Contents

Math 350 - Advanced Calculus
Homework 3

Chan Nguyen

December 11, 2012

Problem 1 Let $\mathbf{F} = \mathbf{Q}$ be the ordered filed of rational numbers. Find the supremum and the infinum of the following subsets of $\mathbf{Q}$ if they exist:

(i) $A = \{1, 1/2, 1/3, 1/4, \ldots, 1/n, 1/(n+1), \ldots\}$

Proof. For $n > 1$, $\dfrac{1}{n} < 1$, thus the supremum of $\mathbf{Q}$ is 1. The series is decreasing as $n \to \infty$, the infinum doesn't exist. □

(ii) $B = \{1/3, 4/9, 13/27, 40/81, \ldots, p/q, (p+q)/3q, \ldots\}$

Proof. First we will show that $\dfrac{1}{3}$ is a lower bound of B, by finding the nth term of the sequence, starting from $\dfrac{p}{q} = \dfrac{1}{3}$. Let's define b_i to be the ith element of B where $i = 0, 1, 2 \ldots$, we have:

$$\begin{aligned}
b_0 &= \frac{p}{q} \\
b_1 &= \frac{q+p}{3q} \\
b_2 &= \frac{3q+q+p}{3^2 q} \\
b_3 &= \frac{3^2 q + 3q + q + p}{3^3 q} \\
\ldots &= \ldots \\
b_n &= \frac{3^{n-1}q + 3^{n-2}q + \ldots 3^0 q + p}{3^n q} \\
\ldots &= \ldots
\end{aligned}$$

Now consider,

$$\begin{aligned}
b_n &= \frac{3^{n-1}q + 3^{n-2}q + \ldots 3^0 q + p}{3^n q} \\
&= \frac{(3^{n-1} + 3^{n-2} + \ldots + 1) \cdot q}{3^n q} + \frac{p}{3^n q} \\
&= \frac{3^{n-1} + 3^{n-2} + \ldots + 1}{3^n} + \frac{p}{3^n q} \\
&= \frac{3^{n-1} + 3^{n-2} + \ldots + 1}{3^n} + \frac{1}{3^{n+1}} \quad \text{substitute } p = 1, q = 3 \\
&= \frac{3^n + 3^{n-1} + \ldots + 3}{3^{n+1}} + \frac{3}{3^{n+1}}
\end{aligned}$$

Therefore

$$b_n = \frac{3^n + 3^{n-1} + \ldots + 3^1 + 3^0}{3^{n+1}}$$

Using geometric sum for the numerator since $3 \neq 1$ to obtain:

$$3^n + 3^{n-1} + \ldots + 3^1 + 3^0 = \sum_{k=0}^{n} 3^k = \frac{1 - 3^{n+1}}{1-3} = \frac{3^{n+1} - 1}{3 - 1}$$

Substituting this back into b_n, we have:

$$b_n = \frac{3^{n+1} - 1}{2} \cdot \frac{1}{3^{n+1}} = \frac{3^{n+1}}{2 \cdot 3^{n+1}} - \frac{1}{2 \cdot 3^{n+1}} = \frac{1}{2} - \frac{1}{2 \cdot 3^{n+1}}$$

Ignore the term $\frac{1}{2}$ in b_n since it is just a constant and take the limit of

$$\lim_{n \to \infty} \left(\frac{1}{2 \cdot 3^{n+1}} \right) = \frac{1}{\infty} = 0$$

which implies

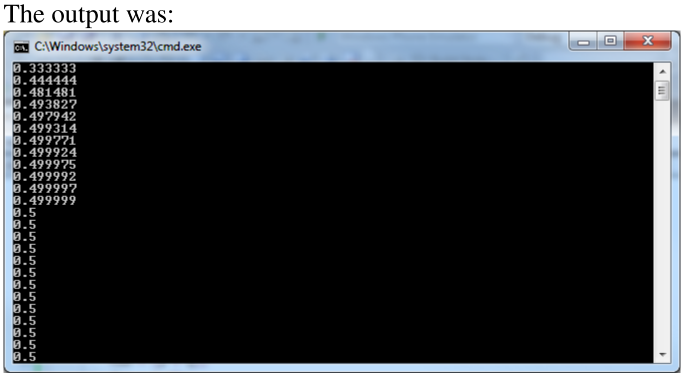
$$\lim_{n \to \infty} b_n = \frac{1}{2}$$

Thus $b_n \geq \frac{1}{3}$ for all $n = 0, 1, 2, 3, \ldots$, so $\frac{1}{3}$ is a lower bound of B. Furthermore, $\frac{1}{3} \in B$, so it must be the greatest lower bound otherwise it's nonsense. Therefore we conclude that $\inf(B) = \frac{1}{3}$

Next we will show that the $\sup(B) = \frac{1}{2}$. Let's ignore the limit above, and try a short C++ solution to find out as what is the value of n,

```
void generate_pq() {
        double p = 1.0, q = 3.0;
        double num, den;
        for (int i = 0; i <= 40; ++i) {
                /* p/q, (p + q)/(3q), ... */
                num = (p + q);
                den = 3.0*q;
                p = num, q = den;
                cout << p/q << '\n';
        }
}
```

The output was:



As we can see, it converges to $\frac{1}{2}$ quite fast which is consistent with our limit result. Thus the upper bound of B is $\frac{1}{2}$. Next we will show that is also the least upper bound by contradiction.

Suppose that there is another upper bound says b that is $< \frac{1}{2}$, we exclude the "=" sign here because the least upper bound must be unique. Since b is the least upper bound of B, we have that for all $b_n \in B, n = 0, 1, 2, \dots$, $b_n \leq b$, and this is clearly a contradiction because as we just have proved above, $b_n \to \frac{1}{2}$ as $n \to \infty$. Therefore the $\sup(B) = \frac{1}{2}$. ☐

Problem 2 Let $\mathbf{F}$ be an ordered field with the least upper bound property. Prove that if $a > 1$ is any number in $\mathbf{F}$, then the set $\{a, a^2, a^3, \dots\}$ is not bounded above.

Proof. We will prove this by contradiction. Suppose that $S = \{a, a^2, a^3, \dots\}$ is bounded above, where $a \in \mathbf{F}$ and $\mathbf{F}$ is an ordered field with the least upper bound property. Now recall **Theorem 4.2** in lecture note 4:

Theorem 4.2. *If $\mathbf{F}$ is an ordered field with the least upper bound property, then the following are true.*

(i) The subset $\mathbf{N} \subset \mathbf{F}$ is not bounded above.

(ii) For each $a \in \mathbf{F}_+$, there is $n \in \mathbf{N}$ such that $\frac{1}{n} = n^{-1} < a$

(iii) For each $a \in \mathbf{F}$, there is $m \in \mathbf{Z}$ such that $m \leq a \leq m + 1$

(iv) For each $a \in \mathbf{F}$ and $\epsilon \in \mathbf{F}_+$, there is a rational number $r \in \mathbf{Q}$ such that $|a - r| < \epsilon$

Hence, if S is bounded above, then the least upper bound must exists. Let's b be the least upper bound of S, then all elements in S is less than or equal to b including all other upper bounds of S of-course. In other words, $a^n \leq b, \ \forall n \in \mathbf{N}$. On the other hand, $\mathbf{F}$ is an ordered field with the least upper bound property, so $b \cdot a^{-1}$ is also an upper bound of S. This leads to the contradiction because we just claim that b is the least upper bound of S or $\sup(\mathrm{S})$, but

$$ba^{-1} = \frac{b}{a} < b \text{ because } a > 1$$

Therefore S is not bounded above. ☐

Problem 3 Let $\mathbf{F}$ be an ordered field with the least upper bound property. Prove that for any number $a \in \mathbf{F}$ and any $\epsilon > 0$ there is natural number n such that $\frac{a}{2^n} < \epsilon$.

Proof. We will prove this by contradiction. Suppose that $\exists a \in \mathbf{F}$ and $\exists \epsilon > 0$, for all natural numbers n, then $\frac{a}{2^n} \geq \epsilon \Leftrightarrow a \geq \epsilon \cdot 2^n$. However, F is an ordered field with the least upper bound property which implies the largest lower bound as well. Look at the expression:

$$a \geq \epsilon \cdot 2^n$$

which tells us that $\epsilon \cdot 2^n, \forall n \in \mathbf{N}$ is clearly a lower bound of a, which implies there exists the largest lower bound. But we can increase n forever, to obtain a larger and larger lower bound which contradicts there exists the largest lower bound. Therefore for any number $a \in \mathbf{F}$ and any $\epsilon > 0$ there is natural number n such that $\frac{a}{2^n} < \epsilon$ ☐

Problem 4

(i) Prove that

$$\max\{\mathrm{a},\mathrm{b}\} = \frac{a+b+|b-a|}{2}$$

(ii) Derive a similar formula for $\max\{\mathrm{a},\mathrm{b},\mathrm{c}\}$ using for example, $\max\{\mathrm{a},\mathrm{b},\mathrm{c}\} = \max\{a, \max\{\mathrm{b},\mathrm{c}\}\}$

(i) To prove this formula, we just need to break it into three cases.

Proof. If $b > a \Rightarrow |b-a| = b-a$, then

$$\frac{a+b+|b-a|}{2} = \frac{a+b+b-a}{2} = \frac{2b}{2} = b$$

If $a < b \Rightarrow |b-a| = -(b-a) = a-b$, then

$$\frac{a+b+|b-a|}{2} = \frac{a+b+a-b}{2} = \frac{2a}{2} = a$$

If $a = b$ then

$$\frac{a+b+0}{2} = \frac{2a}{2} = \frac{2b}{2} = a = b$$

Thus

$$\max\{\mathrm{a},\mathrm{b}\} = \frac{a+b+|b-a|}{2}$$

□

(ii) Using the given formula for two numbers we can derive the max of three numbers as follows:

$$\begin{aligned}
\max\{\mathrm{a},\mathrm{b},\mathrm{c}\} &= \max\{a, \max\{\mathrm{b},\mathrm{c}\}\} \\
&= \frac{a+\max\{b,c\}+|\max\{b,c\}-a|}{2} \\
&= \frac{a+\dfrac{b+c+|c-b|}{2}+\left|\dfrac{b+c+|c-b|}{2}-a\right|}{2} \\
&= \frac{\dfrac{2a+b+c+|c-b|}{2}+\left|\dfrac{b+c+|c-b|-2a}{2}\right|}{2}
\end{aligned}$$

Problem 5 Prove that if

$$|x-x_0| < \frac{\epsilon}{2} \text{ and } |y-y_0| < \frac{\epsilon}{2}$$

then

$$|(x+y)-(x_0+y_0)| < \epsilon \text{ and } |(x-y)-(x_0-y_0)| < \epsilon$$

Proof. Since $|x-x_0| \geq 0$ and $|y-y_0| \geq 0$, we have:

$$|x-x_0| + |y-y_0| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Now consider

$$|(x+y)-(x_0+y_0)| = |(x-x_0)+(y-y_0)| = |x-x_0|+|y-y_0| < \epsilon$$

and

$$|(x-y)-(x_0-y_0)| = |(x-x_0)+(y_0-y)| = |x-x_0|+|y_0-y| = |x-x_0|+|y-y_0| < \epsilon$$

□

Problem 6 Prove that if

$$|x - x_0| < \frac{\epsilon}{2(|y_0| + 1)} \text{ and } |x - x_0| < 1 \text{ and } |y - y_0| < \frac{\epsilon}{2(|x_0| + 1)}$$

then

$$|xy - x_0 y_0| < \epsilon$$

(Hint. Write $xy - x_0 y_0$ in a way that involves $x - x_0$ and $y - y_0$)

Proof. We have that:

$$\begin{aligned}
|xy - x_0 y_0| &= |(xy - xy_0 + xy_0 - x_0 y_0)| \\
&= |x(y - y_0) + y_0(x - x_0)| \\
&\leq |x(y - y_0)| + |y_0(x - x_0)| \text{ by triangle inequality} \\
&= |x| \cdot |y - y_0| + |y_0| \cdot |x - x_0| \\
&< |x| \frac{\epsilon}{2(|x_0| + 1)} + |y_0| \cdot \frac{\epsilon}{2(|y_0| + 1)} \\
&= \frac{\epsilon}{2} \cdot \frac{|x|}{|x_0| + 1} + \frac{\epsilon}{2} \cdot \frac{|y_0|}{|y_0| + 1}
\end{aligned}$$

Now what we need to show is

$$\frac{|x|}{|x_0| + 1} \leq 1 \text{ and } \frac{|y_0|}{|y_0| + 1} \leq 1$$

and we're done.

The second one is trivial since $\frac{|y_0|}{|y_0| + 1} \leq 1 \Leftrightarrow |y_0| \leq |y_0| + 1 \Leftrightarrow 0 \leq 1$ which is true. To show the first inequality is true, we need $|x - x_0| < 1$ from the hypothesis which is equivalent to:

$$-1 < x - x_0 < 1 \Rightarrow x < x_0 + 1$$

Thus,

$$|xy - x_0 y_0| < \frac{\epsilon}{2} \cdot \frac{|x|}{|x_0| + 1} + \frac{\epsilon}{2} \cdot \frac{|y_0|}{|y_0| + 1} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

Problem 7 Prove that if $x_0 \neq 0$ and

$$|x - x_0| < \min\left\{\frac{|x_0|}{2}, \frac{\epsilon |x_0|^2}{2}\right\}$$

then $x \neq 0$ and $\left|\frac{1}{x} - \frac{1}{x_0}\right| < \epsilon$

Proof. We have that,

$$M = \min\left\{\frac{|x_0|}{2}, \frac{\epsilon |x_0|^2}{2}\right\} = \begin{cases} \frac{|x_0|}{2} & \text{, if } \frac{|x_0|}{2} < \frac{\epsilon |x_0|^2}{2} \quad (1) \\ \frac{\epsilon |x_0|^2}{2} & \text{, if } \frac{|x_0|}{2} > \frac{\epsilon |x_0|^2}{2} \quad (2) \end{cases}$$

If $M = \frac{|x_0|}{2} \Rightarrow |x - x_0| < \frac{|x_0|}{2} \Rightarrow x \neq 0$ otherwise $|x_0| < \frac{|x_0|}{2}$ is nonsense because $x_0 \neq 0$.

If $M = \frac{\epsilon |x_0|^2}{2} \Rightarrow |x - x_0| < \frac{\epsilon |x_0|^2}{2} \Rightarrow x \neq 0$ otherwise $\Rightarrow |x_0| < \frac{\epsilon |x_0|^2}{2}$ is also nonsense because $x_0 \neq 0$

On the other hand, we have

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| = \left|\frac{x_0 - x}{x x_0}\right|$$

Now we are ready to prove the other part by considering two cases:

- (1) $|x-x_0| < \dfrac{|x_0|}{2} < \dfrac{\epsilon|x_0|^2}{2}$
  From $|x-x_0| < \dfrac{\epsilon|x_0|^2}{2} \Rightarrow \epsilon > \dfrac{2|x-x_0|}{|x_0|^2} > \left|\dfrac{x_0-x}{xx_0}\right| = \left|\dfrac{1}{x} - \dfrac{1}{x_0}\right|$

- (2) $|x-x_0| < \dfrac{\epsilon|x_0|^2}{2} < \dfrac{|x_0|}{2}$
  From $|x-x_0| < \dfrac{\epsilon|x_0|^2}{2} \Rightarrow \epsilon > \dfrac{2|x-x_0|}{|x_0|^2} > \left|\dfrac{x_0-x}{xx_0}\right| = \left|\dfrac{1}{x} - \dfrac{1}{x_0}\right|$

□

Problem 8 Let $\mathbf{F}$ be an ordered field with the least upper bound property. If $A \neq \emptyset$ is bounded below, let B be the set of all lower bounds of A. Prove that:

(i) $B \neq \emptyset$

(ii) B is bounded above

(iii) $\sup(B) = \inf(A)$

Proof. The set $B \neq \emptyset$ because A is bounded below (any lower bound for A is in B). Because A is nonempty, there is a in A, and this a satisfies $y \leq a$ for all y in B. Because of this, $y \leq x$ for all x in A and all y in B, and thus $\sup(B) \leq \inf(A)$. If $\sup(B) < \inf(A)$, then there is a number x such that $\sup(B) < x < \inf(A)$. The inequality $\sup(B) < x$ implies that $b < x$ for all $b \in B$ and thus that x is not in B. The inequality $x < \inf(A)$ implies that $x < a$ for all a in A, and thus that x is a lower bound for A. Therefore x is in B. But this contradicts the inequality $\sup(B) < x$ as noted. □

Problem 9 Let $\mathbf{F}$ be an ordered field with the least upper bound property. Let $A \subset \mathbf{F}$ be a nonempty set of numbers. Prove that $\alpha = \sup(A)$ if and only if α is an upper bound for A and for any $\epsilon > 0$, there is an x in A such that $\alpha < x + \epsilon$.

Proof. To prove if and only if, we need to prove it for both direction, and we will prove this by contradiction.

- $\Rightarrow$: The contradiction can be written as follows:
  If $\alpha = \sup(A)$ then α is an upper bound for A and $(\exists \epsilon > 0)(\forall x \in A)$ such that $\alpha \geq x + \epsilon$. Note that we don't take the part α is an upper bound for A into the statement, and apply Demorgan's law: $\neg(A \text{ and } B) = \neg(A) \text{ or } \neg(B)$ strictly because it's trivially true when $\alpha = \sup(A)$. On the other hand, this supposition implies that $\alpha - \epsilon$ is also the least upper bound of A because $\alpha - \epsilon \geq x,\ \forall x \in A$. Furthermore $\epsilon > 0$ and $\alpha - \epsilon < \alpha$, this is a contradiction. Therefore if $\alpha = \sup(A)$ then α is an upper bound for A and for any $\epsilon > 0$, there is an x in A such that $\alpha < x + \epsilon$.

- $\Leftarrow$: To prove from this side, we assume that if α is an upper bound for A and for any $\epsilon > 0$, there is an x in A such that $\alpha < x + \epsilon$ then $\alpha \neq \sup(A)$. This side is too obvious and it leads to contradiction immediately.

□

Problem 10 Let $\mathbf{F}$ be an ordered field with the least upper bound property. Let $A \subset \mathbf{F}$ and $B \subset \mathbf{F}$ be the two nonempty sets of numbers that are bounded above, and let $A + B$ denote the set of all numbers of the form $x + y$ with $x \in A$ and $y \in B$. Prove that $\sup(A + B) = \sup(A) + \sup(B)$.

Proof. Since $x \leq \sup(A)$ and $y \leq \sup(B)$ for every x in A and y in B, it follows that $x + y \leq \sup(A) + \sup(B)$. Thus $\sup(A) + \sup(B)$ is an upper bound for $A + B$, so $\sup(A + B) \leq \sup(A) + \sup(B)$. If x and y are chosen in A and B, respectively, so that $\sup(A) - x < \epsilon/2$ and $\sup(B) - y < \epsilon/2$, then $\sup(A) + \sup(B) - (x + y) < \epsilon$. Hence $\sup(A + B) \geq x + y > \sup(A) + \sup(B) - \epsilon$. □

Problem 11 Suppose that A and B are 2 non-empty sets of numbers such that $x \leq y$ for all x in A and all y in B.

(a) Prove that $\sup(A) \leq y$ for all $y \in B$.

Proof. Since any y in B satisfies $y \geq x$ for all x in A, any y in B is an upper bound for A, so $y \geq \sup(A)$. □

(b) Prove that $\sup(A) \leq \inf(B)$

Proof. Part (a) shows that $\sup(A)$ is a lower bound for B, so $\sup(A) \leq \inf(B)$. □