Contents

Problem 1 2
Problem 2 3
Problem 3 4
Problem 4 5
Problem 5 5
Problem 6 6
Problem 7 7
Problem 8 7
Problem 9 9
Problem 10 9
Problem 11 9

Math 350 - Advanced Calculus
Homework 4

Chan Nguyen

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Problem 1

(i) Prove that if $a_n < b_n$ and $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ then $a \le b$

(ii) Prove that if $a_n \le c_n \le b_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$, then $\lim_{n\to\infty} c_n = L$

Solution

(i) First, we introduce two claims:

- Claim 1: If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ then $\lim_{n\to\infty}(a_n - b_n) = a - b$
- Claim 2: If the limit of (a_n) exists and $a_n \ge 0$ then $\lim_{n\to\infty} a_n \ge 0$.

Claim 1:

Proof. By definition of limit, for $\epsilon > 0$, there exist N_1, N_2 such that for all $n > N_1$ and $n > N_2$

$$|a_n - a| \le \frac{\epsilon}{2} \quad \text{and} \quad |b_n - b| \le \frac{\epsilon}{2}$$

Let $N = \max(N_1, N_2)$ then for all $n \ge N$, we have

$$|(a_n - b_n) - (a - b)| = |(a_n - a) - (b_n - b)| \le |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\square$

Claim 2:

Proof. Let $\lim_{n\to\infty} a_n = M$. Suppose that $a_n \ge 0$ but $M < 0$. Let

$$\epsilon = -\frac{M}{2}$$

since $M < 0 \to -\frac{M}{2} > 0 \to \epsilon > 0$. By definition of limit,

$$|a_n - M| < -\frac{M}{2}$$

which implies $a_n - M < -\frac{M}{2} \Leftrightarrow a_n < -\frac{M}{2} + M \Leftrightarrow a_n < \frac{M}{2}$. In other words, $a_n < 0$ which is a contradiction. Therefore,

$$\lim_{n\to\infty} a_n \ge 0$$

$\square$

Now we are ready for the original problem! Consider the sequence (c_n) where

$$c_n = b_n - a_n$$

since $b_n > a_n$, we must have $c_n > 0$, then from **Claim 1** and **Claim 2** we have

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = b - a > 0$$

which implies $b > a$.

(ii) Prove that if $a_n \leq c_n \leq b_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$, then $\lim_{n\to\infty} c_n = L$

Proof. By definition of limit, for all $\epsilon > 0$ there exists $N_1 \in \mathbf{N}$ for all $n \geq N_1$, then $|a_n - L| < \epsilon$ and there exists $N_2 \in mathbfN$ for all $n \geq N_2$ then $|b_n - L| < \epsilon$. Let $N = \max(N_1, N_2)$, then for all $n > N$, we have

$$|a_n - L| < \epsilon \text{ and } |b_n - L| < \epsilon$$

Consider

$$\begin{aligned}
& a_n \leq c_n \leq b_n \\
\Leftrightarrow \quad & a_n - L \leq c_n - L \leq b_n - L \\
\Leftrightarrow \quad & -\epsilon < a_n - L \leq c_n - L \leq b_n - L < \epsilon \\
\Leftrightarrow \quad & -\epsilon < c_n - L < \epsilon
\end{aligned}$$

$$\therefore \lim_{n\to\infty} c_n = L$$

□

Problem 2 Prove that

(i) $\displaystyle\lim_{n\to\infty} \frac{3n^3 + 7n^2 + 1}{4n^3 - 8n + 63} = \frac{3}{4}$

(ii) $\displaystyle\lim_{n\to\infty} \frac{2^n + (-1)^n}{2^{n+1} + (-1)^{n+1}} = \frac{1}{2}$

Solution

(i) $\displaystyle\lim_{n\to\infty} \frac{3n^3 + 7n^2 + 1}{4n^3 - 8n + 63} = \frac{3}{4}$

Proof. Divide both numerator and denominator by n^3

$$\lim_{n\to\infty} \frac{3n^3 + 7n^2 + 1}{4n^3 - 8n + 63} = \lim_{n\to\infty} \frac{3 + \frac{7}{n} + \frac{1}{n^3}}{4 - \frac{8}{n^2} + \frac{63}{n^3}} = \frac{3}{4}$$

□

(ii) $\displaystyle\lim_{n\to\infty} \frac{2^n + (-1)^n}{2^{n+1} + (-1)^{n+1}} = \frac{1}{2}$

Proof. We have,

$$\begin{aligned}
\lim_{n\to\infty} \frac{2^n + (-1)^n}{2^{n+1} + (-1)^{n+1}} &= \lim_{n\to\infty} \frac{2^n}{2^{n+1} + (-1)^{n+1}} + \lim_{n\to\infty} \frac{(-1)^n}{2^{n+1} + (-1)^{n+1}} \\
&= \lim_{n\to\infty} \frac{2^n}{2^{n+1} + (-1)^{n+1}} + 0 \\
&= \lim_{n\to\infty} \frac{2^n}{2^{n+1} + (-1)^{n+1}}
\end{aligned}$$

There are two cases that we need to consider: $(-1)^n$ is either -1 or 1,

- If n is even, then $(-1)^{n+1} = -1$,

$$\lim_{n\to\infty} \frac{2^n}{2^{n+1} + (-1)^{n+1}} = \lim_{n\to\infty} \frac{2^n}{2^{n+1} - 1} \geq \lim_{n\to\infty} \frac{2^n}{2^{n+1}} = \frac{1}{2}$$

- If n is odd, then $(-1)^{n+1} = 1$,

$$\lim_{n\to\infty} \frac{2^n}{2^{n+1} + (-1)^{n+1}} = \lim_{n\to\infty} \frac{2^n}{2^{n+1} + 1} \leq \lim_{n\to\infty} \frac{2^n}{2^{n+1}} = \frac{1}{2}$$

Combine these two cases, we have $\frac{1}{2} \leq \lim_{n\to\infty} \frac{2^n}{2^{n+1} + (-1)^{n+1}} \leq \frac{1}{2}$. By Squeeze's Theorem (Sandwich's Lemma), $\lim_{n\to\infty} \frac{2^n}{2^{n+1} + (-1)^{n+1}} = \frac{1}{2}$ ☐

Problem 3 Prove that if $a > 0$, then $\lim_{n\to\infty} \sqrt[n]{a} = 1$

Solution

Proof. If $a = 1$, there is nothing to prove!

If $0 < a < 1$, the we can consider $a = \frac{1}{b}$ for some $b \neq 0$ then the limit becomes $\lim_{n\to\infty} \sqrt[n]{\frac{1}{b}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\frac{1}{b}}}$ which deduce to case $a > 1$ because if $0 < a < 1$ then $\frac{1}{a} > 1$. If $a > 1$ then $\sqrt[n]{a} > 1$. Let $\sqrt[n]{a} = (1 + x)$ for some $x > 0$. Hence

$$(\sqrt[n]{a})^n = (1+x)^n \Leftrightarrow a = (1+x)^n$$

Look at the binomial expansion of $(1+x)^n$,

$$(1+x)^n = 1 + nx + \underbrace{\binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots \binom{n}{n}x^n}_{\geq 0}$$

which tells us that $(1+x)^n \geq 1 + nx$ for $x \geq 0$. Thus

$$(1+x)^n = a > 1 + nx \Rightarrow \frac{a-1}{n} > x$$

Adding $1 < \sqrt[n]{a}$ to obtain:

$$1 < \sqrt[n]{a} = 1 + x < 1 + \frac{a-1}{n}$$

It's easy to see that

$$\lim_{n\to\infty} \left(1 + \frac{a-1}{n}\right) = 1$$

Applying ”Sandwich” lemma, we have:

$$\lim_{n\to\infty} \sqrt[n]{a} = 1$$

$\square$

Problem 4 Prove that $\lim_{n\to\infty} \sqrt[n]{n} = 1$. (Hint: put $\sqrt[n]{n} = 1 + a_n$, prove that $a_n > 0$ for $n > 1$, deduce that $n - 1 \geq \frac{1}{2}n(n-1)a_n^2$ for $n > 1$ hence $0 \leq a_n^2 \leq \frac{2}{n}$)

Solution

Proof. Let $\sqrt[n]{n} = 1 + a_n \Rightarrow (\sqrt[n]{n})^n = (1 + a_n)^n \Rightarrow n = (1 + a_n)^n$. Using Binomial theorem, we have that:

$$n \geq \frac{n(n-1)}{2}a_n^2 \text{ for } n > 1$$

Hence,

$$a_n^2 \leq \frac{2}{n-1}$$

where $a_n^2 \geq 0$, so $0 \leq a_n^2 \leq \frac{2}{n-1} \Rightarrow 0 \leq a_n \leq \sqrt{\frac{2}{n-1}}$

Next we consider the limit,

$$\lim_{n\to\infty} \sqrt{\frac{2}{n-1}} = 0$$

Apply Squeeze's Theorem (Sandwich Lemma) for a_n,

$$0 \leq a_n \leq 0$$

, then we must have

$$\lim_{n\to\infty} a_n = 0$$

where $\sqrt[n]{n} = a_n + 1$ which implies $\lim_{n\to\infty} \sqrt[n]{n} = 1$ $\square$

Problem 5 Does the sequences $a_1, a_2, a_3 \ldots$ converge or diverge? If it converges, what is the limit?

(i) $a_n = \frac{n}{n+1} - \frac{n+1}{n}$

(ii) $a_n = \frac{2^n}{n!}$

(iii) a_n the nth decimal digit of $\sqrt{2}$ (thus $a_1 = 4, a_2 = 1, a_3 = 4, a_4 = 2 \ldots$ and so on).

Solution

(i) $a_n = \frac{n}{n+1} - \frac{n+1}{n}$ converges.

Proof.

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \left(\frac{n}{n+1} - \frac{n+1}{n}\right) \\
&= \lim_{n\to\infty} \frac{n^2 - (n+1)^2}{n(n+1)} \\
&= \lim_{n\to\infty} \frac{(n-n-1)(n+n+1)}{n(n+1)} \\
&= \lim_{n\to\infty} \frac{-(\frac{2}{n} + \frac{1}{n^2})}{1 + \frac{1}{n}} = \frac{0}{1} = 0
\end{aligned}$$

□

(ii) $a_n = \frac{2^n}{n!}$ converges.

Proof. Consider
$$\frac{2\cdot 2\cdot 2\cdots 2}{1\cdot 2\cdot 3\cdots n} \le \frac{2}{1}\cdot\frac{2}{n} \text{ since } \frac{2\cdot 2\cdots 2}{2\cdot 3\cdots (n-1)} < 1$$
In addition, we have that:
$$\frac{1}{n!} \le \frac{2^n}{n!}$$
where
$$\lim_{n\to\infty}\frac{1}{n!} = 0 = \lim_{n\to\infty} 2\cdot\frac{2}{n}$$
which implies
$$0 \le \frac{2^n}{n!} \le 0$$
Hence, by Squeeze's Theorem (Sandwich's Lemma)
$$\lim_{n\to\infty}\frac{2^n}{n!} = 0$$
□

(iii) a_n the nth decimal digit of $\sqrt{2}$ (thus $a_1 = 4, a_2 = 1, a_3 = 4, a_4 = 2\ldots$ and so on).

Proof. Assume that we know $\sqrt{2}$ is irrational where $a_n \in D = \{0,1,2,3,4,5,6,7,8,9\}$, if a_n converge, then the number would be of the form:
$$d.d_1d_3d_4d_id_id_i\ldots$$
Suppose that it converges, thus the limit exits:
$$\lim_{n\to\infty} a_n = d, \text{ for some } d \in D$$
By definition of limit, for all $\epsilon > 0$, there exits a $N \in \mathbf{N}$ such that for all $n > N$:
$$|a_n - d| < \epsilon$$
Pick $\epsilon = 1 \to |a_n - d| < 1$ where a_n, d are decimal digits which is in D. The only way for $|a_n - d| < 1$ is when $a_n = d$ which is clearly a contradiction because this implies $\sqrt{2}$ is rational. Therefore, nth decimal digit of $\sqrt{2}$ does not converge.

□

Problem 6 Prove or give a counterexample. Let $a_1, a_2, \ldots$ be a sequence such that $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$. Does a_n have to converge?

Solution

This is false. A counterexample could be:
$$a_n = \sqrt{n}$$
The limit of $a_{n+1} - a_n$ is:
$$\begin{aligned}\lim_{n\to\infty}(\sqrt{n+1} - \sqrt{n}) &= \lim_{n\to\infty}\left(\frac{(\sqrt{n+1})^2 - (\sqrt{n})^2}{\sqrt{n+1}+\sqrt{n}}\right)\\ &= \lim_{n\to\infty}\left(\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}\right)\\ &= \lim_{n\to\infty}\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right) = 0\end{aligned}$$
but
$$\lim_{n\to\infty}\sqrt{n} = \infty$$

Problem 7 Prove that the sequence $x_1, x_2, x_3 \ldots$ of real numbers defined by $x_1 = 1$ and $x_{n+1} = x_n + \frac{1}{x_n^2}$ is unbounded.

Solution

Proof. Prove by contradiction:
Suppose that the limit of this sequence exists, then by definition of limit, for all $\epsilon > 0$, there exists a natural number N such that for all $n > N$ then $|x_n - L| < \epsilon$. But there is no such ϵ because $x_{n+1} > x_n$. We will prove this by induction:
Base case,

$$x_2 = 1 + \frac{1}{1^2} = 1 + 1 = 2$$

Inductive hypothesis,

$$x_n > x_{n-1}$$

Consider,

$$x_{n+1} = x_n + \frac{1}{x_n^2}$$

where $\frac{1}{x_n^2} > 0$ Thus

$$x_{n+1} > x_n$$

Therefore the limit doesn't exists.
Another way to prove it is to consider the following fact: "If the limit exists, it must satisfy an algebraic expression". In other words, if the sequence converge we will have $x_n = x_{n-1}$,

$$x_n = x_n + \frac{1}{x_n^2}$$

Cancel x_n from both sides of the equation, we obtain:

$$0 = \frac{1}{x_n^2}$$

which is impossible, thus we have a contradiction.

□

Problem 8 Prove or give a counterexample

(i) If (a_n) is a non-decreasing sequence that is $a_1 \leq a_2 \leq a_3 \leq \ldots$ such that $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$, then (a_n) is convergent.

(ii) If (a_n) is non-decreasing and bounded above, and $\lim_{n\to\infty} a_n = a$ then $a_n \leq a$

Solution

(i) This is false. A counterexample would be the same as problem 6.

$$a_n = \sqrt{n}$$

because a non-decreasing sequence is also an increasing sequence. However if we want to be more precise about the equal sign, we could choose

$$a_n = \sqrt{\left\lfloor \frac{n}{2} \right\rfloor}$$

Apparently,

$$\lim_{n\to\infty} a_n = \infty$$

What we need to show is that

$$\lim_{n\to\infty} a_{n+1} - a_n = 0$$

There are two cases that we need to consider:

- $n \pmod 2 \equiv 0 \to n = 2k$ for some positive integers k. We have,

$$a_n = \sqrt{\left\lfloor \frac{2k}{2} \right\rfloor} = \sqrt{2k/2} = \sqrt{k}$$

$$a_{n+1} = \sqrt{\left\lfloor \frac{2k+1}{2} \right\rfloor} = \sqrt{k}$$

  This case is trivial since

$$\lim_{n\to\infty}(a_{n+1} - a_n) \lim_{n\to\infty}(\sqrt{k} - \sqrt{k}) = 0$$

- $n \pmod 2 \equiv 1 \to n = 2k+1$ for some positive integers k. We have,

$$a_n = \sqrt{\left\lfloor \frac{2k+1}{2} \right\rfloor} = \sqrt{k}$$

$$a_n = \sqrt{\left\lfloor \frac{2k+1+1}{2} \right\rfloor} = \sqrt{k+1}$$

  Note that as $n \to \infty$, so does $k \to infty$, so we can replace the n in the limit as k.

$$\begin{aligned}
\lim_{n\to\infty}(a_{n+1} - a_n) &= \lim_{k\to\infty}(\sqrt{k+1} - \sqrt{k}) \\
&= \lim_{k\to\infty}\left(\frac{(\sqrt{k+1})^2 - (\sqrt{k})^2}{\sqrt{k+1} + \sqrt{k}}\right) \\
&= \lim_{k\to\infty}\left(\frac{k+1-k}{\sqrt{k+1} + \sqrt{k}}\right) \\
&= \lim_{k\to\infty}\left(\frac{1}{\sqrt{k+1} + \sqrt{k}}\right) \\
&= \frac{1}{\infty} = 0
\end{aligned}$$

(ii) If (a_n) is non-decreasing and bounded above, and $\lim_{n\to\infty} a_n = a$ then $a_n \leq a$

Proof. Since (a_n) is bounded above, there exists a least upper bound for (a_n). Let $\sup((a_n)) = M$, we will show that M is actually a. By definition of the least upper bound, we know that $a_n \leq M$ for all $n \in \mathbf{N}$. Let ϵ be arbitrary number in $\mathbf{R}$ such that $\epsilon > 0$, then we can choose a N such that

$$M - \epsilon \leq a_N \leq M$$

In addition, (a_n) is non-decreasing sequence $a_N \leq a_{N+1} \leq a_{N+2} \leq \dots$ which implies

$$M - \epsilon < a_n \leq M, \ \forall n \geq N \tag{1}$$

As we can see, the statement above is just a definition of limit in disguising form. In fact, we can rearrange (1) as:

$$|a_n - M| < \epsilon$$

which implies M is the limit of (a_n). Hence,

$$\sup((a_n)) = \lim_{n\to\infty} a_n = a$$

Therefore $a_n \leq a, \ \forall n$ $\square$

Problem 9

(i) Give an example of a sequence of real numbers with subsequence converging to every integer.

(ii) Give an example of a sequence of real numbers with subsequence converging to every real number.

Solution

(i) Sequence:
$$-1, 0, 1, -2, -1, 0, 1, 2, -3, -2, -1, 0, 1, 2, 3, \ldots$$

(ii) Sequence:
$$-1.x0.x, 1.x, -2.x, -1.x, 0.x, 1.x, 2.x \ldots$$
where x is a string of digit set $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ with length $|x| = w$, then for $0 \leq n, w - n < w$, we define x as following:
$$\underbrace{000}_{m0's} \quad \underbrace{1234}_{\text{all positive integers that is less than } 10^{w-n}}$$

Problem 10 Prove that if the subsequence (a_{2n}) and (a_{2n+1}) of a sequence (a_n) of real numbers both converge to the same limit L then (a_n) converges to L.
Solution

Proof. Consider an $\epsilon > 0$ for both subsequences, we have

- If (a_{2n}) converges to L then there exists a natural number N_0 such that for all $n > N_0$, $|a_n - L| < \epsilon$.
- If (a_{2n+1}) converges to L then there exists a natural number N_1 such that for all $n > N_1$, $|a_n - L| < \epsilon$.

Let $N = \max(N_0, N1)$ then for all $n > N$, $|a_n < L|$ for both subsequences. Therefore both subsequences converge to the same limit L.

□

Problem 11 Let (a_n) be a sequence of real numbers such that $0 < a_1 < 1$, and $a_{n+1} = \dfrac{2}{1 + a_n}$ for all $n > 1$

(i) Prove that the subsequence (a_{2n}) and (a_{2n+1}) are both monotonic, one is decreasing and the other is increasing.

(ii) Prove that (a_n) converges and find its limit.

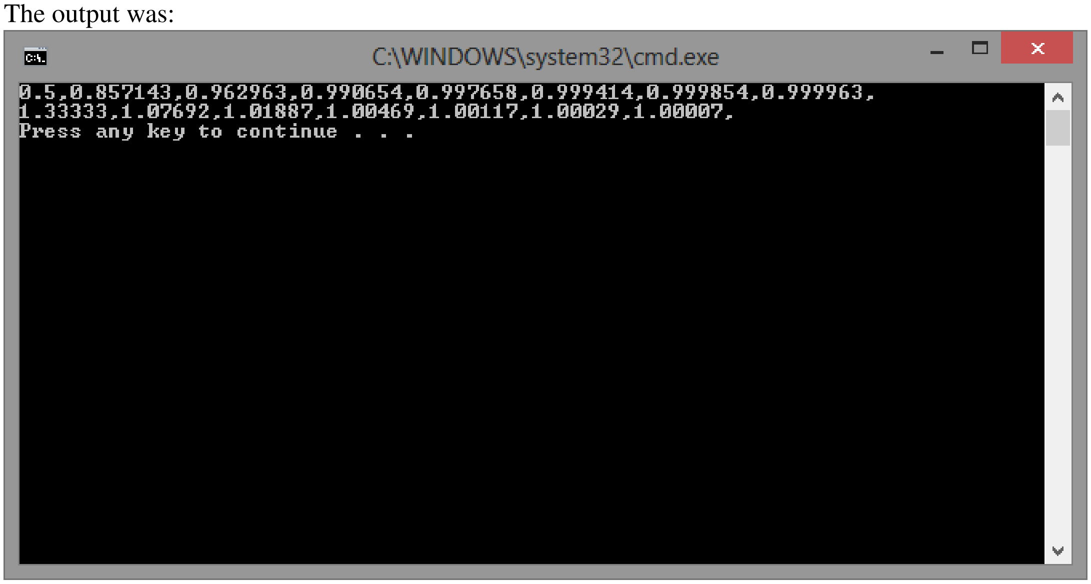
Solution

(i) To make sure that we understand the problem thoroughly, we start with a C++ program to demonstrate the sequence,

```
#include <iostream>
#include <vector>
#include <algorithm>

using namespace std;

double f(double x) {
        return (2.0/(x + 1));
```

```cpp
}

void display_seq(const vector<double>& seq) {
        for_each(seq.begin(), seq.end(), [&](double e) { cout << e << ","; });
        cout << '\n';
}

int main() {
        double init = 1.0/2.0;
        vector<double> evens;
        vector<double> odds;
        for (int i = 1; i <= 15; ++i) {
                if (i % 2 == 1) {
                        odds.push_back(init);
                } else {
                        evens.push_back(init);
                }
                init = f(init);
        }
        display_seq(odds);
        display_seq(evens);
        return 0;
}
```

The output was:



As we can from the output, the odd sequence is going from $0.5 \to 1.0$ and the even sequence is going from $1.3 \to 1.0$ which is quite interesting where $a_1 = \dfrac{1}{2}$

Proof. Consider $\square$

(ii) Prove that (a_n) converges and find its limit.