, P) \leq \int_a^b f(x) \leq U(f, P)$$

where

$$L(f, P) = \sum_{i=1}^{n} \underbrace{\inf\{f(x) : t_{i-1} \leq x \leq t_i\}}_{>0} \cdot \underbrace{(t_i - t_{i-1})}_{>0}$$

which implies

$$0 < L(f, P) \leq \int_a^b f(x) \leq U(f, P)$$

Therefore

$$\int_a^b f(x) > 0$$

$\square$

Problem 13 Suppose that f is continuous on $[a, b]$ and that $\displaystyle\int_a^b fg = 0$ for all continuous g on $[a, b]$. Prove that $f = 0$.

Proof. By contradiction we assume that $f \neq 0$. Since $\displaystyle\int_a^b fg = 0$ for all continuous g on $[a, b]$, let $g = f$, we have

$$\int_a^b f^2 = 0$$

which implies $\sup\{L(f, P)\} = 0 \Rightarrow \sup\left\{\displaystyle\sum_{i=1}^{n} m_i(t_i - t_{i-1})\right\}$. On the other hand, $m_i = \inf\{f(x) : t_{i-1} \leq x \leq t_i\}$, but $f(x) \neq 0$ for all $x \in [t_{i-1}, t_i]$ and for all i.

...not done!

$\square$

Problem 14 For $a, b > 1$. Prove that

$$\int_1^a \frac{1}{x}dx + \int_1^b \frac{1}{x}dx = \int_1^{ab} \frac{1}{x}dx$$

Proof. First we note that the only way for the left hand side to be equal to the right hand side is

$$\int_1^b \frac{1}{x}dx = \int_a^{ab} \frac{1}{x}dx$$

because from Integral Theorem, we know that for $a < c < b$

$$\int_a^c f + \int_c^b f = \int_a^b f$$

So it reduces to show that

$$\int_a^b f = \int_{ac}^{bc} f \text{ for } c > 1$$

Suppose that f is integrable on both $[a, b]$ and $[ac, bc]$, let

$$\begin{aligned}
m_i &= \inf\{f(cx) : t_{i-1} \leq x \leq t_i\} \\
m_i' &= \inf\{f(x) : ct_{i-1} \leq x \leq ct_i\} \\
M_i &= \sup\{f(cx) : t_{i-1} \leq x \leq t_i\} \\
M_i' &= \sup\{f(x) : ct_{i-1} \leq x \leq ct_i\}
\end{aligned}$$

It's easy to see that $m_i = m_i'$ and $M_i = M_i'$ because if $x' \in [ct_{i-1}, ct_i]$, then $f(x') = f(cx)$ for $x \in [t_{i-1}, t_i]$. Now consider

$$\begin{aligned}
P &= \{t_0, t_1, t_2, \ldots, t_n\} \text{ of } [a, b] \\
P' &= \{ct_0, ct_1, ct_2, \ldots, ct_n\} \text{ of } [ac, bc]
\end{aligned}$$

We have

$$\begin{aligned}
L(f, P') &= \sum_{i=1}^n m_i'(ct_i - ct_{i-1}) \\
&= \sum_{i=1}^n cm_i'(t_i - t_{i-1}) \\
&= \sum_{i=1}^n m_i(t_i - t_{i-1}) \\
&= L(f, P)
\end{aligned}$$

Similarly,

$$\begin{aligned}
U(f, P') &= \sum_{i=1}^n M_i'(ct_i - ct_{i-1}) \\
&= \sum_{i=1}^n cM_i'(t_i - t_{i-1}) \\
&= \sum_{i=1}^n M_i(t_i - t_{i-1}) \\
&= U(f, P)
\end{aligned}$$

Moreover,

$$L(f,P') = L(f,P) \leq \int_a^b f \leq U(f,P) = U(f,P')$$

$$L(f,P) = L(f,P') \leq \int_{ac}^{bc} f \leq U(f,P') = U(f,P)$$

Hence,

$$\int_a^b f = \int_{ac}^{bc} f$$

Now back to the original problem. Since $a, b > 1$, we have $1 < a < b < ab$, so

$$\begin{aligned} \int_1^a \frac{1}{a} dx + \int_1^b \frac{1}{x} dx &= \int_1^a \frac{1}{a} dx + \int_a^{ab} \frac{1}{x} dx \\ &= \int_1^{ab} \frac{1}{x} dx \end{aligned}$$

□

Problem 15 Prove that if f is continuous on $[a, b]$ then

$$\int_a^b f = (b-a) f(\xi)$$

for some number ξ in $[a, b]$ and show by example that continuity is essential.

Proof. Since f is continuous on a compact interval $[a, b]$, f attains a maximum and a minimum, so $\min(f) \leq f(x) \leq \max(f)$ for all $x \in [a, b]$. On the other hand, by Theorem 15.3 continuity implies integrable, so f is integrable thus $\int_a^b f$ exists and

$$L(f,P) \leq \int_a^b f \leq U(f,P)$$

for some partition P. Moreover, we have that

$$\min(f)[b-a] \leq L(f,P) \leq \int_a^b f \leq U(f,P) \leq \max(f)[b-a]$$

because $[b-a]$ is the longest length of every subinterval and $\min(f) \leq m_i, \max(f) \geq M_i$ for all i in the summation. Thus

$$\min(f)[b-a] \leq \int_a^b f \leq \max(f)[b-a]$$

By Intermediate Value Theorem, there exists $\xi \in [a, b]$ such that $\min(f) \leq f(\xi) \leq \max(f)$ which implies

$$\int_a^b f = f(\xi)[b-a]$$

To show the continuity is essential, first we notice that if f is injective then $f(\xi)$ must be unique. So if f is discontinuous at only ξ then this function won't work. Thus a choice for f would be

$$f(x) = \begin{cases} x & , x > a \\ -x & , x < a \end{cases}$$

□