Contents

Math 350 - Advanced Calculus
Homework 6

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Problem 1 Let $0 < a < 3$ be a number. Let $x_0 = a$ and $y_0 = a - 1$, and inductively define

$$x_{n+1} = x_n - \frac{x_n y_n}{2}$$

$$y_{n+1} = \frac{y_n^2(y_n - 3)}{4}$$

(a) Prove that $a(1 + y_n) = x_n^2$ for all n.

(b) Prove that the sequence (y_n) converges to 0.

(c) Prove that the sequence (x_n) converges to $\sqrt{a}$

Solution.

(a) *Proof.* We will prove by induction n. Base case:

$$\text{lhs} = x_0^2 = a^2 = \text{rhs} = a(1 + y_0) = a(1 + a - 1) = a^2$$

Suppose $a(1 + y_n) = x_n^2$, we will show that it's also true for $n + 1$. We have

$$\begin{aligned}
x_{n+1}^2 &= \left(x_n - \frac{x_n y_n}{2}\right)^2 \\
&= x_n^2 - 2 \cdot x_n \cdot \frac{x_n y_n}{2} + \frac{x_n^2 y_n^2}{4} \\
&= x_n^2(1 - y_n + \frac{y_n^2}{4}) \\
&= a(1 + y_n)(1 - y_n + \frac{y_n^2}{4}) \\
&= a(1 - y_n^2 + \frac{y_n^2}{4} + \frac{y_n^3}{4}) \\
&= a(1 - \frac{3y_n^2}{4} + \frac{y_n^3}{4}) \\
&= a(1 + \frac{y_n^2(y_n - 3)}{4}) \\
&= a(1 + y_{n+1})
\end{aligned}$$

□

(b) *Proof.* Since $0 < a < 3 \Rightarrow -1 < a - 1 = y_0 < 2$. A brief C++ program shows that:

```
a = -0.5
------------------
-0.5
-0.21875
-0.0385056
-0.00112628
-9.51739e-007
-6.79356e-013
-3.46143e-025
-8.98612e-050
-6.05628e-099
-2.75089e-197
-0
-0
-0
-0
-0

initial a = 0.5
------------------
0.5
-0.15625
-0.0192642
-0.00028012
-5.88559e-008
-2.59801e-015
-5.06225e-030
-1.92198e-059
-2.77049e-118
-5.75673e-236
-0
-0
-0
-0
-0
```

```cpp
#include <iostream>
#include <vector>
#include <algorithm>

using namespace std;

void generate_sequence(double init) {
        cout << "initial a = " << init << endl;
        cout << "------------------\n";
        double y0 = init;
        for (int i = 0; i < 15; ++i) {
                cout << y0 << endl;
                y0 = y0*y0*(y0 - 3)/4.0;
        }
        cout << endl << endl;
}

int main() {
```

```
        double inits[5] = {-0.5, 0.0, 0.5, 1.0, 1.5};
        for (int i = 0; i < 5; ++i) {
                generate_sequence(inits[i]);
        }
        return 0;
}
```

As we can see if the initial value falls in the range between $(-1, 2)$, the sequence is actually increasing and converges to 0 except for the first value. So our goal is to prove that it's bounded by 0 and increasing. First we will prove that (y_n) is bounded by 0 for all $n \geq 1$. Base case $n = 1$, we have

$$y_1 = \frac{y_0^3 - 3y_0^2}{4}$$

where $-1 < y_0 < 2 \Rightarrow -1 < y_0^3 < 8$, and $0 \leq 3y_0^2 < 12$. Thus $\dfrac{y_0^3 - 3y_0^2}{4} < 0$. Suppose that $y_n < 0$, we want to show that it's also true for $n + 1$. Indeed, $y_{n+1} = \dfrac{y_n^3 - 3y_n^2}{4} < 0$ because $y_n^3 < 0$ and $3y_n^2 > 0$. Therefore, (y_n) is bounded above by 0.
Next we want to show that (y_n) is increasing starting from y_1, so we want to show that

$$\begin{aligned} & y_{n+1} \geq y_n \quad ,\forall n \\ \Leftrightarrow \quad & \frac{y_n^2(y_n - 3)}{4} \geq y_n \\ \Leftrightarrow \quad & y_n^3 - 3y_n^2 - 4y_n \geq 0 \end{aligned}$$

We have

$$y(a) = y_n^3 - 3y_n^2 - 4y_n = y_n(y_n - 4)(y_n + 1)$$

where $y_n \leq 0$.

$$\underbrace{y_n}_{\leq 0}\underbrace{(y_n - 4)}_{\leq 0}\underbrace{(y_n + 1)}_{???}$$

To make this expression greater than 0, we need $y_n \geq -1$ which is true because $y_n \leq 0$ for all n, so it will never be able to reach -1 except for the initial value which is consistent with our data from the program. Hence (y_n) is bounded by 0 and it's increasing after the first few terms, so there exists $N \in \mathbb{N}$ such that if $n > N$ then (y_n) is bounded and increasing. By Monotone Theorem, we can conclude that (y_n) converges or the limit of (y_n) exists. To show that this limit is 0, let L be the limit of the sequence. We have

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} = L$$

Hence,

$$\begin{aligned} & \frac{L^2(L - 3)}{4} = L \\ \Leftrightarrow \quad & L^3 - 3L - 4L = 0 \\ \Leftrightarrow \quad & L(L - 4)(L + 1) = 0 \end{aligned}$$

There are 3 solutions to this equation, however only 0 actually works. 4 can be eliminated because $4 \notin (-1, 2)$ and we know that the (y_n) is bounded by 0. The same logic applies to -1, since the sequence is bounded by 0 and increasing, it can't be -1. Therefore the limit of (y_n) must be 0. □

(c) *Proof.* From part (a) we have that:

$$x_n^2 = a(1 + y_n) \Leftrightarrow x_n = \sqrt{a(1 + y_n)}$$

Hence,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \sqrt{a(1+y_n)}$$

By Algebra Limit Theorem, we have

$$\begin{aligned}\lim_{n\to\infty} \sqrt{a(1+y_n)} &= \lim_{n\to\infty} \sqrt{a} + \lim_{n\to\infty} ay_n \\ &= \sqrt{a} + a \cdot 0 \\ &= \sqrt{a}\end{aligned}$$

□

Problem 2 Let $S \subset \mathbb{R}$. A number x is an interior point of the set $S \subset R$ if there is $r > 0$ such that the interval $(x-r, x+r) \subset S$. The set of all interior points of S is denoted by S°. Prove the following properties:

(i) $S^\circ \subset S$

(ii) $(S \cap T)^\circ = S^\circ \cap T^\circ$

(iii) $S^\circ \cup T^\circ \subset (S \cup T)^\circ$, but these two sets are not necessarily equal.

(iv) S° is the largest open set contained in S, that is S° is an open set, and if $U \subset S$ is an open set, then $U \subset S^\circ$.

Solution.

(i) Let $x \in S^\circ$, then by definition of interior point, there exists $r > 0$ such that $(x-r, x+r) \subset S \Rightarrow x \in S \Rightarrow S^\circ \subset S$.

(ii) $\subset$: Let $x \in (S \cap T)^\circ$, then by definition of interior point, there exists $r > 0$ such that $(x-r, x+r) \subset (S \cap T)$ which implies

$$(x-r, x+r) \subset S$$

and

$$(x-r, x+r) \subset T$$

Hence, $x \in S^\circ$ and $x \in T^\circ \Rightarrow x \in (S^\circ \cap T^\circ) \Rightarrow (S \cap T)^\circ \subset (S^\circ \cap T^\circ)$. (1)
$\supset$: Let $x \in (S^\circ \cap T^\circ) \Rightarrow x \in S^\circ$ and $x \in T^\circ$. By definition of interior point, $\exists r_1, r_2 > 0$ such that

$$(x-r_1, x+r_1) \subset S \text{ and } (x-r_2, x+r_2) \subset T$$

Let $r = \min(r_1, r_2) \Rightarrow (x-r, x+r) \subset (S \cap T) \Rightarrow x \in (S \cap T)^\circ$. (2)
From (1) and (2) we have $(S \cap T)^\circ = S^\circ \cap T^\circ$

(iii) Let $x \in (S^\circ \cup T^\circ) \Rightarrow x \in S^\circ$ or $x \in T^\circ$. By definition of interior point, and without loss of generality, we assume that there exists $r > 0$ in such that $(x-r, x+r) \subset S \Rightarrow x \in (S \cup T) \Rightarrow x \in (S \cup T)^\circ$. Thus $(S^\circ \cup T^\circ) \subset (S \cup T)^\circ$.
However, it's not always true that $(S \cup T)^\circ \subset (S^\circ \cup T^\circ)$. For example if $S = (-\infty, 0)$ and $T = (0, +\infty)$

(iv) Recall

Theorem 7.3. *A set $S \in \mathbb{R}$ is open if and only if $\forall x \in S$, $\exists r > 0$ such that $(x-r, x+r) \subset S$.*

By definition of interior sets, $\forall x \in S^\circ$, $\exists r > 0$ such that $(x-r, x+r) \subset S^\circ$. Apply Theorem 7.3 to S°, we have that S° is open. Let U be a set such that $U \subset S$ and U is open, then $\forall x \in U$, $\exists r > 0$ such that $(x-r, x+r) \subset U$, but $U \subset S$, thus $\forall x \in U$, $\exists r > 0$ such that $(x-r, x+r) \subset U \subset S$. In other words, U is also a set that contains interior points of S, where S° is the set that contains "all" interior points of S, thus $U \subset S^\circ$. Since U is arbitrarily chosen, S° is the largest open set contained in S.

Problem 3 The closure of a set of numbers $S \subset \mathbb{R}$ is the set $S^- = S \cup S'$, the union of the set and its set of limit points S'. Prove the following properties.

(i) If $S \subset T$ then $S^- \subset T^-$

(ii) If $(S \cup T)^- = S^- \cup T^-$

(iii) If $(S \cap T)^- \subset S^- \cap T^-$

(iv) S^- is the smallest closed set that contains S.

Problem 4 Define $\lim_{x \to a} f(x) = \infty$ to mean that for every N there is a $\delta > 0$ such that for all x if $0 < |x - a| < \delta$ then $f(x) > N$. Prove that

$$\lim_{x \to 3} \frac{1}{(x-3)^2} = \infty$$

Solution.

Proof. Let $N \in \mathbb{R}+$, and $\delta = \frac{1}{\sqrt{N}} > 0$. Thus if $0 < |x - 3| < \frac{1}{\sqrt{N}} \Rightarrow (|x-3|)^2 < \frac{1}{N} \Rightarrow \frac{1}{(x-3)^2} > N$. Since N is arbitrarily chosen, $\frac{1}{(x-3)^2} > N$ for all N. Therefore,

$$\lim_{x \to 3} \frac{1}{(x-3)^2} = \infty$$

$\square$

Problem 5 Prove that if $\lim_{x \to 0} \frac{f(x)}{x} = L$ and $b \neq 0$, then $\lim_{x \to 0} \frac{f(bx)}{x} = bL$

Solution.

Proof. By definition of limit, we have that there exists $\epsilon > 0$, and $\delta > 0$ such that if $|x - 0| = |x| < \delta$ then

$$\left| \frac{f(x)}{x} - L \right| < \epsilon$$

Now consider,

$$\lim_{x \to 0} \frac{f(bx)}{x} = \lim_{x \to 0} \frac{bf(bx)}{bx} = b \lim_{x \to 0} \frac{f(bx)}{bx} \text{ since } b \neq 0$$

On the other hand, $|x| < \delta \Rightarrow |x| < \frac{\delta}{|b|}$ because $\frac{\delta}{|b|} > 0 \Rightarrow |bx| < \delta \Rightarrow \left| \frac{f(bx)}{bx} - L \right| < \epsilon$.

Therefore,

$$b \lim_{x \to 0} \frac{f(bx)}{bx} = bL$$

$\square$

Problem 6 Let S be a nonempty subset of real numbers that is bounded above but has no greatest element. Prove that $\sup(S)$ is a limit point of S.

Proof. $\square$

Problem 7 A sequence (a_n) is a Cauchy sequence if, for any $\epsilon > 0$, there is a natural number N such that if $n, m > N$, then $|a_n - a_m| < \epsilon$. Prove the following:

(a) Any convergent sequence is a Cauchy sequence.

(b) Any Cauchy sequence is bounded.

(c) Any subsequence of a Cauchy sequence is a Cauchy sequence.

(d) If a subsequence of Cauchy sequence converges, then the whole sequence also converges.

Solution.

(a) *Proof.* Let (a_n) be a convergent sequence to L, then $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that if $n > N$, then $|a_n - L| < \dfrac{\epsilon}{2}$. Given $\epsilon > 0$, we choose $n, m > N$, then

$$|a_n - L| < \frac{\epsilon}{2} \text{ and } |a_m - L| < \frac{\epsilon}{2}$$

Consider, $|a_n - a_m| = |a_n - L + L - a_m| < |a_n - L| + |a_m - L|$ by Triangle Inequality. Hence,

$$|a_n - a_m| = |a_n - L| + |a_m - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

And this shows that a convergent sequence satisfy all Cauchy sequence property which implies it must be a Cauchy sequence. □

(b) *Proof.* Let (a_n) be such a Cauchy sequence, then by definition for any $\epsilon > 0$, there is a natural number N such that if $n, m > N$, then $|a_n - a_m| < \epsilon$. Let $m = N + 1, n > m$ and $\epsilon = 1$, then

$$1 > |a_n - a_{N+1}| > |a_n| - |a_{N+1}| \Leftrightarrow |a_n| < (|a_{N+1}| + 1) \text{ for all } n$$

Let $M = \max\{a_0, a_1, a_2, \dots, |a_{N+1}| + 1\}$, then $a_n < M$ for all $n \in \mathbb{N}$. Thus (a_n) is bounded. □

(c) *Proof.* Let (a_n) be such a Cauchy sequence. From (ii), we know that (a_n) is bounded. By Theorem 6.6

Theorem 6.6. *Every bounded sequence of real numbers has a convergent susbequence.*

We have, (a_n) has a convergent subsequene, and since it converges, it is a Cauchy sequence as proved in (i). □

(d) *Proof.* Let $A = (a_n)$ be such a Cauchy sequence, then given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that if $n, m > N$ then

$$|a_n - a_m| < \frac{\epsilon}{2}$$

Also, let $B = (a_{n_i}) = \{a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, \dots\}$ be a convergent subsequence of A that converges to L, then $\exists M \in \mathbb{N}$ such that

$$|a_M - L| < \frac{\epsilon}{2}$$

Hence, if $n \geq M$, we have

$$\begin{aligned} |a_n - L| &= |a_n - a_M + a_M - L| \\ &\leq |a_n - a_M| + |a_M - L| \text{ (Triangle Inequality)} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

Therefore, (a_n) converges. □

Problem 8

1. Prove that a set $U \subset \mathbb{R}$ is open if and only if for any $x \in U$ there is $r > 0$ such that the interval $(x-r, x+r) \subset U$.
2. Prove that a set $U \subset \mathbb{R}$ is open if and only if U is a union of open intervals.
3. Prove that a set $U \subset \mathbb{R}$ if and only if U is a union of countably many disjoint open intervals.

Problem 9 Give an example of open sets $U_1 \supset U_2 \supset U_2 \dots$ in $\mathbb{R}$ such that the intersection $\bigcap_{n=1}^{\infty} U_n$ is closed and non-empty.

Problem 10 Give an example of closed sets $C_1 \supset C_2 \supset C_3 \dots$ in $\mathbb{R}$ such that the intersection $\bigcup_{n=1}^{\infty} C_n$ is empty.

Problem 11

(a) Suppose that $\lim_{x \to a} f(x) = L \neq 0$. Prove that $\lim_{x \to a} \dfrac{1}{f(x)} = \dfrac{1}{L}$.

Proof. By definition of limit, fix $\epsilon > 0$, there exists $\delta > 0$ such that $|x - a| < \delta \Rightarrow |f(x) - L| < \epsilon \cdot f(x) \cdot L$. Consider,

$$\left|\frac{1}{f(x)} - \frac{1}{L}\right| = \left|\frac{L - f(x)}{f(x) \cdot L}\right| < \frac{\epsilon f(x) \cdot L}{f(x) \cdot L} = \epsilon$$

□

(b) Let $r(x) = \dfrac{p(x)}{q(x)}$ be a rational function, where $p(x)$ and $q(x)$ are polynomials in x. Prove that r is continuous at all x such that $q(x) \neq 0$.

Proof. □

Problem 12 Let I be an interval in $\mathbb{R}$ and let $a \in I$. If f is a function whose domain contains $I \setminus \{a\}$ define

$$\lim_{x \to a+} f(x) = \lim_{x \to a} f_+(a)$$

where f_+ is the function with domain $I \cap (a, \infty)$ given by $f_+(x) = f(x)$. Similarly, define

$$\lim_{x \to a_-} f(x) = \lim_{x \to a} f_-(x)$$

where f_- is the function with domain $I \cap (-\infty, a)$ given by $f_+(x) = f(x)$. Prove that $\lim_{x \to a} f(x)$ exists if and only if both $\lim_{x \to a+}$ and $\lim_{x \to a-} f(x)$ exist and are equal.

Problem 13 Let f be a real valued function defined on (a, ∞), where $a > 0$ is some positive real number. Let $\lim_{x \to \infty+} f(x)$ given by

$$\lim_{x \to \infty+} = \lim_{y \to 0} g(y)$$

where $g : (0, 1/a) \to \mathbb{R}$ is given by $g(y) = f(1/y)$ if the latter limit exists. Prove that $\lim_{x \to \infty+} f(x)$ exists if and only if for any $\epsilon > 0$ there exists a number $N \geq a$ such that $|f(x) - f(y)| < \epsilon$ if $x, y > N$.

Problem 14

(a) If $\lim_{x\to a} g(x)$ does not exist, can $\lim_{x\to a}[f(x)+g(x)]$ exist? Can $\lim_{x\to a} f(x)g(x)$ exist?

Proof. Yes, for example if $g(x) = 1 - f(x)$ then $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a}[f(x)+1-f(x)]$ exists even if $\lim_{x\to a} f(x)$ does not exist; and if $g(x) = \dfrac{1}{f(x)}$ where $f(x) \neq 0$ for all $x \neq a$, then $\lim_{x\to a} f(x)g(x)$ does exists even if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ do not exist. For example, if $f(x) = 1/(x-a)$ for $x \neq a$, and $g(x) = x - a$. $\square$

(b) If $\lim_{x\to a} f(x)$ exist and $\lim_{x\to a}[f(x)+g(x)]$ exists, must $\lim_{x\to a} g(x)$ exists?

Proof. Yes, again we can write $g(x) = (f(x)+g(x)) - f(x)$.Each of the terms $f(x)+g(x)$ and $f(x)$ on the right side has limit when $x \to a$, so their difference also has limit when $x \to a$. $\square$

(c) If $\lim_{x\to a} f(x)$ exists, and $\lim_{x\to a} g(x)$ does not exist, can $\lim_{x\to a}[f(x)+g(x)]$ exist?

Proof. No. This is just another of starting part (b). $\square$

(d) If $\lim_{x\to a} f(x)$ exist, and $\lim_{x\to a} f(x)g(x)$ exists, does it follow that $\lim_{x\to a} g(x)$ exists?

Proof. No. Let $f(x) = 0$ for all x and let $g(x) = 1$ if x is rational and -1 if x is irrational. Then for any a, $f(x)$ and $f(x)g(x) = 0$ both have limit when $x \to a$. but the limit of $g(x)$ does not exist when $x \to a$. $\square$

Problem 15

(a) Prove that if $0 < a < 2$ then $a < \sqrt{2a} < 2$.

Proof. If $0 < a < 2$ then $a^2 < 2a < 4 \Rightarrow a < \sqrt{2a} < 2$. $\square$

(b) Prove that the sequence,,

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$$

converges.

Proof. Part (a) show that $\sqrt{2} < \sqrt{2\sqrt{2}} < \sqrt{2\sqrt{2\sqrt{2}}} < \ldots < 2$. so by Monotonic Convergence Theorem, the sequence converges. $\square$

(c) Find the limit. Hint: Notice that if $\lim_{n\to\infty} a_n = L$ then $\lim_{n\to\infty} \sqrt{2a_n} = \sqrt{2L}$.

Proof. If this sequence is denoted by $\{a_n\}$, then the sequence $\{\sqrt{2a_n}\}$ is the same as $\{a_{n+1}\}$ for all n. So the hint show that $L = \sqrt{2L} \Leftrightarrow L = 2$. $\square$

Problem 16

(a) Prove that a convergent sequence is always bounded.

Proof. Suppose that $\lim_{n\to\infty} a_n = L$. Choose N so that $|a_n - L| < 1$ for $n > N$. Then $|a_n| < \max(|L| + 1, |a_1|, |a_2|, \ldots, |a_N|)$ for all n. $\square$

(b) Suppose that $\lim_{n\to\infty} a_n = 0$, and that some $a_n > 0$. Prove that the sets of all numbers a_n actually has a maximum member.

Proof. Choose N so that $|a_n - 0| < a_1$ for $n > N$. Then the maximum of $a_1, a_2, a_3, \ldots, a_N$ is the maximum of a_n for all n. $\square$

Problem 17 Prove that $\lim_{n\to\infty} a_n = L$, then

$$\lim_{n\to\infty} \frac{a_1 + \ldots + a_n}{n} = L$$

Proof. If $\epsilon > 0$, pick N so that $|a_n - L| < \epsilon$. Then

$$|a_N + a_{N+1} + a_{N+2} + a_{N+M} - ML| < \epsilon M$$

so

$$\left| \frac{1}{N+M} \cdot [a_N + a_{N+1} + \ldots + a_{N+M}] - \frac{ML}{N+M} \right| < \frac{\epsilon M}{N+M} < \epsilon$$

Choose M so that,

$$\left| \frac{ML}{N+M} - L \right| < \epsilon \text{ and } \left| \frac{1}{N+M} [a_1 + a_2 + \ldots + a_N] \right| < \epsilon$$

Then,

$$\left| \frac{1}{N+M} [a_1 + a_2 + \ldots + a_N] - L \right| < 3\epsilon$$

$\square$

Problem 18

(a) Prove that if $\lim_{n\to\infty} (a_{n+1} - a_n) = L$, then $\lim_{n\to\infty} a_n/n = L$.

Proof. Let $b_n = a_{n+1} - a_n$. Then $\{b_n\}$ satisfies the hypothesis of Problem 17 and the conclusion says that

$$L = \lim_{n\to\infty} \frac{b_1 + \ldots + b_n}{n} = \lim_{n\to\infty} \frac{a_{n+1} - a_1}{n} = \lim_{n\to\infty} \frac{a_n}{n}$$

$\square$

(b) Suppose that f is continuous and $\lim_{x\to\infty} [f(x+1) - f(x)] = L$. Prove that $\lim_{x\to\infty} f(x)/x = L$.

Proof. The hypothesis $\lim_{x\to\infty} [f(x+1) - f(x)] = L$ implies that a_n and b_n, the inf and sup of f on $[n, n+1]$ satisfy $\lim_{n\to\infty} [a_{n+1} - a_n] = L$ and $\lim_{n\to\infty} [b_{n+1} - b_n] = L$. So by part (a), we have $\lim_{n\to\infty} a_n/n = \lim_{n\to\infty} b_n/n = L$, which implies that $\lim_{x\to\infty} f(x)/x = L$. $\square$

Problem 19

1. Suppose that $\{a_n\}$ is a convergent sequence of points all in $[0,1]$. Prove that $\lim_{n\to\infty} a_n$ is also in $[0,1]$.

   Proof. Suppose $\lim_{n\to\infty} a_n = L > 1$. Since $L - 1 > 0$, there would be some n with $|L - a_n| < L - 1$, hence $a_n > 1$, a contradiction. Similarly, we cannot have $L < 0$. $\square$

2. Find a convergent sequence $\{a_n\}$ of points all in $(0,1)$ such that $\lim_{n\to\infty} a_n$ is not on $(0,1)$.

   Proof. $a_n = 1/n$. $\square$

Problem 20 Suppose that f is a function on $\mathbb{R}$ such that

$$|f(x) - f(y)| \leq c|x - y| \text{ for all } x, y$$

Prove that f is continuous.

Proof. If $c = 0$ then f is constant, so continuous. If $c \neq 0$, then $\epsilon > 0$, then $|f(x) - f(a)| < \epsilon$ for $|x - a| < \epsilon/c$. $\square$