Contents

Math 350 - Advanced Calculus
Homework 5

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Problem 1

(i) $\lim_{n\to\infty}(\sqrt[8]{n^2+1}-\sqrt[8]{n^2})=0$

Proof. Let $u=\sqrt[4]{n^2+1}, v=\sqrt[4]{n^2}$, then the limit becomes:

$$\lim_{n\to\infty}(\sqrt{u}+\sqrt{v})=\lim_{n\to\infty}\frac{u-v}{\sqrt{u}+\sqrt{v}}=\lim_{n\to\infty}\frac{\sqrt{n^2+1}-\sqrt{n^2}}{\sqrt[8]{n^2+1}+\sqrt[8]{n^2}}$$

Since we already knew $\lim_{n\to\infty}(\sqrt{n^2+1}-\sqrt{n^2})=0$, we must have $\lim_{n\to\infty}(\sqrt[8]{n^2+1}-\sqrt[8]{n^2})=0$ ☐

(ii) $\lim_{n\to\infty}(\sqrt[8]{n^2+1}-\sqrt[4]{n+1})=0$

Proof. We have,

$$\begin{aligned}\sqrt[8]{n^2+1}-\sqrt[4]{n+1} &= \sqrt[8]{n^2+1}-\sqrt[8]{n^2}+\sqrt[8]{n^2}-\sqrt[4]{n+1}\\ &= \underbrace{\sqrt[8]{n^2+1}-\sqrt[8]{n^2}}_{M}+\underbrace{\sqrt[4]{n}-\sqrt[4]{n+1}}_{N}\end{aligned}$$

From part (i), we already knew that $\lim_{n\to\infty}M=0$, thus it remains to show that $\lim_{n\to\infty}N=0$. Again, we can use the same technique from part (i) by letting $u=\sqrt{n}, v=\sqrt{n+1}$ to obtain,

$$\lim_{n\to\infty}\frac{u-v}{\sqrt{u}+\sqrt{v}}=\lim_{n\to\infty}\frac{n-(n+1)}{\sqrt[4]{n}+\sqrt[4]{n+1}}=\lim_{n\to\infty}\frac{-1}{\sqrt[4]{n}+\sqrt[4]{n+1}}$$

Apply Sandwich's lemma to

$$\frac{1}{n+n+1}\leq\frac{-1}{\sqrt[4]{n}+\sqrt[4]{n+1}}\leq\frac{-1}{\sqrt[4]{n}}$$

we then have $\lim_{n\to\infty}(\sqrt[8]{n^2+1}-\sqrt[4]{n+1})=0$ ☐

Problem 2 Prove that if $a,b\geq 0$ then $\lim_{n\to\infty}\sqrt[n]{a^n+b^n}=\max(a,b)$

Proof. Without loss of generality, assume $\max(a,b)=a$. Since $a,b\geq 0$, we have that:

$$\begin{aligned}&\sqrt[n]{a^n}\leq\sqrt[n]{a^n+b^n}\leq\sqrt[n]{2a^n}\\ \Leftrightarrow\quad & a\leq\sqrt[n]{a^n+b^n}\leq\sqrt[n]{2}a\end{aligned}$$

where $\lim_{n\to\infty} a = a$ and $\lim_{n\to\infty} \sqrt[n]{2} = 1$ (from homework 4). Apply "Algebra's of Limits" Theorem,

$$\lim_{n\to\infty} (\sqrt[n]{2} \cdot a) = \lim_{n\to\infty} \sqrt[n]{2} \cdot \lim_{n\to\infty} a = 1 \cdot a = a$$

By Sandwich's Lemma, we must have

$$\lim_{n\to\infty} \sqrt[n]{a^n + b^n} = a = \max(a, b)$$

$\square$

Problem 3

(i) If $r_n = \frac{a_{n+1}}{a_n}$, then prove that $r_{n+1} = 1 + \frac{1}{r_n}$

Proof. Consider $r_{n+1} = \frac{a_{n+2}}{a_{n+1}} = \frac{a_{n+1} + a_n}{a_{n+1}} = 1 + \frac{a_n}{a_{n+1}} = 1 + \frac{1}{r_n}$ $\square$

(ii) Let (a_n) be the Fibonacci sequence, $a_0 = a_1 = 1, a_{n+2} = a_n + a_{n+1}$.

Proof. To have a better understanding of the sequence (r_n), we can write a quick program in C++,

```cpp
#include <iostream>
#include <vector>
#include <algorithm>

using namespace std;

void fibonacci_ratio(int N) {
        int a0 = 1;
        int a1 = 1;
        int a2 = a0 + a1;
        for (int i = 0; i < N; ++i) {
                a2 = a0 + a1;
                cout << i << " : " << static_cast<double>(a2)/(a1) << endl;
                a0 = a1;
                a1 = a2;
        }
}

int main() {
        fibonacci_ratio(20);
        return 0;
}
```

The outputs are:

As we can see the the even terms are decreasing and the odd terms are increasing, but both of them converges to 1.61803. To prove that this sequence is converge, we first prove that it's bounded by 2, then we prove that one subsequence is decreasing, the other is increasing. Apply Monotone Theorem, we can conclude that the sequence converges.

First we prove that $r_n \leq 2$ for all n. We have $a_{n-1} \leq a_n$ for all n, hence $a_{n-1} + a_n \leq a_n + a_n \Leftrightarrow a_n \leq 2a_n$. Thus

$$r_n = \frac{a_{n+1}}{a_n} \leq \frac{2a_n}{a_n} = 2$$

Next we will show that $r_n \leq r_{n+2}$ and $r_{n+1} \geq r_{n+3}$ by induction on n. For base cases, we have:

$$r_0 = \frac{a_1}{a_0} = 1, r_1 = \frac{a_2}{a_1} = 2, r_2 = \frac{a_3}{a_2} = \frac{3}{2}, r_3 = \frac{a_4}{a_3} = \frac{5}{3}$$

So $r_0 < r_2$ and $r_1 > r_3$ are true. Now suppose that $r_n \leq r_{n+2}$ and $r_{n+1} \geq r_{n+3}$, we have

$$r_{n+2} = 1 + \frac{1}{r_{n+1}} \leq 1 + \frac{1}{r_{n+3}} = r_{n+4} \text{ since } r_{n+1} \geq r_{n+3}$$

$$r_{n+3} = 1 + \frac{1}{r_{n+2}} \geq 1 + \frac{1}{r_{n+4}} = r_{n+5} \text{ since } r_{n+2} \leq r_{n+4}$$

Hence, we have one sequence is increasing and one sequence is decreasing, and both are bounded which implies they both converges. In addition, $(r_{\text{odd}}) \cup (r_{\text{even}}) = (r_n)$, so the limit exists. To find this limit, suppose that

$$\lim_{n\to\infty} r_n = r$$

where r is the limit that we are looking for. By property of limit, we have

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} r_{n+1} = r$$

In addition, from part (i) we know that $r_{n+1} = 1 + \frac{1}{r_n}$, so we have $\lim_{n\to\infty} r_{n+1} = \lim_{n\to\infty} \left(1 + \frac{1}{r_n}\right) = 1 + \lim_{n\to\infty} \frac{1}{r_n} = 1 + \frac{1}{r} = r$. So if there is solution to the equation $1 + \frac{1}{r} = r$, then the limit exists. Indeed,

$$\begin{aligned} & 1 + \frac{1}{r} = r \\ \Leftrightarrow\ & r^2 - r - 1 = 0 \\ \Leftrightarrow\ & r = \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

On the other hand, since $a_n > 0$ for all n, $r_n = \dfrac{a_{n+1}}{a_n} > 0$ which implies $r > 0$. Thus

$$r = \frac{1 + \sqrt{5}}{2}$$

$\square$

Problem 4 Prove that the sequence $a_0, a_1, a_2 \dots$ converges to a if and only if the sequence $a_0, a, a_1, a, a_2, a, a_3, \dots$ converges.

Proof. Since this is if only if proof, we have two cases:

- $\Rightarrow$:
  Since $a_0, a_1, a_2, \dots$ converges to a, by definition of limit, for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $n > N$, then $|a_n - a| < \epsilon$. Now consider the sub sequence
  $$a, a, a, a, \dots$$
  We have that $|a - a| < \epsilon, \ \forall \epsilon > 0$, thus $a, a, a, a \dots$ also converges to a. But $\{a_0, a_1, a_2, a_3, \dots\} \cup \{a, a, a, a, \dots\} = \{a_0, a, a_1, a, a_2, a, a_3, \dots\}$. Hence, $a_0, a, a_1, a, a_2, a, a_3, \dots$ converges to a.

- $\Leftarrow$:
  Suppose that $\langle a_0, a, a_1, a, a_2, a, a_3, \dots \rangle$ converges to L, $L \neq \pm\infty$, by definition of limit, for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $n > N$, then $|a_n - L| < \epsilon$, thus there must be a sequence
  $$\langle a_{N+1}, a, a_{N+2}, a, a_{N+3}, a, a_{N+4}, \dots \rangle$$
  that is getting closer and closer to L. But there is always an alternating a between each a_i and a_{i+1}, so $L = a$ otherwise $|a_n - L| < \epsilon$ would make no sense by the choice of $\epsilon = |a - L|$. Hence,
  $$\langle a_0, a, a_1, a, a_2, a, a_3 \dots \rangle$$
  converges to a. Next we will show that $\langle a_0, a_1, a_2, \dots \rangle$ converges to a as well. So far we have, for every $\epsilon > 0$, there exits a $N \in \mathbb{N}$ such that for all $n > N$, $|a_n - a| < \epsilon$.

$\square$

Alternative proof

Proof. Choosing N wisely, we actually can prove it much easier.

- $\Rightarrow$:
  Since $\langle a_0, a_1, a_2, \dots \rangle$ converges to a, by definition of limit, for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $n > N$, then $|a_n - a| < \epsilon$. Now consider the sequence,
  $$(b_n) = \langle a_0, a, a_1, a, a_2, a, a_3, \dots \rangle$$
  For every $n > 2N + 1$, $|a_n - a| < \epsilon$, thus (b_n) converges to a.

- $\Leftarrow$:
  Conversely, if (b_n) converges, then all subsequences must converge to the same limit. Since $\langle a, a, a, \dots \rangle$ converges to a, so does $\langle a_0, a_1, a_2, \dots \rangle$.

$\square$

Problem 5 Prove that $\lim\limits_{n \to \infty} a_n = a$ then set of the numbers $A = \{a, a_0, a_1, a_2, \dots\}$ is closed.

Proof. Recall

Theorem 7.1. *A number a is a limit point of the set $S \subset \mathbf{R}$ if and only there is a sequence of points $x_n \in S$ such that $x_n \neq a$ and $x_n \to a$.*

Since $\lim\limits_{n \to \infty} a_n = a$, there exists a subsequence $b_{i_1}, b_{i_2}, b_{i_3} \dots$ such that $b_n \to a$ and $b_n \neq a$ which implies a is a limit point of the set $B = \{b_{i_1}, b_{i_2}, b_{i_3} \dots\}$. On the other hand, by definition of closed set, we want to show that $A' \subset A$, so the remaining part is to find A'. Since $A = B \cup C$, where C are finitely many that do not lie in B, thus C is either the set that contains at least one a because B does not contains a. Thus $A' \subset A$ which implies A is closed.

$\square$

Problem 6 Let $a_0, a_1, a_2, a_3 \dots$ be a bounded sequence of real numbers, and let

$$b_n = \sup\{a_n, a_{n+1}, a_{n+2}, \dots\}$$

(i) Prove that the sequence $b_0, b_1, b_2, b_3 \dots$ converges. The limit $\lim_{n\to\infty} b_n$ is denoted by $\lim_{n\to\infty} \sup(a_n)$.

(ii) Find $\lim_{n\to\infty} \sup(a_n)$ for each of the following:

(a) $a_n = \dfrac{1}{n+1}$

(b) $a_n = (-1)^n \dfrac{1}{n+1}$

(c) $a_n = (-1)^n \dfrac{n}{n+1}$

(i) First note that, in general if $A < B$ then $\sup(B) \geq \sup(A)$. Since (a_n) is bounded, it's bounded above and below, thus the sup exists. Now consider $b_n = \sup\{a_n, a_{n+1}, a_{n+2} \dots\}$, if we look at it closed enough we see that it is just the least upper bound of the subsequence of (a_n) starting from index n. Hence, $\{a_{n+1}, a_{n+2}, a_{n+3}, \dots\} \subset \{a_n, a_{n+1}, a_{n+2}, a_{n+3}, \dots\}$ which implies b_{n+1} is an upper bound for b_n, thus

$$b_n \geq b_{n+1}$$

for all n which implies (b_n) is a monotonic decreasing sequence.
Next we will show that a bounded sequence and decreasing will converge. Consider the sequence (b_n). Since (a_n) is bounded, we must have (b_n) bounded because b_n contains all least upper bounds of (a_n). Thus the $b = \inf(b_n)$ exists. On the other hand, for all $\epsilon > 0$, $b + \epsilon > b$, so $b + \epsilon$ is not a lower bound for (b_n). Therefore there exists $N \in \mathbf{N}$ such that for all $n > N$, $b_n < b + \epsilon$. Also, since (b_n) is a decreasing sequence, $b_{n+1} \leq b_n$ for all n which implies

$$b - \epsilon < b \leq b_N \leq b_n < b + \epsilon \Leftrightarrow |b_n - b| < \epsilon \ \ ,\forall n > N$$

and this is precisely the definition of limit. Therefore (b_n) converges.

(ii) (a) $a_n = \dfrac{1}{n+1}$

As we can see, the limit of (a_n) is 0 as $n \to 0$. So the larger the index n, the closer the *sup* to 0. In fact, for a sequence starting from index $n+1$:

$$\{a_{n+1}, a_{n+2}, a_{n+3}, \dots\}$$

have the least upper bound is the first term of the sequence a_n because $\dfrac{1}{n+1} \geq \dfrac{1}{n+2} \geq \dfrac{1}{n+3} \geq \dots$.
Hence, in this case:

$$\lim_{n\to\infty} \sup(a_n) = \lim_{n\to\infty} a_n = 0$$

(b) $a_n = (-1)^n \dfrac{1}{n+1}$

This problem is slightly different from part (a) because of $(-1)^n$ which makes (a_n) alternate between negative and positive.

$$(a_n) = \left\{1, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}, \frac{1}{5}, \frac{-1}{6}, \frac{1}{7}, \frac{-1}{8}, \dots\right\}$$

Look at this sequence, we can see that the least upper bound of a subsequence is no longer the first element because it could be a negative term. However the idea remains the same, suppose that we start from index 3:

$$\left\{\frac{-1}{4}, \frac{1}{5}, \frac{-1}{6}, \frac{1}{7}, \frac{-1}{8}, \dots\right\}$$

then the $sup = \frac{1}{5}$. In addition, as we move further and further, the sup is getting smaller and smaller. In fact, it has the same limit as part (i), i.e.

$$\lim_{n\to\infty} \sup(a_n) = \lim_{n\to\infty} a_n = 0$$

(c) $a_n = (-1)^n \dfrac{n}{n+1}$
Let's ignore the $(-1)^n$ for a moment, and try to evaluate this limit. We have,

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \left(\frac{1}{1+1/n}\right) = 1$$

Hence, (a_n) is alternating between -1 and 1. In addition, (a_n) is increasing because

$$\frac{n}{n+1} \leq \frac{n+1}{n+2} \Leftrightarrow (n+1)^2 - n(n+2) = n^2 + 2n + 1 - n^2 - 2n = 1 > 0$$

Thus $a_n \leq a_{n+1}$ for all n, where $a_n \to 1$ so we can show the sup of (a_n) is 1. First we show that 1 is an upper bound for non-empty set $S = (a_n)$. Since $n < n+1 \Rightarrow \dfrac{n}{n+1} < 1$ for all n. Hence 1 is an upper bound for (a_n). Now suppose that there exists another upper bound $u = \dfrac{1}{n+1}$ for some integers n that is smaller than 1. But $\dfrac{1}{n+1} < \dfrac{n}{n+1}$ which is a contradiction. Therefore $\sup(a_n) = 1$. From part (i), we already show that b_n is a monotonic decreasing sequence and it converges. On the other hand since 1 is the least upper bound for (a_n), it is also the least upper bound for (a_{n+1}) because it is decreasing. Therefore

$$\lim_{n\to\infty} \sup(b_n) = 1$$

Problem 7 Find all the limit points of the set $S = \{\frac{1}{n} + \frac{1}{m} : m, n \in \mathbf{N}+\}$

Proof. First if both $m, n \to \infty$, we can prove that 0 is one limit point. Recall the definition of limit point:

Definition. *A number a is a limit point of a set $S \subset \mathbf{R}$ if for every $\epsilon > 0$ there is $x \in S$ such that $0 < |a - x| < \epsilon$, that is the set $S \cap (a - \epsilon, a + \epsilon) \setminus \{a\}$ is not empty.*

To show that 0 is a limit point we can choose $N = \max(m, n)$, and for all $\epsilon > 0$, we have $|0 - (1/N + 1/N)| < \epsilon$. Thus 0 is a limit point. Now if we fix m and increase n, we will show that $\frac{1}{m}$ is also a fix point. Consider the set:

$$S_1 = \{1/m + 1/n : n \in \mathbf{N}\}$$

then for all $\epsilon > 0$, $S_1 \cap (1/m - \epsilon, 1/m + \epsilon) \setminus \{1/m\} \neq \emptyset$. Thus $\frac{1}{m}$ is also a limit point. In fact, without loss of generality, we can fix either m or n and let the other approaches ∞. Therefore, all limits points are:

$$S' = \{0, \frac{1}{n} \mid n \in \mathbf{Z}+\}$$

□

Problem 8 Let (a_n) be a sequence of numbers that is bounded and injective, that is $a_n \neq a_m$ if $n \neq m$.

(i) Prove that if a is the only limit point of the set $A = \{a_n \mid n \in \mathbf{N}\}$, then the sequence a_n converges and $\lim_{n\to\infty} a_n = a$.

Proof. Since a is the only limit point of $A = \{a_n \mid n \in \mathbf{N}\}$, by Theorem 7.1, there exists a sequence points $a_n \in A$ such that $a_n \neq a$ and $a_n \to a$. Also from the definition of limit point, there exists a $\epsilon > 0$, such that the interval $(a - \epsilon, a + \epsilon)$ contains all $a_n \in A$, except for some finite terms that are not in this interval. Let denote this outside sequence as

$$(b_n) = \{b_{i_1}, b_{i_2}, b_{i_3}, \ldots, b_{i_k}\}$$

Note that we can't assume the index is from $1 \to k$, because it's not necessary for there terms to be in a particular order. In other words, the index could be arbitrary as long as $b_n \in A$, and $b_n \notin (a - \epsilon, a + \epsilon)$. Furthermore, since a is the only limit point, there must be only two types of sequence, either $\in (a - \epsilon, a + \epsilon)$ or $\notin (a - \epsilon, a + \epsilon)$. Next let's $N = \max(i_1, i_2, i_3, \ldots, i_k)$ then $N + 1$ must be in $(l - \epsilon, l + \epsilon)$. Hence for all $n \geq N + 1$ $a_n \in (a - \epsilon, a + \epsilon)$ which implies for all $\epsilon > 0$, there exists $N + 1 = M \in \mathbf{N}$ such that $|a_n - a| < \epsilon$ for all $n \geq M$ which is precisely the definition of limit. Therefore $\lim\limits_{n\to\infty} a_n = a$. ☐

Another way to prove it,

Proof. Suppose a_n does not converges to a, then there is $\epsilon > 0$ such that for every natural number k, there is $n_k > k$ such that $|a_{n_k} - a| \geq \epsilon$. The sequence (a_{n_k}) is a subsequence of (a_n) thus is bounded. Therefore it has a subsequence $(a_{n_{kl}})$ which converges to a point $b \neq a$ because $|a_{n_{kl}} - a| > \epsilon$. In addition, the sequence (a_n) is one-to-one, all elements of this sequence are distinct and therefore b is a limit point of (a_n). ☐

(ii) Show by a counterexample that this property is not true for unbounded sequences.

Proof. A counter example could be

$$\{1, 1/2, 2, 1/3, 3, 1/4, 4, 1/5, \ldots\}$$

which have only one limit point 0 because $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$ and the sequence of all natural number is unbounded, thus does not converge. ☐

Problem 9 Prove that a set $S \subset \mathbb{R}$ is bounded if and only if every sequence of points in S has a convergent subsequence.

Proof. Assume that $S \subset \mathbb{R}$ is bounded. If (x_n) is a sequence in S , then (x_n) is bounded and thus it has a convergent subsequence. Assume that S is not bounded, then for any integer n there is an x_n in S such that $|x_n| > n$. Since any convergent sequence is bounded, the sequence (x_n) cannot have a convergent subsequence. ☐